while the electric force acting on Q is

$$\vec{F}_e = Q\vec{E} \qquad (\text{coul}\,)\left(\frac{\text{N}}{\text{coul}}\right) \tag{9.146}$$

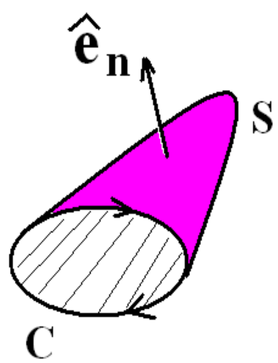
The total force acting on the moving charge Q is

$$\vec{F} = \vec{F}_m + \vec{F}_e = Q\,(\vec{E} + \vec{v}\times\vec{B}) \tag{9.147}$$

The magnetic forces can be summed over lines, surfaces or volumes. This gives rise to a one-dimensional, two-dimensional and three-dimensional representation for the magnetic force. In one-dimension the element of force acting on an element of wire is $d\vec{F}_m = (\vec{v}\times\vec{B})\,\rho_\ell\,ds$ where ρ_ℓ is the charge density per unit length (coul/m) and ds is an element of arc length. In two-dimensions the element of force acting on a surface is $d\vec{F}_m = (\vec{v}\times\vec{B})\,\rho_S\,d\sigma$ where ρ_S is the charge density per unit area (coul/m^2) and $d\sigma$ is an element of area. In three-dimensions the element of force acting within a volume is $d\vec{F}_m = (\vec{v}\times\vec{B})\,\rho_V\,dV$ where ρ_V is the charge density per unit volume (coul/m^3) and dV is an element of volume. An integration gives the total magnetic force as

$$\begin{aligned}
\vec{F}_m &= \int (\vec{v}\times\vec{B})\,\rho_\ell\,ds \quad \text{one-dimension}\\
\vec{F}_m &= \iint (\vec{v}\times\vec{B})\,\rho_S\,d\sigma \quad \text{two-dimension}\\
\vec{F}_m &= \iiint (\vec{v}\times\vec{B})\,\rho_V\,dV \quad \text{three-dimension}
\end{aligned} \tag{9.148}$$

Example 9-11. **The Maxwell-Faraday Equation**



Faraday's law[10] of induction investigates the magnetic flux $\iint_S \vec{B}\cdot d\vec{S}$ across a surface[11] S determined by a simple closed curve C. Think of a simple closed curve in space drawn on a sheet of rubber and then hold the simple closed curve fixed but deform the rubber surface into any kind of continuous surface S having C for its boundary. The direction of the unit normal $\hat{\mathbf{e}}_n$ to the surface S is determined by the right-hand rule of moving the fingers of the right

[10] Michael Faraday (1791-1867) English physicist who studied electricity and magnetism.
[11] Think of a rubber sheet across C and then deform the sheet to form the surface S.

hand in the direction around C so that the thumb points in the direction of the normal. Faraday's law, **obtained experimentally**, states that the line integral of the electric field around the closed curve C equals the negative of the time rate of change of the magnetic flux. This law can be written

$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{S} \tag{9.149}$$

Here $\vec{E}$ is the electric field, $\vec{r}$ is the position vector defining the closed curve C, $\vec{B}$ is the magnetic field and $d\vec{S} = \hat{\mathbf{e}}_n\, d\sigma$ is a vector element of area on the surface S. The Faraday law, given by equation (9.149), assumes the curve C and surface S are fixed and do not change with time. The left-hand side of equation (9.149) is the work done in moving around the curve C within the electric field $\vec{E}$. The right-hand side of equation (9.149) is the negative time rate of change of the magnetic flux across the surface S. One can employ Stokes theorem and express equation (9.149) in the form

$$\iint_S \nabla\times\vec{E}\cdot d\vec{S} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \qquad \text{or} \qquad \iint_S \left[\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right]\cdot d\vec{S} = 0 \tag{9.150}$$

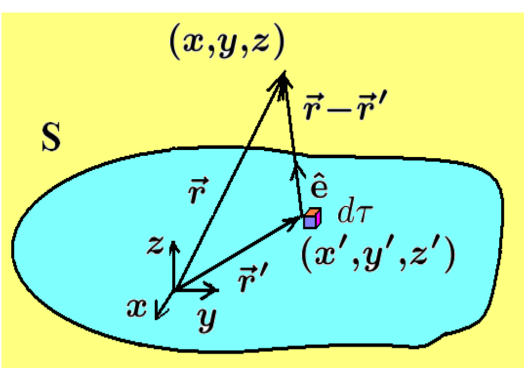
The equation (9.150) holds for all arbitrary surfaces S and consequently the integrand must equal zero giving the Maxwell-Faraday equation

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{9.152}$$

which is the second Maxwell equation.

■

Example 9-12. **The Biot-Savart law**



Consider a volume V enclosed by a surface S as illustrated and let $\vec{J} = \vec{J}(x',y',z')$ denote the current density within this volume. Let (x',y',z') denote a point inside V where an element of volume $dV = dx'dy'dz'$ is constructed. In addition, construct the vectors $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ to a general point (x,y,z) outside of the volume V and

$\vec{r}' = x'\,\hat{\mathbf{e}}_1 + y'\,\hat{\mathbf{e}}_2 + z'\,\hat{\mathbf{e}}_3$ to the point (x', y', z') inside the volume V. The vector $\vec{r} - \vec{r}'$ then points from the point (x', y', z') to the point (x, y, z). The vector $\hat{\mathbf{e}} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$ is a unit vector in this direction as illustrated in the accompanying figure.

The magnetic field $\vec{B} = \vec{B}(x, y, z)$ at the point (x, y, z) due to a current density $\vec{J} = \vec{J}(x', y', z')$ inside V is given by the Biot[12]-Savart[13] law

$$\vec{B} = \vec{B}(x, y, z) = \frac{\mu_0}{4\pi} \iiint\limits_V \nabla \cdot \left[\frac{\vec{J}(x', y', z') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] dx'dy'dz' \tag{9.152}$$

The divergence of this magnetic field is determined by calculating

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \iiint\limits_V \nabla \cdot \left[\frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] dV \tag{9.153}$$

In order to evaluate the divergence as given by equation (9.153) one can employ the vector identities

$$\begin{aligned} \nabla \times (f\vec{A}) &= (\nabla f) \times \vec{A} + f(\nabla \times \vec{A}) \\ \nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \end{aligned} \tag{9.154}$$

Let $\vec{B} = \dfrac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$ and $\vec{A} = \vec{J}$ along with the second of the equations (9.154) to show

$$\nabla \cdot (\vec{J} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot (\nabla \times \vec{B}) = -\vec{J} \cdot (\nabla \times \vec{B}) \tag{9.155}$$

This holds because $\vec{J}$ is a function of the primed coordinates and ∇ involves differentiation with respect to the unprimed coordinates so that $\nabla \times \vec{J}$ is zero. Using the first equation in (9.154) with $f = \dfrac{1}{|\vec{r} - \vec{r}'|^3}$ one can write

$$\nabla \times \vec{B} = \nabla \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{1}{|\vec{r} - \vec{r}'|^3} \nabla \times (\vec{r} - \vec{r}') - (\vec{r} - \vec{r}') \times \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \right) \tag{9.156}$$

One can verify that

$$\nabla \times (\vec{r} - \vec{r}') = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - x') & (y - y') & (z - z') \end{vmatrix} = \vec{0}$$

and if $f = |\vec{r} - \vec{r}'|^{-3}$, then $\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_3$ where

$$\frac{\partial f}{\partial x} = -3|\vec{r} - \vec{r}'|^{-4} \frac{\partial}{\partial x} \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \frac{-3(x - x')}{|\vec{r} - \vec{r}'|^5}$$

[12] Jean Baptiste Biot (1774-1862) A French mathematician.

[13] F elix Savart (1791-1841) A French physician who studied physics.

In a similar fashion one can verify that

$$\frac{\partial f}{\partial y} = \frac{-3(y-y')}{|\vec{r}-\vec{r}'|^5} \qquad \text{and} \qquad \frac{\partial f}{\partial z} = \frac{-3(z-z')}{|\vec{r}-\vec{r}'|^5}$$

Using the above results verify that

$$\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|^3}\right) = \frac{-3}{|\vec{r}-\vec{r}'|^5}(\vec{r}-\vec{r}') \tag{9.157}$$

and show the right-hand side of equation (9.156) is zero because $(\vec{r}-\vec{r}')\times(\vec{r}-\vec{r}') = \vec{0}$. It then follows that

$$\nabla\cdot\vec{B} = 0$$

which is the third equation in the Maxwell's equations (9.134).

■

Example 9-13. If the charge density ρ (coul/m^3) moves with velocity $\vec{v}$ (m/s), then the current density is given by $\vec{J} = \rho\vec{v}$ (amp/m^2). Surround the current density field with a simple closed surface S which encloses a volume V. The flux across the surface S is then given by

$$\iint_S \vec{J}\cdot d\vec{S} = \iint_S \vec{J}\cdot\hat{\mathbf{e}}_n\,dS = \iint_V \nabla\cdot\vec{J}\,dV$$

where the divergence theorem of Gauss has been employed to express the flux surface integral with a volume integral. The charge must be conserved so that the flux out of the volume must be accounted for by the time rate of change of the charge density within the volume so that one can write

$$\iint_S \vec{J}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{J}\,dV = -\frac{d}{dt}\iiint_V \rho\,dV = -\iiint_V \frac{\partial\rho}{\partial t}\,dV$$

or

$$\iiint_V \left[\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right] dV = 0 \tag{9.158}$$

The equation (9.158) must hold for all volumes V and consequently one must require

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{9.159}$$

The equation (9.159) is know as **the continuity equation** for the charge density.

■

Example 9-14. The last Maxwell equation is hard to derive. Historically, Ampere[14] showed that for straight line currents the curl of the magnetic field was proportional to the volume current density $\vec{J}$ so that one could write

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Maxwell realized that this equation did not hold in general because it did not satisfy the property that the divergence of the curl must be zero. Based upon theoretical reasoning Maxwell came up with the modified equation

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{9.160}$$

where the term $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is known as Maxwell's term for Ampere's law. Taking the divergence of equation (9.160) one can show

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \nabla \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial \nabla \cdot \vec{E}}{\partial t} \qquad \text{Use the first Maxwell equation and show} \\
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \nabla \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial (\rho / \epsilon_0)}{\partial t} \\
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \left[\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right] = 0
\end{aligned}$$

where the continuity equation from the previous example has been employed to show the divergence of the curl is zero. The equation (9.160) is the last of the Maxwell equations from (9.134).

■

Example 9-15. If there are no charges or currents in space, then the Maxwell equations (9.134) simplify to the form

$$\begin{aligned}
\nabla \cdot \vec{E} =& 0 \\
\nabla \times \vec{E} =& -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} =& 0 \\
\nabla \times \vec{B} =& \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{9.161}$$

Use the property of the del operator that

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

[14] André Marie Ampère (1775-1836) A French physicist, chemist and mathematician.

and take the curl of the second and fourth of the Maxwell's equations to obtain

$$\begin{aligned}
\nabla \times (\nabla \times \vec{E}) =& \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\nabla \times \vec{B} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla \times (\nabla \times \vec{B}) =& \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\nabla \times \vec{E} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}
\end{aligned}$$

The first and third of the Maxwell equations require that $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ so that the vector fields $\vec{E}$ and $\vec{B}$ must satisfy the wave equations

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

Here the product $\mu_0 \epsilon_0 = \frac{1}{c^2}$, where $c = 3 \times (10)^{10}\,\text{cm/sec}$ is the speed of light.

■

Exercises

▶ **9-1.** Solve each of the one-dimensional Laplace equations

$$\begin{aligned}
\frac{d^2U}{dx^2} =& 0, \qquad U = U(x) \quad \text{rectangular} \\
\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = \frac{1}{r}\frac{d}{dr}\left(r \frac{dU}{dr} \right) =& 0, \qquad U = U(r) \quad \text{polar} \\
\frac{d^2U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dU}{d\rho} \right) =& 0, \qquad U = U(\rho) \quad \text{spherical}
\end{aligned} \tag{9.162}$$

▶ **9-2.** Verify that the velocity field $\vec{V} = V_0 \cos\alpha\, \hat{\mathbf{e}}_1 - V_0 \sin\alpha\, \hat{\mathbf{e}}_2$, $\quad V_0, \alpha \quad$ are constants is both irrotational and solenoidal. Find and sketch the velocity field, streamlines. Find the velocity potential. Note that for $\alpha = 0$ the flow is a parallel flow and for $\alpha = \frac{\pi}{2}$ the flow is a vertical flow.

▶ **9-3.** Verify that the velocity field $\vec{V} = 2x\, \hat{\mathbf{e}}_1 - 2y\, \hat{\mathbf{e}}_2$ is both irrotational and solenoidal. Find and sketch the vector field and the streamlines for $0 < x < 2$, $0 < y < 2$. Also find the velocity potential. The velocity field for this type of fluid motion can be used to describe the flow in the vicinity of a corner.

▶ **9-4.** For the velocity field $\vec{V} = 2y\, \hat{\mathbf{e}}_1 + 2x\, \hat{\mathbf{e}}_2$ find and sketch the vector field and streamlines. Find the velocity potential.

▶ **9-5.** Consider the vector field $\vec{E} = \frac{1}{r^2}\,\hat{\mathbf{e}}_r$ in polar coordinates. (a) Show this vector field is irrotational. (b) Find a potential function $\phi = \phi(r)$ satisfying $\phi(r_0) = 0$, where $r_0 > 0$.

▶ **9-6.** True or false, if both $\vec{A}$ and $\vec{B}$ are irrotational, then the vector $\vec{F} = \vec{A} \times \vec{B}$ is solenoidal.

▶ **9-7.** Show that if $\phi = \phi(x, y, z)$ is a solution of the Laplace equation $\nabla^2\phi = 0$, then (a) Show the vector $\vec{V} = \nabla\phi$ is irrotational. (b) Show the vector $\vec{V} = \nabla\phi$ is solenoidal.

▶ **9-8.**

(a) Show the velocity field $\vec{V} = \dfrac{kx}{x^2+y^2}\,\hat{\mathbf{e}}_1 + \dfrac{ky}{x^2+y^2}\,\hat{\mathbf{e}}_2$ is both irrotational and solenoidal and has the potential function $\Phi = \dfrac{k}{2}\ln(x^2+y^2)$ and the stream function $\Psi = k\tan^{-1}\dfrac{y}{x}$

(b) Show that $$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \quad \text{and} \quad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}$$

(c) Express the potential function and stream function in polar coordinates and sketch the equipotential curves and streamlines. This type of velocity field is said to correspond to a source at the origin if $k > 0$ or a sink at the origin if $k < 0$.

▶ **9-9.** Verify that the velocity field $\vec{V} = \dfrac{-ky}{x^2+y^2}\,\hat{\mathbf{e}}_1 + \dfrac{kx}{x^2+y^2}\,\hat{\mathbf{e}}_2$ is both irrotational and solenoidal. Find the potential and streamlines for this velocity field. This type of flow is termed a circulation about the origin of strength k.

▶ **9-10.** Sketch the field lines and analyze the vector fields defined by:

$$\begin{array}{llll}
(a) & \vec{F} = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 & (d) & \vec{F} = 2xy\,\hat{\mathbf{e}}_1 + (x^2 - y^2)\,\hat{\mathbf{e}}_2 \\
(b) & \vec{F} = y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 & (e) & \vec{F} = (x^2 + y^2)\,\hat{\mathbf{e}}_1 + 2xy\,\hat{\mathbf{e}}_2 \\
(c) & \vec{F} = a\,\hat{\mathbf{e}}_1 + b\,\hat{\mathbf{e}}_2 & (f) & \vec{F} = a\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2
\end{array}$$

▶ **9-11.** Show in polar coordinates that the Cauchy-Riemann equations can be expressed as
$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad \frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

▶ **9-12.** At all points (x, y) between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, the vector function $\vec{F} = \dfrac{-y\ \hat{\mathbf{e}}_1 + x\ \hat{\mathbf{e}}_2}{x^2 + y^2}$ is continuous and equals the gradient of the scalar function

$$\Phi(x, y) = \tan^{-1}\frac{y}{x}.$$

Show that $\displaystyle\int_{(-2,0)}^{(2,0)} \vec{F} \cdot d\vec{r}$ is not independent of the path of integration by computing this line integral along the upper half and then the lower half of the circle $x^2 + y^2 = 4$. Is the region of integration a simply-connected region?

▶ **9-13.** Find a vector potential for

$$(a) \quad \vec{F} = 2y\ \hat{\mathbf{e}}_1 + 2x\ \hat{\mathbf{e}}_2 \qquad\qquad (b) \quad \vec{F} = (x - y)\ \hat{\mathbf{e}}_1 - z\ \hat{\mathbf{e}}_3$$

▶ **9-14.** For the gravity field $\vec{F} = -mg\ \hat{\mathbf{e}}_3$

((a) Show that this vector field is irrotational.

(b) Find the potential function from which this field is derivable.

(c) Show that the work done in moving from a height h_1 to a height h_2 is the change in potential energy.

▶ **9-15.** (**Conservation of Energy**)

(a) If $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$ show that $\vec{F} \cdot \dfrac{d\vec{r}}{dt} = \dfrac{1}{2}m\dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right)^2$

(b) Show $\displaystyle\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F} \cdot d\vec{r} = \frac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)}^{(x,y,z)} = \frac{1}{2}mv^2\Big|_{(x,y,z)} - \frac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)}$

(c) If $\vec{F}$ is a conservative vector field such that $\vec{F} = -\nabla\phi$, show that

$$\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F} \cdot d\vec{r} = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} \nabla\phi \cdot d\vec{r} = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} d\phi = -\phi(x, y, z) + \phi(x_0, y_0, z_0)$$

(d) Show that $\phi(x_0, y_0, z_0) + \dfrac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)} = \phi(x, y, z) + \dfrac{1}{2}mv^2\Big|_{(x,y,z)}$ which states that for a conservative vector field the sum of the potential energy and kinetic energy at point (x_0, y_0, z_0) is the same as the sum of the potential energy and kinetic energy at the point (x, y, z).

▶ **9-16.** A conservative vector field has the family of equipotential curves

$$x^2 - y^2 = c.$$

Find the field lines and vector field associated with this potential.

▶ **9-17.** **(Research project on orbital motion)**

Assume a mass m located at a position $\vec{r} = r\,\hat{\mathbf{e}}_r$ experiences a central force $\vec{F} = mf(r)\,\hat{\mathbf{e}}_r$

(a) Show the equation of motion is given by $m\dfrac{d^2\vec{r}}{dt^2} = \vec{F}$ which can be written in the form $\dfrac{d\vec{v}}{dt} = f(r)\,\hat{\mathbf{e}}_r$

(b) Show that $\vec{r} \times \vec{v} = \vec{h}$ is a constant.

(c) Show the motion of m is in a plane and that the mass m sweeps out an area at a constant rate. (Kepler's law of areas).

(d) Show that in the special case $mf(r) = -\dfrac{GmM}{r^2}$ the mass m is attracted toward mass M, assumed to be at the origin, and $\dfrac{d\vec{v}}{dt} = -\dfrac{k}{r^2}\,\hat{\mathbf{e}}_r$ where $k = GM$ is a constant.

(e) Show that $\vec{h} = r^2\,\hat{\mathbf{e}}_r \times \dfrac{d\,\hat{\mathbf{e}}_r}{dt}$, $\dfrac{d\vec{v}}{dt} \times \vec{h} = k\dfrac{d\,\hat{\mathbf{e}}_r}{dt}$ and $\vec{v} \times \vec{h} = k(\,\hat{\mathbf{e}}_r + \vec{\epsilon})$ where $\vec{\epsilon}$ is a constant vector.

(f) Use the results from part (e) and show

$$\vec{r} \times \vec{v} \cdot \vec{h} = h^2 \quad \text{and} \quad \vec{r} \cdot \vec{v} \times \vec{h} = kr(1 + \epsilon\,\cos\theta)$$

where θ is the angle between $\vec{\epsilon}$ and $\vec{r}$, and consequently

$$r = \frac{\alpha}{1 + \epsilon\cos\theta}, \quad \text{where} \quad \alpha = \frac{h^2}{k}$$

Note r describes a conic section having eccentricity ϵ.

(f) When $\epsilon < 1$ show an ellipse results with m having an orbital period

$$T = \frac{\text{area of ellipse}}{h/2} = \frac{2\pi}{\sqrt{k}}a^{3/2}, \quad \text{where} \quad \frac{T^2}{a^3} = \frac{4\pi^2}{k}$$

This is known as Kepler's third law.

▶ **9-18.** (a) Find the potential associated with the conservative vector field

$$\vec{F} = (y^2\cos x + z^3)\ \hat{\mathbf{e}}_1 + (2y\sin x - 4)\ \hat{\mathbf{e}}_2 + 3xz^2\ \hat{\mathbf{e}}_3$$

(b) Find the differential equation which describes the field lines.

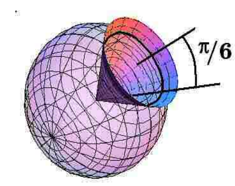
▶ **9-19.** Show that the vector field

$$\vec{F} = (2xyz + y)\ \hat{\mathbf{e}}_1 + (x^2z + x)\ \hat{\mathbf{e}}_2 + x^2y\ \hat{\mathbf{e}}_3$$

is conservative and find its scalar potential.

► **9-20.**

A right circular cone intersects a sphere of radius r as illustrated. Find the solid angle subtended by this cone.



Right circular cone intersecting sphere.

► **9-21.** Evaluate $\iint_S \vec{r}\cdot d\vec{S}$, where S is a closed surface having a volume V.

► **9-22.** In the divergence theorem $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS$ let $\vec{F} = \phi(x,y,z)\,\vec{C}$ where $\vec{C}$ is a nonzero constant vector and show $\iiint_V \nabla\phi\,dV = \iint_S \phi\,\hat{\mathbf{e}}_n\,dS$

► **9-23.** Assume $\vec{F}$ is both solenoidal and irrotational so that $\vec{F}$ is the gradient of a scalar function Φ (a) Show Φ is a solution of Laplace's equation and (b) Show the integral of the normal derivative of Φ over any closed surface must equal zero.

► **9-24.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and show $\nabla|\vec{r}|^\nu = \nu\,|\vec{r}|^{\nu-2}\vec{r} = \nu\,|\vec{r}|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}}$ where $\hat{\mathbf{e}}_{\vec{r}} = \dfrac{\vec{r}}{|\vec{r}|}$ is a unit vector in the direction $\vec{r}$.

► **9-25.** For $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ show that

$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial x} = \frac{x-x_0}{|\vec{r}-\vec{r}_0|}$$
$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial y} = \frac{y-y_0}{|\vec{r}-\vec{r}_0|}$$
$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial z} = \frac{z-z_0}{|\vec{r}-\vec{r}_0|}$$

► **9-26.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to the variable point (x,y,z) and let $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_2 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ denote the position vector to the fixed point (x_0,y_0,z_0).

(a) Show $\nabla|\vec{r}-\vec{r}_0|^\nu = \nu\,|\vec{r}-\vec{r}_0|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}-\vec{r}_0}$ where $\hat{\mathbf{e}}_{\vec{r}-\vec{r}_0}$ is a unit vector in the direction $\vec{r}-\vec{r}_0$.

(b) Show $\nabla^2|\vec{r}-\vec{r}_0|^\nu = \nu(\nu+1)|\vec{r}-\vec{r}_0|^{\nu-2}$

(c) Write out the results from part (b) in the special cases $\nu = -1$ and $\nu = 2$.

▶ **9-27.** In thermodynamics the internal energy U of a gas is a function of pressure P and volume V denoted by $U = U(P,V)$. If a gas is involved in a process where the pressure and volume change with time, then this process can be described by a curve called a P-V diagram of the process. Let $Q = Q(t)$ denote the amount of heat obtained by the gas during the process. From the first law of thermodynamics which states that $dQ = dU + P dV$, show that $dQ = \frac{\partial U}{\partial P}\, dP + \left[\frac{\partial U}{\partial V} + P\right] dV$ and determine whether the line integral $\int_{t_0}^{t_1} dQ$, which represents the heat received during a time interval Δt, is independent of the path of integration or dependent upon the path of integration.

▶ **9-28.**

(a) If x and y are **independent** variables and you are given an equation of the form $F(x) = G(y)$ for all values of x and y what can you conclude if (i) x varies and y is constant and (ii) y varies but x is constant.

(b) Assume a solution to Laplace's equation $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ in Cartesian coordinates of the form $\phi = X(x)Y(y)$, where the variables are separated. If the variables x and y are **independent** show that there results two linear differential equations
$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda,$$
where λ is termed a separation constant.

▶ **9-29.** Evaluate the line integral $I = \int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = yz\,\hat{\mathbf{e}}_1 + xz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ and C is the curve $\vec{r} = \vec{r}(t) = \cos t\ \hat{\mathbf{e}}_1 + (\frac{t}{\pi} + \sin t\,)\,\hat{\mathbf{e}}_2 + \frac{3t}{\pi}\,\hat{\mathbf{e}}_3$ between the points $(1,0,0)$ and $(-1,1,3)$.

▶ **9-30.** A particle moves along the x-axis subject to a restoring force $-Kx$. Find the potential energy and law of conservation of energy for this type of motion.

▶ **9-31.** Evaluate the line integral
$$I = \int_K (2x + y)\,dx + x\,dy, \qquad \text{where } K \text{ consists of straight line segments}$$
$\overline{OA} + \overline{AB} + \overline{BC}$ connecting the points $O(0,0)$, $A(3,3)$, $B(5,-1)$ and $C(7,5)$.

► **9-32.** The problems below are concerned with obtaining a solution of Laplace's equation for temperature T. Chose an appropriate coordinate system and make necessary assumptions about the solution in order to reduce the problem to a one-dimensional Laplace equation.

(a) Find the steady-state temperature distribution along a bar of length L assuming that the sides of the bar are insulated and the ends are kept at temperatures T_0 and T_1. This corresponds to solving $\dfrac{d^2T}{dx^2} = 0$, $T(0) = T_0$ and $T(L) = T_1$.

(b) Find the steady-state temperature distribution in a circular pipe where the inside of the pipe has radius r_1 and temperature T_1, and the outside of the pipe has a radius r_2 and is maintained at a temperature T_2. This corresponds to solving $\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = 0$ such that $T(r_1) = T_1$ and $T(r_2) = T_2$

(c) Find the steady-state temperature distribution between two concentric spheres of radii ρ_1 and ρ_2, if the surface of the inner sphere is maintained at a temperature T_1, whereas the outer sphere is maintained at a temperature T_2. This corresponds to solving $\dfrac{1}{\rho^2}\dfrac{d}{d\rho}\left(\rho^2\dfrac{dT}{d\rho}\right) = 0$ such that $T(\rho_1) = T_1$ and $T(\rho_2) = T_2$.

(d) Find the steady-state temperature distribution between two infinite and parallel plates $z = z_1$ and $z = z_2$ maintained, respectively, at temperatures of T_1 and T_2.

► **9-33.** Find the potential function associated with the conservative vector field

$$\vec{F} = 6xz\,\hat{\mathbf{e}}_1 + 8y\,\hat{\mathbf{e}}_2 + 3x^2\,\hat{\mathbf{e}}_3.$$

► **9-34.** Newton's law of attraction states that two particles of masses m_1 and m_2 attract each other with a force which acts in the direction of the line joining the two masses and whose magnitude is given by $F = Gm_1m_2/r^2$, where r is the distance between the masses and G is a universal constant.

(a) If mass m_1 is at the origin and mass m_2 is at a point (x, y, z), find the vector force of attraction of mass m_1 on mass m_2.

(b) If mass m_1 is at a fixed point $P_1(x_1, y_1, z_1)$ and mass m_2 is at the point (x, y, z), find the vector force of attraction of mass m_1 on mass m_2.

► **9-35.** Show that $u = u(x,t) = f(x-ct)+g(x+ct)$, f, g arbitrary functions, is a solution of the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ Here f and g are wave shapes moving to the left and right.

▶ **9-36.** Express the Maxwell equations (9.161) as a system of partial differential equations.

▶ **9-37.** Assume solutions to the Maxwell equations (9.161) are waves moving in the x-direction only. This is accomplished by assuming exponential type solutions having the form $e^{i(kx-\omega t)}$ where i is an imaginary unit satisfying $i^2=-1$.

(a) Show that $\vec{E}=\vec{E}(x,t)=\vec{E}_0 e^{i(kx-\omega t)}$ and $\vec{B}=\vec{B}(x,t)=\vec{B}_0 e^{i(kx-\omega t)}$ are solutions of Maxwell's equations in this special case.

(b) Show that $\vec{B}_0=\dfrac{k}{\omega}(\,\hat{\mathbf{e}}_1\times\vec{E}_0)$

(c) Show that the waves for $\vec{E}$ and $\vec{B}$ are mutually perpendicular.

▶ **9-38.** Consider the following vector fields:

$\vec{B}$ a magnetic field intensity with units of amp/m

$\vec{E}$ an electrostatic intensity vector with units of volts/m

$\vec{Q}$ a heat flow vector with units of joules/cm² · sec

$\vec{V}$ a velocity vector with units of cm/sec

(a) Assign units of measurement to the following integrals and interpret the meanings of these integrals:

$$(a)\quad \iint_S \vec{E}\cdot d\vec{S} \qquad (b)\quad \iint_S \vec{Q}\cdot d\vec{S} \qquad (c)\quad \iint_S \vec{V}\cdot d\vec{S} \qquad (d)\quad \int_C \vec{B}\cdot d\vec{r}$$

(b) Assign units of measurements to the quantities:

$$(a)\quad \text{curl}\,\vec{H} \quad (b)\quad \text{div}\,\vec{E} \quad (c)\quad div\,\vec{Q} \quad (d)\quad \text{div}\,\vec{V}$$

▶ **9-39.** Solve each the following vector differential equations

$$(a)\quad \frac{d\vec{y}}{dt}=\hat{\mathbf{e}}_1 t+\hat{\mathbf{e}}_3\sin t \quad (b)\quad \frac{d^2\vec{y}}{dt^2}=\hat{\mathbf{e}}_1\sin t+\hat{\mathbf{e}}_2\cos t \quad (c)\quad \frac{d\vec{y}}{dt}=3\vec{y}+6\,\hat{\mathbf{e}}_3$$

▶ **9-40.** Solve the simultaneous vector differential equations $\dfrac{d\vec{y}_1}{dt}=\vec{y}_2,\qquad \dfrac{d\vec{y}_2}{dt}=-\vec{y}_1$

▶ **9-41.** A particle moves along the spiral $r=r(\theta)=r_0e^{\theta\cot\alpha}$, where r_0 and α are constants. If $\theta=\theta(t)$ is such that $\dfrac{d\theta}{dt}=\omega=constant$, find the components of velocity in the direction $\vec{r}$ and in the direction perpendicular to $\vec{r}$.

▶ **9-42.** Couloumb's law states that the force between two charges q_1 and q_2 acts along the line joining the two charges and the magnitude of the force varies directly to the product of charges and inversely as the square of the distance r between the charges. Symbolically the magnitude of this force is $F = q_1 q_2/r^2$ in the appropriate system of units[15]. A charge Q is called a test charge if it is located at a variable point (x, y, z) and experiences a force from another charge q_1, located at a fixed point. The ratio of the force experienced by the test charge to the magnitude of the test charge is called the electrostatic intensity $\vec{E}$ at the point (x, y, z) and is given by $\vec{E} = \frac{\vec{F}}{Q}$. An equivalent statement is that a unit charge has been placed at the point P and the electrostatic intensity is the total force which acts on this test charge.

(a) Show that a charge q located at the origin produces an electrostatic intensity $\vec{E}$ at a point (x, y, z) given by

$$\vec{E} = \frac{q\vec{r}}{r^3}, \text{ where } r = \mid \vec{r} \mid \text{ and } \quad \vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3.$$

(b) Show that (i) $\vec{E} = -\nabla \frac{q}{r}$ (ii) $\vec{E}$ has the scalar potential $\phi = \frac{q}{r}$ (iii) $\operatorname{curl} \vec{E} = 0$ and (iv) $\operatorname{div} \vec{E} = -q\nabla^2 \frac{1}{r} = 0$

c) Let q_1 denote a fixed charge located at the point $P_1(x_1, y_1, z_1)$ and let q_2 denote another fixed charge located at the point $P_2(x_2, y_2, z_2)$. Show the electrostatic intensity on a test charge at (x, y, z) is given by

$$\vec{E} = -\frac{q_1\vec{r}_1}{r_1^2} - \frac{q_2\vec{r}_2}{r_2^3},$$

where $r_i = \mid r_i \mid$ for $i = 1, 2$, and $\vec{r}_i = (x_i - x)\,\hat{\mathbf{e}}_1 + (y_i - y)\,\hat{\mathbf{e}}_2 + (z_i - z)\,\hat{\mathbf{e}}_3$.

(d) Show for n charges q_i, $i = 1, 2, \dots, n$, located respectively at the points $P_i(x_i, y_i, z_i)$, for $i = 1, 2, \dots, n$, the electrostatic intensity at a general point (x, y, z) is given by

$$\vec{E} = -\sum_{i=1}^{n} \frac{q_i}{r_i^3}\vec{r}_i,$$

where $\vec{r}_i = (x_i - x)\,\hat{\mathbf{e}}_1 + (y_i - y)\,\hat{\mathbf{e}}_2 + (z_i - z)\,\hat{\mathbf{e}}_3$ and $r_i = \mid r_i \mid$ for $i = 1, 2, \dots, n$.

[15] If charges are measured in units of statcoulombs and distance is measured in centimeters, then the force has units of dynes.

Chapter 10
Matrix and Difference Calculus

The matrix calculus is used in the study of linear systems and systems of differential equations and occurs in engineering mathematics, physics, statistics, biology, chemistry and many other scientific applications. The difference calculus is used to study discrete events.

The Matrix Calculus

A matrix is **a rectangular array of numbers or functions** and can be expressed in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \tag{10.1}$$

where the quantities a_{ij} for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ are called the elements of the matrix. Here the double subscript notation a_{ij} is used to denote the element in the ith row and jth column. A matrix with m rows and n columns is called a m by n matrix and expressed in the form "A is a $m \times n$ matrix". Matrices are usually denoted using capital letters and whenever it is necessary to emphasize the elements and size of the matrix it is sometimes expressed in the form $A = (a_{ij})_{m\times n}$. The rows of the matrix A are called row vectors and the columns of the matrix A are called column vectors.

For a and b positive integers, then matrices of the form $R = (r_{a1}\ r_{a2} \dots r_{aj} \dots r_{an})$ are called **n-dimensional row vectors** and matrices of the form

$$C = \begin{pmatrix} c_{1b} \\ c_{2b} \\ \vdots \\ c_{ib} \\ \vdots \\ c_{mb} \end{pmatrix} = \text{col}\,(c_{1b}, c_{2b}, \dots, c_{ib}, \dots, c_{mb}) \tag{10.2}$$

are called **m-dimensional column vectors**. The column notation $\text{col}(c_{1b}, \dots, c_{mb})$ is used to conserve space in typesetting the m-dimensional column vector.

Properties of Matrices

1. Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ having the same dimension are equal if $a_{ij} = b_{ij}$ for all values of i and j. Equality is expressed $A = B$.
2. Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ of the same size can be added or subtracted and the resulting matrices are denoted

$$\begin{aligned} C =& A + B \quad \text{where} \quad C = (c_{ij})_{m\times n} \text{ with } c_{ij} = a_{ij} + b_{ij} \\ D =& A - B \quad \text{where} \quad D = (d_{ij})_{m\times n} \text{ with } d_{ij} = a_{ij} - b_{ij} \end{aligned}$$

   Here like elements are added or subtracted.
3. If the matrix $A = (a_{ij})_{m\times n}$ is multiplied by a scalar β, the resulting matrix is

$$\beta\, A = (\beta\, a_{ij})_{m\times n}$$

   That is, each component a_{ij} of A is multiplied by the scalar β.
4. Matrices of the same size obey the following laws.

$$\begin{aligned} A + B =& B + A \qquad \text{commutative law} \\ A + (B + C) =& (A + B) + C \qquad \text{associative law} \end{aligned}$$

   For α and β scalar quantities one can write the

$$\text{scalar distributive laws} \qquad \begin{cases} \alpha(A+B) = & \alpha A + \alpha B \\ (\alpha+\beta)A = & \alpha A + \beta A \\ \alpha\,(\beta A) = & (\alpha\beta)A \end{cases}$$

5. The zero matrix has all zeros for elements and can be expressed in one of the forms $[0]_{m\times n}$ or $[0]$ or $\underline{0}$ or $\tilde{0}$.

6. If the elements of the matrix A are functions of a single variable, say t, one can write $a_{ij} = a_{ij}(t)$ or $A = A(t) = (a_{ij}(t))$ to emphasize this fact, then the derivative of the matrix A is given by

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right) \tag{10.3}$$

   and the integral of the matrix A is

$$\int A(t)\,dt = \left(\int a_{ij}(t)\,dt\right) + C \tag{10.4}$$

   where C is a constant matrix of appropriate size. Here the derivative of a matrix is obtained by differentiating each element of the matrix and the integral of the matrix is obtained by integrating each element within the matrix and the constants of integration are collected into a constant matrix.

Example 10-1. Find the derivative and integral of the matrix $A = \begin{bmatrix} 1 & x \\ \sin x & e^{-2x} \end{bmatrix}$

Solution Taking the derivative of each element one finds

$$\frac{dA}{dx} = \begin{bmatrix} 0 & 1 \\ \cos x & -2e^{-2x} \end{bmatrix}$$

Taking the integral of each element one finds

$$\int A\,dt = \begin{bmatrix} x & \frac{1}{2}x^2 \\ -\cos x & -\frac{1}{2}e^{-2x} \end{bmatrix} + C$$

where $C = (c_{ij})_{2\times 2}$ is an arbitrary constant matrix.

■

The Dot or Inner Product

The **dot or inner product** of a n-dimensional row vector R and n-dimensional column vector C, where

$$R = (r_1, r_2, r_3, \ldots, r_n) \qquad \text{and} \qquad C = \text{col}(c_1, c_2, c_3, \ldots, c_n)$$

is **a single number** written as the matrix product

$$RC = (r_1, r_2, r_3, \ldots, r_n) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = r_1c_1 + r_2c_2 + r_3c_3 + \cdots + r_nc_n = \sum_{m=1}^{n} r_m c_m \tag{10.5}$$

representing **the summation of the products of the mth row vector element with the mth column vector element, as m varies from 1 to n.** In order to calculate an inner product the row vector and column vector **must have the same number of elements**.

Matrix Multiplication

Let $A = (a_{ij})_{m\times n}$ denote an $m \times n$ matrix and let $B = (b_{ij})_{p\times q}$ denote a $p \times q$ matrix. If the dimensions n, p have the proper size, then the matrix A can be right-multiplied by the matrix B to produce a new matrix C. This matrix product is written $C = AB$ and this matrix product **can only occur when the matrices A and B have the proper dimensions**. For the matrix product $AB = A_{m\times n}B_{p\times q}$ to exist it is required that **the dimension p of B** must equal **the dimension n of A** and whenever this condition is satisfied, then the matrices are said to satisfy the **compatibility condition for matrix multiplication to occur**. If the column dimension of A **does not**

equal the row dimension of B, then the matrices A and B **cannot be multiplied**. The matrix product of two matrices A and B, having the proper dimensions, is written $C = AB$ and then one can say either

$$(a) \quad A \text{ premultiplies } B$$
$$(b) \quad \text{or } B \text{ postmultiplies } A$$

If the **row dimension of B equals the column dimension of A** one can write $p = n$, then the two matrices A and B can be multiplied and the resulting matrix product C has dimension $m \times q$. This is sometimes expressed in the form

$$C_{m\times q} = A_{m\times n} B_{n\times q}$$

where attention is drawn to the fact that the matrices satisfy the compatibility condition for matrix multiplication. Expressing the matrices A, B and C in expanded form one can write

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & \boxed{c_{ij}} & \cdots & c_{iq} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mq} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nq} \end{pmatrix}$$

The element c_{ij} belong to the matrix product C is calculated using **the elements from the ith row vector of A and the elements from the jth column vector of B** to represent c_{ij} as a dot or inner product. **The ith row vector of A is dotted with the j column vector from B and the resulting single number is called c_{ij}.** This inner or dot product is defined as above, but now a double subscript notation is in use so that one obtains

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Performing all possible inner products of the ith row vector with the jth column vector as i varies from 1 to m and j varies from 1 to q produces the product matrix $C = (c_{ij})_{m\times q}$.

Example 10-2. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix}_{3\times 4}$ the matrices A and B satisfy the compatibility condition for matrix multiplication and the matrix product $C = AB$ will be a matrix having the dimension of 2 rows and 4 columns. One can write

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix} = AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix}$$

where c_{11} is the inner product of row 1 with column 1 giving

$$c_{11} = 1(7) + 2(11) + 3(15) = 74$$

In a similar fashion one finds

c_{12} is the inner product of row 1 with column 2 giving
$$c_{12} = 1(8) + 2(12) + 3(16) = 80$$
c_{13} is the inner product of row 1 with column 3 giving
$$c_{13} = 1(9) + 2(13) + 3(17) = 86$$
c_{14} is the inner product of row 1 with column 4 giving
$$c_{14} = 1(10) + 2(14) + 3(18) = 92$$
c_{21} is the inner product of row 2 with column 1 giving
$$c_{21} = 4(7) + 5(11) + 6(15) = 173$$
c_{22} is the inner product of row 2 with column 2 giving
$$c_{22} = 4(8) + 5(12) + 6(16) = 188$$
c_{23} is the inner product of row 2 with column 3 giving
$$c_{23} = 4(9) + 5(13) + 6(17) = 203$$
c_{24} is the inner product of row 2 with column 4 giving
$$c_{24} = 4(10) + 5(14) + 6(18) = 218$$

This gives the matrix product

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix} = C = \begin{pmatrix} 74 & 80 & 86 & 92 \\ 173 & 188 & 303 & 218 \end{pmatrix}$$

■

Matrices with the proper dimensions satisfy the properties

$$\begin{aligned} A(B+C) =& AB + AC \qquad \text{left distributive law} \\ (B+C)A =& BA + CA \qquad \text{right distributive law} \\ A(BC) =& (AB)C \qquad \text{associative law} \end{aligned}$$

Example 10-3. Note that only in **special cases** is matrix multiplication commutative. One can say in general $AB \neq BA$. Consider the matrix product of the 2×2 matrices given by $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$. One finds

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix}$$

which shows that in general matrix multiplication is not commutative.

In addition, if the matrix product of A and B produces $AB = [0]$, this **does not mean** $A = [0]$ or $B = [0]$. For example, if $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, then one can show

$$AB = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

■

Special Square Matrices

There are many special matrices which have interesting properties. The following are some definitions of **special square matrices** which arise in applied mathematics, engineering, physics and the sciences.

The identity matrix

The $n \times n$ **identity matrix** can be expressed $I = (\delta_{ij})_{n \times n}$ where **$\boldsymbol{\delta_{ij}}$ is the Kronecker delta** and defined

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This matrix is characterized by having all 1's along the main diagonal and zero's everywhere else. An example of a 3×3 identity matrix is given by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Secondary diagonal

Main diagonal

The identity matrix has the property

$$AI = IA = A \tag{10.6}$$

for all square matrices A where A and I **have the same dimensions**.

The transpose matrix

The **transpose of a matrix** $A = (a_{ij})_{m\times n}$ is obtained by **interchanging the rows and columns of the matrix** A. The transpose matrix is denote $A^T = (a_{ji})_{n\times m}$. That is, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} \qquad \text{then} \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}_{n\times m}$$

Note that $(A^T)^T = A$. If $A^T = A$, then the matrix A is called **a symmetric matrix**. If $A^T = -A$, then A is called **a skew-symmetric matrix**. The matrix transpose of a product satisfies

$$(AB)^T = B^T A^T, \qquad (ABC)^T = C^T B^T A^T$$

so that **the transpose of a product is the product of the transposed matrices in reverse order**.

Lower triangular matrices

Matrices which satisfy

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = 0 \quad \text{for} \quad i < j,$$

are called **lower triangular matrices**. Such matrices have zero for elements everywhere above the main diagonal. Any example of a lower triangular matrix is given in the figure 10-1.

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 1 & -1 & -2 & 4 \end{pmatrix}$$

Figure 10-1. A 4×4 lower triangular matrix.

Upper triangular matrices

If a square matrix A satisfies

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = 0 \quad \text{for} \quad i > j,$$

it is called an **upper triangular matrix**. Such matrices have zero for elements everywhere below the main diagonal. An example upper triangular matrix is illustrated in the figure 10-2.

$$A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Figure 10-2. A 4×4 upper triangular matrix.

Diagonal matrices

A matrix which has zeros for all elements above and below the main diagonal is called a **diagonal matrix**. Such a matrix can be described by

$$D = (d_{ij}), \quad \text{where} \quad d_{ij} = 0 \quad \text{for} \quad i \neq j.$$

Diagonal matrices are sometimes written $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The identity matrix is an example of a diagonal matrix. Another example of a diagonal matrix is given in the figure 10-3.

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 10-3. Example of a 4×4 diagonal matrix.

Tridiagonal matrices

A matrix A satisfying

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = \begin{cases} 0, & i > j + 1 \\ 0, & i < j - 1 \end{cases}$$

is called a **tridiagonal matrix**. Such a matrix is recognized as having elements along the main diagonal and the immediate diagonals above and below the main diagonal. All other elements within the matrix are zero. An example tridiagonal matrix is given in the figure 10-4.

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Figure 10-4. A 5×5 tridiagonal matrix.

The trace of a matrix

The **trace of a $n \times n$ square matrix A** is denoted $\text{Tr}(A)$ and represents a summation of the diagonal elements of the matrix A. One can write

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

If matrices A and B are conformable matrices, then the trace satisfies the properties

$$\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B), \qquad \text{Tr}(AB) = \text{Tr}(BA)$$

The Inverse Matrix

If A and E are square matrices such that their matrix product produces the identity matrix, that is, if $AE = EA = I$, then E is called **the inverse of A** and the matrix E is written

$$E = A^{-1},$$

which is read **"E equals A inverse".** Thus, the inverse matrix has the property that

$$AA^{-1} = A^{-1}A = I.$$

The inverse matrix, if it exists, is unique. This statement can be proven by first assuming that the inverse is not unique and then showing that this assumption is wrong. This type of proof is known as the method of **reductio ad absurdum**[1] to verify something is true.

For example, if A_1 and A_2 are both inverses of the matrix A, then by hypothesis both of the statements

$$AA_1 = A_1A = I \qquad \text{and} \qquad AA_2 = A_2A = I$$

must be true. Consequently, one can write

$$A_2 = A_2I = A_2(AA_1) = (A_2A)A_1 = IA_1 = A_1.$$

Hence, $A_2 = A_1 = A^{-1}$ and the initial assumption is wrong and so the inverse matrix must be unique.

[1] The method of reductio ad absurdum is used to prove a statement in mathematics by assuming initially that the statement is true (or false) and then performing an analysis of this assumption (the reduction of the proposition) to arrive at a conclusion which is obviously absurd and contradicts the initial assumption. The method of reductio ad absurdum was used by the early Greek mathematicians as a method for proving many theorems.

Example 10-4. For A an $n \times n$ square matrix, show that $(A^{-1})^{-1} = A$. That is, show the inverse of an inverse matrix is again the original matrix A.

Solution Let $B = A^{-1}$ so that $B^{-1} = (A^{-1})^{-1}$, then by definition of an inverse matrix one can write

$$AB = AA^{-1} = I.$$

Right-multiply this equation on both sides by B^{-1} to obtain

$$ABB^{-1} = IB^{-1} = B^{-1}.$$

Using the result that $BB^{-1} = I$ and that $AI = A$, this last equation simplifies to

$$AI = A = B^{-1} = (A^{-1})^{-1}$$

which establishes the result.

■

Example 10-5. Show that $(AB)^{-1} = B^{-1}A^{-1}$. That is, show **the inverse of a product of two matrices is the product of the inverses in the reverse order**.

Solution By definition

$$(AB)^{-1}(AB) = I.$$

so that if one postmultiples both sides of this equation by B^{-1} and simplifies the results, one finds

$$\begin{aligned}
(AB)^{-1}(AB)B^{-1} &= IB^{-1} \\
(AB)^{-1}A(BB^{-1}) &= B^{-1}, \quad \text{associative law} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1}
\end{aligned}$$

Now postmultiply both sides of this last equation by A^{-1} to obtain

$$\begin{aligned}
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1}I &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

which establishes the result.

■

Methods for calculating the inverse of a square matrix, if the inverse exists, are developed in a later section. In this section, the emphasis is on definitions, terminology and certain operational properties associated with square matrices.

Matrices with Special Properties

The following is some terminology associated with square matrices A and B.

(1) If $AB = -BA$, then A and B are called **anticommutative**.

(2) If $AB = BA$, then A and B are called **commutative**.

(3) If $AB \neq BA$, then A and B are called **noncommutative**.

(4) If $A^p = \overbrace{AA\cdots A}^{p\ times} = \widetilde{0}$ for some positive integer p, then A is called **nilpotent of order** p.

(5) If $A^2 = A$, then A is called **idempotent**.

(6) If $A^2 = I$, then A is called **involutory**.

(7) If $A^{p+1} = A$, then A is called **periodic with period** p. The smallest integer p for which $A^{p+1} = A$ is called the least period p.

(8) If $A^T = A$, then A is called a **symmetric matrix**.

(9) If $A^T = -A$, then A is called a **skew-symmetric matrix**.

(10) If A^{-1} **exists**, then A is called a **nonsingular matrix**.

(11) If A^{-1} **does not exist**, then A is called **a singular matrix.**

(12) If $A^T A = AA^T = I$, then A is called an **orthogonal matrix** and $A^T = A^{-1}$.

Example 10-6.

The matrix $A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ is periodic with least period 2 because
$A^2 = AA = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $A^3 = A^2A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = A$

■

Example 10-7.

The matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ is nilpotent of index 2 because

$$A^2 = AA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{0}$$

■

Example 10-8.

The matrix

$$B = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

is idempotent because

$$B^2 = BB = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = B$$

■

An orthogonal matrix

If A is an $n \times n$ square matrix satisfying $A^T A = AA^T = I$, then A is called an **orthogonal matrix**, and $A^{-1} = A^T$. An example of an orthogonal matrix is given by

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad AA^T = I$$

Example 10-9. Some examples of special matrices are:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad \text{is lower triangular}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{bmatrix} \qquad \text{is upper triangular}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{is an identity matrix which is also diagonal}$$

$$T = \begin{bmatrix} \beta & \gamma & 0 & 0 & 0 \\ \alpha & \beta & \gamma & 0 & 0 \\ 0 & \alpha & \beta & \gamma & 0 \\ 0 & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \alpha & \beta \end{bmatrix} \qquad \text{is a tridiagonal matrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{is an orthogonal matrix satisfying} \qquad AA^T = I$$

If $f = f(\bar{x}) = f(x_1, x_2, \ldots, x_n)$ is a function of n-variables, then the Hessian matrix associated with f is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1\, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\, \partial x_n} \\ \frac{\partial^2 f}{\partial x_2\, \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2\, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\, \partial x_1} & \frac{\partial^2 f}{\partial x_n\, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

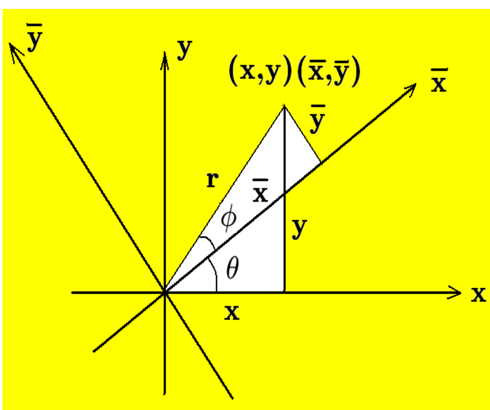
■

Example 10-10. Consider the fixed set of axes x, y and a set of barred axes $\bar{x}, \bar{y}$ where the barred set of axes is rotated about the origin through an angle θ as illustrated below. Consider a general point P having coordinates (x, y) with respect the unbarred axes. This same point P has coordinates $(\bar{x}, \bar{y})$ with respect to the barred set of axes. Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\bar{\hat{\mathbf{e}}}_1$, $\bar{\hat{\mathbf{e}}}_2$ denote unit vectors in the directions of the x, y and $\bar{x}, \bar{y}$-axes respectively. The position vector $\vec{r}$ of the point P can be expressed in either of the forms

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \qquad \text{or} \qquad \vec{r} = \bar{x}\,\bar{\hat{\mathbf{e}}}_1 + \bar{y}\,\bar{\hat{\mathbf{e}}}_2$$

The transformation equations between the coordinates can be obtained by taking the dot product of $\vec{r}$ with the unit vectors $\bar{\hat{\mathbf{e}}}_1$ and $\bar{\hat{\mathbf{e}}}_2$ to obtain

$$\vec{r}\cdot\bar{\hat{\mathbf{e}}}_1 = \bar{x} = x\,\hat{\mathbf{e}}_1\cdot\bar{\hat{\mathbf{e}}}_1 + y\,\hat{\mathbf{e}}_2\cdot\bar{\hat{\mathbf{e}}}_1 = x\cos\theta + y\sin\theta$$
$$\vec{r}\cdot\bar{\hat{\mathbf{e}}}_2 = \bar{y} = x\,\hat{\mathbf{e}}_1\cdot\bar{\hat{\mathbf{e}}}_2 = y\,\hat{\mathbf{e}}_2\cdot\bar{\hat{\mathbf{e}}}_2 = x(-\sin\theta) + y\cos\theta$$



The above transformation equations between the $(\bar{x}, \bar{y})$ axes which have been rotated through an angle θ with respect to a fixed set of (x, y) axes can be represented by the matrix equation

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad \bar{X} = AX \qquad (10.7)$$

where $\bar{X} = \text{col}(\bar{x}, \bar{y})$ and $X = \text{col}(x, y)$ are column vectors. Here the coefficient matrix of the above transformation is $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and its transpose matrix is $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. If one calculates the matrix product of A times it transpose A^T, one finds $AA^T = I$, the identity matrix. Matrices with this property are called **orthogonal matrices.** Left-multiplication of equation (10.7) by $A^{-1} = A^T$ gives the inverse transformation $A^T\bar{X} = A^TAX = IX = X$ which can be expressed in expanded form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}.$$

The row and column vectors which make up the rows and columns of the matrix A are called orthogonal vectors.

■

Example 10-11. Represent the given system of differential equations in matrix form.

$$\frac{dy_1}{dt} = y_1 + y_2 - y_3 + \sin t, \qquad \frac{dy_2}{dt} = 2y_2 + y_3 + \cos t, \qquad \frac{dy_3}{dt} = 3y_3 + \sin 2t$$

Solution The above system of differential equations can be represented in the form

$$\frac{d\overline{y}}{dt} = A\overline{y} + \overline{f}(t) \tag{10.8}$$

where $\overline{y} = \overline{y}(t) = \mathrm{col}(y_1, y_2, y_3)$ denotes a column vector, $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is a coefficient matrix and $\overline{f} = \overline{f}(t) = \mathrm{col}(\sin t, \cos t,\ \sin 2t)$ represents a variable right-hand side to the differential system. Matrix differential equations of the form given by equation (10.8) subject to the initial condition $\overline{y}(0) = \bar{c}$, where $\bar{c}$ is a constant, are called **initial-value problems.**

■

Example 10-12. The nth order linear differential equation

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + a_2(t)\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_{n-2}(t)\frac{d^2y}{dt^2} + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = 0$$

is converted to matrix form by defining

$$\bar{y} = \mathrm{col}(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^{n-2}y}{dt^{n-2}}, \frac{d^{n-1}y}{dt^{n-1}})$$

and

$$A = A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2} & -a_{n-3}(t) & \cdots & -a_2(t) & -a_1(t) \end{pmatrix}$$

The given scalar equation can then be represented by the matrix equation

$$\frac{d\bar{y}}{dt} = A(t)\bar{y}$$

The matrix $A = A(t)$ is called **the companion matrix**.

■

Example 10-13. Represent the differential equation $\frac{d^2y}{dt^2}+\omega^2 y = \sin 2t$ in matrix form.

Solution Let $y_1 = y$ and $y_2 = \frac{dy_1}{dt} = \frac{dy}{dt}$, then

$$\frac{dy_2}{dt} = \frac{d^2y_1}{dt^2} = \frac{d^2y}{dt^2} = -\omega^2 y + \sin 2t = -\omega^2 y_1 + \sin 2t$$

The given scalar differential equation can then be represented in the matrix form

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin 2t \end{pmatrix}$$

or

$$\frac{d\overline{y}}{dt} = A\overline{y} + \overline{f}(t) \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \overline{f}(t) = \begin{pmatrix} 0 \\ \sin 2t \end{pmatrix} \quad \text{and} \quad \overline{y} = \text{col}(y_1, y_2)$$

■

Example 10-14. Associated with the initial value problem

$$\frac{d\bar{y}}{dt} = A(t)\bar{y} + \bar{f}(t), \qquad \bar{y}(0) = \bar{c} \tag{10.9}$$

is the **matrix differential equation**

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I \tag{10.10}$$

where $X = (x_{ij})_{n\times n}$ and I is the $n \times n$ identity matrix. Associated with the matrix differential equation (10.10) is the **adjoint differential equation**

$$\frac{dZ}{dt} = -ZA(t), \qquad Z(0) = I, \quad Z = (z_{ij})_{n\times n} \tag{10.11}$$

The relationship between the three differential equations given by equations (10.9), (10.10), and (10.11), is as follows. Left-multiply equation (10.10) by Z and right-multiply equation (10.11) by X to obtain

$$Z\frac{dX}{dt} = ZA(t)X \tag{10.12}$$

$$\frac{dZ}{dt}X = -\, ZA(t)X \tag{10.13}$$

and then add the equation (10.12) and (10.13) to obtain

$$Z\frac{dX}{dt} + \frac{dZ}{dt}X = \frac{d}{dt}(ZX) = [0] \tag{10.14}$$

This implies that the matrix product $ZX = C$ is a constant. If the matrix X is nonsingular, then X^{-1} exists, so that one can solve for Z as $Z = X^{-1}C$. At time $t = 0$, it is required that $Z(0) = I$ and $X(0) = I$ so that $C = I$ and therefore $Z = X^{-1}$. Consequently, the adjoint equation can be expressed in the form

$$\frac{dX^{-1}}{dt} = -X^{-1}A(t), \qquad X^{-1}(0) = I \tag{10.15}$$

Now multiply equation (10.9) on the left by X^{-1} to obtain

$$X^{-1}\frac{d\bar{y}}{dt} = X^{-1}A\bar{y} + X^{-1}\bar{f}(t) \tag{10.16}$$

and then multiply equation (10.15) on the right by $\bar{y}$ to obtain

$$\frac{dX^{-1}}{dt}\bar{y} = -X^{-1}A\bar{y} \tag{10.17}$$

Sum the equations (10.16) and (10.17) to obtain

$$X^{-1}\frac{d\bar{y}}{dt} + \frac{dX^{-1}}{dt}\bar{y} = \frac{d}{dt}\left(X^{-1}\bar{y}\right) = X^{-1}\bar{f}(t) \tag{10.18}$$

Integrate equation (10.18) from 0 to t and show

$$\int_0^t \frac{d}{dt}\left(X^{-1}(t)\bar{y}(t)\right)\,dt = \int_0^t X^{-1}(t)\bar{f}(t)\,dt$$

which produces the result

$$X^{-1}(t)\bar{y}(t)\Big|_0^t = X^{-1}(t)\bar{y}(t) - X^{-1}(0)\bar{y}(0) = \int_0^t X^{-1}(t)\bar{f}(t)\,dt$$

which indicates that the solution to the matrix equation (10.9) can be represented in the form

$$\bar{y}(t) = X(t)\bar{c} + X(t)\int_0^t X^{-1}(\xi)\bar{f}(\xi)\,d\xi \tag{10.19}$$

■

The Determinant of a Square Matrix

A **fundamental principle from probability and statistics** is that if something can be done in n different ways and after it has been done in one of these ways, a second something can be done in m different ways, then the two somethings can be done in the order stated in $n \cdot m$ different ways. If a third something can be done in p different ways, then the three somethings can be done in $n \cdot m \cdot p$ different ways. This

principle of multiplication by the number of distinct ways a thing can be done can be extended to more than just two or three somethings.

A **permutation of a set of objects** represents some arrangement of the objects. The number of different permutations of n-objects is $n!$ (read n-factorial). This is because there are n choices for the first position of the arrangement, $(n-1)$ choices for the second position of the arrangement, $(n-3)$ choices for the third position of the arrangement, etc, and these quantities are being multiplied.

A **transposition** is **an interchanging of the positions of two objects within an arrangement** of the set of objects. In examining all possible permutations of the integers $(1, 2, 3, \ldots, n)$ one finds these permutations can be divided into a group representing an even number of transpositions and another group representing the odd number of transpositions. For example, in going from $(1234\ldots)$ to $(2134\ldots)$ represents 1 transposition and going from $(1234,\ldots)$ to $(2314\ldots)$ would be two transpositions, etc.

The determinant of a $n \times n$ square matrix $A = (a_{ij})$ is denoted by either of the symbols $\det A$ or $\mid A \mid$. The determinant is **a single number** given by either of the summations

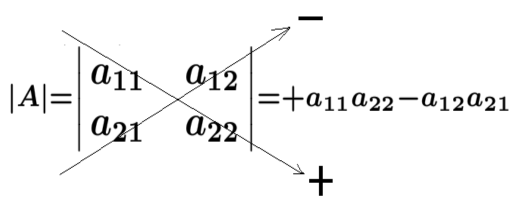
$$
\begin{aligned}
\det A =\mid A \mid = \sum (-1)^m a_{i1} a_{j2} a_{k3} \ldots a_{\ell n} \qquad & \text{column expansion} \\
\det A =\mid A \mid = \sum (-1)^m a_{1i} a_{2j} a_{3k} \cdots a_{n\ell} \qquad & \text{row expansion}
\end{aligned}
$$

The single number $\det A =\mid A \mid$ is **the sum of all possible products** in which there appears one and only one element from each row (or column) multiplied by the appropriate plus or minus sign. The sigma sign Σ denotes a sum over all $n!$ permutations of the numbers $(1, 2, 3, \ldots, n)$ and the integers $(i, j, k, \ldots, \ell)$ represent distinct permutations of the numbers from the set $(1, 2, 3, \ldots, n)$. The appropriate plus or minus sign is assigned to each product within the sum and is based upon whether the permutation $(i, j, k, \ldots, \ell)$ is either even (+1) or odd (-1). That is, $m = +1$ if $(i, j, k, \ldots, \ell)$ represents an even number of transpositions associated with the set $(1, 2, 3, \ldots, n)$ and $m = -1$ if $(i, j, k, \ldots, \ell)$ represents an odd number of transpositions associated with the set $(1, 2, 3, \ldots, n)$.

Example 10-15. The matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has the determinant

$$
\mid A \mid = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}
$$

Here $(1,2)$ is an even permutation of $(1,2)$ and $(2,1)$ represents an odd permutation of $(1,2)$ A mnemonic device to remember this 2×2 determinant is illustrated by the following figure

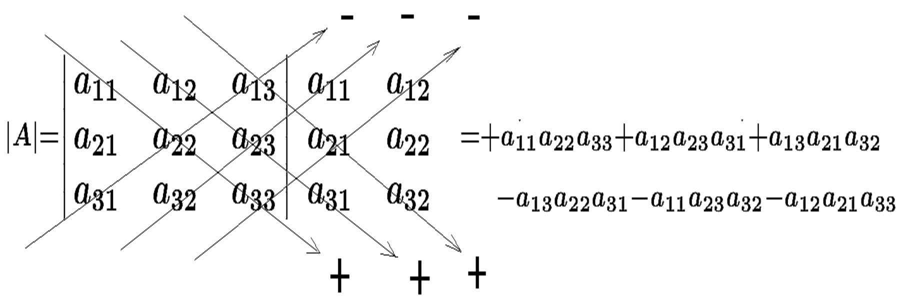


■

Example 10-16. The 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has the determinant

$$| A |= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

A mnemonic device to remember this 3×3 determinant is to append the first two columns to the end of the matrix and draw diagonal lines through the elements to create the following figure, where the elements on each diagonal are multiplied.



Note that the determinant of a $n \times n$ matrix has n-**factorial terms** and consequently if n is large, then mnemonic devices like those above **are not employed because the calculations become cumbersome** and sometimes extremely lengthy. Instead it has been found that by using **row reduction methods**[2] the given matrix can be converted to an equivalent upper triangular or lower triangular matrix having all zeros either below or above the main diagonal. The determinant of these special triangular matrices is then just **a product of the diagonal elements**.

■

Example 10-17. Find the derivative of the determinant

$$y = \det A =| A |= \begin{vmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{vmatrix}$$

[2] Row reduction methods are considered later in this chapter.

Solution

The definition of a determinant gives the relation $y = y(t) = \sum(-1)^m a_{i1}(t)a_{j2}(t)$ where the summation is over all permutations of the integers $(1, 2)$. Differentiating this relation gives

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sum(-1)^m a_{i1}(t)a_{j2}(t)\right) = \sum(-1)^m \left[\frac{da_{i1}(t)}{dt}a_{j2}(t) + a_{i1}(t)\frac{da_{j2}(t)}{dt}\right]$$

which has the expanded form

$$\frac{dy}{dt} = \begin{vmatrix} \frac{da_{11}(t)}{dt} & \frac{da_{12}(t)}{dt} \\ a_{21}(t) & a_{22}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) \\ \frac{da_{21}(t)}{dt} & \frac{da_{22}(t)}{dt} \end{vmatrix}$$

■

Minors and Cofactors

Associated with each element a_{pq} of a $n \times n$ square matrix A are the quantities m_{pq} and c_{pq} called **the minor and cofactor of the element** a_{pq}**.** The minor m_{pq} of an element a_{pq} is the determinant of the $(n-1)\times(n-1)$ matrix formed by **deleting the row and column of** A **which contains the element** a_{pq}. The cofactor of a_{pq} is then defined as $c_{pq} = (-1)^{p+q}m_{pq}$. That is, the cofactor is **the minor with the appropriate plus or minus sign** $(-1)^{p+q}$ **which is determined by the row number** p **and column number** q **of the element** a_{pq}. The matrix containing the cofactor elements c_{pq} of a_{pq} is written $C = (c_{pq})_{n\times n}$ and is called **the cofactor matrix** associated with A. The cofactor matrix has the property that $AC^T = \text{diag}\,[|A|, |A|, \ldots, |A|] = |A|I$, where $|A|$ is the determinant of A.

Example 10-18. (**(Minors and Cofactors)** For the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ calculate the cofactor matrix $C = (c_{ij})_{3\times 3}$ and then calculate AC^T.

Solution The minor of element a_{ij} is obtained by crossing out the row and column containing a_{ij} and then taking the determinant of the remaining elements. One finds

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$m_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad m_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$m_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad m_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad m_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{aligned} c_{11} &= m_{11}, & c_{12} &= -m_{12}, & c_{13} &= m_{13} \\ c_{21} &= -m_{21}, & c_{22} &= m_{22}, & c_{23} &= -m_{23} \\ c_{31} &= m_{31}, & c_{32} &= -m_{32}, & c_{33} &= m_{33} \end{aligned}$$

$$AC^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} c_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

where

$$|A| = \det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

In general for $A_{n\times n}$ one can write

$$|A| = \det A = \sum_{j=1}^{n} a_{ij}c_{ij} \quad \text{or} \quad |A| = \det A = \sum_{j=1}^{n} a_{ij}c_{ij}$$

for the column and row expansion of a determinant. If $n > 3$ these methods for calculating a determinant are ill advised as the method is very time consuming.

■

Properties of Determinants

Many of the properties of determinants are associated with performing elementary row (or column) operations upon the elements of the determinant. The **three basic elementary row operations being performed on determinants** are

(i) The interchange of any two rows.

(ii) The multiplication of a row by a nonzero scalar α

(iii) The replacement of the ith row by the sum of the ith row and α times the jth row, where $i \neq j$ and α is any nonzero scalar quantity.

The following are some properties of determinants stated without proof.

1. If two rows (or columns) of a determinant are equal or one row is a constant multiple of another row, then the determinant is equal to zero.
2. The interchange of any two rows (or two columns) of a determinant changes the numerical sign of the determinant.
3. If the elements of any row (or column) are all zero, then the value of the determinant is zero.
4. If the elements of any row (or column) of a determinant are multiplied by a scalar m and the resulting row vector (or column vector) is added to any other row (or column), then the value of the determinant is unchanged. As an example, take a 3×3 determinant and multiply row 3 by a nonzero constant m and add the result to row 2 to obtain

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ (d+mg) & (e+mh) & (f+mi) \\ g & h & i \end{vmatrix}.$$

5. If all the elements in a row (or column) are multiplied by the same scalar q, then the determinant is multiplied by q. This produces

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ qa_{i1} & qa_{i2} & \cdots & qa_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = q \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

6. The determinant of the product of two matrices is the product of the determinants and $|AB| = |A||B|$.
7. If each element of a row (or column) is expressible as the sum of two (or more) terms, then the determinant may also be expressed as the sum of two (or more) determinants. For example,

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}+b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

8. Let c_{ij} denote the cofactor of a_{ij} in the determinant of A. The value of the determinant $|A|$ is the sum of the products obtained by multiplying each element of a row (or column) of A by its corresponding cofactor and

$$\begin{aligned} |A| =& a_{i1}c_{i1} + \cdots + a_{in}c_{in} = \sum_{k=1}^{n} a_{ik}c_{ik} \quad \text{row expansion} \\ \text{or} \quad |A| =& a_{1j}c_{1j} + \cdots + a_{nj}c_{nj} = \sum_{k=1}^{n} a_{kj}c_{kj} \quad \text{column expansion} \end{aligned}$$

If the elements of a row (or column) are multiplied by the cofactor elements from a different row (or column), then zero is obtained. These results can be used to write $AC^T = |A|I$

Example 10-19.

Show the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$ has the cofactor matrix $C = \begin{bmatrix} -5 & 5 & -5 \\ 2 & -4 & -2 \\ -1 & -3 & 1 \end{bmatrix}$.
If the elements from any row (or column) of A are multiplied by their respective cofactors, then the sum of these products gives us the determinant $|A|$. For example, using row expansions one can verify

$$|A| = (1)(-5) + (0)(5) + (1)(-5) = -10$$
$$|A| = (-1)(2) + (1)(-4) + (2)(-2) = -10$$
$$|A| = (3)(-1) + (2)(-3) + (-1)(1) = -10$$

and using a column expansion there results

$$|A| = (1)(-5) + (-1)(2) + (3)(-1) = -10$$
$$|A| = (0)(5) + (1)(-4) + (2)(-3) = -10$$
$$|A| = (1)(-5) + (2)(-2) + (-1)(1) = -10.$$

Observe also that if the elements from any row (or column) are multiplied by the cofactors from a different row (or column), then the sum of these elements is zero. For example, row 1 multiplied by the cofactors from row 2 gives

$$(1)(2) + (0)(-4) + (1)(-2) = 0.$$

Another example is row 2 multiplied by the cofactors from row 3

$$(-1)(-1) + (1)(-3) + (2)(1) = 0.$$

These results may be further illustrated by calculating the matrix product

$$AC^T = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \, \mathrm{diag}\,(1,1,1) = |A|I \tag{10.20}$$

■

Example 10-20.

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 6 \\ 1 & 1 & 0 & 3 & 9 \\ -2 & 0 & 1 & -3 & -9 \\ 0 & -1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 4 & 12 \end{bmatrix}.$$

Solution: Utilizing property 4, one can multiply any row by a constant and add the result to any other row **without changing the value of the determinant**. Perform the following operations on the above determinant: (a) subtract row 1 from row 5 (b) multiply row 1 by two and add the result to row 3, and (c) subtract row 1 from row 2. Performing these calculations produces

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 6 \end{vmatrix}.$$

Now perform the operations: (a) add row 2 to row 4 and (b) subtract row 3 from row 5. The determinant now has the form

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}.$$

Observe that the row operations performed have produced zeros both above and below the main diagonal. Next perform the operations of (a) subtracting twice row 5 from row 1, (b) subtracting row 5 from row 2, (c) subtracting row 5 from row 3, and (d) subtracting row 5 from row 4. These operations produce

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}.$$

By expanding $|A|$ using cofactors of the first rows and associated subdeterminants, there results

$$|A| = (1)(1)(1) \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5.$$

A much more general procedure for calculating the determinant of a matrix A is to use **row operations and reduce $|A| = \det(A)$ to a triangular form having all zeros below the main diagonal.** For example, reduce A to the form:

$$|A| = \det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix}.$$

The determinant of A is then obtained by **multiplying all the elements on the main diagonal** and

$$|A| = \det(A) = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

■

Rank of a Matrix

The **rank of a** $m \times n$ **matrix** A is denoted using the notation rank(A). The **rank of the matrix** A is a real number defined as the size of the **largest nonzero determinant that can be formed using the elements of** A. If $A = (a_{ij})_{m\times n}$, one can show that the maximum possible rank(A) is the smaller of the numbers m and n.

Calculation of the Inverse Matrix

The following illustrates some methods for calculating the inverse of a square matrix **if such an inverse exists**. Previously it has been shown that if C is the cofactor matrix of A, then

$$AC^T = |A|I. \tag{10.21}$$

By multiplying this equation on the left by A^{-1} and dividing by $|A|$, one can verify the result

$$A^{-1} = \frac{1}{|A|}C^T. \tag{10.22}$$

as a formula for calculating the inverse matrix. Define the **transpose of the cofactor matrix** C^T to be the **adjoint of** A. The notation $\text{Adj}\, A$ is used to denote the adjoint matrix. Using this definition, the above results can be expressed in the form

$$(\text{Adj}\, A)A = A(\text{Adj}\, A) = |A|I \qquad \text{or} \qquad A^{-1} = \frac{1}{|A|}\text{Adj}\, A \tag{10.23}$$

If A is an $n \times n$ square matrix and the determinant satisfies $\det A = |A| = 0$, then A is called **a singular matrix**. If A **is singular**, then the inverse matrix **does not exist**. If $\det A = |A| \neq 0$, then A is called **a nonsingular matrix**, and the inverse matrix A^{-1} exists under these conditions as can be discerned by examining the equation (10.23).

Example 10-21.

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

Solution: The cofactor matrix associated with A is given by $C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ and

$$\text{Adj}\, A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}.$$

This gives $|A| = 10$ so that A is nonsingular and the inverse is given by

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \quad \text{As a check, verify that} \quad AA^{-1} = I$$

■

Elementary Row Operations

A very useful matrix operation is **an elementary row operation** performed on a matrix. These elementary row operations can be used to obtain a wide variety of results.

An **elementary row matrix** E is any matrix **formed from the identity matrix** $I = (\delta_{ij})$ **by performing any of the following elementary row operations upon the identity matrix.**

(a) Interchange any two rows of I

(b) Multiplication of a row of I by any nonzero scalar m

(c) Replacement of the ith row of I by the sum of the ith row and m times the jth row, where $i \neq j$ and m is any scalar.

An **elementary column matrix** E is obtained if column operations are used instead of row operations. An **elementary transformation of a matrix** A is **the multiplication of** A **by an elementary row matrix.**

Example 10-22. Consider the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and the elementary matrices

E_1 where row 1 and 2 of the identity matrix are interchanged.

E_2 where row 1 is interchanged with row 3 and then rows 1 and 2 are interchanged.

E_3 where row 1 of the identity matrix is multiplied by 3.

E_4 where row 2 of the identity matrix is multiplied by 3 and the result added to row 1.

These elementary matrices can be represented

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Observe that multiplication of the matrix A by an elementary matrix E produces the following elementary transformations of the matrix A.

$$E_1 A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

where the first two rows of A are interchanged,

$$E_2 A = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

where simultaneously row 2 is moved to row 1, row 3 is moved to row 2 and row 1 is moved to row 3,

$$E_3 A = \begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{bmatrix}$$

where row 1 is multiplied by the scalar 3, and

$$E_4 A = \begin{bmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{bmatrix}$$

where row 2 of A is multiplied by 3 and the result is added to row 1. Observe that the elementary matrices E_1, E_2, E_3 and E_4 are obtained by performing elementary row operations on the identity matrix. When one of these elementary matrices multiplies the matrix A it has the same effect as performing the corresponding elementary row operation on the matrix A.

■

Let the product of successive elementary row transformations be denoted by

$$E_k E_{k-1} \cdots E_3 E_2 E_1 = P.$$

Similarly, one can define the product of successive elementary column transformations by

$$E_1 E_2 E_3 \cdots E_m = Q.$$

The **equivalence of two matrices A and B** is defined as follows. Let P and Q denote, respectively, the product of successive elementary row and column transformations as defined above. If $B = PA$, then B is said to be **row equivalent to** A. If $B = AQ$, then

B is said to be **column equivalent to** A. If $B = PAQ$, then B is said to be **equivalent to to the matrix** A.

All elementary matrices have inverses and are therefore nonsingular matrices. If A is nonsingular, then A^{-1} exists. For A nonsingular, one can perform a sequence of elementary row transformations on the matrix A and reduce A to an identity matrix. These operations are denoted by

$$E_k E_{k-1} \cdots E_3 E_2 E_1 A = I \quad \text{or} \quad PA = I \tag{10.24}$$

Right-multiplication of equation (10.24) by A^{-1} gives

$$P = E_k E_{k-1} \cdots E_3 E_2 E_1 = A^{-1}. \tag{10.25}$$

This equation suggests how one might build a "machine" for finding the inverse matrix of a nonsingular matrix A. Write down the matrix $A_{n\times n}$ and append to the right of it the identity matrix $I_{n\times n}$. By doing this the identity matrix can then be used like a "recording device," to record all elementary row operations that are performed on A. That is, whatever a row operation is performed upon A you must also perform **the same row operation on the appended identity matrix**. The matrix A with the identity matrix appended to its right-hand side is called **an augmented matrix.** After writing down

$$A \mid I, \tag{10.26}$$

observe that if an elementary row transformation is applied to the matrix A, then it is possible to "record" this transformation on the right-hand side of the equation (10.26). For example, if E_1 is an elementary row transformation applied to the augmented matrix, one obtains

$$E_1 A \mid E_1 I.$$

By performing a sequence of elementary row transformations upon the augmented matrix, given by equation (10.26), one can change the augmented matrix to the form

$$E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1 I, \tag{10.27}$$

where the sequence of elementary transformations has been "recorded" on the right-hand side of the augmented matrix. If one can choose the elementary matrices E_i, $i = 1, \ldots, k$, in such a way that the left-hand side of the augmented matrix (10.27) becomes the identity matrix, there would result the equation (10.24) on the left-hand side of the transformed augmented matrix. Consequently, the right-hand side of equation (10.27) becomes an equation, which gives the inverse matrix. The following example illustrates this "machine."

Example 10-23. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$.

Solution Append to the matrix A the identity matrix I to obtain the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right]. \tag{10.28}$$

Now try to select a sequence of elementary row operations with the goal of reducing the left-hand side of the augmented matrix (10.28) to the identity matrix. Each time an elementary row operation is applied to to the left-hand side of the augmented matrix (10.28) be sure to "record" the operation on the right-hand side. To illustrate, consider the following row operations applied to the augmented matrix (10.28).

- Replace row 2 by adding row 1 to row 2
- Multiply row 1 by (-2) and add the result to row 3. This produces the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 0 & -1 & -4 & -2 & 0 & 1 \end{array}\right].$$

Next perform the elementary row operation of replacing row 3 by adding row 2 to row 3 to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Finally, perform the following row operations:

- Multiply row 3 by (-5) and add the result to row 2.
- Multiply row 3 by (-3) and add the result to row 1. The above row operations produce the desired result of producing the identity matrix on the left-hand side of the augmented matrix. The final form for the augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & 6 & -4 & -5 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

and an examination of the right-hand side of the augmented matrix gives the inverse matrix $A^{-1} = \begin{bmatrix} 4 & -3 & -3 \\ 6 & -4 & -5 \\ -1 & 1 & 1 \end{bmatrix}$. One can readily verify that $AA^{-1} = I$.

■

Eigenvalues and Eigenvectors

Consider the operator box illustrated in the figure 10-5 where the input to the operator box is the $n \times 1$ **nonzero column vector** $x = \text{col}(x_1, x_2, \ldots, x_n)$ and the output from the operator box is the $n \times 1$ column vector $y = Ax$ were A is a $n \times n$ nonzero constant matrix. The operator box is said to transform the **nonzero column vector** x to the column vector y by matrix multiplication. For example, if $n = 3$ one would have the situation illustrated.

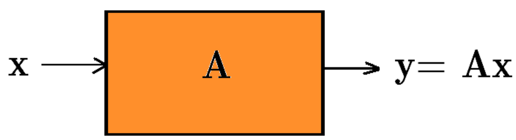


Figure 10-5.
Transformation of vector x to vector y.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

If there are special **nonzero column vectors** x such that **the output** y **is proportional to the input** x, then these special vectors are called **eigenvectors** and the proportionality constants are called **eigenvalues**. If the output y is proportional to the nonzero input x, then the equation $y = Ax = \lambda x$ must be satisfied, where λ is the scalar proportionality constant. If the equation $Ax = \lambda x$ has **nonzero solutions**, then one can write

$$\begin{aligned} Ax =& \lambda x = \lambda I x \\ (A - \lambda I)\, x =& [0]_{n\times 1} \end{aligned}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{10.29}$$

Cramer's[3] rule states that in order for this last equation to have **a nonzero solution** it is required that the determinant of the unknowns $x_1, x_2, \ldots, x_n$ be zero. This requires that

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{13} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{10.30}$$

[3] Gabriel Cramer (1704-1752) A Swiss mathematician who studied determinants.

Solving this equation for the values of λ gives the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ associated with the matrix A. Substituting an eigenvalue λ into the equation (10.29) enables one to solve for the corresponding eigenvector.

Example 10-24.

Find the eigenvalues and eigenvectors associated with the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Solution The eigenvalues and eigenvectors of the matrix A are determined by solving the matrix equation $Ax = \lambda x$ or

$$(A - \lambda I)\, x = \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10.31}$$

In order for this system to have a nonzero solution for the column vector x, Cramer's rule requires that

$$\det(A - \lambda I) = 0$$

or

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

Solving this equation for λ gives

$$(\lambda+1)(\lambda-5) = 0 \quad \text{with roots} \quad \lambda = -1 \quad \text{and} \quad \lambda = 5$$

which are called the eigenvalues of the matrix A. The eigenvector corresponding to the eigenvalue $\lambda = -1$ is found be substituting $\lambda = -1$ into the equation (10.31) to obtain

$$\begin{pmatrix} 1-(-1) & 4 \\ 2 & 3-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the equation $2x_1 + 4x_2 = 0$ or $x_1 = -2x_2$. This result specifies how the first component of the eigenvector is related to the second component of the eigenvector and gives $x = \text{col}(x_1, x_2) = \text{col}(-2x_2, x_2)$ for the eigenvector. Note that x_2 must be **some nonzero constant in order that the eigenvector be nonzero.** For convenience select the value $x_2 = 1$ to obtain the eigenvector $x = \text{col}(-2, 1)$. Note that any nonzero constant times an eigenvector is also an eigenvector. To summarize what has just been done, one can say the solution of the matrix equation (10.31), using the value $\lambda = -1$, tells us that $\text{col}(-2, 1)$ is an eigenvector of the matrix A and any constant times the eigenvector is also an eigenvector. In a similar fashion, substitute the value $\lambda = 5$ into the equation (10.31) to obtain

$$\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies $x_1 = x_2$. This gives the eigenvector $x = \text{col}(x_1, x_2) = \text{col}(x_2, x_2)$ where the component x_2 **must be some nonzero constant.** Selecting the value $x_2 = 1$ gives the eigenvector $x = \text{col}(1, 1)$. This shows that corresponding to the eigenvalue $\lambda = 5$ there is the eigenvector $x = \text{col}(1, 1)$. Note also that any nonzero constant times $\text{col}(1, 1)$ is also and eigenvector.

■

Example 10-25.

Find the eigenvalues and eigenvectors associated with the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & -3 & 7 \end{pmatrix}$$

Solution The eigenvalues and eigenvectors of the matrix A are determined by solving the matrix equation

$$(A - \lambda I)\, x = \begin{pmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & -3 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{10.32}$$

In order for this system to have a nonzero solution for the column vector x, Cramer's rule requires that

$$\det(A - \lambda I) = 0$$

or

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & -3 & 7-\lambda \end{vmatrix} = \lambda^3 + 9\lambda^2 - 26\lambda + 24 = 0$$

Solving this equation for λ one finds the factored form

$$(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0 \quad \text{with roots} \quad \lambda = 2,\ \lambda = 3,\ \text{and } \lambda = 4$$

which are called the eigenvalues of the matrix A. The eigenvector corresponding to the eigenvalue $\lambda = 2$ is found be substituting the value $\lambda = 2$ into the equation (10.32) to obtain

$$\begin{pmatrix} 2-2 & 2 & 2 \\ 0 & -2 & 4 \\ 0 & -3 & 7-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & -2 & 4 \\ 0 & -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives the equations $2x_2 + 2x_3 = 0$, $-2x_2 + 4x_3 = 0$ and $-3x_2 + 5x_3 = 0$. These equations imply $x_2 = x_3 = 0$ giving the eigenvector $\text{col}(x_1, 0, 0)$. Selecting the value of $x_1 = 1$ for convenience, the eigenvector corresponding to the eigenvalue $\lambda = 2$ is given by col$(1,0,0)$. Note that any nonzero constant times this vector is also an eigenvector. Substituting the eigenvalue $\lambda = 3$ into the equation (10.32) gives the equations $-x_1+2x_2+2x_3=0$, $-3x_2+4x_3=0$ and $-3x_2+4x_3=0$. These equations imply that $x_2 = \frac{2}{7}x_1$ and $x_3 = \frac{3}{14}x_1$. This gives the eigenvector col$(x_1, \frac{2}{7}x_1, \frac{3}{14}x_1)$. Selecting the value $x_1 = 14$ for convenience, one finds the eigenvector col$(14,4,3)$ corresponding to the eigenvalue $\lambda = 3$. Note that any nonzero constant times this eigenvector is also an eigenvector. Substituting the eigenvalue $\lambda = 4$ into the equation (10.32) gives the equations $-2x_1+2x_2+2x_3=0$, $-4x_2+4x_3=0$ and $-3x_2+3x_3=0$. These equations imply that $x_3 = \frac{1}{2}x_1$ and $x_2 = \frac{1}{2}x_1$ and so the eigenvector can be expressed col$(x_1, \frac{1}{2}x_1, \frac{1}{2}x_1)$. Selecting the value $x_1 = 2$ for convenience gives the eigenvector col$(2,1,1)$ corresponding to the eigenvalue $\lambda = 4$.

■

Properties of Eigenvalues and Eigenvectors

The following are some important properties concerning the eigenvalues and eigenvectors associated with an $n \times n$ square matrix A.

Property 1: If X is an eigenvector of A, then kX is also an eigenvector of A for any nonzero scalar k.

Assume that the vector X is an eigenvector of A, so that it must satisfy the equation $AX = \lambda X$. If this equation is multiplied by a nonzero constant k there results $kAX = k\lambda X$ which can be written $A(kX) = \lambda(kX)$ and given the interpretation kX is an eigenvector of A.

Property 2: An eigenvector of a square matrix cannot correspond to two different eigenvalues.

Let λ_1, λ_2 with $\lambda_1 \neq \lambda_2$ be two different eigenvalues of A. Assume X_1 is an eigenvector of A corresponding to both λ_1 and λ_2. Our assumption implies that the equations

$$AX_1 = \lambda_1 X_1 \qquad \text{and} \qquad AX_1 = \lambda_2 X_1$$

must be satisfied simultaneously. Subtracting these equations shows us that $(\lambda_1 - \lambda_2)X_1 = [0]$. But, if $\lambda_1 - \lambda_2 \neq 0$, then this equation would imply that $X_1 = [0]$, which contradicts the fact that X_1 must be a nonzero eigenvector. Hence, the original assumption must be false.

Property 3: If a matrix A has one of its eigenvalues as zero and $\lambda = 0$, then A is a singular matrix.

The eigenvalues of A are determined by the characteristic equation

$$C(\lambda) = \det(A - \lambda I) = |A - \lambda I| = (-1)^n \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$$

If $\lambda = 0$ is a root of the characteristic equation, then $C(\lambda) = \det A = 0$ and consequently the matrix A is singular.

Property 4: Two matrices A and B are said **to be similar** if there exists a nonsingular matrix Q such that $B = Q^{-1}AQ$. If the matrices A and B are similar, then they have the same characteristic equation.

The above property is established if it can be shown that the characteristic equation of B equals the characteristic equation of A. For Q nonsingular and $B = Q^{-1}AQ$, one can write

$$\begin{aligned} B - \lambda I &= Q^{-1}AQ - \lambda I \\ &= Q^{-1}AQ - \lambda Q^{-1} I Q \\ &= Q^{-1}(A - \lambda I)Q. \end{aligned}$$

The determinant of this equation gives

$$C(\lambda) = |B - \lambda I| = \left|Q^{-1}(A - \lambda I)Q\right| = \left|Q^{-1}\right| |A - \lambda I| \, |Q| \, .$$

But $QQ^{-1} = I$ and $|Q| \left|Q^{-1}\right| = 1$ hence $C(\lambda) = |B - \lambda I| = |A - \lambda I|$ and the above property is established.

Additional Properties Involving Eigenvalues and Eigenvectors

The following are some additional properties and definitions relating to eigenvalues and eigenvectors of an $n \times n$ square matrix A. The properties are given without proof.

1. If the n eigenvalues $\lambda_1, \ \lambda_2, \ldots, \lambda_n$ of A are all distinct, then there exists n-linearly independent eigenvectors.
2. If an eigenvalue repeats itself, then the characteristic equation is said to have **a multiple root**. In such cases there may or may not exit n linearly independent eigenvectors. If the characteristic equation can be written

$$C(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k} = 0$$

where $\sum_{i=1}^{k} n_i = n$, then n_i is called the multiplicity of the eigenvalue λ_i.

3. If A is a symmetric matrix and λ_i is an eigenvalue of multiplicity r_i, then there are r_i linearly independent eigenvectors.
4. An $n \times n$ square matrix is **similar to a diagonal matrix** if it has n-independent eigenvectors.
5. The set of all eigenvalues of A is called the **spectrum of the matrix A.**
6. The largest (in absolute value) eigenvalue of the matrix A is called the **spectral radius of A.**
7. If A is a real symmetric matrix, then all eigenvalues are real.
8. If A is a real skew symmetric matrix, then all eigenvalues are imaginary.
9. If $a = \max |a_{ij}|$ and λ is an eigenvalue of A, then $|\lambda| \le na$.
10. An eigenvalue of A must lie within one of n circular disks whose centers are a_{ii}, $i = 1, 2, \dots, n$, and whose radii are

$$r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|;$$

that is, the centers of these disks are determined by the elements along the main diagonal of A, and the radius of the disk with center at a_{ii} is obtained by deleting a_{ii} from the ith row and then summing the absolute value of the remaining elements in the ith row.

11. The eigenvalues of a real matrix A satisfy:

$$(a) \qquad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{Trace}\,(A)$$

$$(b) \qquad \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n = \det\,(A)$$

$$(c) \qquad \sum_{i=1}^{n} \lambda_i^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$$

Example 10-26. For the matrix $A = \begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix}$ find matrices Q and Q^{-1} such that $Q^{-1}AQ$ is a diagonal matrix.

