Non-linear Sequences

Yes, **NON**-linear sequences!

Here's one. Can you figure out the nth term?

1, 4, 9, 16, 25...

You may notice that the difference between them is different this time. This is how we know it is non-linear. What else? The difference each time seems to increase by the same amount though!

Difference = 3, 5, 7, 9... so that increases by 2 each time. Hmm, interesting!

Have you figured out the nth term? Confident?

Do you recognise these numbers at all? Seen them anywhere before?

You might recognise them as being square numbers. That is the areas of squares with sides of 1, 2, 3, 4, 5... and so on.

That means that they are the areas of squares from the sequence n.

Therefore this nth term will be n^2

This non-linear sequence has a special name - a quadratic sequence - since we will get a shape with four (quad) sides from it. This is just a complicated way of saying that if we multiply 2 numbers together we get a rectangle or a square (just as we noted in Book 1, Multiplication - In A Minute).

This will not give a straight line when graphed so that's why, as well as not having the same amount between terms, this is called non-linear. This is a circular argument, since this will not give a straight line when graphed because it doesn't have the same amount between terms... and it doesn't have the same amount between each term because it won't give a straight line when graphed...

Any power on n, such as $n^3, n^4 or, n^8$, are non-linear. So this is a big family compared to our linear sequences, of just n^1, or n.

Some examples would be...

n^3

1, 8, 27, 64, 625....

n^8

1, 256, 6561, 65536...

A Man Called Al

A Man Called Al

A couple of thousand years ago, all the above was known about. People were merrily using these sequences and it was all well understood.

A question was asked by a man called Al. What if we could use different numbers in the sequence than just the list in n = 1, 2, 3, 4...?

For example, looking at 2n + 1, this generates
3, 5, 7, 9...

as we use n = 1, 2, 3, 4...

But what would happen if we used 1.5? That value is halfway between 1 and 2. Would the answer we get be halfway between 3 and 5?

Let's see

2(1.5) + 1 = 4

Yes! It is halfway!

So it turns out we can use any number, not just from the sequence n, and get answers that will fit on the line, between our current answers. This was expanded further.

What about negative numbers?

What about negative decimals?

Trying

-2

2(-2) + 1 = - 3

adding 2 to this repeatedly takes us to -1, 1, 3, 5... and yes back on to that sequence! So we know that fits.

What about

- 1.5?

2(-1.5) + 1 = - 2

An answer that lies between our answer above, - 3, and the next term, - 1. This tells us that the decimals work too.

So what happened was that n became expanded to an infinite amount of possible values apart from just positive whole numbers as it was before. This made the discrete (steps) become continuous (no gaps). It’s like the difference between stairs and a flat slope. You are stuck where on the stairs you can step, only at certain points. The possible locations are discrete. On a slope you can place your foot anywhere. The possible locations are continuous.

This discovery led to a new name being given to all possible numbers. They decided they couldn’t use n anymore, so they said that this new ‘any number’ would be called...

x

Sequences like

2n + 1

became

$2x+1$

instead.

Meaning any value can be used for x

This led to ‘true’ straight lines when graphed as the values between the steps were actual values which were known to be true. Before this was ignored when the dots were joined. We’ll see more of this in ‘Gradient/Equation of A Straight Line - In A Minute’.

Oh, this man called Al.

His surname was Gebra.

Hence Algebra!

Ok, not really. Algebra means ‘restoration’ which doesn’t precisely translate to what we are doing now. However we will see later that we could interpret this as using the Third Rule of Maths, doing the reverse.

Child, 12, Amazes Teacher

Child, 12, Amazes Teacher

In a German schoolroom, a tired and bored teacher takes his class for another day. Wanting half an hour of peace, he decides to give the class yet another boring problem to solve.

“Add up all the numbers from 1 to 100. Quietly.”

With a sigh, the class got started.

After 2 minutes one pupil raised his hand.

“What?” asked the teacher.

“I’ve finished”, said the boy.

“You can’t have. Show me.”

Indeed the boy had finished. He showed his work to the teacher. In it was a ultra fast way to add up sequences.

What had the boy figured out?

Can you figure out a fast way to add up 1 to 100, apart from adding 1 + 2 + 3 + 4 and so on?

Have a think!

The boy, Carl, had realised that there’s no rule that says you have to add them up in order. He also knew well that Multiplication was just addition. So if he could turn this addition into a multiplication, it would be an easy task. But how?

He noticed that he had

1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100

to do.

Adding the two outer numbers, gives 1 + 100 = 101
Then the next two 2 + 99 = 101
Then the next two 3 + 98 = 101
And so on.

Each addition gives 101.

As there are one hundred numbers, that makes 50 additions.

50 additions of 101 will be
50 x 101

101
x 50

5050

So the answer is 5 050!

Try this yourself with 1 - 10

1 + 10 = 11
2 + 9 = 11

there will be 5 of these
5 x 11 = 55.

Easy.

So it turns out all we do in general is
Add 1 to the final value (100 + 1, or 10 + 1) and multiply it by half the number there are (50 or 5).

So 1 - 16

will be 17 x 8

17
x 8

136

For odd numbers

1 - 11

It will still work.

12 x 5.5

55
x 12

660

One decimal place (as in ‘Decimals - In A Minute’).

66.0

or just

66

We know 1-10 is 55, so 1-11 will be 55 + 11 = 66. Yes it works!

The boy, Carl, went on to become a famous mathematician and scientist.

His full name was

Carl Friedrich Gauss.

The unit of magnetic flux density, the gauss, is named after him.

Example Questions

In Maths questions you can be asked to find the nth term, such as we have seen, or to write a sequence given the nth term. Here's an example of both.

Find the nth term
2, 8, 18, 32, 50....

Write out the first four terms of the sequence 5 - 3n

Ans:

$2n^2$

2, -1, -4, -7...

Introduction to Gradient

Introduction

Having looked at sequences, we know look at the equation of a straight line. This is the first stepping stone to understanding and being able to do algebra. Follow on from here, understand it well, and you will be well on your way to becoming a mathematician!

What Is Gradient?

Imagine you are a climbing a hill. Is it easy or difficult?

If easy, why?

If difficult, why?

It may be because you're unfit, but assuming not (!), then it may well be due to the steepness of the hill.

The steeper it is, the harder it is to walk up.

Imagine you are climbing this hill.

I call you up and ask you what you are up to.

You say *'I'm climbing a hill! It's very steep.'*

I ask

'How steep?'

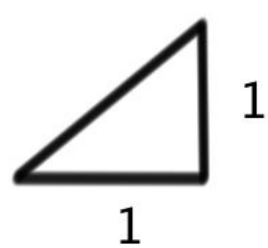
What would you say then? Does 'very' really qualify? What I think is very steep may not be the same for you as for me. So, we need to *attach a number* to this so that we both know how steep it is and what that means.

How to Calculate Steepness

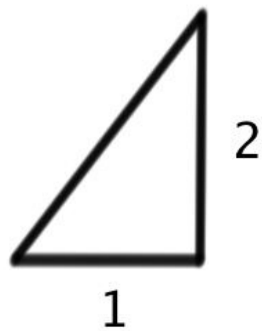
Steepness of a hill, or its gradient, is calculated in a very simple manner.

For every step you go along horizontally, how many steps do you have to go up?