Solution: Calculate the eigenvalues $\lambda_1,\ \lambda_2$ and eigenvectors $X_1,\ X_2$ of the given matrix A and show

$$\lambda_1 = 1, \quad X_1 = \text{col}[\sqrt{3}, 1] \qquad \text{and} \qquad \lambda_2 = 2, \quad X_2 = \text{col}[1, -\sqrt{3}].$$

Let $Q = [X_1, X_2] = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ denote the matrix containing the eigenvectors of A for its column vectors. By definition the eigenvalues and eigenvectors satisfy the equations

$$AX_1 = \lambda_1 X_1 \qquad \text{and} \qquad AX_2 = \lambda_2 X_2,$$

and these equations can be expressed using the above notation as

$$\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} \lambda_1 x_{11} \\ \lambda_1 x_{21} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \lambda_2 x_{12} \\ \lambda_2 x_{22} \end{bmatrix}.$$

These two sets of linear equations can be represented by the single matrix equation

$$\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{10.33}$$

The matrix whose column vectors are n-linearly independent eigenvectors of A is called the **modal matrix** associated with A. Here Q is the modal matrix of A. Denote the diagonal matrix having the eigenvalues of A for the elements on the diagonal as $D = \text{diag}(\lambda_1, \lambda_2)$, then the equation (10.33) can be written as

$$AQ = QD \tag{10.34}$$

Left multiplication by Q^{-1} gives $Q^{-1}AQ = D$, where

$$Q = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \qquad Q^{-1} = \frac{1}{4} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \qquad D = \text{diag}(1, 2)$$

This example illustrates that the modal matrix can be used to reduce a given matrix to a diagonal form.

Changes of variables of the form $D = Q^{-1}AQ$, for the proper choice of the matrix Q, is a transformation often used to produce diagonal matrices in a variety of applications.

■

Example 10-27. Find the eigenvalues and eigenvectors associated with the matrix

$$A = \begin{bmatrix} -1 & 4 & 6 & -6 \\ 1 & 4 & 0 & 2 \\ -4 & 0 & 7 & -8 \\ -1 & -2 & 0 & 0 \end{bmatrix}$$

Solution: The characteristic equation of A can be calculated by evaluating the determinant

$$C(\lambda) = |A - \lambda I| = \begin{vmatrix} -1-\lambda & 4 & 6 & -6 \\ 1 & 4-\lambda & 0 & 2 \\ -4 & 0 & 7-\lambda & -8 \\ -1 & -2 & 0 & -\lambda \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) = 0.$$

The eigenvalues are $\lambda = 1,\ 2,\ 3,\ 4$ and each eigenvector associated with A must be a nonzero solution vector which satisfies the equation

$$AX = \lambda X$$

or

$$\begin{bmatrix} -1-\lambda & 4 & 6 & -6 \\ 1 & 4-\lambda & 0 & 2 \\ -4 & 0 & 7-\lambda & -8 \\ -1 & -2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Substitute successively the values $\lambda = 1,\ 2,\ 3,\ 4$ into this equation and each time solve for X to obtain the eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The **modal matrix** Q, is the matrix having these eigenvectors for its column vectors. The modal matrix Q can be used to produce a diagonal matrix containing the eigenvalues of A such that

$$Q^{-1}AQ = D = \text{diag}(1, 2, 3, 4).$$

The proof of this statement is left as an exercise. It can also be verified that

$$\det(A) = 24 \quad \text{and} \quad (\text{rank}\, A) = 4.$$

As a final note, it should be pointed out that when one or more of the eigenvalues of a matrix A are repeated roots, then a set of n linearly independent eigenvectors may or may not exist. The number N of linearly independent eigenvectors associated with an eigenvalue λ_i is given by the formula

$$N = n - (\text{rank}\,[A - \lambda_i I]),$$

where n is the rank of A.

■

Infinite Series of Square Matrices

In the following discussions it is to be understood that the matrix A is an $n \times n$ constant square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ which are distinct. Consider the infinite series

$$S(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{10.35}$$

and the corresponding matrix infinite series

$$S(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_k A^k + \cdots = \sum_{k=0}^{\infty} c_k A^k. \tag{10.36}$$

where the $n \times n$ matrix A has replaced the value x in equation (10.35) and the identity matrix has replaced the coefficient of the c_0 constant term. Convergence of the matrix infinite series can be defined in a manner analogous to that of the scalar infinite series. Examine the sequence of partial sums

$$S_N = \sum_{k=0}^{N} c_k A^k$$

and if $\lim_{N\to\infty} S_N$ exists, then the matrix series is said to converge, otherwise it is said to diverge. It can be shown that if the series in equation (10.35) is convergent for $x = \lambda_i$ $(i = 1, 2, \ldots, n)$, where λ_i is an eigenvalue of A, then the matrix series in equation (10.36) is convergent.

Some specific examples of series associated with a $n \times n$ constant square matrix A are the following.

1. The Exponential Series Corresponding to the scalar exponential series

$$e^{x\,t} = 1 + x\,t + x^2 \frac{t^2}{2!} + \cdots + x^k \frac{t^k}{k!} + \cdots$$

there is **the exponential matrix** $e^{A\,t}$ defined by the series

$$e^{A\,t} = I + A\,t + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots \tag{10.37}$$

Here A is constant so that one can differentiate equation (10.37) with respect to t to obtain

$$\begin{aligned}
\frac{d}{dt}\left(e^{A\,t}\right) =&A + A^2 t + A^3 \frac{t^2}{2!} + A^4 \frac{t^3}{3!} + \cdots + A^k \frac{t^{k-1}}{(k-1)!} + \cdots \\
\frac{d}{dt}\left(e^{A\,t}\right) =&A\left(I + A\,t + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots\right) \\
\frac{d}{dt}\left(e^{A\,t}\right) =&A\,e^{A\,t} = e^{A\,t}A
\end{aligned} \tag{10.38}$$

The exponential matrix e^X is an important matrix used for solving systems of linear differential equations. The exponential matrix has the following properties which are stated without proofs.

1. If the matrices X and Y commute, so that $XY = YX$, then one can write

$$e^X e^Y = e^Y e^X = e^{X+Y} \tag{10.39}$$

However, if the matrices X and Y **do not commute** so that $XY \neq YX$, then the equation (10.39) is not true.

2. $\left(e^X\right)^{-1} = e^{-X}$
3. $e^{[0]} = I$
4. $e^X e^{-X} = I$
5. $e^{\alpha X} e^{\beta X} = e^{(\alpha+\beta)X}$
6. $e^{X^T} = \left(e^X\right)^T$

2. The Sine Series Corresponding to the scalar sine series

$$\sin(x\,t) = x\,t - \frac{x^3\,t^3}{3!} + \frac{x^5\,t^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}\,t^{2n+1}}{(2n+1)!} + \cdots$$

there is the matrix sine series

$$\sin(A\,t) = A\,t - \frac{A^3\,t^3}{3!} + \frac{A^5\,t^5}{5!} + \cdots + (-1)^n \frac{A^{2n+1}\,t^{2n+1}}{(2n+1)!} + \cdots \tag{10.40}$$

3. The Cosine Series Corresponding to the scalar cosine series

$$\cos(x\,t) = 1 - \frac{x^2\,t^2}{2!} + \frac{x^4\,t^4}{4!} - \cdots + (-1)^n \frac{x^{2n}\,t^{2n}}{(2n)!} + \cdots$$

there is the matrix cosine series

$$\cos(A\,t) = I - \frac{A^2\,t^2}{2!} + \frac{A^4\,t^4}{4!} + \cdots + (-1)^n \frac{A^{2n}\,t^{2n}}{(2n)!} + \cdots \tag{10.41}$$

Differentiate equation (10.40) with respect to t and show

$$\begin{aligned}
\frac{d}{dt}\sin(At) =& A - A^3\frac{t^2}{2!} + A^5\frac{t^4}{4!} + \cdots + (-1)^n A^{2n+1}\frac{t^{2n}}{(2n)!} + \cdots \\
\frac{d}{dt}\sin(At) =& A\left(I - \frac{A^2\,t^2}{2!} + \frac{A^4\,t^4}{4!} + \cdots + (-1)^n\frac{A^{2n}\,t^{2n}}{(2n)!} + \cdots\right) \\
\frac{d}{dt}\sin(At) =& A\cos(At) = \cos(At)\,A
\end{aligned} \tag{10.42}$$

Differentiate equation (10.41) with respect to t and show

$$\begin{aligned}
\frac{d}{dt}\cos(At) =& -A^2\,t + A^4\frac{t^3}{3!} + \cdots + (-1)^n A^{2n}\frac{t^{2n-1}}{(2n-1)!} + \cdots \\
\frac{d}{dt}\cos(At) =& -A\left(At - \frac{A^3\,t^3}{3!} + \frac{A^5\,t^5}{5!} + \cdots + (-1)^n\frac{A^{2n+1}\,t^{2n+1}}{(2n+1)!} + \cdots\right) \\
\frac{d}{dt}\cos(At) =& -A\sin(At) = -\sin(At)\,A
\end{aligned} \tag{10.43}$$

Example 10-28. Show that if A is a constant matrix, then $X(t) = e^{A(t-t_0)}$ is a matrix solution to the initial-value problem to solve

$$\frac{dX}{dt} = AX, \qquad X(t_0) = I$$

Solution Differentiate X and show $\frac{dX}{dt} = Ae^{A(t-t_0)} = AX$ is satisfied. At the initial time $t = t_0$, one finds $X(t_0) = e^{A(0)} = I$.

■

The Hamilton-Cayley Theorem

The Hamilton[4]-Cayley[5] theorem states that **every $n \times n$ constant square matrix A satisfies its own characteristic equation.** That is, if $C(\lambda) = 0$ is the characteristic equation association with a $n \times n$ square matrix A, then the equation $C(A) = [0]$ must be satisfied. This result is known as the Hamilton-Cayley theorem.

---

[4] William Rowan Hamilton (1806–1865),Irish mathematician and physicist.

[5] Arthur Cayley (1821–1895),English mathematician.

Example 10-29. The following is an example illustrating the Hamilton-Cayley theorem. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix},$$

then the characteristic equation associated with the matrix A is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1 = 0.$$

Replacing the scalar λ by the matrix A one obtains $C(A) = A^2 - 4A + I$, where I is the 2×2 identity matrix. The given matrix A when squared gives

$$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}.$$

Substituting I,A and A^2 into $C(A)$ gives

$$C(A) = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [0]$$

and, hence, A satisfies its own characteristic equation.

■

In order to prove the Hamilton-Cayley theorem, assume the $n \times n$ constant square matrix A is given and it has associated with it the characteristic polynomial of the form

$$C(\lambda) = |A - \lambda I| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-2}\lambda^2 + \alpha_{n-1}\lambda + \alpha_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are appropriate scalar constants. Replace the scalar λ by the matrix A and replace the constant term α_n by $\alpha_n I$, to obtain the Hamilton-Cayley matrix equation

$$C(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2}A^2 + \alpha_{n-1}A + \alpha_n I$$

To prove the Hamilton-Cayley theorem it must be demonstrated that $C(A) = [0]$. Toward this purpose replace the matrix A in equation (10.23) by the matrix $A - \lambda I$ to obtain

$$(A - \lambda I)\mathrm{Adj}(A - \lambda I) = |A - \lambda I| I,$$

where the various elements of the matrix $\mathrm{Adj}(A - \lambda I)$ are formed from $A - \lambda I$ by deleting a certain row and column and then taking the determinant of the $(n-1) \times (n-1)$ system that remains. This implies λ^{n-1} is the highest power of λ that can be in any element of the matrix $\text{Adj}(A-\lambda I)$. Observe that the equation $\text{Adj}(A-\lambda I)$ can be written in the form

$$\text{Adj}(A-\lambda I) = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-2}\lambda^2 + B_{n-1}\lambda + B_n,$$

where $B_1,\ B_2,\ldots,B_n$ are $n\times n$ matrices not containing λ and in general depend upon the elements of A. Using the relation

$$(A-\lambda I)\text{Adj}(A-\lambda I) = |A-\lambda I|I = C(\lambda)I$$

Write the right-hand side as

$$C(\lambda)I = \lambda^n I + \alpha_1\lambda^{n-1}I + \cdots + \alpha_{n-2}\lambda^2 I + \alpha_{n-1}\lambda I + \alpha_n I,$$

and write the left-hand side as

$$\begin{aligned}(A-\lambda I)&(B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-2}\lambda^2 + B_{n-1}\lambda + B_n)\\ &= -B_1\lambda^n - B_2\lambda^{n-1} - \ldots - B_{n-2}\lambda^3 - B_{n-1}\lambda^2 - B_n\lambda\\ &\quad + AB_1\lambda^{n-1} + \cdots + AB_{n-3}\lambda^3 + AB_{n-2}\lambda^2 + AB_{n-1}\lambda + AB_n\end{aligned}$$

By comparing the left and right-hand sides of this equation one can equate the coefficients of like powers of λ and obtain the following equations.

$$\begin{aligned}
-B_1 &= I & &: \ A^n\\
AB_1 - B_2 &= \alpha_1 I & &: \ A^{n-1}\\
AB_2 - B_3 &= \alpha_2 I & &: \ A^{n-2}\\
&\vdots & &\vdots\\
AB_{n-3} - B_{n-2} &= \alpha_{n-3}I & &: \ A^3\\
AB_{n-2} - B_{n-1} &= \alpha_{n-2}I & &: \ A^2\\
AB_{n-1} - B_n &= \alpha_{n-1}I & &: \ A\\
AB_n &= \alpha_n I & &: \ I
\end{aligned}$$

Now multiply the first equation by A^n, the second equation by A^{n-1}, the third equation by A^{n-2},..., the second to last equation by A and the last equation by I. The multiplication factors are illustrated to the right-hand side of the equations listed above. After multiplication, the equations are summed. Note that on the right-hand side there results the matrix equation $C(A)$ and on the left-hand side the summation produces the zero matrix. This establishes the Hamilton-Cayley theorem.

Evaluation of Functions

Let $f(x)$ denote a scalar function of x, where all derivatives with respect to x are defined at $x=0$. Functions which satisfy this condition can then be represented as a power series expansion about the point $x=0$. This power series expansion has the form

$$f(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where c_k are the coefficients of the power series. Let A denote an $n \times n$ matrix with characteristic equation $C(\lambda) = 0$ which has the roots (eigenvalues) λ_i, $(i = 1, 2, \dots, n)$.

The infinite power series for $f(x)$ can be represented in the alternative form

$$f(x) = C(x) \sum_{k=0}^{\infty} c_k^* x^k + R(x), \tag{10.44}$$

where c_k^* are new coefficients to be determined, $C(x)$ is the characteristic polynomial associated with the matrix A, and $R(x)$ is a remainder polynomial of degree less than or equal to $(n-1)$ which can be expressed in the form

$$R(x) = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_{n-1} x + \beta_n$$

where $\beta_1, \beta_2, \dots, \beta_n$ are constants. By the Hamiliton–Cayley theorem $C(A) = [0]$, thus, the matrix function $f(A)$ becomes

$$f(A) = R(A), \tag{10.45}$$

which implies the matrix function $f(A)$ can be expressed as some linear combination of the matrices $\{I, A, A^2, \dots, A^{n-1}\}$ and consequently must have the form

$$f(A) = R(A) = \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_{n-1} A + \beta_n I$$

where $\beta_1, \dots, \beta_n$ are constants to be determined. This result is not unexpected since it has been previously shown how one can use the Hamilton–Cayley theorem to express all powers of A, greater than or equal to the dimension n of A, in terms of linear combinations of the integer powers of A less than or equal to $n-1$. Also, from equation (10.44), one can write the n special relations

$$f(\lambda_i) = R(\lambda_i), \quad i = 1, 2, \dots, n, \tag{10.46}$$

which must exist between the functions $f(x)$ and $R(x)$. These equations are n independent relations one can use to solve for the unknown coefficients β_i in the polynomial representation for $R(x)$. If λ_i is a repeated root of $C(\lambda)=0$, then the equations (10.46) do not form a set of n-linearly independent equations. However, if the eigenvalue λ_i is a repeated root of $C(\lambda)=0$, then the derivative relation

$$\frac{dC}{d\lambda}\bigg|_{\lambda=\lambda_i}=0$$

must also be true. By differentiating equation (10.44) and evaluating the result at $\lambda=\lambda_i$, the equations

$$\frac{df(x)}{dx}\bigg|_{x=\lambda_i}=\frac{dR}{dx}\bigg|_{x=\lambda_i}$$

can be used to determine the constants in the representation of $R(A)$. That is, if m_i is the multiplicity of the characteristic root λ_i, then the equations

$$f(\lambda_i)=R(\lambda_i),\quad \frac{df}{dx}\bigg|_{x=\lambda_i}=\frac{dR}{dx}\bigg|_{x=\lambda_i},\ldots,\frac{d^{m_i-1}f}{dx^{m_i-1}}\bigg|_{x=\lambda_i}=\frac{d^{m_i-1}R}{dx^{m_i-1}}\bigg|_{x=\lambda_i}, \tag{10.47}$$

form a set of m_i linear equations. These equations can be used to determine the coefficients in the remainder polynomial $R(x)$ and consequently the matrix $R(A)$ representing $f(A)$ can be determined.

Example 10-30. Given the matrix

$$A=\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

find the matrix function $f(A)=A^k$, where k is a positive integer.

Solution: The characteristic equation of A is

$$C(\lambda)=|A-\lambda I|=\begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix}=\lambda^2-6\lambda+5=(\lambda-5)(\lambda-1)=0.$$

Examination of the above equation one can see that $\lambda_1=1$ and $\lambda_2=5$ are the characteristic roots or eigenvalues of the given matrix A. The Hamilton–Cayley theorem requires that $C(A)=[0]$, which implies

$$A^2=6A-5I$$

Successive multiplications by the matrix A gives

$$A^3=6A^2-5A=6(6A-5I)-5A=31A-30I$$
$$A^4=31A^2-30A=31(6A-5I)-30A=156A-155I,$$

and continuing in this manner, one finds the general form

$$R(A) = f(A) = A^k = \beta_1 A + \beta_2 I$$

for some constants β_1 and β_2 (i.e., $f(A)$ is some linear combination of $\{I, A\}$). For this example,

$$R(x) = \beta_1 x + \beta_2 \quad \text{and} \quad f(x) = x^k$$

and consequently

$$\begin{aligned} R(\lambda_1) &= R(1) = \beta_1 + \beta_2 = (1)^k = 1 = f(1) \\ R(\lambda_2) &= R(5) = 5\beta_1 + \beta_2 = (5)^k = f(5). \end{aligned} \tag{7.11}$$

From these equations the unknown constants β_1 and β_2 can be determined. As an exercise show that

$$\beta_1 = \frac{1}{4}(5^k - 1) \qquad \text{and} \qquad \beta_2 = \frac{1}{4}(5 - 5^k).$$

The matrix relation

$$R(A) = f(A) = A^k = \left(\frac{5^k - 1}{4}\right) A + \left(\frac{5 - 5^k}{4}\right) I$$

is a general formula for expressing the powers of the matrix A as a linear combination of the matrices $\{A, I\}$. Checking this result with the previous calculations obtained by use of the Hamilton–Cayley theorem, one finds

$$\begin{aligned} &\text{for } k = 0, \qquad A^0 = I \\ &\text{for } k = 1, \qquad A^1 = A \\ &\text{for } k = 2, \qquad A^2 = 6A - 5I \\ &\text{for } k = 3, \qquad A^3 = 31A - 30I \\ &\text{for } k = 4, \qquad A^4 = 156A - 155I \end{aligned}$$

which agrees with the previous results.

In general, the Hamilton-Cayley theorem implies that if A is a $n \times n$ square matrix, then powers of A, say A^m, for integer values of m, can be represented in the form

$$A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$$

where $c_0, \ldots, c_{n-1}$ are constants to be determined.

■

Example 10-31. Given the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

find the matrix function $f(A) = e^{At}$.

Solution: Here $f(x) = e^{xt}$, and from the previous example, the eigenvalues of A have been determined as $\lambda_1 = 1$ and $\lambda_2 = 5$. If the matrix equation is to be represented in the form

$$f(A) = R(A) = \gamma_1 A + \gamma_2 I$$

which is a linear combination of $\{I, A\}$ then one can use the equation (10.46) and write

$$\begin{aligned} f(\lambda_1) &= e^{\lambda_1 t} = e^t = \gamma_1(1) + \gamma_2 \\ f(\lambda_2) &= e^{\lambda_2 t} = e^{5t} = \gamma_1(5) + \gamma_2 \end{aligned} \tag{10.48}$$

From these equations it is possible to solve for γ_1 and γ_2 and show

$$\gamma_1 = \frac{e^{5t} - e^t}{4}, \qquad \text{and} \qquad \gamma_2 = \frac{5e^t - e^{5t}}{4}.$$

The matrix function for $f(A)$ can then be represented as

$$f(A) = e^{At} = \left(\frac{e^{5t} - e^t}{4}\right) A + \left(\frac{5e^t - e^{5t}}{4}\right) I$$

which has the equivalent matrix form

$$e^{At} = \frac{1}{4}\begin{bmatrix} (e^{5t} + 3e^t) & (e^t - e^{5t}) \\ (3e^t - 3e^{5t}) & (3e^{5t} + e^t) \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

■

Example 10-32. Find the matrix function

$$f(A) = \sin At \quad \text{for} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation of A is

$$C(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (\lambda^2 - 1)(1 - \lambda) = 0.$$

The eigenvalues of A are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = -1.$$

Express $f(A)$ as a linear combination of the matrices $\{I, A, A^2\}$ and write

$$f(A) = R(A) = \sin At = c_0 I + c_1 A + c_2 A^2,$$

where c_0, c_1 and c_2 are functions of t to be determined. Since there is a repeated root, use differentiation and write the equations

$$\begin{aligned} R(x) &= c_0 + c_1 x + c_2 x^2 \\ R'(x) &= c_1 + 2c_2 x \end{aligned}$$

to obtain the system of equations

$$\begin{aligned} R(\lambda_1) = f(\lambda_1) &= \sin t = c_0 + c_1 + c_2 \\ R'(\lambda_1) = f'(\lambda_1) &= \cos t = c_1 + 2c_2 \\ R(\lambda_3) = f(\lambda_3) &= -\sin t = c_0 - c_1 + c_2 \end{aligned} \tag{10.49}$$

These are three independent equations which can be used to solve for the coefficients c_0, c_1 and c_2. Solving these equations for $c_0,\ c_1$ and c_2 one finds

$$c_0 = \frac{1}{2}(\sin t - \cos t), \qquad c_1 = \sin t, \qquad c_2 = \frac{1}{2}(\cos t - \sin t)$$

and consequently the matrix function $\sin At$ has the representation

$$\sin At = \left(\frac{\sin t - \cos t}{2}\right) I + A \sin t + \frac{1}{2}(\cos t - \sin t) A^2.$$

An alternate form for this result is the matrix form

$$\sin At = \frac{1}{2}\begin{bmatrix} 0 & 2\sin t & \cos t - \sin t \\ \cos t + \sin t & \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\cos t & \cos t + \sin t \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

■

Four-terminal Networks

Consider the electrical networks illustrated in figure 10-6. No matter how complicated the circuit inside the boxes, there are only two input and two output terminals. Such devices are called four terminal networks and are represented by a box like those illustrated in figure 10-6, where the quantities Z, Z_1, and Z_2 are called impedances. Impedances Z are used in alternating current (a.c.) circuits and are analogous to the resistance R use in direct current (d.c.) circuits.

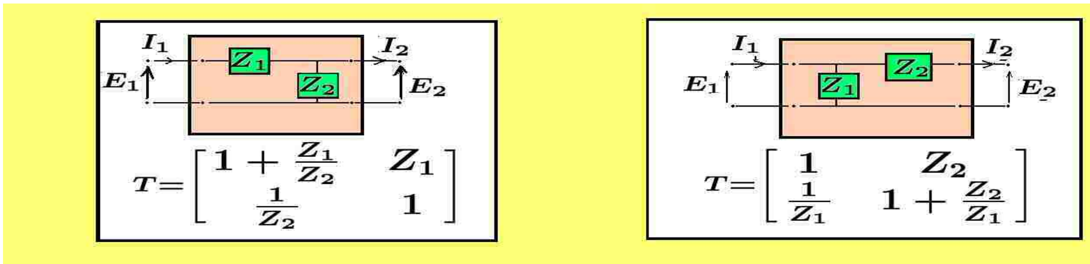


Figure 10-6. Four terminal networks

In figure 10-6 the quantities I_1, E_1 and I_2, E_2 are the input and output current and voltages. The column vectors $S_1 = \text{col}[E_1, I_1]$ and $S_2 = \text{col}[E_2, I_2]$ are called the input state vector and output state vector of the network. The networks are assumed to be linear so that the general relation between the input and output states can be expressed as the matrix equation

$$S_1 = TS_2, \quad \text{where} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is called the transmission matrix . The element $T_{11}, T_{12}, T_{21}, T_{22}$ are in general complex numbers which satisfy the property $\det T = T_{11}T_{22} - T_{21}T_{12} = 1$. Solving for S_2 in terms of S_1 gives

$$S_2 = T^{-1}S_1 = PS_1$$

where the matrix P is called the transfer matrix of the network and is given by

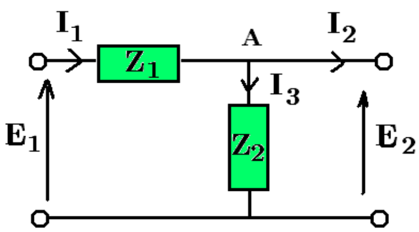
$$P = \begin{bmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix}$$

If the input output current and voltages are linearly related, then it is easy to solve for the currents in terms of the voltages or the voltages in terms of the currents to obtain

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{T_{22}}{T_{12}} & -\frac{1}{T_{12}} \\ \frac{1}{T_{12}} & -\frac{T_{11}}{T_{12}} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \qquad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{T_{11}}{T_{21}} & -\frac{1}{T_{21}} \\ \frac{1}{T_{21}} & -\frac{T_{22}}{T_{21}} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Applying Kirchoff's laws to the short-circuit condition $E_2 = 0$ and the open-circuit condition $I_2 = 0$ allows for the determination of the transmission matrices given in the figure 10-6.

Example 10-33.



For the four terminal network illustrated one must have

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} E_1 =& T_{11}E_2 + T_{12}I_2 \\ I_1 =& T_{21}E_2 + T_{22}I_2 \end{aligned}$$

At the junction labeled A one must have $I_1 = I_2 + I_3$

(i) Examine the open-circuit condition $I_2 = 0$ and show $E_1 = T_{11}E_2$, $I_1 = T_{21}E_2$ and $I_1 = I_3$. This gives the relations

$$T_{11} = \frac{E_1}{E_2} = \frac{I_1Z_1 + I_1Z_2}{I_1Z_2} = 1 + \frac{Z_1}{Z_2} \qquad \text{and} \qquad I_1 = T_{21}(I_1Z_2) \quad \text{or} \quad T_{21} = \frac{1}{Z_2}$$

(ii) Examine the short-circuit condition $E_2 = 0$ and show $I_3 = 0$ so that $I_1 = I_2$, then under these conditions

$$E_1 = T_{12}I_2 \quad \text{or} \quad T_{12} = \frac{E_1}{I_2} = \frac{I_1Z_1}{I_1} = Z_1 \quad \text{and} \quad I_1 = T_{22}I_1 \qquad \Longrightarrow T_{22} = 1$$

Also note the determinant of the transmission matrix is unity.

■

Calculus of Finite Differences

There are many concepts in science and engineering that can be approached from either a **discrete or a continuous** viewpoint. For example, consider how you might record the temperature outside at some specific place as a function of time. One technique would be to purchase a chart recorder capable of measuring and plotting the temperature as a function of time. This would give a continuous record of the temperature over some interval of time. Another way to record the temperature would be to measure the temperature, at the specified place, at discrete time intervals. The contrast between these two methods is that one method measures temperature continuously while the other method measures the temperature in a discrete fashion.

In any laboratory experiment, one must make a decision as to how data from the experiment is to be collected. Whether discrete measurements or continuous measurements are recorded depends upon many factors as well as the type of experiment being considered. The techniques used to analyze the data collected depends upon whether the data is continuous or discrete.

The investment of money at compound interest is an example of a physical problem which requires analysis of discrete values. Say, \$1,000.00 is to be invested at R percent interest compounded quarterly. How does one determine the discrete values representing the amount of money available at the end of each compound period? To solve this problem, let P_0 denote the amount of money initially invested, R the percent interest yearly with $\frac{1}{4}\frac{R}{100} = i$ the quarterly interest and let P_n denote the principal due at the end of the nth compound period. The equations for the determination of P_n can be found by examining the discrete values produced. For P_0 the initial amount invested, one finds

$$
\begin{aligned}
P_1 &= P_0 + P_0 i = P_0(1+i) \\
P_2 &= P_1 + P_1 i = P_1(1+i) = P_0(1+i)^2 \\
P_3 &= P_2 + P_2 i = P_2(1+i) = P_0(1+i)^3 \\
&\vdots \\
P_n &= P_{n-1} + P_{n-1} i = P_{n-1}(1+i) = P_0(1+i)^n
\end{aligned}
$$

For $i = \frac{1}{4}\frac{R}{100}$ and $P_0 = 1,000.00$, figure 10-7 illustrates a graph of P_n vs time, for a 30 year period, where one year represents four payment periods. In this figure values of R for 4%, 5.5%, 7%, 8.5% and 10% were used in the above calculations.

Let us investigate some techniques that can be used in the analysis of discrete phenomena like the compound interest problem just considered.

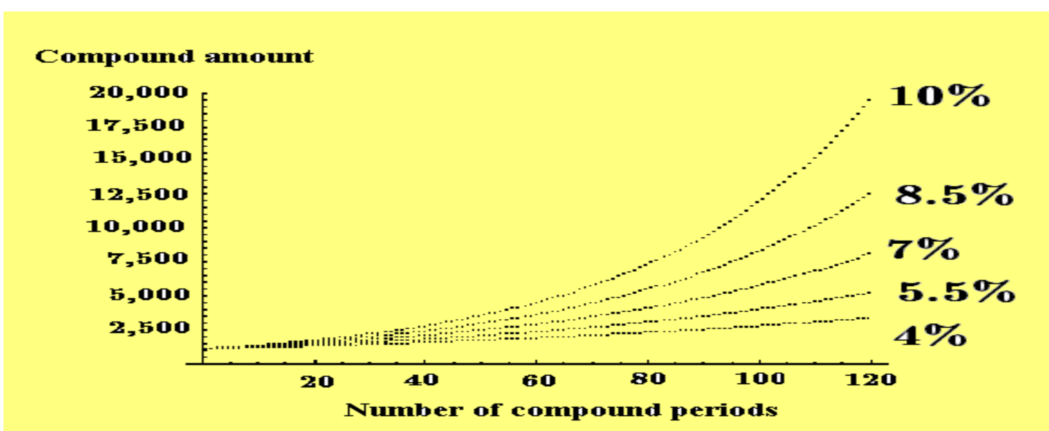


Figure 10-7.

Return from \$1,000 investment compounded quarterly over 30 year period.

The study of calculus has demonstrated that derivatives are the mathematical quantities that represent **continuous change**. If derivatives (continuous change) are replaced by **differences (discrete change)**, then linear ordinary differential equations become **linear difference equations**. Let us begin our study of discrete phenomena by investigating difference equations and determining ways to construct solutions to such equations.

In the following discussions, note that the various techniques developed for analyzing discrete systems are very similar to many of the methods used for studying continuous systems.

Differences and Difference Equations

Consider the function $y = f(x)$ illustrated in the figure 10-8 which is evaluated at the equally spaced $x-$values of $x_0, x_1, x_2, \ldots, x_i, \ldots, x_n, x_{n+1}, \ldots$ where $x_{i+1} = x_i + h$ for $i = 0, 1, 2, \ldots, n$ where h is the distance between two consecutive points.

Let $y_n = f(x_n)$ and consider **the approximation of the derivative** $\frac{dy}{dx}$ **at the discrete value** x_n**.** Use the definition of a derivative and write the approximation as

$$\left.\frac{dy}{dx}\right|_{x=x_n} \approx \frac{y_{n+1} - y_n}{h}.$$

This is called a **forward difference approximation.** By letting $h = 1$ in the above equation one can define the first forward difference of y_n as

$$\Delta y_n = y_{n+1} - y_n. \tag{10.50}$$

There is no loss in generality in letting $h = 1$, as one can always rescale the x-axis by defining the new variable X defined by the transformation equation $x = x_0 + Xh$, then when $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh, \ldots$ the scaled variable X takes on the values $X = 0,\ 1,\ 2, \ldots, n, \ldots$.

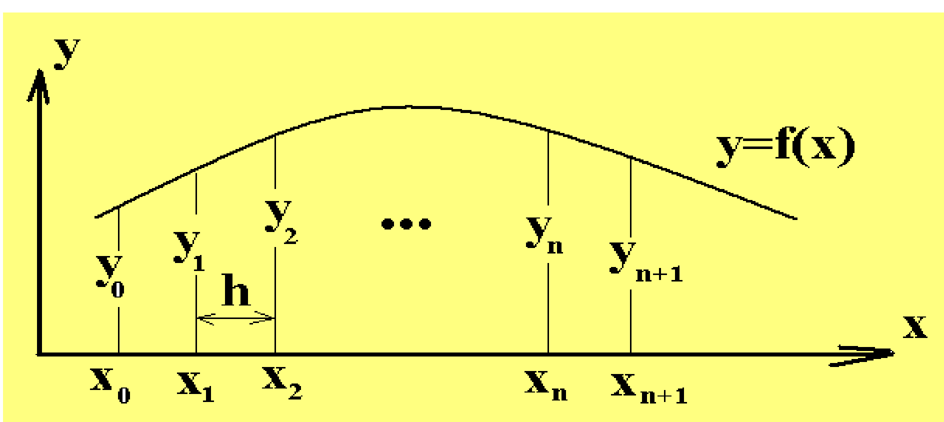


Figure 10-8. Discrete values of $y = f(x)$.

Define the **second forward difference as a difference of the first forward difference.** A second difference is denoted by the notation $\Delta^2 y_n$ and

$$
\begin{aligned}
&\Delta^2 y_n = \Delta(\Delta y_n) = \Delta y_{n+1} - \Delta y_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)\\
\text{or}\quad &\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n.
\end{aligned} \tag{10.51}
$$

Higher ordered difference are defined in a similar manner. A **nth order forward difference is defined as the difference of the $(n-1)$st forward difference, for $n = 2, 3, \ldots$.**

Analogous to the **differential operator $D = \frac{d}{dx}$, there is a stepping operator E** defined as follows:

$$
\begin{aligned}
Ey_n &= y_{n+1}\\
E^2 y_n &= y_{n+2}\\
&\cdots\\
E^m y_n &= y_{n+m}.
\end{aligned} \tag{10.52}
$$

From the definition given by equation (10.50) one can write the first ordered difference

$$
\Delta y_n = y_{n+1} - y_n = Ey_n - y_n = (E-1)y_n
$$

which illustrates that the difference operator Δ can be expressed in terms of the stepping operator E and

$$
\Delta = E - 1. \tag{10.53}
$$

This operator identity, enables us to express the second-order difference of y_n as

$$
\begin{aligned}
\Delta^2 y_n &= (E-1)^2 y_n\\
&= (E^2 - 2E + 1)y_n\\
&= E^2 y_n - 2Ey_n + y_n\\
&= y_{n+2} - 2y_{n+1} + y_n.
\end{aligned}
$$

Higher order differences such as $\Delta^3 y_n = (E-1)^3 y_n,\ \ \Delta^4 y_n = (E-1)^4 y_n, \ldots$ and higher ordered differences are quickly calculated by applying **the binomial expansion** to the operators operating on y_n.

Difference equations are equations which involve differences. For example, the equation

$$
L_2(y_n) = \Delta^2 y_n = 0
$$

is an example of a second-order difference equation, and

$$
L_1(y_n) = \Delta y_n - 3y_n = 0
$$

is an example of a first-order difference equation. The symbols $L_1(),\ L_2()$ are operator symbols representing linear operators. Using the operator E, the above equations can be written as

$$L_2(y_n) = \Delta^2 y_n = (E-1)^2 y_n = y_{n+2} - 2y_{n+1} + y_n = 0 \quad \text{and}$$
$$L_1(y_n) = \Delta y_n - 3y_n = (E-1)y_n - 3y_n = y_{n+1} - 4y_n = 0,$$

respectively.

There are many instances where variable quantities are assigned values at uniformly spaced time intervals. Let us study these discrete variable quantities by using differences and difference equations. **An equation which relates values of a function y and one or more of its differences is called a difference equation.** In dealing with difference equations one assumes that the function y and its differences $\Delta y_n,\ \Delta^2 y_n, \ldots$, evaluated at x_n, are all defined for every number x in some set of values $\{x_0,\ x_0+h,\ x_0+2h, \ldots, x_0+nh, \ldots\}$. A difference equation is called linear and of order m if it can be written in the form

$$L(y_n) = a_0(n)y_{n+m} + a_1(n)y_{n+m-1} + \cdots + a_{m-1}(n)y_{n+1} + a_m(n)y_n = g(n), \tag{10.54}$$

where the coefficients $a_i(n),\ i = 0,\ 1,\ 2, \ldots, m$, **and the right-hand side** $g(n)$ **are known functions of** n. If $g(n) \neq 0$, the difference equation is said to be **nonhomogeneous** and if $g(n) = 0$, the difference equation is called **homogeneous.**

The difference equation (10.54) can be written in the operator form

$$L(y_n) = [a_0(n)E^m + a_1(n)E^{m-1} + \cdots + a_{m-1}(n)E + a_m(n)]y_n = g(n),$$

where E is the stepping operator.

A mth-order **linear initial value problem** associated with a mth-order linear difference equation consists of a linear difference equation of the form given in the equation (10.54) together with a set of m **initial values** of the type

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad \ldots, \quad y_{m-1} = \alpha_{m-1},$$

where $\alpha_0,\ \alpha_1, \ldots, \alpha_{m-1}$ are specified constants.

Example 10-34.

Show $\Delta a^k = (a-1)a^k$, for a constant and k an integer.

Solution: Let $y_k = a^k$, then by definition

$$\Delta y_k = y_{k+1} - y_k = a^{k+1} - a^k = (a-1)a^k.$$

■

Example 10-35.

The function

$$k^{\underline{N}} = k(k-1)(k-2)\cdots[k-(N-2)][k-(N-1)], \qquad k^{\underline{0}} \equiv 1$$

is called **a factorial falling function** which is a polynomial function. Here $k^{\underline{N}}$ is a product of N terms. Show $\Delta k^{\underline{N}} = N\, k^{\underline{N-1}}$ for N a positive integer and fixed.

Solution: Observe that the factorial polynomials are

$$k^{\underline{0}} = 1, \quad k^{\underline{1}} = k, \quad k^{\underline{2}} = k(k-1), \quad k^{\underline{3}} = k(k-1)(k-2), \quad \cdots$$

Use $y_k = k^{\underline{N}}$ and calculate the forward difference

$$\begin{aligned}\Delta y_k = &\; y_{k+1} - y_k = (k+1)^{\underline{N}} - k^{\underline{N}} \\ = &\; (k+1)\underbrace{(k)(k-1)\cdots[k+1-(N-1)]}_{k^{\underline{N-1}}} - \underbrace{k(k-1)(k-2)\cdots[k-(N-2)]}_{k^{\underline{N-1}}}[k-(N-1)]\end{aligned}$$

which simplifies to

$$\Delta y_k = \Delta k^{\underline{N}} = \{(k+1) - [k-(N-1)]\}\, k^{\underline{N-1}} = N k^{\underline{N-1}}.$$

The function $k^{\overline{N}} = k(k+1)(k+2)\cdots[k+(N-2)][k+(N-1)]$ is the **factorial rising function**. As an exercise show $\Delta \dfrac{1}{k^{\overline{N}}} = \dfrac{-N}{k^{\overline{N+1}}}$

■

Example 10-36.

Verify the forward difference relation

$$\Delta(U_k V_k) = U_k \Delta V_k + V_{k+1}\Delta U_k$$

Solution: Let $y_k = U_k V_k$, then write

$$\begin{aligned}\Delta y_k &= y_{k+1} - y_k \\ &= U_{k+1}V_{k+1} - U_k V_k + [U_k V_{k+1} - U_k V_{k+1}] \\ &= U_k[V_{k+1} - V_k] + V_{k+1}[U_{k+1} - U_k] \\ &= U_k \Delta V_k + V_{k+1}\Delta U_k.\end{aligned}$$

■

Special Differences

The table 10.1 contains a list of some well known forward differences which are useful in many applications. The verification of these differences is left as an exercise.

Table 10.1 Some common forward differences		
1.	$\Delta a^k = (a-1)a^k$	
2.	$\Delta k^{\underline{N}} = N\,k^{\underline{N-1}} \qquad N$ fixed	$k^{\underline{N}}$ is factorial falling
3.	$\Delta \sin(\alpha+\beta k) = 2\sin(\beta/2)\cos(\alpha+\beta/2+\beta k)$	α, β constants
4.	$\Delta \cos(\alpha+\beta k) = -2\sin(\beta/2)\sin(\alpha+\beta/2+\beta k)$	α, β constants
5.	$\Delta\binom{k}{N} = \binom{k}{N-1} \qquad N$ fixed	$\binom{k}{N}$ are binomial coefficients
6.	$\Delta(k!) = k(k!)$	
7.	$\Delta(U_kV_k) = U_k\Delta V_k + V_{k+1}\Delta U_k$	
8.	$\Delta\left(\frac{1}{k^{\overline{N}}}\right) = \frac{-N}{k^{\overline{N+1}}}, \qquad N$ fixed	$k^{\overline{N}}$ is factorial rising
9.	$\Delta k^2 = 2k+1$	
10.	$\Delta \log k = \log(1+1/k)$	

Finite Integrals

Associated with **finite differences** are **finite integrals**. If $\Delta y_k = f_k$, then the function y_k, whose difference is f_k, is called **the finite integral of** f_k. The inverse of the difference operation Δ is denoted Δ^{-1} and one can write $y_k = \Delta^{-1} f_k$, if $\Delta y_k = f_k$. For example, consider the difference of the factorial falling function $k^{\underline{N}}$. If $\Delta k^{\underline{N}} = N k^{\underline{N-1}}$, then $\Delta^{-1} N k^{\underline{N-1}} = k^{\underline{N}}$. Associated with the difference table 10.1 is the finite integral table 10.2. The derivation of the entries is left as an exercise.

Table 10.2 Some selected finite integrals		
1.	$\Delta^{-1}a^k = \dfrac{a^k}{a-1} \qquad a \neq 1$	
2.	$\Delta^{-1}k^{\underline{n}} = \dfrac{k^{\underline{n+1}}}{n+1}$	$k^{\underline{n}}$ is factorial falling
3.	$\Delta^{-1}\sin(\alpha+\beta k) = \dfrac{-1}{2\sin(\beta/2)}\cos(\alpha-\beta/2+\beta k)$	α, β constants
4.	$\Delta^{-1}\cos(\alpha+\beta k) = \dfrac{1}{2\sin(\beta/2)}\sin(\alpha-\beta/2+\beta k)$	α, β constants
5.	$\Delta^{-1}\binom{k}{n} = \binom{k}{n+1} \qquad$ n fixed	$\binom{k}{n}$ are binomial coefficients
6.	$\Delta^{-1}(a+bk)^{\underline{n}} = \dfrac{(a+bk)^{\underline{n+1}}}{b(n+1)}$	a, b constants.

Summation of Series

Let $\Delta y_k = y_{k+1} - y_k = f_k$, then one can substitute $k = 0, 1, 2, \ldots$ to obtain

$$\begin{aligned} y_1 - y_0 =& f_0 \\ y_2 - y_1 =& f_1 \\ y_3 - y_2 =& f_2 \\ &\vdots \\ y_n - y_{n-1} =& f_{n-1} \\ y_{n+1} - y_n =& f_n \end{aligned} \tag{10.55}$$

Adding these equations one obtains

$$\sum_{i=0}^{n} f_i = y_{n+1} - y_0 = \Delta^{-1} f_i\big]_{i=0}^{n+1} = y_i]_{i=0}^{n+1} \qquad \text{where} \quad \Delta y_k = f_k.$$

One can verify that by adding the equations (10.55) from some point $i = m$ to n, one obtains the more general result

$$\sum_{i=m}^{n} f_i = y_{n+1} - y_m = \Delta^{-1} f_i\big]_{i=m}^{n+1} = y_i]_{i=m}^{n+1}. \tag{10.56}$$

Example 10-37.

Evaluate the sum

$$S = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$$

Solution: Let $f_k = k(k+1) = k^2 + k$ and show one can write f_k as the factorial falling function $f_k = (k+1)^{\underline{2}}$. Therefore,

$$S = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} (i+1)^{\underline{2}} = \Delta^{-1} f_i\big]_{i=1}^{n+1} = \frac{(i+1)^{\underline{3}}}{3}\Bigg]_{i=1}^{n+1} = \frac{(n+2)^{\underline{3}}}{3} - \frac{2^{\underline{3}}}{3}$$

which simplifies to $S = \dfrac{(n+2)(n+1)n}{3} - \dfrac{2\cdot 1\cdot 0}{3} = \dfrac{1}{3}n(n+1)(n+2)$.

■

Difference Equations with Constant Coefficients

Difference equations arise in a variety of situations. The following are some examples of where difference equations arise in applications. In assuming a power series solution to differential equations, the coefficients must satisfy certain recurrence formula which are nothing more than difference equations. In the study of stability of numerical methods there occurs difference equations which must be analyzed. In the computer simulation of various types of real-world processes, difference equations frequently occur. Difference equations also are studied in the areas of probability, statistics, economics, physics, and biology. We begin our investigation of difference equations by studying those with **constant coefficients** as these are the easiest to solve.

Example 10-38.

Given the difference equation

$$y_{n+1} - y_n - 2y_{n-1} = 0$$

with the initial conditions $y_0 = 1,\ y_1 = 0$. Find values for y_2 through y_{10}.

Solution: In the given difference equation, replace n by $n+1$ in all terms, to obtain

$$y_{n+2} = y_{n+1} + 2y_n,$$

then one can verify

$$\begin{aligned}
n &= 0, & y_2 &= y_1 + 2y_0 = 2 \\
n &= 1, & y_3 &= y_2 + 2y_1 = 2 \\
n &= 2, & y_4 &= y_3 + 2y_2 = 6 \\
n &= 3, & y_5 &= y_4 + 2y_3 = 10 \\
n &= 4, & y_6 &= y_5 + 2y_4 = 22 \\
n &= 5, & y_7 &= y_6 + 2y_5 = 42 \\
n &= 6, & y_8 &= y_7 + 2y_6 = 86 \\
n &= 7, & y_9 &= 7_8 + 2y_7 = 170 \\
n &= 8, & y_{10} &= y_9 + 2y_8 = 342.
\end{aligned}$$

■

The study of difference equations with constant coefficients closely parallels the development of ordinary differential equations. Our goal is to determine functions $y_n = y(n)$, defined over a set of values of n, which reduce the given difference equation to an identity. Such functions are called solutions of the difference equation. For example, the function $y_n = 3^n$ is a solution of the difference equation $y_{n+1} - 3y_n = 0$ because $3^{n+1} - 3 \cdot 3^n = 0$ for all $n = 0,\ 1,\ 2, \ldots$. Recall that for linear differential equations with constant coefficients one can assume a solution of the form $y(x) = \exp(\omega x)$. This assumption leads to producing the characteristic equation and consequently the characteristic roots associated with the differential equation. In the special case $x = n$, there results $y(n) = y_n = \exp(\omega n) = \lambda^n$, where $\lambda = \exp(\omega)$ is a constant. This suggests in our study of difference equations with constant coefficients that one should assume a solution of the form $y_n = \lambda^n$, where λ is a constant. Analogous to ordinary linear differential equations with constant coefficients, a linear, nth-order, homogeneous difference equation with constant coefficients has associated with it a characteristic equation with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. The characteristic equation is found by assuming a solution $y_n = \lambda^n$, where λ is a constant. The various cases that can arise are illustrated by the following examples.

Example 10-39. **(Characteristic equation with real roots)**

Solve the second-order difference equation

$$y_{k+2} - 3y_{k+1} + 2y_k = 0.$$

Solution: Assume solutions of the form $y_k = \lambda^k$, where λ is a constant. This assumed solution produces $y_{k+1} = \lambda^{k+1}$ and $y_{k+2} = \lambda^{k+2}$. Substituting these values into the difference equation produces the equation

$$(\lambda^2 - 3\lambda + 2)\lambda^k = 0,$$

which tells us the required values for λ in order that $y_k = \lambda^k$ satisfy the difference equation. For a nontrivial solution it is required that $\lambda \neq 0$. This produces the characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0.$$

A short cut for writing down the characteristic equation is to observe the form of the given difference equation, when written in an operator form involving the stepping operator E. One can quickly obtain the characteristic equation from this operator form. For example, the given difference equation can be expressed in the form $(E^2 - 3E + 2)y_k = 0$, where the operator $E^2 - 3E + 2$ shows us the general form of the characteristic equation when E is replaced by λ. The characteristic equation has the roots $\lambda_1 = 2$ and $\lambda_2 = 1$, and hence two linearly independent solutions are

$$y_1(k) = 2^k \quad \text{and} \quad y_2(k) = 1^k = 1$$

which is called a **fundamental set of solutions.** The general solution can be written as a linear combination of this fundamental set and so one can write

$$y(k) = y_k = c_1(2)^k + c_2,$$

where c_1 and c_2 are arbitrary constants. Given a set of initial conditions of the form $y_0 = A$ and $y_1 = B$, where A and B are given constants, one can form the equations

$$y_0 = A = c_1 + c_2$$
$$y_1 = B = 2c_1 + c_2,$$

from which the constants c_1 and c_2 can be determined. Solving for these constants produces the solution of the initial value problem which satisfies the given initial conditions. The desired solution is unique and found to be

$$y_k = (B - A)(2)^k + (2A - B).$$

■

Example 10-40. **(Characteristic equation with repeated roots)**

Find the general solution to the difference equation

$$y_{n+2} - 4y_{n+1} + 4y_n = 0.$$

Solution: Write the difference equation in operator form $(E^2 - 4E + 4)y_n = 0$ and assume a solution of the form $y_n = \lambda^n$. By substituting the assumed solution into the difference equation one obtains the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$ which has the repeated roots $\lambda = 2,\ 2$. As with ordinary differential equations, one solution is $y_1(n) = 2^n$ and the second independent solution can be obtained by a multiplication of the first solution by the independent variable n. This is analogous to the case of repeated roots for ordinary differential equations with constant coefficients. A second independent solution is therefore $y_2(n) = n2^n$, and the general solution can be expressed as

$$y(n) = y_n = c_0 2^n + c_1 n 2^n,$$

where c_0 and c_1 are arbitrary constants. To verify that $n2^n$ is a second independent solution, the method of variation of parameters is used. Assume that a second solution has the form $y_n = U_n 2^n$, where U_n is an unknown function of n to be determined. Substituting this assumed solution into the difference equation produces the equation

$$2^{n+2}(U_{n+2} - 2U_{n+1} + U_n) = 0$$

which can be written as

$$(E^2 - 2E + 1)U_n = (E-1)^2 U_n = \Delta^2 U_n = 0. \tag{10.57}$$

It is left as an exercise to verify that the general solution of $\Delta^k U_n = 0$ is given by

$$U_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + \cdots + c_{k-1} n^{k-1},$$

and therefore, equation (10.57) has the solution $U_n = c_0 + c_1 n$, which when substituted into the assumed solution gives the result $y(n) = c_0(2)^n + c_1 n(2)^n$ as the general solution.

■

In general, if the characteristic equation associated with a linear difference equation with constant coefficients has a characteristic root $\lambda = a$ of multiplicity k, then

$$y_1(n) = a^n,\ y_2(n) = na^n,\ y_3(n) = n^2 a^n, \ldots, y_k(n) = n^{k-1} a^n$$

are k linearly independent solutions of the difference equation. To show this, solve the difference equation

$$(E-a)^k y_n = 0, \tag{10.58}$$

as was done previously. Here the characteristic equation is $(\lambda - a)^k = 0$ with $\lambda = a$ as a root of multiplicity k. The method of **variation of parameters** starts by assuming a solution to equation (10.58) of the form $y_n = a^n U_n$. Observe that

$$\begin{aligned} Ey_n &= a^{n+1}U_{n+1} \quad \text{and} \quad (E-a)y_n = a^{n+1}\Delta U_n \\ E(E-a)y_n &= a^{n+2}\Delta U_{n+1} \quad \text{and} \quad (E-a)^2 y_n = a^{n+2}\Delta^2 U_n \\ E(E-a)^2 y_n &= a^{n+3}\Delta^2 U_{n+1} \quad \text{and} \quad (E-a)^3 y_n = a^{n+3}\Delta^3 U_n. \end{aligned}$$

Continuing in this manner, show that U_n must satisfy

$$(E-a)^k y_n = a^{n+k}\Delta^k U_n = 0.$$

For a nonzero solution it is required that a^{n+k} be different from zero, and so U_n must be chosen such that $\Delta^k U_n = 0$. The general solution of this equation is

$$U_n = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1},$$

and hence the general solution of equation (10.58) is $y_n = a^n U_n$.

One can compare the case of repeated roots for difference and differential equations and readily discern the analogies that exist.

Example 10-41. **(Characteristic equation with complex or imaginary roots)**

Solve the difference equation

$$y_{n+2} - 10y_{n+1} + 74y_n = (E^2 - 10E + 74)y_n = 0.$$

Solution: Assume a solution of the form $y_n = \lambda^n$ and obtain the characteristic equation $\lambda^2 - 10\lambda + 74 = 0$ which has the complex roots $\lambda_1 = 5 + 7i$ and $\lambda_2 = 6 - 7i$. Two independent solutions are therefore

$$y_1(n) = (5+7i)^n \quad \text{and} \quad y_2(n) = (6-7i)^n.$$

To obtain solutions in the form of real quantities, represent the roots λ_1, λ_2 in the polar form $re^{i\theta}$, as illustrated in figure 10-9.

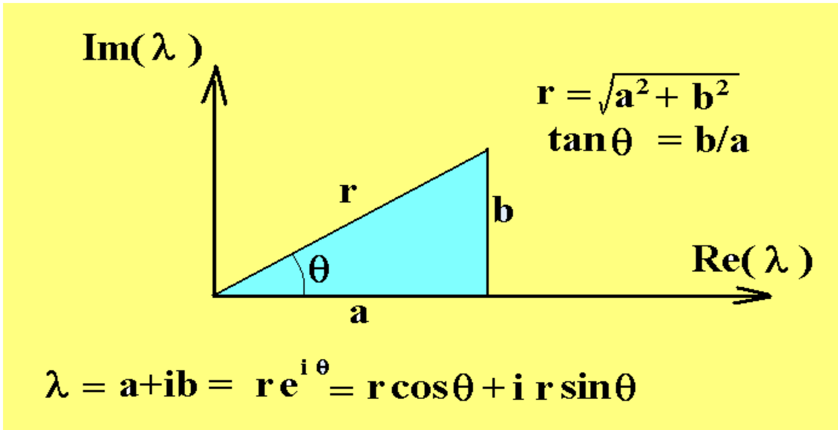


Figure 10-9. Polar form of complex numbers.

In this figure r is called the modulus or length of the complex number λ and θ is called an argument of the complex number λ. Values of 2π can be added to obtain other arguments of λ. A value of θ satisfying $-\pi < \theta \leq \pi$ is called the principal value of the argument of λ. The complex root λ_1 has a modulus and argument of

$$r = \sqrt{5^2 + 7^2} \qquad \text{and} \qquad \theta = \arctan(7/5). \tag{10.59}$$

The polar form of the characteristic roots produce solutions to the difference equations that can then be expressed in the form

$$y_1(n) = r^n e^{in\theta} \qquad \text{and} \qquad y_2(n) = r^n e^{-in\theta}.$$

The Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, is used to write these solutions in the form

$$\begin{aligned} y_1(n) &= r^n(\cos n\theta + i\sin n\theta) \\ y_2(n) &= r^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

The solutions $y_1(n)$ and $y_2(n)$ are independent solutions of the given difference equation and hence any linear combination of these solutions is also a solution. Form the linear combinations

$$\begin{aligned} y_3(n) &= \frac{1}{2}[y_1(n) + y_2(n)] \quad \text{and} \\ y_4(n) &= \frac{1}{2i}[y_1(n) - y_2(n)] \end{aligned}$$

to obtain the real solutions

$$y_3(n) = r^n \cos n\theta$$
$$y_4(n) = r^n \sin n\theta.$$

The general solution is any linear combination of these functions and can be expressed

$$y(n) = r^n[c_1 \cos n\theta + c_2 \sin n\theta],$$

where r and θ are defined by equation (10.59) and c_1 and c_2 are arbitrary constants. Therefore, when complex roots arise, these roots are expressed in polar form in order to obtain a real solution and imaginary solution to the given difference equation. If real solutions are desired, then one can take linear combinations of the real solution and imaginary solution in order to construct a general solution.

Nonhomogeneous Difference Equations

Nonhomogeneous difference equations can be solved in a manner analogous to the solution of nonhomogeneous differential equations and one may use the **method of undetermined coefficients** or the **method of variation of parameters** to obtain particular solutions.

Example 10-42. (Undetermined coefficients)

Solve the first order difference equation

$$L(y_n) = y_{n+1} + 2y_n = 3n.$$

Solution: First solve the homogeneous equation

$$L(y_n) = y_{n+1} + 2y_n = 0$$

Assume a solution $y_n = \lambda^n$ and obtain the characteristic equation $\lambda + 2 = 0$ with characteristic root $\lambda = -2$. The complementary solution is then $y_c(n) = c_1(-2)^n$, where c_1 is an arbitrary constant. Next find any particular solution $y_p(n)$ which produces the right-hand side. Analogous to what has been done with differential equations, examine the differences of the right-hand side of the given equation. Let $r(n) = 3n$, then the first difference is a constant since $\Delta r(n) = r(n+1) - r(n) = 3$. The basic terms occurring in the right-hand side and the difference of the right-hand side are listed as members of the set $S = \{1, n\}$. If any member of S occurs in the complementary solution, then the set S is modified by multiplying each term of the set S by n. If any member of the new set S also occurs in the complementary solution, then members of the set S are modified again. This is analogous to what one does in the study of ordinary differential equations. Here one can assume a particular solution of the given difference equation which is some linear combination of the functions in S. This requires that an assumed particular solution have the form

$$y_p(n) = An + B$$

where A and B are undetermined coefficients. Substitute this assumed particular solution into the difference equation and obtain

$$A(n+1) + B + 2An + 2B = 3n$$

which simplifies to

$$(A + 3B) + 3An = 3n.$$

Comparing like terms produces the equations $3A = 3$ and $A + 3B = 0$. Solving for A and B produces $A = 1$ and $B = -1/3$. Hence, the particular solution becomes

$$y_p(n) = n - \frac{1}{3}.$$

The general solution can be written as the sum of the complementary and particular solutions.

$$y_n = y_c(n) + y_p(n) = c_1(-2)^n + n - \frac{1}{3}.$$

■

Example 10-43. **(Variation of parameters)**

Determine a particular solution to the difference equation

$$y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = f_n, \tag{10.60}$$

where $a_1(n),\ a_2(n),\ f_n$ are given functions of n.

Solution: Assume that two independent solutions to the linear homogeneous equation

$$L(y_n) = y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = 0$$

are known. Denote these solutions by u_n and v_n so that by hypothesis $L(u_n) = 0$ and $L(v_n) = 0$. Assume a particular solution to the nonhomogeneous equation (10.60) of the form

$$y_n = \alpha_n u_n + \beta_n v_n, \tag{10.61}$$

where $\alpha_n,\ \beta_n$ are to be determined. There are two unknowns and consequently two conditions are needed to determine these quantities. As with ordinary differential equations, assume for our first condition the relation

$$\Delta\alpha_n u_{n+1} + \Delta\beta_n v_{n+1} = 0. \tag{10.62}$$

The second condition is obtained by substituting the assumed solution, given by equation (10.61), into the given difference equation. Starting with the assumed solution given by equation (10.61) show

$$\begin{aligned}
y_{n+1} &= y_n + \Delta y_n = \alpha_n u_n + \beta_n v_n + \Delta(\alpha_n u_n + \beta_n v_n)\\
y_{n+1} &= \alpha_n u_n + \beta_n v_n + \alpha_n \Delta u_n + \beta_n \Delta v_n + [(\Delta\alpha_n)u_{n+1} + (\Delta\beta_n)v_{n+1}].
\end{aligned}$$

This equation simplifies since by assumption equation (10.62) must hold. One can then show that y_{n+1} reduces to

$$y_{n+1} = \alpha_n u_{n+1} + \beta_n v_{n+1}. \tag{10.63}$$

In equation (10.63) replace n by $n+1$ everywhere and establish the result

$$y_{n+2} = \alpha_{n+1} u_{n+2} + \beta_{n+1} v_{n+2}. \tag{10.64}$$

By substituting equations (10.61), (10.63) and (10.64) into the equation (10.60), a second condition for determining the unknown constants is found. This second condition is that α_n and β_n must satisfy the equation

$$\alpha_{n+1}u_{n+2} + \beta_{n+1}v_{n+2} + a_1(n)(\alpha_n u_{n+1} + \beta_n v_{n+1}) + a_2(n)(\alpha_n u_n + \beta_n v_n) = f_n.$$

Rearrange terms in this equation, and show it can be written in the form

$$(\alpha_{n+1} - \alpha_n)u_{n+2} + (\beta_{n+1} - \beta_n)v_{n+2} + \alpha_n L(u_n) + \beta_n L(v_n) = f_n. \tag{10.65}$$

By hypothesis $L(u_n) = 0$ and $L(v_n) = 0$, thus simplifying the equation (10.65). The equations (10.62) and (10.65) are produce the two conditions

$$\begin{aligned}
\Delta\alpha_n u_{n+1} + \Delta\beta_n v_{n+1} &= 0\\
\Delta\alpha_n u_{n+2} + \Delta\beta_n v_{n+2} &= f_n
\end{aligned}$$

for determining the constants α_n and β_n. This system of equations can be solve by Cramer's rule and written

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n = -\frac{f_n v_{n+1}}{C_{n+1}}, \qquad \Delta\beta_n = \beta_{n+1} - \beta_n = \frac{f_n u_{n+1}}{C_{n+1}}, \tag{10.66}$$

where $C_{n+1} = u_{n+1}v_{n+2} - u_{n+2}v_{n+1}$ is called the Casoratian (the analog of the Wronskian for continuous systems). It can be shown that C_{n+1} is never zero if $u_n,\ v_n$ are linearly independent solutions of equation (10.60). The first order difference equations are a special case of problem 21 of the exercises at the end of this chapter, where it is demonstrated that the solutions can be written in the form

$$\begin{aligned} \alpha_n &= \alpha_0 - \sum_{i=0}^{n-1} \frac{f_i v_{i+1}}{C_{i+1}} \\ \beta_n &= \beta_0 + \sum_{i=0}^{n-1} \frac{f_i u_{i+1}}{C_{i+1}} \end{aligned} \tag{10.67}$$

and the general solution to equation (10.60) can be expressed as

$$y_n = \alpha_0 u_n + \beta_0 v_n - u_n \sum_{i=0}^{n-1} \frac{f_i v_{i+1}}{C_{i+1}} + v_n \sum_{i=0}^{n-1} \frac{f_i u_{i+1}}{C_{i+1}}. \tag{10.68}$$

■

Analogous to the nth-order linear differential equation with constant coefficients

$$L(D) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)y = r(x) \tag{10.69}$$

with $D = \frac{d}{dx}$ a differential operator, there is the nth-order difference equation

$$L(E) = (E^n + a_1 E^{n-1} + \cdots + a_{n-1}E + a_n)y_k = r(k), \tag{10.70}$$

where E is the stepping operator satisfying $Ey_k = y_{k+1}$. Most theorems and techniques which can be applied to the ordinary differential equation (10.69) have analogous results applicable to the difference equation (10.70).

Exercises

In the following exercises if the size of the matrix is not specified, then assume that the given matrices are square matrices.

▶ **10-1.** Given the matrices:

$$A = \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$$
$$B = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$$

(a) Verify that $AA^{-1} = A^{-1}A = I$
(b) Verify that $BB^{-1} = B^{-1}B = I$
(c) Calculate AB
(d) Calculate $B^{-1}A^{-1}$
(e) Find $(AB)^{-1}$ and check your answer.

▶ **10-2.** Start with the definition $AA^{-1} = I$ and take the transpose of both sides of this equation. Note that $I^T = I$ and show that $(A^{-1})^T = (A^T)^{-1}$

▶ **10-3.** Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
Hint: $ABC = (AB)C$

▶ **10-4.** If A and B are nonsingular matrices which commute, then show that
(a) A^{-1} and B commute
(b) B^{-1} and A commute
(c) A^{-1} and B^{-1} commute Hint: If $AB = BA$, then $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$

▶ **10-5.** If A is nonsingular and symmetric, show that A^{-1} is also symmetric.
Hint: If $AA^{-1} = I$, then $(AA^{-1})^T = (A^{-1})^T A^T = I$

▶ **10-6.** If A is nonsingular and $AB = AC$, show $B = C$

▶ **10-7.** Show that if $AB = A$ and $BA = B$, then A and B are idempotent.
Hint: Examine the products ABA and BAB

▶ **10-8.**
(a) Show $(A^2)^{-1} = (A^{-1})^2$
(b) Show for m a nonzero scalar that $(mA)^{-1} = \frac{1}{m}A^{-1}$

▶ **10-9.** Assume that A is a square matrix show that

$$(a)\ AA^T \text{ is symmetric} \quad (b)\ A+A^T \text{ is symmetric} \quad (c)\ A-A^T \text{ is skew-symmetric}$$

(d) Show A can be written as the sum of a symmetric and skew symmetric matrix.

▶ **10-10.** If A and B are symmetric square matrices
(a) Show that AB is symmetric if $AB = BA$
(b) Show that $AB = BA$ if AB is symmetric.

▶ **10-11.** Show $AA^{-1} = A^{-1}A = I$ when

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

▶ **10-12.** For

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

how should the constants a, b, c and d be chosen in order that A and B commute?

▶ **10-13.** For

$$A = \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} (\sqrt{2}-1) & 2 \\ (\sqrt{2}-1) & -(\sqrt{2}-1) \end{bmatrix},$$

find A^2, A^3, B^2 and B^3 and identify the special matrices A and B.

▶ **10-14.** For

$$A = \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

find A^2, A^3, A^4, A^5, A^6 and A^7 and identify the matrix A.

▶ **10-15.** Assume that $A^2B = I$ and that $A^5 = A$ (A is periodic with period 4). Solve for the square matrix B in terms of A.

▶ **10-16.** Let $AX = B$, where A is an $n \times n$ square matrix, and X and B are $n \times 1$ column vectors. Solve for the column vector X and state what conditions are required for the solution to exist.

▶ **10-17.** *Background material*

Definition: (Congruence) *Two integers I and J are said to be congruent modulo L,(written $I \equiv J(\bmod L)$) if $I - J = nL$ for some integer n*

This definition implies that two integers are congruent modulo L if and only if they have the same remainder when divided by L. Some examples are:

$$13 \equiv 1(\bmod 12) \qquad 7 \equiv 1(\bmod 3) \qquad 31 \equiv 2(\bmod 29)$$
$$26 \equiv 2(\bmod 12) \qquad 34 \equiv 1(\bmod 3) \qquad 77 \equiv 19(\bmod 29)$$
$$54 \equiv 6(\bmod 12) \qquad 305 \equiv 2(\bmod 3) \qquad 46 \equiv 17(\bmod 29)$$

CRYPTOGRAMS (A writing in cipher.) The message, "HOW ARE YOU?"could be written as a matrix of dimension 3×3 in the form

$$A = \begin{bmatrix} H & O & W \\ A & R & E \\ Y & O & U \end{bmatrix}.$$

Associate with each letter of the alphabet an integer as in the following scheme:

$$\begin{array}{cccccc} A=1 & F=6 & K=11 & P=16 & U=21 & Z=26 \\ B=2 & G=7 & L=12 & Q=17 & V=22 & ?=27 \\ C=3 & H=8 & M=13 & R=18 & W=23 & \text{Blank}=28 \\ D=4 & I=9 & N=14 & S=19 & X=24 & !=29 \\ E=5 & J=10 & O=15 & T=20 & Y=25 & \end{array}$$

Here 29 symbols are used as it is desirable to do modulo arithmetic in the modulo 29 system (29 being a prime number) and the blank stands for a blank character. By replacing the letters in the above matrix A, by their number equivalents, there results

$$A = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix}.$$

To disguise this message, the matrix A is multiplied by another matrix C, to form the matrix $B = AC$. In this multiplication, modulo 29 arithmetic is used as it is desired that only numbers between 1 and 29 are needed for our result. For example, using the matrix

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix} \qquad \text{with} \qquad C^{-1} = \begin{bmatrix} 2 & -4 & -1 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

there results

$$B = AC = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 19 \\ 6 & 9 & 2 \\ 17 & 27 & 11 \end{bmatrix} = \begin{bmatrix} B & F & S \\ F & I & B \\ Q & ? & K \end{bmatrix}$$

Here modulo 29 arithmetic has been used. For example,

$$\begin{aligned} 8(1) + 15(0) + 23(1) = 31 &\equiv 2(\bmod 29) \\ 8(1) + 15(1) + 23(-2) = -23 &\equiv 6(\bmod 29) \\ 8(0) + 15(-1) + 23(4) = 77 &\equiv 19(\bmod 29). \end{aligned}$$

with similar results using inner products involving the second and third row vectors of A. Upon receiving the coded message, where you know that the matrix C was used to make up the code, then you can decipher the message by multiplying by $C^{-1}(\bmod 29)$, since $B = AC$ implies that $A = BC^{-1}$. For example,

$$A = BC^{-1} = \begin{bmatrix} 2 & 6 & 19 \\ 6 & 9 & 2 \\ 17 & 27 & 11 \end{bmatrix} \begin{bmatrix} 2 & -4 & -1 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix} = \begin{bmatrix} H & O & W \\ A & R & E \\ Y & O & U \end{bmatrix}.$$

Here are some coded messages which were coded modulo 29 using the matrix C above.

$$\begin{bmatrix} C & T & C & I & ? \\ ! & ! & D & ! & C \\ P & F & E & N & N \end{bmatrix}$$

$$\begin{bmatrix} Y & R & L \\ S & Y & ? \\ T & V & M \\ G & Q & T \\ ! & T & R \\ P & X & M \\ U & N & P \\ Y & J & X \\ W & V & U \\ W & Y & L \\ R & J & Q \\ & & O \\ S & G & M \\ N & E & H \end{bmatrix}$$

▶ **10-18.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} \qquad (c)\quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

▶ **10-19.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \\ 2 & 0 & 3 \end{vmatrix} \qquad (c)\quad \begin{vmatrix} 2 & 0 & -1 \\ -1 & -1 & -1 \\ 2 & 4 & 3 \end{vmatrix}$$

▶ **10-20.** Given

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

(a) Find all minors M_{ij} (b) Find all cofactors C_{ij} (c) Calculate AC^T

▶ **10-21.** Evaluate the determinant

$$\begin{vmatrix} 2x & 3x & 4x \\ y & -y & 0 \\ z & 3z & 2z \end{vmatrix}$$

▶ **10-22.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 0 & 3 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 2 & 0 & 3 \\ -1 & 0 & \pi \\ e & 0 & .32 \end{vmatrix}$$

$$(c)\quad \begin{vmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & 0 & -2 & 5 \end{vmatrix} \qquad (d)\quad \begin{vmatrix} a_1 & \ell_1 & \ell_2 & \ell_3 \\ 0 & a_2 & \ell_4 & \ell_5 \\ 0 & 0 & a_3 & \ell_6 \\ 0 & 0 & 0 & a_4 \end{vmatrix}$$

$$(e)\quad \begin{vmatrix} a_1 & 0 & 0 & 0 \\ \ell_1 & a_2 & 0 & 0 \\ \ell_2 & \ell_3 & a_3 & 0 \\ \ell_4 & \ell_5 & \ell_6 & a_4 \end{vmatrix} \qquad (f)\quad \begin{vmatrix} 25 & 0 & -25 & 75 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 6 & 8 \\ -5 & 0 & -10 & 25 \end{vmatrix}$$

(g) How is the determinant in (f) related to the determinant in (c)?

▶ **10-23.**

(a) Find the inverse of

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

(b) What condition must be satisfied for the inverse to exist?

► **10-24.** Let

$$A = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$$

(a) Calculate $C = AB$ (b) Find $|A|$, $|B|$, $|C|$ (c) Verify that $|C| = |A||B|$

► **10-25.** Show that the equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) can be represented by the determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

► **10-26.** Find A^{-1} and verify that $AA^{-1} = I$.

$$(a) \quad A = \begin{bmatrix} 4 & 1 \\ 11 & 3 \end{bmatrix} \qquad (b) \quad A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} \qquad (c) \quad A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 3 & -3 \\ 2 & 3 & -5 \end{bmatrix}$$

► **10-27.** Verify that the given matrices are orthogonal

$$(a) \quad A = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad (b) \quad U = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$(c) \qquad V = \begin{bmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

► **10-28.** Find the inverse of the following matrices:

(a) A lower triangular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

(b) An upper triangular matrix

$$B = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) A symmetric matrix

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.0 \\ 0.0 & 0.1 & 0.2 & 0.1 \\ 0.0 & 0.0 & 0.1 & 0.2 \end{bmatrix}$$

(d) A diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \qquad \lambda_i \neq 0 \text{ for all } i$$

► **10-29.** Find values of α_1, α_2 and α_3 such that the given matrix is orthogonal

$$A = \begin{bmatrix} \frac{1}{2} & \alpha_2 & 0 \\ \alpha_1 & \frac{1}{2} & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

▶ **10-30.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{ij} = a_{ij}(t)$, $i, j = 1, 2, 3$ are differentiable functions of t.

(a) Show

$$\frac{d}{dt}(\det\ A) = \begin{vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} \end{vmatrix}$$

(b) Evaluate $\frac{d}{dt}(\det\ A)$ when

$$A = \begin{bmatrix} 2 & t \\ t^2 & t^3 \end{bmatrix}$$

▶ **10-31.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{ij} = a_{ij}(t)$, $i, j = 1, 2, 3$ are differentiable functions of t.

(a) Show

$$\frac{d}{dt}(\det\ A) = \begin{vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} & \frac{da_{13}}{dt} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} & \frac{da_{23}}{dt} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{da_{31}}{dt} & \frac{da_{32}}{dt} & \frac{da_{33}}{dt} \end{vmatrix}$$

(b) Evaluate $\frac{d}{dt}(\det\ A)$ when

$$A = \begin{bmatrix} 1 & t & t+1 \\ 0 & t^2 & 2t \\ t-1 & t & 0 \end{bmatrix}$$

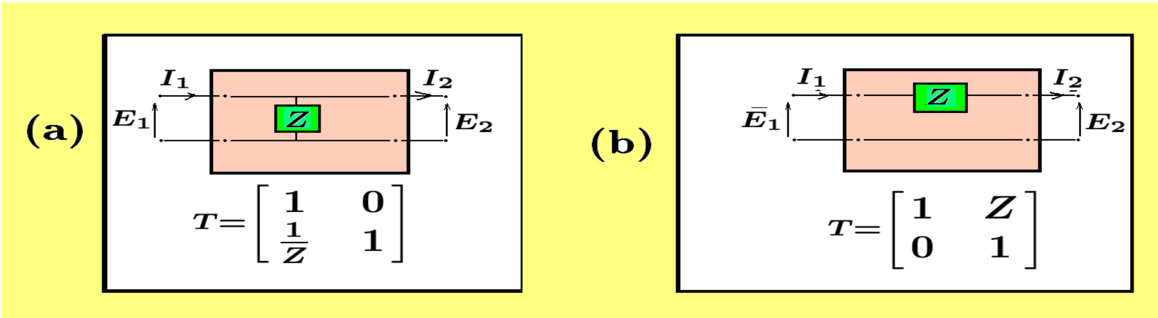
▶ **10-32.** Is the statement

$$\det(A+B) = \det(A) + \det(B)$$

true for arbitrary $n \times n$ square matrices A and B? Test your answer by using the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

▶ **10-33.** Verify the transmission matrices for the four-terminal networks illustrated.



▶ **10-34.** Let $A = \begin{bmatrix} t^3+t & \cos 3t \\ e^{4t} & \tanh 2t \end{bmatrix}$ and find

$$(a)\quad \frac{dA}{dt} \qquad\qquad (b)\quad \int A\,dt$$

▶ **10-35.** Let $C(\lambda) = |A - \lambda I| = \det(A - \lambda I) = 0$ denote the characteristic equation associated with the matrix A having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$.

(a) Show that $C(\lambda) = (-1)^n\lambda^n + c_1\lambda^{n-1} + \cdots c_{n-1}\lambda + c_n = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

(b) Show that $c_n = \lambda_1\lambda_2\cdots\lambda_n = |A| = \det A$

(c) Show A is singular if any eigenvalue is zero.

(c) Use the fact that the matrix A satisfies its own characteristic equation and show

$$A^{-1} = \frac{-1}{c_n}\left[(-1)^n A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I\right]$$

▶ **10-36.**

(a) Show that $A\displaystyle\int_0^t e^{At}\,dt + I = e^{At}$

(b) Show that $\displaystyle\int_0^t e^{At}\,dt = A^{-1}\left[e^{At} - I\right] = \left[e^{At} - I\right]A^{-1}$

▶ **10-37.** Verify the following matrix relations for the $n \times n$ matrix A.

(a) $\sin A = \dfrac{e^{iA} - e^{-iA}}{2i}$ where $i^2 = -1$

(b) $\cos A = \dfrac{e^{iA} + e^{-iA}}{2}$

(c) $\sinh A = \dfrac{e^{A} - e^{-A}}{2}$

(d) $\cosh A = \dfrac{e^{A} + e^{-A}}{2}$

(e) $\sin^2 A + \cos^2 A = I$

(f) $\cosh^2 A - \sinh^2 A = I$

▶ **10-38.** Consider the initial-value matrix differential equation

$$\frac{dX(t)}{dt} = AX(t) + F(t), \qquad\qquad X(t_0) = C$$

where A is a constant matrix.

(a) Left-multiply the given matrix differential equation by the matrix function $e^{-A(t-t_0)}$ and show

$$\frac{d}{dt}\left(e^{-A(t-t_0)}X\right) = e^{-A(t-t_0)}F(t)$$

(b) Integrate both sides of the result from part (a) from $t = t_0$ to t and show

$$X = X(t) = e^{A(t-t_0)}C + e^{At}\int_{t_0}^{t} e^{-A\xi}F(\xi)\,d\xi$$

(c) Show in the special case $F = [0]$, the solution reduces to $X = X(t) = e^{A(t-t_0)}C$

▶ **10-39.** Show the relation between the vector differential equation

$$\frac{d\bar{y}}{dt} = A(t)\bar{y} + \bar{f}(t), \qquad \bar{y}(0) = \bar{c}$$

and the matrix differential equation

$$\frac{dY}{dt} = A(t)Y, \qquad Y(0) = I$$

is given by

$$\bar{y} = Y(t)\bar{c} + Y(t)\int_0^t Y^{-1}(\xi)\bar{f}(\xi)\,d\xi$$

▶ **10-40.** Verify the forward differences in table 10.1.

▶ **10-41.** Verify the finite integrals in table 10.2.

▶ **10-42.** Solve the given difference equations.

(a) $y_{n+2} - 5y_{n+1} + 6y_n = 0$ (d) $y_{n+2} + 4y_{n+1} + 3y_n = 0$

(b) $y_{n+2} - 6y_{n+1} + 9y_n = 0$ (e) $y_{n+2} + 2y_{n+1} + y_n = 0$

(c) $y_{n+2} - 6y_{n+1} + 13y_n = 0$ (f) $y_{n+2} + 2y_{n+1} + 10y_n = 0$

▶ **10-43.** Find the finite integrals

$$(a) \quad \Delta^{-1}x^2 \qquad (b) \quad \Delta^{-1}\frac{1}{(3+2x)^{\overline{n}}} \qquad (c) \quad \Delta^{-1}\frac{1}{x^{\overline{n}}}$$

▶ **10-44.**

(a) For $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, verify the matrix function $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ e^{3t} - e^{2t} & e^{3t} \end{bmatrix}$

(b) For $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, verify the matrix function $e^{Bt} = \begin{bmatrix} e^{2t} & e^{2t} - e^{t} \\ 0 & e^{t} \end{bmatrix}$

▶ **10-45.**

(a) Verify the matrix differential equation

$$\frac{dX(t)}{dt} = AX(t) + F(t), \qquad X(0) = C$$

has the matrix solution

$$X = X(t) = e^{At}C + e^{At}\int_0^t e^{-A\xi}F(\xi)\,d\xi$$

(b) For $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $F(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ solve the matrix differential equation

$$\frac{dX(t)}{dt} = AX + F(t), \qquad X(0) = C$$

Chapter 11
Introduction to Probability and Statistics

The collecting of some type of data, organizing the data, determining how some characteristic of the data is to be presented as well conducting some type of analysis of the data, all comes under the category of **probability and statistics.**

Random Sampling

To determine some characteristic associated with a very large group of objects, called the **population**, it is impractical to examine every member of the group in order to perform an analysis of the population. Instead **a random selection of data associated with objects from the group** is examined. This is called **a random sample** from the population. Populations can be finite or infinite and by selecting a sample from the population one expects that some characteristics of the population can be inferred from an analysis of the sample data.

Analysis of the sample data, without trying to infer conclusions about the population from which the sample data comes, is called **descriptive or deductive statistics.** An analysis of sample data which tries to predict some characteristic of the population is called **inductive statistics or statistical inference**.

Simulations

Consider the figure 11-1 where some complicated system is described by n-input variables, j-parameter values, k-output variables and m-neglected or unknown variables. One replaces the complicated system with a model that in some way mimics or approximates the behavior of the real system. Those quantities that effect the model but whose behavior the model is not designed to study are called **exogenous variables**. These are usually the independent variables such as the input variables $x_1, \dots, x_n$ and parameters $p_1, \dots, p_j$ effecting system behavior. The behavior of those quantities from the complicated system that the model is designed to study are called **endogenous variables**. These are usually the dependent variables such as the outputs $y_1, \dots, y_k$ produced by the system.

Simulation is the process of designing **a mechanical or mathematical model of a real system** and then conducting experiments with this model for various purposes such as (i) obtaining a better understanding of the system (ii) to help construct theories for observed behavior (iii) aid in predicting future behavior (iv) to study how changes in inputs and parameters values effect the behavior of the system

(v) Finding ways to improve the system by experimenting with how inputs and parameter values produce changes in the system. (vi) to aid in making inferences and speculations on future system behavior.

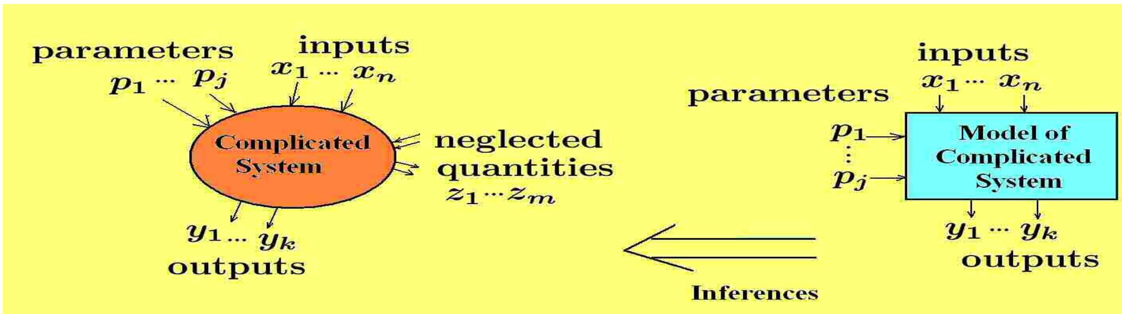


Figure 11-1.

Replacing complicated system with approximating mathematical model.

The model constructed can be continuous or discrete. By constructing a mathematical model one can use a computer to generate thousands of data values representing various outputs under a variety of input scenarios and then one can use statistics on the data values generated to make inferences concerning the behavior of the real system. For example, **Monte Carlo simulations** are discrete models where random numbers are used in specific ways to help simulate the behavior of a system. Monte Carlo methods can be used to study a wide variety of things. A small sampling of disciplines where Monte Carlo techniques are employed are the study areas of aerodynamics, fluid dynamics, atomic physics, radiation analysis, material research, oil exploration, and to verify theoretical predictions.

An example of a Monte Carlo method is the calculation of the value of π using random numbers. It can be shown that generating enough random numbers and using them in the proper way, one can calculate π as accurately as you desire.

The Representation of Data

The data from a population can be either **discrete or continuous**. If Y is a variable representing the characteristic being sampled and Y can take on any value between two given values, then Y is called a continuous variable. If Y is not a continuous variable, then it is called a discrete variable.

Some examples of discrete data is presented in the table 11.1. These numbers can be plotted as vertical or horizontal bar graphs, either stacked or grouped or as a line graph. The figure 11-2 illustrates these type of graphs.

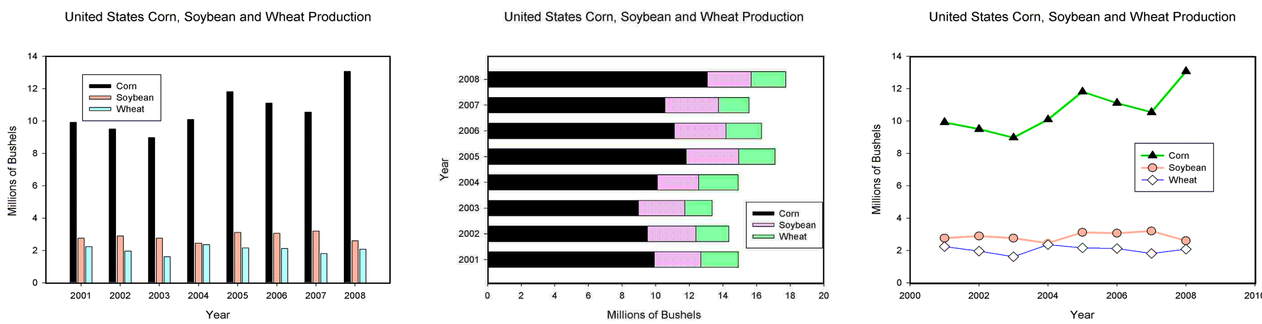


Figure 11-2. United States Corn, Soybean and Wheat Production

Table 11.1
Unite States Production of Corn, Soybeans and Wheat
(in millions of bushels)

Year	Corn	Soybean	Wheat
2001	9.92	2.76	2.23
2002	9.50	2.89	1.95
2003	8.97	2.76	1.61
2004	10.09	2.45	2.35
2005	11.81	3.12	2.16
2006	11.11	3.06	2.11
2007	10.54	3.19	1.81
2008	13.07	2.59	2.07

Tabular Representation of Data

A statistical experiment usually consists of collecting data from a random selection of the population. For example, suppose the systolic blood pressure of two hundred individuals are taken from a random sample of the population. The systolic blood pressure is measured in units of mmHg and is the blood pressure as the heart begins to pump. The diastolic blood pressure being a measure of the blood pressure between heart beats. The data set collected consists of 200 numbers, representing the sample size. A representative set of such numbers is presented in the table 11.2.

127	115	132	117	138	138	152	121	142	120	104	116	139	165	150	132	142	94	124	145
157	137	118	163	138	159	140	87	162	132	156	148	159	136	164	103	125	136	136	146
102	111	142	116	145	156	167	95	148	143	120	130	95	171	115	87	139	119	148	132
169	121	138	128	129	143	143	128	108	77	120	128	157	109	173	125	159	100	97	144
119	129	131	124	161	144	154	119	125	97	123	129	113	119	109	112	156	168	135	136
135	145	156	125	140	130	86	101	139	184	144	118	150	149	142	118	134	124	154	142
186	130	127	168	122	139	156	146	107	168	117	100	134	113	104	115	149	148	133	128
121	148	133	144	127	127	168	102	117	123	156	129	89	138	136	100	153	110	112	150
104	148	124	114	121	126	153	128	114	137	131	104	135	124	146	115	152	127	113	143
139	147	134	142	133	124	149	156	142	109	147	96	142	163	120	118	180	125	157	118

Table 11.2 Systolic Blood Pressure (mmHg)
measurements taken from 200 Random Individuals

Examine the data in table 11.2 and order the data in **a tally sheet** to form **a frequency table**. Show that the smallest value is 74 and the largest value is 186. Divide the data into categories or **class intervals** of equal length by **defining an upper limit and lower limit and midpoint for each class interval**. This is called grouping the data. Examples of class intervals are given in the table 11.3 where 74 is the first midpoint with 16 midpoints to 186. If $74 + 16x = 186$, then $x = 7$ steps between midpoints or the class interval is of size 7. Go through the data and find the number of systolic blood pressures in each class interval. This is called **determining the class frequency** associated with the grouped data. Then calculate the relative frequency column, the cumulative frequency column and cumulative relative frequency column as illustrated in the table 11.3. The cumulative frequency associated with a value x is just the sum of the frequencies less than or equal to x. The cumulative relative frequency is obtained by dividing the cumulative frequency by the sample size. Note that **the cumulative frequency ends in the sample size** and **the cumulative relative frequency ends with 1**.

Table 11.3 Frequency Table						
Class Interval	Class Midpoint	Tallies	Frequency	Relative Frequency	Cumulative Frequency	Cumulative Relative Frequency
71-77	74	/	1	$\frac{1}{200} = 0.005$	1	0.005
78-84	81		0	$\frac{0}{200} = 0.000$	1	0.005
85-91	88	////	4	$\frac{4}{200} = 0.020$	5	0.025
92-98	95	///// /	6	$\frac{6}{200} = 0.030$	11	0.055
98-105	102	///// ///// /	11	$\frac{11}{200} = 0.055$	22	0.110
106-112	109	///// ////	9	$\frac{9}{200} = 0.045$	31	0.155
113-119	116	///// ///// /// ///// /////	23	$\frac{23}{200} = 0.115$	54	0.270
120-126	123	///// ///// /// ///// /////	23	$\frac{23}{200} = 0.115$	77	0.385
128-133	130	///// ///// ///// ///// ///// /	26	$\frac{26}{200} = 0.130$	103	0.515
134-140	137	///// ///// ///// ///// /////	25	$\frac{25}{200} = 0.125$	128	0.640
141-147	144	///// ///// //// ///// /////	24	$\frac{24}{200} = 0.120$	152	0.760
148-154	151	///// ///// ///// ///	18	$\frac{18}{200} = 0.090$	170	0.850
155-161	158	///// //// /////	14	$\frac{14}{200} = 0.070$	184	0.920
162-168	165	///// /////	10	$\frac{10}{200} = 0.050$	194	0.970
168-175	172	///	3	$\frac{3}{200} = 0.015$	197	0.985
176-182	179	/	1	$\frac{1}{200} = 0.005$	198	0.990
183-189	186	//	2	$\frac{2}{200} = 0.010$	200	1.00

A graphical representation of the data in table 11.3 can be presented by defining **a relative frequency function** $f(x)$ and **a cumulative relative frequency function** $F(x)$ associated with the sample. These functions are defined

$$\begin{aligned} f(x) &= \begin{cases} f_j & \text{when } x = X_j \\ 0 & \text{otherwise} \end{cases} \\ F(x) &= \sum_{x_j \le x} f(x_j) = \text{sum of all } f(x_j) \text{ for which } x_j \le x \end{aligned} \tag{11.1}$$

and are illustrated in the figure 11-3.

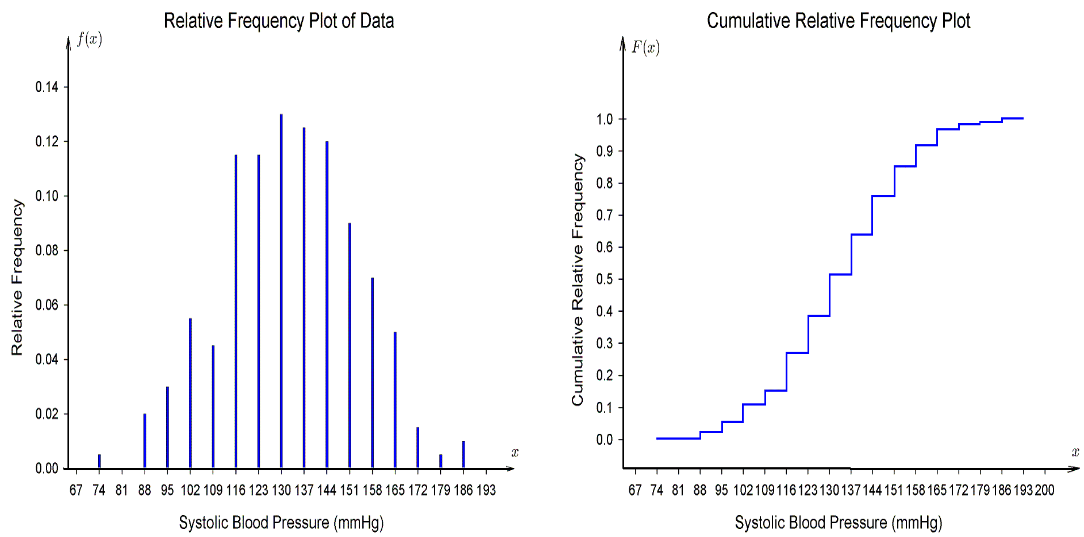


Figure 11-3.

The relative frequency function and cumulative relative frequency function

The results illustrated in the table 11.3 can be generalized. If $X_1, X_2, \ldots, X_k$ are k different numerical values in a sample of size N where X_1 occurs $\widetilde{f}_1$ times and X_2 occurs $\widetilde{f}_2$ times, ..., and X_k occurs $\widetilde{f}_k$ times, then $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k$ are called the frequencies associated with the data set and satisfy

$$\widetilde{f}_1 + \widetilde{f}_2 + \cdots + \widetilde{f}_k = N = \text{sample size}$$

The **relative frequencies associated with the data** are defined by

$$f_1 = \frac{\widetilde{f}_1}{N}, \qquad f_2 = \frac{\widetilde{f}_2}{N}, \qquad \ldots, \qquad f_k = \frac{\widetilde{f}_k}{N}$$

which satisfy the summation property

$$\sum_{i=1}^{k} f_i = f_1 + f_2 + \cdots + f_k = 1$$

Define **a frequency function** associated with the sample using

$$f(x) = \begin{cases} f_j, & \text{when } x = X_j \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, k$$

The frequency function determines **how the numbers in the sample are distributed.**

Also define **a cumulative frequency function** $F(x)$ for the sample, sometimes referred to as **a sample distribution function**. The cumulative frequency function is defined

$$F(x) = \sum_{t \le x} f(t) = \text{sum of all relative frequencies less than or equal to } x$$

Whenever the data has too many numerical values then one usually defines **class intervals** and **class midpoints** with class frequencies as in table 11.3. This is called **grouping of the data** and the corresponding frequency function and cumulative frequency function are associated with the grouped data.

The relative frequency distribution $f(x)$ is also called a **discrete probability distribution** for the sample and the cumulative relative frequency function $F(x)$ or distribution function represents **a probability**. In particular,

$$\begin{aligned} F(x) =& P(X \le x) = \text{Probability that population variable } X \text{ is less than or equal to } x\\ 1 - F(x) =& P(X > x) = \text{Probability that population variable } X \text{ is greater than } x \end{aligned} \tag{11.2}$$

Arithmetic Mean or Sample Mean

Given a set of data points $X_1, X_2, \ldots, X_N$, define the **arithmetic mean or sample mean** of the data set by

$$\text{sample mean } = \overline{X} = \frac{X_1 + X_2 + \cdots + X_N}{N} = \frac{\sum_{j=1}^{N} X_j}{N} \tag{11.3}$$

If the frequency of the data points are known, say $X_1, X_2, \ldots, X_k$ occur with frequencies $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k$, then the arithmetic mean is calculated

$$\overline{X} = \frac{\widetilde{f}_1 X_1 + \widetilde{f}_2 X_2 + \cdots + \widetilde{f}_k X_k}{\widetilde{f}_1 + \widetilde{f}_2 + \cdots + \widetilde{f}_k} = \frac{\sum_{j=1}^{k} \widetilde{f}_j X_j}{\sum_{j=1}^{k} \widetilde{f}_j} = \frac{\sum_{j=1}^{k} \widetilde{f}_j X_j}{N} \tag{11.4}$$

Note that the finite data collected is used to calculate an estimate of the **true population mean** μ associated with the total population.

Median, Mode and Percentiles

After arranging the data from low to high, **the median of the data set** is the middle value or **the arithmetic mean of the two middle values**. This value divides the data set into **two equal numbered parts**. In a similar fashion find those points which divide the data set, arranged in order of magnitude, into four equal parts. These values are usually denoted Q_1, Q_2, Q_3 and are called the **first, second and third**

quartiles. Note that Q_2 will be the same as the median. If the data set is divided into **ten equal parts** by numbers $D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9$, then these numbers are called **deciles**. If the data set is divided into one hundred equal parts by numbers $P_1, P_2, \ldots, P_{99}$, then these numbers are called **percentiles**. In general, if the data is divided up into quartiles, deciles, percentiles or some other equal subdivision, then the subdivisions created are called **quantiles**. The **mode** of the data set is that value which occurs with greatest frequency. Note that the mode may not exist or even if it does exist, it might not be a unique value. A **unimodal data set** is one which has a unique single mode.

The Geometric and Harmonic Mean

The **geometric mean** G associated with the data set $\{X_1, X_2, \ldots, X_N\}$ is the Nth root of the product of the numbers in the set. The geometric mean is denoted

$$G = \sqrt[N]{X_1 X_2 \cdots X_N} \tag{11.5}$$

The **harmonic mean** H associated with the above data set is obtain by first taking the arithmetic mean of the reciprocals and then taking the reciprocal of the result. The harmonic mean can be expressed using either of the relations

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} \frac{1}{X_i}} \qquad \text{or} \qquad \frac{1}{H} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{X_i} \tag{11.6}$$

The arithmetic mean $\overline{X}$, geometric mean G and harmonic mean H, satisfy the inequalities

$$H \le G \le \overline{X} \tag{11.8}$$

The equality sign being used when all the numbers in the data set are equal to one another.

The Root Mean Square (RMS)

The **root mean square** (RMS), sometimes referred to as the **quadratic mean**, of the data set $\{X_1, X_2, \ldots, X_n\}$ is defined for a discrete set of values as

$$RMS = \sqrt{\frac{\sum_{j=1}^{N} X_j^2}{N}} \tag{11.8}$$

and for a continuous set of values $f(x)$ over an interval (a,b) it is defined

$$RMS = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx} \tag{11.9}$$

The root mean square is used to measure the **average magnitude** of a quantity that varies.

Mean Deviation and Sample Variance

The **mean deviation** (MD), sometimes called the **average deviation**, of a set of numbers $X_1, X_2, \dots, X_N$ represents **a measure of the data spread from the mean** and is defined

$$MD = \frac{\sum_{j=1}^N \mid X_j - \overline{X} \mid}{N} = \frac{1}{N}\left[\mid X_1 - \overline{X} \mid + \mid X_2 - \overline{X} \mid + \cdots + \mid X_n - \overline{X} \mid\right] \tag{11.10}$$

where $\overline{X}$ is the arithmetic mean of the data. The mean deviation associated with numbers $X_1, X_2, \dots, X_k$ occurring with frequencies $\widetilde{f}_1, \widetilde{f}_2, \dots, \widetilde{f}_k$ can be calculated

$$MD = \frac{\sum_{j=1}^k \widetilde{f}_j \mid X_j - \overline{X} \mid}{N} = \frac{1}{N}\left[\widetilde{f}_1 \mid X_1 - \overline{X} \mid + \widetilde{f}_2 \mid X_2 - \overline{X} \mid + \cdots + \widetilde{f}_k \mid X_k - \overline{X} \mid\right] \tag{11.11}$$

The **sample variance** of the data set $\{X_1, X_2, \dots, X_n\}$ is denoted s^2 and is calculated using the relation

$$s^2 = \frac{1}{n-1}\sum_{j=1}^n (X_j - \overline{X})^2 = \frac{1}{n-1}\left[(x_1 - \overline{X})^2 + (x_2 - \overline{X})^2 + \cdots + (x_n - \overline{X})^2\right] \tag{11.12}$$

where $\overline{X}$ is the sample mean or arithmetic mean. The sample variance is a measure of how much **dispersion or spread** there is in the data. The positive square root of the sample variance is denoted s, which is called **the standard deviation** of the sample.

Note that some textbooks define the sample variance as

$$\mathcal{S}^2 = \frac{1}{n}\sum_{j=1}^n (X_j - \overline{X})^2 \tag{11.13}$$

where n is used as a divisor instead of $n-1$. This is confusing to beginning students who often ask, "Why the different definitions for sample variance in different textbooks?" The reason is that for small values for n, say $n < 30$, then equation (11.12) will produce a better estimate of **the true standard deviation σ associated with the total population** from which the sample is taken. For sample sizes larger than $n = 30$

there will be very little difference in calculation of the standard deviation from either definition. To convert the standard deviation $\mathcal{S}$, calculated using equation (11.13) one need only multiply $\mathcal{S}$ by $\sqrt{\dfrac{n}{n-1}}$ to obtain the value s as specified by equation (11.12).

The sample variance given by equation (11.12) requires that one first calculate $\overline{X}$, then one must calculate all the terms $X_j - \overline{X}$. All this preliminary calculation introduces **roundoff errors** into the final result. The sample variance can be calculated using a **short cut method** of computing without having to do preliminary calculations. The short cut method is derived using the expansion

$$(X_j - \overline{X})^2 = X_j^2 - 2X_j\overline{X} + \overline{X}^2$$

and substituting it into equation (11.12). A summation of terms gives

$$\sum_{j=1}^{n}(X_j - \overline{X})^2 = \sum_{j=1}^{n} X_j^2 - 2\overline{X}\sum_{j=1}^{n} X_j + \overline{X}^2\sum_{j=1}^{n}(1)$$

The substitution $\overline{X} = \dfrac{1}{n}\sum_{j=1}^{n} X_j$ and using $\sum_{j=1}^{n}(1) = n$, gives the result

$$\begin{aligned}
\sum_{j=1}^{n}(X_j - \overline{X})^2 =& \sum_{j=1}^{n} X_j^2 - 2\frac{1}{n}\sum_{j=1}^{n} X_j\sum_{j=1}^{n} X_j + \left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)^2 n\\
=& \sum_{j=1}^{n} X_j^2 - \frac{2}{n}\left(\sum_{j=1}^{n} X_j\right)^2 + \frac{1}{n}\left(\sum_{j=1}^{n} X_n\right)^2\\
=& \sum_{j=1}^{n} X_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} X_j\right)^2
\end{aligned}$$

This produces the shortcut formula for the sample variance

$$s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} X_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} X_j\right)^2\right] \tag{11.14}$$

If $X_1, \ldots, X_m$ are m sample values occurring with frequencies $\widetilde{f}_1, \ldots, \widetilde{f}_m$, the equation (11.14) can be expressed in the form

$$s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{m} X_j\widetilde{f}_j - \frac{1}{n}\left(\sum_{j=1}^{m} X_j\widetilde{f}_j\right)^2\right] \tag{11.15}$$

In general, if **the true population mean μ is known exactly**, so that $\mu = \frac{1}{N}\sum_{j=1}^{N} X_j$, where N is the population size, then the population standard deviation is given by

$$\sigma = \sqrt{\frac{\sum_{j=1}^{N}(X_j - \mu)^2}{N}} \tag{11.16}$$

Use N if the exact population mean is known and use $n-1$ if samples of size $n << N$ are selected from a population where the true mean μ is unknown.

Probability

An experiment or observation produces samples from a population where the outcome recorded either belongs or does not belong to a prescribed collection of events being studied. A **sample space S is the set of all possible outcomes from an experiment.** A sample space can be either finite or infinite. An example of a sample space with a finite collection of events is the roll of a single die. Here the sample space is $S = \{1, 2, 3, 4, 5, 6\}$ corresponding to the numbers on the six faces of the die. An example of an infinite sample space is that of an experiment where the outcome from an single event can be a real number within a specified range.

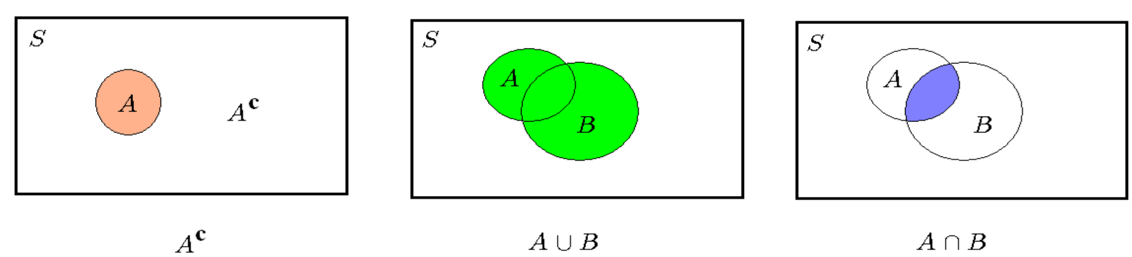


Figure 11-4. Venn diagrams

A Venn diagram consists of representing the sample space by a rectangle, then any event within S can be represented by the interior of a circle A within the rectangle. The set of all events not in A is called **the complement of A** and is denoted using the notation A^c. The **null set, empty set or impossible event** is denoted by the symbol $\emptyset$. The **union of two events** A and B is denoted $A \cup B$ and represents all events or experiments of S contained in A or B or both. The **intersection of two events** A and B is denoted $A \cap B$ and represents all events in S contained in both A and B. The concepts of a complement, union and intersection of sets is illustrated in the figure 11-4. If two sets A and B have no events in common, then this is denoted by $A \cap B = \emptyset$. In this case, the sets A and B are said to be **mutually exclusive events** or **disjoint events**. The notation $A \subset B$ is used to denote "all elements of A are contained in the set B" This can also be expressed $B \supset A$, which is read " B contains A". If the sample space contains n-sets $\{A_1, A_2, \dots, A_n\}$, then the union of these sets is denoted

$$A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{or} \qquad \mathop{\cup}_{j=1}^{n} A_j$$

The intersection of these sets is denoted

$$A_1 \cap A_2 \cap \cdots \cap A_n \qquad \text{or} \qquad \mathop{\cap}_{j=1}^{n} A_j$$

If $A_j \cap A_k = \emptyset$, for all values of j and k, with $k \neq j$, then the sets $\{A_1, A_2, \dots, A_n\}$ are said to represent **mutually exclusive events**.

Probability Fundamentals

Assuming that there are h **ways an event can happen** and f **ways for the event to fail** and these ways are **all equally likely to happen**, then the probability p that an event will happen in a given trial is

$$p = \frac{h}{h+f} \tag{11.17}$$

and the probability q that the event will fail is given by

$$q = \frac{f}{h+f} \tag{11.18}$$

These probabilities satisfy $p + q = 1$.

In general, given a finite sample space $S = \{e_1, e_2, \dots, e_n\}$ containing n simple events $e_1, \dots, e_n$, assign to each element of S a number $P(e_i)$, $i = 1, \dots, n$ called the probability assigned to event e_i of S. The probability numbers $P(e_i)$ assigned must satisfy the following conditions.

1. Each probability is a nonnegative number satisfying $0 \le P(e_i) \le 1$
2. The sum of the probabilities assigned to all simple events of the sample space must sum to unity or

$$\sum_{j=1}^{n} P(e_j) = P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

3. If each event is **equally likely to happen**, then one usually assigns a probability value $P(e_i) = \frac{1}{n}$ to each event as then $\sum_{i=1}^{n} P(e_i) = 1$.
4. The probability assigned to the entire sample space is unity and one writes $P(S) = 1$.

Probability of an Event

After assigning probabilities to each simple event of S, it is then possible to determine the probability of any event E associated with events from S. Consider the following cases.

1. The event $E = \emptyset$ is the empty set.
2. The event E is one of the simple events e_i from S $\big(i$ fixed, with $1 \le i \le n\big)$
3. The event E is the union of two or more events from S.

For the case 1, define the probability of the empty set $\emptyset$ as zero and write $P(\emptyset) = 0$. In case 2, the probability of event E is the same as the probability $P(e_i)$ so that, $P(E) = P(e_i)$. Consider now the case 3. If E is an event associated with the sample space S and $\overline{E}$ is its complement, then

$$P(E) = 1 - P(\overline{E}) \tag{11.19}$$

This is known as the **complementation rule** for probabilities.

If E_1 and E_2 are **mutually exclusive events** associated with S, then $E_1 \cap E_2 = \emptyset$ and

$$P(E_1 \cup E_2) = P(E_1) + P(E_2), \qquad E_1 \cap E_2 = \emptyset \tag{11.20}$$

In general, if $E = E_1 \cup E_2 \cup \cdots \cup E_m$ is an event and $E_1, E_2, \ldots, E_m$ are mutually exclusive events associated with S, then the intersection gives $E_1 \cap E_2 \cap \cdots \cap E_m = \emptyset$ and the probability of E is

$$P(E) = P(E_1 \cup E_2 \cup \cdots \cup E_m) = P(E_1) + P(E_2) + \cdots + P(E_m) \tag{11.21}$$

This is known as the **addition rule** for mutually exclusive events.

If E_1 and E_2 are arbitrary events associated with a sample space S, and these events **are not mutually exclusive**, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \tag{11.22}$$

Here $P(E_1)$ is the sum of all the simple events defining E_1 and $P(E_2)$ is the sum of all the simple events defining E_2. If the events are not mutually exclusive, then the sum of the probabilities associated with the simple events common to both the events E_1 and E_2 are counted twice. The sum of these common probabilities of simple events which are counted twice is $P(E_1 \cap E_2)$ and so this value is subtracted from the sum $P(E_1) + P(E_2)$.

Example 11-1.

Two coins are tossed. What is the probability that at least one tail occurs? Assume the coins are not trick coins so that there are four equally likely events that can occur. Let H denote heads and T denote tails, then the sample space for the experiment is $S = \{HH, HT, TH, TT\}$. Because each event is equally likely, assign a probability of $1/4$ to each event. For this example, the event E to be investigated is

$$E = \{HT\} \cup \{TH\} \cup \{TT\}$$

That is, at least one tail occurs. Consequently,

$$P(E) = P(HT) + P(TH) + P(TT) = 1/4 + 1/4 + 1/4 = 3/4$$

■

Example 11-2.

A pair of fair dice are rolled. The sample space associated with this experiment is a representation of all possible outcomes.

$$\begin{array}{rllllll} S = \{ & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ & (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & (6,6)\,\} \end{array}$$

There are 36 equally likely possible outcomes. Assign a probability of $1/36$ to each simple event.

If E_1 is the event that a 7 is rolled, then

$$P(E_1) = P((1,6)) + P((2,5)) + P((3,4)) + P((4,3)) + P((5,2)) + P((6,1)) = 6/36 = 1/6$$

If E_2 is the event that an 11 is rolled, then

$$P(E_2) = P((5,6)) + P((6,5)) = 2/36 = 1/18$$

If E_3 is the event doubles are rolled, then

$$P(E_3) = P((1,1)) + P((2,2)) + P((3,3)) + P((4,4)) + P((5,5)) + P((6,6)) = 6/36 = 1/6$$

If E_4 is the event that a 10 is rolled, then

$$P(E_4) = P((4,6)) + P((5,5)) + P((6,4)) = 3/36 = 1/12$$

If E_5 is the event a 10 is rolled or a double is rolled, the $E_5 = E_4 \cup E_3$. Note that the event $(5,5)$ is common to both events E_4 and E_3 with $E_4 \cap E_3 = (5,5)$ and $P(E_4 \cap E_3) = P((5,5)) = 1/36$. Hence,

$$P(E_5) = P(E_4 \cup E_3) = P(E_4) + P(E_3) - P(E_4 \cap E_3) = 3/36 + 6/36 - 1/36 = 8/36 = 2/9$$

If E_6 is the event that a 10 is rolled or a 7 is rolled, then $E_6 = E_4 \cup E_1$. Here $E_4 \cap E_1 = \emptyset$ and so these events are mutually exclusive. Consequently,

$$P(E_6) = P(E_4 \cup E_1) = P(E_4) + P(E_1) = 3/36 + 6/36 = 9/36 = 1/4$$

■

Note that there are many situations where **the sample space is not finite**. In such cases the probabilities assigned to the events in S are **based upon employment of relative frequencies observed from taking a large number of trials from the population being studied. For a large number of trials the relative frequency of an event happening is approximately the same as the probability of the event happening**. Observation of this fact should be made by examining the earlier development of relative frequency tables associated with our analysis of data collected. Also note that one can think of equation (11.20) is a special case of the more general equation (11.22), the special case occurring when E_1 and E_2 are mutually exclusive events.

If p is a probability of success assign to a single trial of an event, then the **expected number** of successes in n-trials is the product np.

Conditional Probability

If two events E_1 and E_2 are related in some manner such that the probability of occurrence of event E_1 depends upon whether E_2 has or has not occurred, then this is called **the conditional probability** of E_1 given E_2 and it is denoted using the notation $P(E_1 \mid E_2)$. Here the vertical line is read as "given" and events to the right of the vertical line are treated as events which have occurred. The conditional probability of E_1 given E_2 is

$$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \qquad P(E_2) \neq 0 \tag{11.23}$$

The conditional probability of E_2 given E_1 is

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \qquad P(E_1) \neq 0 \tag{11.24}$$

The equations (11.23) and (11.24) imply that the probability of both events E_1 and E_2 occurring is given by

$$P(E_1 \cap E_2) = P(E_1)P(E_2 \mid E_1) = P(E_2)P(E_1 \mid E_2), \qquad P(E_1) \neq 0, \quad P(E_2) \neq 0 \tag{11.25}$$

If the events E_1 and E_2 are independent events, then

$$P(E_1 \cap E_2) = P(E_1)P(E_2) \tag{11.26}$$

and consequently

$$P(E_1 \mid E_2) = P(E_1), \qquad \text{and} \qquad P(E_2 \mid E_1) = P(E_2)$$

This condition occurs whenever the probability of E_1 **does not depend upon the event** E_2 and similarly, the probability of event E_2 **does not depend upon** E_1.

Two events E_1 and E_2 are said to be **independent events** if and only if the probability of occurrence of E_1 and E_2 is given by $P(E_1 \cap E_2) = P(E_1)P(E_2)$. That is, the probability of both E_1 and E_2 occurring is the product of the probabilities of occurrence of each event. Two events that are **not independent** are called **dependent events**.

In general, if $E_1, E_2, \dots, E_m$ are all independent events, then

$$P(E_1 \cap E_2 \cap \cdots \cap E_m) = P(E_1)P(E_2)\cdots P(E_m) \tag{11.27}$$

This is sometime written in the form

$$P(\bigcap_{k=1}^{m} E_k) = \prod_{k=1}^{m} P(E_k) \tag{11.28}$$

and is known as **the multiplication principle for independent events**.

Example 11-3. Given an ordinary deck of 52 cards, suppose it is required to find the probability of selecting two cards and they are both aces. Here the probability of selecting an ace on the first draw is $P_1 = 4/52$. If the first card selected is an ace and it is not put back into the deck, then on the second draw there are only 3 aces left in the deck which now has only 51 cards. Consequently the probability of getting an ace on the second draw is $P_2 = 3/51$. The required probability of obtaining an ace on both the first and second drawing of cards is given by the multiplication principle for independent events so that one can write $P = P_1P_2 = \dfrac{4}{52} \cdot \dfrac{3}{51} = \dfrac{1}{221}$

■

The above discussions can be generalized. In studying the occurrence or non-occurrence of three events E_1, E_2, E_3 the probability is denoted

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2) \tag{11.29}$$

and for independent events

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3) \tag{11.30}$$

with similar extensions to a higher number of events taking place.

Example 11-4.

The dice table is completely surrounded with players so that you can see only a part of the table. A player rolls the dice and you see one die comes up a 6, but you can't see the other die. What is the probability the player has rolled a 7 or 11?

Solution: Here there are two events E_1, E_2 with

$$E_1 = \text{event one die is a 6}$$

$$E_2 = \text{event sum of dice is 7 or 11}$$

and we are to find $P(E_2 \mid E_1)$. To solve this problem write down the simple events as

$$E_1 = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$E_2 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(5,6),(6,5)\}$$

$$\text{with} \quad E_1 \cap E_2 = \{(6,1),(6,5)\}$$

Recall the simple events all have equal probabilities of $1/36$ and consequently

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{2/36}{6/36} = 1/3$$

Observe that $P(E_2) = 8/36 = 2/9 \neq P(E_2 \mid E_1) = 1/3$. These events are not independent. That is, knowing one die is a 6 does effect the probability of the sum being 7 or 11.

■

Example 11-5.

Two cards are selected at random from an ordinary deck of 52 cards. Find the probability that both cards are spades. Find the probability that the first card is a spade and the second card is a heart. In performing this experiment assume that the first card selected is not replaced in the deck.

Solution: Examine the events

$$
\begin{aligned}
E_1 =& \text{ The event that the first card selected is a spade.}\\
E_2 =& \text{ The event that the second card selected is a spade.}\\
E_3 =& \text{ The event that the second card selected is a heart.}
\end{aligned}
$$

Using elementary probability theory

$$P(E_1) = \frac{13}{52} = \frac{\text{Number of spades in deck}}{\text{Total number of cards in deck}}$$

Now if event E_1 has occurred, the deck now has only 51 cards with 12 spades. Consequently, the conditional probability is

$$
\begin{aligned}
P(E_2 \mid E_1) =& \frac{12}{51} = \frac{\text{Number of spades in deck}}{\text{Total number of cards in deck}}\\
\text{and} \quad P(E_3 \mid E_1) =& \frac{13}{51} = \frac{\text{Number of hearts in deck}}{\text{Total number of cards in deck}}
\end{aligned}
$$

Calculate the probabilities

$$
\begin{aligned}
P(E_1 \cap E_2) =& P(E_1)P(E_2 \mid E_1) = \frac{13}{52} \cdot \frac{12}{51} = \frac{3}{51}\\
\text{and} \quad P(E_1 \cap E_3) =& P(E_1)P(E_3 \mid E_1) = \frac{13}{52} \cdot \frac{13}{51} = \frac{13}{204}
\end{aligned}
$$

■

Permutations

Assume something can be done in ℓ different ways and one of these ℓ-ways has been done. Now if a second something can be done in m different ways, then the number of ways that the two somethings can be done is given by the product $\ell \cdot m$. If a third something can be done in n ways, then the three somethings can be done in $\ell \cdot m \cdot n$-ways. This **multiplication principle** can be extended if more than three somethings are involved in the study.

Example 11-6. How many three digit even numbers can be formed using the digits $\{1, 2, 3, 5, 7\}$ if repetition of any digits is not allowed?

Solution A three digit number has the representation (hundreds place)(tens place)(units place). If the three digit number is to be an even number, then the units place must be filled with the number 2. Hence there are $\ell = 1$ ways to perform this task. The tens place can be filled with any of the numbers 1,3,5,7 and so $m = 4$ ways to perform this task. Finally, if one of the numbers $1, 3, 5, 7$ is selected for the tens place and there is to be no repetition of numbers, then only 3 numbers are left for the hundreds place. This gives $n = 3$ ways for the hundreds place. This shows that there are $n \cdot m \cdot \ell = 3 \cdot 4 \cdot 1 = 12$ ways to complete the task.

■

Each arrangement in an ordered set of items is called **a permutation of the set of items**. For example, how many ways can you arrange three books on a shelf? There are 3 choices for the first book, 2 choices for the second book and 1 selection for the last book. This gives $3 \cdot 2 \cdot 1 = 3! = 6$ ways to arrange three books on a shelf. If the books are labeled a,b and c, then the 6 arrangements are

$$\begin{array}{ccc} abc & cab & bca \\ bac & acb & cba \end{array}$$

In general, the **number of permutations of n things is n-factorial**, written $n!$. That is, there are n choices for the first item, $(n-1)$ choices for the second item, $(n-2)$ choices for the third item, etc. This gives the number of permutations as

$$_nP_n = n(n-1)(n-2)\cdots(3)(2)(1) = n! \tag{11.31}$$

The grouping of a selection of m items from a collection of n items, $m < n$, is called **the number of permutations of n items taken m at a time**. For example, to determine the number of permutations of the letters a,b,c,d taken two at a time, note that there are 4 choices for the first letter leaving three choices for the second letter. This gives 12 such arrangements. These arrangements are

$$\begin{array}{cccc} ab & ba & ca & da \\ ac & bc & cb & db \\ ad & bd & cd & dc \end{array}$$

In general, **the number of permutations of n things taken m at a time** is given by the formula

$$ {}_nP_m = n(n-1)(n-2)\cdots(n-m+1) \tag{11.32} $$

which is a product of m-factors. In the special case $m = n$, **the number of permutations of n things taken all the time** is ${}_nP_n$ as given by equation (11.31).

If in the collection of items there are n_1 repeats on one item, n_2 repeats of another item, ..., n_m repeats of still another item, then many of the total number of permutations will be repeats. To remove these repeats just divide by factorial associated with the number of repeats. This gives the permutation of n things taken all at a time as

$$ P = \frac{n!}{n_1!n_2!\cdots n_m!} \qquad \text{where } n = n_1 + n_2 + n_3 + \cdots + n_m \tag{11.33} $$

Combinations

A collection of items **without regard to the order of arrangement** is called **a combination**. For example, abc,bca,cab,bac,cba,acb all represent the same collection of the letters a,b and c, where the different permutations are ignored. The **number of combination of n things taken m at time** is denoted by using either of the notations $\binom{n}{m}$ or ${}_nC_m$ and is calculated as follows. If ${}_nC_m$ or $\binom{n}{m}$ is a collection of m items from a set of n items, then for each combination of m things there are $m!$ permutations so that one can write

$$ \begin{aligned} {}_nP_m =& m!\binom{n}{m} = m!{}_nC_m \\ \text{or } {}_nC_m = \binom{n}{m} =& \frac{{}_nP_m}{m!} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!} \\ =& \frac{n!}{m!(n-m)!} \qquad n \ge 0, \quad 0 \le m \le n \end{aligned} \tag{11.34} $$

The term $\binom{n}{m}$ represents the mth binomial coefficient.

Note that binomial coefficients $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ and that $\binom{n}{m} = \binom{n}{n-m}$ The binomial coefficients also satisfy the recursive property

$$ \binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1} \qquad m \ge 0 \text{ and integral} $$

Binomial Coefficients

The binomial expansion $(p+q)^n$, for n an integer, can be expressed in the form

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{m}p^{n-m}q^m + \cdots + \binom{n}{n}q^n \tag{11.35}$$

Note in the special case $p = q = 1$ one obtains

$$(1+1)^n = 2^n = 1 + \sum_{m=1}^{n} \binom{n}{m} = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \tag{11.36}$$

and rearranging terms one finds

$$\sum_{m=1}^{n} \binom{n}{m} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1 \tag{11.37}$$

Let p denote the probability that an event will happen and $q = 1 - p$ denote the probability that the event will fail in a single trial and examine the probability that the event will happen in n-trials. In one trial $(p+q) = 1$. In two trials, the expansion

$$(p+q)^2 = \binom{2}{0}p^2 + \binom{2}{1}pq + \binom{2}{2}q^2 = p^2 + 2pq + q^2$$

represents all possible outcomes. For example, if the trial is flipping a coin and p is the probability of heads H and q is the probability of tail T, then p^2 is the probability of getting two successive heads HH, $2pq$ is the probability of getting a head and tail or tail and head HT or TH and q^2 is the probability of getting two tails TT. Similarly, in three trials of flipping a coin the expansion

$$(p+q)^3 = \binom{3}{0}p^3 + \binom{3}{1}p^2q + \binom{3}{2}pq^2 + \binom{3}{3}q^3$$

gives the probabilities

$\binom{3}{0}p^3 = p^3$ is probability of getting three successive heads HHH

$\binom{3}{1}p^2q = 3p^2q$ is the probability of getting HHT or HTH or THH

and represents the probability of getting two heads in 3 trials.

$\binom{3}{2}pq^2 = 3pq^2$ is the probability of getting HTT or THT or TTH

and represents the probability of getting one head in 3 trials.

$\binom{3}{3}q^3 = q^3$ is the probability of getting TTT

and represents the probability of not getting a head in 3 trials.

In general, in studying n-trials associate with a two event happening, one would examine the binomial expansion

$$(p+q)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-j} q^j$$

and the term within this expansion having the form

$$\binom{n}{m} p^m q^{n-m} = {}_nC_m \, p^m q^{n-m} = \frac{n!}{m!\,(n-m)!} p^m q^{n-m} \tag{11.38}$$

represents **the probability that the event will happen m times in n trials**.

Discrete and Continuous Probability Distributions

The estimated probability of an event is taken as the relative frequency of occurrence of the event. As the number of observations upon which the relative frequency is based increases, then the discrete probability is replace by a continuous function $f(x)$ called the probability function or probability density function of the distribution with the condition that the total area under the probability density function must equal unity. The figure 11-5 is a graphical representation illustrating this conversion.

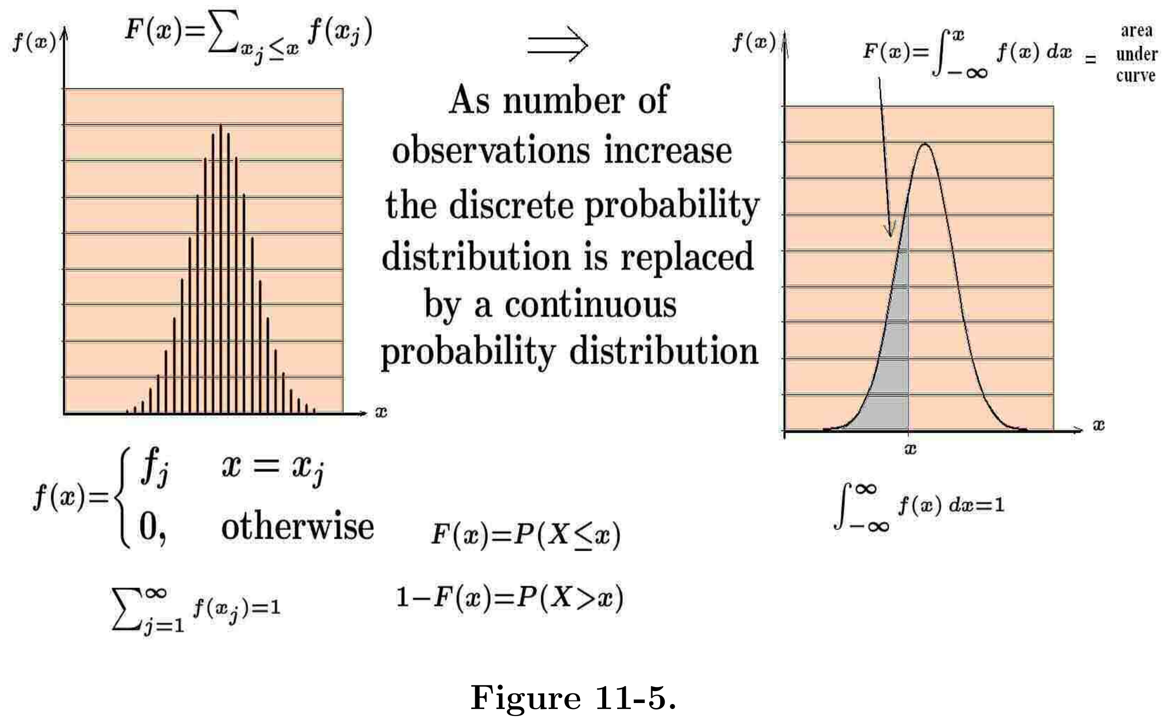


Figure 11-5.

Discrete frequency function replaced by continuous probability distribution.

The function $F(x) = \sum_{x_j \le x} f(x_j)$ is called the **cumulative frequency function** associated with the discrete sample. In the continuous case it is called the distribution function $F(x)$ and calculated by the integral $F(x) = \int_{-\infty}^{x} f(x)\,dx$ which represents the area from $-\infty$ to x under the probability density function. These summation processes are illustrated in the figure 11-6.

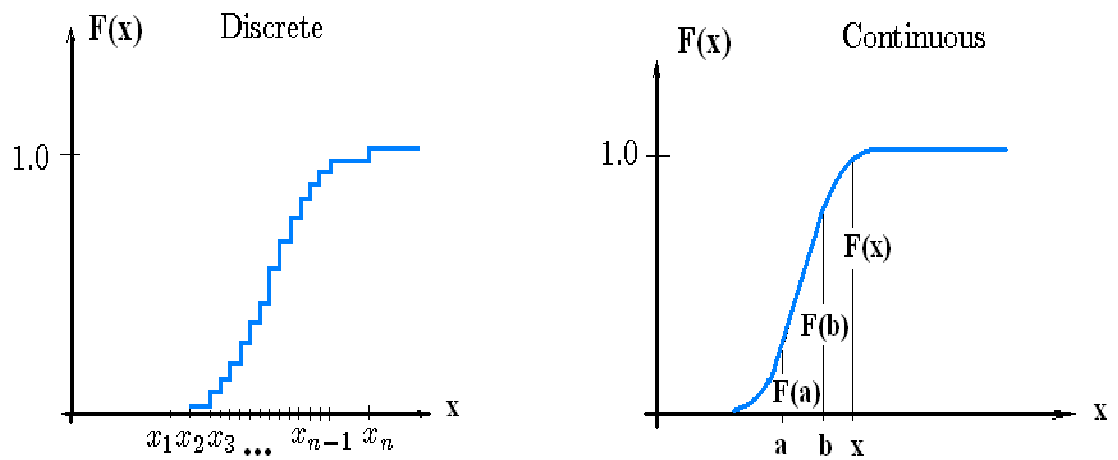


Figure 11-6. Cumulative frequency function for discrete and continuous case.

In both the discrete and continuous cases the cumulative frequency function represents the probability

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{with} \quad 1 - F(x) = P(X > x) = \int_{x}^{\infty} f(x)\,dx \tag{11.39}$$

with the property that if a, b are points x with $a < b$, then

$$P(a < x \le b) = P(X \le b) - P(X \le a) = F(b) - F(a) \tag{11.40}$$

which represents the probability that a random variable X lies between a and b. In the discrete case

$$P(a < X \le b) = \sum_{a < x_j \le b} f(x_j) = F(b) - F(a) \tag{11.41}$$

and in the continuous case

$$P(a < X \le b) = \int_{a}^{b} f(x)\,dx = F(b) - F(a) \tag{11.42}$$

Table 11.4 Mean and Variance for Discrete and Continuous Distributions		
Discrete		**Continuous**
$\mu = E[x] = \sum_{j=1}^{n} x_j f(x_j)$	**population** **mean** μ	$\mu = E[x] = \int_{-\infty}^{\infty} x f(x)\,dx$
$\sigma^2 = E[(x-\mu)^2] = \sum_{j=1}^{n} (x_j-\mu)^2 f(x_j)$	**population** **variance** σ^2	$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

The continuous cumulative frequency function satisfies the properties

$$\frac{dF(x)}{dx} = f(x), \qquad F(-\infty) = 0, \qquad F(+\infty) = 1, \quad \text{and} \quad F(a) < F(b) \text{ if } a < b \tag{11.43}$$

The table 11.4 illustrates the relationships of the mean and variance associated with the discrete and continuous probability densities.

If X is a real random variable and $g(X)$ is any continuous function of X, then the numbers

$$\begin{aligned} E[g(X)] &= \sum_{j=1}^{n} g(x_j) f(x_j) \qquad \text{discrete} \\ E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x)\,dx \qquad \text{continuous} \end{aligned} \tag{11.44}$$

associated with the probability density $f(x)$ are defined as **the mathematical expectation of the function** $g(X)$. In the special case $g(X) = X^k$ for $k = 1, 2, \ldots, n$ an integer, the equations (11.44) become

$$\begin{aligned} E[X^k] &= \sum_{j} x_j^k f(x_j) \qquad \text{discrete} \\ E[X^k] &= \int_{-\infty}^{\infty} x^k f(x)\,dx \qquad \text{continuous} \end{aligned}$$

These expectation equations are referred to as **the kth moment of X**. In the special case $g(X) = (X-\mu)^k$, the equations (11.44) become

$$\begin{aligned} E[(X-\mu)^k] &= \sum_{j} (x_j-\mu)^k f(x_j) \quad \text{discrete} \\ E[(X-\mu)^k] &= \int_{-\infty}^{\infty} (x-\mu)^k f(x)\,dx \quad \text{continuous} \end{aligned}$$

and these quantities are called **the kth central moments of X**. Note the special cases

$$E[1] = 1, \qquad \mu = E[X], \qquad \sigma^2 = E[(X-\mu)^2] \tag{11.45}$$

The **expectation of a sum of random variables** $X_1, X_2, \ldots, X_n$ **equals the sum of the expectations** and consequently

$$E(X_1 + X_2 + \cdots X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) \tag{11.46}$$

The **expectation of a product of independent random variables equals the product of the expectations** which is expressed

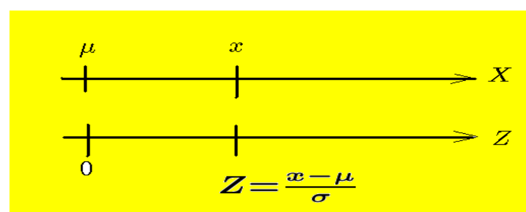
$$E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n) \tag{11.47}$$

Scaling

The probability density function $f(x)$ is said to be symmetric with respect a number $x = \mu$ if for all values of x the density function satisfies the relation

$$f(\mu + x) = f(\mu - x) \tag{11.48}$$

A random variable X having a mean μ and variance σ^2 can be **scaled** by introducing the new variable $Z = (X-\mu)/\sigma$. The variable Z is referred to as **the standardized variable** corresponding to X.



Let $f(x)$ denote the probability density function associated with the random variable X and define the function $f^*(z) = \sigma f(x) = \sigma f(\sigma z + \mu)$ as the probability function associated with the random variable Z. Using the scaling illustrated in the figure above, observe that $x = \sigma z + \mu$ with $dx = \sigma\, dz$ so that

$$f(x)\,dx = f(\sigma z + \mu)\sigma\,dz = f^*(z)\,dz$$

then the mean value on the Z-scale is given by

$$\begin{aligned}
\mu^* = & \int_{-\infty}^{\infty} z f^*(z)\,dz = \int_{-\infty}^{\infty} \left(\frac{x}{\sigma} - \frac{\mu}{\sigma}\right) f(x)\,dx \\
= & \frac{1}{\sigma}\int_{-\infty}^{\infty} x f(x)\,dx - \frac{\mu}{\sigma}\int_{-\infty}^{\infty} f(x)\,dx \\
= & \frac{1}{\sigma}\mu - \frac{\mu}{\sigma}(1) = 0
\end{aligned}$$

and the variance on the Z-scale is given by

$$\sigma^{*2} = \int_{-\infty}^{\infty} (z-\mu^*)^2 f^*(z)\,dz = \int_{-\infty}^{\infty} z^2 f^*(z)\,dz \quad \text{since } \mu^* = 0$$
$$\sigma^{*2} = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^2 f(x)\,dx = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \frac{1}{\sigma^2}\sigma^2 = 1$$

This demonstrates that the introduction of a scaled variable Z centers the mean at zero and introduces a variance of unity.

The Normal Distribution

The **normal probability distribution** is a continuous function with two parameters called μ and $\sigma > 0$. The parameter σ is called the **standard deviation** and σ^2 is called **the variance of the distribution**. The normal probability distribution has the form

$$f(x) = N(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\mu)^2/\sigma^2\right], \qquad -\infty < x < \infty \tag{11.49}$$

and is illustrated in the figure 11-7. The parameter μ is known as **the mean of the distribution** and represents a location parameter for positioning the curve on the x-axis. Note the normal probability curve is **symmetric about the line** $x = \mu$. The parameter σ is sometimes called a scale parameter which is associated with the **spread and height** of the probability curve. The quantity σ^2 represents the variance of the distribution and σ represents the **standard deviation of the distribution**. The total area under this curve is 1 with approximately 68.27% of the area between the lines $\mu \pm \sigma$, 95.45% of the total area is between the lines $\mu \pm 2\sigma$ and 99.73% of the total area is between the lines $\mu \pm 3\sigma$. The area bounded by the curve $N(x;\mu,\sigma^2)$ and the x-axis is unity. The area under the curve $N(x;\mu,\sigma^2)$ between $X = b$ and $X = a < b$ represents the probability $P(a < X \le b)$. For example, one can write the probabilities

$$\begin{aligned} P(\mu-\sigma < X \le \mu+\sigma) =& .6827 \\ P(\mu-2\sigma < X \le \mu+2\sigma) =& .9545 \\ P(\mu-3\sigma < X \le \mu+3\sigma) =& .9973 \end{aligned} \tag{11.50}$$

The function

$$\phi(z) = N(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{11.51}$$

is called a normalized probability distribution with mean $\mu = 0$ and standard deviation of $\sigma = 1$.

The cumulative distribution function $F(x)$ associated with the normal probability density function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$ is given by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = P(X \le x) \tag{11.52}$$

and represents the area under the probability curve form $-\infty$ to x. Note that this integral cannot be evaluated in a closed form and one must use numerical methods to calculate the value of the integral for a given value of x. The area calculated represents the probability $P(X \le x)$. The total area under the normal probability density function is unity and so the area

$$1 - F(x) = \int_{x}^{\infty} f(x)\,dx = P(X > x) \tag{11.53}$$

represents the probability $P(X > x)$. These areas are illustrated in the figure 11-7.

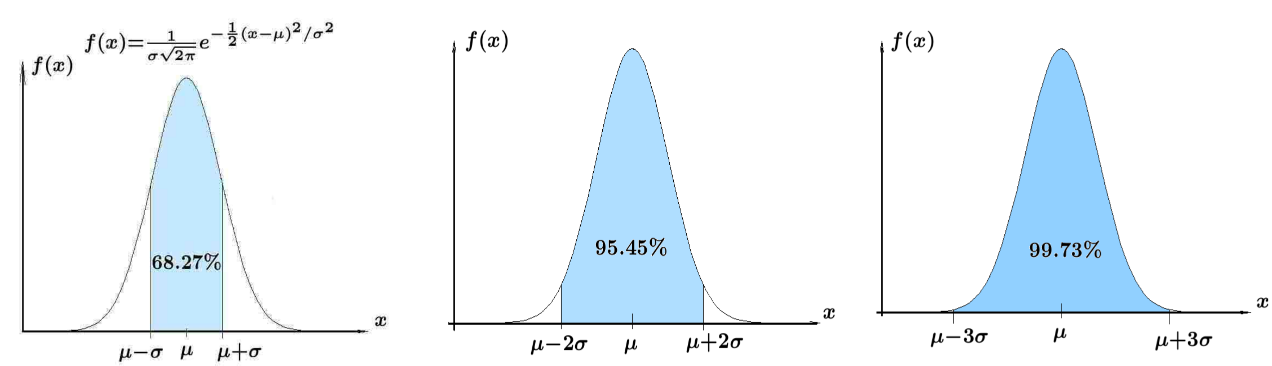


Figure 11-7. Percentage of total area under normal probability curve.

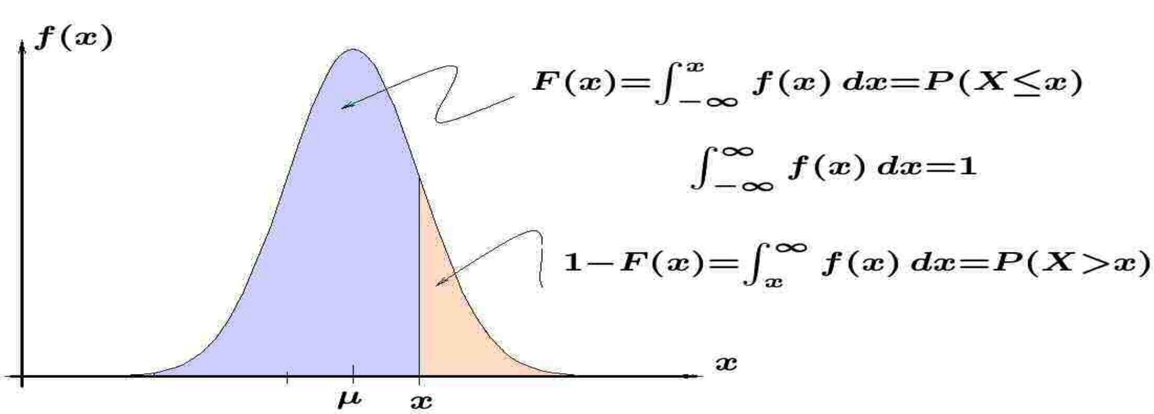


Figure 11-8. Area under normal probability curve representing probabilities.

Standardization

The normal probability density function, sometimes called **the Gaussian distribution**, has the form

$$N(x;\mu,\sigma^2) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{11.54}$$

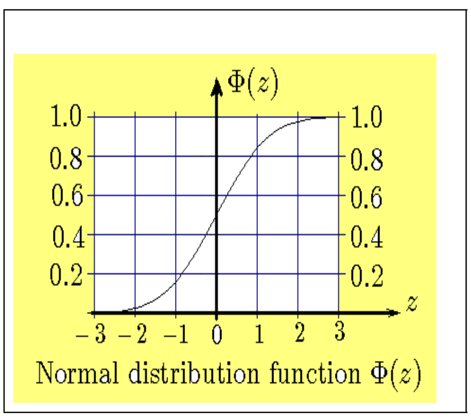
and the area under this curve between the values $x = a$ and $x = b$, where $a < b$, represents **the probability that a random variable X lies between the values a and b.** This probability is represented

$$P(\,a < X < b\,) = \int_a^b f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \tag{11.55}$$

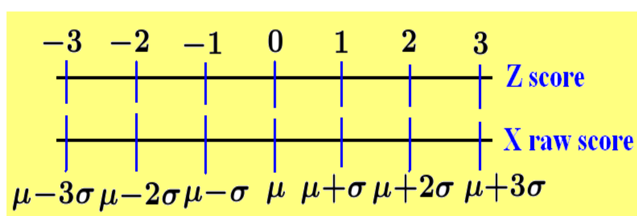
Note that this integral cannot be integrated in closed form and so numerical integration techniques are used to create tables for a normalized form or standard form associated with the above integral. See for example the table 11.5. The distribution function associated with the normal probability density function is given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}\left(\frac{\xi-\mu}{\sigma}\right)^2}\,d\xi \tag{11.56}$$

Introducing the standardized variable $z = \dfrac{x-\mu}{\sigma}$, with $dz = \dfrac{dx}{\sigma}$, the distribution function, given by equation (11.56), with variable x is converted to a normalized form with variable z. The normalized form and associated scaling is illustrated below,



$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-z^2/2}\,dz, \tag{11.57}$$



and equation (11.54) is replaced by the standard form for the probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{11.58}$$

which is the integrand of the integral given by equation (11.57). In the representations (11.58) and (11.57) the variable z is called a **random normal number**.

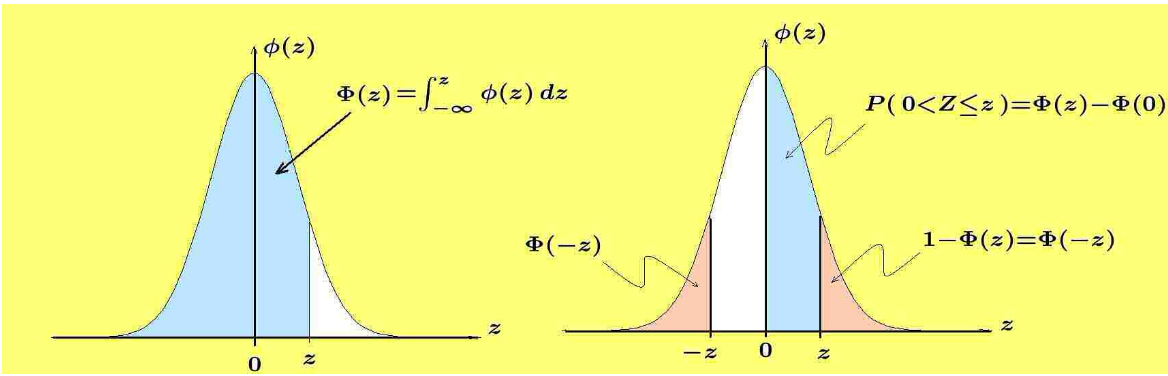


Figure 11-9.

Standard normal probability curve and distribution function as area.

Note that with a change of variable $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ so that in terms of probabilities

$$P(a < X \le b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \tag{11.59}$$

In figure 11-9, the normal probability curve is symmetric about $z=0$ and so the area under the curve between 0 and z represents the probability $P(0 < Z \le z) = \Phi(z) - \Phi(0)$ and quantities like $\Phi(-z)$, by symmetry, have the value $\Phi(-z) = 1 - \Phi(z)$. The standard normal curve has the properties that $\Phi(-\infty) = 0$, $\Phi(0) = 1/2$ and $\Phi(\infty) = 1$. The table 11.5 gives **the area under the standard normal curve** for values of $z \ge 0$, then it is possible to employ the **symmetry of the standard normal curve** to calculated specific areas associated with probabilities. For example, to find the area from $-\infty$ to -1.65 examine the table of values and find $\Phi(1.65) = .9505$ so that $\phi(-1.65) = 1 - .9505 = .0495$ or the area from -1.65 to $+\infty$ is .9505, then one can write the probability statement $P(Z > -1.65) = .9505$. As another example, to find the area under the standard normal curve between $z = -1.65$ and $z = 1$, first find the following values

$$\begin{aligned}
&\text{Area from } 0 \text{ to } 1 = \Phi(1) - \Phi(0) = .8413 - 0.5000 = .3413\\
&\text{Area from } 0 \text{ to } 1.65 = \Phi(1.65) - \Phi(0) = .9505 - .5000 = .4505\\
&\text{Area from } -1.65 \text{ to } 0 = .4505\\
&\text{Area from } -1.65 \text{ to } 1 = .4505 + .3413 = .7918 = P(-1.65 < Z \le 1)
\end{aligned}$$

Sketch a graph of the above values as areas under the normal probability density function to get a better understanding of the values presented.

The **normal distribution function** $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$ and the **error function**[1] which is defined

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du \tag{11.60}$$

can be related by writing

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\xi^2/2}\, d\xi + \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\xi^2/2}\, d\xi = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\xi^2/2}\, d\xi$$

and then making the substitutions $u = \xi/\sqrt{2}$, $du = d\xi/\sqrt{2}$ to obtain

$$\Phi(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{z/\sqrt{2}} e^{-u^2} \sqrt{2}\, du = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{z}{\sqrt{2}}\right) \tag{11.61}$$

The normal probability functions given by equations (11.49) and (11.57) are known by other names such as Gaussian distribution, normal curve, bell shaped curve, etc. The normal distribution occurs in the study of various types of errors such as measurements in the quality and precision control of tools and equipment. The normal distribution arises in many different applied areas of the physical and social sciences because of **the central limit theorem**. The central limit theorem, sometimes called **the law of large numbers**, involves consequences of taking large samples from any kind of distribution and can be described as follows. Perform an experiment and select n independent random variables X from some population. If $x_1, x_2, \ldots, x_n$ represents the set of n independent random variables selected, then the mean m_1 of this sample can be constructed. Perform the experiment again and calculate the mean m_2 of the second set of n random independent variables. Continue doing this same experiment a large number of times and collect all the mean values from each experiment. This gives a set of average values $S = \{m_1, m_2, \ldots, m_N\}$ created from performing the experiment N times. The central limit theorem says that the distribution of the set of average values S approaches a normal probability distribution with mean μ_s and variance σ_s^2 given by

$$\begin{aligned} \mu_s =& \text{Mean of the set of averages from a large number of samples} = \mu \\ \sigma_s^2 =& \text{Variance of set of averages from large number of samples} = \frac{\sigma^2}{n} \end{aligned}$$

where μ and σ^2 represent the true mean and true variance of the population being sampled. The central limit theorem always holds and does not depend upon the

[1] There are alternative definitions of the error function due to scaling.

shape of the original distribution being sampled. The normal distribution is also related to **least-square estimation**. It is also used as the theoretical basis for **the chi-square, student-t and F-distributions**. The normal distribution is used in many Monte Carlo simulation computer programs.

The Binomial Distribution

The binomial probability distribution is given by

$$b(x;n,p) = f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0,1,2,\dots,n \\ 0, & \text{otherwise} \end{cases} \qquad q = 1-p \tag{11.62}$$

It is a discrete probability distribution with parameters n and p where n represents the number of trials and p represents the probability of success in a single trial with $q = 1-p$ the probability of failure in a single trial. For large values of n the binomial distribution approaches the normal distribution. In equation (11.62), the function $f(x)$ represents the probability of x successes and $n-x$ failures in n-trials. The cumulative probabilities are given by

$$F(x) = B(x;n,p) = \sum_{k=0}^{x} b(k;n,p), \qquad \text{for } x = 0,1,2,\dots,n \tag{11.63}$$

As an exercise verify that

$$b(x;n,p) = b(n-x;n,1-p), \qquad B(x;n,p) = 1 - B(n-x-1;n,1-p) \tag{11.64}$$

The **binomial probability law**, sometimes called **the Bernoulli distribution**, occurs in those application areas where one of two possible outcomes can result in a single trial. For example, (yes, no), (success, failure), (left, right), (on, off), (defective, nondefective), etc. For example, if there are d defective items in a bin of N items and an item is selected at random from the bin, then the probability of obtaining a defective item in a single trial is $p = d/N$. The binomial probability distribution involves sampling with replacement. Consequently, each time a sample of n items is selected from the bin containing N items, the probability of obtaining x defective items is given by equation (11.62) with $p = d/N$ and $q = 1 - d/N$.

In the equation (11.62), the term

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}, \qquad \binom{n}{0} = 1, \quad \text{and} \quad \binom{0}{0} = 1 \tag{11.65}$$

represent the binomial coefficients in the binomial expansion

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{x}p^x q^{n-x} + \cdots + \binom{n}{n}q^n = 1 \tag{11.66}$$

In equations (11.62) and (11.66) the term $\binom{n}{x}$ **represents the number of different way of selecting x-objects from a collection of n-objects** and the term

$$p^x q^{n-x} = \underbrace{pp\cdots p}_{\text{x times}}\ \underbrace{qq\cdots q}_{\text{n-x times}} \tag{11.67}$$

represents the probability of x successes and $n-x$ failures in n-trials without regard to any ordering of the arrangements of how the successes or failures occur. Consequently, the equation (11.67) must be multiplied by the number of different arrangements of the successes and failures and this is what produces the binomial probability distribution.

The binomial distribution has the following properties

$$\text{mean} = \mu = np \qquad \text{and} \qquad \text{variance} = \sigma^2 = npq \tag{11.68}$$

The figure 11-10 illustrates the binomial distribution for the parameter values $n=10$ and $p=0.2, 0.5$ and 0.9.

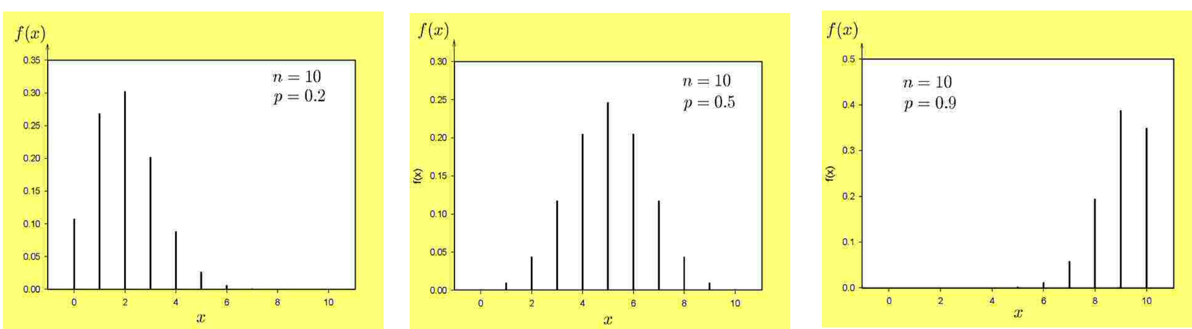


Figure 11-10. Selected binomial distributions for $n=10$.

The Multinomial Distribution

The **multinomial distribution** occurs when **many events can happen during a single trial**. If only one event can result from m mutually exclusive events $E_1, E_2, \ldots, E_m$ occurring in a single trial, where $p_1, p_2, \ldots, p_m$ are the probabilities assigned to the m-events, then the probability of getting n_1 $E_1's$, n_2 $E_2's, \ldots, n_m$ $E_m's$ is given by the multinomial probability function

$$f(n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

where $n_1 + n_2 + \cdots + n_m = n$ and $p_1 + p_2 + \cdots + p_m = 1$.

The Poisson Distribution

The **Poisson probability distribution** has the form

$$f(x;\lambda) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, 3, \ldots \tag{11.69}$$

with parameter $\lambda > 0$. Here x is an integer which can increase without bound. The Poisson probability distribution has the following properties,

1. $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right) = e^{-\lambda}e^{\lambda} = 1$
2. mean$=\mu = \lambda$
3. variance $\sigma^2 = \lambda$

The cumulative probability function is given by

$$F(x;\lambda) = \sum_{k=0}^{x} f(k;\lambda) \tag{11.70}$$

The Poisson distribution occurs in application areas which record isolated events over a period of time. For example, the number of cars entering an intersection in a ten minute interval, the number of telephone lines in use during different periods of the day, the number of customers waiting in line, the life expectancy of a light bulb, the number of transistors that fail in one year, etc.

The figure 11-11 illustrates the Poisson distribution for the parameter values $\lambda = 1/2, 1, 2$ and 3.

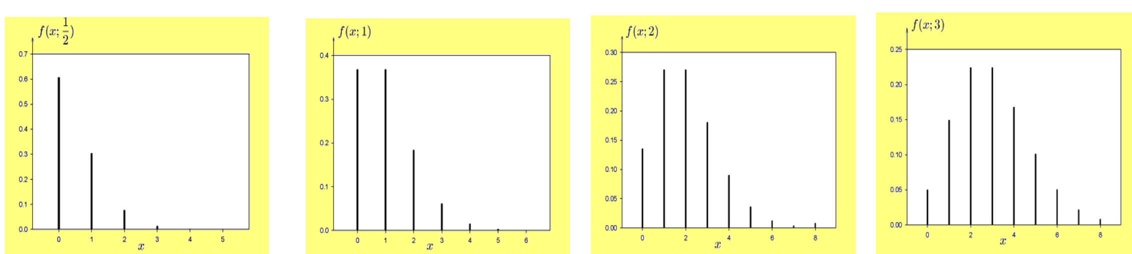


Figure 11-11. Selected Poisson distributions for $\lambda = \frac{1}{2}, 1, 2, 3$.

In general, the Poisson distribution is **a discrete probability distribution** used to determine **the number of events occurring in a fixed interval of time**.

The Hypergeometric Distribution

The **hypergeometric probability distribution** has the form

$$f(x) = h(x; n, n_1, n_2) = \frac{\binom{n_1}{x}\binom{n_2}{n-x}}{\binom{n_1+n_2}{n}}, \qquad x = 0, 1, 2, 3, \ldots, n \tag{11.71}$$

where x is an integer satisfying $0 \le x \le n$, n_1 represents the number of successes and n_2 represents the number of failures, where n items are selected from $(n_1 + n_2)$ items without replacement. This is **a probability distribution with three parameters**, n, n_1 and n_2. The hypergeometric probability distribution is used in quality control, estimates of animal population size from capture-recapture data, the spread of an infectious disease when a fixed number of individuals are exposed to an illness.

Note that the binomial distribution is used in **sampling with replacement** while the hypergeometric distribution is applicable for problems where there is **sampling without replacement**. The hypergeometric distribution has mean

$$\mu = \frac{nn_1}{n_1 + n_2}$$

and variance given by

$$\sigma^2 = \frac{nn_1n_2(n_1 + n_2 - n)}{(n_1 + n_2)^2(n_1 + n_2 - 1)}$$

The equation (11.71) represents the probability of x successes and $n - x$ failures selected from $n_1 + n_2$ items where the sampling is without replacement. For example, to find the probability of selecting two aces from a standard deck of 52 playing cards in 6 draws with no replacement of cards selected one would select the following parameters for the hypergeometric distribution. Here there are 6 draws so that $n = 6$. There are 4 aces in the deck so $n_1 = 4$ is the number of successes in the deck and $n_2 = 48$ is the number of failures in the deck, with $n_1 + n_2 = 52$ the total number of cards in the deck. The hypergeometric distribution gives the probability of $x = 2$ successes in $n = 6$ draws as

$$h(2; 6, 4, 48) = \frac{\binom{4}{2}\binom{48}{4}}{\binom{52}{6}} = \frac{621}{10829} = 0.0573$$

The Exponential Distribution

The **exponential probability distribution** is a continuous probability distribution with parameter $\lambda > 0$ and is defined

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{11.72}$$

The exponential distribution is used in studying time to failure of a piece of equipment , waiting time for next event to occur, like waiting time for an elevator, or time waiting in line to be served. This distribution has the mean

$$\mu = \lambda$$

and the variance is given by

$$\sigma^2 = \lambda^2$$

Note that the area under the probability curve $f(x)$, for $-\infty < x < \infty$ is equal to 1 or

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = 1$$

The Gamma Distribution

The **gamma probability distribution** is defined

$$f(x) = \begin{cases} \dfrac{1}{\theta^\alpha\,\Gamma(\alpha)} x^{\alpha-1}\, e^{-x/\theta}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases} \tag{11.73}$$

where $\Gamma(\alpha)$ is the gamma function. This probability density function has the two parameters $\alpha > 0$ and $\theta > 0$. It is a continuous probability distribution with a shape parameter α and scale parameter θ. The gamma distribution is used frequently in econometrics.

This probability distribution arises in determining the waiting time for a given number of events to occur. For example, waiting for 10 calls to a switch board, or life testing until a failure occurs. It also occurs in weather prediction of precipitation processes. The gamma distribution has mean

$$\mu = \alpha\theta \qquad \text{and variance} \qquad \sigma^2 = \alpha\,\theta^2$$

The gamma distribution with parameters $\alpha = 1$ and $\theta = 1/\lambda$ produces the exponential distribution. The figure 11-12 illustrates the gamma distribution for selected values of the parameters α and θ.

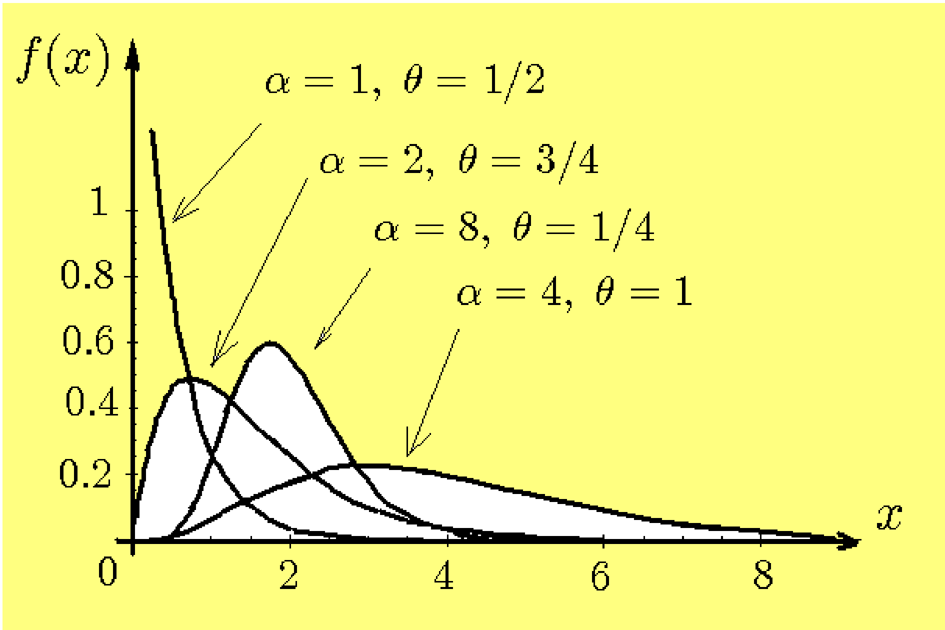


Figure 11-12. The gamma distribution for selected values of α and θ

Chi-Square χ^2 Distribution

The **chi-square probability distribution** has the form

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu-2)/2}\, e^{-x/2}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{11.74}$$

where $\Gamma(\ )$ represents the gamma function[2] and $\nu = 1, 2, 3, \ldots$ is a parameter called the number of degrees of freedom. Note that the chi-square distribution is sometimes written as the χ^2-distribution. It is a special case of the gamma distribution when the parameters of the gamma distribution take on the values $\alpha = \nu/2$ and $\theta = 2$. This distribution has the mean $\mu = \nu$ and variance $\sigma^2 = 2\nu$.

The chi-square distribution is used in testing of hypothesis, determining confidence intervals and testing differences in various statistics associated with independent samples. The tables 11.6(a) and (b) give values for areas under the probability density function.

[2] Recall the gamma function is defined $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ with the property $\Gamma(x+1) = x\Gamma(x)$.

Student's t-Distribution

The **student's**[3] **t-distribution** with n degrees of freedom is given by the probability density function

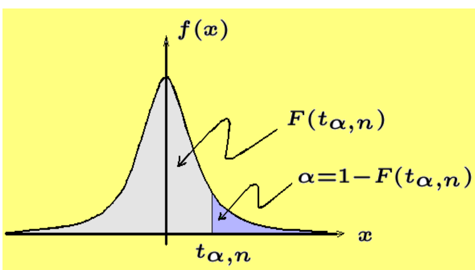
$$f(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \qquad -\infty < x < \infty \tag{11.75}$$

where $\Gamma(\ )$ denotes the gamma function and $n = 1, 2, 3, \dots$ is a parameter. The student's t-distribution has the mean 0 for $n > 1$, otherwise the mean is undefined. Similarly, the variance is given by $\frac{n}{n-2}$ for $n > 2$, otherwise the variance is undefined.

The cumulative distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx \tag{11.76}$$

The table 11.6 contains values of $t_{\alpha,n}$ which satisfy the equation



$$\int_{t_{\alpha,n}}^{\infty} f(x)\,dx = \alpha = 1 - F(t_{\alpha,n}) \tag{11.77}$$

The normal distribution is related to the student's t-distribution as follows. If $\overline{x}$ and s are the mean and standard deviation associated with a random sample of size n from a normal distribution $N(x; \mu, \sigma^2)$, then the quantity $\frac{(\overline{x} - \mu)\sqrt{n}}{s}$ has a student-t-distribution with $n - 1$ degrees of freedom.

The student's t-distribution is a continuous probability distribution used to estimate the mean of a population where (i) the population has a normal distribution (ii) the sample size from the population is small and (iii) the standard deviation of the population is unknown. The table 11.7 gives values for area under this probability density function.

[3] Developed by W.S. Gosset who used the name "Student" as a pseudonym.

The F-Distribution

The **F-distribution** has the probability density function

$$f(x) = f_{n,m}(x) = \begin{cases} \dfrac{\Gamma\left(\dfrac{m+n}{2}\right)}{\Gamma\left(\dfrac{m}{2}\right)\Gamma\left(\dfrac{n}{2}\right)} n^{n/2} m^{m/2} \dfrac{x^{n/2-1}}{(m+nx)^{(m+n)/2}}, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{11.78}$$

which is sometimes given in the form

$$f(x) = f_{n,m}(x) = \begin{cases} \dfrac{\Gamma\left(\dfrac{m+n}{2}\right)}{\Gamma\left(\dfrac{m}{2}\right)\Gamma\left(\dfrac{n}{2}\right)} (n/m)^{n/2} \dfrac{x^{n/2-1}}{(1+\frac{n}{m}x)^{(m+n)/2}}, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{11.79}$$

where $\Gamma(\ )$ denotes the gamma function. The F-distribution has the parameters $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

If X_1 and X_2 are independent random variables associated with a chi-square distribution having respectively the degrees of freedom n and m, then the quantity $Y = \dfrac{X_1/n}{X_2/m}$ will have a F-distribution with n and m degrees of freedom.

The tables 11.6 (a)(b)(c)(d)(e) contain values of $F_{\alpha,n,m}$ such that

$$\int_{F_{(\alpha,n,m)}}^{\infty} f_{n,m}(x)\, dx = \alpha$$

for α having the values $0.1, 0.05, .025, .01$, and $.005$. Observe the symmetry of the F-distribution and note that in the use of the upper tail values from the tables it is customary to employ the relation

$$F(df_m, df_n, 1-\alpha/2) = \frac{1}{F(df_n, df_m, \alpha/2)} \tag{11.80}$$

where df_m and df_n denote the degrees of freedom for m and n.

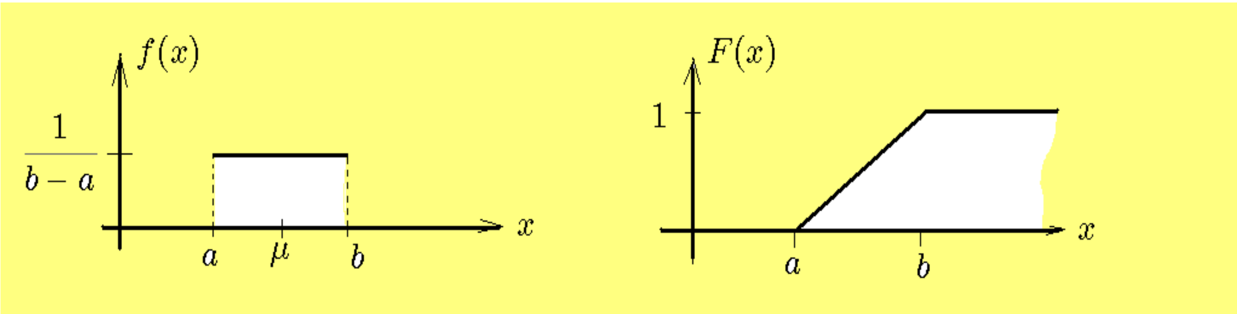
The chi-square, student t and F distributions are used in testing of hypothesis, confidence intervals and testing differences or ratios of various statistics associated with independent samples. The degrees of freedom associated with these distributions can be thought of as a parameter representing an increase in reliability of the calculated statistic. That is, a statistic associated with one degree of freedom is less reliable than the same statistic calculated using a higher degree of freedom. In some cases the degrees of freedom are related to the number of data points used to calculate the statistic. In some cases the degrees of freedom is obtained by subtracting 1 from the sample size n.

The Uniform Distribution

The uniform probability density function $f(x)$ and the associated distribution function $F(x)$ are given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_{a}^{x} f(x)\, dx$$

It is sometimes referred to as the rectangular distribution on the interval $a < x < b$.



This distribution has the mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{a}^{b} x \frac{1}{b-a}\, dx = \frac{1}{2}(a+b)$$

and variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \frac{1}{12}(b-a)^2$$

The cumulative distribution function is given by $F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$

The uniform probability density function is used in pseudo-random number generators with sampling is over the interval $0 \le x \le 1$.

Confidence Intervals

Sampling theory is a study of the various relationships that exist between properties of a population and information obtained based upon samples from the population. For example, each sample collected from a population has associated with it a sample mean $\overline{x} = \mu_{\overline{x}}$ and sample variance $s^2 = \sigma^2_{\overline{x}}$. How do these quantities compare with the true population mean μ and true population variance σ^2 ? It would be nice to put limits, like numbers γ_1, γ_2, associated with the value $\mu_{\overline{x}}$ so that one can write a statement like

$$\mu_{\overline{x}} - \gamma_1 < \mu < \mu_{\overline{x}} + \gamma_2$$

It would also be nice to be able to adjust γ_1 and γ_2 so that one could say that there is a 90% probability that the true mean lies within the specified limits. It would be better still if one could change the 90% value to obtain limits for say a 95%,97%,or 99% probability that the true mean lie within the bounds specified. The probability values $90\%, 95\%, 97\%$ or 99% are called **confidence levels** associated with the calculated mean value. To determine such limits one can employ **the central limit theorem** from statistics which says that if (i) the number n of independent random variables in each sample (the sample size) is large with a finite mean and variance for each sample and (ii) the number of samples taken is large. Then the mean value associated with the large set of sample means will be normally distributed.

Another way to state the above is as follows. For X a continuous random variable which comes from some kind of probability distribution having a well defined mean μ and variance σ^2, the **central limit theorem** states that if a **large number of sample means are collected**, and one forms a table of these mean values and does an analysis of the collected set of n means and forms a frequency table, just like table 11.3, then one finds that these sample means are approximately normally distributed. The central limit theorem also states that the distribution of the sample means can be made as close to a normal distribution as desired, by taking larger and larger sample sizes. It can be shown that the distribution of the sample means $\bar{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. The normal distribution can be scaled to standard form by making an appropriate change of variables.

To use the central limit theorem select a confidence level $\gamma = 1 - \alpha$ which represents the area between the limits $-z_{\alpha/2}$ and $z_{\alpha/2}$ associated with the normalized normal probability density function as illustrated in the figure 11-13. This determines values α and $\alpha/2$ and the values $\pm z_{\alpha/2}$ can be obtained from the normalized probability table 11.5. Some example values for $1 - \alpha$ are

$1-\alpha$	.90	.95	.99	.999
α	.10	.05	.01	.001
$\alpha/2$	.05	.025	.005	.0005
$z_{\alpha/2}$	1.645	1.960	2.576	3.291

If $\overline{x}$ is the mean of a sample $\{x_1, x_2, \ldots, x_n\}$ of size n, then confidence limits on the value $\overline{x}$ are determined as follows.

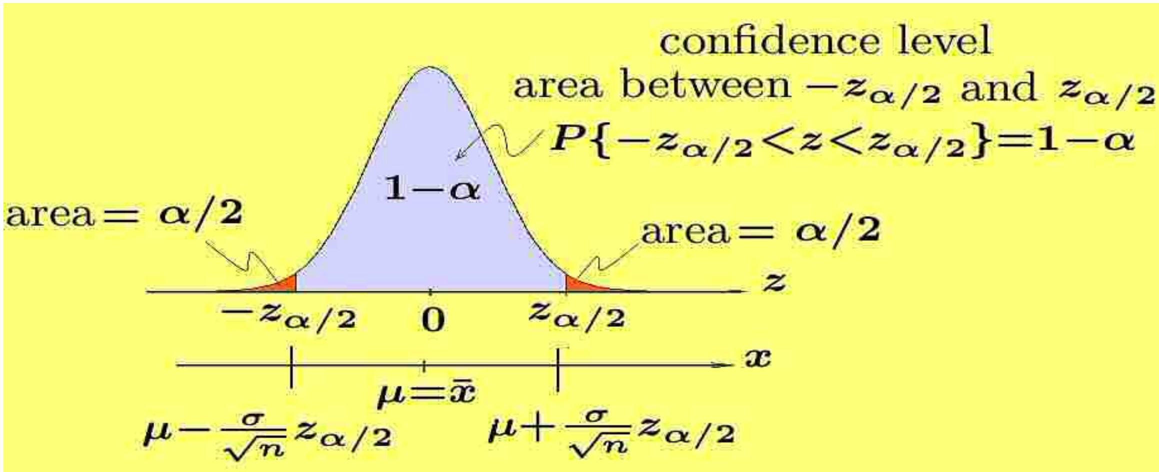


Figure 11-13. Raw scores scaled to normal probability density function.

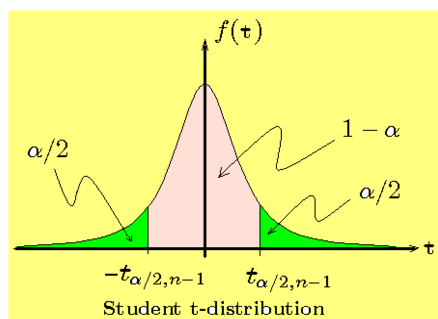
Normal distribution with known variance σ^2

If the variance of the population is known then use the central limit theorem to construct the following confidence interval for the mean μ of the population based upon a $1-\alpha=\gamma$ level of confidence

$$CONF\{\overline{x}-\frac{\sigma}{\sqrt{n}}z_{\alpha/2}\le\mu\le\overline{x}+\frac{\sigma}{\sqrt{n}}z_{\alpha/2}\} \tag{11.81}$$

Normal distribution with unknown variance σ^2

In the case where the population variance is unknown, then make use of the fact that $t=\dfrac{|\overline{x}-\mu|}{s/\sqrt{n}}$ follows **a student t-distribution** to construct a confidence interval. From the student t-distribution determine the value $t_{\alpha/2,n-1}$ based upon $n-1$ degrees of freedom, n being the sample size, such that the right tailed area under equals $\alpha/2$ as illustrated in the accompanying figure.



Some examples for a sample size of $n=11$ and degrees of freedom $n-1=10$ are given in the following table.

$1-\alpha$	.90	.95	.99	.999
α	.10	.05	.01	.001
$\alpha/2$	.05	.025	.005	.0005
$t_{\alpha/2,10}$	1.812	2.228	3.169	4.144

The confidence interval for the mean μ of the population uses the computed variance

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})^2 \tag{11.82}$$

to produce the $\gamma = 1 - \alpha$ confidence level

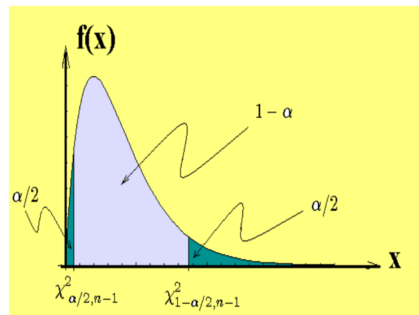
$$CONF\,\{\overline{x} - \frac{s}{\sqrt{n}}\,t_{\alpha/2,n-1} \le \mu \le \overline{x} + \frac{s}{\sqrt{n}}\,t_{\alpha/2,n-1}\} \tag{11.83}$$

where n is the sample size.

Confidence interval for the variance σ^2

The confidence interval for the variance σ^2 of the population having a normal distribution is based upon the fact that the variable $Y = (n-1)s^2/\sigma^2$ follows **a chi-square distribution with $n-1$ degrees of freedom**, where again n represents the sample size.

First select a level of confidence $\gamma = 1 - \alpha$ and then from a chi-square distribution table with $n-1$ degrees of freedom determine the $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ values which represent the points corresponding to the tail areas of the chi-square probability density function as illustrated.



Secondly, one must calculate the variance squared s^2 using equation (11.82), then construct the confidence interval for the variance of a normal distribution given by

$$CONF\,\{(n-1)\frac{s^2}{\chi^2_{1-\alpha/2,n-1}} \le \sigma^2 \le (n-1)\frac{s^2}{\chi^2_{\alpha/2,n-1}}\} \tag{11.84}$$

Least Squares Curve Fitting

A set of data points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_i, y_i), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ can be plotted on ordinary graph paper and then a line $y = \beta_0 + \beta_1 x$ can also be plotted to obtain a figure such as illustrated in the figure 11-14.

Assume that the data points are normally distributed about the straight line and that errors $e_1, e_2, \ldots, e_n$ occur in the y-variable, where the errors are defined as **the differences between the y-value on the line and the y-value of the data point**. What would be the "best" straight line to represent the given data points? There are many ways to define "best". By defining the error e_i associated with the ith data point (x_i, y_i) as

$$\begin{aligned} e_i =& (y \text{ of line at } x_i) - (y \text{ data value at } x_i) \\ e_i =& \beta_0 + \beta_1 x_i - y_i \end{aligned} \tag{11.85}$$

then associated with the given set of data are the errors

$$\begin{aligned} e_1 =& y(x_1) - y_1 = \beta_0 + \beta_1 x_1 - y_1 \\ e_2 =& y(x_2) - y_2 = \beta_0 + \beta_1 x_2 - y_2 \\ e_3 =& y(x_3) - y_3 = \beta_0 + \beta_1 x_3 - y_3 \\ &\vdots \\ e_n =& y(x_n) - y_n = \beta_0 + \beta_1 x_n - y_n. \end{aligned} \tag{11.86}$$

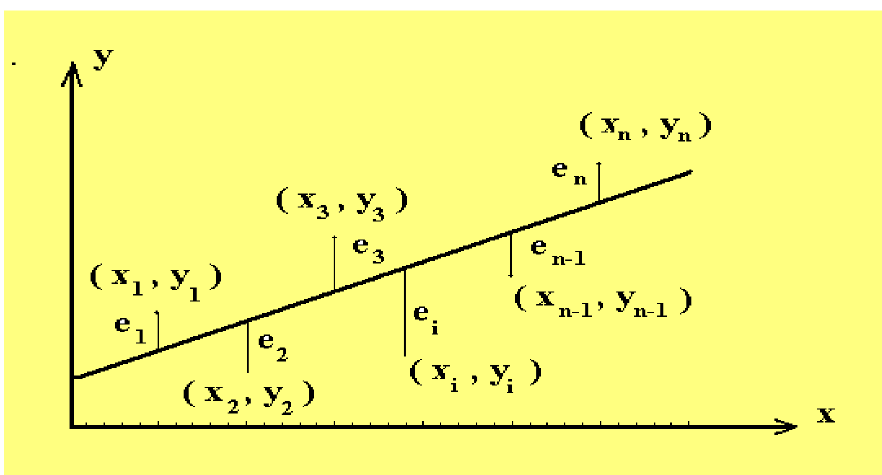


Figure 11-14. Straight line approximation to represent data points.

One way to define the "best" straight line $y = \beta_0 + \beta_1 x$ is to select the constants β_0 and β_1 which **minimize the sum of squares of the errors** associated with the data set. That is, if

$$E = E(\beta_0, \beta_1) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)^2 \tag{11.87}$$

denotes the sum of squares of the errors, then E has a minimum value when the conditions

$$\frac{\partial E}{\partial \beta_0} = 0 \quad \text{and} \quad \frac{\partial E}{\partial \beta_1} = 0 \tag{11.88}$$

are satisfied simultaneously. Hence, the constants β_0 and β_1 must be selected to satisfy the simultaneous equations

$$\begin{aligned}\frac{\partial E}{\partial \beta_0} =& 2\sum_{i=1}^{n}(\beta_0+\beta_1 x_i - y_i)(1) = 0\\ \frac{\partial E}{\partial \beta_1} =& 2\sum_{i=1}^{n}(\beta_0+\beta_1 x_i - y_i)(x_i) = 0.\end{aligned} \tag{11.89}$$

The equations (11.89) simplify to the 2×2 linear system of equations

$$\begin{aligned}n\beta_0 + \left(\sum_{i=1}^{n} x_i\right)\beta_1 =& \sum_{i=1}^{n} y_i\\ \left(\sum_{i=1}^{n} x_i\right)\beta_0 + \left(\sum_{i=1}^{n} x_i^2\right)\beta_1 =& \sum_{i=1}^{n} x_i y_i\end{aligned} \tag{11.90}$$

which can then be solved for the coefficients β_0 and β_1. This gives the "best" least squares straight line $y = y(x) = \beta_0 + \beta_1 x$.

Alternatively, set all of the equations (11.86) equal to zero, to obtain a system of equations having the matrix form

$$\begin{aligned}A\overline{\beta} =& \overline{\mathbf{y}}\\ \begin{bmatrix}1 & x_1\\ 1 & x_2\\ 1 & x_3\\ \vdots & \vdots\\ 1 & x_n\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} =& \begin{bmatrix}y_1\\ y_2\\ y_3\\ \vdots\\ y_n\end{bmatrix}.\end{aligned} \tag{11.91}$$

By doing this the data set of errors, calculated from the difference in the data set y values and the straight line y values, is represented as an over determined system of equations for determining the constants β_0 and β_1. That is, there are more equations than there are unknowns and so the unknowns β_0, β_1 are selected to minimize the sum of squares error associated with the over determined system of equations. Observe that left multiplying both sides of equation (11.91) by the transpose matrix A^T gives the new set of equations $A^T A\overline{\beta} = A^T\overline{\mathbf{y}}$ or

$$\begin{bmatrix}1 & 1 & 1 & \dots & 1\\ x_1 & x_2 & x_3 & \dots & x_n\end{bmatrix}\begin{bmatrix}1 & x_1\\ 1 & x_2\\ 1 & x_3\\ \vdots & \vdots\\ 1 & x_n\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & \dots & 1\\ x_1 & x_2 & x_3 & \dots & x_n\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ \vdots\\ y_n\end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix} \tag{11.92}$$

which is the matrix form of the equations (11.90). This presents an alternative way to solve for the coefficients β_0 and β_1

Linear Regression

The previous least squares method applied to a straight line fit of data. The ideas presented can be generalized to fitting data to **any linear combination of functions**. Given a set of data points (x_i, y_i), for $i = 1, 2, \ldots, n$, assume a curve fit function of the form

$$y = y(x) = \beta_0 f_0(x) + \beta_1 f_1(x) + \beta_2 f_2(x) + \cdots + \beta_k f_k(x) \tag{11.93}$$

where $\beta_0, \beta_1, \ldots, \beta_k$ are unknown coefficients and $f_0(x), f_1(x), f_2(x), \ldots, f_k(x)$ represent linearly independent functions, called the basis of the representation. Note that for the previous straight line fit the independent functions $f_0(x) = 1$ and $f_1(x) = x$ were used. In general, select any set of independent functions and select the β coefficients such that the sum of squares error

$$\begin{aligned} E = E(\beta_0, \beta_1, \ldots, \beta_k) =& \sum_{i=1}^n (y(x_i) - y_i)^2 \\ E = E(\beta_0, \beta_1, \ldots, \beta_k) =& \sum_{i=1}^n \left[\beta_0 f_0(x_i) + \beta_1 f_1(x_i) + \beta_2 f_2(x_i) + \cdots + \beta_k f_k(x_i) - y_i\right]^2 \end{aligned} \tag{11.94}$$

is a minimum. The determination of the β-values requires a solution be found from the set of **simultaneous least square equations**

$$\frac{\partial E}{\partial \beta_0} = 0, \quad \frac{\partial E}{\partial \beta_1} = 0, \quad \cdots, \quad \frac{\partial E}{\partial \beta_k} = 0. \tag{11.95}$$

Another way to obtain the system of equations (11.95) is to first represent the data in the matrix form

$$A\overline{\beta} = \overline{\mathbf{y}}$$

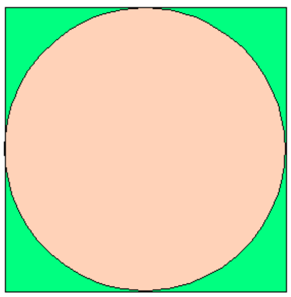
$$\begin{bmatrix} f_0(x_1) & f_1(x_1) & f_2(x_1) & \cdots & f_k(x_1) \\ f_0(x_2) & f_1(x_2) & f_2(x_2) & \cdots & f_k(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & f_2(x_n) & \cdots & f_k(x_n) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{11.96}$$

Both sides of the equation (11.96) can be left multiplied by the transpose matrix A^T and the resulting system can be solved for the unknown coefficients. In matrix notation write

$$\begin{aligned} A\overline{\beta} =& \overline{\mathbf{y}} \\ A^T A\overline{\beta} =& A^T\overline{\mathbf{y}} \\ \overline{\beta} =& (A^T A)^{-1} A^T \overline{\mathbf{y}}. \end{aligned} \tag{11.97}$$

The solution of the system of equations (11.95) or (11.97) will produce the coefficients β_i, $i = 0, 1, \ldots, k$, which minimizes the sum of square error.

Monte Carlo Methods



Monte Carlo methods is a term used to describe a wide variety of computer techniques which employ **random number generators** to simulate an event or events and then perform a statistical analysis of the results. Sometimes Monte Carlo methods are constructed to solve difficult problems where deterministic methods fail. If performed properly, Monte Carlo methods can give very accurate answers. The only drawback is that **some** Monte Carlo techniques take a very long time to run on the computer.

An example of a simple Monte Carlo method is the calculation of the area A of a circle using random numbers. Consider a circle with radius 1/2 which is placed inside the unit square having vertices $(0,0), (1,0), (1,1)$ and $(0,1)$. The area of this circle is $\pi/4 = 0.7853981634....$

Most computer languages have **a uniform random number generator** which generates pseudo-random numbers lying between 0 and 1. Construct a computer program which employs the uniform random number generator to generate two random numbers (x_r, y_r), where $0 < x_r < 1$ and $0 < y_r < 1$, then imagine the circle inside the unit square as a circular dart board and the random number generated by the computer program (x_r, y_r) is where the dart lands. Construct the computer program to perform a test as to whether the point (x_r, y_r) is on or off the circular dart board. Perform this test N-times and record the number of hits which land on or inside the circle. To calculate the area of the circle assume the ratio of hits inside circle to total number of points generated is in the same proportion as the area of the circle is to the area of the square. One can then use the ratio

$$\frac{\text{Number hits inside circle}}{\text{Total number of darts thrown}} = \frac{\text{Area of circle}}{\text{Area of square}}$$

to determine the area A of the circle. If H denotes the number of hits inside the circle and N represents the total number of darts thrown, the area of the circle is determined by $\dfrac{H}{N} = \dfrac{A}{1} = A$.

Class midpoint for Average Area	Frequency
0.78027	0
0.780783	0
0.781295	0
0.781808	2
0.782321	4
0.782834	12
0.783347	19
0.78386	44
0.784372	63
0.784885	69
0.785398	79
0.785911	62
0.786424	65
0.786937	44
0.78745	20
0.787962	9
0.788475	6
0.788988	2
0.789501	0
0.790014	0
0.790527	0

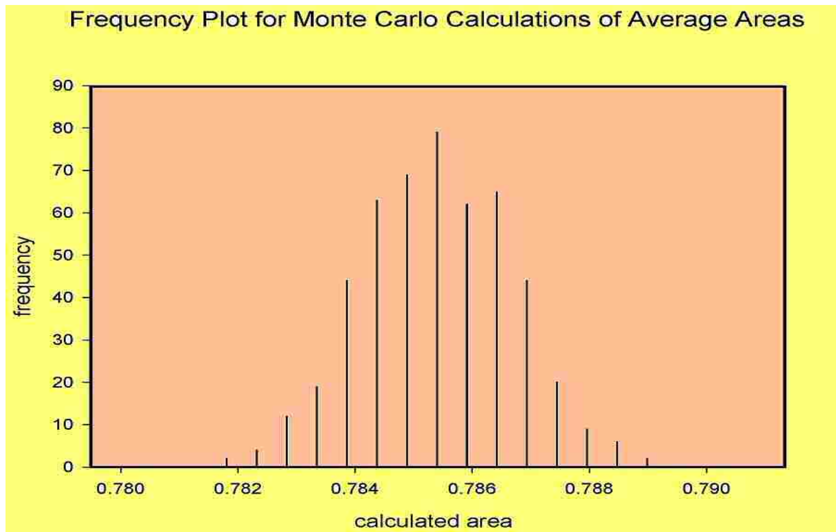


Figure 11-15. Average areas from Monte Carlo simulation.

Perform the above experiment K-times to calculate a set of approximate areas $\{A_1, A_2, \ldots, A_K\}$ having an average area $\overline{A} = \frac{1}{K}\sum_{i=1}^{K} A_i$. Put all of the above computer code in a loop and calculate M-averages $\{\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_M\}$. The central limit theorem tells us that the set of averages must be normally distributed. By calculating the mean and standard deviation associated with all these averages it is possible to determine very accurate bounds on the area of the circle. Using the values $N = 1000$ throws, $K = 100$ areas, and $M = 500$ area averages, modern laptop computers can calculate the results in less than one minute.

The data generated for the above values of N,K and M is presented in figure 11-15 as a bar chart having a mean 0.7853974 and standard deviation 0.001282.

Obtaining a Uniform Random Number Generator

Some form of a uniform random number generator, called a pseudorandom number generator and abbreviated (PRNG), can usually be found as an intrinsic function within many of the more popular computer programming languages. If the computer language you are using does not have a uniform random number generator, then you can obtain one from off the internet. Pseudorandom number generators generate a sequence of numbers $\{x_n\}$ satisfying $0 \le x_n \le 1$. The sequence of numbers generated is not truly random because of technical issues involving the mathematical methods used to generate the uniform random numbers. Most of the PRNG programs in use have passed many statistical tests which guarantee that the sequence generated is random enough for Monte Carlo studies and other statistical applications.

Linear Interpolation

In obtaining a specific numerical value from a table of (x, y) values it is sometimes necessary to use **linear interpolation** where a line is constructed between two known numerical values and then values along the line are used as estimates for the tabular values between the known values. In one dimension, one can say that if (x_1, y_1) and (x_2, y_2) are known values, then if x is a value between x_1 and x_2, the corresponding value for y is given by $y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$. This result can be expressed in a variety of forms. One form is to make the substitution $x_2 - x_1 = h$ with $x = x_1 + \beta h$, then $y = y_1 + \beta(y_2 - y_1)$ or $y = (1 - \beta)y_1 + \beta y_2$.

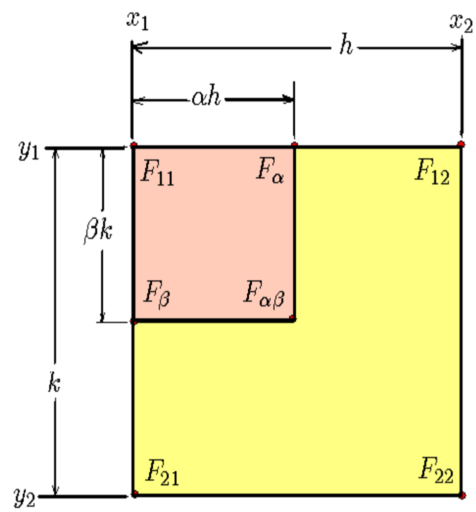
Interpolation in two-dimension

Consider the set of values in a table as illustrated in the accompanying figure. Let F_{11} denote the value in the table corresponding to the position (x_1, y_1). Similarly, define the values F_{12}, F_{21} and F_{22} corresponding respectively to the points (x_1, y_2), (x_2, y_1) and (x_2, y_2). Interpolation over this two dimensional array is the problem of determining the values F_α, F_β and $F_{\alpha,\beta}$ which are positioned on the boundaries and interior to the box connecting the known data values $F_{11}, F_{12}, F_{22}, F_{21}$.

		x_1	x_2	
	7.956	7.471	7.134	6.885
	7.343	6.872	6.545	6.302
	6.881	6.422	6.102	5.865
	6.521	6.071	5.757	5.525
y_1	6.233	5.791	5.482	5.253
y_2	5.998	5.562	5.257	5.031
	5.803	5.372	5.071	4.847
	5.638	5.212	4.913	4.692

Interpolate first in the x-direction and then in the y-direction or vice-versa and show that

$$\begin{aligned} F_\alpha =&(1-\alpha)F_{11}+\alpha F_{12}, \quad F_\beta=(1-\beta)F_{11}+\beta F_{21}\\ F_{\alpha,\beta} =&(1-\alpha)(1-\beta)F_{11}+\alpha(1-\beta)F_{12}+\beta(1-\alpha)F_{21}+\alpha\beta F_{22} \end{aligned}$$



Note how the F_α and F_β values vary as the parameters α and β vary from 0 to 1. This is a straight forward linear interpolation between the given values. The value $F_{\alpha,\beta}$ is obtained by first doing a linear interpolation in the y direction at the columns x_1 and x_2, which is then followed by a linear interpolation in the x-direction.

An alternative method of interpolation is to use the Taylor series expansion in both the x and y directions to obtain the alternative interpolation formula

$$F_{\alpha,\beta}=(1-\alpha-\beta)F_{11}+\beta F_{21}+\alpha F_{12}$$

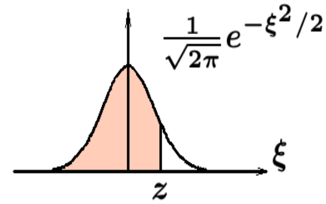
Sometimes it is necessary to modify the above interpolation formulas for application to entries in a three-dimensional array of numbers. The interpolation result is obtained by applying the one-dimensional interpolation formulas in each of the x, y and z directions.

Statistical Tables

This introduction to the study of statistics concludes with some well known statistical tables. These tables are employed in various types of statistics testing. Statistical tables in many forms where extensively used prior to the advent of computers. The internet provides the access to a much larger variety of statistical tables.

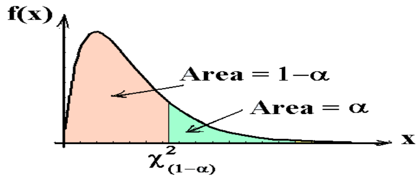
Table 11.5 Area Under Standard Normal Curve

$$Area = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$$



z	Area	z	Area	z	Area	z	Area	z	Area	z	Area	z	Area
0.00	.5000	.50	.6915	1.00	.8413	1.50	.9332	2.00	.9772	2.50	.9938	3.00	.9987
.01	.5040	.51	.6950	1.01	.8438	1.51	.9345	2.01	.9778	2.51	.9940	3.01	.9987
.02	.5080	.52	.6985	1.02	.8461	1.52	.9357	2.02	.9783	2.52	.9941	3.02	.9987
.03	.5120	.53	.7019	1.03	.8485	1.53	.9370	2.03	.9788	2.53	.9943	3.03	.9988
.04	.5160	.54	.7054	1.04	.8508	1.54	.9382	2.04	.9793	2.54	.9945	3.04	.9988
.05	.5199	.55	.7088	1.05	.8531	1.55	.9394	2.05	.9798	2.55	.9946	3.05	.9989
.06	.5239	.56	.7123	1.06	.8554	1.56	.9406	2.06	.9803	2.56	.9948	3.06	.9989
.07	.5279	.57	.7157	1.07	.8577	1.57	.9418	2.07	.9808	2.57	.9949	3.07	.9989
.08	.5319	.58	.7190	1.08	.8599	1.58	.9429	2.08	.9812	2.58	.9951	3.08	.9990
.09	.5359	.59	.7224	1.09	.8621	1.59	.9441	2.09	.9817	2.59	.9952	3.09	.9990
.10	.5398	.60	.7257	1.10	.8643	1.60	.9452	2.10	.9821	2.60	.9953	3.10	.9990
.11	.5438	.61	.7291	1.11	.8665	1.61	.9463	2.11	.9826	2.61	.9955	3.11	.9991
.12	.5478	.62	.7324	1.12	.8686	1.62	.9474	2.12	.9830	2.62	.9956	3.12	.9991
.13	.5517	.63	.7357	1.13	.8708	1.63	.9484	2.13	.9834	2.63	.9957	3.13	.9991
.14	.5557	.64	.7389	1.14	.8729	1.64	.9495	2.14	.9838	2.64	.9959	3.14	.9992
.15	.5596	.65	.7422	1.15	.8749	1.65	.9505	2.15	.9842	2.65	.9960	3.15	.9992
.16	.5636	.66	.7454	1.16	.8770	1.66	.9515	2.16	.9846	2.66	.9961	3.16	.9992
.17	.5675	.67	.7486	1.17	.8790	1.67	.9525	2.17	.9850	2.67	.9962	3.17	.9992
.18	.5714	.68	.7517	1.18	.8810	1.68	.9535	2.18	.9854	2.68	.9963	3.18	.9993
.19	.5753	.69	.7549	1.19	.8830	1.69	.9545	2.19	.9857	2.69	.9964	3.19	.9993
.20	.5793	.70	.7580	1.20	.8849	1.70	.9554	2.20	.9861	2.70	.9965	3.20	.9993
.21	.5832	.71	.7611	1.21	.8869	1.71	.9564	2.21	.9864	2.71	.9966	3.21	.9993
.22	.5871	.72	.7642	1.22	.8888	1.72	.9573	2.22	.9868	2.72	.9967	3.22	.9994
.23	.5910	.73	.7673	1.23	.8907	1.73	.9582	2.23	.9871	2.73	.9968	3.23	.9994
.24	.5948	.74	.7704	1.24	.8925	1.74	.9591	2.24	.9875	2.74	.9969	3.24	.9994
.25	.5987	.75	.7734	1.25	.8944	1.75	.9599	2.25	.9878	2.75	.9970	3.25	.9994
.26	.6026	.76	.7764	1.26	.8962	1.76	.9608	2.26	.9881	2.76	.9971	3.26	.9994
.27	.6064	.77	.7794	1.27	.8980	1.77	.9616	2.27	.9884	2.77	.9972	3.27	.9995
.28	.6103	.78	.7823	1.28	.8997	1.78	.9625	2.28	.9887	2.78	.9973	3.28	.9995
.29	.6141	.79	.7852	1.29	.9015	1.79	.9633	2.29	.9890	2.79	.9974	3.29	.9995
.30	.6179	.80	.7881	1.30	.9032	1.80	.9641	2.30	.9893	2.80	.9974	3.30	.9995
.31	.6217	.81	.7910	1.31	.9049	1.81	.9649	2.31	.9896	2.81	.9975	3.31	.9995
.32	.6255	.82	.7939	1.32	.9066	1.82	.9656	2.32	.9898	2.82	.9976	3.32	.9995
.33	.6293	.83	.7967	1.33	.9082	1.83	.9664	2.33	.9901	2.83	.9977	3.33	.9996
.34	.6331	.84	.7995	1.34	.9099	1.84	.9671	2.34	.9904	2.84	.9977	3.34	.9996
.35	.6368	.85	.8023	1.35	.9115	1.85	.9678	2.35	.9906	2.85	.9978	3.35	.9996
.36	.6406	.86	.8051	1.36	.9131	1.86	.9686	2.36	.9909	2.86	.9979	3.36	.9996
.37	.6443	.87	.8078	1.37	.9147	1.87	.9693	2.37	.9911	2.87	.9979	3.37	.9996
.38	.6480	.88	.8106	1.38	.9162	1.88	.9699	2.38	.9913	2.88	.9980	3.38	.9996
.39	.6517	.89	.8133	1.39	.9177	1.89	.9706	2.39	.9916	2.89	.9981	3.39	.9997
.40	.6554	.90	.8159	1.40	.9192	1.90	.9713	2.40	.9918	2.90	.9981	3.40	.9997
.41	.6591	.91	.8186	1.41	.9207	1.91	.9719	2.41	.9920	2.91	.9982	3.41	.9997
.42	.6628	.92	.8212	1.42	.9222	1.92	.9726	2.42	.9922	2.92	.9982	3.42	.9997
.43	.6664	.93	.8238	1.43	.9236	1.93	.9732	2.43	.9925	2.93	.9983	3.43	.9997
.44	.6700	.94	.8264	1.44	.9251	1.94	.9738	2.44	.9927	2.94	.9984	3.44	.9997
.45	.6736	.95	.8289	1.45	.9265	1.95	.9744	2.45	.9929	2.95	.9984	3.45	.9997
.46	.6772	.96	.8315	1.46	.9279	1.96	.9750	2.46	.9931	2.96	.9985	3.46	.9997
.47	.6808	.97	.8340	1.47	.9292	1.97	.9756	2.47	.9932	2.97	.9985	3.47	.9997
.48	.6844	.98	.8365	1.48	.9306	1.98	.9761	2.48	.9934	2.98	.9986	3.48	.9997
.49	.6879	.99	.8389	1.49	.9319	1.99	.9767	2.49	.9936	2.99	.9986	3.49	.9998

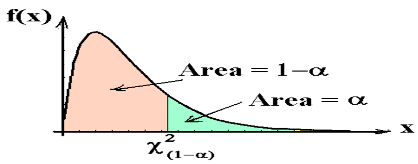
Table 11.6(a). Critical Values for the Chi-square Distribution with ν Degrees of Freedom



$$\int_0^{\chi^2_{(1-\alpha)}} \frac{x^{(\nu/2)-1}}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2}\, dx = 1-\alpha$$

α	0.995	0.990	0.975	0.950	0.900
$1-\alpha$	0.005	0.010	0.025	0.050	0.100
ν	$\chi^2_{0.005}$	$\chi^2_{0.010}$	$\chi^2_{0.025}$	$\chi^2_{0.050}$	$\chi^2_{0.100}$
1	0.0000	0.0002	0.0010	0.0039	0.0158
2	0.0100	0.0201	0.0506	0.1026	0.2107
3	0.0717	0.1148	0.2158	0.3518	0.5844
4	0.2070	0.2971	0.4844	0.7107	1.0636
5	0.4117	0.5543	0.8312	1.1455	1.6103
6	0.6757	0.8721	1.2373	1.6354	2.2041
7	0.9893	1.2390	1.6899	2.1673	2.8331
8	1.3444	1.6465	2.1797	2.7326	3.4895
9	1.7349	2.0879	2.7004	3.3251	4.1682
10	2.1559	2.5582	3.2470	3.9403	4.8652
11	2.6032	3.0535	3.8157	4.5748	5.5778
12	3.0738	3.5706	4.4038	5.2260	6.3038
13	3.5650	4.1069	5.0088	5.8919	7.0415
14	4.0747	4.6604	5.6287	6.5706	7.7895
15	4.6009	5.2293	6.2621	7.2609	8.5468
16	5.1422	5.8122	6.9077	7.9616	9.3122
17	5.6972	6.4078	7.5642	8.6718	10.0852
18	6.2648	7.0149	8.2307	9.3905	10.8649
19	6.8440	7.6327	8.9065	10.1170	11.6509
20	7.4338	8.2604	9.5908	10.8508	12.4426
21	8.0337	8.8972	10.2829	11.5913	13.2396
22	8.6427	9.5425	10.9823	12.3380	14.0415
23	9.2604	10.1957	11.6886	13.0905	14.8480
24	9.8862	10.8564	12.4012	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1602	12.1981	13.8439	15.3792	17.2919
27	11.8076	12.8785	14.5734	16.1514	18.1139
28	12.4613	13.5647	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7084	19.7677
30	13.7867	14.9535	16.7908	18.4927	20.5992
40	20.7065	22.1643	24.4330	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7643	37.6886
60	35.5345	37.4849	40.4817	43.1880	46.4589
70	43.2752	45.4417	48.7576	51.7393	55.3289
80	51.1719	53.5401	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2911
100	67.3276	70.0649	74.2219	77.9295	82.3581

Table 11.6(b). Critical Values for the Chi-square Distribution with ν Degrees of Freedom

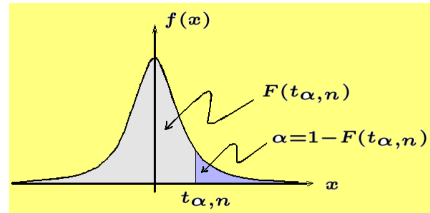


$$\int_0^{\chi^2_{(1-\alpha)}} \frac{x^{(\nu/2)-1}}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2}\,dx = 1-\alpha$$

α	0.100	0.050	0.025	0.010	0.005
$1-\alpha$	0.900	0.950	0.975	0.990	0.995
ν	$\chi^2_{0.900}$	$\chi^2_{0.950}$	$\chi^2_{0.975}$	$\chi^2_{0.990}$	$\chi^2_{0.995}$
1	2.7055	3.8415	5.0239	6.6349	7.8794
2	4.6052	5.9915	7.3778	9.2103	10.5966
3	6.2514	7.8147	9.3484	11.3449	12.8382
4	7.7794	9.4877	11.1433	13.2767	14.8603
5	9.2364	11.0705	12.8325	15.0863	16.7496
6	10.6446	12.5916	14.4494	16.8119	18.5476
7	12.0170	14.0671	16.0128	18.4753	20.2777
8	13.3616	15.5073	17.5345	20.0902	21.9550
9	14.6837	16.9190	19.0228	21.6660	23.5894
10	15.9872	18.3070	20.4832	23.2093	25.1882
11	17.2750	19.6751	21.9200	24.7250	26.7568
12	18.5493	21.0261	23.3367	26.2170	28.2995
13	19.8119	22.3620	24.7356	27.6882	29.8195
14	21.0641	23.6848	26.1189	29.1412	31.3193
15	22.3071	24.9958	27.4884	30.5779	32.8013
16	23.5418	26.2962	28.8454	31.9999	34.2672
17	24.7690	27.5871	30.1910	33.4087	35.7185
18	25.9894	28.8693	31.5264	34.8053	37.1565
19	27.2036	30.1435	32.8523	36.1909	38.5823
20	28.4120	31.4104	34.1696	37.5662	39.9968
21	29.6151	32.6706	35.4789	38.9322	41.4011
22	30.8133	33.9244	36.7807	40.2894	42.7957
23	32.0069	35.1725	38.0756	41.6384	44.1813
24	33.1962	36.4150	39.3641	42.9798	45.5585
25	34.3816	37.6525	40.6465	44.3141	46.9279
26	35.5632	38.8851	41.9232	45.6417	48.2899
27	36.7412	40.1133	43.1945	46.9629	49.6449
28	37.9159	41.3371	44.4608	48.2782	50.9934
29	39.0875	42.5570	45.7223	49.5879	52.3356
30	40.2560	43.7730	46.9792	50.8922	53.6720
40	51.8051	55.7585	59.3417	63.6907	66.7660
50	63.1671	67.5048	71.4202	76.1539	79.4900
60	74.3970	79.0819	83.2977	88.3794	91.9517
70	85.5270	90.5312	95.0232	100.4252	104.2149
80	96.5782	101.8795	106.6286	112.3288	116.3211
90	107.5650	113.1453	118.1359	124.1163	128.2989
100	118.4980	124.3421	129.5612	135.8067	140.1695

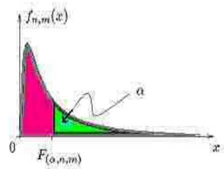
Table 11.7 Critical Values $t_{\alpha,n}$ for the Student's t Distribution with n Degrees of Freedom

$$\int_{t_{\alpha,n}}^{\infty} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-(n+1)/2} dx = \alpha = 1 - F(t_{\alpha,n})$$



n	$t_{0.1,n}$	$t_{0.05,n}$	$t_{0.025,n}$	$t_{0.01,n}$	$t_{0.005,n}$	$t_{0.001}$
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
50	1.299	1.676	2.009	2.403	2.678	3.261
60	1.296	1.671	2.000	2.390	2.660	3.232
70	1.294	1.667	1.994	2.381	2.648	3.211
80	1.292	1.664	1.990	2.374	2.639	3.195
90	1.291	1.662	1.987	2.368	2.632	3.183
100	1.290	1.660	1.984	2.364	2.626	3.174
∞	1.282	1.645	1.960	2.326	2.576	3.098

Table 11.8 (a) Critical Values of the F-Distribution for $\alpha = 0.1$

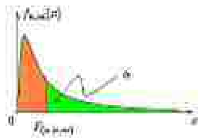


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	39.863	49.500	53.593	55.833	57.240	58.204	58.906	59.439	59.858	60.195	61.220	61.740	62.055	62.265	62.529	62.794	63.061
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392	9.425	9.441	9.451	9.458	9.466	9.475	9.483
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230	5.200	5.184	5.175	5.168	5.160	5.151	5.143
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920	3.870	3.844	3.828	3.817	3.804	3.790	3.775
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297	3.238	3.207	3.187	3.174	3.157	3.140	3.123
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937	2.871	2.836	2.815	2.800	2.781	2.762	2.742
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703	2.632	2.595	2.571	2.555	2.535	2.514	2.493
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538	2.464	2.425	2.400	2.383	2.361	2.339	2.316
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416	2.340	2.298	2.272	2.255	2.232	2.208	2.184
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323	2.244	2.201	2.174	2.155	2.132	2.107	2.082
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.274	2.248	2.167	2.123	2.095	2.076	2.052	2.026	2.000
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188	2.105	2.060	2.031	2.011	1.986	1.960	1.932
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.164	2.138	2.053	2.007	1.978	1.958	1.931	1.904	1.876
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.122	2.095	2.010	1.962	1.933	1.912	1.885	1.857	1.828
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059	1.972	1.924	1.894	1.873	1.845	1.817	1.787
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.055	2.028	1.940	1.891	1.860	1.839	1.811	1.782	1.751
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.028	2.001	1.912	1.862	1.831	1.809	1.781	1.751	1.719
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977	1.887	1.837	1.805	1.783	1.754	1.723	1.691
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.984	1.956	1.865	1.814	1.782	1.759	1.730	1.699	1.666
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937	1.845	1.794	1.761	1.738	1.708	1.677	1.643
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.948	1.920	1.827	1.776	1.742	1.719	1.689	1.657	1.623
22	2.949	2.561	2.351	2.219	2.128	2.060	2.008	1.967	1.933	1.904	1.811	1.759	1.726	1.702	1.671	1.639	1.604
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.919	1.890	1.796	1.744	1.710	1.686	1.655	1.622	1.587
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.906	1.877	1.783	1.730	1.696	1.672	1.641	1.607	1.571
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.895	1.866	1.771	1.718	1.683	1.659	1.627	1.593	1.557
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819	1.722	1.667	1.632	1.606	1.573	1.538	1.499
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763	1.662	1.605	1.568	1.541	1.506	1.467	1.425
50	2.809	2.412	2.197	2.061	1.966	1.895	1.840	1.796	1.760	1.729	1.627	1.568	1.529	1.502	1.465	1.424	1.379
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707	1.603	1.543	1.504	1.476	1.437	1.395	1.348
100	2.756	2.356	2.139	2.002	1.906	1.834	1.778	1.732	1.695	1.663	1.557	1.494	1.453	1.423	1.382	1.336	1.282
120	2.748	2.347	2.130	1.992	1.896	1.824	1.767	1.722	1.684	1.652	1.545	1.482	1.440	1.409	1.368	1.320	1.265

Table 11.8 (b) Critical Values of the F-Distribution for $\alpha = 0.05$

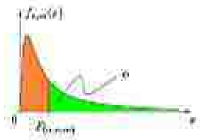


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.883	240.543	241.882	245.950	248.013	249.260	250.095	251.143	252.196	253.253
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396	19.429	19.446	19.456	19.462	19.471	19.479	19.487
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.703	8.660	8.634	8.617	8.594	8.572	8.549
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.858	5.803	5.769	5.746	5.717	5.688	5.658
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.619	4.558	4.521	4.496	4.464	4.431	4.398
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	3.938	3.874	3.835	3.808	3.774	3.740	3.705
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.511	3.445	3.404	3.376	3.340	3.304	3.267
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.218	3.150	3.108	3.079	3.043	3.005	2.967
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.006	2.936	2.893	2.864	2.826	2.787	2.748
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.845	2.774	2.730	2.700	2.661	2.621	2.580
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.719	2.646	2.601	2.570	2.531	2.490	2.448
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.617	2.544	2.498	2.466	2.426	2.384	2.341
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671	2.533	2.459	2.412	2.380	2.339	2.297	2.252
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.463	2.388	2.341	2.308	2.266	2.223	2.178
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.403	2.328	2.280	2.247	2.204	2.160	2.114
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.352	2.276	2.227	2.194	2.151	2.106	2.059
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.308	2.230	2.181	2.148	2.104	2.058	2.011
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.269	2.191	2.141	2.107	2.063	2.017	1.968
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.234	2.155	2.106	2.071	2.026	1.980	1.930
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.203	2.124	2.074	2.039	1.994	1.946	1.896
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321	2.176	2.096	2.045	2.010	1.965	1.916	1.866
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.151	2.071	2.020	1.984	1.938	1.889	1.838
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275	2.128	2.048	1.996	1.961	1.914	1.865	1.813
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.108	2.027	1.975	1.939	1.892	1.842	1.790
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.236	2.089	2.007	1.955	1.919	1.872	1.822	1.768
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.015	1.932	1.878	1.841	1.792	1.740	1.683
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	1.924	1.839	1.783	1.744	1.693	1.637	1.577
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026	1.871	1.784	1.727	1.687	1.634	1.576	1.511
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.836	1.748	1.690	1.649	1.594	1.534	1.467
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927	1.768	1.676	1.616	1.573	1.515	1.450	1.376
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.910	1.750	1.659	1.598	1.554	1.495	1.429	1.352

Table 11.8 (c) Critical Values of the F-Distribution for $\alpha = 0.025$

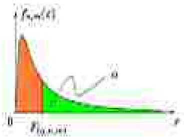


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	647.789	799.500	864.163	899.583	921.848	937.111	948.217	956.656	963.285	968.627	984.867	993.103	998.081	1001.414	1005.598	1009.800	1014.020
2	38.506	39.000	39.165	39.248	39.298	39.331	39.355	39.373	39.387	39.398	39.431	39.448	39.458	39.465	39.473	39.481	39.490
3	17.443	16.044	15.439	15.101	14.885	14.735	14.624	14.540	14.473	14.419	14.253	14.167	14.115	14.081	14.037	13.992	13.947
4	12.218	10.649	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.657	8.560	8.501	8.461	8.411	8.360	8.309
5	10.007	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.428	6.329	6.268	6.227	6.175	6.123	6.069
6	8.813	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.269	5.168	5.107	5.065	5.012	4.959	4.904
7	8.073	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.568	4.467	4.405	4.362	4.309	4.254	4.199
8	7.571	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.101	3.999	3.937	3.894	3.840	3.784	3.728
9	7.209	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.769	3.667	3.604	3.560	3.505	3.449	3.392
10	6.937	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.522	3.419	3.355	3.311	3.255	3.198	3.140
11	6.724	5.256	4.630	4.275	4.044	3.881	3.759	3.664	3.588	3.526	3.330	3.226	3.162	3.118	3.061	3.004	2.944
12	6.554	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.177	3.073	3.008	2.963	2.906	2.848	2.787
13	6.414	4.965	4.347	3.996	3.767	3.604	3.483	3.388	3.312	3.250	3.053	2.948	2.882	2.837	2.780	2.720	2.659
14	6.298	4.857	4.242	3.892	3.663	3.501	3.380	3.285	3.209	3.147	2.949	2.844	2.778	2.732	2.674	2.614	2.552
15	6.200	4.765	4.153	3.804	3.576	3.415	3.293	3.199	3.123	3.060	2.862	2.756	2.689	2.644	2.585	2.524	2.461
16	6.115	4.687	4.077	3.729	3.502	3.341	3.219	3.125	3.049	2.986	2.788	2.681	2.614	2.568	2.509	2.447	2.383
17	6.042	4.619	4.011	3.665	3.438	3.277	3.156	3.061	2.985	2.922	2.723	2.616	2.548	2.502	2.442	2.380	2.315
18	5.978	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.667	2.559	2.491	2.445	2.384	2.321	2.256
19	5.922	4.508	3.903	3.559	3.333	3.172	3.051	2.956	2.880	2.817	2.617	2.509	2.441	2.394	2.333	2.270	2.203
20	5.871	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.573	2.464	2.396	2.349	2.287	2.223	2.156
21	5.827	4.420	3.819	3.475	3.250	3.090	2.969	2.874	2.798	2.735	2.534	2.425	2.356	2.308	2.246	2.182	2.114
22	5.786	4.383	3.783	3.440	3.215	3.055	2.934	2.839	2.763	2.700	2.498	2.389	2.320	2.272	2.210	2.145	2.076
23	5.750	4.349	3.750	3.408	3.183	3.023	2.902	2.808	2.731	2.668	2.466	2.357	2.287	2.239	2.176	2.111	2.041
24	5.717	4.319	3.721	3.379	3.155	2.995	2.874	2.779	2.703	2.640	2.437	2.327	2.257	2.209	2.146	2.080	2.010
25	5.686	4.291	3.694	3.353	3.129	2.969	2.848	2.753	2.677	2.613	2.411	2.300	2.230	2.182	2.118	2.052	1.981
30	5.568	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.307	2.195	2.124	2.074	2.009	1.940	1.866
40	5.424	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.182	2.068	1.994	1.943	1.875	1.803	1.724
50	5.340	3.975	3.390	3.054	2.833	2.674	2.553	2.458	2.381	2.317	2.109	1.993	1.919	1.866	1.796	1.721	1.639
60	5.286	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.061	1.944	1.869	1.815	1.744	1.667	1.581
100	5.179	3.828	3.250	2.917	2.696	2.537	2.417	2.321	2.244	2.179	1.968	1.849	1.770	1.715	1.640	1.558	1.463
120	5.152	3.805	3.227	2.894	2.674	2.515	2.395	2.299	2.222	2.157	1.945	1.825	1.746	1.690	1.614	1.530	1.433

Table 11.8 (d) Critical Values of the F-Distribution for $\alpha = 0.01$

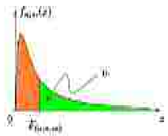


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	4052.181	4999.500	5403.352	5624.583	5763.650	5858.986	5928.356	5981.070	6022.473	6055.847	6157.285	6208.730	6239.825	6260.649	6286.782	6313.030	6339.391
2	98.503	99.000	99.166	99.249	99.299	99.333	99.356	99.374	99.388	99.399	99.433	99.449	99.459	99.466	99.474	99.482	99.491
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345	27.229	26.872	26.690	26.579	26.505	26.411	26.316	26.221
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659	14.546	14.198	14.020	13.911	13.838	13.745	13.652	13.558
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051	9.722	9.553	9.449	9.379	9.291	9.202	9.112
6	13.745	10.925	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.559	7.396	7.296	7.229	7.143	7.057	6.969
7	12.246	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.314	6.155	6.058	5.992	5.908	5.824	5.737
8	11.259	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.515	5.359	5.263	5.198	5.116	5.032	4.946
9	10.561	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	4.962	4.808	4.713	4.649	4.567	4.483	4.398
10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.558	4.405	4.311	4.247	4.165	4.082	3.996
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.251	4.099	4.005	3.941	3.860	3.776	3.690
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.010	3.858	3.765	3.701	3.619	3.535	3.449
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100	3.815	3.665	3.571	3.507	3.425	3.341	3.255
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939	3.656	3.505	3.412	3.348	3.266	3.181	3.094
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.522	3.372	3.278	3.214	3.132	3.047	2.959
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691	3.409	3.259	3.165	3.101	3.018	2.933	2.845
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.312	3.162	3.068	3.003	2.920	2.835	2.746
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.227	3.077	2.983	2.919	2.835	2.749	2.660
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434	3.153	3.003	2.909	2.844	2.761	2.674	2.584
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.088	2.938	2.843	2.778	2.695	2.608	2.517
21	8.017	5.780	4.874	4.369	4.042	3.812	3.640	3.506	3.398	3.310	3.030	2.880	2.785	2.720	2.636	2.548	2.457
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	2.978	2.827	2.733	2.667	2.583	2.495	2.403
23	7.881	5.664	4.765	4.264	3.939	3.710	3.539	3.406	3.299	3.211	2.931	2.781	2.686	2.620	2.535	2.447	2.354
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	2.889	2.738	2.643	2.577	2.492	2.403	2.310
25	7.770	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.217	3.129	2.850	2.699	2.604	2.538	2.453	2.364	2.270
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.700	2.549	2.453	2.386	2.299	2.208	2.111
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.522	2.369	2.271	2.203	2.114	2.019	1.917
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698	2.419	2.265	2.167	2.098	2.007	1.909	1.803
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.352	2.198	2.098	2.028	1.936	1.836	1.726
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503	2.223	2.067	1.965	1.893	1.797	1.692	1.572
120	6.851	4.787	3.949	3.480	3.174	2.956	2.792	2.663	2.559	2.472	2.192	2.035	1.932	1.860	1.763	1.656	1.533

Table 11.8 (e) Critical Values of the F-Distribution for $\alpha = 0.005$



$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	16210.723	19999.500	21614.741	22499.583	23055.798	23437.111	23714.566	23925.406	24091.004	24224.487	24630.205	24835.971	24960.340	25043.628	25148.153	25253.137	25358.573
2	198.501	199.000	199.166	199.250	199.300	199.333	199.357	199.375	199.388	199.400	199.433	199.450	199.460	199.466	199.475	199.483	199.491
3	55.552	49.799	47.467	46.195	45.392	44.838	44.434	44.126	43.882	43.686	43.085	42.778	42.591	42.466	42.308	42.149	41.989
4	31.333	26.284	24.259	23.155	22.456	21.975	21.622	21.352	21.139	20.967	20.438	20.167	20.002	19.892	19.752	19.611	19.468
5	22.785	18.314	16.530	15.556	14.940	14.513	14.200	13.961	13.772	13.618	13.146	12.903	12.755	12.656	12.530	12.402	12.274
6	18.635	14.544	12.917	12.028	11.464	11.073	10.786	10.566	10.391	10.250	9.814	9.589	9.451	9.358	9.241	9.122	9.001
7	16.236	12.404	10.882	10.050	9.522	9.155	8.885	8.678	8.514	8.380	7.968	7.754	7.623	7.534	7.422	7.309	7.193
8	14.688	11.042	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	6.814	6.608	6.482	6.396	6.288	6.177	6.065
9	13.614	10.107	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.032	5.832	5.708	5.625	5.519	5.410	5.300
10	12.826	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.471	5.274	5.153	5.071	4.966	4.859	4.750
11	12.226	8.912	7.600	6.881	6.422	6.102	5.865	5.682	5.537	5.418	5.049	4.855	4.736	4.654	4.551	4.445	4.337
12	11.754	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.721	4.530	4.412	4.331	4.228	4.123	4.015
13	11.374	8.186	6.926	6.233	5.791	5.482	5.253	5.076	4.935	4.820	4.460	4.270	4.153	4.073	3.970	3.866	3.758
14	11.060	7.922	6.680	5.998	5.562	5.257	5.031	4.857	4.717	4.603	4.247	4.059	3.942	3.862	3.760	3.655	3.547
15	10.798	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.070	3.883	3.766	3.687	3.585	3.480	3.372
16	10.575	7.514	6.303	5.638	5.212	4.913	4.692	4.521	4.384	4.272	3.920	3.734	3.618	3.539	3.437	3.332	3.224
17	10.384	7.354	6.156	5.497	5.075	4.779	4.559	4.389	4.254	4.142	3.793	3.607	3.492	3.412	3.311	3.206	3.097
18	10.218	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.683	3.498	3.382	3.303	3.201	3.096	2.987
19	10.073	7.093	5.916	5.268	4.853	4.561	4.345	4.177	4.043	3.933	3.587	3.402	3.287	3.208	3.106	3.000	2.891
20	9.944	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.502	3.318	3.203	3.123	3.022	2.916	2.806
21	9.830	6.891	5.730	5.091	4.681	4.393	4.179	4.013	3.880	3.771	3.427	3.243	3.128	3.049	2.947	2.841	2.730
22	9.727	6.806	5.652	5.017	4.609	4.322	4.109	3.944	3.812	3.703	3.360	3.176	3.061	2.982	2.880	2.774	2.663
23	9.635	6.730	5.582	4.950	4.544	4.259	4.047	3.882	3.750	3.642	3.300	3.116	3.001	2.922	2.820	2.713	2.602
24	9.551	6.661	5.519	4.890	4.486	4.202	3.991	3.826	3.695	3.587	3.246	3.062	2.947	2.868	2.765	2.658	2.546
25	9.475	6.598	5.462	4.835	4.433	4.150	3.939	3.776	3.645	3.537	3.196	3.013	2.898	2.819	2.716	2.609	2.496
30	9.180	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.006	2.823	2.708	2.628	2.524	2.415	2.300
40	8.828	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	2.781	2.598	2.482	2.401	2.296	2.184	2.064
50	8.626	5.902	4.826	4.232	3.849	3.579	3.376	3.219	3.092	2.988	2.653	2.470	2.353	2.272	2.164	2.050	1.925
60	8.495	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.570	2.387	2.270	2.187	2.079	1.962	1.834
100	8.241	5.589	4.542	3.963	3.589	3.325	3.127	2.972	2.847	2.744	2.411	2.227	2.108	2.024	1.912	1.790	1.652
120	8.179	5.539	4.497	3.921	3.548	3.285	3.087	2.933	2.808	2.705	2.373	2.188	2.069	1.984	1.871	1.747	1.606

Exercises

▶ **11-1.** A box contains 10 white balls, 3 black balls and 2 red balls.
(a) What is the probability of drawing a white ball?
(b) What is the probability of drawing a black ball?
(c) What is the probability of drawing a red ball?

▶ **11-2.** A box contains 10 white balls, 3 black balls and 2 red balls.
(a) What is the probability of drawing a white ball and then drawing a black ball?
(b) What is the probability of drawing two white balls?
(c) What is the probability of drawing a red ball and then a black ball?

▶ **11-3.** A bowl of fruit contains 3 apples, 5 oranges and 3 pears. If two fruits are selected at random,
(a) What is the probability of getting 2 pears?
(b) What is the probability of getting 2 apples?
(c) What is the probability of getting 2 oranges?

▶ **11-4.** Calculate the mean, variance and standard deviations of the following.
(a) G=grade of student on 6 exams. $G = \{84, 91, 72, 68, 87, 78\}$
(b) T=test scores for class of 17 students.
$$T = \{71, 82, 66, 88, 100, 97, 96, 100, 77, 77, 84, 89, 93, 98, 100, 100, 75\}$$
(c) A= Average absenteeism rate in days missed per 100 working days over 6 year period taken from a certain factory. $A = \{8.05, 13.35, 5.10, 4.43, 6.22, 7.81\}$

▶ **11-5.** Show that $s^2 = \dfrac{1}{n-1}\displaystyle\sum_{j=1}^{m}(x_j - \bar{x})^2 nf_j$ can be written

$$s^2 = \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 nf_j - \frac{1}{n}\left(\sum_{j=1}^{m} x_j nf_j\right)^2\right\}$$

▶ **11-6.** If a pair of fair dice are rolled and X denote the sum of the upward numbers, then find the probabilities of rolling a 2,3,4,5,6,7,8,9,10,11 and 12.

▶ **11-7.** Find the arithmetic mean, geometric mean and harmonic mean of the given numbers.

$$(a)\quad 1,2,3,4,5,6,7 \qquad (b)\quad 1,3,5,7,9,11,13 \qquad (c)\quad 2,4,6,8,10,12,14$$

▶ **11-8.** What is the probability of getting 10 consecutive heads in the toss of a fair coin?

▶ **11-9.** Given the following sample of ball bearing diameters, in inches, taken over one production cycle.

Ball bearing diameters in inches									
.738	.729	.743	.740	.736	.728	.735	.741	.737	.740
.735	.730	.736	.733	.745	.736	.742	.735	.734	.738
.734	.737	.732	.744	.741	.738	.732	.737	.742	.746
.739	.740	.735	.730	.744	.733	.727	.732	.734	.735
.724	.730	.739	.739	.733	.726	.735	.746	.731	.737
.738	.739	.735	.727	.735	.736	.744	.740	.736	.740

(a) Make a tally sheet and use the class marks $\{.725, .728, .731, .734, .737, .740, .743, .746\}$ with class intervals of length $\pm.0015$ added to the class marks.

(b) Determine and sketch the frequency distribution and cumulative frequency distribution as well as the relative frequency distribution and cumulative relative frequency distribution.

(c) Find the mean and variance directly.

(d) Find the mean and variance using the class marks and frequencies.

(e) Using the results from parts (a) and (b), approximate the following probabilities if X represents a random variable representing the diameter of a ball bearing.

$$(i)\ P(X \le .737) \qquad (ii)\ P(.728 < X \le .734) \qquad (iii)\ P(X > .734)$$

(f) In the absence of other information how can the relative frequency distribution be interpreted?

▶ **11-10.** A box contains 10 identical balls. Six of the balls are white and 4 of the balls are black.

(a) What is the probability of drawing a white ball from the box?

(b) What is the probability of drawing a black ball from the box?

▶ **11-11. (Binomial Distribution)**

The discrete binomial distribution is given by $f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$

If $q = 1 - p$ show that

(a) $(p+q)^n = \sum_{x=0}^{n} f(x) = 1$, $\quad (b)$ $\sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} = (p+q)^{n-1}$, $\quad (c)$ $x\binom{n}{x} = n\binom{n-1}{x-1}$

(d) Use parts (b) and (c) to show the mean of the binomial distribution is given by

$$\mu = E[x] = \sum_{x=0}^{n} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np$$

▶ **11-12.**

(a) Show that $s^2 = \dfrac{1}{N-1} \sum_{j=1}^{N} (x_j - \bar{x})^2$ can be written in the shortcut form

$$s^2 = \frac{1}{N(N-1)} \left\{ N \sum_{j=1}^{N} x_j^2 - \left(\sum_{j=1}^{N} x_j \right)^2 \right\}$$

(b) Illustrate the use of the above two formulas by completing the table below and evaluating s^2 by two different methods.

x	x^2	$x - \bar{x}$	$(x - \bar{x})^2$
6			
3			
8			
5			
2			
$\sum_{j=1}^{5} x_j =$	$\sum_{j=1}^{5} x^2 =$		$\sum_{j=1}^{5} (x - \bar{x})^2 =$
$\bar{x} =$			
$\left(\sum_{j=1}^{5} x_j \right)^2 =$			

▶ **11-13.** For the given probability density functions $f(x)$, find the cumulative distribution function $F(x) = \int_{-\infty}^{x} f(x)\,dx$ and then plot graphs of both $f(x)$ and $F(x)$.

(a) $f(x) = \begin{cases} \alpha\, e^{-\alpha x}, & x > 0 \text{ and } \alpha > 0 \text{ constant} \\ 0, & x \le 0 \end{cases}$

(b) $f(x) = \begin{cases} 0, & x \le -x_0 \\ \frac{1}{2x_0}, & -x_0 < x < x_0 \\ 0, & x \ge x_0 \quad \text{where } x_0 > 0 \text{ is a constant} \end{cases}$

(c) $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \; -\infty < x < \infty$ Leave $F(x)$ in integral form.

(d) $f(x) = \begin{cases} \frac{1}{2}\, e^{x}, & -\infty < x < 0 \\ \frac{1}{2}\, e^{-x}, & 0 < x < \infty \end{cases}$

▶ **11-14.** Use factorials to show

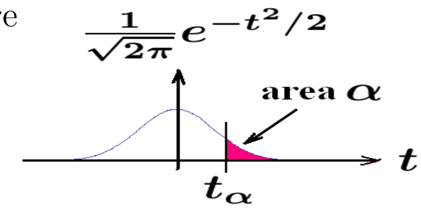
(a) $\dbinom{n}{m} = \dbinom{n-1}{m-1} + \dbinom{n-1}{m}$

(b) $\dbinom{n}{m+1} = \dfrac{n-m}{m+1}\dbinom{n}{m}$

▶ **11-15.**

(a) Use a table of areas to find values of t_α given the are

Do for $\alpha = 0.001,\ 0.01,\ 0.025,\ 0.05,\ 0.1$



(b) Explain how you would use the table of areas to calculate the probability $P(\alpha < X < \beta)$ associated with a normal distribution $(\mu = 0, \sigma = 1)$.

(c) Use the table of areas to verify (i) $P(-1 < X < 1) \approx 0.68$, (ii) $P(-2 < x < 2) \approx 0.955$, (iii) $P(-3 < X < 3) \approx 0.997$

▶ **11-16.** Given an ordinary deck of 52 playing cards.

(a) What is the probability of drawing a black ace?

(b) What is the probability of drawing an ace or a king?

▶ **11-17.** Given an ordinary deck of 52 playing cards. Let E_1 denote the event of drawing an ace and E_2 the event of drawing a heart.

(a) Are the events E_1 and E_2 mutually exclusive?

(b) What is the probability of drawing either an ace or a heart or both?

▶ **11-18. (Computer Problem for Normal Distribution)**

There are numerous web sites which use numerical methods to calculate the area $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ under the normalized probability curve $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Use one of these web sites to verify the values given in the table 11.4.

▶ **11-19. (Binomial Distribution)**

Show the variance of the binomial distribution $f(x)$, given in the problem 11-11, is $\sigma^2 = npq$ by verifying the following relations.

$$
\begin{aligned}
(i) &\quad \text{Show } \sigma^2 = E[(x-\mu)^2 f(x)] = E[x^2] - (E[x])^2 \\
(ii) &\quad \text{Show } E[x^2] = \sum_{x=1}^{n} [x(x-1) + x] f(x) = n(n-1)p^2 + np \\
(iii) &\quad \text{Show } \sigma^2 = npq
\end{aligned}
$$

▶ **11-20.** Sketch the bell shaped probability density curve $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ associated with the normalized normal distribution. Find and show sketches of the given probabilities as areas of shaded regions on this curve.

$$
\begin{array}{lll}
(a)\ P(\, z \le 0.5\,) & (d)\ P(\, z > 0.5\,) & (g)\ P(\, z < 2.3\,) \\
(b)\ P(\, z \le -2.2\,) & (e)\ P(\, -1.3 \le x \le 0.76\,) & (h)\ P(\, z > 2.3) \\
(c)\ P(\, |z| \le 2\,) & (f)\ P(\, |z| \le 3) & (i)\ P(\, |z| \le 1\,)
\end{array}
$$

▶ **11-21.** (Hypergeometric distribution)

A certain shipment of transistors contains 12 transistors, 3 of which are defective. If a batch of 5 transistors is drawn from the shipment, then

(a) determine $f(x)$, $x = 0, 1, 2, 3$ which represents the probability that x of the 5 items selected are defective.

(b) determine the minimum number of transistors that must be drawn to make the probability of obtaining at least 5 nondefective transistors greater than 0.8.

▶ **11-22.** Let X denote a random variable normally distributed with mean $\mu = 12$ and standard deviation $\sigma = 4$. Find values of and show with sketches the probabilities as areas of shaded regions for representing the probabilities

$$
\begin{array}{ll}
(a)\quad P(X \le 14) & (c)\quad P(X \le 10) \\
(b)\quad P(8 \le X \le 16) & (d)\quad P(0 \le X \le 24)
\end{array}
$$

▶ **11-23.** Assume that a given State has regulations specifying that the fluoride levels in water may not exceed 1.5 milligrams per liter. Your are given the assignment to analyze the following sample of fluoride levels, in milligrams per liter, taken over a 45 day period.

.753	.945	.883	.721	.812	.731	.833	.891	.792
.860	.890	.782	.923	.858	1.01	.842	.890	.825
.843	.849	.772	1.05	.972	.910	.732	.799	.897
.855	.830	.761	.934	.942	.890	.782	.835	.899
.977	.891	.824	.837	.792	.843	.844	.803	.943

(a) Use class intervals about the class marks $M = \{.73, .78, .83, .88, .93, .98, 1.03\}$ where $\pm .025$ is added to each class mark to form the class interval. Find and plot the frequency and cumulative frequency distribution for this data.

(b) Find the mean and variance associated with the given data.

(c) If X is a random variable representing the fluoride level from the above sample, then approximate the following probabilities.

$$(i)\ P(\,X \le .88\,) \qquad (ii)\ P(\,.78 < X \le .93\,) \qquad (iii)\ P(\,X > .83\,)$$

▶ **11-24.** (Binomial distribution)

Let p denote the probability of an event happening (success) and $q = 1-p$ denote the probability of an event not happening (failure) in a single trial. To study success or failure of an event in n-trials one usually first calculates $(p+q)^n$.

(a) Show that

$$(p+q)^n = \binom{n}{0} q^n + \binom{n}{1} pq^{n-1} + \cdots + \binom{n}{x} p^x q^{n-x} + \cdots + \binom{n}{n} p^n$$

where the term $f(x) = \binom{n}{x} p^x q^{n-x}$ denotes the probability density function representing the probability that the event will happen exactly x times in n trials and there are $n-x$ failures, with $x = 0, 1, 2, \ldots, n$ an integer.

(b) Find the probability of getting exactly 2 heads in 5 tosses of a fair coin.

(c) Find the probability of getting at least 2 heads in 5 tosses of a fair coin.

(d) Find the probability of getting at least 4 heads in 6 tosses of a fair coin.

▶ **11-25.** (Binomial distribution)

Forty identical transistors are placed on life tests simultaneously and are operated for T hours. The probability that any transistor survives to time T is 0.8. Let X denote a random variable that represents the number of transistors which are operational at time T. The distribution function for X is needed to compute probabilities. The binomial distribution is applicable if (a) each transistor is identical and has the same chance of failure as any other and (b) life testing of each transistor is identical and is accomplished under separate independent conditions. Let success mean survival of transistor to time T, then $p = 0.8$ and $n = 40$. The probability density function for the random variable X is

$$f(x) = \binom{40}{x}(0.8)^x(0.2)^{40-x}, \qquad x = 0, 1, 2, \dots, 40$$

(a) Find the probability that exactly 33 transistors are operational at time T.
(b) Find the probability that 3 transistors have failed by time T.
(c) Find the probability that at least 3 transistors have failed by time T (i.e. the probability that no more than 37 have survived equals $\sum_{k=0}^{37} f(k)$)
(d) Find the probability that 80% of transistors survive.

▶ **11-26.** (Poisson distribution)

Many random experiments involve time. In an experiment, at any instant of time, either something happens or it does not happen and only one thing can occur. The number of these things that happen in a prescribed time interval is observed and recorded. The Poisson distribution describes such situations. Let events which occur randomly in time be called random points. A random variable X will then represent the number of random points that occur in the interval between times $t = 0$ and time $t > 0$. The probability of observing exactly x random points between 0 and t is given by the probability density function

$$f(x) = f(x;t) = \frac{(\lambda t)^x\, e^{-\lambda t}}{x!}, \qquad x = 0, 1, 2, \dots$$

where $\lambda t > 0$ and $\lambda > 0$ represents the average number of random points per unit of time. If $\lambda t = 9$ and X denotes a random variable

(a) Show that $f(x) = \frac{9^x e^{-9}}{x!}$ and $f(x+1) = \frac{9}{x+1} f(x)$
(b) Find $P(X > 4)$ i.e. 5 or more random points occur.
(c) Find $P(X \leq 8)$ i.e. no more than 8 random points occur.
(d) Find $P(8 \leq X \leq 12)$ i.e. between 8 and 12 random points occur.

▶ **11-27.** (Poisson distribution)

Let X denote a random variable representing the number of light bulbs which fail during a specified time interval T. The random variable X is assumed to have a Poisson probability density function where an average of 2 bulbs fail during the time interval T. Use the Poisson probability density function

$$f(x) = \frac{2^x}{x!}e^{-2}, \qquad x = 0, 1, 2, \ldots$$

and find the probability

(a) of no failures during time interval T

(b) of more than one failure during time interval T

(c) of more than five failures during time interval T

▶ **11-28.** The error function or Gauss error function is defined[1]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

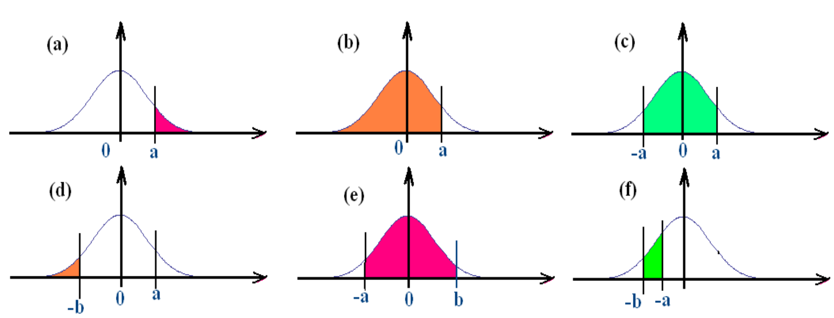
(a) Show that

$$\int_{-\infty}^{x} N(x;0,1)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

(b) Show that

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\left(\frac{t-\mu}{\sigma}\right)^2}\,dt = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

▶ **11-29.** Each of the curves below represent graphs of the normal probability distribution $N(x;0,1)$. Explain how one would use areas associated with the normal probability tables to find the areas of the shaded regions.



Find the area associated with the shaded areas.

[1] There are alternative definitions for the error function.

▸ **11-30.** If $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$, find the value of and illustrate with sketches the representations of the following probabilities as shaded area under the normal probability density curve.

(a)	$P(Z \le 1)$	(d)	$P(Z \le -3.2)$	(g)	$P(Z \le 2)$
(b)	$P(Z > 1)$	(e)	$P(-1.2 \le Z \le 0.75)$	(h)	$P(\lvert Z\rvert \le 2$
(c)	$P(Z \le 3.2)$	(f)	$P(\lvert Z\rvert \le z)$	(i)	$P(\lvert Z\rvert \le 3)$

▸ **11-31.** (Monte Carlo computer problem)

(a) Give a physical interpretation to the integral $I_1 = \frac{1}{b-a}\int_a^b f(x)\,dx$

(b) Give a physical interpretation to the summation $I_2 = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$ where $a \le x_i \le b$ for all integers i.

(c) If $I_1 = I_2$, show estimate for $I = \int_a^b f(x)\,dx$ is given by $I = \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)$

(d) Calculate 500 random numbers[2] x_i with $x_i \in (-1, 2)$ and estimate the integral $I = \frac{1}{\sqrt{2\pi}}\int_{-1}^{2} e^{-x^2/2}\,dx$. Do this over and over again and calculate 1000 estimates for I and then do descriptive statistics on your results and compare your computer answer with the answer obtained from table lookup.

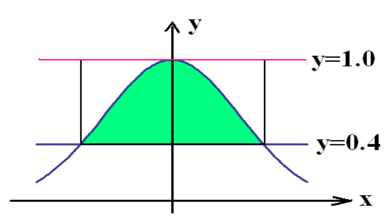
▸ **11-32.** (Monte Carlo computer problem)

Write a Monte Carlo computer program to calculate the area bounded by the curves $y = e^{-x^2/2}$ and $y = 0.4$ as illustrated below. Set up as an integral (see previous problem) or throw darts at area.

Hint 1: $y = e^{-x^2/2} = 0.4$ when $x = x^* = \pm\sqrt{-2\ln(0.4)}$

Hint 2: Construct a rectangle where $-x^* < x < x^*$ and $0.4 \le y \le 1.0$ about area to be calculated by Monte Carlo method.

Hint 3: Generate random numbers (x_r, y_r) with $-x^* \le x_r \le x^*$ and $0.4 \le y_r \le 1.0$ and determine if the point (x_r, y_r) is inside or outside the area to be determined.



Be sure to perform descriptive statistics on your results and if your instructor gives you extra credit, put confidence intervals on your answer for the area.

Chapter 12
Introduction to more Advanced Material

The following is a potpourri of selected topics involving mathematical applications of calculus together with an introduction to advanced calculus techniques and mathematical methods related to calculus. The material selected presents applications and topics that you might encounter in your scientific investigations in other courses. The material presented will also give you some idea of what to expect in more advanced mathematical courses beyond calculus.

An integration method

To integrate the definite integral $\int_a^b f(x)\,dx$ you can use the following integration method **if you know the inverse function** $f^{-1}(x)$ **associated with** $f(x)$.

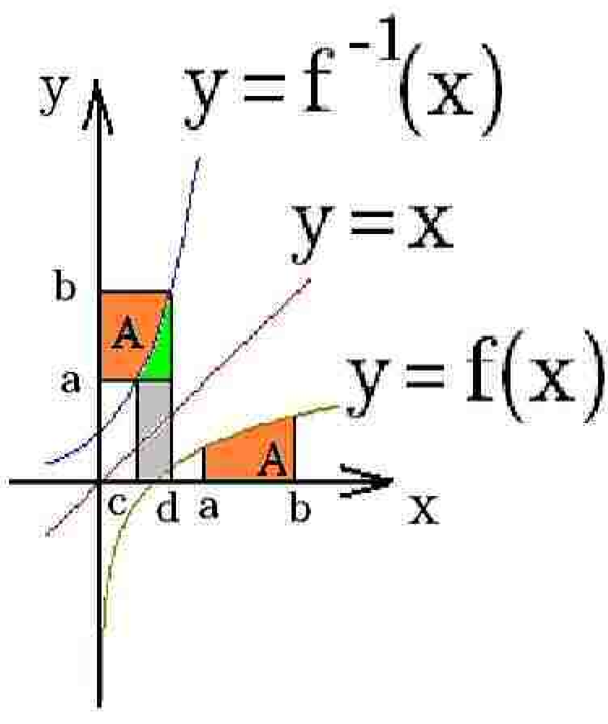


Figure 12-1. A function and its inverse function.

We know that the functions $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=x$ as illustrated in the figure 12-1. Examine the figure 12-1 and note that one can express the area $A=\int_a^b f(x)\,dx$ as

$$\begin{aligned} A=\int_a^b f(x)\,dx = & \underbrace{(b-a)d}_{orange+green\ area} - \underbrace{\int_c^d [f^{-1}(x)-a]\,dx}_{green\ area} \\ = & (b-a)d-\int_c^d f^{-1}(x)\,dx+(d-c)a \end{aligned} \tag{12.1}$$

which shows that the area A is given by the rectangular area (orange plus green area) minus the area under the inverse curve (green plus grey area) corrected by the rectangular grey area. Hence, if you know $\displaystyle\int_a^b f(x)\,dx$, then you can find $\displaystyle\int_c^d f^{-1}(x)\,dx$ and vice-versa.

The use of integration to sum infinite series

There is a definite relation between certain infinite series and definite integrals. For example, consider the relationship between the problem of finding the sum of the alternating infinite series

$$\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\frac{1}{a+4b}-\cdots+(-1)^n\frac{1}{a+nb}+\cdots \tag{12.2}$$

where $a>0$, $b>0$ and the associated problem of evaluating the definite integral

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt \tag{12.3}$$

Use the well known series expansion

$$\frac{1}{1+x}=1-x+x^2+x^3-x^4+\cdots \tag{12.4}$$

with x replaced by t^b to write the equation (12.3) in the form

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt=\int_0^1 t^{a-1}\left[1-t^b+t^{2b}-t^{3b}+t^{4b}-\cdots\right]\,dt \tag{12.5}$$

and then integrate each term to produce the result

$$\begin{aligned} \int_0^1 \frac{t^{a-1}}{1+t^b}\,dt = & \left[\frac{t^a}{a}-\frac{t^{a+b}}{a+b}+\frac{t^{a+2b}}{a+2b}+\cdots+(-1)^n\frac{t^{a+nb}}{a+nb}+\cdots\right]_0^1 \\ \int_0^1 \frac{t^{a-1}}{1+t^b}\,dt = & \frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}+\cdots+(-1)^n\frac{1}{a+nb}+\cdots \end{aligned} \tag{12.6}$$

This demonstrates that if the infinite series on the right-hand side converges, then it can be evaluated by calculating the integral on the left-hand side.

Example 12-1. (**Sum of series**)

Find the sum of the infinite series

$$\frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \cdots \tag{12.7}$$

which is an example of the series (12.2) when $a = 1$ and $b = 3$.

Solution

Use the equation (12.6) to find the sum of the given infinite series by evaluating the integral

$$I = \int_0^1 \frac{dt}{1+t^3} \tag{12.8}$$

Use partial fractions and write

$$\frac{1}{1+t^3} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$$

and show that $A = 1/3$, $B = -1/3$ and $C = 2/3$. Using some algebra and completing the square on the denominator term the required integral can be reduced to the following standard forms

$$\begin{aligned} I = \int_0^1 \frac{dt}{1+t^3} &= \frac{1}{3}\int_0^1 \frac{dt}{1+t} + \int_0^1 \frac{-t/3+2/3}{1-t+t^2}\,dt \\ &= \frac{1}{3}\int_0^1 \frac{dt}{1+t} - \frac{1}{6}\int_0^1 \frac{2t-1}{1-t+t^2}\,dt + \frac{1}{2}\int_0^1 \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + (t-1/2)^2} \end{aligned}$$

where each integral is in a standard form which can be easily integrated. If you don't recognize these integrals then look them up in an integration table. Integration of each term produces

$$I = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + \frac{1}{3}\ln(1+t) - \frac{1}{6}\ln(1-t+t^2)\right]_0^1 = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln(2)\right) \tag{12.9}$$

giving the final result

$$I = \frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \cdots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln(2)\right) \tag{12.10}$$

■

Example 12-2. (**Sum of series**)

Show that

$$\frac{1}{2\cdot 5}+\frac{1}{8\cdot 11}+\frac{1}{14\cdot 17}+\frac{1}{20\cdot 23}+\cdots=\frac{1}{9}(\frac{\pi}{3}+\ln 2)$$

Solution

Let S denote the sum of the series and use partial fractions to write

$$\frac{1}{n\cdot(n+3)}=\frac{A}{n}+\frac{B}{n+3}$$

for $n=2,8,14,20,\ldots$ to show that

$$S=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\frac{1}{14}-\frac{1}{17}+\frac{1}{20}-\frac{1}{23}+\cdots\right]$$

Note that the sum S is a special case of the Taylor's series

$$S(x)=\frac{1}{3}\left[\frac{x^2}{2}-\frac{x^5}{5}+\frac{x^8}{8}-\frac{x^{11}}{11}+\frac{x^{14}}{14}-\frac{x^{17}}{17}+\frac{x^{20}}{20}-\frac{x^{23}}{23}+\cdots\right]$$

with $S=S(1)$ the desired sum. The derivative of $S(x)$ produces

$$\frac{dS}{dx}=\frac{1}{3}\left[x-x^4+x^7-x^{10}+x^{13}-x^{16}+x^{19}-x^{22}+\cdots\right]$$

The derivative series is recognized as a geometric series with sum $\frac{x}{x^3+1}$ so that one can write

$$\frac{dS}{dx}=\frac{1}{3}\frac{x}{x^3+1}$$

The desired series sum can now be expressed in terms of an integral

$$S=S(1)=\frac{1}{3}\int_0^1\frac{x}{x^3+1}\,dx$$

As an exercise, use partial fractions and show

$$S=S(1)=\frac{1}{3}\int_0^1\frac{x}{x^3+1}\,dx=\frac{1}{3}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)-\frac{1}{3}\ln(x+1)+\frac{1}{6}\ln(1-x+x^2)\right]_0^1$$

which simplifies to

$$S=S(1)=\frac{1}{9}\left(\frac{\pi}{\sqrt{3}}-\ln 2\right)$$

■

Refraction through a prism

Refraction through a prism is often encountered in physics courses and the calculus needed to analyze the physical problem is messy. Let's investigate this problem.

Consider the prism illustrated in the figure 12-2. The angle A of a prism is known as the apex angle or refracting angle. The angles associated with a ray of light entering or leaving the sides of a prism are measured with respect to a normal line constructed to a side of the prism and the ray of light is governed by Snell's law[1]

$$\begin{array}{ll} \text{entering} & \text{leaving} \\ n_a \sin x = n_g \sin \alpha & n_g \sin \beta = n_a \sin \xi \end{array} \tag{12.11}$$

where n_m denotes the refractive index for light in medium m ($m = a$ air and $m = g$ glass). Here x is called the angle of incidence and α is called the angle of refraction. The refracted ray travels across the prism and again undergoes Snell's law and becomes the exiting ray. There is a reversibility principle whereby light can travel in either direction along the path illustrated in the figure 12-2. In figure 12-2 the incident ray is extended and the exiting ray also has been extended and these extended rays intersect in the angle y called **the angle of deviation**.

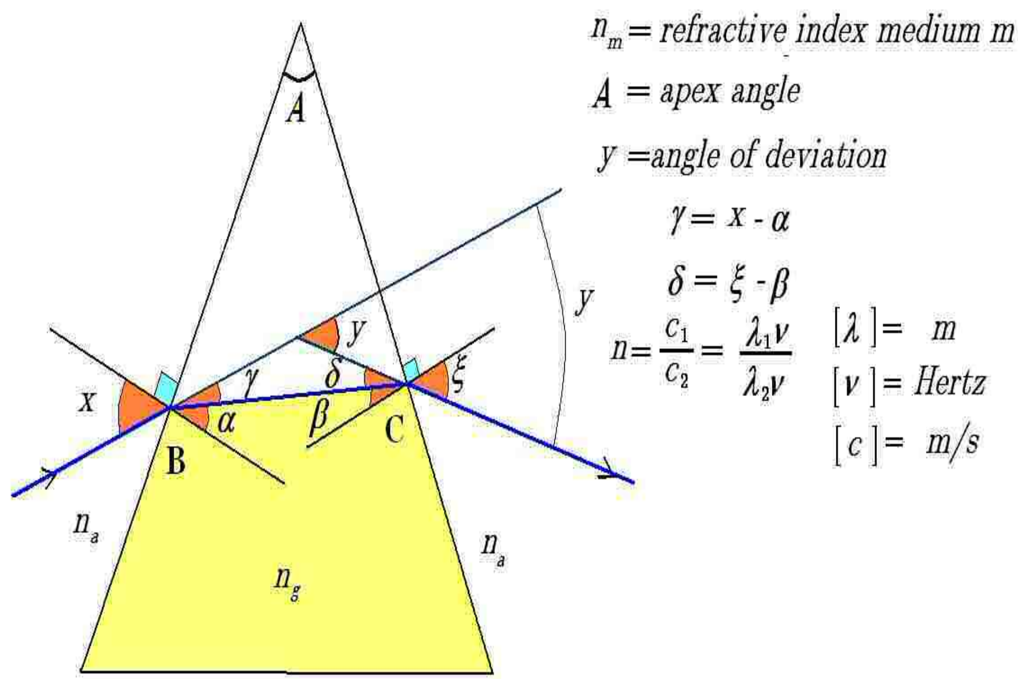


Figure 12-2. Ray of light through a prism.

[1] See Chapter 2 , pages 122-124.

Observe that the sum of the angles of the triangle ABC with top angle A and base angles $\frac{\pi}{2}-\alpha$ and $\frac{\pi}{2}-\beta$ is π radians so that one can sum the angles of the triangle and write

$$\frac{\pi}{2} - \alpha + \frac{\pi}{2} - \beta + A = \pi \quad \text{or} \quad A = \alpha + \beta \tag{12.12}$$

Here the deviation angle y is the exterior angle of a triangle with γ and δ the two opposite interior angles. Consequently one can write

$$y = \gamma + \delta = (x - \alpha) + (\xi - \beta) \quad \text{or} \quad y = x - A + \xi \tag{12.13}$$

Use equation (12.12) to show

$$\sin\beta = \sin(A - \alpha) = \sin A \cos\alpha - \sin\alpha \cos A \tag{12.14}$$

and then use the equations (12.11) in the form

$$\sin\beta = \frac{n_g}{n_a}\sin\xi, \qquad \sin\alpha = \frac{n_a}{n_g}\sin x, \qquad \cos\alpha = \sqrt{1 - \left(\frac{n_a}{n_g}\right)^2 \sin^2 x}$$

to express equation (12.14) in the form

$$\begin{aligned} \frac{n_a}{n_g}\sin\xi =& \sin A\sqrt{1-\sin^2\alpha} - \frac{n_a}{n_g}\sin x\cos A \\ \frac{n_a}{n_g}\sin\xi =& \sin A\sqrt{1 - \left(\frac{n_a}{n_g}\right)^2 \sin^2 x} - \frac{n_a}{n_g}\sin x\cos A \\ \sin\xi =& \frac{1}{n_a}\sin A\sqrt{n_g^2 - n_a^2\sin^2 x} - \sin x\cos A \end{aligned} \tag{12.15}$$

The equations (12.15) together with equation (12.13) can be used to express y as a function of x. One finds that the angle of deviation in terms of the incident angle x can be expressed in the form

$$y = x - A + \sin^{-1}\left[\frac{1}{n_a}\sin A\sqrt{n_g^2 - n_a^2\sin^2 x} - \cos A\sin x\right] \tag{12.16}$$

The figure 12-3 illustrates a graph of the angle of deviation as a function of the incident value for selected nominal values of A, n_a, n_g. Observe that there is some incident angle where the angle of deviation is a minimum. To find out where this minimum value occurs one can differentiate equation (12.16) to obtain

$$\frac{dy}{dx} = 1 - \frac{\frac{n_a\sin(A)\sin(x)\cos(x)}{\sqrt{{n_g}^2 - {n_a}^2\sin^2(x)}} + \cos(A)\cos(x)}{\sqrt{1 - \left(\cos(A)\sin(x) - \frac{\sin(A)\sqrt{{n_g}^2 - {n_a}^2\sin^2(x)}}{n_a}\right)^2}} \tag{12.17}$$

At a minimum value the derivative must equal zero and so equation (12.17) must be set equal to zero and x must be solved for. This is not an easy task and so numerical methods must be resorted to.

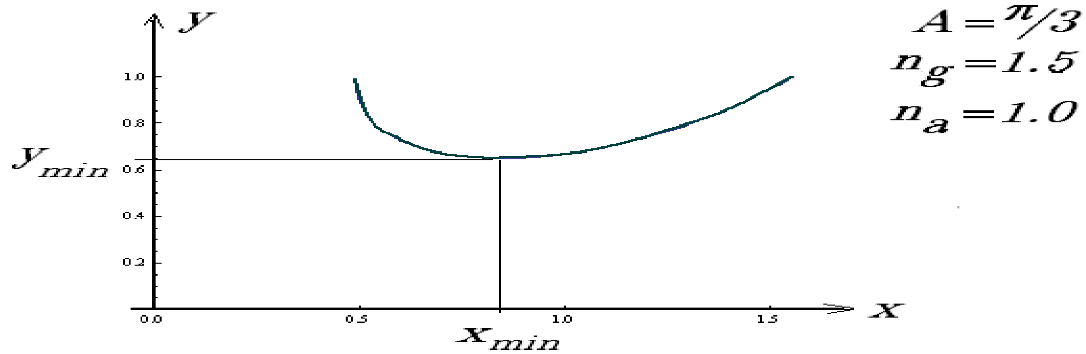


Figure 12-3.

Deviation angle versus incident angle for prism using nominal values above.

An alternative approach of finding the value of x which produces a minimum value is as follows. Observe that when $y = y_{min}$, then $\frac{dy}{dx} = 0$. Assume that y has the value $y = y_{min}$ and differentiate the equations (12.12) and (12.13) with respect to x and show

$$\frac{dA}{dx} = \frac{d\alpha}{dx} + \frac{d\beta}{dx} = 0 \qquad \text{or} \qquad \frac{d\beta}{dx} = -\frac{d\alpha}{dx} \tag{12.18}$$

because the angle A is a constant. One also finds that when $y = y_{min}$, then

$$\frac{dy}{dx} = 1 + \frac{d\xi}{dx} = 0 \qquad \text{or} \qquad \frac{d\xi}{dx} = -1 \tag{12.19}$$

because of our assumption that $y = y_{min}$ and hence $\frac{dy}{dx} = 0$. Next differentiate the equations (12.11) with respect to x and show

$$n_a \cos x = n_g \cos\alpha \frac{d\alpha}{dx} \tag{12.20}$$

and

$$n_g \cos\beta \frac{d\beta}{dx} = n_a \cos\xi \frac{d\xi}{dx} \tag{12.21}$$

The form of the equations (12.20) and (12.21) can be changed by multiplying equation (12.20) by $\cos\beta$ and multiplying equation (12.21) by $\cos\alpha$. The resulting equations are further simplified by using the results from equations (12.18) and (12.19) to produce the equations

$$n_a \cos x \cos\beta = n_g \cos\alpha \cos\beta \frac{d\alpha}{dx} \tag{12.22}$$

$$-n_a \cos\xi \cos\alpha = -\, n_g \cos\alpha \cos\beta \frac{d\alpha}{dx} \tag{12.23}$$

which must hold when $y = y_{min}$.

Addition of the equations (12.22) and (12.23) after simplification produces the condition

$$\cos x \cos\beta = \cos\xi \cos\alpha \tag{12.24}$$

Now square both sides of equation (12.24) and verify that

$$\begin{aligned} \cos^2 x \cos^2\beta =& \cos^2\xi \cos^2\alpha \\ (1-\sin^2 x)(1-\sin^2\beta) =&(1-\sin^2\xi)(1-\sin^2\alpha) \\ (1-\frac{n_g^2}{n_a^2}\sin^2\alpha)(1-\sin^2\beta) =&(1-\frac{n_a^2}{n_g^2}\sin^2\beta)(1-\sin^2\alpha) \end{aligned} \tag{12.25}$$

Expand the last equation from equation (12.25) and simplify the result to show equation (12.25) reduces to the condition that

$$|\sin\beta| = |\sin\alpha| \tag{12.26}$$

under the condition that $y = y_{min}$. The equation (12.26) implies that under the condition that $y = y_{min}$ there must be symmetry in the light ray moving through the prism. That is, the conditions

$$\beta = \alpha = \frac{A}{2} \qquad \text{and} \qquad x = \frac{y_{min} + A}{2} \tag{12.27}$$

must be satisfied when a minimum angular deviation is achieved.

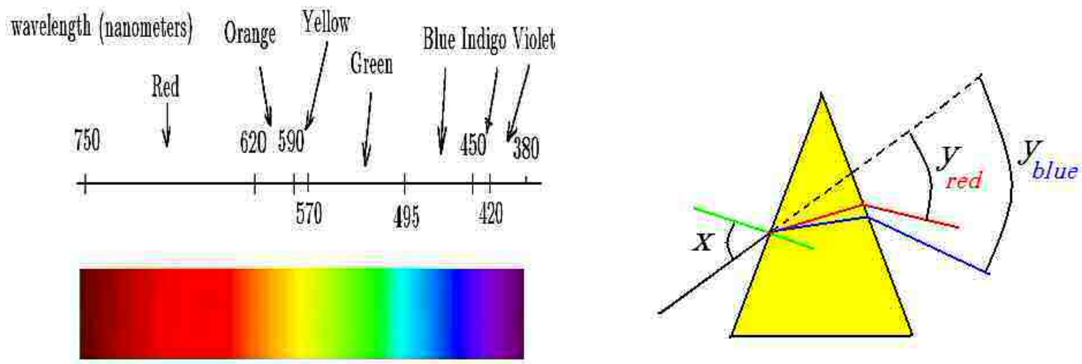


Figure 12-4.
White light splits into colors.

Recall that white light gets split into the colors Red, Orange, Yellow, Green, Blue, Indigo, Violet, remembered using the acronym (ROY G BIV). This is because the deviation angle for red light is less than the deviation angle for violet light. One can observe this by expressing the index of refraction in terms of the frequency (ν) and wavelength (λ) of the light. The refraction index being given by

$$n = c_1/c_2 = \lambda_1\nu/\lambda_2\nu$$

See the figures 12-2 and 12-3.

Differentiation of Implicit Functions

The following is a prsentation of various ways of representing implicit functions and the resulting techniques used to obtain derivatives from these representations. The ideas presented can be extended to cover systems of m-equations in n-unknowns, with $m < n$. The given m-equations define implicitly m of the variables in terms of the remaining $n - m$ variables. Make note in the following representations that if you are given m-equations, then there will always be m dependent variables.

one equation, two unknowns

Any equation of the form

$$F(x,y)=0 \tag{12.28}$$

implicitly defines y as one or more functions of x. Treating y as a function of x differentiate the equation (12.28) with respect to x and show

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \tag{12.29}$$

from which one can solve for the first derivative to obtain

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \tag{12.30}$$

provided that $\frac{\partial F}{\partial y}\neq 0$. Higher derivatives are obtained by differentiating the first derivative. For example, differentiate the equation (12.29) with respect to x and show

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial x\,\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial y}\frac{d^2 y}{dx^2}+\frac{dy}{dx}\left[\frac{\partial^2 F}{\partial y\,\partial x}+\frac{\partial^2 F}{\partial y^2}\frac{dy}{dx}\right]=0 \tag{12.31}$$

One can then solve for the second derivative term. Higher ordered derivatives are obtained by differentiating the equation (12.31).

one equation, three unknowns

An equation of the form

$$F(x,y,z)=0 \tag{12.32}$$

implicitly defines z as one or more functions of x and y provided that $\frac{\partial F}{\partial z}\neq 0$. Treating z as a function of x and y one can differentiate the equation (12.32) with respect to x and obtain

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \tag{12.33}$$

Solving for $\frac{\partial z}{\partial x}$ one finds

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{12.34}$$

An alternative representation of the derivative of z with respect to x can be obtained as follows. Take the differential of equation (12.32) to obtain

$$dF=\frac{\partial F}{\partial x}\,dx+\frac{\partial F}{\partial y}\,dy+\frac{\partial F}{\partial z}\,dz=0 \tag{12.35}$$

If y is held constant, then dy is zero and equation (12.35) yields the result

$$\left(\frac{dz}{dx}\right)_y = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{12.36}$$

Here the symbol on the left of equation (12.36) is used to emphasize that y is being held constant during the differentiation process. Some engineering texts feel this notation is less ambiguous than the use of the partial derivative symbol occurring in equation (12.34).

Differentiating the equation (12.32) with respect to y produces the result

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \tag{12.37}$$

from which one finds the partial derivative

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial F}{\partial z} \neq 0 \tag{12.38}$$

Alternatively, set x equal to a constant so that $dx = 0$ in equation (12.35), then equation (12.35) produces the result

$$\left(\frac{dz}{dy}\right)_x = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial F}{\partial z} \neq 0 \tag{12.39}$$

Here the derivative is represented using the alternative notation $\left(\frac{dz}{dy}\right)_x$ emphasizing the derivative is obtained holding x constant.

one equation, four unknowns

Any equation of the form

$$F(x, y, z, w) = 0 \tag{12.40}$$

implicitly defines w as one or more functions of x, y and z. Differentiate equation (12.40) with respect to x and show

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial w}} \tag{12.41}$$

Differentiate equation (12.40) with respect to y and show

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial w}} \tag{12.42}$$

Differentiate equation (12.40) with respect to z and show

$$\frac{\partial F}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial w}} \tag{12.43}$$

provided $\frac{\partial F}{\partial w} \neq 0$.

one equation, n-unknowns

Any equation of the form

$$F(x_1, x_2, \ldots, x_n, w) = 0 \tag{12.44}$$

implicitly defines w as one or more functions of the n-variables $(x_1, x_2, \ldots, x_n)$. It is left as an exercise to show that for a fixed integer value of j between 1 and n that

$$\frac{\partial w}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial w}}, \quad \text{provided} \quad \frac{\partial F}{\partial w} \neq 0 \tag{12.45}$$

two equations, three unknowns

Given two equations having the form

$$F(x,y,z) = 0 \quad \text{and} \quad G(x,y,z) = 0 \tag{12.46}$$

then these equations define implicitly (a) z as a function of x and (b) y as a function of x. Treat $z = z(x)$ and $y = y(x)$ and differentiate each of the equations (12.46) with respect to x and show

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} &= 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} &= 0 \end{aligned} \tag{12.47}$$

The equations (12.47) represent two equations in the two unknowns $\frac{dy}{dx}$ and $\frac{dz}{dx}$ which can be solved. Use Cramers rule[2] and show these equations have the solutions

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \tag{12.48}$$

where

$$\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} \neq 0$$

[2] See Cramers rule in Appendix B

The determinants in the equations (12.48) are called Jacobian determinants of F and G and are often expressed using the shorthand notation

$$\frac{\partial(F,G)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial G}{\partial y}\frac{\partial F}{\partial z} \tag{12.49}$$

In terms of Jacobian determinants the derivatives represented by the equations (12.48) can be represented

$$\frac{dy}{dx} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} \tag{12.50}$$

where $\dfrac{\partial(F,G)}{\partial(y,z)} \neq 0$. Note the patterns associated with the partial derivatives and the Jacobian determinants. We will make use of these patterns to calculate derivatives directly from the given transformation equations in later presentations and examples.

two equations, four unknowns

Two equations of the form

$$\begin{aligned} F(x,y,u,v) =& 0 \\ G(x,y,u,v) =& 0 \end{aligned} \tag{12.51}$$

implicitly define u and v as functions of x and y so that one can write $u = u(x,y)$ and $v = v(x,y)$. The derivatives of the equations(12.51) with respect to x can then be expressed

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} =& 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} =& 0 \end{aligned} \tag{12.54}$$

The equations (12.54) represent two equations in the two unknowns $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. One can use Cramers rule to solve this system of equations and obtain the solutions

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} \tag{12.53}$$

This solution is valid provided the Jacobian determinant $\dfrac{\partial(F,G)}{\partial(u,v)}$ is different from zero.

In a similar fashion one can differentiate the equations (12.51) with respect to the variable y and obtain

$$\begin{aligned} \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} =&0 \\ \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} =&0 \end{aligned} \tag{12.54}$$

which produces two equations in the two unknowns $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$. Solving this system of equations using Cramers rule produces the solutions

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} \tag{12.55}$$

This solution is valid provided the Jacobian determinant $\dfrac{\partial(F,G)}{\partial(u,v)}$ is different from zero.

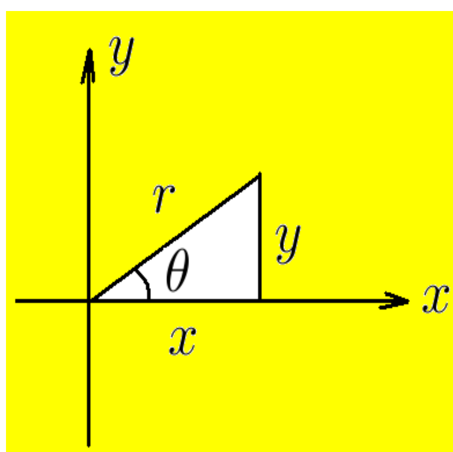
Example 12-3. (Conversion of the Laplace equation)

Transform the Laplace equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{12.56}$$

from rectangular (x, y) coordinates to (r, θ) polar coordinates.

Solution 1



$$x = r\cos\theta \quad y = r\sin\theta$$

Figure 12-5.

Rectangular (x, y) and polar (r, θ) coordinates

If $U = U(x,y)$ is converted to polar coordinates to become $U = U(r,\theta)$ one can treat $r = r(x,y)$ and $\theta = \theta(x,y)$ to calculate the following derivatives of U with respect x and y

$$\begin{aligned}\frac{\partial U}{\partial x} =& \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\\ \frac{\partial^2 U}{\partial x^2} =& \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial r\,\partial \theta}\frac{\partial \theta}{\partial x}\right]\\ &+ \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\left[\frac{\partial^2 U}{\partial \theta\,\partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right]\end{aligned} \tag{12.57}$$

and

$$\begin{aligned}\frac{\partial U}{\partial y} =& \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y}\\ \frac{\partial^2 U}{\partial y^2} =& \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial r}{\partial x}\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 U}{\partial r\,\partial \theta}\frac{\partial \theta}{\partial y}\right]\\ &+ \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial \theta}{\partial y}\left[\frac{\partial^2 U}{\partial \theta\,\partial r}\frac{\partial r}{\partial y} + \frac{\partial^2 U}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right]\end{aligned} \tag{12.58}$$

The transformation equations from rectangular to polar coordinates is performed using the transformation equations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{12.59}$$

One can solve for r and θ in terms of x and y to obtain

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{12.60}$$

Differentiate the equations (12.60) with respect to x and show

$$2r\frac{\partial r}{\partial x} = 2x \qquad \text{and} \qquad \sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \tag{12.61}$$

which simplifies using the transformation equations (12.59) to the values

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \text{and} \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \tag{12.62}$$

Differentiate the the equations (12.60) with respect to y and show

$$2r\frac{\partial r}{\partial y} = 2y \qquad \text{and} \qquad \sec^2\theta\frac{\partial \theta}{\partial y} = \frac{1}{x} \tag{12.63}$$

Use the transformation equations (12.59) and simplify the equations (12.63) and show

$$\frac{\partial r}{\partial y} = \sin\theta \qquad \text{and} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \tag{12.64}$$

It is now possible to differentiate the first derivatives given by equations (12.62) and (12.64) to obtain the second derivatives

$$\begin{array}{ll} \dfrac{\partial^2 r}{\partial x^2} = -\sin\theta \dfrac{\partial \theta}{\partial x} & \dfrac{\partial^2 r}{\partial y^2} = \cos\theta \dfrac{\partial \theta}{\partial y} \\ \dfrac{\partial^2 r}{\partial x^2} = \dfrac{\sin^2\theta}{r} \quad , & \dfrac{\partial^2 r}{\partial y^2} = \dfrac{\cos^2\theta}{r} \end{array} \tag{12.65}$$

and

$$\begin{array}{ll} \dfrac{\partial^2 \theta}{\partial x^2} = -\dfrac{\cos\theta}{r}\dfrac{\partial \theta}{\partial x} + \dfrac{\sin\theta}{r}\dfrac{\partial r}{\partial x} & \dfrac{\partial^2 \theta}{\partial y^2} = -\dfrac{\sin\theta}{r}\dfrac{\partial \theta}{\partial y} - \dfrac{\cos\theta}{r^2}\dfrac{\partial r}{\partial y} \\ \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{2\sin\theta\cos\theta}{r^2} \quad , & \dfrac{\partial^2 \theta}{\partial y^2} = -\dfrac{2\sin\theta\cos\theta}{r^2} \end{array} \tag{12.66}$$

Substitute the first and second derivatives from equations (12.62), (12.64), (12.66) into the equations (12.57) and (12.58) to show that after simplification the equation (12.56) becomes

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0 \tag{12.67}$$

Solution 2

Write the transformation equations (12.59) as

$$F(x,y,r,\theta) = x - r\cos\theta = 0 \qquad \text{and} \qquad G(x,y,r,\theta) = y - r\sin\theta = 0 \tag{12.68}$$

and use the notation for Jacobian determinants to write

$$\frac{\partial r}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,\theta)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} 1 & r\sin\theta \\ 0 & -r\cos\theta \end{vmatrix}}{\begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(r,x)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} -sin\theta & -r\cos\theta \\ -\sin\theta & 0 \end{vmatrix}}{r} = -\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,\theta)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} 0 & r\sin\theta \\ 1 & -r\cos\theta \end{vmatrix}}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(r,y)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} -\cos\theta & 0 \\ -\sin\theta & 1 \end{vmatrix}}{r} = \frac{\cos\theta}{r}$$

These derivatives can be compared with the previous results given in the equations (12.62) and (12.64).

■

three equations, five unknowns

A set of equations having the form

$$\begin{aligned} F(x,y,u,v,w) =&0\\ G(x,y,u,v,w) =&0 \\ H(x,y,u,v,w) =&0 \end{aligned} \tag{12.69}$$

implicitly defines u,v,w as functions of x and y so that one can write

$$u = u(x,y) \qquad v = v(x,y) \qquad w = w(x,y) \tag{12.70}$$

The partial derivatives of u,v and w with respect to x and y are calculated in a manner similar to the previous representations presented.

In order to save space in typesetting sometimes the notation for partial derivatives is shortened to the use of subscripts. For example, one can define

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad F_x = \frac{\partial F}{\partial x}, \quad F_{xx} = \frac{\partial^2 F}{\partial x^2}, \quad F_w = \frac{\partial F}{\partial w}, \quad F_{xy} = \frac{\partial^2 F}{\partial x\,\partial y}, \quad \text{etc}$$

We now employ this notation and take the partial derivatives of equations F,G and H with respect to x and write

$$\begin{aligned} F_x + F_u u_x + F_v v_x + F_w w_x =&0\\ G_x + G_u u_x + G_v v_x + G_w w_x =&0\\ H_x + H_u u_x + H_v v_x + H_w w_x =&0 \end{aligned} \tag{12.71}$$

The equations (12.71) represent three equations in the three unknowns u_x, v_x and w_x which can be solved using Cramers rule. This can be accomplished by defining the 3 by 3 Jacobian determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix} \tag{12.72}$$

If this Jacobian determinant is different from zero, the system of equations (12.71) has a unique solution for u_x, v_x, w_x given by

$$u_x = -\frac{\frac{\partial(F,G,H)}{\partial(x,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad v_x = -\frac{\frac{\partial(F,G,H)}{\partial(u,x,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad w_x = -\frac{\frac{\partial(F,G,H)}{\partial(u,v,x)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}} \tag{12.73}$$

DIfferentiate the equations (12.69) with respect to y to obtain

$$\begin{aligned} F_y + F_u u_y + F_v v_y + F_w w_y =&0 \\ G_y + G_u u_y + G_v v_y + G_w w_y =&0 \\ H_y + H_u u_y + H_v v_y + H_w w_y =&0 \end{aligned} \tag{12.74}$$

This produces three equations in the three unknowns u_y, v_y, w_y which can be solved using Cramers rule. If the Jacobian determinant of F, G, H is different from zero, then the unique solution is given by

$$u_y = -\frac{\frac{\partial(F,G,H)}{\partial(y,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad v_y = -\frac{\frac{\partial(F,G,H)}{\partial(u,y,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad w_y = -\frac{\frac{\partial(F,G,H)}{\partial(u,v,y)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}} \tag{12.75}$$

Generalization

The system of equations

$$\begin{aligned} F_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =&0 \\ F_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =&0 \\ \vdots \qquad\qquad\qquad & \vdots \\ F_m(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =&0 \end{aligned} \tag{12.76}$$

in $m+n$ unknowns implicitly defines the functions

$$\begin{aligned} y_1 =&y_1(x_1, x_2, \ldots, x_n) \\ y_2 =&y_2(x_1, x_2, \ldots, x_n) \\ \vdots \quad & \vdots \\ y_m =&y_m(x_1, x_2, \ldots, x_n) \end{aligned} \tag{12.77}$$

If the Jacobian determinant

$$\frac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, y_2, \ldots, y_m)} = \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix}$$

is different from zero at a point $(x_1^0, x_2^0, \ldots, x_n^0, y_1^0, y_2^0, \ldots, y_m^0)$, then one can calculate the partial derivatives $\dfrac{\partial y_i}{\partial x_j}$ at this point for any combination of integer values for i, j satisfying $1 \le i \le m$ and $1 \le j \le n$. The above is true because if one calculates the derivatives of the functions in equation (12.76) with respect to say x_j, $1 \le j \le n$, one finds

$$\begin{aligned}\frac{\partial F_1}{\partial x_j} + \frac{\partial F_1}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_1}{\partial y_m}\frac{\partial y_m}{\partial x_j} =&0\\ \frac{\partial F_2}{\partial x_j} + \frac{\partial F_2}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_2}{\partial y_m}\frac{\partial y_m}{\partial x_j} =&0\\ \vdots \qquad \vdots \qquad\qquad &\vdots\\ \frac{\partial F_n}{\partial x_j} + \frac{\partial F_n}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_n}{\partial y_m}\frac{\partial y_m}{\partial x_j} =&0\end{aligned} \tag{12.78}$$

The system of equations (12.78) can be solved by Cramers rule and because the Jacobian determinant is different from zero the system of equations (12.78) has a unique solution for the various first partial derivatives.

Higher derivatives can be obtained by differentiating the first order partial derivatives.

The Gamma Function

One definition of the Gamma function is given by the integral

$$\Gamma(x) = \int_0^\infty \xi^{x-1} e^{-\xi}\, d\xi \tag{12.79}$$

The value $x = 1$ substituted into the equation (12.79) produces the result

$$\Gamma(1) = \int_0^\infty e^{-\xi}\, d\xi = -e^{-\xi}\big]_0^\infty = 1 \tag{12.80}$$

so that $\Gamma(1) = 1$. Substitute $x = n+1$, a positive integer, into equation (12.79) and then integrate by parts to obtain

$$\Gamma(n+1) = \int_0^\infty e^{-\xi}\xi^n\, d\xi = \left[-\xi^n e^{-\xi}\right]_0^\infty + \int_0^\infty n\xi^{n-1}e^{-\xi}\, d\xi$$

which simplifies to the recurrence relation

$$\Gamma(n+1) = n\Gamma(n), \qquad n > 0, \text{ an integer} \tag{12.81}$$

In general, for any real positive value for x which is less than unity, one can show that $\Gamma(x)$ is a particular solution of the functional equation

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0 \tag{12.82}$$

Replacing x by $-x$, the equation (12.82) is sometimes represented

$$\Gamma(1-x) = -x\Gamma(-x) \tag{12.83}$$

Using the recurrence relation (12.81) one can show that

$$\begin{aligned}
\Gamma(n) =&(n-1)\Gamma(n-1)\\
\Gamma(n-1) =&(n-2)\Gamma(n-2)\\
\Gamma(n-2) =&(n-3)\Gamma(n-3)\\
\vdots \quad =& \quad \vdots\\
\Gamma(3) =&2\,\Gamma(2)\\
\Gamma(2) =&1\,\Gamma(1) = 1
\end{aligned} \tag{12.84}$$

The equations (12.81) and (12.84) demonstrate that

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n! \tag{12.85}$$

Observe that when $n = 0$ the equation (12.85) becomes $\Gamma(1) = 0!$, but we know $\Gamma(1) = 1$, hence this is one of the reasons for the convention of defining $0!$ as 1.

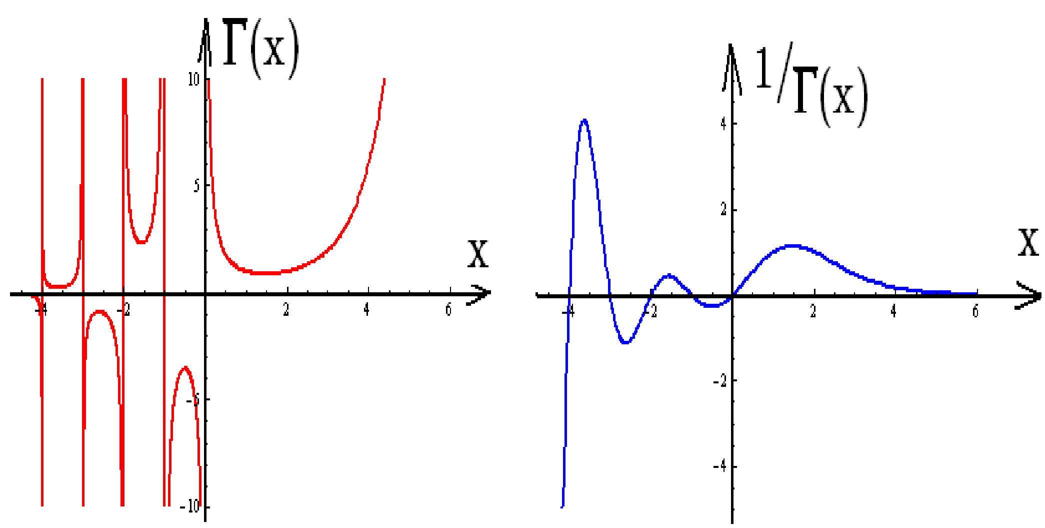


Figure 12-6.

The Gamma function $\Gamma(x)$ and $1/\Gamma(x)$.

Write equation (12.81) in the form $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ to show that for $n = 0, -1, -2, \ldots$ the function $\Gamma(n)$ becomes infinite. The function $\Gamma(x)$ and $1/\Gamma(x)$ are illustrated in the figure 12-6.

Example 12-4.

Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Solution

Substitute $x = 1/2$ into equation (12.79) and show

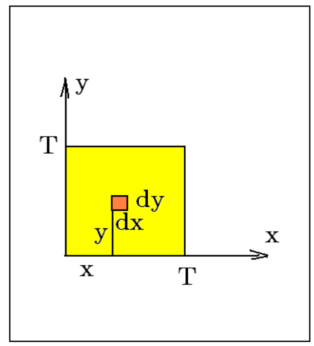
$$\Gamma(\frac{1}{2}) = \int_0^\infty \xi^{-1/2} e^{-\xi}\, d\xi = \int_0^\infty \frac{e^{-\xi}}{\sqrt{\xi}}\, d\xi \tag{12.86}$$

In equation (12.86) make the substitution $\xi = x^2$ with $d\xi = 2x\,dx$ to obtain

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-x^2}}{x} 2x\,dx = 2\int_0^\infty e^{-x^2}\,dx \tag{12.87}$$

Let I denote the integrals

$$I = \int_0^\infty e^{-x^2}\,dx \qquad \text{and} \qquad I = \int_0^\infty e^{-y^2}\,dy$$



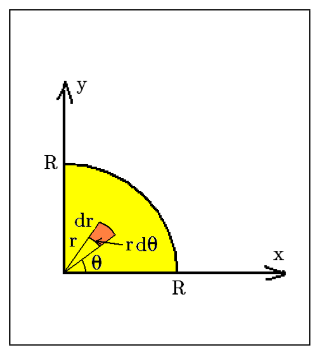
and form the double integral

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \lim_{T\to\infty} \int_0^T \int_0^T e^{-(x^2+y^2)}\,dxdy$$

and observe that as T increases without bound the area of integration fills up the first quadrant.

Change the double integral for I^2 from rectangular to polar coordinates where

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dxdy = rdrd\theta$$



and write

$$I^2 = \lim_{R\to\infty} \int_{r=0}^R \int_0^{\pi/2} e^{-r^2}\, rdrd\theta$$

and observe that as R increases without bound the area of integration is still over the first quadrant. Now integrate with respect to θ and then integrate with respect to r to show

$$I^2 = \frac{\pi}{4} \qquad \text{or} \qquad I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Substitute this result into the equation (12.87) to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

■

Product of odd and even integers

One can now apply the previous results

$$\Gamma(n) = (n-1)\Gamma(n-1), \qquad \Gamma(1) = 1, \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{12.88}$$

to show that for n a positive integer

$$\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\cdots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi} \tag{12.89}$$

This result demonstrates that an alternative representation for **the product of the odd integers** is given by

$$1\cdot 3\cdot 5\cdots(2n-5)(2n-3)(2n-1) = \frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right) \tag{12.90}$$

Using the equations (12.88) one can demonstrate

$$\Gamma\left(\frac{2n+2}{2}\right) = \left(\frac{2n}{2}\right)\left(\frac{2n-2}{2}\right)\left(\frac{2n-4}{2}\right)\cdots\left(\frac{6}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{2}\right) \tag{12.91}$$

which shows that **the product of the even integers** can be represented in the form

$$2\cdot 4\cdot 6\cdots(2n-4)(2n-2)(2n) = 2^n\Gamma(n+1) \tag{12.92}$$

Example 12-5.

Let $S_n = \int_0^{\pi/2} \sin^n x\, dx$ and integrate by parts using

$$\begin{aligned} U &= \sin^{n-1} x & dV &= \sin x\, dx \\ dU &= (n-1)\sin^{n-2} x\, \cos x\, dx & V &= -\cos x \end{aligned}$$

to obtain

$$\begin{aligned} S_n &= -\sin^{n-1} x\, \cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{n-2} x\, \cos^2 x\, dx \\ S_n &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\, dx = (n-1)S_{n-2} - (n-1)S_n \end{aligned}$$

which simplifies to the recurrence formula

$$S_n = \frac{n-1}{n} S_{n-2} \tag{12.93}$$

The above recurrence relation implies that

$$S_n = \frac{n-1}{n} S_{n-2} = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) S_{n-4} = \cdots \tag{12.94}$$

If n is odd, say $n = 2m-1$, then equation (12.94) eventually becomes

$$S_{2m-1} = \left(\frac{2m-2}{2m-1}\right)\left(\frac{2m-4}{2m-3}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\cdot S_1 \tag{12.95}$$

where

$$S_1 = \int_0^{\pi/2} \sin x\,dx = -\cos x\big]_0^{\pi/2} = 1$$

The numerator of equation (12.95) is a **product of even integers** and the denominator of equation (12.95) is a **product of odd integers** so that one can employ the results from equations (12.90) and (12.92) to write equation (12.95) in the form

$$S_{2m-1} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(m)}{\Gamma\left(\frac{2m+1}{2}\right)} \tag{12.96}$$

If n is even, say $n = 2m$, then equation (12.94) eventually becomes

$$S_{2m} = \left(\frac{2m-1}{2m}\right)\left(\frac{2m-3}{2m-2}\right)\cdots\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\,S_0 \tag{12.97}$$

where

$$S_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \tag{12.98}$$

Note that the numerator in equation (12.97) is a **product of odd integers** and the denominator is a **product of even integers**. Using the results from equations (12.90) and (12.92) the above result can be expressed in the form

$$S_{2m} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)}\,\frac{\pi}{2} \tag{12.99}$$

■

Example 12-6.

Let $C_n = \displaystyle\int_0^{\pi/2} \cos^n x\,dx$ and follow the step-by-step analysis as in the previous example and demonstrate that

$$C_n = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{\sqrt{\pi}}{2}\frac{\Gamma(m)}{\Gamma\left(\frac{2m+1}{2}\right)} & \text{if } n = 2m-1 \text{ is odd} \\ \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)}\frac{\pi}{2} & \text{if } n = 2m \text{ is even} \end{cases} \tag{12.100}$$

■

Various representations for the Gamma function

The integral representation of the Gamma function

$$\Gamma(x) = \int_0^\infty \xi^{x-1}\, e^{-\xi}\, d\xi \qquad x > 0 \tag{12.101}$$

can be transformed to many alternative representations for use in special situations.

(i) The substitution $\xi = \ln(\frac{1}{y})$ or $y = e^{-\xi}$, converts the equation (12.101) to the form

$$\Gamma(x) = \int_0^1 \left(\ln\frac{1}{y}\right)^{x-1} dy \tag{12.102}$$

which is the form Euler originally studied.

(ii) The substitution $\xi = zt$ converts equation (12.101) to the form

$$\Gamma(x) = \int_0^\infty z^{x-1} t^{x-1} e^{-zt}\, z\, dt$$

and replacing x by $x+1$ there results

$$\Gamma(x+1) = z^{x+1} \int_0^\infty t^x e^{-zt}\, dt \tag{12.103}$$

The above change of variables are just a sampling of forms for obtaining alternative integral representations of the Gamma function.

Euler's constant γ, defined by the limit

$$\gamma = \lim_{n\to\infty}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln(n)\right] = 0.5772156649\cdots \tag{12.104}$$

occurs in many alternative representations of the Gamma function. When γ is represented as a continued fraction (See chapter 4, page 337) one finds the list notation given by

$$\gamma = [0\,; 1, 1, 2, 1, 2, 1, , 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 40, 1, \ldots]$$

The first nine convergents are

$$\begin{array}{lll} \gamma_1 = 1 & \gamma_4 = \frac{4}{7} = 0.571428571 & \gamma_7 = \frac{71}{123} = 0.577235772 \\ \gamma_2 = \frac{1}{2} = 0.5 & \gamma_5 = \frac{11}{19} = 0.578947368 & \gamma_8 = \frac{228}{395} = 0.577215190 \\ \gamma_3 = \frac{3}{5} = 0.6 & \gamma_6 = \frac{15}{26} = 0.5769233077 & \gamma_9 = \frac{3035}{5258} = 0.577215671 \end{array} \tag{12.105}$$

where γ_9 is accurate to seven decimal places.

Sometime around 1729 Euler defined the Gamma function in the form

$$\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{x(1+x)(2+x)(3+x)\cdots(n-1+x)} = \lim_{n\to\infty} \frac{n^x}{x(1+\frac{x}{1})(1+\frac{x}{2})\cdots(1+\frac{x}{n-1})} \tag{12.106}$$

Karl Weierstrass modified Euler's form for the Gamma function and represented it in the form

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right] \tag{12.107}$$

where γ is Euler's constant from equation (12.104). Here the infinite product $\prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right]$ is convergent for all values of z positive, negative, real or complex.

Other forms of the Gamma function can be found in the mathematical literature. The representation of the Gamma function in the complex plane provides new incites into properties of the Gamma function. As an interesting exercise check out some textbooks on the Gamma function to see how all of the above forms of the Gamma function are equivalent. This type of exercise is one example illustrating the concept that functions and ideas, which occur in mathematical studies, can be presented in a variety of ways.

Euler formula for the Gamma function

Having a variety of forms for representing the Gamma function provides one the opportunity to seek out and discover other properties of the Gamma function. For example, employ the Weirstrass representation

$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty} \left[\left(1+\frac{x}{n}\right) e^{-x/n}\right] \tag{12.108}$$

and show

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(-x)} = -x^2 e^{\gamma x} e^{-\gamma x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}\right) e^{-x/n} e^{x/n} \tag{12.109}$$

This equation simplifies using the property $\Gamma(1-x) = -x\Gamma(-x)$. One can verify equation (12.109) simplifies to

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = x \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right)$$

or

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(1-x)} = x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots \tag{12.110}$$

Recall Euler's infinite product formula for $\sin\theta$ (see Example 4-38)

$$\sin(\pi x) = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots \tag{12.111}$$

and compare this infinite product with the one occurring in equation (12.110) to show

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{12.112}$$

which is known as Euler's reflection formula for the Gamma function. Make note of the fact that if the value $x = 1/2$ is substituted into equation (12.112) one obtains

$$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \pi \qquad \text{or} \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{12.114}$$

which agrees with our previous result.

Using the previous result $n\Gamma(n) = \Gamma(1+n)$, the equation (12.112) is sometimes written in the form

$$\Gamma(1+n)\Gamma(1-n) = \frac{n\pi}{\sin n\pi} \tag{12.114}$$

The Zeta function related to the Gamma function

The Gamma function $\Gamma(z)$ and the Riemann Zeta function $\zeta(z)$ are related. Recall that one definition of the Zeta function is (see Example 4-38)

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad z > 1 \tag{12.115}$$

and the integral form for representing the Gamma function is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad z > 0 \tag{12.116}$$

In equation (12.116) make the change of variable $t = rx$ and show

$$\Gamma(z) = \int_0^\infty (rx)^{z-1} e^{-rx}\, r\, dx = r^z \int_0^\infty x^{z-1} e^{-rx}\, dx \tag{12.117}$$

One can express equation (12.117) in the form

$$\frac{1}{r^z} = \frac{1}{\Gamma(z)} \int_0^\infty x^{z-1} e^{-rx}\, dx \tag{12.118}$$

A summation of equation (12.118) over integer values for r produces the result

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z} = \frac{1}{\Gamma(z)} \sum_{r=1}^{\infty} \int_0^\infty x^{z-1} e^{-rx}\, dx \tag{12.119}$$

Now interchange the roles of summation and integration on the right hand side of equation (12.119) to obtain

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} x^{z-1} \sum_{r=1}^{\infty} e^{-rx}\, dx \tag{12.120}$$

where now the summation on the right hand side of equation (12.120) is the geometric series

$$\sum_{r=1}^{\infty} e^{-rx} = e^{-x} + e^{-2x} + e^{-3x} + \cdots = \frac{e^{-x}}{1-e^{-x}}$$

Consequently, the equation (12.120) simplifies to the form

$$\zeta(z)\Gamma(z) = \int_0^{\infty} x^{z-1} \frac{e^{-x}}{1-e^{-x}}\, dx \tag{12.121}$$

Observe that the Gamma function and Zeta function properties dictate that z be restricted such that $z \neq 1, 0, -1, -2, -3, \ldots$ in using equation (12.121).

Product property of the Gamma function

The Gamma function satisfies the product property that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}} \tag{12.122}$$

In order to derive this result we develop the following background material.

Example 12-7.

Show that

$$\sin n\theta = 2^{n-1} \sin\theta \sin(\theta + \frac{\pi}{n}) \sin(\theta + \frac{2\pi}{n}) \sin(\theta + \frac{3\pi}{n}) \cdots \sin(\frac{\theta + (n-1)\pi}{n}) \tag{12.123}$$

and

$$\lim_{\theta\to 0} \frac{\sin n\theta}{\sin\theta} = n = 2^{n-1} \sin(\frac{\pi}{n}) \sin(\frac{2\pi}{n}) \sin(\frac{3\pi}{n}) \cdots \sin(\frac{(n-1)\pi}{n}) \tag{12.124}$$

Solution

The following proof follows that presented in the reference Hobson[3]

First show that the function

$$x^{2n} - 2x^n \cos n\theta + 1 \tag{12.125}$$

[3] Ernest William Hobson, **A treatise on plane trigonometry**, 5th Edition, Cambridge University Press, 1921, Pages 117-119.

can be written as a product of factors having the form

$$x^{2n} - 2x^n \cos n\theta + 1 = 2^{n-1} \prod_{r=0}^{n-1} (x^2 - 2x\cos(\theta + \frac{2r\pi}{n}) + 1) \tag{12.126}$$

This is accomplished by considering the function $x^{2n} - 2x^n \cos n\theta + 1$ and then dividing it by x^n and defining

$$u_n = x^n - 2\cos n\theta + x^{-n} \tag{12.127}$$

and then verifying that u_n can be written as

$$\begin{aligned} u_n =&(x^{n-1} + x^{-n+1})(x - 2\cos\theta + x^{-1}) \\ &+ 2\cos\theta(x^{n-1} - 2\cos[(n-1)\theta] + x^{-n+1}) - (x^{n-2} - 2\cos[(n-2)\theta] + x^{-(n-2)}) \end{aligned} \tag{12.128}$$

or in terms of the u_n definition

$$u_n = (x^{n-1} + x^{-n+1})u_1 + 2u_{n-1}\cos\theta - u_{n-2} \tag{12.129}$$

Observe that u_n is divisible by u_1 if both u_{n-1} and u_{n-2} are also divisible by u_1. To show this is true verify that

$$u_2 = x^2 - 2\cos 2\theta + x^{-2} = (x - 2\cos\theta + x^{-1})(x + 2\cos\theta + x^{-1})$$

and consequently u_2 is divisible by u_1. Using equation (12.129) write

$$u_3 = (x^2 + x^{-2})u_1 + 2u_2\cos\theta - u_1$$

to show u_3 is divisible by u_1. Continuing in this fashion $u_4, u_5, u_6, \ldots, u_{n-2}, u_{n-1}$, are all divisible by u_1. This demonstrates that $x^2 - 2x\cos\theta + 1$ is a factor of $x^{2n} - 2x^n \cos n\theta + 1$. Since θ is an arbitrary angle, replace θ by $\theta + 2r\pi/n$, r an integer constant, to show

$$x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1 \quad \text{is a factor of} \quad x^{2n} - 2x^n \cos[n\left(\theta + \frac{2r\pi}{n}\right)] + 1$$

for $r = 0, 1, 2, \ldots, n-1$.

Using the trigonometric identity

$$\cos n\theta = \cos[n(\theta + \frac{2r\pi}{n})]$$

for r an integer, one can say that the factors of

$$x^{2n} - 2x^n \cos n\theta + 1$$

are given by $x^2 - 2x\cos(\theta + \frac{2r\pi}{n}) + 1$ for $r = 0, 1, 2, \ldots, n-1$. This implies $x^{2n} - 2x^n \cos n\theta + 1$ can be expressed as

$$x^{2n} - 2x^n \cos n\theta + 1 = \prod_{r=0}^{n-1} (x^2 - 2x\cos(\theta + \frac{2r\pi}{n}) + 1) \tag{12.130}$$

In the special case $x = 1$ the equation (12.130) simplifies to

$$1 - \cos n\theta = 2^{n-1} \prod_{r=0}^{n-1} \left(1 - \cos\left(\theta + \frac{2r\pi}{n}\right)\right) \tag{12.131}$$

Replacing θ by 2θ in equation (12.131) and simplifying one obtains

$$2\sin^2 n\theta = 2^{n-1} 2^n \sin^2\theta \sin^2(\theta + \frac{\pi}{n}) \sin^2(\theta + \frac{2\pi}{n}) \cdots \sin^2(\theta + \frac{(n-1)\pi}{n}) \tag{12.132}$$

Further simplify equation (12.132) and then take the square root of both sides to obtain

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1} \sin(\theta + \frac{\pi}{n}) \sin(\theta + \frac{2\pi}{n}) \cdots \sin(\theta + \frac{(n-1)\pi}{n}) \tag{12.133}$$

where the positive square root is taken when each term is positive. In equation (12.133) take the limit as $\theta \to 0$ and verify

$$n = 2^{n-1} \sin(\frac{\pi}{n}) \sin(\frac{2\pi}{n}) \cdots \sin(\frac{(n-1)\pi}{n}) \tag{12.134}$$

■

Now consider the product

$$y = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$$

and reverse the terms within the product to show

$$y^2 = \left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right)\right]\cdots\left[\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right]$$

followed by writing equation (12.112) in the form

$$\Gamma\left(\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right) = \frac{\pi}{\sin(\frac{m\pi}{n})}$$

for $m = 1, 2, \ldots, n-1$. This identity produces

$$y^2 = \frac{\pi}{\sin(\frac{\pi}{n})} \frac{\pi}{\sin(\frac{2\pi}{n})} \frac{\pi}{\sin(\frac{3\pi}{n})} \cdots \frac{\pi}{\sin(\frac{(n-1)\pi}{n})}$$

One can now employ the result from Example 12-6, equation (12.134), to show

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$$

Derivatives of $\ln\Gamma(z)$

Using the Weierstrass definition of the Gamma function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{z}\right)e^{-z/n}\right] \tag{12.135}$$

take the natural logarithm of both sides and show

$$\ln\Gamma(z) = -\ln z - \gamma z - \sum_{k=1}^{\infty}\left[\ln\left(1+\frac{z}{k}\right) - \frac{z}{k}\right] \tag{12.136}$$

Take the derivative of each term in this equation to show

$$\frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right) \tag{12.137}$$

Differentiate equation (12.137) to obtain

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}\frac{1}{(z+k)^2} = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots \tag{12.138}$$

Continuing differentiating the function $\ln\Gamma(z)$ and use the fact that

$$\begin{aligned}
\frac{d}{dz}(z+k)^{-2} =&(-2)(z+k)^{-3}\\
\frac{d^2}{dz^2}(z+k)^{-2} =&(-2)(-3)(z+k)^{-4}\\
\frac{d^3}{dz^3}(z+k)^{-2} =&(-2)(-3)(-4)(z+k)^{-5}\\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

and demonstrate that

$$\begin{aligned}
\frac{d^n}{dz^n}\ln\Gamma(z) =&(-1)^n(n-1)!\sum_{k=0}^{\infty}\frac{1}{(z+k)^n}\\
\frac{d^{n+1}}{d^{n+1}}\ln\Gamma(z) =&(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}
\end{aligned} \tag{12.139}$$

Make note of the following definitions. The function $\zeta(n,z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^n}$, **where any term where $(z+k)=0$ is understood to be excluded from the summation process,** is defined as **the Hurwitz[4] Zeta function** and satisfies $\zeta(n,0)=\zeta(n)$, the Zeta function. The function

$$\psi_n(z) = (-1)^{n+1} n! \zeta(n+1, z) = \frac{d^{n+1}}{d^{n+1}} \ln[\Gamma(z)] \tag{12.140}$$

is referred to as **the polygamma function of order n.** Observe that

$$\psi_0(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{and} \qquad \psi_n(z) = \frac{d^n}{dz^n}\psi_0(z)$$

The polygamma function of order zero $\psi_0(x)$ is called **the digamma function** and is illustrated in the figure 12-7.

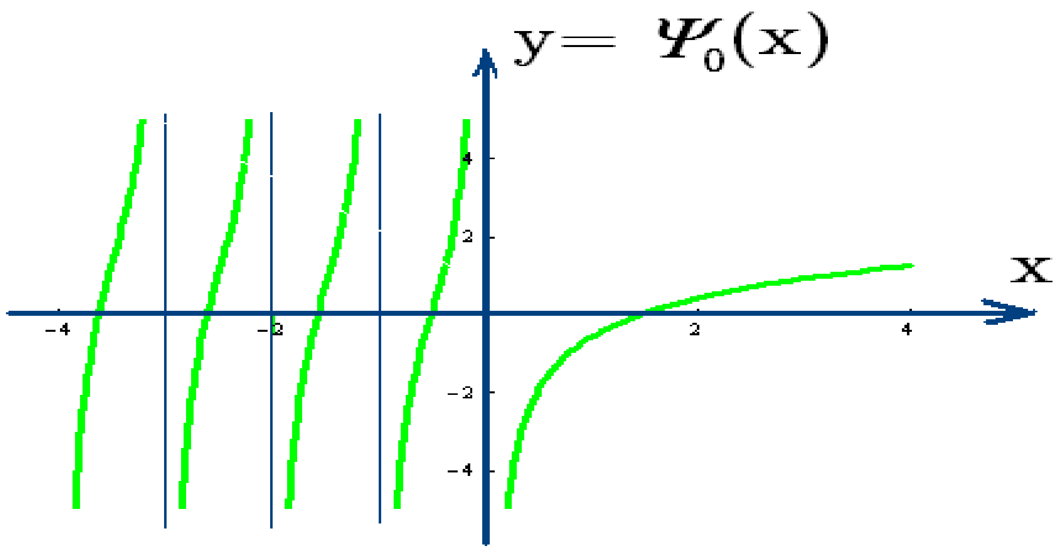


Figure 12-7. The polygamma function of order zero.

[4] Adolf Hurwitz (1859-1919) German professor of mathematics.

Taylor series expansion for $\ln\Gamma(x+1)$

Make reference to the equations (12.136), (12.137), (12.138), (12.139), and verify that when z is replaced by $(x+1)$ in these equations, one obtains the following values.

$$
\begin{aligned}
\ln\Gamma(x+1)\Big|_{x=0} =& \ln\Gamma(1) = 0\\
\frac{d}{dx}\ln\Gamma(x+1)\Big|_{x=0} =& -\gamma \quad \text{because (12.137) is a telescoping series}\\
\frac{d^2}{dx^2}\ln\Gamma(x+1)\Big|_{x=0} =& \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \zeta(2)\\
\vdots \qquad & \qquad \vdots\\
\frac{d^n}{dx^n}\ln\Gamma(x+1)\Big|_{x=0} =& (-1)^n(n-1)!\zeta(n) \qquad n \ge 2
\end{aligned}
\tag{12.141}
$$

where

$$
\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}
$$

is the Riemann Zeta function. The above values of the derivatives of $\ln\Gamma(x+1)$, evaluated at $x=0$, produces the Taylor series expansion for $\ln\Gamma(x+1)$ as

$$
\ln\Gamma(x+1) = -\gamma x + \zeta(2)\frac{x^2}{2} - \zeta(3)\frac{x^3}{3} + \zeta(4)\frac{x^4}{4} + \cdots + (-1)^n\zeta(n)\frac{x^n}{n} + \cdots \tag{12.142}
$$

which converges if x is less than unity.

Another product formula

Define the function

$$
\phi(z) = \frac{n^z\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{(n-1)}{n}\right)}{n\Gamma(nz)} \tag{12.143}
$$

and use the equation (12.106) to write

$$
\Gamma\left(z+\frac{r}{n}\right) = \lim_{m\to\infty}\frac{(m-1)!\,m^{z+r/n}}{\left(z+\frac{r}{n}\right)\left(z+\frac{r}{n}+1\right)\left(z+\frac{r}{n}+2\right)\cdots\left(z+\frac{r}{n}+m-1\right)} \tag{12.144}
$$

for $r = 0, 1, 2, \ldots, n-1$ and

$$
\Gamma(nz) = \lim_{m\to\infty}\frac{(nm-1)!(nm)^{nz}}{(nz)(nz+1)(nz+2)\cdots(nz+nm-1)} \tag{12.145}
$$

to express equation (12.143) in the form

$$
\phi(z) = \frac{n^{nz}\displaystyle\prod_{r=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!\,m^{z+r/n}}{\left(z+\frac{r}{n}\right)\left(z+\frac{r}{n}+1\right)\left(z+\frac{r}{n}+2\right)\cdots\left(z+\frac{r}{n}+m-1\right)}}{n\displaystyle\lim_{m\to\infty}\frac{(nm-1)!(nm)^{nz}}{nz(nz+1)(nz+2)\cdots(nz+nm-1)}} \tag{12.146}
$$

The product nm is used in the definition of $\Gamma(nz)$ to show that the equation (12.146) simplifies after a lot of careful algebra to

$$\phi(z) = \lim_{m\to\infty} \frac{n^{nz-1}\,[(m-1)!]^n\, m^{\frac{n-1}{2}}\, m^{mn}}{(nm-1)!\,(nm)^{nz}} = \lim_{m\to\infty} \frac{[(m-1)!]^n\, m^{\frac{n-1}{2}}\, m^{mn-1}}{(nm-1)!} \tag{12.147}$$

The equation (12.147) shows that $\phi(z)$ is independent of z and is a constant. To find the value of the constant, select a value of z where equation (12.143) can be evaluated. Selecting the value $z = 1/n$ one finds after simplification the product formula first derived by Gauss[5] and Legendre[6]

$$n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right) = n^{1/2}\,(2\pi)^{\frac{n-1}{2}}\,\Gamma(nz) \tag{12.148}$$

Using equation (12.148) one can produce the special cases

$$\begin{aligned} n=2 \qquad\qquad & \Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \Gamma(2z)\,(2\pi)^{1/2}\,2^{1/2-2z} \\ n=3 \qquad\qquad & \Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right) = \Gamma(3z)(2\pi)\,3^{1/2-3z} \end{aligned} \tag{12.149}$$

Example 12-8. (Summation)

The function $\Psi(x)$ defined by $\Psi(x) = \dfrac{d}{dx}\ln\Gamma(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ occurs in representing the summation of many finite and infinite convergent series. The Gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x)$$

so that

$$\ln\Gamma(x+1) = \ln x + \ln\Gamma(x) \tag{12.150}$$

Differentiate equation (12.150) and show

$$\Psi(x+1) = \frac{1}{x} + \Psi(x) \qquad \text{or} \qquad \Psi(x+1) - \Psi(x) = \frac{1}{x} \tag{12.151}$$

This demonstrates that $\Psi(x)$ satisfies the difference equation

$$\Delta\Psi(x) = \frac{1}{x} \qquad \text{or} \qquad \Delta\Psi(a+n) = \frac{1}{a+n}, \quad \text{a is constant} \tag{12.152}$$

---

[5] Carl Friedrich Gauss (1777-1855) A famous German mathematician.
[6] Adrien-Marie Legendre (1752-1833) A famous French mathematician.

Using the results from pages 361-362, one can write

$$\sum_{i=1}^{n} \Delta\Psi(a+i) = \sum_{i=1}^{n} \frac{1}{a+i} = \Psi(a+n+1) - \Psi(a+1) \tag{12.153}$$

As an example of how equation (12.153) can be employed, examine the finite sum

$$S = \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+nb} \tag{12.154}$$

This finite series can be expressed

$$S = \frac{1}{b}\sum_{i=1}^{n} \frac{1}{\frac{a}{b}+i} = \frac{1}{b}\sum_{i=1}^{n} \Delta\Psi(\frac{a}{b}+i) = \frac{1}{b}\left[\Psi(\frac{a}{b}+n+1) - \Psi(\frac{a}{b}+1)\right] \tag{12.155}$$

■

Use differential equations to find series

Another way to find the series representation of a given function is to first differentiate the function and then form a differential equation satisfied by the given function. One can then substitute a power series into the differential equation and determine the coefficients of the power series by comparing like terms as illustrated in the following examples.

Example 12-9. Determination of series

Find a power series expansion to represent the function $y = a^x$, where a is a constant.

Solution Write $y = y(x) = a^x = e^{x \ln a}$ and differentiate this function to obtain $\frac{dy}{dx} = e^{x \ln a} \ln a = a^x \ln a$, so that $y = a^x$ is a solution of the differential equation

$$\frac{dy}{dx} = y \ln a \tag{12.156}$$

satisfying the initial condition at $x = 0$, $y(0) = a^0 = 1$.

Assume that y has the power series representation

$$y = y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \tag{12.157}$$

with derivative

$$\frac{dy}{dx} = y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots \tag{12.158}$$

where $c_0, c_1, c_2, c_3, \ldots$ are constants to be determined. Substitute the representations (12.157) and (12.158) into the differential equation (12.156) to obtain

$$c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} + \cdots = \ln a \left[c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots\right] \tag{12.159}$$

In equation (12.159) **equate the coefficients of like powers of** x and show

$$\begin{aligned} c_1 =& c_0 \ \ln a \\ 2c_2 =& c_1 \ \ln a \\ 3c_3 =& c_2 \ \ln a \\ \vdots \quad & \quad \vdots \\ (n+1)c_{n+1} =& c_n \ \ln a \\ \vdots \quad & \quad \vdots \end{aligned} \tag{12.160}$$

The general equation

$$(n+1)c_{n+1} = c_n \ \ln a \qquad \text{or} \qquad c_{n+1} = \frac{c_n \ \ln a}{(n+1)} \tag{12.161}$$

which holds for $n = 0, 1, 2, 3, \ldots$ is called **a recurrence relation** or **recurrence formula** associated with the given series and tells one how to select the coefficients in order to satisfy the differential equation. Recall that $c_0 = y(0) = 1$ is determined from the initial value $x = 0$. Using the recurrence formula (12.161) and the equations (12.160) one finds

$$\begin{aligned} n =& 0 & c_1 =& \ln a \\ n =& 1 & c_2 =& \frac{1}{2!}(\ln a)^2 \\ n =& 2 & c_3 =& \frac{1}{3!}(\ln a)^3 \\ \vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots \\ n =& m & c_m =& \frac{1}{m!}(\ln a)^m \end{aligned} \tag{12.162}$$

and consequently the power series expansion for $y = a^x$ is given by

$$y = a^x = 1 + x \ \ln a + \frac{x^2}{2!} (\ln a)^2 + \frac{x^3}{3!} (\ln a)^3 + \cdots + \frac{x^m}{m!} (\ln a)^m + \cdots \tag{12.163}$$

■

Example 12-10. **Determination of series**

Find a power series expansion to represent the function $y = y(x) = (h+x)^n$, where h is a constant.

Solution Differentiate the function $y = y(x) = (h+x)^n$ and show

$$\frac{dy}{dx} = y'(x) = n(h+x)^{n-1} \tag{12.164}$$

Multiply equation (12.164) by $(h+x)$ and show y is a solution of the differential equation

$$(h+x)\frac{dy}{dx} = n\, y \tag{12.165}$$

with initial condition at $x = 0$ given by $y(0) = h^n$. Assume $y = (h+x)^n$ has the power series representation

$$y = (h+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_m x^m + \cdots \tag{12.166}$$

with derivative

$$\frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + m c_m x^m + \cdots \tag{12.167}$$

Note that the index m has been selected for the general term of the series as the value n occurs in the differential equations and we don't want these values to become confused with one another. Substitute the power series (12.166) and (12.167) into the differential equation (12.165) to obtain

$$(h+x)\left[c_1 + 2c_2 x + 3c_3 x^2 + \cdots + m c_m x^{m-1} + \cdots\right] = n[c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots] \tag{12.168}$$

Expand the lefthand side of equation (12.168) and show

$$\begin{aligned} hc_1 + &2hc_2 x + 3hc_3 x^2 + \cdots + hm c_m x^{m-1} + \cdots \\ &+ c_1 x + 2c_2 x^3 + 3c_3 x^3 + \cdots + m c_m x^m + \cdots \\ &= \; n\, c_0 + n\, c_1 x + n\, c_2 x^2 + n\, c_3 x^3 + \cdots + n\, c_m x^m + \cdots \end{aligned} \tag{12.169}$$

In equation (12.169) **equate the coefficients of like powers of** x to obtain a **recurrence relation** or **recurrence formula**. One finds

$$\begin{aligned} hc_1 =& n\, c_0 \\ (2hc_2 + c_1) =& n\, c_1 \\ (3hc_2 + 2c_2) =& n\, c_2 \\ \vdots \quad & \quad \vdots \\ [(m+1)hc_{m+1} + nc_m] =& n\, c_m \end{aligned} \tag{12.170}$$

Here the recurrence formula is

$$(m+1)h\,c_{m+1} + m\,c_m = n\,c_m \qquad \text{or} \qquad c_{m+1} = \frac{(n-m)}{(m+1)h}\,c_m \tag{12.171}$$

for $m = 0, 1, 2, \ldots$. If $y = y(x) = (h+x)^n$, then $y(0) = c_0 = h^n$. Substitute the values $m = 0, 1, 2, 3, \ldots, n, n+1, \ldots$ into the recurrence formula (12.171) to obtain

$$\begin{aligned}
m =&0 & c_1 =&\frac{n}{h}\,c_0 = nh^{n-1} = \binom{n}{1}\,h^{n-1}\\
m =&1 & c_2 =&\frac{(n-1)}{2h}\,c_1 = \frac{n(n-1)}{2!}h^{n-2} = \binom{n}{2}h^{n-2}\\
m =&2 & c_3 =&\frac{(n-2)}{3h}\,c_2 = \frac{n(n-1)(n-2)}{3!}h^{n-3} = \binom{n}{3}h^{n-3}\\
\vdots\ & \ \vdots & \vdots\ & \ \vdots\\
m =&n-1 & c_n =&\frac{1}{nh}\,c_{n-1} = \frac{n!}{n!}h^0 = 1 = \binom{n}{n}h^0\\
m =&n & c_{n+1} =&0\\
m =&n+1 & c_{n+2} =&0
\end{aligned} \tag{12.172}$$

and $c_n = 0$ for all integer values of m satisfying $m \ge n$. In the equations (12.172) the terms

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!}, & m \le n \\ 0, & m > n \end{cases} \tag{12.173}$$

are the binomial coefficients. Substituting the values given by equations (12.172) into the power series (12.166) one obtains the finite series of terms

$$y = (h+x)^n = \binom{n}{0}\,h^n + \binom{n}{1}\,h^{n-1}x + \binom{n}{2}\,h^{n-2}x^2 + \cdots + \binom{n}{n-1}\,hx^{n-1} + \binom{n}{n}\,x^n \tag{12.175}$$

which is the well-known binomial expansion.

■

The Laplace Transform

Consider the mathematical operator box labeled $\mathcal{L}\{f(t)\}$ or $\mathcal{L}\{\quad; t \to s\}$ as illustrated in the figure 12-8. This mathematical operator $\mathcal{L}$ is called **the Laplace transform operator and is defined**

$$\begin{aligned}
&\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)\\
\text{or} \qquad &\mathcal{L}\{f(t);\ t \to s\} = \lim_{T\to\infty}\int_0^T f(t)e^{-st}\,dt = F(s), \qquad s > 0
\end{aligned} \tag{12.175}$$

and represents a transformation from a function $f(t)$ in the t-domain to a function $F(s)$ in the s-domain (frequently called the frequency domain). The Laplace transform[7] has many applications in mathematics, statistics, physics and engineering.

[7] This operator is named after Pierre Simon Laplace (1749-1857) A famous French mathematician.

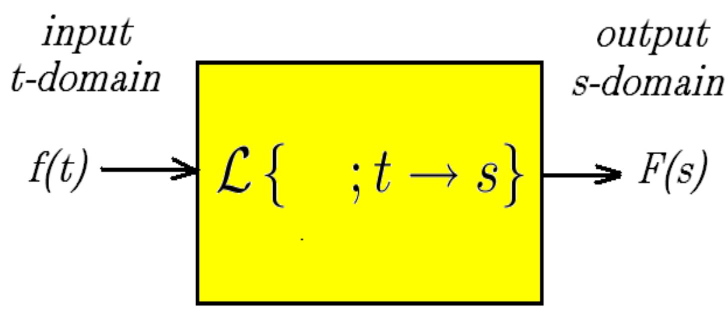


Figure 12-8. **The Laplace transform operator.**

In the defining equation (12.175) the parameter s is selected such that the integral exists and many times it is expressed as a complex variable $s = \sigma + i\,\omega$ where σ and ω are real and i is an imaginary component satisfying $i^2 = -1$. The Laplace transform can be studied with or without employing knowledge of complex variables.

Example 12-11. (Laplace transform)

Find the Laplace transform of $\sin(\alpha\,t)$ where α is a nonzero constant.

Solution

By definition $\mathcal{L}\{\sin(\alpha\,t);\ t \to s\} = \int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt$

Integrate by parts with $u = \sin(\alpha\,t)$ and $dv = e^{-st}\,dt$ to obtain

$$I = \int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt = -\sin(\alpha\,t)\,\frac{e^{-st}}{s}\Big|_0^\infty + \frac{\alpha}{s}\int_0^\infty \cos(\alpha\,t)\,e^{-st}\,dt$$

Integrate by parts again with $u = \cos(\alpha\,t)$ and $dv = e^{-st}\,dt$ and show

$$I = \int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt = \frac{\alpha}{s}\left[\cos(\alpha\,t)\,\frac{e^{-st}}{-s}\Big|_0^\infty - \frac{\alpha}{s}\,I\right]$$

This last equation simplifies to

$$(1 + \frac{\alpha^2}{s^2})I = \frac{\alpha}{s^2} \qquad \text{or} \qquad I = \int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt = \mathcal{L}\{\sin(\alpha\,t); t \to s\} = \frac{\alpha}{s^2 + \alpha^2}$$

■

Example 12-12. **Laplace transform**

Find the Laplace transform of $e^{\alpha t}$

Solution

By definition $\mathcal{L}\left\{e^{\alpha t}\ ;\ t \to s\right\} = \int_0^\infty e^{\alpha t} e^{-st}\, dt = \int_0^\infty e^{-(s-\alpha)t}\, dt$ Scale the integral to obtain

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha}\int_0^\infty e^{-u}\, du, \qquad u = (s-\alpha)\, t$$

to obtain

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha}\left[-e^{-u}\right]_0^\infty = \frac{1}{s-\alpha}\lim_{T\to\infty}\left[-e^{-T} - (-e^0)\right]$$

which produces the result

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha} \qquad \text{provided that } s > \alpha$$

■

Using various integration techniques one can verify the following short table of Laplace transforms.

Short Table of Laplace Transforms	
Function $f(t)$	**Laplace Transform $\mathcal{L}\{f(t)\} = F(s)$**
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$
t^{n-1}	$\frac{(n-1)!}{s^n} \qquad n = 1, 2, 3, \ldots$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$t\, e^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$
$t^{n-1}\, e^{\alpha t}$	$\frac{(n-1)!}{(s-\alpha)^n} \qquad n = 1, 2, 3, \ldots$
$t^{k-1}\, e^{\alpha t}$	$\frac{\Gamma(k)}{(s-\alpha)^k}, \qquad k > 0$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$\sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}$
$\cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}$

Inverse Laplace Transformation $\mathcal{L}^{-1}$

The symbol $\mathcal{L}^{-1}$ is used to denote the inverse Laplace transform operator with the property that $\mathcal{L}^{-1}$ undoes what $\mathcal{L}$ does. That is, if the inverse Laplace transform operator is applied to both sides of the equation $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}^{-1}\mathcal{L}\{f(t)\} = f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{12.176}$$

This indicates that the table of Laplace transforms given above is to be interpreted in either of two ways. Reading the above table left to right indicates $\mathcal{L}\{f(t)\} = F(s)$ and reading the table from right to left indicates $f(t) = \mathcal{L}^{-1}\{F(s)\}$. In general, one can say

$$\mathcal{L}\{f(t)\} = F(s) \quad \text{if and only if} \quad f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{12.177}$$

One use of the Laplace transform operator is to take a difficult problem in the t−domain and transform it to an easier problem in the s−domain. Solve the easier problem in the s−domain and convert the answer back to the t−domain. There are two ways by which one can convert a Laplace transform back to the t−domain. One conversion method is to have an extensive table of Laplace transforms so that one can use table lookup to convert a function $F(s)$ back to the correct function $f(t)$ by using the property (12.176) or (12.177). Table lookup is the preferred method for now.

Another method used to find the inverse Laplace transform is more advanced and requires knowledge of complex variable theory. This more advanced method is expressed in the language of complex variables as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s);\, s \to t\ \} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds$$

where the integration is part of a line integral in the complex plane. Once you learn all the theory involved, the inverse Laplace transform is really simple to use. The difficulty in using the complex form for the inverse transform is that you will need to take a complete course in complex variable theory to use and understand how to employ it. The inverse Laplace transformation techniques often used are illustrated in the figure 12-9.

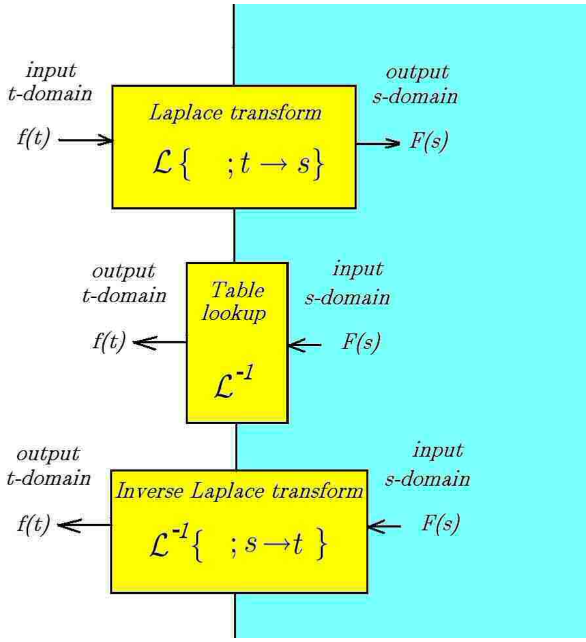


Figure 12-9. Inverting the Laplace transform.

Properties of the Laplace transform

Using the definition of the Laplace transform and applying various integration techniques one can develop a table of Laplace transform properties.

Example 12-13. (Laplace transform of derivative)

Show that if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^+) \text{ or } f'(t) = \mathcal{L}^{-1}\left\{sF(s) - f(0^+)\right\}$$

Solution

By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrate by parts with $u = e^{-st}$ and $dv = f'(t)\, dt$ to obtain

$$\mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t) e^{-st}\, dt = sF(s) - f(0^+)$$

Note that $f(t)$ need only be defined for $t \geq 0$ so that the 0^+ is to remind you that the function $f(t)$ evaluated at zero is just the right-hand limit as $t \to 0$.

■

Example 12-14. First shift property

If $F(s) = \mathcal{L}\{f(t)\}$, show that $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$ or $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$

Solution

By definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

so that if one replaces s by $s-a$ there results

$$F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\,dt = \int_0^\infty e^{at}f(t)\,e^{-st}\,dt = \mathcal{L}\left\{e^{at}f(t)\right\}$$

or

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

■

Example 12-15. (Second shift property)

If $F(s) = \mathcal{L}\{f(t)\}$, show $\mathcal{L}\{f(t-\alpha)H(t-\alpha)\} = e^{-\alpha s}F(s)$ or

$$f(t-\alpha)H(t-\alpha) = \mathcal{L}^{-1}\left\{e^{-\alpha s}F(s)\right\}$$

where $H(t)$ is the Heaviside step function defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Solution

By definition

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

so that replacing t by u and then multiplying by $e^{-\alpha s}$ one finds

$$e^{-\alpha s}F(s) = \int_0^\infty f(u)e^{-su}e^{-\alpha s}\,du = \int_0^\infty f(u)e^{-s(u+\alpha)}\,du$$

Make the change of variables $t = u+\alpha$ with $dt = du$ and then calculate the appropriate limits of integration to show

$$e^{-\alpha s}F(s) = \int_{t=\alpha}^\infty f(t-\alpha)e^{-st}\,dt = \int_0^a (0)\cdot f(t-\alpha)e^{-st}\,dt + \int_\alpha^\infty (1)\cdot f(t-\alpha)\,dt$$

This last integral has the more compact form

$$e^{-\alpha s}F(s) = \int_0^\infty f(t-\alpha)H(t-\alpha)e^{-st}\,dt = \mathcal{L}\{f(t-\alpha)H(t-\alpha)\}$$

or

$$\mathcal{L}^{-1}\left\{e^{-\alpha s}F(s)\right\} = f(t-\alpha)H(t-\alpha)$$

■

As an integration exercise one can verify the various transform properties listed on the next page.

Properties of the Laplace Transform		
Function $f(t)$	**Laplace Transform $F(s)$**	**Comment**
$c_1 f(t)$	$c_1 F(s)$	linearity property
$f'(t)$	$sF(s) - f(0^+)$	Derivative property
$f''(t)$	$s^2F(s) - sf(0^+) - f'(0+)$	Derivative property
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$	Derivative property
$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}\,F(s) \quad s>0$	Integration transform
$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$	linearity property
$tf(t)$	$-F'(s)$	multiplication by t property
$t^2 f(t)$	$(-1)^2 F''(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$e^{\alpha t} f(t)$	$F(s-\alpha)$	First shift property
$\frac{1}{t}\,f(t)$	$\int_s^\infty F(s)\,ds$	Division by t property
$f(t-\alpha)H(t-\alpha)$	$e^{-\alpha s}F(s)$	Second shift property
$\frac{1}{\alpha}\,f(\frac{t}{\alpha})$	$F(\alpha s)$	scaling property
$\frac{1}{\alpha}\,e^{\beta t/\alpha} f(\frac{t}{\alpha})$	$f(\alpha s - \beta)$	shifting scaling
$f(t+p) = f(t)$	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$	periodic property
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Convolution property

Note the following repetitive properties exhibited by the above table.

(i) The derivative property, when expressed in words states, **Differentiation of function in the t-domain is represented in the s-domain by multiplication of the transform function by s and subtracting the initial value of the function differentiated.** Note that the second derivative and higher derivatives follow this rule. For example, if $f'(t)$ and $f''(t)$ are the functions differentiated, then

$$\mathcal{L}\left\{f''(t)\right\} = \underbrace{s[sF(s) - f(0^+)] - f'(0^+)}_{\substack{\text{s times transform of}\\ \text{function differentiated}\\ \text{minus initial value}\\ \text{of function differentiated}}}$$

$$\mathcal{L}\left\{f'''(t)\right\} = \underbrace{s[s^2F(s) - sf(0^+) - f'(0^+)] - f''(0^+)}_{\substack{\text{s times transform of}\\ \text{function differentiated}\\ \text{minus initial value}\\ \text{of function differentiated}}}$$

(ii) Multiplication by t in the t-domain corresponds to a differentiation in the s-domain multiplied by a -1.

(iii) Division by t in the t-domain corresponds to an integration from s to ∞ in the s-domain.

Example 12-16. Laplace transform

Use Laplace transform techniques to solve the differential equation $\dfrac{dy}{dt} = \alpha y$ with initial condition $y(0) = 1$, where α is a known constant.

Solution

Here $y = y(t)$ is a function of time t and taking the Laplace transform of both sides of the given differential equation produces

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\left\{\alpha\, y\right\}$$

Let $Y(s) = \mathcal{L}\left\{y(t)\right\}$ denote the transform in the s-domain and make note that

$$\mathcal{L}\left\{y'(t)\right\} = \mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) \quad \text{and} \quad \mathcal{L}\left\{\alpha\, y(t)\right\} = \alpha\, \mathcal{L}\left\{y(t)\right\} = \alpha\, Y(s)$$

so that the differential equation in the t-domain becomes **an algebraic equation** in the s-domain. The resulting algebraic equation is

$$sY(s) - 1 = \alpha Y(s)$$

One can now solve this algebraic equation for the transform function $Y(s)$ to obtain

$$Y(s) = \frac{1}{s-\alpha}$$

Using table lookup one finds the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

One can verify the correctness of the solution by showing the function $y(t) = e^{\alpha t}$ satisfies the given differential equation and given initial condition.

Warning—The Laplace transform technique for solving differential equations only works on linear differential equations. It is not applicable in dealing with nonlinear differential equations. Also note that when dealing with more difficult linear equations one needs to develop more advanced methods for obtaining an inverse Laplace transform.

■

Introduction to Complex Variable Theory

Consider the figure 12-10, where S represents a nonempty set of points in the $z = x + i\,y$ complex z-plane, where i is an imaginary unit with the property that $i^2 = -1$. If there exists a rule f which assigns to each value $z = x + i\,y$ belonging to S, one and only one complex number $\omega = u + i\,v$, then the correspondence is called a function or mapping of the point z to the point ω and this correspondence is denoted using the notation

$$\omega = f(z) = f(x + i\,y) = u + i\,v = u(x,y) + i\,v(x,y)$$

Here $\omega = u + i\,v$ is the image point of $z = x + i\,y$ and is represented in a plane called the $\omega-$plane. Functions of a complex variable $\omega = f(z)$ are represented as mappings from the $z-$plane to the $\omega-$plane as illustrated in the figure 12-10. Note that if S denotes a region in the $z-$plane and $f(z)$ is a single-valued, then the image of S under the mapping $\omega = f(z)$ is the region S' in the $\omega-$plane. The boundary curve C of S in the $z-$plane has the image curve C' in the $\omega-$plane.

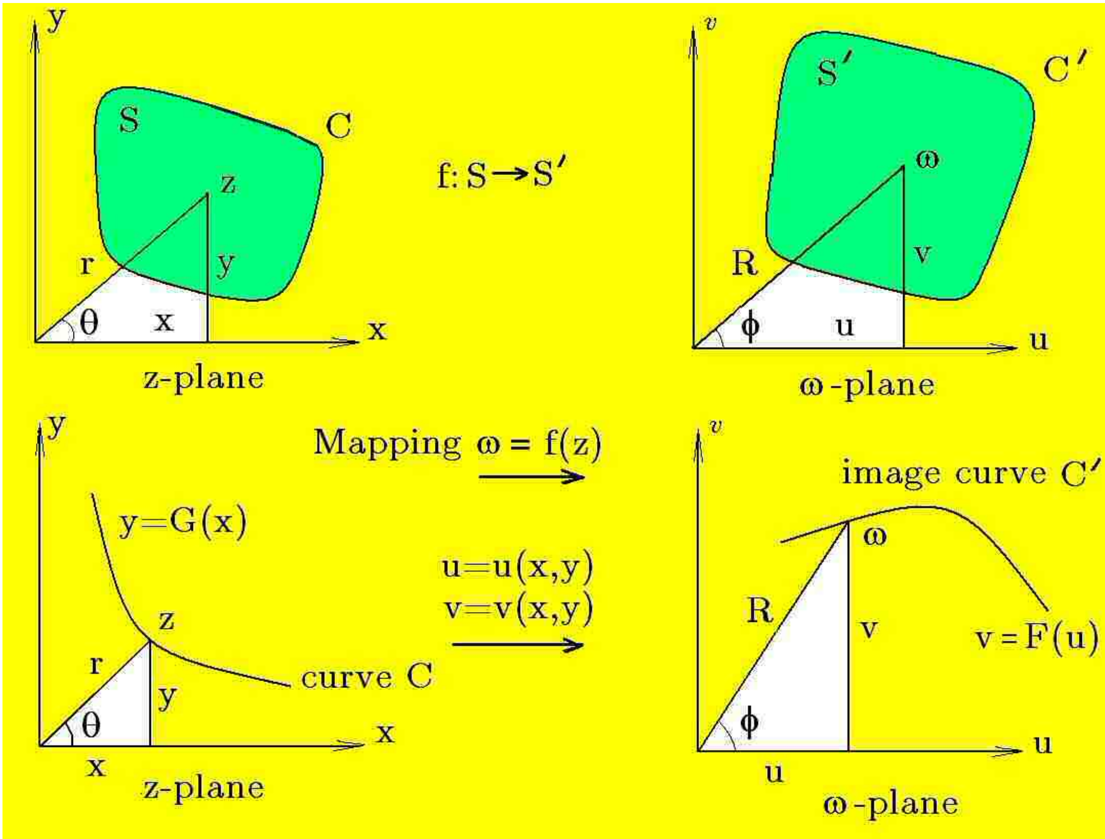


Figure 12-10. **Representing function of complex variable as mapping.**

Polar coordinates (r, θ) in the $z-$plane correspond to polar coordinates (R, ϕ) in the $\omega-$plane, where $x = r\cos\theta,\ y = r\sin\theta$ in the $z-$plane and $u = R\cos\phi,\ v = R\sin\phi$ in the $\omega-$plane. A curve $y = G(x)$ in the $z-$plane has the image curve with parametric form

$$u = u(x, G(x)), \quad v = v(x, G(x))$$

in the $\omega-$plane, which produces the image curve $v = F(u)$.

Functions of a complex variable $\omega = f(z)$ represent a mapping from the $z-$plane to the $\omega-$plane. One usually selects special regions S and curves C to illustrate the mappings. For example, circles, squares, triangles, etc.

Example 12-17. **(Representing function of a complex variable as mapping)**

Consider the complex function

$$\omega = f(z) = z^2 = (x + i\,y)^2 = x^2 + 2ixy + i^2y^2 = (x^2 - y^2) + i\,(2xy) = u + i\,v$$

The complex function $\omega = f(z) = z^2$ defines the mapping

$$u = u(x, y) = x^2 - y^2 \quad \text{and} \quad v = v(x, y) = 2xy$$

In polar coordinates with $z = re^{i\theta}$, the image point is $\omega = f(z) = z^2 = r^2e^{i2\theta} = Re^{i\phi}$ so that the mapping in polar form is given by

$$R = r^2, \quad \text{and} \quad \phi = 2\theta$$

Knowing the transformation equations one can then construct special figures to give various interpretations of this mapping. The figure 12-11 illustrates four selected interpretations to illustrate the mapping $\omega = f(z) = z^2$.

The first mapping illustrates two circles with radius r_1 and r_2 and their image circles r_1^2 and r_2^2. The rays $\theta = \alpha$ and $\theta = \beta$ map to the image rays $\phi = 2\alpha$ and $\phi = 2\beta$. The green region S maps to the green region S'.

The second mapping illustrates the hyperpola $x^2 - y^2 = u$ and $2xy = v$ for the selected values of u and v given by u_0, u_1 and v_0, v_1. These hyperbola map to straight lines in the $\omega-$plane.

The third mapping illustrates the line $x = 0$ mapping to the line segment $\overline{A'B'}$, where $v = 0$, with $u = -y^2$, $B < y < A$. The line $y = 0$ maps to the line segment $\overline{B'C'}$,where $v = 0$ and $u = x^2$,for $0 < x < x_1$. The line $x = x_1$, $C < y < D$ maps to the parabola with parametric equations $u = x_1^2 - y^2$, $v = 2x_1y$, $C < y < D$.

The fourth mapping illustrates the hyperbola $x^2 - y^2 = u$, for $u = 0, 2, 4, 6$ mapping to the lines $u = 0, 2, 4, 6$ and the hyperbola $2xy = v$, for $v = 2, 4, 6$ are mapped to the lines $v = 2, 4, 6$.

It is left as an exercise for you to make up additional figures to illustrate the mapping $\omega = f(z) = z^2$.

■

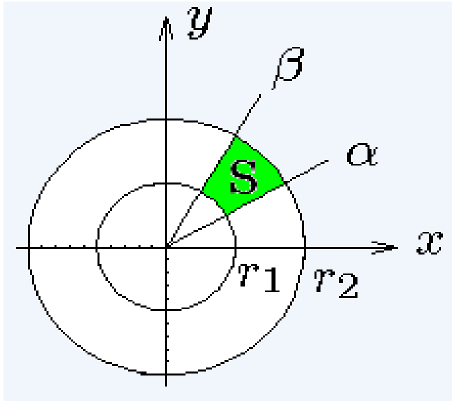
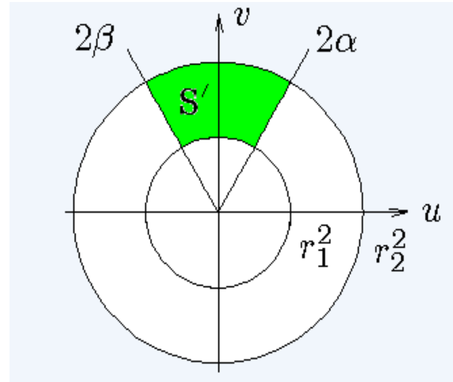
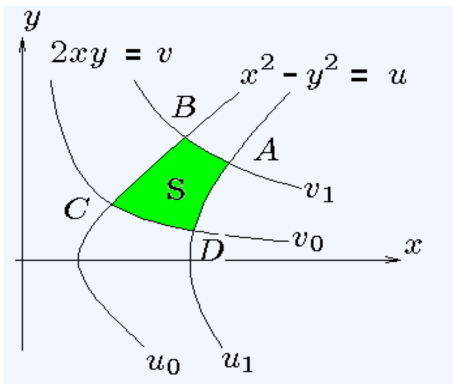
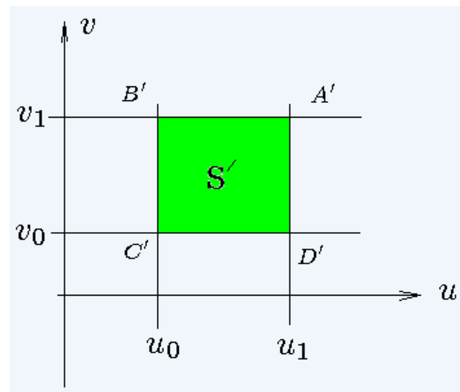
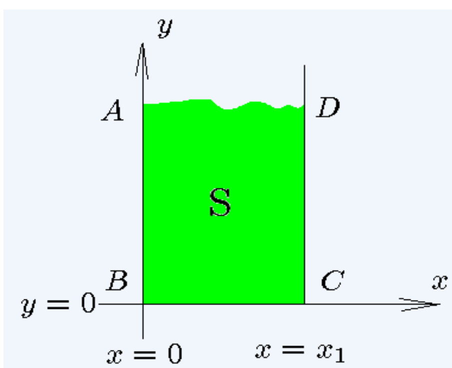
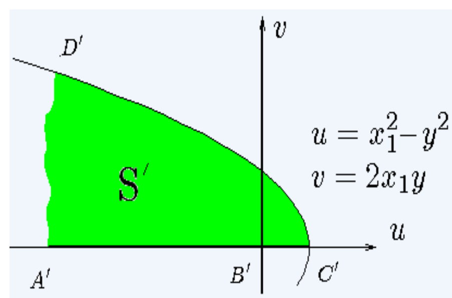
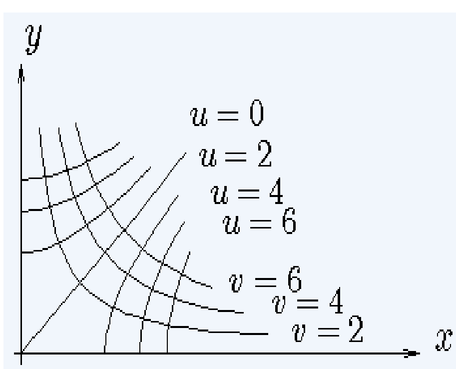
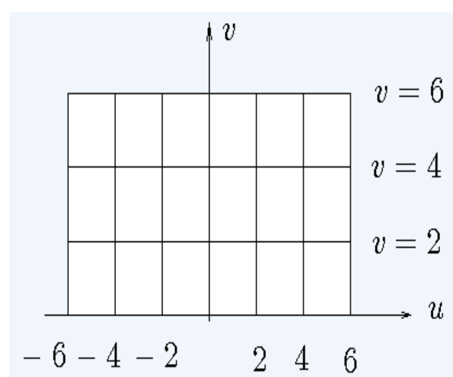
<table>
<tr><th></th><th>z-plane</th><th>Mapping</th><th>ω-plane</th></tr>
<tr><td>1.</td><td></td><td>$\omega = z^2$</td><td></td></tr>
<tr><td>2.</td><td></td><td>$\omega = z^2$</td><td></td></tr>
<tr><td>3.</td><td></td><td>$\omega = z^2$</td><td></td></tr>
<tr><td>4.</td><td></td><td>$\omega = z^2$</td><td></td></tr>
</table>

Figure 12-11. Selected images illustrating the mapping $\omega = f(z) = z^2$

Derivative of a Complex Function

The derivative of a complex function $\omega = f(z) = u(x,y) + i\,v(x,y)$, where $z = x + i\,y$, is defined in the exact same way as that of a real function $y = f(x)$. That is,

$$\begin{aligned} \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta z, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - [u(x,y) + i\,v(x,y)]}{\Delta x + i\,\Delta y} \end{aligned} \tag{12.178}$$

if this limit exists. In the definition of a derivative, equation (12.178), **the limit must be independent of the path taken as Δz tends toward zero.**

Consider the points $z = x + i\,y$ and $z + \Delta z = (x + \Delta x) + i\,(y + \Delta y)$ in the $z-$plane as illustrated in the figure 12-11.

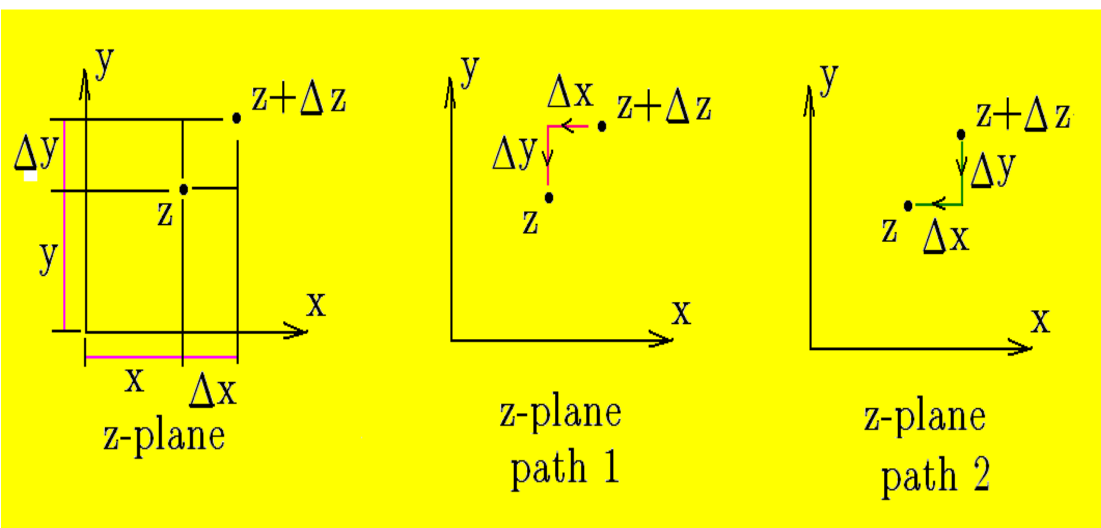


Figure 12-12. Delta z approaching zero along different paths.

If the limit in equation (12.178) approaches zero along the path 1 of figure 12-12, then the definition given by equation (12.178) becomes, after first setting $\Delta x = 0$

$$\begin{aligned} \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - [u(x,y) + i\,v(x,y)]}{i\,\Delta y} \\ \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \\ \frac{d\omega}{dz} &= f'(z) = \; -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{See page 158, Volume I} \end{aligned} \tag{12.179}$$

If the limit in equation (12.178) approaches zero along the path 2 of figure 12-12, then the definition given by equation (12.178) becomes, after first setting $\Delta y = 0$

$$
\begin{aligned}
\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\, v(x+\Delta x, y) - [u(x,y) + i\, v(x,y)]}{\Delta x} \\
\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \\
\frac{d\omega}{dz} &= f'(z) = \frac{\partial u}{\partial x} + i\, \frac{\partial v}{\partial x} \qquad \text{See page 158, Volume I}
\end{aligned}
\tag{12.180}
$$

If the derivative of the complex function is to exist with the limit of equation (12.178) being independent of how Δz approaches zero, then it is necessary that the derivative results from the equations (12.179) and (12.180) must equal one another or

$$
\frac{d\omega}{dz} = f'(z) = \frac{\partial u}{\partial z} + i\, \frac{\partial v}{\partial x} = -i\, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\tag{12.181}
$$

Equating the real and imaginary parts in equation (12.181) one finds that a necessary condition for the existence of a complex derivative associated with the complex function given by $\omega = f(z) = u(x,y) + i\, v(x,y)$ is that the following equations must be satisfied simultaneously

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\tag{12.182}
$$

These simultaneous conditions are known as the **Cauchy-Riemann equations** for the existence of a derivative associated with a function of a complex variable.

A function $\omega = f(z) = u(x,y) + i\, v(x,y)$ which is both single-valued and differential at a point z_0 and all points in some small region around the point z_0, is said to be **analytic or regular at the point z_0.**

Integration of a Complex Function

An introductory calculus course introduces the concepts of differentiation and integration associated with functions of a real variable. Indefinite integration of a real function was defined as the inverse operation of differentiation of the real function. A definite integral was defined as the limit of a summation process representing area under a curve. In complex variable theory one can have indefinite and definite integrals but their physical interpretation is not quite the same as when dealing with real quantities. In addition to indefinite and definite integrations one must know properties of contour integrals.

Contour integration

Let C denote a curve in the z-plane connecting two points $z = a$ and $z = b$ as illustrated in figure 12-13.

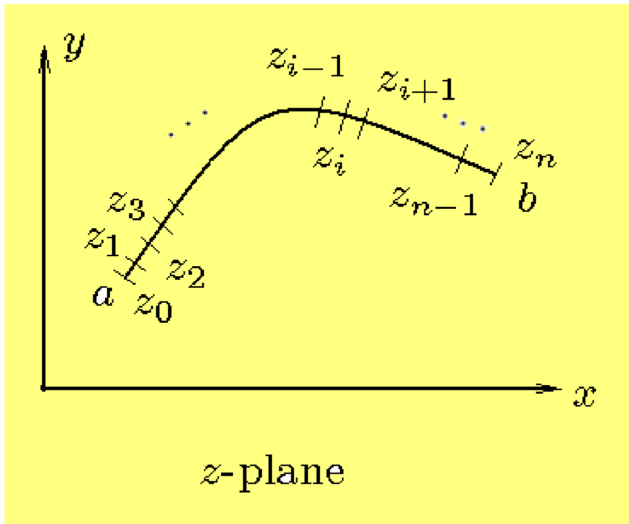


Figure 12-13. Curve in the z-plane.

The curve C is assumed to be a smooth curve represented by a set of parametric equations

$$x = x(t), \quad y = y(t), \quad t_a \le t \le t_b$$

The equation $z = z(t) = x(t) + iy(t)$, for $t_a \le t \le t_b$ represents points on the curve C with the end points given by

$$z(t_a) = x(t_a) + iy(t_a) = a \quad \text{and} \quad z(t_b) = x(t_b) + iy(t_b) = b.$$

Divide the interval (t_a, t_b) into n parts by defining a step size $h = \dfrac{t_b - t_a}{n}$ and letting $t_0 = t_a,\ t_1 = t_a + h,\ t_2 = t_a + 2h, \ldots, t_n = t_a + nh = t_a + n\dfrac{(t_b - t_a)}{n} = t_b$. Each of the values $t_i,\ i = 0, 1, 2, \ldots, n$, gives a point $z_i = z(t_i)$ on the curve C. For $f(z)$, a continuous function at all points z on the curve C, let $\Delta z_i = z_{i+1} - z_i$ and form the sum

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\Delta z_i = \sum_{i=0}^{n-1} f(\xi_i)(z_{i+1} - z_i), \tag{12.183}$$

where ξ_i is an arbitrary point on the curve C between the points z_i and z_{i+1}. Now let n increase without bound, while $|\Delta z_i|$ approaches zero. The limit of the summation in equation (12.183) is called **the complex line integral of $f(z)$ along the curve C** and is denoted

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\xi_i)\Delta z_i. \tag{12.184}$$

If $f(z) = u(x,y) + iv(x,y)$ is a function of a complex variable, then we can express the complex line integral of $f(z)$ along a curve C in the form of **a real line integral** by writing

$$\begin{aligned}\int_C f(z)\,dz =& \int_C [u(x,y) + iv(x,y)]\,(dx + i\,dy)\\ =& \int_C [u(x,y)\,dx - v(x,y)\,dy] + i \int_C [v(x,y)\,dx + u(x,y)\,dy]\end{aligned} \tag{12.185}$$

where $x = x(t),\ y = y(t),\ dx = x'(t)\,dt$ and $dy = y'(t)\,dt$ are substituted for the $x,\ y,\ dx$ and dy values and the limits of integration on the parameter t go from t_a to t_b. This gives the integral

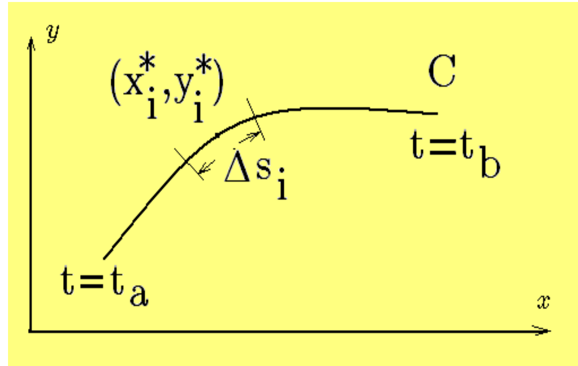
$$\int_C f(z)\,dz = \int_{t_a}^{t_b} \Big[u(x(t),y(t))\,x'(t) - v(x(t),y(t))y'(t)\Big]\,dt + i\int_{t_a}^{t_b} \Big[v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t)\Big]\,dt$$

Now both the real part and imaginary parts are evaluated just like the real integrals you studied in calculus of real variables.

If the **parametric equations defining the curve C are not given, then you must construct the parametric equations defining the contour C over which the integration occurs.** Complex line integrals along a curve C involve a summation process where values of the function being integrated must be known on a specified path C connecting points a and b. In special cases the value of the complex integral is very much dependent upon the path of integration while in other special circumstances the value of the line integral is independent of the path of integration joining the end points. In some special circumstances the path of integration C can be continuously deformed into other paths C^* without changing the value of the complex integral. If you take a course in complex variable theory you will be introduced to various theorems and properties associated with integration involving analytic functions $f(z)$ which are well defined over specific regions of the z-plane.

The integration of a function along a curve is called a line integral. A familiar line integral is the calculation of arc length between two points on a curve. Let $ds^2 = dx^2 + dy^2$ denote an element of arc length squared and let C denote a curve defined by the parametric equations $x = x(t),\ y = y(t)$, for $t_a \le t \le t_b$, then the arc length L between two points $a = [x(t_a), y(t_a)]$ and $b = [x(t_b), y(t_b)]$ on the curve is given by the integral

$$L = \int_C ds = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



If the curve C represents a wire with variable density $f(x,y)$ [gm/cm], then the total mass m of the wire between the points a and b is given by $m = \int_C f(x,y)\, ds$ which can be thought of as the limit of a summation process. If the curve C is partitioned into n pieces of lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_i, \ldots$, then in the limit as n increases without bound and Δs_i approaches zero, one can express the total mass m of the wire as the limiting process

$$m = \int_C f(x,y)\, ds = \lim_{\substack{\Delta s_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i = \int_{t_a}^{t_b} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where (x_i^*, y_i^*) is a general point on the Δs_i arc length.

Whenever the values of x and y are restricted to lie on a given curve defined by $x = x(t)$ and $y = y(t)$ for $t_a \le t \le t_b$, then integrals of the form

$$I = \int_C P(x,y)\, dx + Q(x,y)\, dy = \int_{t_a}^{t_b} P(x(t), y(t))x'(t)\, dt + Q(x(t), y(t))y'(t)\, dt \tag{12.186}$$

are **called line integrals and are defined by a limiting process such as above.** Line integrals are reduced to ordinary integrals by substituting the parametric values $x = x(t)$ and $y = y(t)$ associated with the curve C and integrating with respect to the parameter t. The above line integral is sometimes written in the form

$$\int_C f(z)\, dz = \int_{t_a}^{t_b} f(z(t))\, z'(t)\, dt \tag{12.187}$$

where $z = z(t)$ is a parametric representation of the curve C over the range $t_a \le t \le t_b$.

Whenever the curve C is not a smooth curve, but is composed of a finite number of arcs which are smooth, then the curve C is called piecewise smooth. If $C_1, C_2, \ldots, C_m$ denote the finite number of arcs over which the curve is smooth and

$C = C_1 \cup C_2 \cup \cdots \cup C_m$, then the line integral can be broken up and written as a summation of the line integrals over each section of the curve which is smooth and one would express this by writing

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_m} f(z)\,dz \tag{12.188}$$

Indefinite integration

If $F(z)$ is a function of a complex variable such that

$$\frac{dF(z)}{dz} = F'(z) = f(z)$$

then $F(z)$ is called an anti-derivative of $f(z)$ or an indefinite integral of $f(z)$. The indefinite integral is denoted using the notation $\int f(z)\,dz = F(z) + c$ where c is a constant and $F'(z) = f(z)$ Note that the addition of a constant of integration is included because the derivative of a constant is zero. Consequently, any two functions which differ by a constant will have the same derivatives.

Example 12-18. (Indefinite integration)

Let $F(z) = 3\sin z + z^3 + 5z^2 - z$ with $\frac{dF}{dz} = F'(z) = 3\cos z + 3z^2 + 10z - 1$, then one can write $\int (3\cos z + 3z^2 + 10z - 1)\,dz = 3\sin z + z^3 + 5z^2 - z + c$ where c is an arbitrary constant of integration

■

The table 12.1 gives a short table of indefinite integrals associated with selected functions of a complex variable. Note that the results are identical with those derived in a standard calculus course.

Table 12.1 Short Table of Integrals			
1.	$\int z^n\,dz = \dfrac{z^{n+1}}{n+1} + c, \quad n \neq -1$	11.	$\int \sinh z\,dz = \cosh z + c$
2.	$\int \dfrac{dz}{z} = \log z + c$	12.	$\int \cosh z\,dz = \sinh z + c$
3.	$\int e^z\,dz = e^z + c$	13.	$\int \tanh z\,dz = \log(\cosh z) + c$
4.	$\int k^z\,dz = \dfrac{k^z}{\log k} + c, \quad k$ is a constant	14.	$\int \operatorname{sech}^2 z\,dz = \tanh z + c$
5.	$\int \sin z\,dz = -\cos z + c$	15.	$\int \frac{dz}{\sqrt{z^2+\alpha^2}} = \log(z + \sqrt{z^2+\alpha^2}) + c$
6.	$\int \cos z\,dz = \sin z + c$	16.	$\int \frac{dz}{z^2+\alpha^2} = \frac{1}{\alpha}\tan^{-1}\frac{z}{\alpha} + c$
7.	$\int \tan z\,dz = \log\sec z + c = -\log\cos z + c$	17.	$\int \frac{dz}{z^2-\alpha^2} = \frac{1}{2\alpha}\log\left(\frac{z-\alpha}{z+\alpha}\right) + c$
8.	$\int \sec^2 z\,dz = \tan z + c$	18.	$\int \frac{dz}{\sqrt{\alpha^2-z^2}} = \sin^{-1}\frac{z}{\alpha} + c$
9.	$\int \sec z \tan z\,dz = \sec z + c$	19.	$\int e^{\alpha z}\sin\beta z\,dz = e^{\alpha z}\dfrac{\alpha\sin\beta z - \beta\cos\beta z}{\alpha^2+\beta^2} + c$
10.	$\int \csc z \cot z\,dz = -\csc z + c$	20.	$\int e^{\alpha z}\cos\beta z\,dz = e^{\alpha z}\dfrac{\alpha\cos\beta z + \beta\sin\beta z}{\alpha^2+\beta^2} + c$
c denotes an arbitrary constant of integration			

Definite integrals

The definite integral of a complex function $f(t) = u(t) + i\,v(t)$ which is continuous for $t_a \le t \le t_b$ has the form

$$\int_{t_a}^{t_b} f(t)\,dt = \int_{t_a}^{t_b} u(t)\,dt + i\int_{t_a}^{t_b} v(t)\,dt \tag{12.189}$$

and has the following properties.

1. The integral of a linear combination of functions is a linear combination of the integrals of the functions or

$$\int_{t_a}^{t_b} [c_1 f(t) + c_2 g(t)]\,dt = c_1\int_{t_a}^{t_b} f(t)\,dt + c_2\int_{t_a}^{t_b} g(t)\,dt$$

where c_1 and c_2 are complex constants.

2. If $f(t)$ is continuous and $t_a < t_c < t_b$, then

$$\int_{t_a}^{t_b} f(t)\,dt = \int_{t_a}^{t_c} f(t)\,dt + \int_{t_c}^{t_b} f(t)\,dt$$

3. The modulus of the integral is less than or equal to the integral of the modulus

$$\left|\int_{t_a}^{t_b} f(t)\,dt\right| \le \int_{t_a}^{t_b} |f(t)|\,dt$$

4. If $F = F(t)$ is such that $\dfrac{dF}{dt} = F'(t) = f(t)$ for $t_a \le t \le t_b$, then

$$\int_{t_a}^{t_b} f(t)\,dt = F(t)\big]_{t_a}^{t_b} = F(t_b) - F(t_a)$$

5. If $G(t)$ is defined $G(t) = \displaystyle\int_{t_a}^{t} f(t)\,dt$, then $\dfrac{dG}{dt} = G'(t) = f(t)$
6. The conjugate of the integral is equal to the integral of the conjugate

$$\overline{\int_{t_a}^{t_b} f(t)\,dt} = \int_{t_a}^{t_b} \overline{f(t)}\,dt$$

7. Let $f(t,\tau)$ denote a function of the two variables t and τ which is defined and continuous everywhere over the rectangular region $R = \{(t,\tau)\,|\, t_a \le t \le t_b,\ \tau_c \le \tau \le \tau_d\}$. If $g(\tau) = \displaystyle\int_{t_a}^{t_b} f(t,\tau)\,dt$ and the partial derivatives of f exist and are continuous on R, then

$$\frac{dg}{d\tau} = \int_{t_a}^{t_b} \frac{\partial f(t,\tau)}{\partial \tau}\,dt$$

which shows that differentiation under the integral sign is permissible.

Assume $F(z)$ is an analytic function with derivative $f(z) = \frac{dF}{dz}$ and $z = z(t)$ for $t_1 \le t \le t_2$ is a piecewise smooth arc C in a region R of the z-plane, then one can write

$$\int_C f(z)\,dz = \int_{t_1}^{t_1} \frac{dF}{dz}\,dz = F(z(t))\Big|_{t_1}^{t_2} = F(z(t_2)) - F(z(t_1))$$

This is a fundamental integration property in the z-plane. Note that if $F(z) = U + iV$ is analytic and $f(z) = u + iv$, then one can write

$$F'(z) = \frac{\partial U}{\partial x} + i\,\frac{\partial V}{\partial x} = u + iv = \frac{\partial V}{\partial y} - i\,\frac{\partial U}{\partial y}$$

and consequently,

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C (u+iv)(dx+i\,dy)\\
&= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\\
&= \int_{t_1}^{t_2} \frac{\partial U}{\partial x} x'(t)\,dt + \frac{\partial U}{\partial y} y'(t)\,dt + i\int_{t_1}^{t_2} \frac{\partial V}{\partial x} x'(t)\,dt + \frac{\partial V}{\partial y} y'(t)\,dt\\
&= \int_{z(t_1)}^{z(t_2)} dU + i\int_{z(t_1)}^{z(t_2)} dV = \int_{z(t_1)}^{z(t_2)} dF = F(z)\Big|_{z(t_1)}^{z(t_2)} = F(z(t_2)) - F(z(t_1))
\end{aligned}$$

Example 12-19. Evaluate the integral $I = \int_C (z-z_0)^n\,dz$ where n is an integer and C is the arc of the circle defined by $z = z(t) = z_0 + r\,e^{it}$ for $t_1 \le t \le t_2$ and $r > 0$ constant. For $n \ne -1$ we have

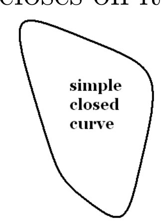
$$I = \int_{z(t_1)}^{z(t_2)} (z-z_0)^n\,dz = \frac{(z-z_0)^{n+1}}{n+1}\Big|_{z(t_1)}^{z(t_2)} \qquad n \ne -1$$

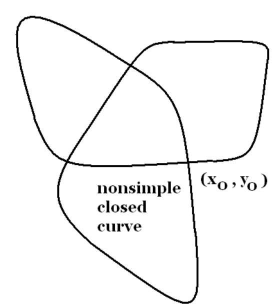
For $n = -1$, substitute $z - z_0 = r\,e^{it}$ with $dz = r\,e^{it}\,i\,dt$ to obtain

$$I = \int_C \frac{dz}{z-z_0} = \int_{t_1}^{t_2} \frac{r\,e^{it}\,i\,dt}{r\,e^{it}} = \int_{t_1}^{t_2} i\,dt = i\,(t_2 - t_1)$$

Note that a special case $z(t_1) = z(t_2)$ occurs when the arc of the circle closes on itself.

Closed curves





A plane curve C in the z-plane can be defined by a set of parametric equations $\{x(t), y(t)\}$ for the parameter t ranging over a set of values $t_1 \le t \le t_2$. A curve C is called a closed curve if the end points coincide. If C is a closed curve, then the end conditions satisfy $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. If (x_0, y_0) is a point on the curve C, which is not an end point, and there exists more than one value of the parameter t such that $\{x(t), y(t)\} = \{x_0, y_0\}$, then the point (x_0, y_0) is called a multiple point or point where the curve C crosses itself. A curve C is called a simple closed curve if the end points meet and it has no multiple points.

Whenever a curve C is a simple closed curve, the line integral of $f(z)$ around C or contour integral around the curve C is represented by an integral having one of the forms

$$\circlearrowleft\!\!\!\!\!\!\int_C f(z)\,dz. \qquad \text{or} \qquad \circlearrowright\!\!\!\!\!\!\int_C f(z)\,dz \tag{12.190}$$

The arrow on the circle indicating the direction of integration as being clockwise or counterclockwise as viewed looking down on the z-plane. A simple closed curve C is said to be traversed in the positive sense if the direction of integration is in a counterclockwise direction around the boundary and it is said to be traversed in the negative sense if the direction of integration is in the clockwise direction around the boundary. One can write $\displaystyle\oint_{C} f(z)\,dz = -\oint_{C} f(z)\,dz$. Observe that by changing the direction of integration one changes the sign of the integral.

Example 12-20. (Contour integration)

For n an integer, and z_0, ρ constants, integrate the function $f(z) = (z-z_0)^n$ around a circle of radius ρ centered at the point z_0. Perform the integration in the positive sense.

Solution: Let C denote the circle $|z-z_0| = \rho$ of radius ρ centered at the point z_0. The curve C can be represented in the parametric form

$$z = z(t) = z_0 + \rho e^{it}, \quad 0 \le t \le 2\pi \quad \text{with} \quad dz = \rho e^{it} i dt.$$

We then have

$$\oint_{C} f(z)\,dz = \oint_{C} (z-z_0)^n\,dz = \int_0^{2\pi} \rho^n e^{int} i\rho e^{it}\,dt = i\rho^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt.$$

For n different from -1 we have $\displaystyle\oint_{C} (z-z_0)^n\,dz = i\rho^{n+1}\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} = 0$. For n equal to -1 we have $\displaystyle\oint_{C} (z-z_0)^n\,dz = \oint_{C} \frac{dz}{z-z_0} = i\int_0^{2\pi} dt = 2\pi i$. Hence, the line or contour integral of the function $f(z) = (z-z_0)^n$, with n an integer, which is taken around a circle C centered at z_0 with radius ρ, can be expressed

$$\oint_{C} (z-z_0)^n\,dz = \begin{cases} 2\pi i & \text{when } n = -1 \\ 0 & \text{when } n \ne -1 \text{ and an integer.} \end{cases} \tag{12.191}$$

This result will be used quite frequently throughout the remainder of this text.

■

The Laurent series

For $z = x+iy$ a complex variable and $z_0 = x_0 + iy_0$ a fixed point in the complex z-plane, one must deal with the following quantities. (i) The magnitude of z denoted by $|z| = \sqrt{x^2+y^2}$ which represents the distance of the point z from the origin in the $z-$plane. (ii) The quantity $|z-z_0| = R$ represents a circle of radius R since

$$|z-z_0| = |x+iy-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2} = R$$
$$\text{or} \quad (x-x_0)^2+(y-y_0)^2 = R^2$$

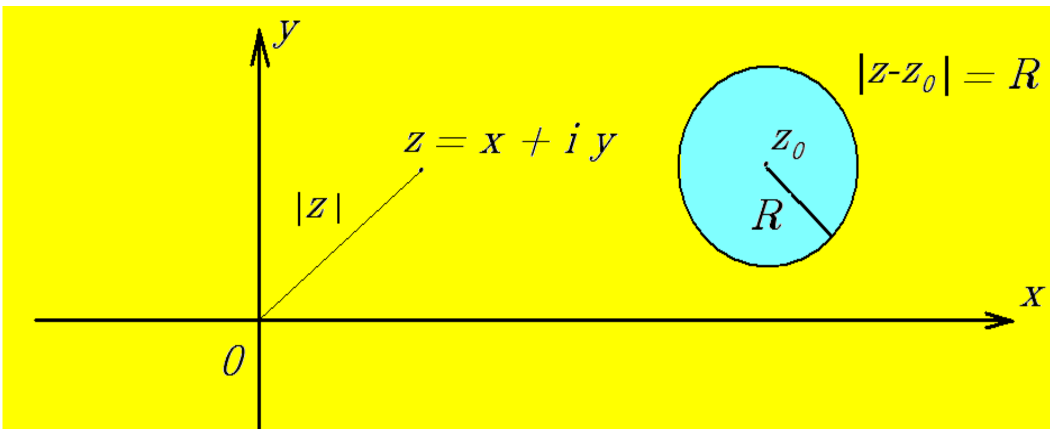


Figure 12-14. The complex $z-$plane.

In the theory of complex variables there is an important type of series called **the Laurent**[8] **series** which represents a function $f(z)$ in an expansion about a singular point z_0 having the form of a power series having both positive and negative powers of $(z-z_0)$. **The point z_0 is called the center of the Laurent series**. The Laurent series has the form

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n \tag{12.192}$$

where the quantities $c_1, c_2, \dots$ and $\alpha_0, \alpha_1, \dots$ are constants. In expanded form the Laurent series (12.192) becomes

$$f(z) = \cdots + \frac{c_3}{(z-z_0)^3} + \frac{c_2}{(z-z_0)^2} + \frac{c_1}{(z-z_0)} + \alpha_0 + \alpha_1(z-z_0) + \alpha_2(z-z_0)^2 + \cdots \tag{12.193}$$

The Laurent series converges in some annular region $R_2 < |z-z_0| < R_1$. It can be shown that the series $\sum_{n=0}^{\infty} \alpha_n (z-z_0)^n$ converges for z in the circular region $|z-z_0| < R_1$ and the series $\sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n}$ converges for the circular region $|z-z_0| > R_2$ Here R_1 and

[8] Pierre Alphonse Laurent (1813-1854) A French mathematician who studied complex analysis.

R_2 are positive constants with $R_2 < R_1$. The annular region of convergence is the intersection of these two regions.

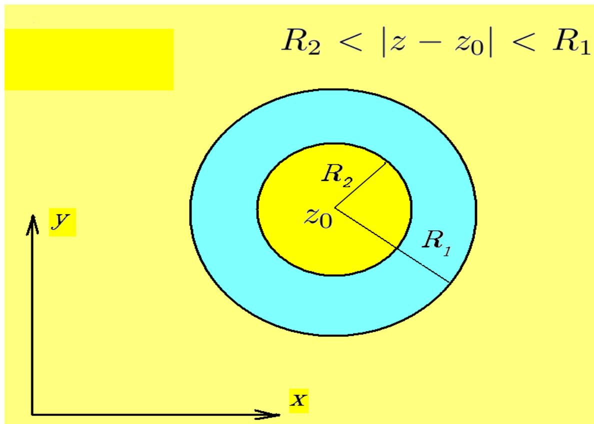


Figure 12-15. Annular region of convergence for Laurent series.

The special Laurent series where the point z_0 is the only singular point of $f(z)$ inside the disk $|z-z_0| < R_2$ is of extreme importance in the study of complex variables. In this special case the point z_0 is **called an isolated singular point.** This special series with negative powers of $z - z_0$ having the form

$$\sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n} = \cdots + \frac{c_m}{(z-z_0)^m} + \cdots + \frac{c_2}{(z-z_0)^2} + \frac{c_1}{(z-z_0)} \tag{12.194}$$

is called **the principal part of the Laurent series** and associated with the principal part of the series is the following terminology

(i) The term c_1 is called **the residue of $f(z)$ at the isolated singular point z_0.**

(ii) If there is only one term in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order 1 or a simple pole.**

(iii) If there are only two terms in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order 2.**

(iv) If there are $m-$terms in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order m.**

(v) If there is an infinite number of terms in the principal part of the Laurent series, then the singular point z_0 is called an **an essential singularity.**

If you get involved with complex variable theory, the binomial expansion

$$(a+b)^{-1} = a^{-1} + (-1)\,a^{-2}b + (-1)(-2)\,a^{-3}b^2/2! + \cdots \quad \text{converges for } |b| < |a| \tag{12.195}$$

will be of great assistance in dealing with Laurent series.

Example 12-21. (**Laurent series**)

Express the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ as a Laurent series centered at the singular point $z=1$.

Solution

The problem is to express $f(z)$ as a series involving both negative and positive powers of $(z-1)$. To accomplish this task analyze the following two representations of $f(z)$

$$(i) \quad f(z) = \frac{[(z-1)+1]}{(z-1)[(z-1)-2]} = \left[1+\frac{1}{z-1}\right][(z-1)-2]^{-1}$$

$$(ii) \quad f(z) = \frac{[(z-1)+1]}{(z-1)[(z-1)-2]} = -\left[1+\frac{1}{z-1}\right][2-(z-1)]^{-1}$$

The last term in the representations (i) and (ii) have binomial expansions having the form of equation (12.195). The binomial expansion for the last term in representation (i) converges for $2<|z-1|$ and the binomial expansion for the last term in representation (ii) converges for $|z-1|<2$. The term $(z-1)>0$ is assumed to hold in both the representations (i) and (ii). Therefore, to get the annular region isolating the singular point at $z=1$ the representation (ii) is used. Expanding the last term in the representation (ii) gives

$$f(z) = -\left[1+\frac{1}{z-1}\right]\left[\frac{1}{2}+\frac{1}{2^2}(z-1)+\frac{1}{2^3}(z-1)^2+\frac{1}{2^4}(z-1)^3+\cdots+\frac{1}{2^{n+1}}(z-1)^n+\cdots\right]$$

(a) Multiply the first term in the representation (ii) by the expanded second term and then collect like terms to obtain the Laurent series expansion

$$f(z) = \frac{-1/2}{z-1} - \frac{3}{2^2} - \frac{3}{2^3}(z-1) - \frac{3}{2^4}(z-1)^2 - \frac{3}{2^5}(z-1)^3 - \frac{3}{2^6}(z-1)^4 - \cdots \tag{12.196}$$

which converges in the annular region defined by **the intersection** of the regions (1) and (2) defined by

$$Region(1) = |z-1|>0 \qquad \text{and} \qquad Region(2) = |z-1|<2$$

The region(1) represents region of convergence of **the principal part of the Laurent series** and the region(2) represents the **region of convergence of that part of the Laurent series with terms** $(z-1)$ **with positive exponents.** The intersection of these two regions is given by $Region(1) \cap Region(2)$ produces the annular region $0<|z-1|<2$

illustrated in the figure 12-16.
(b) If one had used the binomial expansion on the last term in the representation (i) for $f(z)$, one obtains a series which converges for $|z-1| > 2$. After multiplication by the first term, the resulting series would converge in the annular region $r_1 < |z-1| < r_2$ where $r_1 = 2$ and $r_2 = lim_{r\to\infty} r$. This is **not the annular region** which **isolates the singular point at** $z = 1$ because the singular point $z = 3$ is also inside this region.

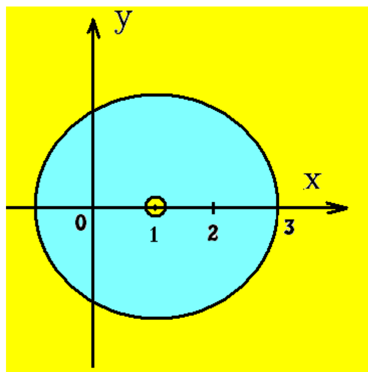


Figure 12-16. Annular region of convergence.

In general, the **radius of convergence for the power series with positive powers** or non-principal part of the Laurent series, is represented by the **distance from the center of the series to the nearest other singular point of the function being studied.** The correct Laurent series expansion, which isolates the singular point at $z = 1$, shows that $f(z)$ has a simple pole at $z = 1$ and the residue of $f(z)$ at $z = 1$ has the value $-1/2$.

■

The above examples represent a small fraction of the many concepts presented in the study of functions of a complex variable.

APPENDIX A
Units of Measurement

The following units, abbreviations and prefixes are from the Système International d'Unitès (designated SI in all Languages.)

Prefixes.

Abbreviations		
Prefix	Multiplication factor	Symbol
exa	10^{18}	W
peta	10^{15}	P
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	K
hecto	10^{2}	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^\circ$K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Frequency	hertz	Hz (s^{-1})
Density	kilogram per cubic meter	$\mathrm{kg/m^3}$
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	$\mathrm{m/s^2}$
Angular acceleration	radian per second squared	$\mathrm{rad/s^2}$
Force	newton	N $(\mathrm{kg\cdot m/s^2})$
Pressure	newton per square meter	$\mathrm{N/m^2}$
Kinematic viscosity	square meter per second	$\mathrm{m^2/s}$
Dynamic viscosity	newton second per square meter	$\mathrm{N\cdot s/m^2}$
Work, energy, quantity of heat	joule	J $(\mathrm{N\cdot m})$
Power	watt	W (J/s)
Electric charge	coulomb	C $(\mathrm{A\cdot s})$
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F $(\mathrm{A\cdot s/V})$
Magnetic flux	weber	Wb $(\mathrm{V\cdot s})$
Inductance	henry	H $(\mathrm{V\cdot s/A})$
Magnetic flux density	tesla	T $(\mathrm{Wb/m^2})$
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical Constants:

- $4\arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643\,\ldots$
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287\,\ldots$
- Euler's constant $\gamma = 0.57721\,56649\,01532\,86060\,6512\,\ldots$
- $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$ Euler's constant
- Speed of light in vacuum $= 2.997925(10)^8\; m\; s^{-1}$
- Electron charge $= 1.60210(10)^{-19}\, C$
- Avogadro's constant $= 6.0221415(10)^{23}\, mol^{-1}$
- Plank's constant $= 6.6256(10)^{-34}\, J\, s$
- Universal gas constant $= 8.3143\, J\, K^{-1}\, mol^{-1} = 8314.3\, J\, Kg^{-1}\, K^{-1}$
- Boltzmann constant $= 1.38054(10)^{-23}\, J\, K^{-1}$
- Stefan–Boltzmann constant $= 5.6697(10)^{-8}\, W\, m^{-2}\, K^{-4}$
- Gravitational constant $= 6.67(10)^{-11}\, N\, m^2 kg^{-2}$

APPENDIX B

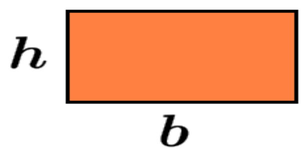
Background Material

Geometry

Rectangle

Area $=$ (base)(height) $= bh$

Perimeter $= 2b + 2h$

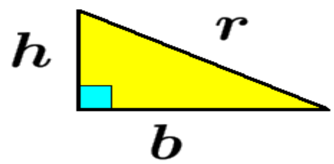


Right Triangle

Area $= \frac{1}{2}$(base)(height) $= \frac{1}{2}bh$

Perimeter $= b + h + r$

where $r^2 = b^2 + h^2$ is the Pythagorean theorem



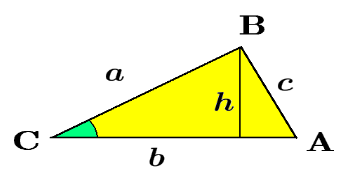
Triangle with sides a, b, c and angles A, B, C

Area $= \frac{1}{2}$(base)(height) $= \frac{1}{2}bh = \frac{1}{2}b(a \sin C)$

Perimeter $= a + b + c$

Law of Sines $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines $\quad c^2 = a^2 + b^2 - 2ab\cos C$

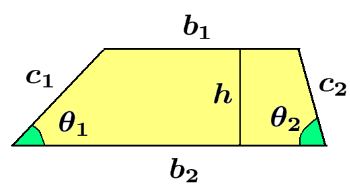


Trapezoid

Area $= \frac{1}{2}(b_1 + b_2)h$

Perimeter $= b_1 + b_2 + c_1 + c_2$

$c_1 = \dfrac{h}{\sin\theta_1} \qquad c_2 = \dfrac{h}{\sin\theta_2}$

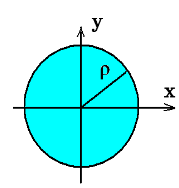


Circle

Area $= \pi\rho^2$

Perimeter $= 2\pi\rho$

Equation $\quad x^2 + y^2 = \rho^2$

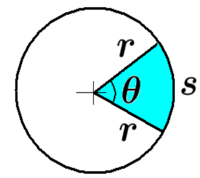


Sector of Circle

Area $= \frac{1}{2}r^2\theta, \quad \theta$ in radians

$s =$ arclength $= r\theta, \quad \theta$ in radians

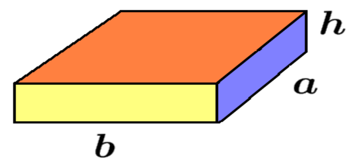
Perimeter $= 2r + s$



Rectangular Parallelepiped

$V =$ Volume $= abh$

$S =$ Surface area $= 2(ab + ah + bh)$



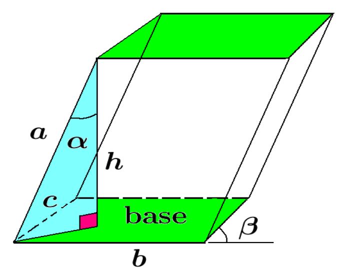
Parallelepiped

Composed of 6 parallelograms

$V =$ Volume $= (Area\ of\ base)(height)$

$A =$ Area of base $= bc\sin\beta$

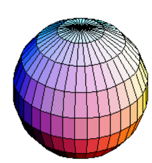
height $= h = a\cos\alpha$



Sphere of radius ρ

$V =$ Volume $= \frac{4}{3}\pi\rho^3$

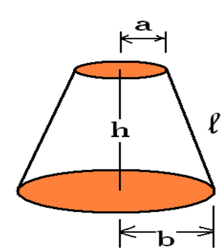
$S =$ Surface area $= 4\pi\rho^2$



Frustum of right circular cone

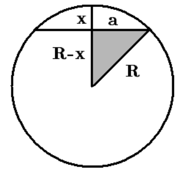
$V =$ Volume $= \frac{\pi}{3}(a^2 + ab + b^2)h$

Lateral surface area $= \pi\ell(a + b)$



Chord Theorem for circle

$a^2 = x(2R - x)$

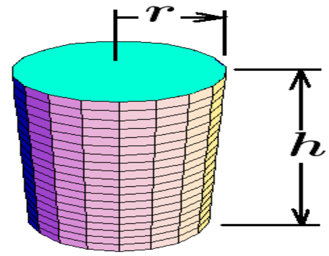


Right Circular Cylinder

$V =$ Volume $=$ (Area of base)(height) $= (\pi r^2)h$

Lateral surface area $= 2\pi r h$

Total surface area $= 2\pi r h + 2(\pi r^2)$

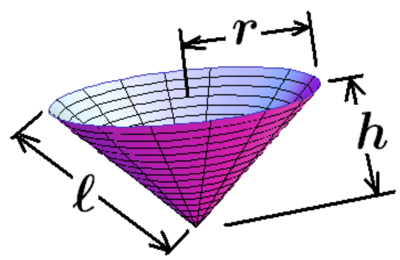


Right Circular Cone

$V =$ Volume $= \dfrac{1}{3}\pi r^2 h$

Lateral surface area $= \pi r \ell = \pi r \sqrt{h^2 + r^2}$

height $= h$, base radius r



Algebra

Products and Factors
$(x+a)(x+b) = x^2 + (a+b)x + ab$
$(x+a)^2 = x^2 + 2ax + a^2$
$(x-b)^2 = x^2 - 2bx + b^2$
$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ac+bc+ab)x + abc$
$x^2 - y^2 = (x-y)(x+y)$
$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
$x^4 - y^4 = (x-y)(x+y)(x^2 + y^2)$
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Binomial Expansion

For $n = 1, 2, 3, \ldots$ an integer, then

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

where $n!$ is read n factorial and is defined

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \text{ and } 0! = 1 \text{ by definition.}$$

Binomial Coefficients

The binomial coefficients can also be defined by the expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where } n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

where for $n = 1, 2, 3, \ldots$ is an integer. The binomial expansion has the alternative representation

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 \cdots + \binom{n}{n}y^n$$

Laws of Exponents

Let s and t denote real numbers and let m and n denote positive integers. For nonzero values of x and y

$$\begin{array}{lll} x^0 = 1, \quad x \neq 0 & (x^s)^t = x^{st} & x^{1/n} = \sqrt[n]{x} \\ x^s x^t = x^{s+t} & (xy)^s = x^s y^s & x^{m/n} = \sqrt[n]{x^m} \\ \dfrac{x^s}{x^t} = x^{s-t} & x^{-s} = \dfrac{1}{x^s} & \left(\dfrac{x}{y}\right)^{1/n} = \dfrac{x^{1/n}}{y^{1/n}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}} \end{array}$$

Laws of Logarithms

If $x = b^y$ and $b \neq 0$, then one can write $y = \log_b x$, where y is called the logarithm of x to the base b. For $P > 0$ and $Q > 0$, logarithms satisfy the following properties

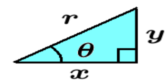
$$\begin{aligned} \log_b(PQ) &= \log_b P + \log_b Q \\ \log_b \frac{P}{Q} &= \log_b P - \log_b Q \\ \log_b Q^P &= P \log_b Q \end{aligned}$$

Trigonometry

Pythagorean identities

Using the Pythagorean theorem $x^2 + y^2 = r^2$ associated with a right triangle with sides x, y and hypotenuse r, there results the following trigonometric identities, known as the Pythagorean identities.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \qquad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2,$$



$$\cos^2\theta + \sin^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

Angle Addition and Difference Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Double angle formulas

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cot A}{\cot^2 A - 1}$$

Half angle formulas

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \qquad \begin{array}{c|c} + & + \\ \hline - & - \end{array}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \begin{array}{c|c} - & + \\ \hline - & + \end{array}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} \qquad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

The sign depends upon the quadrant $A/2$ lies in.

Multiple angle formulas

$$\sin 3A = 3\sin A - 4\sin^3 A, \qquad \sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$

$$\cos 3A = 4\cos^3 A - 3\cos A, \qquad \cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \qquad \tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

Multiple angle formulas

$$\begin{aligned}
\sin 5A =& 5\sin A - 20\sin^3 A + 16\sin^5 A\\
\cos 5A =& 16\cos^5 A - 20\cos^3 A + 5\cos A\\
\tan 5A =& \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}\\
\sin 6A =& 6\cos^5 A\sin A - 20\cos^3 A\sin^3 A + 6\cos A\sin^5 A\\
\cos 6A =& \cos^6 A - 15\cos^4 A\sin^2 A + 15\cos^2 A\sin^4 A - \sin^6 A\\
\tan 6A =& \frac{6\tan A - 20\tan^3 A + 6\tan^5 A}{1 - 15\tan^2 A + 15\tan^4 A - \tan^6 A}
\end{aligned}$$

Summation and difference formula

$$\begin{aligned}
\sin A + \sin B =& 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \sin A - \sin B =& 2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})\\
\cos A + \cos B =& 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \cos A - \cos B =& -2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})\\
\tan A + \tan B =& \frac{\sin(A+B)}{\cos A\cos B}, & \tan A - \tan B =& \frac{\sin(A-B)}{\cos A\cos B}
\end{aligned}$$

Product formula

$$\begin{aligned}
\sin A\sin B =& \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
\cos A\cos B =& \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)\\
\sin A\cos B =& \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)
\end{aligned}$$

Additional relations

$$\begin{aligned}
\sin(A+B)\sin(A-B) &= \sin^2 A - \sin^2 B,\\
-\sin(A+B)\sin(A-B) &= \cos^2 A - \cos^2 B,\\
\cos(A+B)\cos(A-B) &= \cos^2 A - \sin^2 B,
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A \pm \sin B}{\cos A + \cos B} &= \tan(\frac{A \pm B}{2})\\
\frac{\sin A \pm \sin B}{\cos A - \cos B} &= -\cot(\frac{A \mp B}{2})\\
\frac{\sin A + \sin B}{\sin A - \sin B} &= \frac{\tan(\frac{A+B}{2})}{\tan(\frac{A-B}{2})}
\end{aligned}$$

Powers of trigonometric functions

$$\begin{aligned}\sin^2 A &= \frac{1}{2} - \frac{1}{2}\cos 2A, & \cos^2 A &= \frac{1}{2} + \frac{1}{2}\cos 2A\\ \sin^3 A &= \frac{3}{4}\sin A - \frac{1}{4}\sin 3A, & \cos^3 A &= \frac{3}{4}\cos A + \frac{1}{4}\cos 3A\\ \sin^4 A &= \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A, & \cos^4 A &= \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A\end{aligned}$$

Inverse Trigonometric Functions

$$\begin{aligned}\sin^{-1} x &= \frac{\pi}{2} - \cos^{-1} x & \sin^{-1}\frac{1}{x} &= \csc^{-1} x\\ \cos^{-1} x &= \frac{\pi}{2} - \sin^{-1} x & \cos^{-1}\frac{1}{x} &= \sec^{-1} x\\ \tan^{-1} x &= \frac{\pi}{2} - \cot^{-1} x & \tan^{-1}\frac{1}{x} &= \cot^{-1} x\end{aligned}$$

Symmetry properties of trigonometric functions

$$\sin\theta = -\sin(-\theta) = \cos(\pi/2-\theta) = -\cos(\pi/2+\theta) = +\sin(\pi-\theta) = -\sin(\pi+\theta)$$

$$\cos\theta = +\cos(-\theta) = \sin(\pi/2-\theta) = +\sin(\pi/2+\theta) = -\cos(\pi-\theta) = -\cos(\pi+\theta)$$

$$\tan\theta = -\tan(-\theta) = \cot(\pi/2-\theta) = -\cot(\pi/2+\theta) = -\tan(\pi-\theta) = +\tan(\pi+\theta)$$

$$\cot\theta = -\cot(-\theta) = \tan(\pi/2-\theta) = -\tan(\pi/2+\theta) = -\cot(\pi-\theta) = +\cot(\pi+\theta)$$

$$\sec\theta = +\sec(-\theta) = \csc(\pi/2-\theta) = +\csc(\pi/2+\theta) = -\sec(\pi-\theta) = -\sec(\pi+\theta)$$

$$\csc\theta = -\csc(-\theta) = \sec(\pi/2-\theta) = +\sec(\pi/2+\theta) = +\csc(\pi-\theta) = -\csc(\pi+\theta)$$

Transformations

The following transformations are sometimes useful in simplifying expressions.

1. If $\tan\frac{u}{2} = A$, then

$$\sin u = \frac{2A}{1+A^2}, \qquad \cos u = \frac{1-A^2}{1+A^2}, \qquad \tan u = \frac{2A}{1-A^2}$$

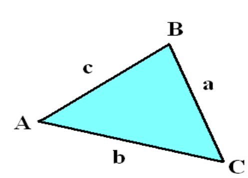
2. The transformation $\sin v = y$, requires $\cos v = \sqrt{1-y^2}$, and $\tan v = \dfrac{y}{\sqrt{1-y^2}}$

Law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc\cos A\\ b^2 &= c^2 + a^2 - 2ac\cos B\\ c^2 &= a^2 + b^2 - 2ab\cos C\end{aligned}$$



Special Numbers

Rational Numbers

All those numbers having the form p/q, where p and q are integers and q is understood to be different from zero, are called rational numbers.

Irrational Numbers

Those numbers that cannot be written as the ratio of two numbers are called irrational numbers.

The Number π

The Greek letter π (pronounced pi) is an irrational number and can be defined as the limiting sum[1] of the infinite series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots + \frac{(-1)^n}{2n+1} + \cdots\right)$$

Using a computer one can verify that the numerical value of π to 50 decimal places is given by

$$\pi = 3.14159265358979323846264338327950288419716939937 51\ldots$$

The number π has the physical significance of representing the circumference C of a circle divided by its diameter D. The symbol π for the ratio C/D was introduced by William Jones (1675-1749), a Welsh mathematician. It became a standard notation for representing C/D after Euler also started using the symbol π for this ratio sometime around 1737.

The Number e

The limiting sum

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is an irrational number which by agreement called the number e. Using a computer this number, to 50 decimal places, has the numerical value

$$e = 2.71828182845904523536028747135266249775724709369996\ldots$$

The number e is referred to as the base of the natural logarithm and the function $f(x) = e^x$ is called the exponential function.

[1] Limits are very important in the study of calculus.

Greek Alphabet

Letter		Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu

Letter		Name
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Notation

By convention letters from the beginning of an alphabet, such as $a, b, c, \ldots$ or the Greek letters $\alpha, \beta, \gamma, \ldots$ are often used to denote quantities which have a constant value. Subscripted quantities such as $x_0, x_1, x_2, \ldots$ or $y_0, y_1, y_2, \ldots$ can also be used to represent constant quantities. A variable is a quantity which is allowed to change its value. The letters u, v, w, x, y, z or the Greek letters ξ, η, ζ are most often used to denote variable quantities.

Inequalities

The mathematical symbols $=$ (equals), $\neq$ (not equal), $<$ (less than), $<<$ (much less than), $\leq$ (less than or equal), $>$ (greater than), $>>$ (much greater than) $\geq$ (greater than or equal), and $|\ |$ (absolute value) occur frequently in mathematics to compare real numbers $a, b, c, \ldots$. The law of trichotomy states that if a and b are real numbers, then exactly one of the following must be true. Either a equals b, a is less than b or a is greater than b. These statements are expressed using the mathematical notations[2]

$$a = b, \qquad a < b, \qquad a > b$$

[2] In mathematical notation, the statement $b > a$, read "b is greater than a", can also be represented $a < b$ or "a is less than b"depending upon your way of looking at things.

Inequalities can be defined in terms of addition or subtraction. For example, one can define

$a < b$ if and only if $a - b < 0$

$a > b$ if and only if $a - b > 0$, or alternatively

$a > b$ if and only if there exists a positive number x such that $b + x = a$.

In dealing with inequalities be sure to observe the following properties associated with real numbers $a, b, c, \ldots$

1. A constant can be added to both sides of an inequality without changing the inequality sign.

$$\text{If } a < b, \text{ then } a + c < b + c \text{ for all numbers c}$$

2. Both sides of an inequality can be multiplied or divided by a positive constant without changing the inequality sign.

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \quad \text{or} \quad a/c < b/c$$

3. If both sides of an inequality are multiplied or divided by a negative quantity, then the inequality sign changes.

$$\text{If } b > a \text{ and } c < 0, \text{ then } bc < ac \quad \text{or} \quad b/c < a/c$$

4. The transitivity law

$$\text{If } a < b, \text{ and } b < c, \text{ then } a < c$$
$$\text{If } a = b \text{ and } b = c, \text{ then } a = c$$
$$\text{If } a > b, \text{ and } b > c, \text{ then } a > c$$

5. If $a > 0$ and $b > 0$, then $ab > 0$
6. If $a < 0$ and $b < 0$, then $ab > 0$ or $0 < ab$
7. If $a > 0$ and $b > 0$ with $a < b$, then $\sqrt{a} < \sqrt{b}$

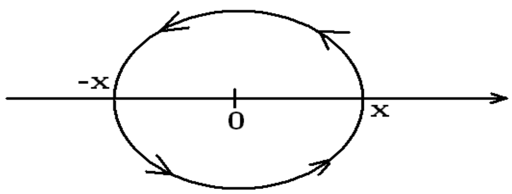
A negative times a negative is a positive

To prove that a real negative number multiplied by another real negative number gives a positive number start by assuming a and b are real numbers satisfying $a < 0$ and $b < 0$, then one can write

$$-a + a < -a \quad \text{or } 0 < -a \qquad \text{and} \qquad -b + b < -b \quad \text{or } 0 < -b$$

since equals can be added to both sides of an inequality without changing the inequality sign. Using the fact that both sides of an inequality can be multiplied by a positive number without changing the inequality sign, one can write

$$0 < (-a)(-b) \qquad \text{or} \qquad (-a)(-b) > 0$$



Another way to show a negative times a negative is a positive is as follows. Think of a number line with the number 0 separating the positive numbers and negative numbers. By agreement, if a number on this number line is multiplied by -1, then the number is to be rotated counterclockwise 180 degrees. If the positive number x is multiplied by -1, then it is rotated counterclockwise 180 degrees to produce the number $-x$. If the number $-x$ is multiplied by -1, then it is to be rotated 180 degrees counterclockwise to produce the positive number x. If $a > 0$ and $b > 0$, then the product $a(-b)$ scales the number $-b$ to produce the negative number $-ab$. If the number $-ab$ is multiplied by -1, which is equivalent to the product $(-a)(-b)$, one obtains by rotation the number $+ab$.

Absolute Value

The absolute value of a number x is defined

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The symbol $\iff$ is often used to represent equivalence of two equations. For example, if a and b are real numbers the statements

$$|x-a| \leq b \quad \iff \quad -b \leq x-a \leq b \quad \iff \quad a-b \leq x \leq a+b$$

are all equivalent statements involving restrictions on the real number x.

An important inequality known as the triangle inequality is written

$$|x+y| \leq |x| + |y| \tag{1.1}$$

where x and y are real numbers. To prove this inequality observe that $|x|$ satisfies $-|x| \leq x \leq |x|$ and also $-|y| \leq y \leq |y|$, so that by adding these results one obtains

$$-(|x|+|y|) \leq x+y \leq |x|+|y| \qquad \text{or} \qquad |x+y| \leq |x|+|y| \tag{1.2}$$

Related to the inequality (1.2) is the reverse triangle inequality

$$|x-y| \geq |x| - |y| \tag{1.3}$$

a proof of which is left as an exercise.

Cramer's Rule

The system of two equations in two unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y =& \gamma_1 \\ \alpha_2 x + \beta_2 y =& \gamma_2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

has a unique solution if $\alpha_1\beta_2 - \alpha_2\beta_1$ is nonzero. The unique solution is given by

$$x = \frac{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}} \quad \text{where} \quad \overset{-\alpha_2\beta_1}{\underset{+\alpha_1\beta_2}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}} = \alpha_1\beta_2 - \alpha_2\beta_1$$

is a single number called the determinant of the coefficients.

The system of three equations in three unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y + \gamma_1 z =& \delta_1 \\ \alpha_2 x + \beta_2 y + \gamma_2 z =& \delta_2 \\ \alpha_3 x + \beta_3 y + \gamma_3 x =& \delta_3 \end{aligned}$$

has a unique solution if the determinant of the coefficients

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

is nonzero. A mnemonic device to aid in calculating the determinant of the coefficients is to append the first two columns of the coefficients to the end of the array and then draw diagonals through the coefficients. Multiply the elements along an arrow and place a plus sign on the products associated with the down arrows and a minus sign associated with the products of the up arrows. This gives the figure

$$\left|\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix}\right| \begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

The solution of the three equations, three unknown system of equations is given by the determinant ratios

$$x = \frac{\begin{vmatrix} \delta_1 & \beta_1 & \gamma_1 \\ \delta_2 & \beta_2 & \gamma_2 \\ \delta_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \alpha_1 & \delta_1 & \gamma_1 \\ \alpha_2 & \delta_2 & \gamma_2 \\ \alpha_3 & \delta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} \alpha_1 & \beta_1 & \delta_1 \\ \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}$$

and is known as Cramer's rule for solving a system of equations.

Appendix C
Table of Integrals
Indefinite Integrals

General Integration Properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int f(x)\,dx = F(x) + C$

2. If $\displaystyle\int f(x)\,dx = F(x) + C$, then the substitution $x = g(u)$ gives $\displaystyle\int f(g(u))\,g'(u)\,du = F(g(u)) + C$
For example, if $\displaystyle\int \frac{dx}{x^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{x}{\beta} + C$, then $\displaystyle\int \frac{du}{(u+\alpha)^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{u+\alpha}{\beta} + C$

3. Integration by parts. If $v_1(x) = \displaystyle\int v(x)\,dx$, then $\displaystyle\int u(x)v(x)\,dx = u(x)v_1(x) - \int u'(x)v_1(x)\,dx$

4. Repeated integration by parts or generalized integration by parts.
If $v_1(x) = \int v(x)\,dx, \quad v_2(x) = \int v_1(x)\,dx, \ldots, v_n(x) = \int v_{n-1}(x)\,dx$, then

$$\int u(x)v(x)\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots + (-1)^{n-1}u^{n-1}v_n + (-1)^n \int u^{(n)}(x)v_n(x)\,dx$$

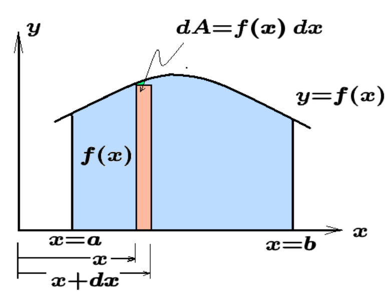
5. If $f^{-1}(x)$ is the inverse function of $f(x)$ and if $\displaystyle\int f(x)\,dx$ is known, then

$$\int f^{-1}(x)\,dx = zf(z) - \int f(z)\,dz, \qquad \text{where} \quad z = f^{-1}(x)$$

6. Fundamental theorem of calculus.
If the indefinite integral of $f(x)$ is known, say $\displaystyle\int f(x)\,dx = F(x) + C$, then the definite integral

$$\int_a^b dA = \int_a^b f(x)\,dx = F(x)\big]_a^b = F(b) - F(a)$$

represents the area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$.



7. Inequalities.
(i) If $f(x) \le g(x)$ for all $x \in (a,b)$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
(ii) If $|f(x) \le M|$ for all $x \in (a,b)$ and $\int_a^b f(x)\,dx$ exists, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b f(x)\,dx \le M(b-a)$$

8. $\displaystyle\int \frac{u'(x)\,dx}{u(x)} = \ln|u(x)| + C$

9. $\displaystyle\int (\alpha u(x)+\beta)^n u'(x)\,dx = \frac{(\alpha u(x)+\beta)^{n+1}}{\alpha(n+1)} + C$

10. $\displaystyle\int \frac{u'(x)v(x)-v'(x)u(x)}{v^2(x)}\,dx = \frac{u(x)}{v(x)} + C$

11. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u(x)v(x)}\,dx = \ln|\frac{u(x)}{v(x)}| + C$

12. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u^2(x)+v^2(x)}\,dx = \tan^{-1}\frac{u(x)}{v(x)} + C$

13. $\displaystyle\int \frac{u'(x)v(x)-u(x)v'(x)}{u^2(x)-v^2(x)}\,dx = \frac{1}{2}\ln|\frac{u(x)-v(x)}{u(x)+v(x)}| + C$

14. $\displaystyle\int \frac{u'(x)\,dx}{\sqrt{u^2(x)+\alpha}} = \ln|u(x)+\sqrt{u^2(x)+\alpha}| + C$

15. $\displaystyle\int \frac{u(x)\,dx}{(u(x)+\alpha)(u(x)+\beta)} = \begin{cases} \dfrac{\alpha}{\alpha-\beta}\displaystyle\int \frac{dx}{u(x)+\alpha} - \frac{\beta}{\alpha-\beta}\int \frac{dx}{u(x)+\beta}, & \alpha \neq \beta \\ \displaystyle\int \frac{dx}{u(x)+\alpha} - \alpha \int \frac{dx}{(u(x)+\alpha)^2}, & \beta = \alpha \end{cases}$

16. $\displaystyle\int \frac{u'(x)\,dx}{\alpha u^2(x)+\beta u(x)} = \frac{1}{\beta}\ln|\frac{u(x)}{\alpha u(x)+\beta}| + C$

17. $\displaystyle\int \frac{u'(x)\,dx}{u(x)\sqrt{u^2(x)-\alpha^2}} = \frac{1}{\alpha}\sec^{-1}\frac{u(x)}{\alpha} + C$

18. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2+\beta^2 u^2(x)} = \frac{1}{\alpha\beta}\tan^{-1}\frac{\beta u(x)}{\alpha} + C$

19. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2 u^2(x)-\beta^2} = \frac{1}{2\alpha\beta}\ln|\frac{\alpha u(x)-\beta}{\alpha u(x)+\beta}| + C$

20. $\displaystyle\int f(\sin x)\,dx = 2\int f\left(\frac{2u}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u = \tan\frac{x}{2}$

21. $\displaystyle\int f(\sin x)\,dx = \int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad u = \sin x$

22. $\displaystyle\int f(\cos x)\,dx = 2\int f\left(\frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u = \tan\frac{x}{2}$

23. $\displaystyle\int f(\cos x)\,dx = -\int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad u = \cos x$

24. $\displaystyle\int f(\sin x, \cos x)\,dx = \int f(u, \sqrt{1-u^2})\frac{du}{\sqrt{1-u^2}}, \qquad u = \sin x$

25. $\displaystyle\int f(\sin x, \cos x)\,dx = 2\int f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \qquad u = \tan\frac{x}{2}$

26. $\displaystyle\int f(x, \sqrt{\alpha+\beta x})\,dx = \frac{2}{\beta}\int f\left(\frac{u^2-\alpha}{\beta}, u\right) u\,du, \qquad u^2 = \alpha+\beta x$

27. $\displaystyle\int f(x, \sqrt{\alpha^2-x^2})\,dx = \alpha\int f(\alpha\sin u, a\cos u)\,\cos u\,du, \qquad x = \alpha\sin u$

General Integrals

28. $\int c\,u(x)\,dx = c\int u(x)\,dx$

29. $\int [u(x)+v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx$

30. $\int u(x)\,u'(x)\,dx = \frac{1}{2}\mid u(x)\mid^2 + C$

31. $\int [u(x)-v(x)]\,dx = \int u(x)\,dx - \int v(x)\,dx]$

32. $\int u^n(x)\,u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + C$

33. $\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$

34. $\int F'[u(x)]\,u'(x)\,dx = F[u(x)] + C$

35. $\int \frac{u'(x)}{u(x)}\,dx = \ln\mid u(x)\mid + C$

36. $\int \frac{u'}{2\sqrt{u}}\,dx = \sqrt{u} + C$

37. $\int 1\,dx = x + C$

38. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

39. $\int \frac{1}{x}\,dx = \ln\mid x\mid + C$

40. $\int e^{au}\,u'\,dx = \frac{1}{a}e^{au} + C$

41. $\int a^u\,u'\,dx = \frac{1}{\ln a}a^u + C$

42. $\int \sin u\,u'\,dx = \cos u + C$

43. $\int \cos u\,u'\,dx = -\sin u + C$

44. $\int \tan u\,u'\,dx = \ln\mid \sec u\mid + C$

45. $\int \cot u\,u'\,dx = \ln\mid \sin u\mid + C$

46. $\int \sec u\,u'\,dx = \ln\mid \sec u + \tan u\mid + C$

47. $\int \csc u\,u'\,dx = \ln\mid \csc u - \cot u\mid + C$

48. $\int \sinh u\,u'\,dx = \cosh u + C$

49. $\int \cosh u\,u'\,dx = \sinh u + C$

50. $\int \tanh u\,u'\,dx = \ln\cosh u + C$

51. $\int \coth u\,u'\,dx = \ln\sinh u + C$

52. $\int \operatorname{sech} u\,u'\,dx = \sin^{-1}(\tanh u) + C$

53. $\int \operatorname{csch} u\,u'\,dx = \ln\tanh\frac{u}{2} + C$

54. $\int \sin^2 u\,u'\,dx = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

55. $\int \cos^2 u\,u'\,dx = \frac{u}{2} + \frac{1}{4}\sin 2u + C$

56. $\int \tan^2 u\,u'\,dx = \tan u - u + C$

57. $\int \cot^2 u\,u'\,dx = -\cot u - u + C$

58. $\int \sec^2 u\,u'\,dx = \tan u + C$

59. $\int \csc^2 u\,u'\,dx = -\cot u + C$

60. $\int \sinh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u - \frac{1}{2}u + C$

61. $\int \cosh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u + \frac{1}{2}u + C$

62. $\int \tanh^2 u\,u'\,dx = u - \tanh u + C$

63. $\int \coth^2 u\,u'\,dx = u - \coth u + C$

64. $\int \operatorname{sech}^2 u\,u'\,dx = \tanh u + C$

65. $\int \operatorname{csch}^2 u\,u'\,dx = -\coth u + C$

66. $\int \sec u\tan u\,u'\,dx = \sec u + C$

67. $\int \csc u\cot u\,u'\,dx = -\csc u + C$

68. $\int \operatorname{sech} u\tanh u\,u'\,dx = -\operatorname{sech} u + C$

69. $\int \operatorname{csch} u\coth u\,u'\,dx = -\operatorname{csch} u + C$

Integrals containing $X = a + bx$, $a \neq 0$ and $b \neq 0$

70. $\int X^n\,dx = \frac{X^{n+1}}{b(n+1)} + C, \quad n \neq -1$

71. $\int xX^n\,dx = \frac{X^{n+2}}{b^2(n+2)} - \frac{aX^{n+1}}{b^2(n+1)} + C, \quad n \neq -1, n \neq -2$

72. $\int X(x+c)^n\,dx = \frac{b}{n+2}(x+c)^{n+2} + \frac{a-bc}{n+1}(x+c)^{n+1} + C$

73. $$\int x^2 X^n\,dx = \frac{1}{b^3}\left[\frac{X^{n+3}}{n+3} - \frac{2aX^{n+2}}{n+2} + \frac{a^2X^{n+1}}{n+1}\right] + C$$

74. $$\int x^{n-1}X^m\,dx = \frac{1}{n+m}x^nX^m + \frac{am}{m+n}\int x^{n-1}X^{m-1}\,dx$$

75. $$\int \frac{X^m}{x^{n+1}}\,dx = -\frac{1}{na}\frac{X^{m+1}}{x^n} + \frac{m-n+1}{n}\frac{b}{a}\int \frac{X^m}{x^n}\,dx$$

76. $$\int \frac{dx}{X} = \frac{1}{b}\ln X + C$$

77. $$\int \frac{x\,dx}{X} = \frac{1}{b^2}\left(X - a\ln|\,X\,|\right) + C$$

78. $$\int \frac{x^2\,dx}{X} = \frac{1}{2b^3}\left(X^2 - 4aX + 2a^2\ln|\,X\,|\right) + C$$

79. $$\int \frac{dx}{xX} = \frac{1}{a}\ln\left|\,\frac{x}{X}\,\right| + C$$

80. $$\int \frac{dx}{x^3X} = -\frac{a+2bx}{a^2xX} + \frac{2b}{a^3}\ln\left|\frac{X}{x}\right| + C$$

81. $$\int \frac{dx}{X^2} = -\frac{1}{bX} + C$$

82. $$\int \frac{x\,dx}{X^2} = \frac{1}{b^2}\left[\ln|\,X\,| + \frac{a}{X}\right] + C$$

83. $$\int \frac{x^2\,dx}{X^2} = \frac{1}{b^3}\left[X - 2a\ln|\,X\,| - \frac{a^2}{X}\right] + C$$

84. $$\int \frac{dx}{xX^2} = \frac{1}{aX} - \frac{1}{a^2}\ln\left|\,\frac{X}{x}\,\right| + C$$

85. $$\int \frac{dx}{x^2X^2} = -\frac{a+2bx}{a^2xX} + \frac{2b}{a^3}\ln\left|\,\frac{X}{x}\,\right| + C$$

86. $$\int \frac{dx}{X^3} = -\frac{1}{2bX^2} + C$$

87. $$\int \frac{x\,dx}{X^3} = \frac{1}{b^2}\left[\frac{-1}{X} + \frac{a}{2X^2}\right] + C$$

88. $$\int \frac{x^2\,dx}{X^3} = \frac{1}{b^3}\left[\ln|\,X\,| + \frac{2a}{X} - \frac{a^2}{2X^2}\right] + C$$

89. $$\int \frac{dx}{xX^3} = \frac{1}{2aX^2} + \frac{1}{aX} - \ln\left|\,\frac{X}{x}\,\right| + C$$

90. $\int \frac{dx}{x^2X^3} = \frac{-b}{2a^2X} - \frac{2b}{a^3X} - \frac{1}{a^3x} + \frac{3b}{a^4}\ln\left|\frac{X}{x}\right|$

91. $\int \frac{x\,dx}{X^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)X^{n-2}} + \frac{a}{(n-1)X^{n-1}}\right] + C, \quad n \neq 1,2$

92. $\int \frac{x^2\,dx}{X^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)X^{n-3}} + \frac{2a}{(n-2)X^{n-2}} - \frac{a^2}{(n-1)X^{n-1}}\right] + C, \quad n \neq 1,2,3$

93. $\int \sqrt{X}\,dx = \frac{2}{3b}X^{3/2} + C$

94. $\int x\sqrt{X}\,dx = \frac{2}{15b^2}(3bx - 2a)X^{3/2} + C$

95. $\int x^2\sqrt{X}\,dx = \frac{2}{105b^3}(8a^2 - 12abx + 15b^2x^2)X^{3/2} + C$

96. $\int \frac{\sqrt{X}}{x}\,dx = 2\sqrt{X} + a\int \frac{dx}{x\sqrt{X}}$

97. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$

98. $\int \frac{dx}{\sqrt{X}} = \frac{2}{b}\sqrt{X} + C$

99. $\int \frac{x\,dx}{\sqrt{X}} = \frac{2}{3b^2}(bx - 2a)\sqrt{X} + C$

100. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2}{15b^3}(8a^2 - 4abx + 3b^2x^2)\sqrt{X} + C$

101. $\int \frac{dx}{x\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{X}-\sqrt{a}}{\sqrt{X}+\sqrt{a}}\right| + C_1, & a > 0 \\ \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{X}{-a}} + C_2, & a < 0 \end{cases}$

102. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$

103. $\int x^n\sqrt{X}\,dx = \frac{2}{(2n+3)b}x^nX^{3/2} - \frac{2na}{(2n+3)b}\int x^{n-1}\sqrt{X}\,dx$

104. $\int \frac{\sqrt{X}}{x^n}\,dx = \frac{-1}{(n-1)a}\frac{X^{3/2}}{x^{n-1}} - \frac{(2n-5)b}{2(n-1)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx$

105. $\int x^{m-1}X^n\,dx = \frac{x^mX^n}{m+n} + \frac{an}{m+n}\int x^{m-1}X^{n-1}\,dx + C$

106. $\int \frac{X^n}{x^{m+1}}\,dx = -\frac{X^{n+1}}{ma\,x^m} + \frac{n-m+1}{m}\frac{b}{a}\int \frac{X^n}{x^m}\,dx$

107. $\displaystyle\int \frac{X^n}{x}\,dx = \frac{X^n}{n} + a\int \frac{X^{n-1}}{x}\,dx$

Integrals containing $X = a + bx$ and $Y = \alpha + \beta x$, $(b \neq 0,\ \beta \neq 0,\ \Delta = a\beta - \alpha b \neq 0\ )$

108. $\displaystyle\int \frac{dx}{XY} = \frac{1}{\Delta}\ln\left|\ \frac{Y}{X}\ \right| + C$

109. $\displaystyle\int \frac{x\,dx}{XY} = \frac{1}{\Delta}\left[\frac{a}{b}\ln|\ X\ | - \frac{\alpha}{\beta}\ln|\ Y\ |\right] + C$

110. $\displaystyle\int \frac{x^2\,dx}{XY} = \frac{x}{b\beta} = \frac{a^2}{b^2\Delta}\ln|X| + \frac{\alpha^2}{\beta^2\Delta}\ln|Y| + C$

111. $\displaystyle\int \frac{dx}{X^2Y} = \frac{1}{\Delta}\left(\frac{1}{X} + \frac{\beta}{\Delta}\ln|\frac{Y}{X}|\right) + C$

112. $\displaystyle\int \frac{x\,dx}{X^2Y} = -\frac{a}{b\Delta X} - \frac{\alpha}{\Delta^2}\ln|\frac{Y}{X}| + C$

113. $\displaystyle\int \frac{x^2\,dx}{X^2Y} = \frac{a^2}{b^2\Delta X} + \frac{1}{\Delta^2}\left[\frac{\alpha^2}{\beta}\ln|Y| + \frac{a(a\beta - 2\alpha b)}{b^2}\ln|X|\right] + C$

114. $\displaystyle\int \frac{X}{Y}\,dx = \frac{b}{\beta}x + \frac{\Delta}{\beta^2}\ln|\frac{Y}{X}| + C$

115. $\displaystyle\int \sqrt{XY}\,dx = \frac{\Delta + 2bY}{4b\beta}\sqrt{XY} - \frac{\Delta^2}{8b\beta}\int \frac{dx}{\sqrt{XY}}$

116. $\displaystyle\int \frac{dx}{X^nY^m} = \frac{-1}{(m-1)\Delta X^{n-1}Y^{m-1}} + \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{X^nY^{m-1}}, \qquad m \neq 1$

117. $$\int \frac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-\Delta\beta}}\tan^{-1}\dfrac{\beta\sqrt{X}}{\sqrt{-\Delta\beta}}, + C_1 & \Delta\beta < 0 \\[2ex] \dfrac{1}{\sqrt{\Delta\beta}}\ln|\dfrac{\beta\sqrt{X} - \sqrt{\Delta\beta}}{\beta\sqrt{X} + \sqrt{\Delta\beta}}| + C_2, & \Delta\beta > 0 \end{cases}$$

118. $$\int \frac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-b\beta}}\tan^{-1}\sqrt{\dfrac{-\beta X}{bY}} + C_1, & b\beta < 0,\ bY > 0 \\[2ex] \dfrac{2}{\sqrt{b\beta}}\tanh^{-1}\sqrt{\dfrac{\beta X}{bY}} + C_2, & b\beta > 0,\ bY > 0 \end{cases}$$

119. $\displaystyle\int \frac{x\,dx}{\sqrt{XY}} = \frac{1}{b\beta}\sqrt{XY} - \frac{(b\alpha + a\beta)}{2b\beta}\int \frac{dx}{\sqrt{XY}}$

120. $\displaystyle\int \frac{\sqrt{Y}}{\sqrt{X}}\,dx = \frac{1}{b}\sqrt{XY} - \frac{\Delta}{2b}\int \frac{dx}{\sqrt{XY}}$

121. $\displaystyle\int \frac{\sqrt{X}}{Y}\,dx = \frac{2}{\beta}\sqrt{X} + \frac{\Delta}{\beta}\int \frac{dx}{Y\sqrt{X}}$

Integrals containing terms of the form $a + bx^n$

122. $$\int \frac{dx}{a+bx^2} = \begin{cases} \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\dfrac{b}{a}}x\right) + C, & ab > 0 \\ \dfrac{1}{2\sqrt{-ab}} \ln\left|\dfrac{a+\sqrt{-ab}x}{a-\sqrt{-ab}x}\right| + C, & ab < 0 \end{cases}$$

123. $$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln|x^2 + \frac{a}{b}| + C$$

124. $$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}$$

125. $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$$

126. $$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right| + C$$

127. $$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$$

128. $$\int \frac{dx}{(a+bx^2)^{n+1}} = \frac{1}{2na}\frac{x}{(a+bx^2)^n} + \frac{2n-1}{2na}\int \frac{dx}{(a+bx^2)^n}$$

129. $$\int \frac{dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha^2\beta}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) + \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

130. $$\int \frac{x\,dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha\beta^2}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) - \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

If $X = a + bx^n$, then

131. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx$$

132. $$\int x^{m-1}X^p\,dx = -\frac{x^mX^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}X^{p+1}\,dx$$

133. $$\int x^{m-1}X^P\,dx = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{(m-n)\,a}{b(m+pn)}\int x^{m-n-1}X^p\,dx$$

134. $$\int x^{m-1}X^p\,dx = \frac{x^mX^{p+1}}{am} - \frac{(m+pn+n)\,b}{am}\int x^{m+n-1}X^p\,dx$$

135. $$\int x^{m-1}X^p\,dx = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}X^{p+1}\,dx$$

136. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m} - \frac{bpn}{m}\int x^{m+n-1}X^{p-1}\,dx$$

Integrals containing $X = 2ax - x^2, \quad a \neq 0$

137. $\displaystyle\int \sqrt{X}\,dx = \frac{(x-a)}{2}\sqrt{X} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

138. $\displaystyle\int \frac{dx}{\sqrt{X}} = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

139. $\displaystyle\int x\sqrt{X}\,dx = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

140. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + a\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

141. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{x-a}{a^2\sqrt{X}} + C$

142. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{x}{a\sqrt{X}} + C$

143. $\displaystyle\int \frac{dx}{X} = \frac{1}{2a}\ln|\frac{x}{x-2a}| + C$

144. $\displaystyle\int \frac{x\,dx}{X} = -\ln|x-2a| + C$

145. $\displaystyle\int \frac{dx}{X^2} = -\frac{1}{4ax} - \frac{1}{4a^2(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$

146. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{1}{2a(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$

147. $\displaystyle\int x^n\sqrt{X}\,dx = -\frac{1}{n+2}x^{n-1}X^{3/2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{X}\,dx, \quad n \neq -2$

148. $\displaystyle\int \frac{\sqrt{X}\,dx}{x^n} = \frac{1}{(3-2n)a}\frac{X^{3/2}}{x^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx, \quad n \neq 3/2$

Integrals containing $X = ax^2 + bx + c$ with $\Delta = 4ac - b^2, \quad \Delta \neq 0, \quad a \neq 0$

149. $$\int \frac{dx}{X} = \begin{cases} \dfrac{1}{\sqrt{-\Delta}}\ln\left(\dfrac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}}\right) + C_1, & \Delta < 0 \\ \dfrac{2}{\sqrt{\Delta}}\tan^{-1}\dfrac{2ax+b}{\sqrt{\Delta}} + C_2, & \Delta > 0 \\ -\dfrac{1}{a(x+b/2a)} + C_3, & \Delta = 0 \end{cases}$$

150. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2c}\ln|\,X\,| - \frac{b}{2a}\int \frac{1}{X}\,dx$

151. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{x}{a} - \frac{b}{2a^2}\ln|X| + \frac{2ac-\Delta}{2a^2}\int \frac{dx}{X}$

152. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2c}\ln|\frac{x^2}{X}| - \frac{b}{2c}\int \frac{dx}{X}$

153. $\displaystyle\int \frac{dx}{x^2X} = \frac{b}{2c^2}\ln|\frac{X}{x^2}| - \frac{1}{cx} + \frac{2ac-\Delta}{2c^2}\int \frac{dx}{X}$

154. $\displaystyle\int \frac{dx}{X^2} = \frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

155. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

156. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{(2ac-\Delta)x+bc}{a\Delta X} + \frac{2c}{\Delta}\int \frac{dx}{X}$

157. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{2cX} - \frac{b}{2c}\int \frac{dx}{X^2} + \frac{1}{c}\int \frac{dx}{xX}$

158. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{cxX} - \frac{3a}{c}\int \frac{dx}{X^2} - \frac{2b}{c}\int \frac{dx}{xX^2}$

159. $\displaystyle\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}}\ln|2\sqrt{aX}+2ax+b| + C_1, & a>0 \\ \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax+b}{\sqrt{\Delta}}\right) + C_2, & a>,\ \Delta>0 \\ -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{-\Delta}}\right) + C_3, & a<0,\ \Delta<0 \end{cases}$

160. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \frac{1}{a}\sqrt{X} - \frac{b}{2a}\int \frac{dx}{\sqrt{X}}$

161. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{2b^2-\Delta}{8a^2}\int \frac{dx}{\sqrt{X}}$

162. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{1}{\sqrt{c}}\ln|\frac{2\sqrt{cX}}{x} + \frac{2c}{x} + b| + C_1, & c>0 \\ -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx+2c}{x\sqrt{\Delta}}\right) + C_2, & c>0,\ \Delta>0 \\ \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{x\sqrt{-\Delta}}\right) + C_3, & c<0,\ \Delta<0 \end{cases}$

163. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$

164. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{4a}(2ax+b)\sqrt{X} + \frac{\Delta}{8a}\int \frac{dx}{\sqrt{X}}$

165. $\displaystyle\int x\sqrt{X}\,dx = \frac{1}{3a}X^{3/2} - \frac{b(2ax+b)}{8a^2}\sqrt{X} - \frac{b\Delta}{16a^2}\int \frac{dx}{\sqrt{X}}$

166. $\displaystyle\int x^2\sqrt{X}\,dx = \frac{6ax-5b}{24a^2}X^{3/2} + \frac{4b^2-\Delta}{16a^2}\int \sqrt{X}\,dx$

167. $\displaystyle\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + c\int \frac{dx}{x\sqrt{X}}$

168. $\displaystyle\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + a\int \frac{dx}{\sqrt{X}} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$

169. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$

170. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{-2(bx+2c)}{\Delta\sqrt{X}} + C$

171. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$

172. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{1}{x\sqrt{X}} + \frac{1}{c}\int \frac{dx}{x\sqrt{X}} - \frac{b}{2c}\int \frac{dx}{X^{3/2}}$

173. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{X}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{X}}$

174. $\displaystyle\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$

175. $\displaystyle\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax+b)}{3\Delta\sqrt{X}}\left(\frac{1}{X} + \frac{8a}{\Delta}\right) + C$

176. $\displaystyle\int X\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X + \frac{3\Delta}{8a}\right) + \frac{3\Delta^2}{128a^2}\int \frac{dx}{\sqrt{X}}$

177. $\displaystyle\int X^2\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X^2 + \frac{5\Delta}{16a}X + \frac{15\Delta^2}{128a^2}\right) + \frac{5\Delta^3}{1024a^3}\int \frac{dx}{\sqrt{X}}$

178. $\displaystyle\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2c)}{\Delta\sqrt{X}} + C$

179. $\displaystyle\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$

180. $\displaystyle\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$

181. $\displaystyle\int f(x,\sqrt{ax^2+bx+c})\,dx$ Try substitutions (i) $\sqrt{ax^2+bx+c} = \sqrt{a}(x+z)$

(ii) $\sqrt{ax^2+bx+c} = xz+\sqrt{c}$ and if $ax^2+bx+c = a(x-x_1)(x-x_2)$, then (iii) let $(x-x_2) = z^2(x-x_1)$

Integrals containing $X = x^2 + a^2$

182. $\displaystyle\int \frac{dx}{X} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ or $\displaystyle\frac{1}{a}\cos^{-1}\frac{a}{\sqrt{x^2+a^2}} + C$ or $\displaystyle\frac{1}{a}\sec^{-1}\frac{\sqrt{x^2+a^2}}{a} + C$

183. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$

184. $\int \frac{x^2\,dx}{X} = x - a\tan^{-1}\frac{x}{a} + C$

185. $\int \frac{x^3\,dx}{X} = \frac{x^2}{2} - \frac{a^2}{2}\ln|x^2+a^2| + C$

186. $\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

187. $\int \frac{dx}{x^2X} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a} + C$

188. $\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

189. $\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

190. $\int \frac{x\,dx}{X^2} = -\frac{1}{2X} + C$

191. $\int \frac{x^2\,dx}{X^2} = -\frac{x}{2X} + \frac{1}{2a}\tan^{-1}\frac{x}{a} + C$

192. $\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$

193. $\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x}{X}| + C$

194. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4X} - \frac{x}{2a^4X} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a} + C$

195. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4X} - \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

196. $\int \frac{dx}{X^3} = \frac{x}{4a^2X^2} + \frac{3x}{8a^4X} + \frac{3}{8a^5}\tan^{-1}\frac{x}{a} + C$

197. $\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{(2(n-1)a^2}\int\frac{dx}{X^{n-1}}, \qquad n > 1$

198. $\int \frac{x\,dx}{X^n} = -\frac{1}{2(n-1)X^{n-1}} + C$

199. $\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int\frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = x^2 + a^2$

200. $\int \sqrt{X}\,dx = \frac{1}{2}xX + \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

201. $\displaystyle\int x\sqrt{X}\,dx = \frac{1}{3}X^{3/2} + C$

202. $\displaystyle\int x^2\sqrt{X}\,dx = \frac{1}{4}xX^{3/2} - \frac{1}{8}a^2x\sqrt{X} - \frac{a^2}{8}\ln|x+\sqrt{X}| + C$

203. $\displaystyle\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{a^2}{3}X^{3/2} + C$

204. $\displaystyle\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$

205. $\displaystyle\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x+\sqrt{X}| + C$

206. $\displaystyle\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$

207. $\displaystyle\int \frac{dx}{\sqrt{X}} = \ln|x+\sqrt{X}| + C \quad \text{or} \quad \sinh^{-1}\frac{x}{a} + C$

208. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

209. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

210. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

211. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

212. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

213. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

214. $\displaystyle\int X^{3/2}\,dx = \frac{1}{4}X^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x+\sqrt{X}| + C$

215. $\displaystyle\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

216. $\displaystyle\int x^2X^{3/2}\,dx = \frac{1}{6}X^{5/2} - \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} - \frac{1}{16}a^6\ln|x+\sqrt{X}| + C$

217. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

218. $$\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} + a^2\sqrt{X} - a^3 \ln|\frac{a+\sqrt{X}}{x}| + C$$

219. $$\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2 \ln|x+\sqrt{X}| + C$$

220. $$\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a \ln|\frac{a+\sqrt{x}}{x}| + C$$

221. $$\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$$

222. $$\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$$

223. $$\int \frac{x^2\,dx}{X^{3/2}} = \frac{-x}{\sqrt{X}} + \ln|x+\sqrt{X}| + C$$

224. $$\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$$

225. $$\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$$

226. $$\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} - \frac{x}{a^4\sqrt{X}} + C$$

227. $$\int \frac{dx}{x^3X^{3/2}} = \frac{-1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\ln|\frac{a+\sqrt{X}}{x}| + C$$

228. $$\int f(x,\sqrt{X})\,dx = a\int f(a\tan u, a\sec u)\,\sec^2 u\,du, \qquad x = a\tan u$$

Integrals containing $X = x^2 - a^2$ with $x^2 > a^2$

229. $$\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C \quad \text{or} \quad -\frac{1}{a}\coth^{-1}\frac{x}{a} + C \quad \text{or} \quad -\frac{1}{a}\tanh^{-1}\frac{a}{x} + C$$

230. $$\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$$

231. $$\int \frac{x^2\,dx}{X} = x + \frac{a}{2}\ln|\frac{x-a}{x+a}| + C$$

232. $$\int \frac{x^3\,dx}{X} = \frac{x^2}{2} + \frac{a^2}{2}\ln|X| + C$$

233. $$\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{X}{x^2}| + C$$

234. $$\int \frac{dx}{x^2X} = \frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{x-a}{x+a}| + C$$

235. $$\int \frac{dx}{x^3 X} = \frac{1}{2a^2 x} - \frac{1}{2a^4} \ln|\frac{x^2}{X}| + C$$

236. $$\int \frac{dx}{X^2} = \frac{-x}{2a^2 X} - \frac{1}{4a^3} \ln|\frac{x-a}{x+a}| + C$$

237. $$\int \frac{x\,dx}{X^2} = \frac{-1}{2X} + C$$

238. $$\int \frac{x^2\,dx}{X^2} = \frac{-x}{2X} + \frac{1}{4a} \ln|\frac{x-a}{x+a}| + C$$

239. $$\int \frac{x^3\,dx}{X^2} = \frac{-a^2}{2X} + \frac{1}{2} \ln|X| + C$$

240. $$\int \frac{dx}{xX^2} = \frac{-1}{2a^2 X} + \frac{1}{2a^4} \ln|\frac{x^2}{X}| + C$$

241. $$\int \frac{dx}{x^2 X^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4 X} - \frac{3}{4a^5} \ln|\frac{x-a}{x+a}| + C$$

242. $$\int \frac{dx}{x^3 X^2} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^4 X} + \frac{1}{a^6} \ln|\frac{x^2}{X}| + C$$

243. $$\int \frac{dx}{X^n} = \frac{-x}{2(n-1)a^2 X^{n-1}} - \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{X^{n-1}}, \qquad n > 1$$

244. $$\int \frac{x\,dx}{X^n} = \frac{-1}{2(n-1)X^{n-1}} + C$$

245. $$\int \frac{dx}{xX^n} = \frac{-1}{2(n-1)a^2 X^{n-1}} - \frac{1}{a^2} \int \frac{dx}{xX^{n-1}}$$

Integrals containing the square root of $X = x^2 - a^2$ with $x^2 > a^2$

246. $$\int \sqrt{X}\,dx = \frac{1}{2} x\sqrt{X} - \frac{a^2}{2} \ln|x + \sqrt{X}| + C$$

247. $$\int x\sqrt{X}\,dx = \frac{1}{3} X^{3/2} + C$$

248. $$\int x^2\sqrt{X}\,dx = \frac{1}{4} xX^{3/2} + \frac{1}{8} a^2 x\sqrt{X} - \frac{a^4}{8} \ln|x + \sqrt{X}| + C$$

249. $$\int x^3\sqrt{X}\,dx = \frac{1}{5} X^{5/2} + \frac{1}{3} a^2 X^{3/2} + C$$

250. $$\int \frac{X}{x}\,dx = \sqrt{X} - a\sec^{-1}|\frac{x}{a}| + C$$

251. $\displaystyle\int \frac{X}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x+\sqrt{X}| + C$

252. $\displaystyle\int \frac{X}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\sec^{-1}|\frac{x}{a}| + C$

253. $\displaystyle\int \frac{dx}{\sqrt{X}} = \ln|x+\sqrt{X}| + C$

254. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

255. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

256. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} + a^2\sqrt{X} + C$

257. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\sec^{-1}|\frac{x}{a}| + C$

258. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2x} + C$

259. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}|\frac{x}{a}| + C$

260. $\displaystyle\int X^{3/2}\,dx = \frac{x}{4}X^{3/2} - \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x+\sqrt{X}| + C$

261. $\displaystyle\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

262. $\displaystyle\int x^2X^{3/2}\,dx = \frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\ln|x+\sqrt{X}| + C$

263. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} + \frac{1}{5}a^2X^{5/2} + C$

264. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + a^3\sec^{-1}|\frac{x}{a}| + C$

265. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\ln|x+\sqrt{X}| + C$

266. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a\sec^{-1}|\frac{x}{a}| + C$

267. $\displaystyle\int \frac{dx}{X^{3/2}} = -\frac{x}{a^2\sqrt{X}} + C$

268. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$

269. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = -\frac{x}{\sqrt{X}} - \frac{a^2}{\sqrt{X}} + C$

270. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \ln|x + \sqrt{X}| + C$

271. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{-1}{a^2\sqrt{X}} - \frac{1}{a^3}\sec^{-1}|\frac{x}{a}| + C$

272. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} - \frac{x}{a^4\sqrt{X}} + C$

273. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\sec^{-1}|\frac{x}{a}| + C$

Integrals containing $X = a^2 - x^2$ with $x^2 < a^2$

274. $\displaystyle\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{or} \quad \frac{1}{a}\tanh^{-1}\frac{x}{a} + C$

275. $\displaystyle\int \frac{x\,dx}{X} = -\frac{1}{2}\ln X + C$

276. $\displaystyle\int \frac{x^2\,dx}{X} = -x + \frac{a}{2}\ln|\frac{a+x}{a-x}| + C$

277. $\displaystyle\int \frac{x^3\,dx}{X} = -\frac{1}{2}x^2 - \frac{a^2}{2}\ln|X| + C$

278. $\displaystyle\int \frac{d}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

279. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{a+x}{a-x}| + C$

280. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

281. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{4a^3}\ln|\frac{a+x}{a-x}| + C$

282. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{1}{2X} + C$

283. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{x}{2X} - \frac{1}{4a}\ln|\frac{a+x}{a-x}| + C$

284. $\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$

285. $\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

286. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} + \frac{x}{2a^4X} + \frac{3}{4a^5}\ln|\frac{a+x}{a-x}| + C$

287. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} + \frac{1}{2a^4X} + \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

288. $\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{X^{n-1}}$

289. $\int \frac{x\,dx}{X^n} = \frac{1}{2(n-1)X^{n-1}} + C$

290. $\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int \frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = a^2 - x^2$ with $x^2 < a^2$

291. $\int \sqrt{X}\,dx = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

292. $\int x\sqrt{X}\,dx = -\frac{1}{3}X^{3/2} + C$

293. $\int x^2\sqrt{X}\,dx = -\frac{1}{4}xX^{3/2} + \frac{1}{8}a^2x\sqrt{X} + \frac{1}{8}a^4\sin^{-1}\frac{x}{a} + C$

294. $\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{1}{3}a^2X^{3/2} + C$

295. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$

296. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} - \sin^{-1}\frac{x}{a} + C$

297. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$

298. $\int \frac{dx}{\sqrt{X}} = \sin^{-1}\frac{x}{a} + C$

299. $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + C$

300. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

301. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

302. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

303. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

304. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

305. $\displaystyle\int X^{3/2}\,dx = \frac{1}{4}xX^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a} + C$

306. $\displaystyle\int xX^{3/2}\,dx = -\frac{1}{5}X^{5/2} + C$

307. $\displaystyle\int x^2X^{3/2}\,dx = -\frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} + \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\sin^{-1}\frac{x}{a} + C$

308. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

309. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2}a^2\sqrt{X} - a^3\ln|\frac{a+\sqrt{X}}{x}| + C$

310. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a} + C$

311. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} - \frac{3}{2}\sqrt{X} + \frac{3}{2}a\ln|\frac{a+\sqrt{X}}{x}| + C$

312. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$

313. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{1}{\sqrt{X}} + C$

314. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{x}{\sqrt{X}} - \sin^{-1}\frac{x}{a} + C$

315. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$

316. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

317. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} + \frac{x}{a^4\sqrt{X}} + C$

318. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\ln|\frac{a+\sqrt{X}}{x}| + C$

Integrals Containing $X = x^3 + a^3$

319. $\displaystyle\int \frac{dx}{X} = \frac{1}{6a^2}\ln|\frac{(x+a)^3}{X}| + \frac{1}{\sqrt{3}a^2}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

320. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{6a}\ln|\frac{X}{(x+a)^3}| + \frac{1}{\sqrt{3}a}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

321. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{2}\ln|X| + C$

322. $\displaystyle\int \frac{dx}{xX} = \frac{1}{3a^3}\ln|\frac{x^3}{X}| + C$

323. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} - \frac{1}{6a^4}\ln|\frac{X}{(x+a)^3}| - \frac{1}{\sqrt{3}a^4}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

324. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{1}{9a^5}\ln|\frac{(x+a)^3}{X}| + \frac{2}{3\sqrt{3}a^5}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

325. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3X} + \frac{1}{18a^4}\ln|\frac{X}{(x+a)^3}| + \frac{1}{3\sqrt{3}a^4}\tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

326. $\displaystyle\int \frac{x^2\,dx}{X^2} = -\frac{1}{3X} + C$

327. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{3a^2X} + \frac{1}{3a^6}\ln|\frac{x^3}{X}| + C$

328. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6X} - \frac{4}{3a^6}\int\frac{x\,dx}{X}$

329. $\displaystyle\int \frac{dx}{X^3} = \frac{1}{54a^3}\left[\frac{9a^5x}{X^2} + \frac{15a^2x}{X} + 10\sqrt{3}\tan^{-1}(\frac{2x-a}{\sqrt{3}a}) + 10\ln|x+a| - 5\ln|x^2-ax+a^2|\right] + C$

Integrals containing $X = x^4 + a^4$

330. $\displaystyle\int \frac{dx}{X} = \frac{1}{4\sqrt{2}a^3}\ln|\frac{X}{(x^2-\sqrt{2}ax+a^2)^2}| - \frac{1}{2\sqrt{2}a^3}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

331. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

332. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{4\sqrt{2}a}\ln|\frac{X}{(x^2+\sqrt{2}ax+a^2)^2}| - \frac{1}{2\sqrt{2}a}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

333. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{1}{4}\ln|X| + C$

334. $\displaystyle\int \frac{dx}{xX} = \frac{1}{4a^4}\ln|\frac{x^4}{X}| + C$

335. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^4x} - \frac{1}{\sqrt{2}4a^5}\ln|\frac{(x^2-\sqrt{2}ax+a^2)^2}{X}| + \frac{1}{2\sqrt{2}a^5}\tan^{-1}\left(\frac{\sqrt{2}ax}{x^2-a^2}\right) + C$

336. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

Integrals containing $X = x^4 - a^4$

337. $\displaystyle\int \frac{dx}{X} = \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C$

338. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C$

339. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{4a}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C$

340. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{1}{4}\ln|X| + C$

341. $\displaystyle\int \frac{dx}{xX} = \frac{1}{4a^4}\ln\left|\frac{X}{x^4}\right| + C$

342. $\displaystyle\int \frac{dx}{x^2X} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a^5}\tan^{-1}\left(\frac{x}{a}\right) + C$

343. $\displaystyle\int \frac{dx}{x^3X} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C$

Miscelaneous algebraic integrals

344. $\displaystyle\int \frac{dx}{b^2+(x+a)^2} = \frac{1}{b}\tan^{-1}\frac{x+a}{b} + C$

345. $\displaystyle\int \frac{dx}{b^2-(x+a)^2} = \frac{1}{b}\tanh^{-1}\frac{x+a}{b} + C$

346. $\displaystyle\int \frac{dx}{(x+a)^2-b^2} = -\frac{1}{b}\coth^{-1}\frac{x+a}{b} + C$

347. $\displaystyle\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} + C$

348. $\displaystyle\int \frac{dx}{\sqrt{x(a+x)}} = 2\sinh^{-1}\sqrt{\frac{x}{a}} + C$

349. $\displaystyle\int \frac{dx}{\sqrt{x(x-a)}} = 2\cosh^{-1}\sqrt{\frac{x}{a}} + C$

350. $\displaystyle\int \frac{dx}{(b+x)(a-x)} = 2\tan^{-1}\sqrt{\frac{b+x}{a-x}} + C, \qquad a > x$

351. $\displaystyle\int \frac{dx}{(x-b)(a-x)} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C, \qquad a > x > b$

352. $\displaystyle\int \frac{dx}{(x+b)(x+a)} = \begin{cases} 2\tanh^{-1}\sqrt{\frac{x+b}{x+a}} + C_1, & a > b \\ 2\tanh^{-1}\sqrt{\frac{x+a}{x+b}} + C_2, & a < b \end{cases}$

353. $\displaystyle\int \frac{dx}{x\sqrt{x^{2n}-a^{2n}}} = -\frac{1}{na^n}\sin^{-1}\left(\frac{a^n}{x^n}\right) + C$

354. $\displaystyle\int \sqrt{\frac{x+a}{x-a}}\,dx = \sqrt{x^2-a^2} + a\cosh^{-1}\frac{x}{a} + C$

355. $\displaystyle\int \sqrt{\frac{a+x}{a-x}}\,dx = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$

356. $\displaystyle\int x\sqrt{\frac{a-x}{a+x}}\,dx = \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{(x-2a)}{2}\sqrt{a^2-x^2} + C, \quad a > x$

357. $\displaystyle\int x\sqrt{\frac{a+x}{a-x}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x+2a}{2}\sqrt{a^2-x^2} + C$

358. $\displaystyle\int (x+a)\sqrt{\frac{x+b}{x-b}}\,dx = (x+a+b)\sqrt{x^2-b^2} + \frac{b}{2}(2a+b)\cosh^{-1}\frac{x}{b} + C$

359. $\displaystyle\int \frac{dx}{\sqrt{2ax+x^2}} = \ln|x+a+\sqrt{2ax+x^2}| + C$

360. $\displaystyle\int \sqrt{ax^2+c}\,dx = \begin{cases} \frac{1}{2}x\sqrt{ax^2+c} + \frac{c}{2\sqrt{a}}\ln|\sqrt{a}x+\sqrt{ax^2+c}| + c, & a > 0 \\ \frac{1}{2}x\sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(\sqrt{\frac{-a}{c}}\,x\right) + C, & a < 0 \end{cases}$

361. $\displaystyle\int \sqrt{\frac{1+ax}{1-ax}}\,dx = \frac{1}{a}\sin^{-1}x - \frac{1}{a}\sqrt{1-x^2} + C$

362. $\displaystyle\int \frac{dx}{(ax+b)^2+(cx+d)^2} = \frac{1}{ad-bc}\tan^{-1}\left[\frac{(a^2+c^2)x+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$

363. $\displaystyle\int \frac{dx}{(ax+b)^2-(cx+d)^2} = \frac{1}{2(bc-ad)}\ln\left|\frac{(a+c)x+(b+d)}{(a-c)x+(b-d)}\right| + C, \quad ad-bc \neq 0$

364. $\displaystyle\int \frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2} = \frac{1}{2(ad-bc)}\tan^{-1}\left[\frac{(a^2+c^2)x^2+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$

365. $\displaystyle\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$

366. $\displaystyle\int \frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}\,dx = x + \frac{1}{d^2-c^2}\left[\frac{(a^2-c^2)(b^2-c^2)}{c}\tan^{-1}\frac{x}{c} - \frac{(a^2-d^2)(b^2-d^2)}{d}\tan^{-1}\frac{x}{d}\right] + C$

367. $\displaystyle\int \frac{ax^2+b}{(cx^2+d)(ex^2+f)}\,dx = \frac{1}{\sqrt{cd}}\left(\frac{ad-bc}{ed-fc}\right)\tan^{-1}\left(\sqrt{\frac{c}{d}}\,x\right) + \frac{1}{\sqrt{ef}}\left(\frac{af-be}{fc-ed}\right)\tan^{-1}\left(\sqrt{\frac{e}{f}}\,x\right) + C$

368. $\displaystyle\int \frac{x\,dx}{(ax^2+bx+c)^2+(ax^2-bx+c)^2} = \frac{1}{4b\sqrt{b^2+4ac}}\ln\left|\frac{2a^2x^2+2ac+b^2-b\sqrt{b^2+4ac}}{2a^2x^2+2ac+b^2+b\sqrt{b^2+4ac}}\right| + C, \quad b^2+4ac > 0$

369. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$$

370. $$\int \frac{(x^2+\alpha^2)(x^2+\beta^2)}{(x^2+\gamma^2)(x^2+\delta^2)}\,dx = x + \frac{1}{\delta^2-\gamma^2}\left[\frac{(\alpha^2-\gamma^2)(\beta^2-\gamma^2)}{\gamma}\tan^{-1}\frac{x}{\gamma} - \frac{(\alpha^2-\delta^2)(\beta^2-\delta^2)}{\delta}\tan^{-1}\frac{x}{\delta}\right] + C$$

371. $$\int \frac{\alpha x^2+\beta}{(\gamma x^2+\delta)(\epsilon x^2+\zeta)}\,dx = \frac{1}{\sqrt{\gamma\delta}}\frac{\alpha\delta-\beta\gamma}{\epsilon\delta-\zeta\gamma}\tan^{-1}\left(\sqrt{\frac{\gamma}{\delta}}\,x\right) + \frac{1}{\sqrt{\epsilon\zeta}}\frac{\alpha\zeta-\beta\epsilon}{\zeta\gamma-\epsilon\delta}\tan^{-1}\left(\sqrt{\frac{\epsilon}{\zeta}}\,x\right) + C$$

372. $$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \cosh^{-1}\left(\frac{2x+a+b}{a-b}\right) + C, \quad a \neq b$$

373. $$\int \frac{dx}{\sqrt{(x-b)(a-x)}} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C$$

374. $$\int \frac{dx}{(\alpha x+\beta)^2+(\gamma x+\delta)^2} = \frac{1}{\alpha\delta-\beta\gamma}\tan^{-1}\left[\frac{(\alpha^2+\gamma^2)x+(\alpha\beta+\gamma\delta)}{\alpha\delta-\beta\gamma}\right] + C$$

375. $$\int \frac{x\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2ab}\sin^{-1}\left[\frac{(a^2+b^2)x^2-(a^4+b^4)}{(a^2-b^2)(a^2+b^2-x^2)}\right] + C$$

376. $$\int \frac{(x+b)\,dx}{(x^2+a^2)\sqrt{x^2+c^2}} = \frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{x^2+c^2}{x^2+a^2}} + \frac{b}{a\sqrt{a^2-c^2}}\cosh^{-1}\left[\frac{a}{c}\sqrt{\frac{x^2+c^2}{x^2+a^2}}\right] + C$$

377. $$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{2a}\ln|ax^2+bx+c| + \left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

378. $$\int \frac{(\sqrt{a}-\sqrt{x})^2}{(a^2+ax+x^2)\sqrt{x}}\,dx = \frac{2\sqrt{3}}{\sqrt{a}}\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}} - \frac{2}{\sqrt{3a}}\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}} + C$$

379. $$\int (a+x)\sqrt{a^2+x^2}\,dx = \frac{1}{6}(2x^2+3ax+2a^2)\sqrt{a^2+x^2} + \frac{1}{2}a^2\sinh^{-1}\frac{x}{a} + C$$

380. $$\int \frac{x^2+a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{a\sqrt{3}}\tan^{-1}\frac{ax\sqrt{3}}{a^2-x^2} + C$$

381. $$\int \frac{x^2-a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{2a^3}\ln\frac{x^2-ax+a^2}{x^2+ax+a^2} + C$$

Integrals containing $\sin ax$

382. $$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

383. $$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$

384. $$\int x^2\sin ax\,dx = \frac{2}{a^2}x\sin ax + \left(\frac{2}{a^3}-\frac{x^2}{a}\right)\cos ax + C$$

385. $\displaystyle\int x^3 \sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a} - \frac{x^3}{a}\right)\cos ax + C$

386. $\displaystyle\int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx$

387. $\displaystyle\int \frac{\sin ax}{x}\,dx = ax - \frac{a^3x^3}{3\cdot 3!} + \frac{a^5x^5}{5\cdot 5!} - \frac{a^7x^7}{7\cdot 7!} + \cdots + \frac{(-1)^n x^{2n+1}x^{2n+1}}{(2n+1)\cdot(2n+1)!} + \cdots$

388. $\displaystyle\int \frac{\sin ax}{x^2}\,dx = -\frac{1}{a}\sin ax + a\int \frac{\cos ax}{x}\,dx$

389. $\displaystyle\int \frac{\sin ax}{x^3}\,dx = -\frac{a}{2x}\cos ax - \frac{1}{2x^2}\sin ax - \frac{a^2}{2}\int \frac{\sin ax}{x}\,dx$

390. $\displaystyle\int \frac{\sin ax}{x^n}\,dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$

391. $\displaystyle\int \frac{dx}{\sin ax} = \frac{1}{a}\ln|\csc as - \cot ax| + C$

392. $\displaystyle\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left[ax + \frac{a^3x^3}{18} + \frac{7a^5x^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$

where $\mathfrak{B}_n$ is the nth Bernoulli number $\mathfrak{B}_1 = 1/6, \mathfrak{B}_2 = 1/30, \ldots$ Note scaling and shifting

393. $\displaystyle\int \frac{dx}{x\sin ax} = -\frac{1}{ax} + \frac{ax}{6} + \frac{7a^3x^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n-1)(2n)!} + \cdots + C$

394. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

395. $\displaystyle\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2} + C$

396. $\displaystyle\int x^2\sin^2 ax\,dx = \frac{1}{6a} - \frac{1}{4a^2}\cos 2ax + \frac{1}{24a^3}(3 - 6a^2x^2)\sin 2ax + C$

397. $\displaystyle\int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^2 ax}{3a} + C$

398. $\displaystyle\int x\sin^3 ax\,dx = \frac{1}{12a}x\cos 3ax - \frac{1}{36a^2}\sin 3ax - \frac{3}{4a}x\cos ax + \frac{3}{4a^2}\sin ax + C$

399. $\displaystyle\int \sin^4 ax\,dx = \frac{3}{8}x - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} + C$

400. $\displaystyle\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax + C$

401. $$\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\cot ax + \frac{1}{a^2}\ln|\sin ax| + C$$

402. $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln|\tan\frac{ax}{2}| + C$$

403. $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

404. $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

405. $$\int \frac{dx}{a-\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{a\tan(ax/2)-1}{\sqrt{a^2-1}}\right] + C, \qquad a > 1$$

406. $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$$

407. $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

408. $$\int \frac{dx}{a+\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{1+a\tan(ax/2)}{\sqrt{a^2-1}}\right] + C, \qquad a > 1$$

409. $$\int \frac{x\,dx}{1+\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$$

410. $$\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$$

411. $$\int \frac{dx}{1-\sin^2 x} = \tan x + C$$

412. $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

413. $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$$

414. $$\int \frac{dx}{\alpha+\beta\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\alpha\tan\dfrac{ax}{2} + \beta\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan\frac{ax}{2} + \beta - \sqrt{\beta^2-\alpha^2}}{\alpha\tan\frac{ax}{2} + \beta + \sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \\ \dfrac{1}{a\alpha}\tan\left(\dfrac{ax}{2} \pm \dfrac{\pi}{4}\right) + C, & \beta = \pm\alpha \end{cases}$$

415. $$\int \frac{dx}{\alpha^2+\beta^2\sin^2 ax} = \frac{1}{a\alpha\sqrt{\beta^2+\alpha^2}}\tan^{-1}\left(\frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\tan ax\right) + C$$

416. $$\int \frac{dx}{\alpha^2 - \beta^2 \sin^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\alpha^2-\beta^2}} \tan^{-1}\left(\dfrac{\sqrt{\alpha^2-\beta^2}}{\alpha}\tan ax\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}} \ln\left|\dfrac{\sqrt{\beta^2-\alpha^2}\tan ax + \alpha}{\sqrt{\beta^2-\alpha^2}\tan ax - \alpha}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

417. $$\int \sin^n ax\,dx = -\frac{1}{an}\sin^{n-1} ax \cos ax + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$$

418. $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

419. $$\int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$$

420. $$\int \frac{\alpha + \beta\sin ax}{1 \pm \sin ax}\,dx = \beta x + \frac{\alpha \mp \beta}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + C$$

421. $$\int \frac{\alpha + \beta\sin ax}{a + b\sin ax}\,dx = \frac{\beta}{b}x + \frac{\alpha b - a\beta}{b}\int \frac{dx}{a + b\sin ax}$$

422. $$\int \frac{dx}{\alpha + \frac{\beta}{\sin ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int \frac{dx}{\beta + \alpha\sin ax}$$

Integrals containing $\cos ax$

423. $$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

424. $$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

425. $$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax + C$$

426. $$\int x^n \cos ax\,dx = \frac{1}{a}x^n \sin ax + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx$$

427. $$\int \frac{\cos ax}{x}\,dx = \ln|x| - \frac{a^2x^2}{2\cdot 2!} + \frac{a^4x^4}{4\cdot 4!} - \frac{a^6x^6}{6\cdot 6!} + \cdots + \frac{(-1)^n a^{2n}x^{2n}}{(2n)\cdot(2n)!} + \cdots + C$$

428. $$\int \frac{\cos ax\,dx}{x^n} = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}\,dx$$

429. $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

430. $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^2x^2}{2} + \frac{a^4x^4}{4\cdot 2!} + \frac{5a^6x^6}{6\cdot 4!} + \cdots + \frac{\mathfrak{E}_n a^{2n+2}x^{2n+2}}{(2n+2)\cdot(2n)!} + \cdots\right] + C$$

431. $$\int \frac{dx}{x\cos ax} = \ln|x| + \frac{a^2x^2}{4} + \frac{5a^4x^4}{96} + \cdots + \frac{\mathfrak{E}_n a^{2n}x^{2n}}{2n(2n)!} + \cdots + C$$

where $\mathfrak{E}_n$ is the nth Euler number $\mathfrak{E}_1 = 1, \mathfrak{E}_2 = 5, \mathfrak{E}_3 = 61, \ldots$ Note scaling and shifting

432. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

433. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

434. $\displaystyle\int \sqrt{1-\cos ax}\,dx = -2\sqrt{2}\cos\frac{ax}{2} + C$

435. $\displaystyle\int \sqrt{1+\cos ax}\,dx = 2\sqrt{2}\sin\frac{ax}{2} + C$

436. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

437. $\displaystyle\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{1}{4a}x\sin 2ax + \frac{1}{8a^2}\cos 2ax + C$

438. $\displaystyle\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a} + C$

439. $\displaystyle\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax + C$

440. $\displaystyle\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax + C$

441. $\displaystyle\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln|\cos ax| + C$

442. $\displaystyle\int \frac{dx}{\cos^3 ax} = \frac{1}{2a}\frac{\sin ax}{\cos^2 ax} + \frac{1}{2a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| + C$

443. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

444. $\displaystyle\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right| + C$

445. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

446. $\displaystyle\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right| + C$

447. $\displaystyle\int \frac{dx}{1+\cos^2 ax} = -\frac{1}{\sqrt{2}a}\tan^{-1}(\sqrt{2}\cot ax) + C$

448. $\displaystyle\int \frac{dx}{1-\cos^2 ax} = -\frac{1}{a}\cot ax + C$

449. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2} + C$$

450. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^2\frac{ax}{2} + C$$

451. $$\int \frac{dx}{\alpha+\beta\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\sqrt{\dfrac{\alpha-\beta}{\alpha+\beta}}\tan\dfrac{ax}{2}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\sqrt{\beta+\alpha}+\sqrt{\beta-\alpha}\tan\frac{ax}{2}}{\sqrt{\beta+\alpha}-\sqrt{\beta-\alpha}\tan\frac{ax}{2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

452. $$\int \frac{dx}{\alpha+\frac{\beta}{\cos ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int\frac{dx}{\beta+\alpha\cos ax}$$

453. $$\int \frac{dx}{(\alpha+\beta\cos ax)^2} = \frac{\alpha\sin ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cos ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int\frac{dx}{\alpha+\beta\cos ax}, \qquad \alpha\neq\beta$$

454. $$\int \frac{dx}{\alpha^2+\beta^2\cos^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tan^{-1}\left(\frac{\alpha\tan ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$$

455. $$\int \frac{dx}{\alpha^2-\beta^2\cos^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\dfrac{\alpha\tan ax}{\sqrt{\alpha^2-\beta^2}}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan ax-\sqrt{\beta^2-\alpha^2}}{\alpha\tan ax+\sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

456. $$\int \frac{dx}{\cos^n ax} = \frac{\sec^{(n-2)} ax\tan ax}{(n-1)a} + \frac{n-2}{n-1}\int\sec^{n-2} ax\,dx + C$$

Integrals containing both sine and cosine functions

457. $$\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$$

458. $$\int \frac{dx}{\sin ax\cos ax} = -\frac{1}{a}\ln|\cot ax| + C$$

459. $$\int \sin ax\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C, \qquad a\neq b$$

460. $$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C$$

461. $$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C$$

462. $$\int \sin^n ax\cos ax\,dx = \frac{sin^{n+1}ax}{(n+1)a} + C$$

463. $$\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C$$

464. $$\int \frac{\sin ax\, dx}{\cos ax} = \frac{1}{a} \ln|\sec ax| + C$$

465. $$\int \frac{\cos ax\, dx}{\sin ax} = \frac{1}{a} \ln|\sin ax| + C$$

466. $$\int \frac{x \sin ax\, dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^3x^3}{3} + \frac{a^5x^5}{5} + \frac{2a^7x^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!}\right] + C$$

467. $$\int \frac{x \cos ax\, dx}{\sin ax} = \frac{1}{a^2}\left[ax - \frac{a^3x^3}{9} - \frac{a^5x^5}{225} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} - \cdots\right] + C$$

468. $$\int \frac{\cos ax\, dx}{x \sin ax} = -\frac{1}{ax} - \frac{ax}{2} - \frac{a^3x^3}{135} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} - \cdots + C$$

469. $$\int \frac{\sin ax}{x \cos ax}\, dx = ax + \frac{a^3x^3}{9} + \frac{2a^5x^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

470. $$\int \frac{\sin^2 ax}{\cos^2 ax}\, dx = \frac{1}{a}\tan ax - x + C$$

471. $$\int \frac{\cos^2 ax}{\sin^2 ax}\, dx = -\frac{1}{a}\cot ax - x + C$$

472. $$\int \frac{x \sin^2 ax}{\cos^2 ax}\, dx = \frac{1}{a} x \tan ax + \frac{1}{a^2}\ln|\cos ax| - \frac{1}{2}x^2 + C$$

473. $$\int \frac{x \cos^2 ax}{\sin^2 ax}\, dx = -\frac{1}{a} x \cot ax + \frac{1}{a^2}\ln|\sin ax| - \frac{1}{2}x^2 + C$$

474. $$\int \frac{\cos ax}{\sin ax}\, dx = \frac{1}{a}\ln|\sin ax| + C$$

475. $$\int \frac{\sin^3 ax}{\cos^3 ax}\, dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln|\cos ax| + C$$

476. $$\int \frac{\cos^3 ax}{\sin^3 ax}\, dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\ln|\sin ax| + C$$

477. $$\int \sin(ax+b)\sin(ax+\beta)\, dx = \frac{x}{2}\cos(b-\beta) - \frac{1}{4a}\sin(2ax+b+\beta) + C$$

478. $$\int \sin(ax+b)\cos(ax+\beta)\, dx = \frac{x}{2}\sin(b-\beta) - \frac{1}{4a}\cos(2ax+b+\beta) + C$$

479. $$\int \cos(ax+b)\cos(ax+\beta)\, dx = \frac{x}{2}\cos(b-\beta) + \frac{1}{4a}\sin(2ax+b+\beta) + C$$

480. $$\int \sin^2 ax \cos^2 bx\, dx = \begin{cases} \dfrac{x}{4} - \dfrac{\sin 2ax}{8a} + \dfrac{\sin 2bx}{8b} - \dfrac{\sin 2(a-b)x}{16(a-b)} - \dfrac{\sin 2(a+b)x}{16(a+b)} + C, & b \neq a \\ \dfrac{x}{8} - \dfrac{\sin 4ax}{32a} + C, & b = a \end{cases}$$

481. $$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \ln|\tan ax| + C$$

482. $$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a} \ln|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)| - \frac{1}{a \sin ax} + C$$

483. $$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a} \ln|\tan \frac{ax}{2}| + \frac{1}{a \cos ax} + C$$

484. $$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2 \cos 2ax}{a} + C$$

485. $$\int \frac{\sin^2 ax}{\cos ax}\, dx = -\frac{\sin ax}{a} + \frac{1}{a} \ln|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)| + C$$

486. $$\int \frac{\cos^2 ax}{\sin ax}\, dx = \frac{\cos ax}{a} + \frac{1}{a} \ln|\tan \frac{ax}{2}| + C$$

487. $$\int \frac{dx}{\cos ax\,(1 + \sin ax)} = \frac{1}{2a(1 + \sin ax)} \left[-1 + (1 + \sin ax) \ln\left|\frac{\cos \frac{ax}{2} + \sin \frac{ax}{2}}{\cos \frac{ax}{2} - \sin \frac{ax}{2}}\right|\right] + C$$

488. $$\int \frac{dx}{\sin ax\,(1 + \cos ax)} = \frac{1}{4a} \sec^2 \frac{ax}{2} + \frac{1}{2a} \ln|\tan \frac{ax}{2}| + C$$

489. $$\int \frac{dx}{\sin ax\,(\alpha + \beta \sin ax)} = \frac{1}{a\alpha} \ln|\tan \frac{ax}{2}| - \frac{\beta}{\alpha} \int \frac{dx}{\alpha + \beta \sin ax}$$

490. $$\int \frac{dx}{\cos ax\,(\alpha + \beta \sin ax} = \frac{1}{\alpha^2 - \beta^2} \left[\frac{\alpha}{a} \ln|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)| - \frac{\beta}{\alpha} \ln\left|\frac{\alpha + \beta \sin ax}{\cos ax}\right|\right] + C, \qquad \beta \neq \alpha$$

491. $$\int \frac{dx}{\sin ax\,(\alpha + \beta \cos ax)} = \frac{1}{\alpha^2 - \beta^2} \left[\frac{\alpha}{a} \ln|\tan \frac{ax}{2}| + \frac{\beta}{a} \ln|\frac{\alpha + \beta \cos ax}{\sin ax}|\right] + C, \qquad \beta \neq \alpha$$

492. $$\int \frac{dx}{\cos ax\,(\alpha + \beta \cos ax)} = \frac{1}{a\alpha} \ln|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)| - \frac{\beta}{\alpha} \int \frac{dx}{\alpha + \beta \cos ax}$$

493. $$\int \frac{dx}{\alpha + \beta \cos ax + \gamma \sin ax} = \begin{cases} \dfrac{2}{a\sqrt{-R}} \tan^{-1}\left(\dfrac{\gamma + (\alpha - \beta)\tan \frac{ax}{2}}{\sqrt{-R}}\right) + C, & \begin{array}{c}\alpha^2 > \beta^2 + \gamma^2 \\ R = \beta^2 + \gamma^2 - \alpha^2\end{array} \\ \dfrac{1}{a\sqrt{R}} \ln\left|\dfrac{\gamma - \sqrt{R} + (\alpha - \beta)\tan \frac{ax}{2}}{\gamma + \sqrt{R} + (\alpha - \beta)\tan \frac{ax}{2}}\right| + C, & \alpha^2 < \beta^2 + \gamma^2 \\ \dfrac{1}{a\beta} \ln\left|\beta + \gamma \tan \dfrac{ax}{2}\right| + C, & \alpha = \beta \\ \dfrac{1}{a\beta} \ln\left|\dfrac{cos \frac{ax}{2} + \sin \frac{ax}{2}}{(\beta + \gamma)\cos \frac{ax}{2} + (\gamma - \beta)\sin \frac{ax}{2}}\right| + C, & \alpha = \gamma \\ \dfrac{1}{a\gamma} \ln|1 + \tan \dfrac{ax}{2}| + C, & \alpha = \beta = \gamma \end{cases}$$

494. $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{\sqrt{2}a} \ln|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)| + C$$

495. $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \ln|\sin ax \pm \cos ax| + C$$

496. $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln|\sin axx \pm \cos ax| + C$$

497. $$\int \frac{\sin ax\,dx}{\alpha + \beta \sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta\sin ax| + C$$

498. $$\int \frac{\cos ax\,dx}{\alpha + \beta \sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta\sin ax| + C$$

499. $$\int \frac{\sin ax \cos ax\,dx}{\alpha^2\cos^2 ax + \beta^2 ax} = \frac{1}{2a(\beta^2 - \alpha^2)}\ln|\alpha^2\cos^2 ax + \beta^2\sin^2 ax| + C, \quad \beta \neq \alpha$$

500. $$\int \frac{dx}{\alpha^2\sin^2 ax + \beta^2\cos^2 ax} = \frac{1}{a\alpha\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\tan ax\right) + C$$

501. $$\int \frac{dx}{\alpha^2\sin^2 ax - \beta^2\cos^2 ax} = \frac{1}{2a\alpha\beta}\ln\left|\frac{\alpha\tan ax - \beta}{\alpha\tan ax + \beta}\right| + C$$

502. $$\int \frac{\sin^n ax}{\cos^{(n+2} ax}\,dx = \frac{\tan^{n+1} ax}{(n+1)a} + C$$

503. $$\int \frac{\cos^n ax}{\sin^{(n+2)} ax}\,dx = -\frac{\cot^{(n+1)} ax}{(n+1)a} + C$$

504. $$\int \frac{dx}{\alpha + \beta\frac{\sin ax}{\cos ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} + \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\beta\sin ax + \alpha\cos ax| + C$$

505. $$\int \frac{dx}{\alpha + \beta\frac{\cos ax}{\sin ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} - \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\alpha\sin ax + \beta\cos ax| + C$$

506. $$\int \frac{\cos^n ax}{\sin^n ax}\,dx = -\frac{\cot^{(n-1)} ax}{(n-1)a} - \int \cot^{(n-2)} ax\,dx$$

507. $$\int \frac{\sin^n ax}{\cos^n ax}\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \frac{\sin^{n-2} ax}{\cos^{n-2} ax}\,dx$$

508. $$\int \frac{\sin ax}{\cos^{(n+1)} ax}\,dx = \frac{1}{na}\sec^n ax + C$$

509. $$\int \frac{\alpha\sin x + \beta\cos x}{\gamma\sin x + \delta\cos x}\,dx = \frac{[(\alpha\gamma + \beta\delta)x + (\beta\gamma - \alpha\gamma)\ln|\gamma\sin x + \delta\cos x|]}{\gamma^2 + \delta^2} + C$$

510. $$\int \frac{\alpha + \beta\sin x}{a + b\cos x}\,dx = \begin{cases} \dfrac{2\alpha}{\sqrt{a^2 - b^2}}\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2} - \dfrac{\beta}{b}\ln|a + b\cos x| + C, & a > b \\ \dfrac{2\alpha}{\sqrt{b^2 - a^2}}\tanh^{-1}\sqrt{\dfrac{b-a}{b+a}}\tan\dfrac{x}{2} - \dfrac{\beta}{b}\ln|a + b\cos x| + C, & a < b \end{cases}$$

511. $$\int \frac{dx}{a^2 - b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{a}{\sqrt{a^2-b^2}}\tan x\right) + C, & a > b \\ \frac{-1}{a\sqrt{b^2-a^2}}\tanh^{-1}\left(\frac{a}{\sqrt{b^2-a^2}}\tan x\right) + C, & b > a \end{cases}$$

512. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = \frac{1}{a^2+b^2}\tan\left(x - \tan^{-1}\frac{b}{a}\right) + C$$

513. $$\int \frac{\sin x\, dx}{\sqrt{a\cos^2 x + 2b\cos x + c}} = \begin{cases} \dfrac{-1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, & b^2 > ac,\ a < 0 \\ \dfrac{-1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, & b^2 > ac,\ a > 0 \\ \dfrac{-1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{ac-b^2}}\right) + C, & b^2 < ac,\ a > 0 \end{cases}$$

514. $$\int \frac{\cos x\, dx}{\sqrt{a\sin^2 x + 2b\sin x + c}} = \begin{cases} \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, & b^2 > ac,\ a < 0 \\ \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, & b^2 > ac,\ a > 0 \\ \dfrac{1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{ac-b^2}}\right) + C, & b^2 < ac,\ a > 0 \end{cases}$$

Integrals containing $\tan ax$, $\cot ax$, $\sec ax$, $\csc ax$

Write integrals in terms of $\sin ax$ and $\cos ax$ and see previous listings.

Integrals containing inverse trigonmetric functions

515. $$\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + C$$

516. $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$$

517. $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln|x^2+a^2| + C$$

518. $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2+a^2| + C$$

519. $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ x\sec^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

520. $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ x\csc^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

521. $$\int x\sin^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{a^2-x^2} + C$$

522. $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{a^2-x^2} + C$$

523. $$\int x \tan^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln|x^2+a^2| + C$$

524. $$\int x \cot^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{a}{2}x + C$$

525. $$\int x \sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\sec^{-1}\frac{x}{a} - \frac{a}{2}\sqrt{x^2-a^2} + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{2}x^2\sec^{-1}\frac{x}{a} + \frac{a}{2}\sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

526. $$\int x \csc^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\csc^{-1}\frac{x}{a} + \frac{a}{2}\sqrt{x^2-a^2} + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{2}x^2\csc^{-1}\frac{x}{a} - \frac{a}{2}\sqrt{x^2-a^2} + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

527. $$\int x^2 \sin^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\sin^{-1}\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

528. $$\int x^2 \cos^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\cos^{-1}\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2} + C$$

529. $$\int x^2 \tan^{-1}\frac{x}{a}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln|x^2+a^2| + C$$

530. $$\int x^2 \cot^{-1}\frac{x}{a}\,dx = \frac{1}{3}\cot^{-1}\frac{x}{a} + \frac{a}{6}x^2 - \frac{a^3}{6}\ln|a^2+x^2| + C$$

531. $$\int x^2 \sec^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\sec^{-1}\frac{x}{a} - \frac{a}{6}x\sqrt{x^2-a^2} - \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{3}x^3\sec^{-1}\frac{x}{a} + \frac{a}{6}x\sqrt{x^2-a^2} + \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + c, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

532. $$\int x^2 \csc^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\csc^{-1}\frac{x}{a} + \frac{a}{6}x\sqrt{x^2-a^2} + \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ \frac{1}{3}x^3\csc^{-1}\frac{x}{a} - \frac{a}{6}x\sqrt{x^2-a^2} - \frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}| + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

533. $$\int \frac{1}{x}\sin^{-1}\frac{x}{a}\,dx = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots + C$$

534. $$\int \frac{1}{x}\cos^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| + - \int \frac{1}{x}\sin^{-1}\frac{x}{a}\,dx$$

535. $$\int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx = \frac{x}{a} - \frac{1}{3^2}\left(\frac{x}{a}\right)^3 + \frac{1}{5^2}\left(\frac{x}{a}\right)^5 - \frac{1}{7^2}\left(\frac{x}{a}\right)^7 + \cdots + C$$

536. $$\int \frac{1}{x}\cot^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| - \int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$$

537. $$\int \frac{1}{x}\sec^{-1}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| + \frac{a}{x} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\left(\frac{x}{a}\right)^7 + \cdots + C$$

538. $\displaystyle\int \frac{1}{x}\csc^{-1}\frac{x}{a}\,dx = -\left(\frac{a}{x} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\left(\frac{x}{a}\right)^7 + \cdots\right) + C$

539. $\displaystyle\int \frac{1}{x^2}\sin^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\sin^{-1}\frac{x}{a} - \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right| + C$

540. $\displaystyle\int \frac{1}{x^2}\cos^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cos^{-1}\frac{x}{a} + \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right| + C$

541. $\displaystyle\int \frac{1}{x^2}\tan^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left|\frac{x^2+a^2}{a^2}\right| + C$

542. $\displaystyle\int \frac{1}{x^2}\cot^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cot^{-1}\frac{x}{a} + \frac{1}{2a}\int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$

543. $\displaystyle\int \frac{1}{x^2}\sec^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{x}\sec^{-1}\dfrac{x}{a} + \dfrac{1}{ax}\sqrt{x^2-a^2} + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ -\dfrac{1}{x}\sec^{-1}\dfrac{x}{a} - \dfrac{1}{ax}\sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$

544. $\displaystyle\int \frac{1}{x^2}\csc^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{x}\csc^{-1}\dfrac{x}{a} - \dfrac{1}{ax}\sqrt{x^2-a^2} + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ -\dfrac{1}{x}\csc^{-1}\dfrac{x}{a} + \dfrac{1}{ax}\sqrt{x^2-a^2} + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$

545. $\displaystyle\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = (a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

546. $\displaystyle\int \cos^{-1}\sqrt{\frac{x}{a+x}}\,dx = (2a+x)\tan^{-1}\sqrt{\frac{x}{2a}} - \sqrt{2ax} + C$

Integrals containing the exponential function

547. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

548. $\displaystyle\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + C$

549. $\displaystyle\int x^2e^{ax}\,dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} + C$

550. $\displaystyle\int x^ne^{ax}\,dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

551. $\displaystyle\int \frac{1}{x}e^{ax}\,dx = \ln|x| + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots + C$

552. $\displaystyle\int \frac{1}{x^n}e^{ax}\,dx = -\frac{1}{(n-1)x^{n-1}}e^{ax} + \frac{a}{n-1}\int \frac{1}{x^{n-1}}e^{ax}\,dx$

553. $\displaystyle\int \frac{e^{ax}}{\alpha+\beta e^{ax}}\,dx = \frac{1}{a\beta}\ln|\alpha+\beta e^{ax}| + C$

554. $\displaystyle\int e^{ax}\sin bx\,dx = \left(\frac{a\sin bx - b\cos bx}{a^2+b^2}\right)e^{ax}+C$

555. $\displaystyle\int e^{ax}\cos bx\,dx = \left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)e^{ax}+C$

556. $\displaystyle\int e^{ax}\sin^n bx\,dx = \left(\frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}\right)e^{ax}\sin^{n-1}bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx$

557. $\displaystyle\int e^{ax}\cos^n bx\,dx = \left(\frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}\right)e^{ax}\cos^{n-1}bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx$

Another way to express the above integrals is to define

$C_n = \displaystyle\int e^{ax}\cos^n bx\,dx$ and $S_n = \displaystyle\int e^{ax}\sin^n bx\,dx$, then one can write the reduction formulas

$$\begin{aligned}C_n =& \frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}e^{ax}\cos^{n-1}bx + \frac{n(n-1)b^2}{a^2+n^2b^2}C_{n-2}\\ S_n =& \frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}e^{ax}\sin^{n-1}bx + \frac{n(n-1)b^2}{a^2+n^2b^2}S_{n-2}\end{aligned}$$

558. $\displaystyle\int xe^{ax}\sin bx\,dx = \left(\frac{[2ab - b(a^2+b^2)x]\cos bx + [a(a^2+b^2)x - a^2+b^2]\sin bx}{(a^2+b^2)^2}\right)e^{ax}+C$

559. $\displaystyle\int xe^{ax}\cos bx\,dx = \left(\frac{[a(a^2+b^2)x - a^2+b^2]\cos bx + [b(a^2+b^2)x - 2ab]\sin bx}{(a^2+b^2)^2}\right)e^{ax}+C$

560. $\displaystyle\int e^{ax}\ln x\,dx = \frac{1}{a}e^{ax}\ln x - \frac{1}{a}\int \frac{1}{x}e^{ax}\,dx$

561. $\displaystyle\int e^{ax}\sinh bx\,dx = \left[\frac{a\sinh bx - b\cosh bx}{(a-b)(a+b)}\right]e^{ax}+C, \qquad a\neq b$

562. $\displaystyle\int e^{ax}\sinh ax\,dx = \frac{1}{4a}e^{2ax} - \frac{x}{2}+C$

563. $\displaystyle\int e^{ax}\cosh bx\,dx = \left[\frac{a\cosh bx - b\sinh bx}{(a-b)(a+b)}\right]e^{ax}+C, \qquad a\neq b$

564. $\displaystyle\int e^{ax}\cosh ax\,dx = \frac{1}{4a}e^{2ax} + \frac{x}{2}+C$

565. $\displaystyle\int \frac{dx}{\alpha+\beta e^{ax}} = \frac{x}{\alpha} - \frac{1}{a\alpha}\ln|\alpha+\beta e^{ax}|+C$

566. $\displaystyle\int \frac{dx}{(\alpha+\beta e^{ax})^2} = \frac{x}{\alpha^2} + \frac{1}{a\alpha(\alpha+\beta e^{ax})} - \frac{1}{a\alpha^2}\ln|\alpha+\beta e^{ax}|+C$

567. $\displaystyle\int \frac{dx}{\alpha e^{ax}+\beta e^{-ax}} = \begin{cases}\dfrac{1}{a\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\dfrac{\alpha}{\beta}}e^{ax}\right)+C, & \alpha\beta>0\\[2ex] \dfrac{1}{2a\sqrt{-\alpha\beta}}\ln\left|\dfrac{e^{ax}-\sqrt{-\beta/\alpha}}{e^{ax}+\sqrt{-\beta/\alpha}}\right|+C, & \alpha\beta<0\end{cases}$

568. $$\int e^{ax}\sin^2 bx\,dx = \left(\frac{a^2+4b^2-a^2\cos(2bx)-2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$$

569. $$\int e^{ax}\cos^2 bx\,dx = \left(\frac{a^2+4b^2+a^2\cos(2bx)+2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$$

Integrals containing the logarithmic function

570. $$\int \ln x\,dx = x\ln|x|+C$$

571. $$\int x\ln x\,dx = \frac{1}{2}x^2\ln|x|-\frac{1}{4}x^2+C$$

572. $$\int x^n\ln x\,dx = \frac{1}{(n+1)^2}x^{n+1}+\frac{1}{n+1}x^{n+1}\ln|x|+C,\qquad n\neq -1$$

573. $$\int \frac{1}{x}\ln x\,dx = \frac{1}{2}(\ln|x|)^2+C$$

574. $$\int \frac{dx}{x\ln x} = \ln|\ln|x||+C$$

575. $$\int \frac{1}{x^2}\ln x\,dx = -\frac{1}{x}-\frac{1}{x}\ln|x|+C$$

576. $$\int (\ln|x|)^2\,dx = x(\ln|x|)^2-2x\ln|x|+2x+C$$

577. $$\int \frac{1}{x}\left(\ln|x|\right)^n\,dx = \frac{1}{n+1}\left(\ln|x|\right)^{n+1}+C,\qquad n\neq -1$$

578. $$\int (\ln|x|)^n\,dx = x(\ln|x|)^n - n\int(\ln|x|)^{n-1}\,dx$$

579. $$\int \ln|x^2+a^2|\,dx = x\ln|x^2+a^2|-2x+2a\tan^{-1}\frac{x}{a}+C$$

580. $$\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2|-2x+a\ln|\frac{x+a}{x-a}|+C$$

581. $$\int (ax+b)\ln(\beta x+\gamma)\,dx = \frac{\beta^2(ax+b)^2-(b\beta-a\gamma)^2}{2a\beta^2}\ln(\beta x+\gamma)-\frac{a}{4\beta^2}(\beta x+\gamma)^2-\frac{1}{\beta}(b\beta-a\gamma)x+C$$

582. $$\int (\ln ax)^2\,dx = x(\ln ax)^2-2x\ln ax+2x+C$$

Integrals containing the hyperbolic function $\sinh ax$

583. $$\int \sinh ax\,dx = \frac{1}{a}\cosh ax+C$$

584. $$\int x \sinh ax\,dx = \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C$$

585. $$\int x^2 \sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right) \cosh ax - \frac{2x}{a^2} \sinh ax + C$$

586. $$\int x^n \sinh ax\,dx = \frac{1}{a} x^n \cosh ax - \frac{n}{a} \int x^{n-1} \cosh ax\,dx$$

587. $$\int \frac{1}{x} \sinh ax\,dx = ax + \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{4 \cdot 5!} + \cdots + C$$

588. $$\int \frac{1}{x^2} \sinh ax\,dx = -\frac{1}{x} \sinh ax + a \int \frac{1}{x} \cosh ax\,dx$$

589. $$\int \frac{1}{x^n} \sinh ax\,dx = -\frac{\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{1}{x^{n-1}} \cosh ax\,dx$$

590. $$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln|\tanh \frac{ax}{2}| + C$$

591. $$\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left[ax - \frac{(ax)^3}{18} + frac7(ax)^5 1800 + \cdots + (-1)^n \frac{2(2^{2n}-1)\mathfrak{B}_n\, a^{2n+1} x^{2n+1}}{(2n+1)!} + \cdots\right] + C$$

592. $$\int \sinh^2 ax\,dx = \frac{1}{2a} x \sinh 2ax - \frac{1}{2} x + C$$

593. $$\int \sinh^n ax\,dx = \frac{1}{na} \sinh^{n-1} ax \cosh ax - \frac{n-1}{n} \int \sinh^{n-2} ax\,dx$$

594. $$\int x \sinh^2 ax\,dx = \frac{1}{4a} x \sinh 2ax - \frac{1}{8a^2} \cosh 2ax - \frac{1}{4} x^2 + C$$

595. $$\int \frac{dx}{\sinh^2 ax} = -\frac{1}{a} \coth ax + C$$

596. $$\int \frac{dx}{\sinh^3 ax} = -\frac{1}{2a} \operatorname{csch} ax \coth ax - \frac{1}{2a} \ln|\tanh \frac{ax}{2}| + C$$

597. $$\int \frac{x\,dx}{\sinh^2 ax} = -\frac{1}{a} x \coth ax + \frac{1}{a^2} \ln|\sinh ax| + C$$

598. $$\int \sinh ax \sinh bx\,dx = \frac{1}{2(a+b)} \sinh(a+b)x - \frac{1}{2(a-b)} \sinh(a-b)x + C$$

599. $$\int \sinh ax \sin bx\,dx = \frac{1}{a^2+b^2}\left[a \cosh ax \sin bx - b \sinh ax \cos bx\right] + C$$

600. $$\int \sinh ax \cos bx\,dx = \frac{1}{a^2+b^2}\left[a \cosh ax \cos bx + b \sinh ax \sin bx\right] + C$$

601. $$\int \frac{dx}{\alpha + \beta \sinh ax} = \frac{1}{\sqrt{\alpha^2+\beta^2}} \ln \left| \frac{\beta e^{ax} + \alpha - \sqrt{\alpha^2+\beta^2}}{\beta e^{ax} + \alpha + \sqrt{\alpha^2+\beta^2}} \right| + C$$

602. $$\int \frac{dx}{(\alpha + \beta \sinh ax)^2} = \frac{-\beta}{a(\alpha^2+\beta^2)} \frac{\cosh ax}{\alpha + \beta \sinh ax} + \frac{\alpha}{\alpha^2+\beta^2} \int \frac{dx}{\alpha + \beta \sinh ax}$$

603. $$\int \frac{dx}{\alpha^2 + \beta^2 \sinh^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\beta^2-\alpha^2}} \tan^{-1}\left(\dfrac{\sqrt{\beta^2-\alpha^2}\tanh ax}{\alpha} \right) + C, & \beta^2 > \alpha^2 \\ \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}} \ln \left| \dfrac{\alpha + \sqrt{\alpha^2-\beta^2}\tanh ax}{\alpha - \sqrt{\alpha^2-\beta^2}\tanh ax} \right| + C, & \beta^2 < \alpha^2 \end{cases}$$

604. $$\int \frac{dx}{\alpha^2 - \beta^2 \sinh^2 ax} = \frac{1}{2a\alpha\sqrt{\alpha^2+\beta^2}} \ln \left| \frac{\alpha + \sqrt{\alpha^2+\beta^2}\tanh ax}{\alpha - \sqrt{\alpha^2+\beta^2}\tanh ax} \right| + C$$

Integrals containing the hyperbolic function cosh *ax*

605. $$\int \cosh ax\, dx = \frac{1}{a} \sinh ax + C$$

606. $$\int x \cosh ax\, dx = \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C$$

607. $$\int x^2 \cosh ax\, dx = -\frac{2}{a^2} x \cosh ax + \left(\frac{x^2}{a} + \frac{2}{a^3} \right) \sinh ax + C$$

608. $$\int x^n \cosh ax\, dx = \frac{1}{a} x^n \sinh ax - \frac{n}{a} \int x^{n-1} \sinh ax\, dx$$

609. $$\int \frac{1}{x} \cosh ax\, dx = \ln|x| + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} + \frac{(ax)^6}{6 \cdot 6!} + \cdots + C$$

610. $$\int \frac{1}{x^2} \cosh ax\, dx = -\frac{1}{x} \cosh ax + a \int \frac{1}{x} \sinh ax\, dx$$

611. $$\int \frac{1}{x^n} \cosh ax\, dx = -\frac{1}{n-1} \frac{\cosh ax}{x^{n-1}} + \frac{a}{n-1} \int \frac{\sinh ax}{x^{n-1}}\, dx, \qquad n > 1$$

612. $$\int \frac{dx}{\cosh ax} = \frac{2}{a} \tan^{-1} e^{ax} + C$$

613. $$\int \frac{x\, dx}{\cosh ax} = \frac{1}{a^2} \left[\frac{a^2x^2}{2} - \frac{a^4x^4}{8} + \frac{5a^6x^6}{144} + \cdots + (-1)^n \frac{\mathfrak{E}_n a^{2n+2} x^{2n+2}}{(2n+2) \cdot (2n)!} + \cdots \right] + C$$

614. $$\int \cosh^2 ax\, dx = \frac{1}{2} x + \frac{1}{2} \sinh ax \cosh ax + C$$

615. $$\int \cosh^n ax\, dx = \frac{1}{na} \cosh^{n-1} ax \sinh ax + \frac{n-1}{n} \int \cosh^{n-2} ax\, dx$$

616. $$\int x \cosh^2 ax\, dx = \frac{1}{4} x^2 + \frac{1}{4a} x \sinh 2ax - \frac{1}{8a^2} \cosh 2ax + C$$

617. $\displaystyle\int \frac{dx}{\cosh^2 ax} = \frac{1}{a}\tanh ax + C$

618. $\displaystyle\int \frac{x\,dx}{\cosh^2 ax} = \frac{1}{a}x\tanh ax - \frac{1}{a^2}\ln|\cosh ax| + C$

619. $\displaystyle\int \frac{dx}{\cosh^n ax} = \frac{1}{(n-1)a}\frac{x\sinh ax}{\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax}, \quad n > 1$

620. $\displaystyle\int \cosh ax \cosh bx\,dx = \frac{1}{2(a-b)}\sinh(a-b)x + \frac{1}{2(a+b)}\sinh(a+b)x + C$

621. $\displaystyle\int \cosh ax \sin bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax\sin bx - b\cosh ax\cos bx\right] + C$

622. $\displaystyle\int \cosh ax \cos bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax\cos bx + b\cosh ax\sin bx\right] + C$

623. $$\int \frac{dx}{\alpha+\beta\cosh ax} = \begin{cases} \dfrac{2}{\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\beta e^{ax}+\alpha}{\sqrt{\beta^2-\alpha^2}} + C, & \beta^2 > \alpha^2 \\[2ex] \dfrac{1}{a\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\beta e^{ax}+\alpha-\sqrt{\alpha^2-\beta^2}}{\beta e^{ax}+\alpha+\sqrt{\alpha^2-\beta^2}}\right| + C, & \beta^2 < \alpha^2 \end{cases}$$

624. $\displaystyle\int \frac{dx}{1+\cosh ax} = \frac{1}{a}\tanh ax + C$

625. $\displaystyle\int \frac{x\,dx}{1+\cosh ax} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln|\cosh\frac{ax}{2}| + C$

626. $\displaystyle\int \frac{dx}{-1+\cosh ax} = -\frac{1}{a}\coth\frac{ax}{2} + C$

627. $\displaystyle\int \frac{dx}{(\alpha+\beta\cosh ax)^2} = \frac{\beta\sinh ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cosh ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int \frac{dx}{\alpha+\beta\cosh ax}$

628. $$\int \frac{dx}{\alpha^2-\beta^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\alpha\tanh ax+\sqrt{\alpha^2-\beta^2}}{\alpha\tanh ax-\sqrt{\alpha^2-\beta^2}}\right| + C, & \alpha^2 > \beta^2 \\[2ex] \dfrac{-1}{a\alpha\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\alpha\tanh ax}{\sqrt{\beta^2-\alpha^2}} + C, & \alpha^2 < \beta^2 \end{cases}$$

629. $\displaystyle\int \frac{dx}{\alpha^2+\beta^2\cosh^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tanh^{-1}\left(\frac{\alpha\tanh ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$

Integrals containing the hyperbolic functions $\sinh ax$ and $\cosh ax$

630. $\displaystyle\int \sinh ax\cosh ax\,dx = \frac{1}{2a}\sinh^2 ax + C$

631. $\displaystyle\int \sinh ax\cosh bx\,dx = \frac{1}{2(a+b)}\cosh(a+b)x + \frac{1}{2(a-b)}\cosh(a-b)x + C$

632. $$\int \sinh^2 ax \cosh^2 ax\, dx = \frac{1}{32a} \sinh 4ax - \frac{1}{8}x + C$$

633. $$\int \sinh^n ax \cosh ax\, dx = \frac{1}{(n+1)a} \sinh^{n+1} ax + C, \quad n \neq -1$$

634. $$\int \cosh^n ax \sinh ax\, dx = \frac{1}{(n+1)a} \cosh^{n+1} ax + C, \quad n \neq -1$$

635. $$\int \frac{\sinh ax}{\cosh ax}\, dx = \frac{1}{a} \ln|\cosh ax| + C$$

636. $$\int \frac{\cosh ax}{\sinh ax}\, dx = \frac{1}{a} \ln|\sinh ax| + C$$

637. $$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a} \ln|\tanh ax| + C$$

638. $$\int \frac{x \sinh ax}{\cosh ax}\, dx = \frac{1}{a^2} \left[\frac{a^3x^3}{3} - \frac{a^5x^5}{15} + \cdots + (-1)^n \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1} x^{2n+1}}{(2n+1)!} + \cdots \right] + C$$

639. $$\int \frac{x \cosh ax}{\sinh ax}\, dx = \frac{1}{a^2} \left[ax + \frac{a^3x^3}{9} - \frac{a^5x^5}{225} + \cdots + (-1)^{n-1} \frac{2^{2n}\mathfrak{B}_n a^{2n+1} x^{2n+1}}{(2n+1)!} + \cdots \right] + C$$

640. $$\int \frac{\sinh^2 ax}{\cosh^2 ax}\, dx = x - \frac{1}{a} \tanh ax + C$$

641. $$\int \frac{\cosh^2 ax}{\sinh^2 ax}\, dx = x - \frac{1}{a} \coth ax + C$$

642. $$\int \frac{x \sinh^2 ax}{\cosh^2 ax}\, dx = \frac{1}{2}x^2 - \frac{1}{a}x \tanh ax + \frac{1}{a^2} \ln|\cosh ax| + C$$

643. $$\int \frac{x \cosh^2 ax}{\sinh^2 ax}\, dx = \frac{1}{2}x^2 - \frac{1}{a}x \coth ax + \frac{1}{a^2} \ln|\sinh ax| + C$$

644. $$\int \frac{\sinh ax}{x \cosh ax}\, dx = ax - \frac{a^3x^3}{9} + \cdots + (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1} x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

645. $$\int \frac{\cosh ax}{x \sinh ax}\, dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{a^3x^3}{135} + \cdots + (-1)^n \frac{2^{2n}\mathfrak{B}_n a^{2n-1} x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

646. $$\int \frac{\sinh^3 ax}{\cosh^3 ax}\, dx = \frac{1}{a} \ln|\cosh ax| - \frac{1}{2a} \tanh^2 ax + C$$

647. $$\int \frac{\cosh^3 ax}{\sinh^3 ax}\, dx = \frac{1}{a} \ln|\sinh ax| - \frac{1}{2a} \coth^2 ax + C$$

648. $$\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{1}{a}\mathrm{sech}\, ax + \frac{1}{a} \ln \tanh \frac{ax}{2}| + C$$

649. $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a}\tan^{-1}(\sinh ax) - \frac{1}{a}\text{csch}\, ax + C$$

650. $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2}{a}\coth ax + C$$

651. $$\int \frac{\sinh^2 ax}{\cosh ax}\, dx = \frac{1}{a}\sinh ax - \frac{1}{a}\tan^{-1}(\sinh ax) + C$$

652. $$\int \frac{\cosh^2 ax}{\sinh ax}\, dx = \frac{1}{a}\cosh ax + \frac{1}{a}\ln|\tanh\frac{ax}{2}| + C$$

653. $$\int \frac{dx}{\cosh ax\,(1+\sinh ax)} = \frac{1}{2a}\ln\left|\frac{1+\sinh ax}{\cosh ax}\right| + \frac{1}{a}\tan^{-1} e^{ax} + C$$

654. $$\int \frac{dx}{\sinh ax\,(\cosh ax + 1)} = \frac{1}{2a}\ln|\tanh\frac{ax}{2}| + \frac{1}{2a(\cosh ax + 1)} + C$$

655. $$\int \frac{dx}{\sinh ax\,(\cosh ax - 1)} = -\frac{1}{2a}\ln|\tanh\frac{ax}{2}| - \frac{1}{2a(\cosh ax - 1)} + C$$

656. $$\int \frac{dx}{\alpha + \beta\frac{\sinh ax}{\cosh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} - \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\beta\sinh ax + \alpha\cosh ax| + C$$

657. $$\int \frac{dx}{\alpha + \beta\frac{\cosh ax}{\sinh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} + \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\alpha\sinh ax + \beta\cosh ax| + C$$

658. $$\int \frac{dx}{b\cosh ax + c\sinh ax} = \begin{cases} \dfrac{1}{a\sqrt{b^2-c^2}}\sec^{-1}\left[\dfrac{b\cosh ax + c\sinh ax}{\sqrt{b^2-c^2}}\right] + C, & b^2 > c^2 \\ \dfrac{-1}{a\sqrt{c^2-b^2}}\text{csch}^{-1}\left[\dfrac{b\cosh ax + c\sinh ax}{\sqrt{c^2-b^2}}\right] + C, & b^2 < c^2 \end{cases}$$

Integrals containing the hyperbolic functions tanh *ax*, coth *ax*, sech *ax*, csch *ax*

Express integrals in terms of $\sinh ax$ and $\cosh ax$ and see previous listings.

Integrals containing inverse hyperbolic functions

659. $$\int \sinh^{-1}\frac{x}{a}\, dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2} + C$$

660. $$\int \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

661. $$\int \tanh^{-1}\frac{x}{a}\, dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2| + C$$

662. $$\int \coth^{-1}\frac{x}{a}\, dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2-a^2| + C$$

663. $$\int \text{sech}^{-1}\frac{x}{a}\, dx = \begin{cases} x\,\text{sech}^{-1}\dfrac{x}{a} + a\sin^{-1}\dfrac{x}{a} + C, & \text{sech}^{-1}(x/a) > 0 \\ x\,\text{sech}^{-1}\dfrac{x}{a} - a\sin^{-1}\dfrac{x}{a} + C, & \text{sech}^{-1}(x/a) < 0 \end{cases}$$

664. $\displaystyle\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a}, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

665. $\displaystyle\int x\sinh^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{1}{4}xx\sqrt{x^2+a^2}+C$

666. $\displaystyle\int x\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

667. $\displaystyle\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}+C$

668. $\displaystyle\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}+C$

669. $\displaystyle\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a)>0 \\ \frac{1}{2}x\operatorname{sech}^{-1}\frac{x}{a} + \frac{1}{2}a\sqrt{a^2-x^2}+C, & \operatorname{sech}^{-1}(x/a)<0 \end{cases}$

670. $\displaystyle\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{2}x^2\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{2}\sqrt{x^2+a^2}+C, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

671. $\displaystyle\int x^2\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\sinh^{-1}\frac{x}{a} + \frac{1}{9}(2a^2-x^2)\sqrt{x^2+a^2}+C$

672. $\displaystyle\int x^2\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

673. $\displaystyle\int x^2\tanh^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\tanh^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|a^2-x^2|+C$

674. $\displaystyle\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\coth^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|x^2-a^2|+C$

675. $\displaystyle\int x^2\operatorname{sech}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{3}\int\frac{x^3\,dx}{\sqrt{x^2+a^2}}$

676. $\displaystyle\int x^2\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{3}\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}$

677. $\displaystyle\int x^n\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\sinh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}$

678. $\displaystyle\int x^n\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)>0 \\ \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)<0 \end{cases}$

679. $\displaystyle\int x^n\tanh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\tanh^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2-x^2}$

680. $\displaystyle\int x^n \coth^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\coth^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2-x^2}$

681. $\displaystyle\int x^n \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\dfrac{x}{a} + \dfrac{a}{n+1}\displaystyle\int\frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\dfrac{x}{a} - \dfrac{a}{n+1}\displaystyle\int\frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

682. $\displaystyle\int x^n \operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{n+1}\int\frac{x^n\,dx}{\sqrt{x^2+a^2}},\quad + \text{ for } x>0,\ - \text{ for } x<0$

683. $\displaystyle\int \frac{1}{x}\sinh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot3\cdot3} + \dfrac{1\cdot3(x/a)^5}{2\cdot4\cdot4\cdot5} - \dfrac{1\cdot3\cdot5(x/a)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots + C, & |x|>a \\ \dfrac{1}{2}\left(\ln\left|\dfrac{2x}{a}\right|\right)^2 - \dfrac{(a/x)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} - \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots + C, & x>a \\ -\dfrac{1}{2}\left(\ln\left|\dfrac{-2x}{a}\right|\right)^2 + \dfrac{(a/x)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots + C, & x<-a \end{cases}$

684. $\displaystyle\int \frac{1}{x}\cosh^{-1}\frac{x}{a}\,dx = \pm\left[\frac{1}{2}\left(\ln\left|\frac{2x}{a}\right|\right)^2 + \frac{(a/x)^2}{2\cdot2\cdot2} + \frac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \frac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots\right] + C$

$+$ for $\cosh^{-1}(x/a) > 0$, $-$ for $\cosh^{-1}(x/a) < 0$

685. $\displaystyle\int \frac{1}{x}\tanh^{-1}\frac{x}{a}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots + C$

686. $\displaystyle\int \frac{1}{x}\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a} + C$

687. $\displaystyle\int \frac{1}{x}\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} -\dfrac{1}{2}\ln\left|\dfrac{a}{x}\right|\,\ln\left|\dfrac{4a}{x}\right| - \dfrac{(x/a)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} - \cdots + C, & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{1}{2}\ln\left|\dfrac{a}{x}\right|\,\ln\left|\dfrac{4a}{x}\right| + \dfrac{(x/a)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

688. $\displaystyle\int \frac{1}{x}\operatorname{csch}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{x}{a}\right|\,\ln\left|\dfrac{4a}{x}\right| + \dfrac{(x/a)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} + \cdots + C, & 0<x<a \\ \dfrac{1}{2}\ln\left|\dfrac{-x}{a}\right|\,\ln\left|\dfrac{-x}{4a}\right| - \dfrac{(x/a)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(x/a)^4}{2\cdot4\cdot4\cdot4} - \cdots, & -a<x<0 \\ -\dfrac{a}{x} + \dfrac{(a/x)^3}{2\cdot3\cdot3} - \dfrac{1\cdot3(a/x)^5}{2\cdot4\cdot5\cdot5} + \cdots + C, & |x|>a \end{cases}$

Integrals evaluated by reduction formula

689. If $S_n = \displaystyle\int \sin^n x\,dx$, then $S_n = -\dfrac{1}{n}\sin^{n-1}x\cos x + \dfrac{n-1}{n}S_{n-2}$

690. If $C_n = \displaystyle\int \cos^n x\,dx$, then $C_n = \dfrac{1}{n}\sin x\cos^{n-1}x + \dfrac{n-1}{n}C_{n-2}$

691. If $I_n = \displaystyle\int \frac{\sin^n ax}{\cos ax}\,dx$, then $I_n = \dfrac{-1}{(n-1)\,a}\sin^{n-1}ax + I_{n-2}$

692. If $I_n = \displaystyle\int \frac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \dfrac{1}{(n-1)\,a}\cos^{n-1}ax + I_{n-2}$

693. If $S_m = \int x^m \sin nx\,dx$ and $C_m = \int x^m \cos nx\,dx$, then

$$S_m = \frac{-1}{n} x^m \cos nx + \frac{m}{n} C_{m-1} \qquad \text{and} \qquad C_m = \frac{1}{n} x^m \sin nx - \frac{m}{n} S_{m-1}$$

694. If $I_1 = \int \tan x\,dx$, and $I_n = \int \tan^n x\,dx$, then $I_n = \dfrac{1}{n-1} \tan^{n-1} x - I_{n-2}, \quad n = 2, 3, 4, \ldots$

695. If $I_n = \int \dfrac{\sin^n ax}{\cos ax}\,dx$, then $I_n = -\dfrac{\sin^{n-1} ax}{(n-1)a} + I_{n-2}$

696. If $I_n = \int \dfrac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \dfrac{\cos^{n-1} ax}{(n-1)a} + I_{n-2}$

697. If $I_{n,m} = \int \sin^n x \cos^m x\,dx$, then

$$\begin{aligned}
I_{n,m} =& \frac{-1}{n+m} \sin^{n-1} x \cos^{m+1} x + \frac{n-1}{n+m} I_{n-2,m} \\
I_{n,m} =& \frac{1}{n+1} \sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{n+1} I_{n+2,m} \\
I_{n,m} =& \frac{1}{n+m} \sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+m} I_{n,m+2} \\
I_{n,m} =& \frac{-1}{m+1} \sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{m+1} I_{n,m+2} \\
I_{n,m} =& \frac{-1}{m+1} \sin^{n-1} x \cos^{m+1} x + \frac{n-1}{m+1} I_{n-2,m+2} \\
I_{n,m} =& \frac{1}{n+1} \sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+1} I_{n+2,m-2}
\end{aligned}$$

698. If $S_n = \int e^{ax} \sin^n bx\,dx$ and $C_n = \int e^{ax} \cos^n bx\,dx$, then

$$\begin{aligned}
C_n =& e^{ax} \cos^{n-1} bx \left[\frac{a \cos bx + nb \sin bx}{a^2 + n^2 b^2} \right] + \frac{n(n-1)\,b^2}{a^2 + n^2 b^2} C_{n-2} \\
S_n =& e^{ax} \sin^{n-1} ax \left[\frac{a \sin bx - nb \cos nx}{a^2 + n^2 b^2} \right] + \frac{n(n-1)\,b^2}{a^2 + n^2 b^2} S_{n-2}
\end{aligned}$$

699. If $I_n = \int x^m (\ln x)^n\,dx$, then $I_n = \dfrac{1}{m+1} x^{m+1} (\ln x)^n - \dfrac{n}{m+1} I_{n-1}$

Integrals involving Bessel functions

700. $\int J_1(x)\,dx = -J_0(x) + C$

701. $\int x J_1(x)\,dx = -x J_0(x) + \int J_0(x)\,dx$

702. $\int x^n J_1(x)\,dx = -x^n J_0(x) + n \int x^{n-1} J_0(x)\,dx$

703. $\displaystyle\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$

704. $\displaystyle\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + C$

705. $\displaystyle\int x^{-\nu} J_{\nu+1}(x)\,dx = x^{-\nu} J_\nu(x) + C$

706. $\displaystyle\int \frac{J_1(x)}{x^n}\,dx = \frac{-1}{n}\frac{J_1(x)}{x^{n-1}} + \frac{1}{n}\int \frac{J_0(x)}{x^{n-1}}\,dx$

707. $\displaystyle\int xJ_0(x)\,dx = xJ_1(x) + C$

708. $\displaystyle\int x^2 J_0(x)\,dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx$

709. $\displaystyle\int x^n J_0(x)\,dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2}J_0(x)\,dx$

710. $\displaystyle\int \frac{J_0(x)}{x^n}\,dx = \frac{J_1(x)}{(n-1)^2 x^{n-2}} - \frac{J_0(x)}{(n-1)x^{n-1}} - \frac{1}{(n-1)^2}\int \frac{J_0(x)}{x^{n-2}}\,dx$

711. $\displaystyle\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x)$

712. $\displaystyle\int xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{x}{\beta^2-\alpha^2}\left[\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n'(\beta x)J_n(\alpha x)\right] + C$

713. If $\displaystyle I_{m,n} = \int x^m J_n(x)\,dx, \quad m \geq -n$, then

$$I_{m,n} = -x^m J_{n-1}(x) + (m+n-1)\,I_{m-1,n-1}$$

714. If $\displaystyle I_{n,0} = \int x^n J_0(x)\,dx$, then $I_{n,0} = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 I_{n-2,0}$ Note that $\displaystyle I_{1,0} = \int xJ_0(x)\,dx = xJ_1(x) + C$ and $\displaystyle I_{0,1} = \int J_1(x)\,dx = -J_0(x) + C$ Note also that the integral $\displaystyle I_{0,0} = \int J_0(x)\,dx$ cannot be given in closed form.

Definite integrals

General integration properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$

2.

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx, \qquad \int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b f(x)\,dx$$

3. If $f(x)$ has a singular point at $x = b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)\,dx$

4. If $f(x)$ has a singular point at $x = a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f(x)\,dx$

5. If $f(x)$ has a singular point at $x = c$, $a < c < b$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx$

6.

$$\begin{aligned} \int_a^b cf(x)\,dx &= c\int_a^b f(x)\,dx, \quad c \text{ constant} \\ \int_a^a f(x)\,dx &= 0, \\ \int_0^b f(x)\,dx &= \int_0^b f(b-x)\,dx \end{aligned}$$

$$\begin{aligned} \int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx, \\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \end{aligned}$$

7. Mean value theorems

$$\int_a^b f(x)\,dx = f(c)(b-a), \quad a \le c \le b$$

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx, \quad g(x) \ge 0,\ a \le c \le b$$

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx \qquad \int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx$$

$$a < \xi < b \qquad\qquad a < \eta < b$$

The last mean value theorem requires that $f(x)$ be monotone increasing and nonnegative throughout the interval (a, b)

8. Numerical integration

Divide the interval (a, b) into n equal parts by defining a step size $h = \frac{b-a}{n}$.

Two numerical integration schemes are

(a) Trapezoidal rule with global error $-\frac{(b-a)}{12}h^2 f''(\xi)$ for $a < \xi < b$.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\right]$$

(b) Simpson's 1/3 rule with global error $-\frac{(b-a)}{90}h^4 f^{(iv)}(\xi)$ for $a < \xi < b$.

$$\int_a^b f(x)\,dx = \frac{2h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

9. If $f(x)$ is periodic with period L, then $f(x+L) = f(x)$ for all x and $\int_0^{nL} f(x)\,dx = n\int_0^L f(x)\,dx$, for integer values of n.

10.

$$\underbrace{\int_0^x dx \int_0^x dx \cdots \int_0^x}_{n \text{ integration signs}} dx\, f(x) = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)\,du$$

Integrals containing algebraic terms

11. $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$

12. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left[\Gamma(\frac{1}{4})\right]^2$

13. $\displaystyle\int_0^1 \frac{dx}{(1-x^{2n})^{n/2}} = \frac{\pi}{2n\sin\frac{\pi}{2n}}$

14. $\displaystyle\int_0^1 \frac{1}{\beta-\alpha x}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{\beta(\beta-\alpha)}}$

15. $\displaystyle\int_0^1 \frac{x^p - x^{-p}}{x^q - x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\tan\frac{p\pi}{2q}, \qquad |p|<q$

16. $\displaystyle\int_0^1 \frac{x^p + x^{-p}}{x^q + x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\sec\frac{p\pi}{2q}, \qquad |p|<q$

17. $\displaystyle\int_0^1 \frac{x^{p-1} - x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{p\pi}{2}, \qquad 0<p<2$

18. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$

19. $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{4}a^2$

20. $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$

21. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

22. $\displaystyle\int_0^1 \frac{x^{\alpha-1}+x^{-\alpha}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

23. $\displaystyle\int_0^\infty \frac{x^m\,dx}{1+x^2} = \frac{\pi}{2}\sec\frac{m\pi}{2}$

24. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{\alpha\pi}{2}$

25. $$\int_0^\infty \frac{dx}{1-x^n} = \frac{\pi}{n}\cot\frac{\pi}{n}$$

26. $$\int_0^\infty \frac{dx}{(a^2x^2+c^2)(x^2+b^2)} = \frac{\pi}{2bc}\frac{1}{c+ab}$$

27. $$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2}\frac{1}{ab\,(a+b)}$$

28. $$\int_0^\infty \frac{dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2p}\frac{1}{a^2+p^2}$$

29. $$\int_0^\infty \frac{x^2\,dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2}\frac{p}{a^2+p^2}$$

30. $$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$$

31. $$\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}$$

32. $$\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$$

Integrals containing trigonometric terms

33. $$\int_0^1 \frac{\sin^{-1}x}{x}\,dx = \frac{\pi}{2}\ln 2$$

34. $$\int_0^{\pi/2} \frac{\tan^{-1}(\frac{b}{a}\tan\theta)\,d\theta}{\tan\theta} = \frac{\pi}{2}\ln|1+\frac{b}{a}|$$

35. $$\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$$

36. $$\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$$

37. $$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2-b^2}}$$

38. $$\int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$$

39. $$\int_0^{\pi/2} \sin^p x\cos^q x\,dx = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q}{2}+1)}$$

40. $$\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$$

41. $$\int_0^{\pi} \cos p\theta\cos q\theta\,d\theta = \begin{cases} 0, & p\neq q \\ \frac{\pi}{2}, & p=q \end{cases}$$

42. $\displaystyle\int_0^\pi \sin p\theta \sin q\theta \, d\theta = \begin{cases} 0, & p \neq q \\ \dfrac{\pi}{2}, & p = q \end{cases}$

43. $\displaystyle\int_0^\pi \sin p\theta \cos q\theta \, d\theta = \begin{cases} 0, & p + q \text{ even} \\ \dfrac{2p}{p^2 - q^2}, & p + q \text{ odd} \end{cases}$

44. $\displaystyle\int_0^\pi \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2 - 1}}$

45. $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$

46. $\displaystyle\int_0^\pi \frac{\sin\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a}\tanh^{-1} a$

47. $\displaystyle\int_0^\pi \frac{\sin 2\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a^2}(1 + a^2)\tanh^{-1} a - \frac{2}{a}$

48. $\displaystyle\int_0^\pi \frac{x\sin x\,dx}{1 - 2a\cos x + a^2} = \begin{cases} \dfrac{\pi}{a}\ln(1 + a), & |a| < 1 \\ \pi\ln\left(1 + \dfrac{1}{a}\right), & |a| > 1 \end{cases}$

49. $\displaystyle\int_0^\pi \frac{\cos p\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \begin{cases} \dfrac{\pi a^p}{1 - a^2}, & a^2 < 1 \\ \dfrac{\pi a^{-p}}{a^2 - 1}, & a^2 > 1 \end{cases}$

50. $\displaystyle\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^2} = \begin{cases} \dfrac{\pi a^p}{(1 - a^2)^3}[(p + 1) - (p - 1)a^2], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{(a^2 - 1)^3}[(1 - p) + (1 + p)a^2], & a^2 > 1 \end{cases}$

51. $\displaystyle\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^3} = \begin{cases} \dfrac{\pi a^p}{2(1 - a^2)^5}\left[(p + 2)(p + 1) + 2(p + 2)(p - 2)a^2 + (p - 2)(p - 1)a^4\right], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{2(a^2 - 1)^5}\left[(1 - p)(2 - p) + 2(2 - p)(2 + p)a^2 + (2 + p)(1 + p)a^4\right], & a^2 > 1 \end{cases}$

52. $\displaystyle\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$

53. $\displaystyle\int_0^{2\pi} \frac{dx}{a + b\sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$

54. $\displaystyle\int_0^{2\pi} \frac{dx}{a + b\cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$

55. $\displaystyle\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$

56. $\displaystyle\int_0^{2\pi} \frac{dx}{(a + b\cos x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$

57. $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \quad m, n \text{ integers} \\ \frac{L}{2}, & m = n \end{cases}$

58. $\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$ for all integer m, n values

59. $\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0 \end{cases}$

60. $\int_{0}^{\infty} \frac{x^m\,dx}{1+2x\cos\beta+x^2} = \frac{\pi}{\sin m\pi}\frac{\sin m\beta}{\sin\beta}$

61. $\int_{0}^{\infty} \frac{\sin\alpha x}{x}\,dx = \begin{cases} \pi/2, & \alpha > 0 \\ 0, & \alpha = 0 \\ -\pi/2, & \alpha < 0 \end{cases}$

62. $\int_{0}^{\infty} \frac{\sin\alpha x\sin\beta x}{x}\,dx = \begin{cases} 0, & \alpha > \beta > 0 \\ \pi/2, & 0 < \alpha < \beta \\ \pi/4, & \alpha = \beta > 0 \end{cases}$

63. $\int_{0}^{\infty} \frac{\sin\alpha x\sin\beta x}{x^2}\,dx = \begin{cases} \frac{\pi\alpha}{2}, & 0 < \alpha \leq \beta \\ \frac{\pi\beta}{2}, & \alpha \geq \beta > 0 \end{cases}$

64. $\int_{0}^{\infty} \frac{\sin^2\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$

65. $\int_{0}^{\infty} \frac{1-\cos\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$

66. $\int_{0}^{\infty} \frac{\cos\alpha x}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-\alpha a}$

67. $\int_{0}^{\infty} \frac{x\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2}e^{-\alpha a}$

68. $\int_{0}^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}$

69. $\int_{0}^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}$

70. $\int_{0}^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$

71. $\int_{0}^{\infty} \frac{\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-\alpha a})$

72. $\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

73. $\int_{0}^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

74. $\int_{0}^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

75. $$\int_0^\infty \sin ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$$

76. $$\int_0^\infty \cos ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$$

77. $$\int_0^\infty \frac{dx}{x^4 + 2a^2x^2\cos 2\beta + a^4} = \frac{\pi}{4a^3\cos\beta}$$

78. $$\int_0^\infty \cos\left(x^2 + \frac{a^2}{x^2}\right)dx = \frac{\sqrt{\pi}}{2}\cos(\frac{\pi}{4} + 2a)$$

79. $$\int_0^\infty \sin\left(x^2 + \frac{a^2}{x^2}\right)dx = \frac{\sqrt{\pi}}{2}\sin(\frac{\pi}{4} + 2a)$$

80. $$\int_0^\infty \frac{\tan bx\,dx}{x(p^2 + x^2)} = \frac{\pi}{2p^2}\,\tanh bp$$

81. $$\int_0^\infty \frac{x\tan bx\,dx}{p^2 + x^2} = \frac{\pi}{2} - \frac{\pi}{2}\tanh bp$$

82. $$\int_0^\infty \frac{x\cot bx\,dx}{p^2 + x^2} = \frac{\pi}{2}\coth bp$$

83. $$\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p}\frac{\sinh ap}{\sinh bp}, \qquad a < b$$

84. $$\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p}\frac{\cosh ap}{\cosh bp}, \qquad a < b$$

85. $$\int_0^\infty \frac{\sin ax}{\cos bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p^2}\frac{\sinh ap}{\cosh bp}, \qquad a < b$$

86. $$\int_0^\infty \frac{\sin ax}{\cos bx}\frac{x\,dx}{(x^2 + p^2)} = -\frac{\pi}{2}\frac{\sinh ap}{\cosh bp}, \qquad a < b$$

87. $$\int_0^\infty \frac{\cos ax}{\sin bx}\frac{x\,dx}{(p^2 + x^2)} = \frac{\pi}{2}\frac{\cosh ap}{\sinh bp}, \qquad a < b$$

Integrals containing exponential and logarithmic terms

88. $$\int_0^1 \frac{\ln\frac{1}{x}}{1 + x}\,dx = \frac{\pi^2}{12}$$

89. $$\int_0^1 \frac{\ln\frac{1}{x}}{(1 - x)}\,dx = \frac{\pi^2}{6}$$

90. $$\int_0^1 \frac{\left(\ln\frac{1}{x}\right)^3}{1 - x}\,dx = \frac{\pi^4}{15}$$

91. $$\int_0^1 \frac{\ln(1 + x)}{x}\,dx = \frac{\pi^2}{12}$$

92. $\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

93. $\int_0^1 (ax^2+bx+c)\frac{\ln\frac{1}{x}}{1-x}\,dx = (a+b+c)\frac{\pi^2}{6} - (a+b) - \frac{a}{4}$

94. $\int_0^1 \frac{\ln\frac{1}{x}}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln 2$

95. $\int_0^1 \frac{1-x^{p-1}}{(1-x)(1-x^p)}(\ln\frac{1}{x})^{2n-1}\,dx = \frac{1}{4n}(1-\frac{1}{p^{2n}})(2\pi)^{2n}\,\mathfrak{B}_{2n-1}$

96. $\int_0^1 \frac{x^m-x^n}{\ln x}\,dx = \ln\left|\frac{1+m}{1+n}\right|$

97. $\int_0^1 x^p(\ln x)^n\,dx = \begin{cases} (-1)^n\frac{n!}{(p+1)^{n+1}}, & \text{n an integer} \\ (-1)^n\frac{\Gamma(n+1)}{(p+1)^{n+1}}, & \text{n noninteger} \end{cases}$

98. $\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

99. $\int_0^{\pi/2} \ln\sin\theta\,d\theta = \frac{\pi}{2}\ln(\frac{1}{2})$

100. $\int_0^{\pi} \ln(a+b\cos x)\,dx = \pi\ln\left|\frac{a+\sqrt{a^2+b^2}}{2}\right|$

101. $\int_0^{2\pi} \ln(a+b\cos x)\,dx = 2\pi\ln|a+\sqrt{a^2-b^2}|$

102. $\int_0^{2\pi} \ln(a+b\sin x)\,dx = 2i\ln|a+\sqrt{a^2-b^2}|$

103. $\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}$

104. $\int_0^{\infty} x^n e^{-ax}\,dx = \frac{\Gamma(n+1}{a^{n+1}}$

105. $\int_0^{\infty} e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma(\frac{1}{2})$

106. $\int_0^{\infty} x^n e^{-a^2x^2}\,dx = \frac{\Gamma(\frac{m+1}{2})}{2a^{m+1}}$

107. $\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

108. $\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

109. $\int_0^\infty e^{-ax}\frac{\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$

110. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$

111. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/4a^2}$

112. $\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$

113. $\int_0^\infty x^{2n}e^{-\beta x^2}\,dx = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^{n+1}\beta^n}\sqrt{\frac{\pi}{\beta}}$

114. $\int_0^\infty e^{-k\left(\frac{x^2}{a^2}+\frac{b^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}\frac{a}{\sqrt{k}}e^{-2kb/a}$

115. $\int_0^\infty \frac{\sin rx\,dx}{x(x^4+2a^2x^2\cos 2\beta+a^4)} = \frac{\pi}{2a^4}\left[1-\frac{\sin(ar\sin\beta+2\beta)}{\sin 2\beta}\,e^{-\beta r\cos\beta}\right]$

116. $\int_0^\infty \frac{\cos rx\,dx}{x^4+2a^2x^2\cos 2\beta+a^4} = \frac{\pi}{2a^3}\frac{\sin(\beta+ar\sin\beta)}{\sin 2\beta}\,e^{-ar\cos\beta}$

117. $\int_0^\infty \frac{\sin rx\,dx}{x(x^6+a^6)} = \frac{\pi}{6a^6}\left[3-e^{-ar}-2e^{-ar/2}\cos\frac{ar\sqrt{3}}{2}\right]$

118. $\int_0^\infty \frac{\cos rx\,dx}{x^6+a^6} = \frac{\pi}{6a^5}\left[e^{-ar}-2e^{-ar/2}\cos(\frac{ar\sqrt{3}}{2}+\frac{2\pi}{3})\right]$

119. $\int_0^\infty \frac{\sin\pi x\,dx}{x(1-x^2)} = \pi$

120. $\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\cos bx\,dx = \frac{1}{2}\ln\left|\frac{p^2+b^2}{q^2+b^2}\right|$

121. $\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\sin bx\,dx = \tan^{-1}\frac{p}{b}-\tan^{-1}\frac{q}{b}$

122. $\int_0^\infty e^{-ax}\frac{\sin px-\sin qx}{x}\,dx = \tan^{-1}\frac{p}{a}-\tan^{-1}\frac{q}{b}$

123. $\int_0^\infty e^{-ax}\frac{\cos px-\cos qx}{x}\,dx = \frac{1}{2}\ln\left|\frac{a^2+a^2}{a^2+p^2}\right|$

124. $\int_0^\infty xe^{-x^2}\sin ax\,dx = \frac{a\sqrt{\pi}}{4}e^{-a^2/4}$

125. $\int_0^\infty x^2e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{4}\left(1-\frac{a^2}{2}\right)\,e^{-a^2/4}$

126. $\int_0^\infty x^3e^{-x^2}\sin ax\,dx = \frac{\sqrt{\pi}}{8}\left(3a-\frac{a^3}{2}\right)\,e^{-a^2/4}$

127. $\displaystyle\int_0^\infty x^4 e^{-x^2} \cos ax\, dx = \frac{\sqrt{\pi}}{8}\left(3 - 3a^2 + \frac{a^4}{4}\right) e^{-a^2/4}$

128. $\displaystyle\int_0^\infty \left(\frac{\ln x}{x-1}\right)^3 dx = \pi^2$

129. $\displaystyle\int_{-\infty}^\infty \frac{x \sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a}(a \cos br + b \sin br)\, e^{-ar}$

130. $\displaystyle\int_{-\infty}^\infty \frac{\sin rx\, dx}{x[(x-b)^2 + a^2]} = \frac{\pi}{a(a^2+b^2)}\left[a - (\cos br - b \sin br)\, e^{-ar}\right]$

131. $\displaystyle\int_{-\infty}^\infty \frac{\cos rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \cos br$

132. $\displaystyle\int_{-\infty}^\infty \frac{\sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \sin br$

133. $\displaystyle\int_{-\infty}^\infty e^{-x^2} \cos 2nx\, dx = \sqrt{\pi}\, e^{-n^2}$

134. $\displaystyle\int_0^\infty \frac{x^{p-1} \ln x}{1+x}\, dx = \frac{-\pi^2}{\sin p\pi} \cot p\pi, \qquad 0 < p < 1$

135. $\displaystyle\int_0^\infty e^{-x} \ln x\, dx = -\gamma$

136. $\displaystyle\int_0^\infty e^{-x^2} \ln x\, dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2 \ln 2)$

137. $\displaystyle\int_0^\infty \ln\left(\frac{e^x + 1}{e^x - 1}\right) dx = \frac{\pi^2}{4}$

138. $\displaystyle\int_0^\infty \frac{x\, dx}{e^x - 1} = \frac{\pi^2}{6}$

139. $\displaystyle\int_0^\infty \frac{x\, dx}{e^x + 1} = \frac{\pi^2}{12}$

Integrals containing hyperbolic terms

140. $\displaystyle\int_0^1 \frac{\sinh(m \ln x)}{\sinh(\ln x)}\, dx = \frac{\pi}{2} \tan \frac{m\pi}{2}, \qquad |m| < 1$

141. $\displaystyle\int_0^\infty \frac{\sin ax}{\sinh bx}\, dx = \frac{\pi}{2b} \tanh(\frac{\pi a}{2b})$

142. $\displaystyle\int_0^\infty \frac{\cos ax}{\cosh bx}\, dx = \frac{\pi}{2b} \text{sech}\,(\frac{\pi a}{2b})$

143. $\displaystyle\int_0^\infty \frac{x\, dx}{\sinh ax} = \frac{\pi^2}{4a^2}$

144. $$\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx = \frac{\pi}{2q}\tan(\frac{\pi p}{2q}), \quad |p| < q$$

145. $$\int_0^\infty \frac{\cosh ax - \cosh bx}{\sinh \pi x}\,dx = \ln\left|\frac{\cos\frac{b}{2}}{\cos\frac{a}{2}}\right|, \quad -\pi < b < a < \pi$$

146. $$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos mx\,dx = \frac{\pi}{2q}\,\frac{\sin\frac{\pi p}{q}}{\cos\frac{\pi p}{q} + \cosh\frac{\pi m}{q}}, \quad q > 0, p^2 < q^2$$

147. $$\int_0^\infty \frac{\sinh px}{\cosh qx}\sin mx\,dx = \frac{\pi}{q}\,\frac{\sin\frac{p\pi}{2q}\sinh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$$

148. $$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos mx\,dx = \frac{\pi}{q}\,\frac{\cos\frac{p\pi}{2q}\cosh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$$

Miscellaneous Integrals

149. $$\int_0^x \xi^{\lambda-1}[1-\xi^\mu]^\nu\,d\xi = \frac{x^\lambda}{\lambda}F(-\nu, \frac{\lambda}{\mu}; \frac{\lambda}{\mu}+1; x^\mu)$$ See hypergeometric function

150. $$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi\,J_n(x)$$

151. $$\int_{-a}^a (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

152. If $f'(x)$ is continuous and $\int_1^\infty \frac{f(x)-f(\infty)}{x}\,dx$ converges, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = [f(0)-f(\infty)]\,\ln\frac{b}{a}$$

153. If $f(x) = f(-x)$ so that $f(x)$ is an even function, then

$$\int_0^\infty f\left(x - \frac{1}{x}\right)\,dx = \int_0^\infty f(x)\,dx$$

154. Elliptic integral of the first kind

$$\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\theta,k), \quad 0 < k < 1$$

155. Elliptic integral of the second kind

$$\int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta = E(\theta,k)$$

156. Elliptic integral of the third kind

$$\int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \Pi(\theta,k,n)$$

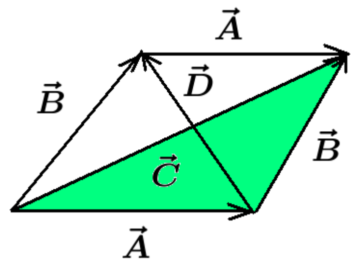
Appendix D
Solutions to Selected Problems

Chapter 6

► **6-1.** (a) $\vec{A}+\vec{B}=9\,\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2+3\,\hat{\mathbf{e}}_3$ (b) $6\vec{A}-3\vec{B}=15\,\hat{\mathbf{e}}_2$ (c) $\vec{A}+2\vec{B}=15\,\hat{\mathbf{e}}_1+5\,\hat{\mathbf{e}}_3$

► **6-2.**

$$\vec{A}+\vec{B}=\vec{C}$$
$$\vec{A}+\vec{D}=\vec{B}$$



Since the vectors are coplaner there exists scalar constants α and β such that $\vec{A}+\alpha\vec{D}=\beta\vec{C}$ This implies

$$\vec{A}+\alpha(\vec{B}-\vec{A})=\beta(\vec{A}+\vec{B}) \quad \text{or} \quad \vec{A}(1-\alpha-\beta)+\vec{B}(\alpha-\beta)=\vec{0}$$

Since $\vec{A}$ and $\vec{B}$ are linearly independent and noncolinear one can state that

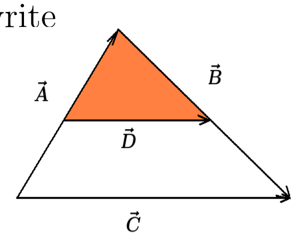
$$1=\alpha+\beta \quad \text{and} \quad 0=\alpha-\beta$$

Solving these simultaneous equations gives $\alpha=1/2$ and $\beta=1/2$ which demonstrates that the diagonals bisect one another.

► **6-3.**

Let $\vec{A}+\vec{B}=\vec{C}$ and then by construction write

$$\frac{1}{2}\vec{A}+\vec{D}+\frac{1}{2}\vec{B}=\vec{C}=\vec{A}+\vec{B}$$

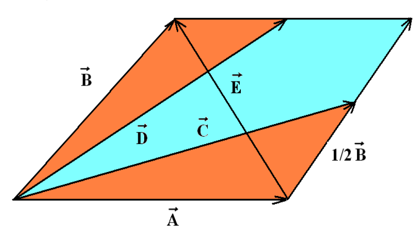


so that

$$\vec{D}=\frac{1}{2}\vec{A}+\frac{1}{2}\vec{B}=\frac{1}{2}\vec{C}$$

▶ **6-4.** By construction

$$\vec{A}+\frac{1}{2}\vec{B}=\vec{C}, \qquad \vec{B}+\frac{1}{2}\vec{A}=\vec{D}, \qquad \vec{A}+\vec{E}=\vec{B}$$



All these vectors are coplaner so that there exists scalar constants $\alpha,\beta,\gamma,\delta$ such that

$$\vec{A}+\alpha\vec{E}=\beta\vec{C} \qquad \text{and} \qquad \vec{A}+\gamma\vec{E}=\delta\vec{D}$$

or

$$\vec{A}+\alpha(\vec{B}-\vec{A})=\beta(\vec{A}+\frac{1}{2}\vec{B}) \qquad \text{and} \qquad \vec{A}+\gamma(\vec{B}-\vec{A})=\delta(\vec{B}+\frac{1}{2}\vec{A})$$

This implies that

$$\vec{A}(1-\alpha-\beta)+\vec{B}(\alpha-\frac{1}{2}\beta)=\vec{0} \qquad \text{and} \qquad \vec{A}(1-\gamma-\frac{1}{2}\delta)+\vec{B}(\gamma-\delta)=\vec{0}$$

This produces the simultaneous equations

$$\alpha+\beta=1, \quad \alpha-\frac{1}{2}\beta=0, \quad \gamma+\frac{1}{2}\delta=1, \quad \gamma-\delta=0$$

Solving for $\alpha,\beta,\gamma,\delta$ one finds

$$\alpha=1/3, \qquad \beta=2/3, \qquad \gamma=2/3, \qquad \delta=2/3$$

▶ **6-5.** (a)$c_1\vec{A}+c_2\vec{B}+c_3\vec{C}=\vec{0}$ gives the system of equations

$$\begin{aligned} c_1-4c_2+7c_3 &=0 \\ c_1-3c_2+6c_3 &=0 \qquad \Longrightarrow \qquad c_1=-3c_2, \qquad c_3=c_2 \\ -2c_1 \qquad -6c_3 &=0 \end{aligned}$$

Since $c_2\neq 0$, select $c_2=1$ for convenience, then $c_1=-3,\ c_2=1,\ c_3=1$, so vectors are linearly dependent.

(b) Linearly independent

(c) Linearly independent

▶ **6-6.** The vectors $\vec{A},\vec{B},\vec{C}$ are linearly dependent if and only if $\vec{A}\cdot(\vec{B}\times\vec{C})=0$
If $\vec{A}\cdot(\vec{B}\times\vec{C})=0$ and $\vec{A}\neq 0$, then $\vec{B}\times\vec{C}=\vec{0}$ which implies the vectors $\vec{B}$ and $\vec{C}$ are colinear. If the determinant $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}=0$, then two rows of the determinant are proportional which implies two of the vectors are colinear.

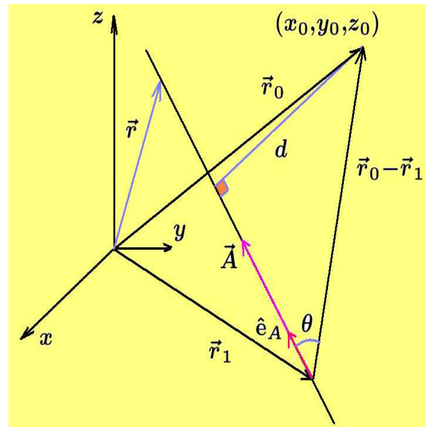
▶ **6-7.** (a) $\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos 0 = |\vec{A}|^2 = C^2$

(b) $\dfrac{d}{dt}(\vec{A}\cdot\vec{A}) = \dfrac{d}{dt}C^2 \implies \vec{A}\cdot\dfrac{d\vec{A}}{dt} + \dfrac{d\vec{A}}{dt}\cdot\vec{A} = 0 \implies \vec{A}\cdot\dfrac{d\vec{A}}{dt} = 0$ which shows $\vec{A}$ is perpendicular to $\dfrac{d\vec{A}}{dt}$

▶ **6-8.**
Equation of line is $\vec{r} = \vec{r}_1 + t\vec{A}$ where t is a parameter. Use the property of right triangles and write

$$d = |\vec{r}_0 - \vec{r}_1|\sin\theta = |(\vec{r}_0 - \vec{r}_1)\times \hat{\mathbf{e}}_A|$$

where the absolute value sign insures that d is positive and $\hat{\mathbf{e}}_A$ is a unit vector in the direction of $\vec{A}$.



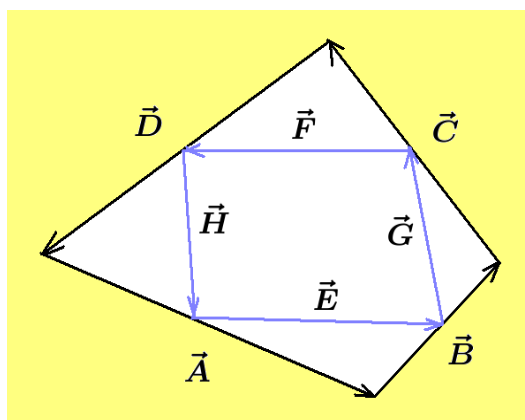
▶ **6-9.** Area is 54 square units

▶ **6-10.**

$$\vec{A}+\vec{B}+\vec{C}+\vec{D} = \vec{0}$$

By construction

$$\vec{E} = \frac{1}{2}\vec{A}+\frac{1}{2}\vec{B}$$
$$\vec{F} = \frac{1}{2}\vec{C}+\frac{1}{2}\vec{D}$$
$$\vec{G} = \frac{1}{2}\vec{B}+\frac{1}{2}\vec{C}$$
$$\vec{H} = \frac{1}{2}\vec{D}+\frac{1}{2}\vec{A}$$



To show $\vec{E}$ is parallel to $\vec{F}$, show $\vec{E}\times\vec{F} = \vec{0}$ and to show $\vec{G}$ is parallel to $\vec{H}$, show $\vec{G}\times\vec{H} = \vec{0}$

$$\vec{E}\times\vec{F} = (\frac{1}{2}\vec{A}+\frac{1}{2}\vec{B})\times(\frac{1}{2}\vec{C}+\frac{1}{2}\vec{D}) = \frac{1}{2}(\vec{A}+\vec{B})\times(-\frac{1}{2})(\vec{A}+\vec{B}) = \vec{0}$$
$$\vec{G}\times\vec{H} = (\frac{1}{2}\vec{B}+\frac{1}{2}\vec{C})\times(\frac{1}{2}\vec{D}+\frac{1}{2}\vec{A}) = \frac{1}{2}(\vec{B}+\vec{C})\times(-\frac{1}{2})(\vec{B}+\vec{C}) = \vec{0}$$

▶ **6-11.**

$$\vec{r}-\vec{r}_0 = (x-x_0)\,\hat{\mathbf{e}}_1 + (y-y_0)\,\hat{\mathbf{e}}_2 + (z-z_0)\,\hat{\mathbf{e}}_3$$
$$(\vec{r}-\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) = \rho^2$$
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \rho^2$$

► **6-12.** (c) 23/9 (d) 23/3

► **6-13.** (a) $\hat{\mathbf{e}}_C = \frac{\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{2}}$ also $-\hat{\mathbf{e}}_C$ (b) $-1/\sqrt{3}$

► **6-14.** (b) $\vec{A} \cdot \hat{\mathbf{e}}_\alpha = -\cos\alpha + \sqrt{3}\sin\alpha$
(c) $\alpha = \pi/6$ or $7\pi/6$
(d) $y(\alpha) = \sqrt{3}\sin\alpha - \cos\alpha$ and $\frac{dy}{d\alpha} = 0$ when $\alpha = -\pi/3$ or $2\pi/3$
$y''(\alpha) = -\sqrt{3}\sin\alpha + \cos\alpha$, $y''(-\pi/3) > 0$ and $y''(2\pi/3) < 0$, Maximum +2, Minimum -2

► **6-15.**

$$\begin{aligned}
\vec{A}(t) =& \vec{A}_0 + \vec{A}_1(t-t_0) + \vec{A}_2\frac{(t-t_0)^2}{2!} + \vec{A}_3\frac{(t-t_0)^3}{3!} + \cdots + \vec{A}_n\frac{(t-t_0)^n}{n!} + \cdots \\
\vec{A}'(t) =& \vec{A}_1 + \vec{A}_2(t-t_0) + \cdots + \vec{A}_n\frac{(t-t_0)^{n-1}}{(n-1)!} + \cdots \\
\vec{A}''(t) =& \vec{A}_2 + \vec{A}_3(t-t_0) + \cdots + \vec{A}_n\frac{(t-t_0)^{n-2}}{(n-2)!} + \cdots \\
\vdots \quad & \quad \vdots
\end{aligned}$$

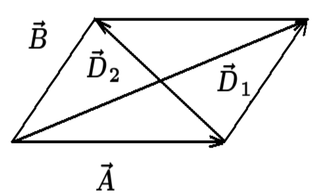
Evaluate the derivatives at $t = t_0$ and show

$$\vec{A}(t_0) = \vec{A}_0, \quad \vec{A}'(t_0) = \vec{A}_1, \quad \vec{A}''(t_0) = \vec{A}_2, \quad \ldots \quad , \quad \vec{A}^{(n)}(t_0) = \vec{A}_n, \quad \ldots$$

► **6-16.** (a) $\vec{A} \times \vec{B} = -16\,\hat{\mathbf{e}}_1 + 8\,\hat{\mathbf{e}}_3$ (b) $16\,\hat{\mathbf{e}}_1 - 8\,\hat{\mathbf{e}}_3$ (e) $\theta = \cos^{-1}(11/21) \approx 58.41°$

► **6-17.**

$$\begin{aligned}
\vec{D}_1 =& \vec{A} + \vec{B} = 3\,\hat{\mathbf{e}}_1 + 11\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3 \\
\vec{D}_2 =& \vec{B} - \vec{A} = \hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 \\
\text{Area} =& |\vec{A} \times \vec{B}| = 15
\end{aligned}$$

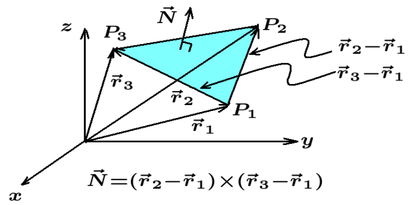


► **6-18.**

$$\begin{aligned}
\cos\alpha =& \sqrt{2}/2 \\
\cos\beta =& 1/2 \\
\cos\gamma =& -1/2
\end{aligned}
\qquad
\vec{e} = \frac{\vec{\rho}}{|\vec{\rho}|} = \cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2/4 + 1/4 + 1/4 = 1$$

► **6-19.** If $\vec{A} \times \vec{B} = \vec{0}$, then $\vec{A}$ is parallel to $\vec{B}$ or $\vec{A} = c\vec{B}$ for some constant c.



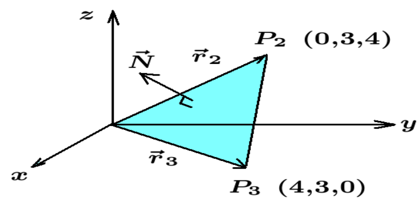
▶ **6-20.**

(d) $(\vec{r}-\vec{r}_1)\cdot\vec{N}=0$

is equation of plane

▶ **6-21.** $\vec{T}=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2$ is tangent to line. $x=3+\lambda,\ y=4-\lambda,\ z=2$

▶ **6-22.**

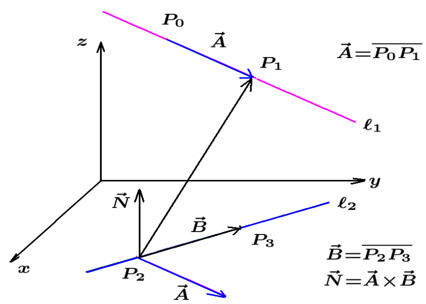


$\vec{N}=12\,\hat{\mathbf{e}}_1-16\,\hat{\mathbf{e}}_2+12\,\hat{\mathbf{e}}_3$

Equation of plane is

$(\vec{r}-r_1)\cdot\vec{N}=0$

or $3(x-4)-4(y-3)+z=0$



▶ **6-23.**

Plane through $\overline{P_0P_1}$ and perpendicular to $\vec{N}$

and plane through $\overline{P_2P_3}$ also perpendicular to $\vec{N}$

are parallel planes.

The vector $\overline{P_2P_1}$ is vector from one plane

to the other and its projection onto $\vec{N}$ is distance between planes

and also equal to the minimum distance between the skew lines.

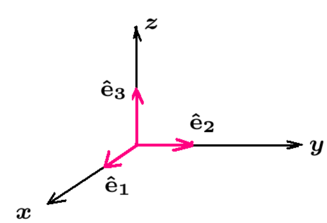
▶ **6-24.** $x=1+2t,\ y=5t,\ z=1+2t$ is parametric equation of line. The point $(6,13,12)$ is not on the line.

▶ **6-25.** If $\vec{r}-\vec{r}_1$ is colinear with $(\vec{r}_2-\vec{r}_1)$, then $(\vec{r}-\vec{r}_1)\times(\vec{r}_2-\vec{r}_1)=\vec{0}$ By vector addition $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$ where λ is a parameter.

▶ **6-27.**

even permutations

$$i=1,\ j=2,\ k=3,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2=\hat{\mathbf{e}}_3$$
$$i=2,\ j=3,\ k=1,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3=\hat{\mathbf{e}}_1$$
$$i=3,\ j=1,\ k=2,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1=\hat{\mathbf{e}}_2$$



odd permutations

$$i=3,\ j=2,\ k=1,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2=-\,\hat{\mathbf{e}}_1$$
$$i=2,\ j=1,\ k=3,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1=-\,\hat{\mathbf{e}}_3$$
$$i=1,\ j=3,\ k=2,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3=-\,\hat{\mathbf{e}}_2$$

► **6-28.** (a) If $\hat{\mathbf{e}}_{\ell_1} = \cos\alpha_1\,\hat{\mathbf{e}}_1 + \cos\beta_1\,\hat{\mathbf{e}}_2 + \cos\gamma_1\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_{\ell_2} = \cos\alpha_2\,\hat{\mathbf{e}}_1 + \cos\beta_2\,\hat{\mathbf{e}}_2 + \cos\gamma_2\,\hat{\mathbf{e}}_3$, then

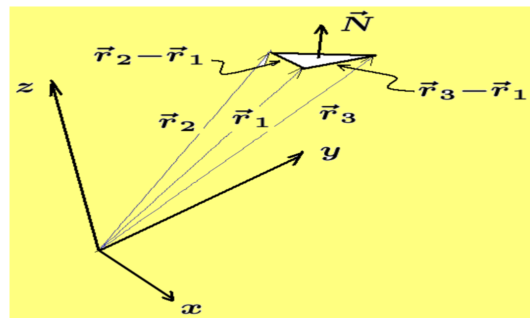
$$\hat{\mathbf{e}}_{\ell_1}\cdot\hat{\mathbf{e}}_{\ell_2} = \cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$$

(b) $\theta = \cos^{-1}(8/9) = .475882$ radians $\approx 27.266°$

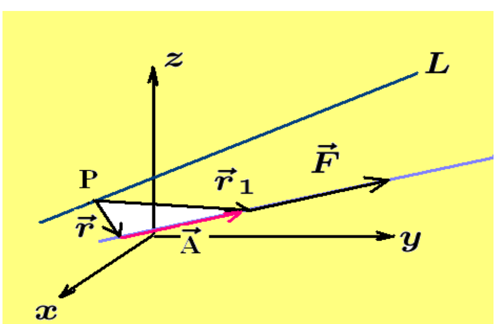
► **6-29.** Shortest distance is 9 units.

► **6-31.** If $\vec{A}\times\vec{B} = \vec{0}$, then $\vec{A}$ is colinear with $\vec{B}$ and if $\vec{B}\times\vec{C} = \vec{0}$, then $\vec{B}$ is colinear with $\vec{C}$. Therefore, $\vec{A}$ is colinear with $\vec{C}$ so that $\vec{A}\times\vec{C} = \vec{0}$.

► **6-32.** Normal to plane is $\vec{N} = (\vec{r}_3 - \vec{r}_1)\times(\vec{r}_2 - \vec{r}_1)$
Equation of plane is $(\vec{r} - \vec{r}_1)\cdot\vec{N} = 0$ or
$(x-3) - 2(y-10) + 2(z-13) = 0$ The distance from given point to plane is projection of $(\vec{r}_0 - \vec{r}_1)$ onto $\hat{\mathbf{e}}_N$ giving 9 units for the distance.
Here $\vec{r}_0 = 6\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 + 18\,\hat{\mathbf{e}}_3$



► **6-33.**



$$\vec{M}_P = \vec{r}_1\times\vec{F}$$
$$\vec{r}_1 = \vec{r} + \vec{A}$$
$$\vec{M}_P = (\vec{r}+\vec{A})\times\vec{F} = \vec{r}\times\vec{F} + \vec{A}\times\vec{F} = \vec{r}\times\vec{F}$$

because $\vec{A}\times\vec{F} = \vec{0}$, $\vec{A}$ and $\vec{F}$ are colinear

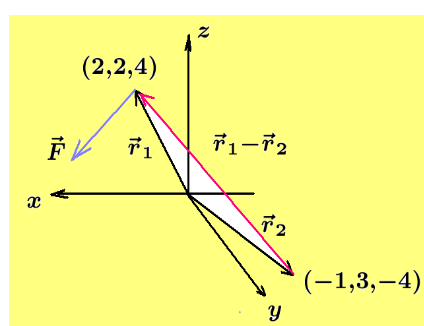
$\vec{M}_P\cdot\hat{\mathbf{e}}_L =$ projection of $\vec{M}_P$ on the line L

► **6-34.** (a) $\vec{M}_0 = \vec{r}_1\times\vec{F} = -1200\,\hat{\mathbf{e}}_1 + 800\,\hat{\mathbf{e}}_2 + 200\,\hat{\mathbf{e}}_3$

(b) $\vec{M}_{P_2} = (\vec{r}_1 - \vec{r}_2)\times\vec{F} = -1400\,\hat{\mathbf{e}}_1 + 1400\,\hat{\mathbf{e}}_2 + 700\,\hat{\mathbf{e}}_3$

(c) $\hat{\mathbf{e}}_L = \dfrac{-\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 - 4\,\hat{\mathbf{e}}_3}{\sqrt{26}}$

$$M_L = \vec{M}_{P_2}\cdot\hat{\mathbf{e}}_L = \frac{2800}{\sqrt{26}}$$



▶ **6-35.** (a) $\frac{t^2}{2}\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 - \frac{t^3}{3}\,\hat{\mathbf{e}}_3 + \vec{c}$

▶ **6-36.**

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \cos t\,\hat{\mathbf{e}}_1 + \sin t\,\hat{\mathbf{e}}_2 \\
\vec{v} = \vec{v}(t) &= \sin t\,\hat{\mathbf{e}}_1 - \cos t\,\hat{\mathbf{e}}_2 + \vec{c} \\
\vec{v}(0) &= -\,\hat{\mathbf{e}}_2 + \vec{c} = 2\,\hat{\mathbf{e}}_3 \quad\Longrightarrow\quad \vec{c} = 2\,\hat{\mathbf{e}}_3 + \,\hat{\mathbf{e}}_2 \\
\vec{v} = \frac{d\vec{r}}{dt} &= \sin t\,\hat{\mathbf{e}}_1 + (1-\cos t)\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 \\
\vec{r} = \vec{r}(t) &= -\cos t\,\hat{\mathbf{e}}_1 + (t - \sin t)\,\hat{\mathbf{e}}_2 + 2t\,\hat{\mathbf{e}}_3
\end{aligned}$$

▶ **6-39.** (a) $\omega = 5\,\hat{\mathbf{e}}_3$

(b) $\vec{v} = \omega \times \vec{r} = -5\sin 5t\ \hat{\mathbf{e}}_1 + 5\cos 5t\ \hat{\mathbf{e}}_2$

▶ **6-40.** $\vec{r} = e^t\ \hat{\mathbf{e}}_1 + \cos t\ \hat{\mathbf{e}}_2 + \sin t\ \hat{\mathbf{e}}_3$ with $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

▶ **6-41.**

$$\vec{C} = \vec{C}(x) = \vec{r}(x) + \alpha\vec{N}(x) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + \frac{(1+(y')^2)}{y''}\,[-y'\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2]$$

If $x = x(t)$ and $y = y(t)$, then

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)}{\frac{dx}{dt}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}
\end{aligned}$$

Substitute these derivatives into $\vec{C} = \vec{C}(x)$ and simplify.

▶ **6-42.** (b) $\vec{C} = \vec{C}(x) = x\,\hat{\mathbf{e}}_1 + e^x\,\hat{\mathbf{e}}_2 + \frac{(1+e^{2x})}{e^x}\,[-e^x\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2]$

▶ **6-43.** When $\hat{\mathbf{e}} = \hat{\mathbf{e}}_A = \frac{1}{|\vec{A}|}\vec{A}$, then $\hat{\mathbf{e}}_A \cdot \vec{A} = |\vec{A}|$

▶ **6-47.** $\frac{\partial U}{\partial x} = (4xy + y^2)\,\hat{\mathbf{e}}_1 + (y + 6xy)\,\hat{\mathbf{e}}_2$ and $\frac{\partial^2 U}{\partial x^2} = 4y\,\hat{\mathbf{e}}_1 + 6y\,\hat{\mathbf{e}}_2$

▶ **6-48.** If $\vec{v} = \frac{d\vec{r}}{dt} = \omega \times \vec{r}$, then

$$\frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = (z\omega_2 - y\omega_3)\,\hat{\mathbf{e}}_1 + (x\omega_3 - z\omega_1)\,\hat{\mathbf{e}}_2 + (y\omega_1 - x\omega_2)\,\hat{\mathbf{e}}_3$$

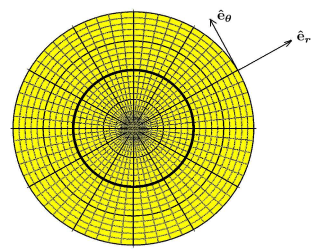
Equate like components to show result.

▶ **6-50.** If $\frac{d}{dt}(\vec{B}\times\vec{C}) = \vec{B}\times\frac{d\vec{C}}{dt} + \frac{d\vec{B}}{dt}\times\vec{C}$, then

$$\begin{aligned}\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] =& \vec{A}\times\frac{d}{dt}(\vec{B}\times\vec{C}) + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\\ =&\vec{A}\times\left[\vec{B}\times\frac{d\vec{C}}{dt} + \frac{d\vec{B}}{dt}\times\vec{C}\right] + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\\ =&\vec{A}\times(\vec{B}\times\frac{d\vec{C}}{dt}) + \vec{A}\times(\frac{d\vec{B}}{dt}\times\vec{C}) + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\end{aligned}$$

▶ **6-51.** The curves $\vec{r} = \vec{r}(r_0,\theta) = r_0\cos\theta\,\hat{\mathbf{e}}_1 + r_0\sin\theta\,\hat{\mathbf{e}}_2$ are coordinate curves which are circles of radius r_0. The curves $\vec{r} = \vec{r}(r,\theta_0) = r\cos\theta_0\,\hat{\mathbf{e}}_1 + r\sin\theta_0\,\hat{\mathbf{e}}_2$ are coordinate curves which are the rays $\theta = \theta_0 = a\ constant$

$$\begin{aligned}\frac{\partial\vec{r}}{\partial r} &= \cos\theta + \sin\theta\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_r\\ \frac{\partial\vec{r}}{\partial\theta} &= -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2 = r\,\hat{\mathbf{e}}_\theta\end{aligned}$$



▶ **6-52.** $\int_C \vec{F}\times d\vec{r} = \int_{(1,3)}^{(2,6)} \hat{\mathbf{e}}_1(y-x)\,dz - \hat{\mathbf{e}}_2\,xy\,dz + \hat{\mathbf{e}}_3(xy\,dy - (y-x)\,dx)$

On the line $y = 3x,\ z = 0,\ dz = 0,\ dy = 3dx$

so that $\int_C \vec{F}\times d\vec{r} = \int_1^2 [x(3x)\,3dx - (3x-x)\,dx]\,\hat{\mathbf{e}}_3 = 18\,\hat{\mathbf{e}}_3$

▶ **6-53.** $\int_C \vec{F}\cdot d\vec{r} = \int_C [(xy+1)dx + (x+z+1)dy + (z+1)dz]$

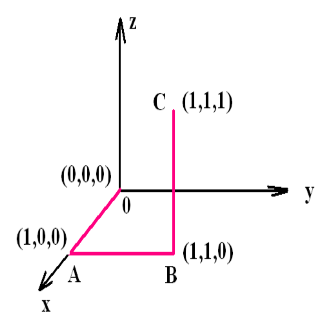
$\int_C \vec{F}\cdot d\vec{r} = \int_{0A}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3$

On 0A, $y = 0, dy = 0, z = 0, dz = 0$ and $I_1 = \int_0^1 dx = 1$

On AB, $x = 1, dx = 0, z = 0, dz = 0$ and $I_2 = \int_0^1 2dy = 2$

On BC, $x = 1, dx = 0, y = 1, dy = 0$ and $I_3 = \int_0^1 (z+1)dz = 3/2$

Therefore $\int_C \vec{F}\cdot d\vec{r} = 9/2$



▶ **6-55.** $\int_C \vec{F}\cdot d\vec{r} = \int_{0A}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} = I_1 + I_2$

On 0A $y = x, dy = dx, z = 0, dz = 0, 0 \le x \le 1,\ I_1 = \int_0^1 (x + 2x^2)\,dx = 7/6$

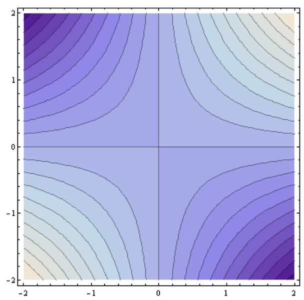
On AB $x = 1, y = 1, dx = dy = 0, 0 \le z \le 2\quad I_2 = \int_0^2 dz = 2$

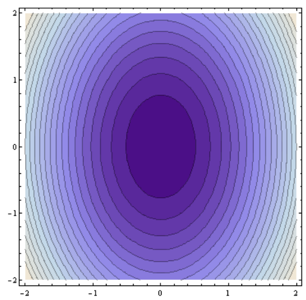
Therefore, $\int_C \vec{F}\cdot d\vec{r} = 19/6$

► **6-57.** On circle $x = \cos\theta,\ y = \sin\theta,\ dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C yz\,dx + 2x\,dy + y\,dz$$
$$= \int_0^{2\pi} -2\sin^2\theta\, d\theta + 2\cos^2\theta\, d\theta = 0$$

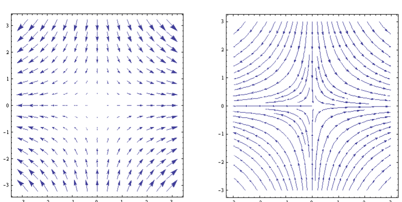
► **6-60.** (b) (d)



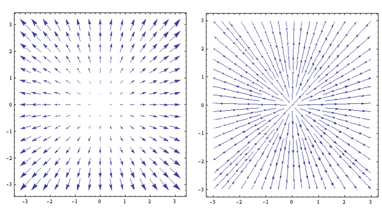


► **6-61.**

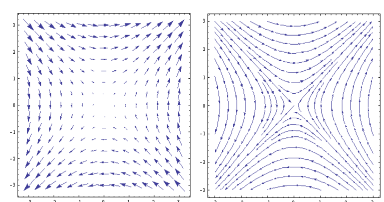
(a) $\dfrac{dx}{x} = \dfrac{dy}{-y} \quad\Longrightarrow\quad xy = c$



(b) $\dfrac{dx}{2x} = \dfrac{dy}{2y} \quad\Longrightarrow\quad y = cx$



(c) $\dfrac{dx}{2y} = \dfrac{dy}{2x} \quad\Longrightarrow\quad x^2 - y^2 = c$



► **6-62.** (c) Vectors $\vec{A}$ and $\vec{B}$ form a plane. $\vec{N} = \vec{A}\times\vec{B}$ is normal to plane and has the same direction as $\vec{r} - \vec{r}_0$.

(d) $(\vec{r}_2 - \vec{r}_1)\times(\vec{r}_3 - \vec{r}_1) = \vec{N}$ is normal to plane and $(\vec{r} - \vec{r}_1)\times\vec{N} = \vec{0}$ is the equation of the line.

► **6-63.** (b)

$$\vec{r} = \cos 2t\ \hat{\mathbf{e}}_1 + \sin 2t\ \hat{\mathbf{e}}_2$$
$$\vec{v} = \frac{d\vec{r}}{dt} = -2\sin 2t\ \hat{\mathbf{e}}_1 + 2\cos 2t\ \hat{\mathbf{e}}_2$$
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = -4\cos 2t\ \hat{\mathbf{e}}_1 - 4\sin 2t\ \hat{\mathbf{e}}_2$$

► **6-65.**

$$(a)\quad \frac{\partial\vec{F}}{\partial x} = 2x\,e1 + yz\ \hat{\mathbf{e}}_2 + 2xy^2z^2\ \hat{\mathbf{e}}_3$$
$$(d)\quad \frac{\partial^2\vec{F}}{\partial x^2} = 2\,\hat{\mathbf{e}}_1 + 2y^2z^2$$

▶ **6-68.** If $\vec{r}_0$ is center of sphere , $\vec{r}_1$ is point on sphere where tangent plane is constructed and $\vec{r}$ is a general point on the tangent plane, then the vector $(\vec{r} - \vec{r}_1)$ must be perpendicular to the vector $\vec{r}_1 - \vec{r}_0$.

▶ **6-69.**

$$\begin{aligned}
\frac{\partial F}{\partial x} =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \\
\frac{\partial^2 F}{\partial x^2} =& \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial u\,\partial v}\frac{\partial v}{\partial x}\right] \\
&+ \frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\left[\frac{\partial^2 F}{\partial v\,\partial u}\frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial x}\right]
\end{aligned}$$

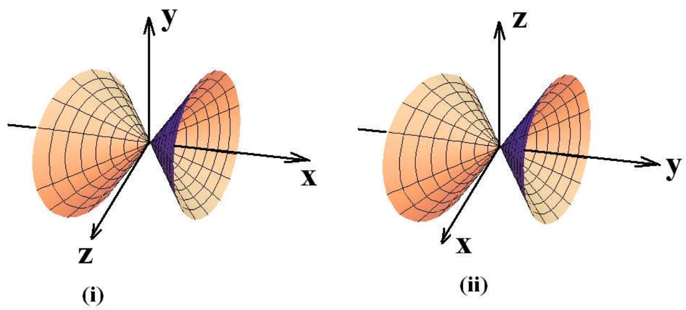
▶ **6-70.** (a) Use area of parallelogram $\vec{A} \times \vec{B}$ so that the area of 1/2 of parallelogram is $\frac{1}{2}\vec{A} \times \vec{B}$.

▶ **6-72.** (b) 58

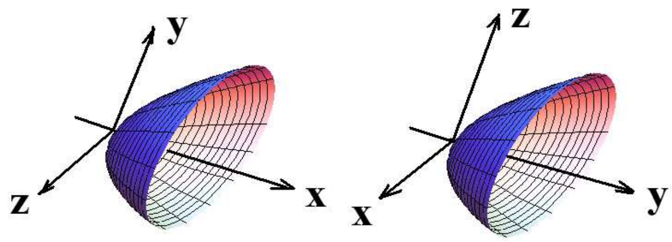
▶ **6-74.** (b) -12

Chapter 7

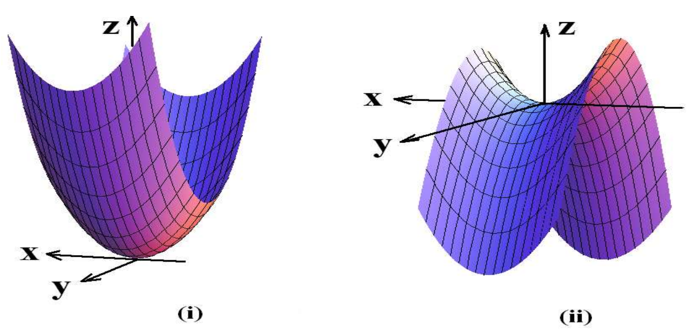
▶ **7-1.**



▶ **7-2.**



▶ **7-3.**



▶ **7-4.**

$$\vec{r} = \alpha \cos\omega t\ \hat{\mathbf{e}}_1 + \alpha \sin\omega t\ \hat{\mathbf{e}}_2 + \beta t\ \hat{\mathbf{e}}_3$$

$$\frac{d\vec{r}}{dt} = -\alpha\omega \sin\omega t\ \hat{\mathbf{e}}_1 + \alpha\omega\cos\omega t\ \hat{\mathbf{e}}_2 + \beta\ \hat{\mathbf{e}}_3$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\alpha^2\omega^2 + \beta^2}$$

$$\text{unit normal} \quad \hat{\mathbf{e}}_t = \frac{-\alpha\omega\sin\omega t\ \hat{\mathbf{e}}_1 + \alpha\omega\cos\omega t\ \hat{\mathbf{e}}_2 + \beta\ \hat{\mathbf{e}}_3}{\sqrt{\alpha^2\omega^2+\beta^2}} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}$$

$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{-\alpha\omega^2\cos\omega t\ \hat{\mathbf{e}}_1 - \alpha\omega^2\sin\omega t\ \hat{\mathbf{e}}_2}{\sqrt{\alpha^2\omega^2+\beta^2}}$$

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_t}{dt}}{\frac{ds}{dt}} = \frac{-\alpha\omega^2\cos\omega t\ \hat{\mathbf{e}}_1 - \alpha\omega^2\sin\omega t\ \hat{\mathbf{e}}_2}{\alpha^2\omega^2+\beta^2} = \kappa\ \hat{\mathbf{e}}_n, \qquad \hat{\mathbf{e}}_n \text{ is unit normal}$$

►7-4 (Continued)

$$\kappa^2 = \frac{d\,\hat{\mathbf{e}}_t}{ds}\cdot\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\alpha^2\omega^4}{(\alpha^2\omega^2+\beta^2)^2}$$

$$\text{Curvature} \qquad \kappa = \frac{\alpha\omega^2}{\alpha^2\omega^2+\beta^2}$$

$$\text{radius of curvature} \quad \rho = \frac{1}{\kappa} = \frac{\alpha^2\omega^2+\beta^2}{\alpha\omega^2}$$

$$\text{unit normal} \quad \hat{\mathbf{e}}_n = \frac{1}{\kappa}\frac{d\,\hat{\mathbf{e}}_t}{ds} = -\cos\omega t\;\hat{\mathbf{e}}_1 - \sin\omega t\;\hat{\mathbf{e}}_2$$

$$\text{unit binormal} \quad \hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t\times\hat{\mathbf{e}}_n = \beta\sin\omega t\;\hat{\mathbf{e}}_1 - \beta\cos\omega t\;\hat{\mathbf{e}}_2 + \frac{\alpha\omega}{\sqrt{\alpha^2\omega^2+\beta^2}}\;\hat{\mathbf{e}}_3$$

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_b}{dt}}{\frac{ds}{dt}} = \frac{\beta\omega\cos\omega t\;\hat{\mathbf{e}}_1 + \beta\omega\sin\omega t\;\hat{\mathbf{e}}_2}{\alpha^2\omega^2+\beta^2} = -\tau\;\hat{\mathbf{e}}_n$$

$$\tau^2 = \frac{d\,\hat{\mathbf{e}}_b}{ds}\cdot\frac{d\,\hat{\mathbf{e}}_b}{ds} = \frac{\beta^2\omega^2}{(\alpha^2\omega^2+\beta^2)^2}$$

$$\tau = \frac{\beta\omega}{\alpha^2\omega^2+\beta^2}, \qquad \sigma = \frac{1}{\tau} = \frac{\alpha^2\omega^2+\beta^2}{\beta\omega}$$

► 7-5.

$$\vec{r}\,' = \frac{d\vec{r}}{ds} \qquad \left(\frac{ds}{dt}\right)^2 = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \vec{r}\,'\cdot\vec{r}\,', \qquad \frac{ds}{dt} = (\vec{r}\,'\cdot\vec{r}\,')^{1/2}$$

$$\hat{\mathbf{e}}_t = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}, \qquad \frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{\frac{ds}{dt}\frac{d^2\vec{r}}{dt^2} - \frac{d\vec{r}}{dt}\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^2}$$

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_t}{dt}}{\frac{ds}{dt}} = \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\vec{r}\,'\frac{d^2s}{dt^2}}{\vec{r}\,'\cdot\vec{r}\,'\;\frac{ds}{dt}} = \kappa\;\hat{\mathbf{e}}_n$$

$$= \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\frac{\vec{r}\,'(\vec{r}\,'\cdot\vec{r}\,'')}{(\vec{r}\,'\cdot\vec{r}\,')^{1/2}}}{\vec{r}\,'\cdot\vec{r}\,'\,(\vec{r}\,'\cdot\vec{r}\,')^{1/2}} = \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\vec{r}\,'\,(\vec{r}\,'\cdot\vec{r}\,'')}{(\vec{r}\,'\cdot\vec{r}\,')^2} = \kappa\;\hat{\mathbf{e}}_n$$

$$\kappa^2 = (\kappa\;\hat{\mathbf{e}}_n)\cdot(\kappa\;\hat{\mathbf{e}}_n) = \frac{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'') - (\vec{r}\,'\cdot\vec{r}\,'')^2}{(\vec{r}\,'\cdot\vec{r}\,')^3}$$

► 7-6. Special case of previous problem

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y(x)\,\hat{\mathbf{e}}_2, \quad \vec{r}\,' = \hat{\mathbf{e}}_1 + y'\;\hat{\mathbf{e}}_2, \quad \vec{r}\,'' = y''\;\hat{\mathbf{e}}_2$$

and

$$\vec{r}\,'\cdot\vec{r}\,' = 1 + (y')^2, \qquad \vec{r}\,'\cdot\vec{r}\,'' = y'y'', \qquad \vec{r}\,''\cdot\vec{r}\,'' = (y'')^2$$

giving

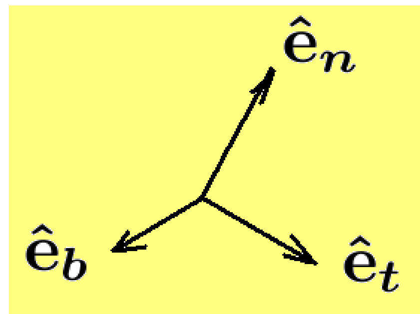
$$\kappa = \frac{\sqrt{(1+(y')^2)(y'')^2 - (y')^2(y'')^2}}{(1+(y')^2)^{3/2}} \qquad \Longrightarrow \qquad \kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

▶ **7-7.**

$$\frac{d\vec{r}}{dt} = \vec{r}\,', \qquad \frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{d\vec{r}}{dt}, \qquad \frac{ds}{dt} = s' = (\vec{r}\,'\cdot\vec{r}\,')^{1/2}$$
$$\frac{d\vec{r}}{ds} = \hat{\mathbf{e}}_t = \frac{\vec{r}\,'}{s'}, \qquad \frac{d^2s}{dt^2} = s'' = \frac{\vec{r}\,'\cdot\vec{r}\,''}{(\vec{r}\,'\cdot\vec{r}\,')^{1/2}}$$
$$\frac{d^2\vec{r}}{ds^2} = \frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n = \frac{s'\vec{r}\,'' - \vec{r}\,'s''}{(s')^3}, \qquad \frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{ds^2} = \kappa\,\hat{\mathbf{e}}_t\times\hat{\mathbf{e}}_n = \kappa\,\hat{\mathbf{e}}_b$$

▶ **7-8.** See derivatives from previous problem.

$$\frac{d^3\vec{r}}{ds^3} = \kappa\frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa(\tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t) + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$
$$\frac{d^3\vec{r}}{ds^3} = \kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$
$$\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3} = \kappa\,\hat{\mathbf{e}}_n\times(\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n) = \kappa^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b$$
$$\frac{d\vec{r}}{ds}\cdot(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}) = \hat{\mathbf{e}}_t\cdot(\kappa^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b) = \kappa^2\tau$$
$$\tau = \frac{1}{\kappa^2}\frac{d\vec{r}}{ds}\cdot(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3})$$



In terms of a parameter t one can write

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}, \qquad \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{r}}{ds}\frac{d^2s}{dt^2} + \frac{d^2\vec{r}}{ds^2}(\frac{ds}{dt})^2$$
$$\frac{d^3\vec{r}}{dt^3} = \frac{d\vec{r}}{ds}\frac{d^3s}{dt^3} + \frac{d^2\vec{r}}{ds^2}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{d^2\vec{r}}{ds^2}2(\frac{ds}{dt})\frac{d^2s}{dt^2} + \frac{d^3\vec{r}}{ds^3}(\frac{ds}{dt})^3$$

which can be express

$$\vec{r}\,'' = \hat{\mathbf{e}}_t\frac{d^2s}{dt^2} + \kappa\,\hat{\mathbf{e}}_n(\frac{ds}{dt})^2$$
$$\vec{r}\,''' = \hat{\mathbf{e}}_t\frac{d^3s}{dt^3} + \kappa\,\hat{\mathbf{e}}_n\frac{ds}{dt}\frac{d^2s}{dt^2} + \kappa\,\hat{\mathbf{e}}_n2\frac{ds}{dt}\frac{d^2s}{dt^2} + (\frac{ds}{dt})^3[\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n]$$
$$\vec{r}\,''\times\vec{r}\,''' = (stuff_1)\,\hat{\mathbf{e}}_b + (stuff_2)\,\hat{\mathbf{e}}_n + \kappa^2\tau(\frac{ds}{dt})^5\,\hat{\mathbf{e}}_t$$
$$\vec{r}\,'\cdot(\vec{r}\,''\times\vec{r}\,''') = \kappa^2\tau(\frac{ds}{dt})^6, \qquad \text{but} \quad (\frac{ds}{dt})^6 = (\vec{r}\,'\cdot\vec{r}\,')^3$$

and κ^2 can be obtained from problem 7-5 to obtain

$$\tau = \frac{\vec{r}\,'\cdot(\vec{r}\,''\times\vec{r}\,''')}{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'') - (\vec{r}\,'\cdot\vec{r}\,'')^2}$$

▶ **7-9.** (a) zero (b) zero

▶ **7-10.**

$$\frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{d\vec{r}}{dt} = \vec{r}\,' \quad \text{or} \quad \hat{\mathbf{e}}_t \frac{ds}{dt} = \vec{r}\,', \qquad |\vec{r}\,'| = \frac{ds}{dt}$$

$$\frac{d}{dt}\left(\hat{\mathbf{e}}_t \frac{ds}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \vec{r}\,''$$

$$\hat{\mathbf{e}}_t \frac{d^2 s}{dt^2} + \frac{ds}{dt}\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \vec{r}\,''$$

$$\hat{\mathbf{e}}_t \frac{d^2 s}{dt^2} + (\frac{ds}{dt})^2 \kappa\, \hat{\mathbf{e}}_n = \vec{r}\,''$$

$$\vec{r}\,' \times \vec{r}\,'' = \hat{\mathbf{e}}_t \frac{ds}{dt} \times (\,\hat{\mathbf{e}}_t \frac{d^2 s}{dt^2} + (\frac{ds}{dt})^2 \kappa\, \hat{\mathbf{e}}_n) = (\frac{ds}{dt})^3 \kappa\, \hat{\mathbf{e}}_b = |\vec{r}\,'|^3 \kappa\, \hat{\mathbf{e}}_b$$

$$|\vec{r}\,' \times \vec{r}\,''| = \kappa |\vec{r}\,'|^3$$

▶ **7-11.** (i)

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot \hat{\mathbf{e}}\,\Big|_{(1,1,1)}$$

$$= [(2xy^2z)\,\hat{\mathbf{e}}_1 + 2yx^2z\,\hat{\mathbf{e}}_2 + (x^2y^2 + x^3)\,\hat{\mathbf{e}}_3] \cdot \left(\frac{3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3}{7}\right)\Big|_{(1,1,1)} = \frac{23}{7}$$

▶ **7-12.**

$$(i) \qquad \text{grad}\,\phi = 2xy\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2$$

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot\, \hat{\mathbf{e}}_\alpha = (2xy\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2) \cdot (\cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2)\Big|_{(2,\sqrt{3})}$$

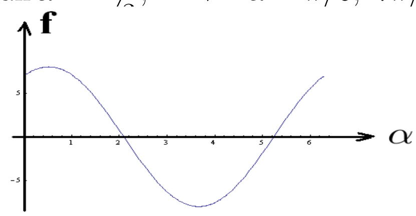
$$\frac{d\phi}{ds} = 4\sqrt{3}\cos\alpha + 4\sin\alpha = f(\alpha)$$

$$(ii) \qquad \frac{df}{d\alpha} = -\,4\sqrt{3}\sin\alpha + 4\cos\alpha = 0, \quad \Longrightarrow \quad \tan\alpha = \frac{1}{\sqrt{3}}, \quad \Longrightarrow \quad \alpha = \pi/6,\ 7\pi/6$$

$$f''(\alpha) = -\,4\sqrt{3}\cos\alpha - 4\sin\alpha$$

$f''(\pi/6) < 0$ maximum at $\pi/6$

$f''(7\pi/6) > 0$ minimum at $7\pi/6$



▶ **7-13.** Show derivative equals

$$\vec{r} \cdot (\vec{r}\,' \times \vec{r}\,''') + \vec{r} \cdot (\vec{r}\,'' \times \vec{r}\,'') + \vec{r}\,' \cdot (\vec{r}\,' \times \vec{r}\,'')$$

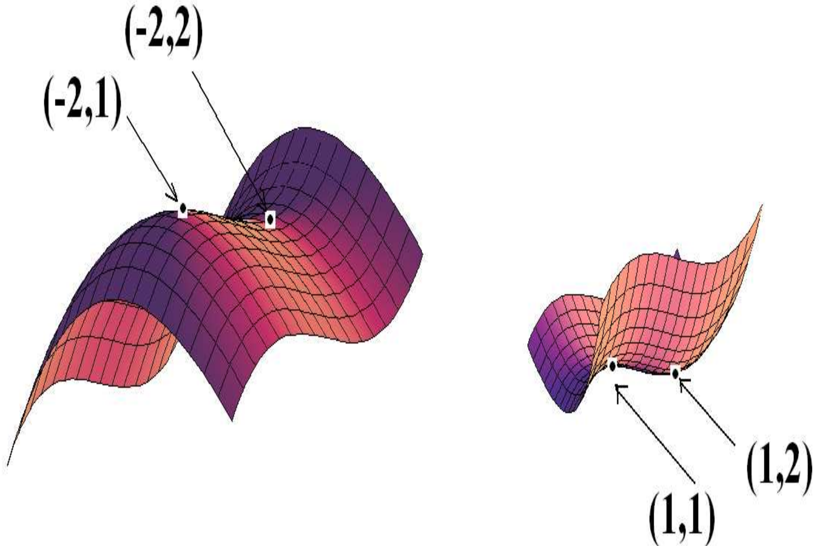
where last two terms are zero. Use triple scalar product relation on last term.

▶ **7-14.** Use the vector identity $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$ and show

$$\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C}) =&\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})\\ \vec{B}\times(\vec{C}\times\vec{A}) =&\vec{C}(\vec{B}\cdot\vec{A})-\vec{A}(\vec{B}\cdot\vec{C})\\ \vec{C}\times(\vec{A}\times\vec{B}) =&\vec{A}(\vec{C}\cdot\vec{B})-\vec{B}(\vec{C}\cdot\vec{A})\end{aligned}$$

Add both sides to obtain desired result.

▶ **7-15.**



$$\frac{\partial z}{\partial x} =x^2+x+2, \qquad \frac{\partial z}{\partial y} = y^2-3y+2$$

critical points are where $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$ simultaneously

critical points $(-2,2)$, $\quad(-2,1)$, $\quad(1,2)$, $(1,1)$

$A=\dfrac{\partial^2 z}{\partial x^2}=2x+1=A, \qquad \dfrac{\partial^2 z}{\partial x\,\partial y}=0=B, \qquad \dfrac{\partial^2 z}{\partial y^2}=2y-3=C, \qquad \Delta=AC-B^2$

At $(2,2)$, $\Delta=-3<0$, saddle point

At $(-2,1)$, $\Delta=3>0$, $A<0$, relative minimum

At $(1,2)$, $\Delta=3>0$, $A>0$, relative minimum

At $(1,1)$, $\Delta=-3<0$, saddle point

▶ **7-16.** $\vec{\omega}=\alpha\,\hat{\mathbf{e}}_t+\beta\,\hat{\mathbf{e}}_n+\gamma\,\hat{\mathbf{e}}_b$ and if

$$\omega\times\,\hat{\mathbf{e}}_t=\kappa\,\hat{\mathbf{e}}_n,\quad \omega\times\,\hat{\mathbf{e}}_b=-\tau\,\hat{\mathbf{e}}_n,\quad \omega\times\,\hat{\mathbf{e}}_n=\tau\,\hat{\mathbf{e}}_b-\kappa\,\hat{\mathbf{e}}_t$$

show that $\alpha=\tau,\beta=0,\gamma=\kappa$ so that $\omega=\tau\,\hat{\mathbf{e}}_t+\kappa\,\hat{\mathbf{e}}_b$

▶ **7-17.** Vector equation of plane is $(\vec{r}-\vec{r}_0)\cdot\vec{N}=0$ where $\vec{r}=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$ is variable point in plane, $\vec{r}_0$ is fixed point in plane and $\vec{N}$ is normal to plane. Note that

$$\begin{aligned}\frac{d\vec{r}}{ds} &= \hat{\mathbf{e}}_t \text{ is normal to normal plane}\\ \frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{ds^2} &= \hat{\mathbf{e}}_t\times\kappa\,\hat{\mathbf{e}}_n=\kappa\,\hat{\mathbf{e}}_b \text{ is normal to osculating plane}\\ \frac{d^2\vec{r}}{ds^2} &= \kappa\,\hat{\mathbf{e}}_n \text{ is normal to rectifying plane}\end{aligned}$$

▶ **7-18.** If $\vec{r}(u,v)=x(u,v)\,\hat{\mathbf{e}}_1+y(u,v)\,\hat{\mathbf{e}}_2+z(u,v)\,\hat{\mathbf{e}}_3$, then

$$\begin{aligned}&\frac{\partial\vec{r}}{\partial u}=\frac{\partial x}{\partial u}\,\hat{\mathbf{e}}_1+\frac{\partial y}{\partial u}\,\hat{\mathbf{e}}_2+\frac{\partial z}{\partial u}\,\hat{\mathbf{e}}_3 \text{ is tangent to coordinate curve } \vec{r}(u,v_0)\\ &\frac{\partial\vec{r}}{\partial v}=\frac{\partial x}{\partial v}\,\hat{\mathbf{e}}_1+\frac{\partial y}{\partial v}\,\hat{\mathbf{e}}_2+\frac{\partial z}{\partial v}\,\hat{\mathbf{e}}_3 \text{ is tangent to coordinate curve } \vec{r}(u_0,v)\\ &\vec{N}=\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} \text{ is normal to surface and}\\ &\hat{\mathbf{e}}_n=\frac{\vec{N}}{|\vec{N}|}=\ell_1\,\hat{\mathbf{e}}_1+\ell_2\,\hat{\mathbf{e}}_2+\ell_3\,\hat{\mathbf{e}}_3 \text{ is unit normal to surface where}\\ &|\vec{N}|=\sqrt{(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v})\cdot(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v})}=\sqrt{EG-F^2}\end{aligned}$$

▶ **7-19.** $\vec{N}=\operatorname{grad}F=\dfrac{\partial F}{\partial x}\,\hat{\mathbf{e}}_1+\dfrac{\partial F}{\partial y}\,\hat{\mathbf{e}}_2+\dfrac{\partial F}{\partial z}\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_n=\frac{\vec{N}}{|\vec{N}|}$ where $|\vec{N}|=H$

▶ **7-20.** If surface is $\vec{r}=\vec{r}(x,y)=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z(x,y)\,\hat{\mathbf{e}}_3$, then

$$\frac{\partial\vec{r}}{\partial x}=\hat{\mathbf{e}}_1+\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \frac{\partial\vec{r}}{\partial y}=\hat{\mathbf{e}}_2+\frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

and $\vec{N}=\dfrac{\partial\vec{r}}{\partial x}\times\dfrac{\partial\vec{r}}{\partial y}=-\dfrac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1-\dfrac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3$ with $\hat{\mathbf{e}}_n=\dfrac{\vec{N}}{|\vec{N}|}$ and $|\vec{N}|=\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}$

▶ **7-21.** $\hat{\mathbf{e}}_n=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2$ or $\hat{\mathbf{e}}_n=\cos\theta\,\hat{\mathbf{e}}_1+\sin\theta\,\hat{\mathbf{e}}_2$

▶ **7-22.** $\hat{\mathbf{e}}_n=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$ or $\hat{\mathbf{e}}_n=\sin\theta\cos\phi\,\hat{\mathbf{e}}_1+\sin\theta\sin\phi\,\hat{\mathbf{e}}_2+\cos\theta\,\hat{\mathbf{e}}_3$

▶ **7-23.** $\phi=ax+by+cz-d=0$ and $\vec{N}=\operatorname{grad}\phi=a\,\hat{\mathbf{e}}_1+b\,\hat{\mathbf{e}}_2+c\,\hat{\mathbf{e}}_3$ so that

$$\hat{\mathbf{e}}_n=\frac{a\,\hat{\mathbf{e}}_1+b\,\hat{\mathbf{e}}_2+c\,\hat{\mathbf{e}}_3}{\sqrt{a^2+b^2+c^2}}$$

▶ **7-24.** $\dfrac{1}{2}\,\hat{\mathbf{e}}_1+\dfrac{\pi}{8}\,\hat{\mathbf{e}}_2$

▶ **7-25.** $\dfrac{1}{15}\,\hat{\mathbf{e}}_1+\dfrac{1}{15}\,\hat{\mathbf{e}}_2+\dfrac{1}{15}\,\hat{\mathbf{e}}_3$

▶ **7-26.** 2π

▶ **7-27.** $\dfrac{19}{24}$

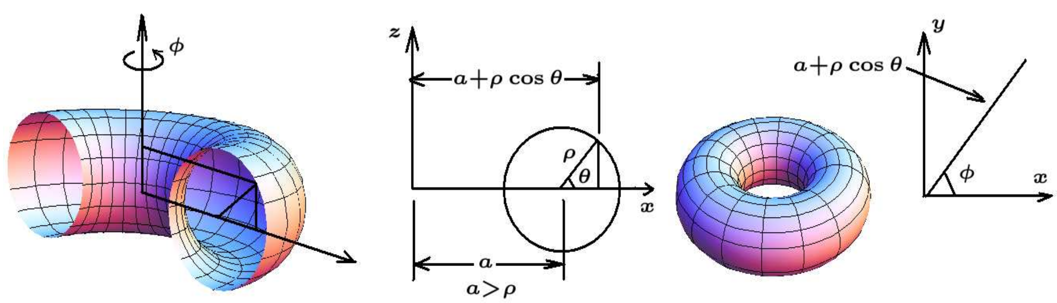
▶ **7-28.** π

▶ **7-30.** If $\vec{r} = \vec{r}(u, v)$, then $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv$ and

$$ds^2 = d\vec{r} \cdot d\vec{r} = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}\,(du)^2 + 2\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\,du\,dv + \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\,(dv)^2$$

▶ **7-31.** (b) $S = \pi R\sqrt{R^2 + H^2}$

▶ **7-32.**



$dS = \rho(a + \rho\cos\theta)\,d\theta\,d\phi$ and $dV = \rho(a + \rho\cos\theta)\,d\rho\,d\theta\,d\phi$ Volume is $V = 2\pi^2 a\rho^2$ and surface area is $S = 4\pi^2 a\rho$

What does the Pappus theorem tell you about the volume?

▶ **7-33.** (d) $ds = \sqrt{1 + (y')^2}\,dx$ where $y = x^2$, $\quad y' = 2x$ so that

$$S = \int_0^2 \sqrt{1 + (2x)^2}\,dx$$

To evaluate this integral make the substitution $2x = \sinh u$ with $2dx = \cosh u\,du$ and show

$$S = \int_0^{\sinh^{-1}(4)} \frac{1}{2}\cosh^2 u\,du = \frac{1}{2}\int_0^{\sinh^{-1}(4)} [\frac{1}{2}\cosh 2u + \frac{1}{2}]\,du$$

Show that

$$S = \sqrt{17} + \frac{1}{4}\sinh^{-1}(4)$$

▶ **7-34.** (a) x, y-plane, coordinate curves $\vec{r}(u_0, v)$ vertical lines, $\vec{r}(u, v_0)$ horizontal lines

(b) x, y-plane polar coordinates, $\vec{r}(u_0, v)$ circles of radius u_0, $\vec{r}(u, v_0)$ rays at angle v_0.

▶ **7-35.** $I = 4\int_0^1 \int_0^{1-x} dydx = 2$

▶ **7-36.**

$$I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\, dS = \left[\iint_{OCDG} + \iint_{GFA0} + \iint_{FABE} + \iint_{BEDC} + \iint_{EDGF} + \iint_{ABC0}\right]\vec{F}\cdot\hat{\mathbf{e}}_n\, dS$$

On face 0CDG $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_1,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -x^2\Big|_{x=0} = 0,\ dS = dydz$

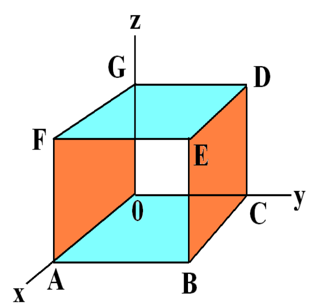
On face GFA0 $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_2,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -y^2\Big|_{y=0} = 0,\ dS = dxdz$

On face FABE $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_1,\ \vec{F}\cdot\hat{\mathbf{e}}_n = x^2\Big|_{x=1} = 1,\ dS = dydz$

On face BEDC, $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_2,\ \vec{F}\cdot\hat{\mathbf{e}}_n = y^2\Big|_{y=1} = 1,\ dS = dxdz$

On face EDGF, $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_3,\ \vec{F}\cdot\hat{\mathbf{e}}_n = z^2\Big|_{z=1} = 1,\ dS = dxdy$

On face ABC0, $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_3,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -z^2\Big|_{z=0} = 0,\ dS = dxdy$



Add the above surface integrals over each face and show $I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\, dS = 1{+}1{+}1 = 3$

▶ **7-37.**

$$\phi = x^2 + y^2 - 1 = 0, \quad 0 \le z \le 3, \quad \vec{N} = \operatorname{grad}\phi = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$$

$$\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \quad \text{since } x^2 + y^2 = 1, \qquad dS = \frac{dxdz}{|\,\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_2|} = \frac{dxdz}{y}$$

$$I = \iint_S f(x,y,z)\, dS = \int_0^3 \int_0^1 2(x+1)y\,\frac{dxdz}{y} = 9$$

▶ **7-38.** Method I Use cartesian coordinates

$$\hat{\mathbf{e}}_n = \frac{x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3}{3} \qquad \frac{dS}{=}\frac{dxdy}{|\,\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_3|} = 3\frac{dxdy}{z}$$

$$\vec{F}\cdot\hat{\mathbf{e}}_n = \frac{x(x+z) + y(y+z) - (x+y)z}{3} = \frac{x^2+y^2}{3}$$

$$I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\, dS = \int_{-3}^{3} \int_{y=-\sqrt{9-x^2}}^{+\sqrt{9-x^2}} \frac{x^2+y^2}{\sqrt{9-x^2-y^2}}\, dydx = 36\pi$$

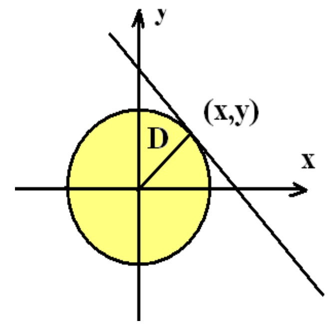
► **7-38.** (continued)

Method II Use spherical coordinates $x = 3\sin\theta\cos\phi,\ y = 3\sin\theta\sin\phi,\ z = 3\cos\theta$

$$dS = 3d\theta\, 3\sin\theta\, d\phi, \qquad \vec{F}\cdot\hat{\mathbf{e}}_n = \frac{x^2+y^2}{3}$$

$$I = \iint_S (3)(x^2+y^2)\sin\theta\, d\theta d\phi = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (3)(9\sin^2\theta\cos^2\phi + 9\sin^2\theta\sin^2\phi)\,\sin\theta\, d\theta d\phi$$

$$I = 27\int_0^{\pi/2}\int_0^{2\pi} \sin^3\theta\, d\theta d\phi = 36\pi$$



► **7-40.** $G = x+y-2 = 0$ and $D^2 = F(x,y) = x^2+y^2$ one finds $\text{grad}\, F = \nabla F = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$ and $\text{grad}\, G = \nabla G = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$ Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ denote position vector to point on circle, then $\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2$ is tangent to circle. Differentiate equation of circle to show $2x + 2y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{x}{y}$, then one can write $\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 - \frac{x}{y}\hat{\mathbf{e}}_2$ is tangent to circle. Observe that $\frac{d\vec{r}}{dx}\cdot\nabla F = 2x + 2y\frac{-x}{y} = 0$ which shows ∇F is perpendicular to $\frac{d\vec{r}}{dx}$ or ∇F is perpendicular to line. The slope of the line is -1 and the vectors $\vec{r}_1 = 2\,\hat{\mathbf{e}}_1, \quad \vec{r}_2 = 2\,\hat{\mathbf{e}}_2$ point to points on the line so that $\vec{\alpha} = \vec{r}_1 - \vec{r}_2$ is a direction vector of the line. Note that $\vec{\alpha}\cdot\nabla G = (2\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2)\cdot(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2) = 0$ so that ∇G is perpendicular to line.

The quantity D^2 is a minimum when the circle just touches the line and at this point of contact the normal to the circle is ∇G and this vector is also perpendicular to the line and has the same direction as ∇F. Consequently, there exists a scalar λ such that $\nabla F + \lambda\nabla G = 0$ at this point of contact.

Here $H = F + \lambda G = x^2 + y^2 + \lambda(x+y-2)$ with

$$\begin{aligned}
\frac{\partial H}{\partial \lambda} &= x + y - 2 = 0 \quad \text{constraint equation}\\
\frac{\partial H}{\partial x} &= 2x + \lambda = 0\\
\frac{\partial H}{\partial y} &= 2y + \lambda = 0
\end{aligned}$$

where the last two equations are a necessary condition for H to have an extreme value. Solve the above system of equations and show $x = 1,\ y = 1,\ \lambda = -2$ so that the minimum distance from the origin to the line is $\sqrt{2}$.

► **7-41.**

$$
\begin{aligned}
F =&\omega + \lambda_1 g + \lambda_2 h = x^2 + y^2 + z^2 + \lambda_1 g + \lambda_2 h \\
\frac{\partial F}{\partial \lambda_1} =&g = x + y + z - 6 = 0 \\
\frac{\partial F}{\partial \lambda_2} =&h = 3x + 5y + 7z - 34 = 0 \\
\frac{\partial F}{\partial x} =&2x + \lambda_1 + 3\lambda_2 = 0 \\
\frac{\partial F}{\partial y} =&2y + \lambda_1 + 5\lambda_2 = 0 \\
\frac{\partial F}{\partial z} =&2z + \lambda_1 + 7\lambda_2 = 0
\end{aligned}
$$

Solve this system of equations and show

$$x = 1, \quad y = 2, \quad z = 3, \quad \lambda_1 = 1, \quad \lambda_2 = -1$$

► **7-42.**

$$\nabla F = -\, 2z\,\hat{\mathbf{e}}_2 + (3y - 1)\,\hat{\mathbf{e}}_3 \quad \text{and} \quad \hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3, \quad \text{with } dS = \frac{dxdy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \frac{dxdy}{z}$$

$$I = \int_1^1 \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} (y-1)dydx = -\pi$$

Problem can also be solved using spherical coordinates $x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta$

► **7-44.** (b) $\dfrac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_2, \quad \dfrac{\partial \vec{r}}{\partial z} = \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ giving

$$
\begin{aligned}
E =&\frac{\partial \vec{r}}{\partial x} \cdot \frac{\partial \vec{r}}{\partial x} = 1 + (\frac{\partial y}{\partial x})^2 \\
F =&\frac{\partial \vec{r}}{\partial x} \cdot \frac{\partial \vec{r}}{\partial z} = \frac{\partial y}{\partial x}\frac{\partial y}{\partial z} \\
G =&\frac{\partial \vec{r}}{\partial z} \cdot \frac{\partial \vec{r}}{\partial z} = (\frac{\partial y}{\partial z})^2 + 1
\end{aligned}
$$

Show

$$dS = \sqrt{EG - F^2} = \sqrt{1 + (\frac{\partial y}{\partial x})^2 + (\frac{\partial y}{\partial z})^2}$$

Normal to surface is $\vec{N} = \dfrac{\partial \vec{r}}{\partial z} \times \dfrac{\partial \vec{r}}{\partial x} = -\dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_3$ Note that if you use $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial z}$ you get $-\vec{N}$

The vector element of area is $d\vec{S} = \hat{\mathbf{e}}_n\, dS = (-\dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_3)\, dxdz$ Take the dot product of both sides of this equation with $\hat{\mathbf{e}}_2$ and show

$$|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|\, dS = dxdz, \qquad \text{or} \qquad dS = \frac{dxdz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|}$$

Here the absolute value sign is used to insure that the element of surface area dS is positive.

▶ **7-45.** $\vec{r}_c = \dfrac{\sum_{j=1}^{n} m_j \vec{r}_j}{\sum_{j=1}^{n} m_j}$ Note that this is a weighted sum of the vectors $\vec{r}_i$ where the weight factors are $m_1, m_2, \ldots, m_n$.

▶ **7-46.** We have verified the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ Change symbols and write $\vec{X}\cdot(\vec{C}\times\vec{D}) = \vec{D}\cdot(\vec{X}\times\vec{C})$ Substitute $\vec{X} = \vec{A}\times\vec{B}$ to show $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{D}\left\{(\vec{A}\times\vec{B})\times\vec{C}\right\}$ Next one can employ the triple vector product relation

$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})$$

to obtain

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

▶ **7-48.** (a) At extremum value for $E(\alpha,\beta) = \sum_{i=1}^{N}(\alpha x_i - \beta - y_i)^2$ require that

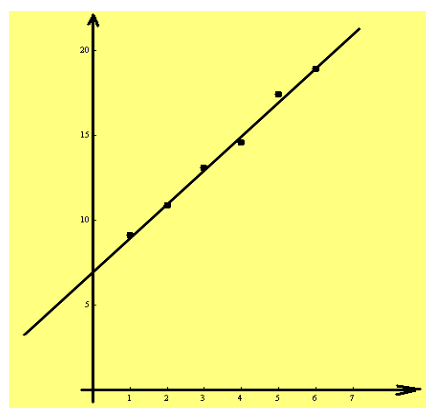
$$\frac{\partial E}{\partial \alpha} = \sum_{i=1}^{N} 2(\alpha x_i - \beta - y_i)x_i = 0$$
$$\frac{\partial E}{\partial \beta} = \sum_{i=1}^{N} 2(\alpha x_i - \beta - y_i)(-1) = 0$$

The above equations can be expressed in the form

$$\alpha\sum_{i=1}^{N} x_i^2 - \beta\sum_{i=1}^{n} x_i = \sum_{i=1}^{N} x_i y_i$$
$$\alpha\sum_{i=1}^{N} x_i - \beta\sum_{i=1}^{N} 1 = \sum_{i=1}^{N} y_i \qquad \text{where} \qquad \sum_{i=1}^{N} 1 = N$$

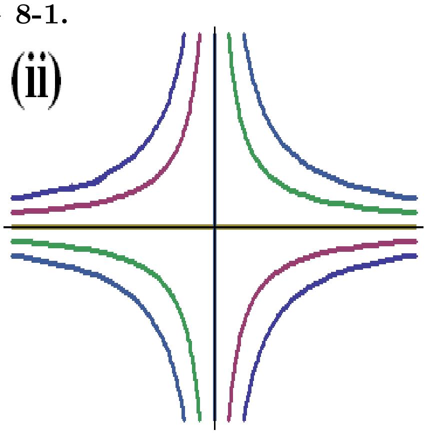
Solve this system of equations to obtain desired result.

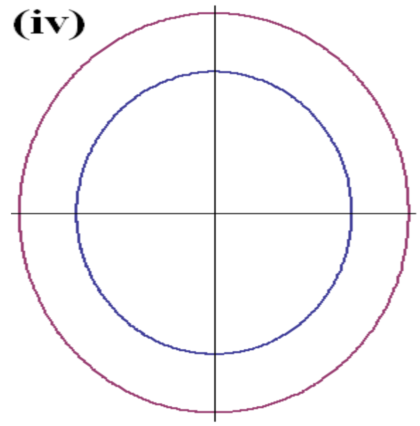
(b) $y = 7 + 2x$



Chapter 8

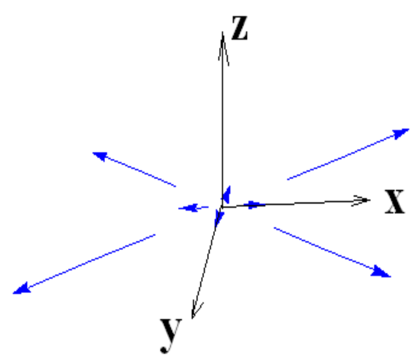
▶ **8-1.**





▶ **8-2.** (i) $\operatorname{grad}\phi = 4\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2$

(iii) $\operatorname{grad}\phi = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$ (ii)



▶ **8-3.** (iii) $z = x^2 + y^2$ paraboloid $\vec{N} = -2x\,\hat{\mathbf{e}}_1 - 2y\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\Big|_{3,4,25} = -6\,\hat{\mathbf{e}}_1 - 8\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

(iv) $z - xy = 0$ hyperbolic paraboloid $\vec{N} = -y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\Big|_{2,3,6} = -3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

▶ **8-4.** $\dfrac{\partial z}{\partial x} = y = 0$ and $\frac{\partial z}{\partial y} = x = 0$ simultaneously at the origin.

▶ **8-5.** Normal to sphere $\vec{N}_1 = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 + 2z\,\hat{\mathbf{e}}_3$ and normal to plane is $\vec{N}_2 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

At the point $(3,2,6)$ one finds $\vec{N}_1 = 6\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 12\,\hat{\mathbf{e}}_3$ and $\vec{N}_2 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

A tangent vector to the curve of intersection is

$$\vec{T} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 6 & 4 & 12 \\ 1 & 1 & 1 \end{vmatrix} = -8\,\hat{\mathbf{e}}_1 + 6\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 \qquad \text{or} \qquad -\vec{T}$$

▶ **8-6.** (i) $r = \sqrt{x^2+y^2+z^2}$ and

$$\begin{aligned}\nabla r^n =& \frac{\partial r^n}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r^n}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r^n}{\partial z}\,\hat{\mathbf{e}}_3\\ =& nr^{n-1}\left[\frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3\right] = nr^{n-1}\left[\frac{x}{r}\,\hat{\mathbf{e}}_1 + \frac{y}{r}\,\hat{\mathbf{e}}_2 + \frac{z}{r}\,\hat{\mathbf{e}}_3\right]\\ =& nr^{n-2}\vec{r}\end{aligned}$$

(iii)

$$\begin{aligned}\nabla f(r) =& \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3\\ =& f'(t)\left[\frac{x}{r}\,\hat{\mathbf{e}}_1 + \frac{y}{r}\,\hat{\mathbf{e}}_2 + \frac{z}{r}\,\hat{\mathbf{e}}_3\right] = f'(r)\frac{\vec{r}}{|\vec{r}|} = f'(r)\frac{\vec{r}}{r}\end{aligned}$$

▶ **8-7.** Let $\vec{r}_1(\tau) = (\tau-1)\,\hat{\mathbf{e}}_1 + (16-\tau)\,\hat{\mathbf{e}}_2 + (2\tau-2)\,\hat{\mathbf{e}}_3$ and $\vec{r}_2(t) = -t\,\hat{\mathbf{e}}_1 + 2t\,\hat{\mathbf{e}}_2 + 3t\,\hat{\mathbf{e}}_3$ and define vector from line 1 to line 2 as $\vec{r}_2 - \vec{r}_1$. Minimize the distance squared

$$f(t,\tau) = |\vec{r}_2 - \vec{r}_1|^2 = (-t-\tau+1)^2 + (2t+\tau-16)^2 + (3t-2\tau+2)^2$$

by examining the critical points where $\dfrac{\partial f}{\partial \tau} = 0$ and $\dfrac{\partial f}{\partial t} = 0$ and show minimum occurs where $t=3$ and $\tau=5$ giving minimum distance $\sqrt{75}$

▶ **8-8.** Method I: Normal to plane is $\vec{N} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and unit normal to plane is $\hat{\mathbf{e}}_N = \frac{\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3}{\sqrt{3}}$. Consider vector $\vec{r}_1 = \hat{\mathbf{e}}_1$ projected onto $\hat{\mathbf{e}}_N$ to obtain distance $\frac{1}{\sqrt{3}}$

Method II: $f = d^2 = x^2+y^2+z^2$ is distance from origin to (x,y,z) squared. Here $z = 1-x-y$ so that $f = d^2 = x^2+y^2+(1-x-y)^2$ Show f has critical points at $x=1/3$, $y=1/3$ and $z=1/3$ giving $d^2 = 1/3$ or $d = \frac{1}{\sqrt{3}}$

▶ **8-9.** (iii) $\dfrac{d\phi}{dn} = [(3x^2y+y^2)\,\hat{\mathbf{e}}_1 + (x^3+2xy)\,\hat{\mathbf{e}}_2]\cdot\hat{\mathbf{e}}_n$ where

$$\begin{aligned}&\text{on bottom of square } \hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_2,\ y=0 \implies \frac{d\phi}{dn} = -x^3\\ &\text{on right side of square } \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_1,\ x=1 \implies \frac{d\phi}{dn} = 3y+y^2\\ &\text{on top of square } \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_2,\ y=1 \implies \frac{d\phi}{dn} = x^3+2x\\ &\text{on left side of square } \hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_2,\ x=0 \implies \frac{d\phi}{dn} = 0\end{aligned}$$

▶ **8-10.** (ii) $z = (x-2)^2-(y-3)^2$ has critical points at $x=2$ and $y=3$. Here $A = \frac{\partial^2 z}{\partial x^2} = 2$, $C = \frac{\partial^2 z}{\partial y^2} = -2$ and $B = \frac{\partial^2 z}{\partial x\,\partial y} = 0$ gives $AC - B^2 = -4 < 0$ so there is a saddle point at the critical value.

▶ **8-13.** $V = \frac{1}{2}kx^2$ with $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. Use $\vec{F} = \text{grad}\, V = -kx\,\hat{\mathbf{e}}_1$. That is, if the spring is stretched in the positive direction a distance x, then the restoring force is in the negative direction and proportional to the displacement. This gives the equation of motion for the spring mass system as

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \omega^2 x = 0, \qquad \omega^2 = \frac{k}{m}$$

▶ **8-14.**

$$\begin{aligned}\vec{F}(x+\Delta x, y, z) =& \vec{F}(x,y,z) + \frac{\partial \vec{F}}{\partial x}\,\Delta x + h.o.t.\\ \vec{F}(x, y+\Delta y, z) =& F(x,y,z) + \frac{\partial \vec{F}}{\partial y}\,\Delta y + h.o.t.\\ \vec{F}(x, y, z+\Delta z) =& \vec{F}(x,y,z) + \frac{\partial \vec{F}}{\partial z}\,\Delta z + h.o.t.\end{aligned}$$

where $h.o.t.$ denotes "higher order terms" which are neglected.

The flux in the x-direction on face CGBF is $\vec{F}\cdot(-\,\hat{\mathbf{e}}_1)\,\Delta S = -F_1\,\Delta y\Delta z$ and the flux in the x-direction of the face DHAE is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial x}\,\Delta x)\cdot(\,\hat{\mathbf{e}}_1)\,\Delta S = (F_1 + \frac{\partial F_1}{\partial x}\,\Delta x)\,\Delta y\Delta z$$

The flux in the y-direction on face DCBA is $\vec{F}\cdot(-\,\hat{\mathbf{e}}_2)\,\Delta S = -F_2\Delta x\Delta z$ and the flux in the y-direction on face HGEF is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial y}\,\Delta y)\cdot(\,\hat{\mathbf{e}}_2)\,\Delta S = (F_2 + \frac{\partial F_2}{\partial y}\,\Delta y)\,\Delta x\Delta z$$

The flux in the z-direction on face AEFB is $\vec{F}\cdot(-\,\hat{\mathbf{e}}_3)\,\Delta S = F_3\,\Delta x\Delta y$ and the flux in the z-direction on face HGCD is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial z}\,\Delta z)\cdot\,\hat{\mathbf{e}}_3\,\Delta S = (F_3 + \frac{\partial F_3}{\partial z}\,\Delta z)\,\Delta x\Delta y$$

Add the flux over each surface and show

$$Total\ \ Flux = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\,\Delta x\Delta y\Delta z$$

so that

$$\frac{Flux}{Volume} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{div}\,\vec{F}$$

▶ **8-15.** (i) $\text{div}\,\vec{F} = 2yz - 2x, \qquad \text{curl}\,\vec{F} = \vec{0}$

(iii) $\text{div}\,\vec{F} = 2y - 2z \qquad \text{curl}\,\vec{F} = \vec{0}$

▶ **8-19.** If $\vec{V} = \nabla\phi$, then $\operatorname{div}\vec{V} = \nabla^2\phi = 0$ and $\operatorname{curl}\vec{V} = \operatorname{curl}(\nabla\phi) = \vec{0}$

▶ **8-21.** Only flux is from top surface and $\iint\limits_S \vec{F}\cdot d\vec{S} = 16\pi$ and from divergence theorem $\iiint\limits_V \operatorname{div}\vec{F}\,dV = 16\pi$

▶ **8-22.** (i) Evaluating the left and right-hand sides of the Green's theorem one finds the value -16/3. (ii) See page 192

▶ **8-23.** Both sides of the Stokes theorem yield the value $\pi/4$

▶ **8-24.** (i) $\phi = x^2y + xy^2 = C$ is family of solution curves.

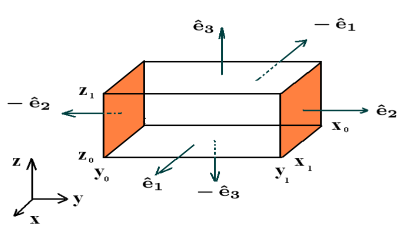
▶ **8-25.** Area$=\pi$

▶ **8-26.** $\iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = \iiint\limits_V \operatorname{div}\vec{F}\,dV = 4\pi a^3$

▶ **8-27.** On sphere with radius $r=1$, $\iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = -4\pi/3$ and on sphere with radius $r=2$ one finds $\iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = 32\pi/3$ Total flux $= 32\pi/3 - 4\pi/3 = 28\pi/3$

▶ **8-28.** On inner surface flux is -2π and on outer surface flux is 8π. Zero flux across top and bottom surfaces. Total flux $= 8\pi - 2\pi = 6\pi$

▶ **8-29.** The divergence of $\vec{F}$ is zero and so the sum of the fluxes associated with the $\pm\hat{\mathbf{e}}_n$ faces must sum to zero. For example, $\int_{z_0}^{z_1}\int_{y_0}^{y_1} y\,dydz - \int_{z_0}^{z_1}\int_{y_0}^{y_1} y\,dydz = 0$



▶ **8-31.** $3V$

▶ **8-32.** If origin outside of S, then use the divergence theorem and show

$$\iint_S \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r^3}\, dS = \iiint_V \nabla \cdot \left(\frac{\vec{r}}{r^3}\right)\, dV$$

and then show $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$

If origin is inside of S, then place a sphere of radius ϵ about the origin and show

$$\iint\limits_S \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS + \iint\limits_{S_\epsilon} \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS = \iiint_V \nabla \cdot \frac{\vec{r}}{r^3}\, dV = 0$$

On sphere of radius ϵ show that $\iint_{S_\epsilon} \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS = -4\pi$

▶ **8-33.** $5x^2yz^3 + 3xy^2z^2$

▶ **8-35.** Area $= \frac{1}{2}$ (base) (height)

▶ **8-36.** -162π

▶ **8-37.** $I = 216$

▶ **8-38.** 4

Problems 8-40 to 8-49 See equations (8.74) to (8.82)

▶ **8-50.** (c) If $\vec{r} = \vec{r}(u, v, w)$, then $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw$ is the diagonal of a volume element in the shape of a parallelepiped having sides $\vec{A} = \frac{\partial \vec{r}}{\partial u}\, du$, $\vec{B} = \frac{\partial \vec{r}}{\partial v}\, dv$ and $\vec{C} = \frac{\partial \vec{r}}{\partial w}\, dw$. The volume of this elemental parallelepiped is

$$dV = |\vec{A} \cdot (\vec{B} \times \vec{C})| = |\frac{\partial \vec{r}}{\partial u} \cdot (\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w})|\, du dv dw$$

Use vector identity and orthogonality property $\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial w} = 0$ to show

$$|\frac{\partial \vec{r}}{\partial u} \cdot (\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w})| = h_u h_v h_w$$

▶ **8-51.** Calculate $\operatorname{grad} v \times \operatorname{grad} w$ and then make use of the fact that mixed partial derivatives are equal to show $\operatorname{div}(\operatorname{grad} v \times \operatorname{grad} w) = 0$

► **8-52.** If $\vec{A} = \alpha\vec{E}_1 + \beta\vec{E}_2 + \gamma\vec{E}_3$ and $\vec{E}_1 \cdot \vec{E}_2 = 0$, $\vec{E}_1 \cdot \vec{E}_3 = 0$, $\vec{E}_2 \cdot \vec{E}_3 = 0$, then show

$$\begin{aligned}
\vec{A} \cdot \vec{E}_1 =& \alpha\vec{E}_1 \cdot \vec{E}_1 \quad \text{or} \quad \alpha = \frac{\vec{A} \cdot \vec{E}_1}{\vec{E}_1 \cdot \vec{E}_1}\\
\vec{A} \cdot \vec{E}_2 =& \beta\vec{E}_2 \cdot \vec{E}_2 \quad \text{or} \quad \beta = \frac{\vec{A} \cdot \vec{E}_2}{\vec{E}_2 \cdot \vec{E}_2}\\
\vec{A} \cdot \vec{E}_3 =& \gamma\vec{E}_3 \cdot \vec{E}_3 \quad \text{or} \quad \gamma = \frac{\vec{A} \cdot \vec{E}_3}{\vec{E}_3 \cdot \vec{E}_3}
\end{aligned}$$

► **8-53.** If $\vec{E}^1$ had the same direction as $\vec{E}_2 \times \vec{E}_3$, then $\vec{E}^1 = \alpha\vec{E}_2 \times \vec{E}_3$. If $\vec{E}^1 \cdot \vec{E}_1 = 1$, then $1 = \alpha\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$ or $\alpha = \dfrac{1}{V}$ for $V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$

Do the same type of arguments for $\vec{E}^2$ and $\vec{E}^3$.

Reverse the roles of $\vec{E}_1, \vec{E}_2, \vec{E}_3$ with $\vec{E}^1, \vec{E}^2, \vec{E}^3$ to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V}(\vec{E}_2 \times \vec{E}_3) \cdot \left[\frac{1}{V}(\vec{E}_3 \times \vec{E}_1) \times \frac{1}{V}(\vec{E}_1 \times \vec{E}_2)\right]$$

then use the triple scalar product relation to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V^3}(\vec{E}_1 \times \vec{E}_2)\left[(\vec{E}_2 \times \vec{E}_3) \times (\vec{E}_3 \times \vec{E}_1)\right]$$

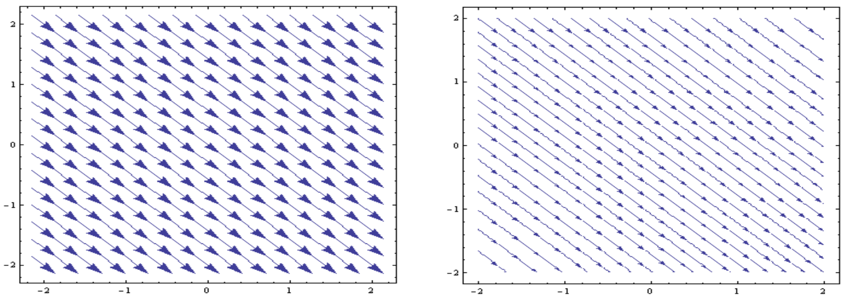
Use another vector identity to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V^3}[\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)]^2 = \frac{1}{V^3}V^2 = \frac{1}{V}$$

Chapter 9

▶ **9-1.** $U = U(x) = c_1 x + c_2, \quad U = U(r) = c_1 \ln r + c_2, \quad U = U(\rho) = -c_1/\rho + c_2$

▶ **9-2.**

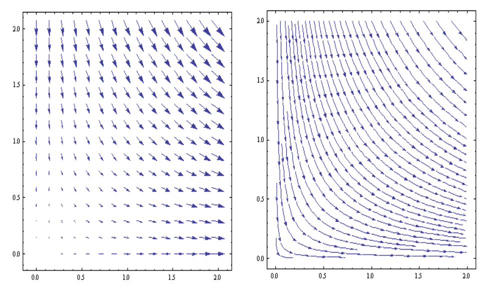


Streamlines $\sin \alpha x + \cos \alpha y = c$ and velocity field is derivable from potential function $\phi = V_0 \cos \alpha\, x - V_0 \sin \alpha\, y$

▶ **9-3.**

Streamlines $xy = c$

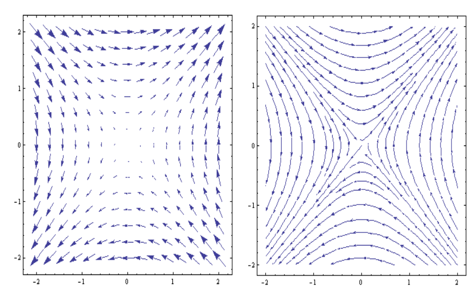
Potential $\phi = x^2 - y^2$



▶ **9-4.**

Streamlines $y^2 - x^2 = c$

Potential $\phi = 2xy$

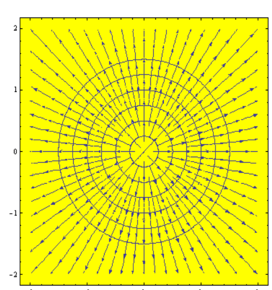


▶ **9-5.** $\phi = \phi(r) = \ln(r_0/r)$

▶ **9-6.** True

▶ **9-7.** Write $\nabla^2 \phi = \nabla \cdot (\nabla \phi) = 0$ and $\nabla \times (\nabla \phi) = 0$

▶ **9-8.** (c)



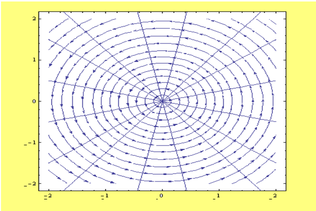
▶ **9-9.** Integrate the equations $\frac{\partial \phi}{\partial x} = \frac{-ky}{x^2+y^2}$ and $\frac{\partial \phi}{\partial y} = \frac{kx}{x^2+y^2}$ to obtain

$$\phi = -k\tan^{-1}\left(\frac{x}{y}\right) + C_1 \qquad \text{and} \qquad \phi = k\tan^{-1}\left(\frac{y}{x}\right) + C_2$$

Show that $\tan^{-1} h + \tan^{-1}(\frac{1}{h}) = \frac{\pi}{2}$ for $h > 0$ and then let $c_1 = \frac{\pi}{2}k$, $c_2 = 0$ to obtain the potential

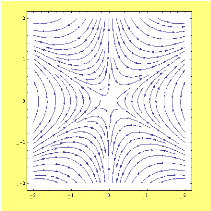
$$\phi = k\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right)\right] = k\tan^{-1}\left(\frac{y}{x}\right)$$

Streamlines are circles $\psi = x^2 + y^2 = c$

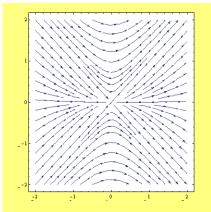


▶ **9-10.**

(d) $2xy \quad x^2 - y^2$



(e) $x^2 + y^2 \quad 2xy$



▶ **9-11.** Show that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \implies \left(\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\cos\theta = \left(\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\sin\theta$$

and

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \implies \left(\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\sin\theta = -\left(\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\cos\theta$$

If the above equations are to hold for all values of θ, then the coefficient of the $\sin\theta$ and $\cos\theta$ terms must equal zero.

▶ **9-12.**

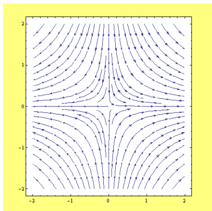
On upper semi-circle $\int_{C_1} \vec{F} \cdot d\vec{r} = -\pi/4$

On the lower semi-circle $\int_{C_2} \vec{F} \cdot d\vec{r} = \pi/4$

► **9-13.** (b) $\vec{V} = [-(x-y)z - yz_0 + x_0z_0]\,\hat{\mathbf{e}}_2$

► **9-14.** $\phi = -mgz, \quad W = \int_{h_1}^{h_2} \vec{F}\cdot d\vec{r} = \int_{h_1}^{h_2} -mg\,dz = -mg(h_2 - h_1) = \phi(h_2) - \phi(h_1)$

► **9-16.** If $\phi = x^2 - y^2$, then $\operatorname{grad}\phi = \vec{F} = 2x\,\hat{\mathbf{e}}_1 - 2y\,\hat{\mathbf{e}}_2$ with field lines $xy = c$



► **9-18.** Show

$$\phi = y^2 \sin x + xz^3 + f_1(y,z)$$
$$\phi = y^2 \sin x - 4y + f_2(x,z)$$
$$\phi = zx^3 + f_3(x,y)$$

Select f_1, f_2, f_3 such that all three integrations are the same to obtain $\phi = y^2 \sin x + xz^3 - 4y$

► **9-19.** $\phi = x^2yz + xy$

► **9-20.** $\pi(2-\sqrt{3})$

► **9-21.** $3V$

► **9-22.** Use the results $\nabla\cdot(\phi\,\vec{C}) = \nabla\phi\cdot\vec{C} = \vec{C}\cdot\nabla\phi$ and $\phi\vec{C}\cdot\hat{\mathbf{e}}_n = \vec{C}\cdot(\phi\,\hat{\mathbf{e}}_n)$ to show

$$\vec{C}\cdot\left[\iiint_V \nabla\phi\,dV - \iint_S \phi\,\hat{\mathbf{e}}_n\,dS\right] = 0$$

and since $\vec{C}$ is arbitrary the term in the brackets must equal zero.

► **9-23.** (a) $\vec{F} = \nabla\phi$ so that $\operatorname{div}\vec{F} == \nabla\cdot(\nabla\phi) = \nabla^2\phi = 0$

(b) Use the Gauss divergence theorem to show

$$\iiint_V \operatorname{div}\vec{F}\,dV = 0 = \iint_S \nabla\phi\cdot\hat{\mathbf{e}}_n\,dS = \iint_S \frac{\partial\phi}{\partial n}\,dS$$

▶ **9-24.** $|\vec{r}| = (x^2+y^2+z^2)^{1/2} = (\vec{r}\cdot\vec{r})^{1/2}$ so that $|\vec{r}|^\nu = (x^2+y^2+z^2)^{\nu/2} = \phi$

$$\frac{\partial\phi}{\partial x} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2x) \quad \frac{\partial\phi}{\partial y} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2y \quad \frac{\partial\phi}{\partial z} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2z)$$

so that

$$\begin{aligned}\nabla|\vec{r}|^\nu =& \frac{\partial\phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{e}}_3 = \nu|\vec{r}|^{\nu-2}(x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3)\\ =&\nu|\vec{r}|^{\nu-2}\vec{r} = \nu|\vec{r}|^{\nu-2}|\vec{r}|\frac{\vec{r}}{|\vec{r}|} = \nu|\vec{r}|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}}\end{aligned}$$

▶ **9-26.** In problem 9-24 replace $\vec{r}$ by $\vec{r}-\vec{r}_0$ and then use the results from problem 9-25 to obtain the result for part (b).

▶ **9-27.** Let $M = \frac{\partial U}{\partial p}$ and $N = \frac{\partial U}{\partial V} + P$ and show $\frac{\partial M}{\partial V} = \frac{\partial^2 U}{\partial p\,\partial v}$ and $\frac{\partial N}{\partial p} = \frac{\partial^2 U}{\partial v\,\partial p} + 1$
Here $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial p}$ so that the line integral is path dependent.

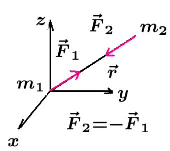
▶ **9-29.** $I = -\frac{3}{4}(5+\pi)$

▶ **9-30.** Potential energy $\frac{1}{2}Kx^2$
Kinetic +potential energy is constant or $\frac{1}{2}mv^2 + \frac{1}{2}Kx^2 = constant$

▶ **9-32.** (a) $T = T(x) = c_1x + c_2$,
$T(0) = c_2 = T_0$ and $T(L) = c_1L + c_2 = T_1 \implies T = T(x) = (\frac{T_1-T_0}{L})x + T_0$

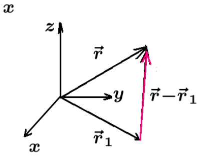
▶ **9-33.** $\phi = 3x^2z + 4y^2$

▶ **9-34.**



(a) $\vec{F}_1 = \dfrac{Gm_1m_2}{|\vec{r}|^3}\vec{r}$ where $|\vec{r}|^2 = x^2+y^2+z^2$

(b) $\vec{F}_1 = \dfrac{Gm_1m_2}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1)$
where $|\vec{r}-\vec{r}_1|^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$



▶ **9-39.** $\vec{y} = \vec{y}_c + \vec{y}_p = \vec{C}_1(1) + \vec{C}_2(t) - \sin t\;\hat{\mathbf{e}}_1 + \cos t\;\hat{\mathbf{e}}_2$

▶ **9-40.** $\vec{y}_1 = \vec{C}_1\,\sin t - \vec{C}_2\,\cos t, \qquad \vec{y}_2 = \vec{C}_1\,\cos t + \vec{C}_2\,\sin t$

▶ **9-41.** $\dfrac{d\vec{r}}{dt} = \underbrace{r_0e^{\theta\cot\alpha}\omega}_{r\omega}\;\hat{\mathbf{e}}_\theta + \underbrace{r_0e^{\theta\cot\alpha}\frac{d\theta}{dt}\cot\alpha}_{r\omega\cot\alpha}\;\hat{\mathbf{e}}_r$

Chapter 10

▶ **10-1.** (c) $AB = \begin{bmatrix} 36 & 59 \\ 11 & 18 \end{bmatrix}$ (d) $B^{-1}A^{-1} = \begin{bmatrix} -18 & 59 \\ 11 & -36 \end{bmatrix}$

▶ **10-2.**

$$\begin{aligned} AA^{-1} =& I \\ (AA^{-1})^T =& I^T = I \\ (A^{-1})^T A^T =& I \quad \text{multiply by } (A^T)^{-1} \\ (A^{-1})^T =& (A^T)^{-1} \end{aligned}$$

▶ **10-4.** (a)

$$\begin{aligned} \text{If} \qquad AB =& BA, \text{ left multipley by } A^{-1} \\ A^{-1}AB =& A^{-1}BA \\ B =& A^{-1}BA \text{ right multiply by } A^{-1} \\ BA^{-1} =& A^{-1}BAA^{-1} \\ BA^{-1} =& A^{-1}B \end{aligned}$$

▶ **10-5.** If A is symmetric, then $A^T = A$ so that if $(A^{-1})TA^T = I$ one can write $(A^{-1})^T A = I$. Multiply this last equation on the right by A^{-1} to obtain $(A^{-1})^T = A^{-1}$ which shows A^{-1} is symmetric.

▶ **10-6.** Left multiply both sides of equation by A^{-1}

▶ **10-7.** If $AB = A$ and $BA = B$, then one can write

$$\begin{aligned} AB =& A \text{ right multiply by A} \\ (AB)A =& A^2 \text{ associative property} \\ A(BA) =& A^2 \text{ properties } BA = B \text{ and } AB = A \\ AB =& A^2 \\ A =& A^2 \end{aligned}$$

$$\begin{aligned} BA =& B \text{ right multiply by B} \\ (BA)B =& B^2 \text{ associative property} \\ B(AB) =& B^2 \text{ properties } AB = A \text{ and } BA = B \\ BA =& B^2 \\ B =& B^2 \end{aligned}$$

▶ **10-8.** (a) Let $Y = A^2 = AA$ with $Y^{-1} = (A^2)^{-1}$, then one can write

$$\begin{aligned} I =& YY^{-1} = (A^2)(A^2)^{-1} \text{ left multiply by } A^{-1} \\ A^{-1} =& A^{-1}AA(A^2)^{-1} \text{ left mulitply by } A^{-1} \\ (A^{-1})^2 =& A^{-1}A(A^2)^{-1} = (A^2)^{-1} \end{aligned}$$

▶ **10-9.** (a) Let $B = AA^T$, then $B^T = (AA^T)^T = AA^T = B$

(d) $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

▶ **10-10.** (a) If $A^T = A$ and $B^T = B$, then $(AB)^T = B^T A^T = BA = AB$
(b) If $((AB)^T = (AB)$, then $B^T A^T = AB$ which implies $BA = AB$

▶ **10-12.** If $AB = BA$, then one must have $c = 0$ and $d = a$.

▶ **10-13.** Show $A^2 = A$ and $A^3 = A$, then show $B^2 = I$ and $B^3 = B$ so A is idempotnt and $B$is involutory.

▶ **10-14.** Show $A^2 = I$ and $A^3 = A$ so that A is involutory.

▶ **10-15.** Show $B = A^2$

▶ **10-16.** $X = A^{-1}B$

▶ **10-18.** (a) -11 (b) 6 (c) -2

▶ **10-19.** (a) 8 (b) 5 (c) 4

▶ **10-20.** (a) $(m_{ij}) = \begin{pmatrix} 14 & -5 & -8 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$

(b) $(c_{ij}) = \begin{pmatrix} 14 & -5 & -8 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix}$

(c) $AC^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

▶ **10-21.** $6xyz$

▶ **10-22.** (a) 0 (b) 0 (c) 36
(d) $a_1a_2a_3a_4$ (e) $a_1a_2a_3a_4$ (f) $45,000 = (36)(25)(5)(2)(5)$

▶ **10-23.** (a) $Z^{-1} = \dfrac{1}{(z_{11}z_{22} - z_{12}z_{21})} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$

▶ **10-24.** $|A| = 3, \quad |B| = 1, \quad |A| \cdot |B| = 3, \qquad |AB| = 3$

▶ **10-25.** $(x_2 - x_1)y - (y_2 - y_1)x = x_2y_1 - x_1y_2$

▶ **10-28.** $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11 & -5 & 1 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 1 & 2 & -7 & -8 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 8 & -6 & 4 & -2 \\ -6 & 12 & -8 & -4 \\ 4 & -8 & 12 & -6 \\ -2 & 4 & -6 & 8 \end{bmatrix}$$

▶ **10-29.** If $AA^T = \begin{bmatrix} \alpha_2^2 + 1/4 & \frac{\alpha_1+\alpha_2}{4} & 0 \\ \frac{\alpha_1+\alpha_2}{2} & \alpha_1^2 + 1/4 & 0 \\ 0 & 0 & \alpha_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then one must select $\alpha_1^2 = \alpha_2^2 = 3/4$, $\alpha_1 = -\alpha_2$ and $\alpha_3^2 = 1$

▶ **10-30.** (b) $\dfrac{d|A|}{dt} = -4t^3 + 6t^2 - 6t$

▶ **10-32.** $|A| = 1, \quad |B| = 1, \quad |A+B| = 3, \quad |A| + |B| = 2$

▶ **10-36.** (a)

$$\begin{aligned} e^{At} =& I + At + A^2\frac{t^2}{2!} + \cdots \\ \int_0^t e^{At}\,dt =& It + A\frac{t^2}{2!} + A^2\frac{t^3}{3!} + \cdots \\ A\int_0^t e^{At}\,dt =& At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots \\ A\int_0^t e^{At}\,dt + I =& e^{At} \end{aligned}$$

(b)

$$\begin{aligned} \int_0^t e^{At}\,dt =& It + A\frac{t^2}{2!} + A^2\frac{t^3}{3!} + \cdots \\ \int_0^t e^{At}\,dt =& A^{-1}\left[At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots\right] = [At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots]A^{-1} \\ \int_0^t e^{At}\,dt =& A^{-1}\left[e^{At} - I\right] = [e^{At} - I]A^{-1} \end{aligned}$$

▶ **10-42.** (b) $y_{n+2} - 6y_{n+1} + 9y_n = 0$ Assume a solution $y_n = A^n$ with $y_{n+1} = A^{n+1}$ and $y_{n+2} = A^{n+2}$, then substitute the assumed solution into the difference equation and obtain the characteristic equation $(A-3)^2 = 0$ with repeated roots $A = 3, 3$. The fundamental set of solutions is $\{3^n, n3^n\}$ and the general solution is $y_n = c_0(3)^n + c_1 n(3)^n$

Chapter 11

▶ **11-1.** (a) 10/15 (b) 3/15 (c) 2/15.

▶ **11-2.** (a) (10/15)(3/14) (b) (10/15)(9/14) (c) (2/15)(3/14)

▶ **11-3.**

Method I Label the fruit A1,A2,A3,O1,O2,O3,O4,O5,P1,P2

Next collect all possible pairs — Note the pair (A1,A2) is the same as the pair (A2,A1) These are all possible event pairs

(A1,A2)	(A1,A3)	(A1,O1)	(A1,O2)	(A1,O3)	(A1,O4)	(A1,O5)	(A1,P1)	(A1,P2)	9
(A2,A3)	(A2,O1)	(A2,O2)	(A2,O3)	(A2,04)	(A2,05)	(A2,P1)	(A2,P2)		8
(A3,01)	(A3,02)	(A3,03)	(A3,04)	(A3,05)	(A3,P1)	(A3,P2)			7
(O1,02)	(O1,03)	(O1,04)	(O1,05)	(O1,P1)	(O1,P2)				6
(O2,03)	(02,04)	(O2,05)	(O2,P1)	(O2,P2)					5
(03,04)	(03,05)	(O3,P1)	(O3,P2)						4
(04,05)	(O4,P1)	(O4,P2)							3
(O5,P1)	(O5,P2)								2
(P1,P2)									1

Total of 45 possible two event pairs. Assign an equal probability to each event of 1/45

Since the event (P1,P2) can only happen once —its probability is 1/45
The probability of getting two oranges is 10/45 since there are 10 (0,0) pairs above
The probability of getting two apples is 3/45 since there are 3 (A,A) pairs above.

Method 2 From AAA, OOOOO, PP assign a probability of 1/10 to each single event of selecting one fruit. (All have same probability)

Probability of getting a pear is $1/10 + 1/10 = 2/10 = 1/5$

If a pear is selected, then we are left with AAA,OOOOO,P Now each single event has probability of 1/9

Probability of getting a pear on second selection is 1/9.

The product $(1/5)(1/9) = 1/45$ is probability of getting a pear plus another pear.

Similarly, the probability of getting an orange is $5/10 = 1/2$ the first time and 4/9 the second time, so that $(1/2)(4/9) = 2/9$ is probability of getting two oranges.

The probability of getting two apples is $(3/10)(2/9) = 6/90 = 1/15$

▶ **11-4.** (a) mean 80, variance 79.6, standard deviation 8.922

▶ **11-5.** $(x_j - \bar{x})^2 = x_j^2 - 2x_j\bar{x} + \bar{x}^2$ so that

$$s^2 = \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - 2\bar{x}\sum_{j=1}^{m} x_j n f_j + \bar{x}^2 n \sum_{j=1}^{m} f_j\right\}$$

But $\sum_{j=1}^{m} f_j = 1$ and $n\sum_{j=1}^{m} x_j f_j = \bar{x}n$ so that

$$\begin{aligned} s^2 =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - 2n\bar{x}^2 + n\bar{x}^2\right\} \\ =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - n\bar{x}^2\right\} \\ =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - n\left(\frac{1}{n^2}\right)\left(\sum_{j=1}^{m} x_j f_j n\right)^2\right\} \end{aligned}$$

▶ **11-6.** $P(2) = P(12) = 1/36$, $P(3) = P(11) = 2/16$, $P(4) = P(10) = 3/36$, $P(5) = P(9) = 4/36$, $P(6) = P(8) = 5/36$, $P(7) = 6/36$

▶ **11-8.** Binomial distribution $(p+q)^n = p^n + np^{n-1}q + \cdots$ where $p = 1/2$, $q = 1/2$ and $n = 10$ gives $(1/2)^{10} = 0.00097956$

▶ **11-9.** (a)

x_j	$\tilde{f}_j$	f_j	$F(x)$	$x_j\tilde{f}_j$	$x_j^2\tilde{f}_j$
0.725	2	$0.0333 = 2/60$	$0.0333 = 2/60$	1.450	1.0513
0.728	7	$0.1167 = 7/60$	$0.1500 = 9/60$	5.096	3.7099
0.731	11	$0.1833 = 11/60$	$0.3333 = 20/60$	8.041	5.878
0.734	14	$0.2333 = 14/60$	$0.5666 = 34/60$	10.276	7.5426
0.737	13	$0.2167 = 13/60$	$0.7833 = 47/60$	9.581	7.0612
0.740	7	$0.1167 = 7/60$	$0.9000 = 54/60$	5.180	3.8332
0.743	3	$0.05 = 3/60$	$0.95 = 57/60$	2.229	1.6561
0.746	3	$0.05 = 3/60$	$1.00 = 60/60$	2.238	1.6695

(c) $mean = 0.73488$, $variance = 0.00002461$, $Standard\ deviation = 0.00496$

(e) (i) $P(X \le 0.737) = 0.783$,

(ii) $P(0.728 < X < 0.734) = P(X \le 0.734) - P(X \le 0.728) = 0.42$,

(iii) $P(X > 0.734) = 1 - P(X \le 0.734) = 0.4334$

▶ **11-11.** (a)

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \cdots + \binom{n}{n-x}p^x q^{n-x} + \cdots \binom{n}{n}q^n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

so that $$(p+q)^n = \sum_{x=0}^{n} \binom{n}{n-x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = 1 = \sum_{x=0}^{n} f(x)$$

(b)

$$(p+q)^{n-1} = \sum_{x=0}^{n-1} \binom{n-1}{n-1-x} p^x q^{n-1-x}$$

shift summation index by letting $x = X - 1$ so that

$$(p+q)^{n-1} = \sum_{X=1}^{n} \binom{n-1}{n-1-X+1} p^{X-1} q^{n-X} = \sum_{X=1}^{n} \binom{n-1}{X-1} p^{X-1} q^{n-X} = 1$$

since $\binom{n-1}{n-1-X+1} = \binom{n-1}{X-1}$

(c) $x\binom{n}{x} = x\dfrac{n!}{x!(n-x)!} = \dfrac{n(n-1)!}{(x-1)!(n-1-(x-1))!} = n\binom{n-1}{x-1}$

(d)

$$\mu = \sum_{x=0}^{n} x f(x) = \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x q^{n-x} = np$$

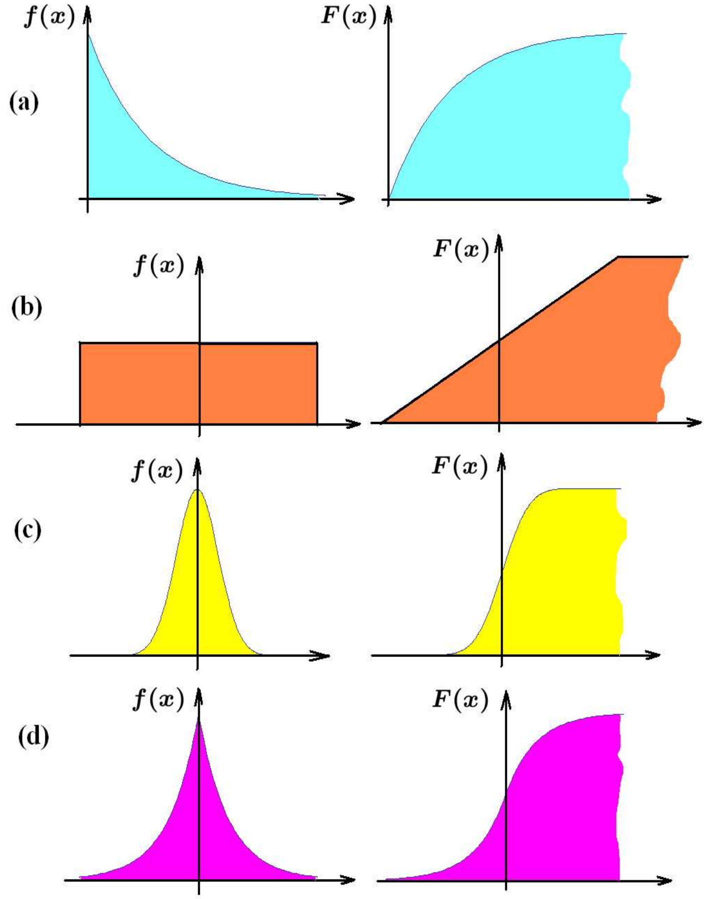
▶ **11-12.**

$$\begin{aligned}
s^2 =& \frac{1}{N-1}\sum_{j=1}^{N}(x_j - \bar{x})^2 = \frac{1}{N-1}\sum_{j=1}^{N}(x_j^2 - 2\bar{x}x_j + \bar{x}^2) \\
=& \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - 2\bar{x}\sum_{j=1}^{N} x_j + N\bar{x}^2\right) \\
=& \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - 2\bar{x}N\bar{x} + N\bar{x}^2\right) = \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - N\bar{x}^2\right)
\end{aligned}$$

Use the fact that $\bar{x} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} x_j$ and write

$$s^2 = \frac{1}{N(N-1)}\left\{ N\sum_{j=1}^{N} x_j^2 - \left(\sum_{j=1}^{N} x_j\right)^2 \right\}$$

▶ **11-13.**



(a) $F(x) = \int_0^x \alpha e^{-\alpha x}\, dx = 1 - e^{-\alpha x}$

(c) $F(x) = \int_{-\infty}^x f(x)\, dx$

▶ **11-14.** (b) $\binom{n}{n+1} = \dfrac{n!}{(m+1)!(n-(m+1))!} = \dfrac{(n-m)n!}{(m+1)m!(n-m)!} = \dfrac{n-m}{m+1}\binom{n}{m}$

▶ **11-15.** (c) (i) Area from $-\infty$ to $z=1$ is 0.8413 and area from $-\infty$ to $z=0$ is 0.5. Therefore area between 0 and 1 is $0.8413 - 0.5 = 0.3413$ By symmetry, twice this is 0.6826 which represents $P(-1 < X \le 1)$

▶ **11-16.** (b) $P(Ace) = 4/52$ *and* $P(king) = 4/52$ *so that*
$P(Ace\ or\ King) = P(Ace) + P(King) = 8/52 = 2/13$

▶ **11-17.** (b) $P(E_1) = 4/52$ *and* $P(E_2) = 13/52$ *and*
$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 4/52 + 13/52 - 1/52 = 16/52 = 4/13$

► **11-19.** (i)

$$\begin{aligned}\sum_{x=0}^{n} x^2 f(x) =& \sum_{x=0}^{n}[x(x-1)+x]f(x)\\ =& \sum_{x=0}^{n} x(x-1)f(x) + \sum_{x=0}^{n} xf(x)\\ =& \sum_{x=0}^{n} x(x-1)f(x) + np\end{aligned}$$

Use $x(x-1)\binom{n}{x} = x(x-1)\dfrac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} = n(n-1)\binom{n-1}{x-2}$ and write

$$\sum_{x=0}^{n} x^2 f(x) = \sum_{x=2}^{n} n(n-1)\binom{n-2}{x-2}p^x q^{n-x} + np = n(n-1)p^2 \sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}q^{n-x} + np$$

Note that by shifting the summation index

$$\sum_{X=0}^{n-2}\binom{n-2}{X}p^X q^{n-2-X} = \sum_{X=0}^{n-2}\binom{n-2}{n-2-X}p^X q^{n-2-X} = (p+q)^{n-2} = 1$$

so that $\sum\limits_{x=0}^{n} x^2 f(x) = n(n-1)p^2 + np$

$$\begin{aligned}\sigma^2 =& \sum_{x=0}^{n}(x-\mu)^2 f(x) = \sum_{x=0}^{n}(x^2 - 2x\mu + \mu^2)f(x)\\ =& \sum_{x=0}^{n} x^2 f(x) - 2\mu \sum_{x=0}^{n} xf(x) + \mu^2 \sum_{x=0}^{n} f(x)\\ =& \sum_{x=0}^{n} x^2 f(x) - \mu^2 = E[(x^2)] - (E[x])^2\\ =& n(n-1)p^2 + np - n^2p^2\\ =& np(1-p) = npq\end{aligned}$$

► **11-21.** (a) $f(x) = \dfrac{\binom{3}{x}\binom{9}{5-x}}{\binom{12}{5}}$ $\quad \sum_{x=1}^{5} f(x) = 1$

$$f(x) = 7/44, \quad f(1) = 21/44, \quad f(2) = 7/22, \quad f(3) = 1/12, \quad f(4) = f(5) = 0$$

(b) The probability that $n = 6$ items are selected with zero nondefectives is $f(x) = \dfrac{\binom{3}{x}\binom{9}{6-x}}{\binom{12}{6}}$ evaluated at $x = 0$ or $f(0) = 1/11$. The probability that 6 items are selected and there is 1 defective is $f(1) = 9/22$. We are not interested in $f(x)$ for x

larger than 1 because if 2 items or more are defective out of 6, we cannot obtain 5 nondefectives. $P(X \leq 1) = 1/11 + 9/22 = 1/2$

If $n = 7$ items are selected $f(x) = \dfrac{\binom{3}{x}\binom{9}{7-x}}{\binom{12}{7}}$. Here

$$f(0) = 1/22, \quad f(1) = 7/22, \quad f(2) = 21/44$$

then $P(X \leq 2) = \dfrac{1}{22} + \dfrac{7}{22} + \dfrac{21}{44} = \dfrac{37}{44} = 0.84 > 0.8$

▶ **11-24.** (b) $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

(c)

$\dfrac{1}{32} = p^5 = (\dfrac{1}{2})^5$ probability of 5 heads

$\dfrac{5}{32} = 5p^4q = 5(\dfrac{1}{2})^4(\dfrac{1}{2})$ probability of 4 heads, 1 tail

$\dfrac{10}{32} = 10p^3q^2 = 10(\dfrac{1}{2})^3(\dfrac{1}{2})^2$ probability of 3 heads, 2 tails

$\dfrac{10}{32} = 10p^2q^3 = 10(\dfrac{1}{2})^2(\dfrac{1}{2})^3$ porobability of 2 heads, 1 tail

$\dfrac{13}{16} = p^5_5 p^4q + 10p^3q^2 + 10p^2q^3$ probability of getting at least 2 heads

▶ **11-25.**

x	$f(x) = \binom{40}{x}(0.8)^x(0.2)^{40-x}$
40	0.000132923
39	0.00132923
38	0.00647999
37	0.02052
36	0.0474524
35	0.0854143
34	0.124563
33	0.151255
32	0.155981
31	0.13865

(a) $f(33) = P(X = 33) = 0.151255$

(b) $P(X = 37) = f(37) = 0.02052$

(c) $\sum_{k=0}^{37} = 1 - f(40) - f(39) - f(38) = 0.992057963$

(d) $\sum_{k=32}^{40} f(k) = f(32) + \cdots + f(40) = 0.59312713$

▶ **11-26.**

x	$f(x) = \frac{9^x e^{-9}}{x!}$
0	0.00012341
1	0.00111069
2	0.0049981
3	0.0149943
4	0.0337372
5	0.0607269
6	0.0910903
7	0.117116
8	0.131756
9	0.131756
10	0.11858
11	0.0970201
12	0.072765

(a) $P(X > 4) = 1 - \sum_{k=0}^{4} f(k) = 0.94503636$

(b) $P(X \le 8) = \sum_{k=0}^{\infty} f(k) = 0.45565260$

(c) $P(8 < X \le 12) = \sum_{k=8}^{12} f(k) = 0.5518747$

▶ **11-27.** $f(k) = \dfrac{2^k e^{-2}}{k!}$

$$(a) \quad f(0) = e^{-2} = 0.1353$$

$$(b) \quad \sum_{k=6}^{\infty} f(k) = 1 - \sum_{k=0}^{5} f(k) = 0.0166$$

$$(c) \quad 1 - \sum_{k=0}^{1} f(k) = 0.5940$$

▶ **11-28.** (a) Write $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2}\, dt = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2}\, dt + \dfrac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-t^2/2}\, dt$

The first integral equals $1/2$ and in the second integral make the substitution $\tau = t/\sqrt{2}$ to obtain result.

(b) Similar to part (a) but make the substitution $\tau = \frac{1}{\sqrt{2}} \left(\frac{t-\mu}{\sigma}\right)$ with $d\tau = \frac{1}{\sigma\sqrt{2}}\, dt$.

▶ **11-33.** $Area \approx 0.982923$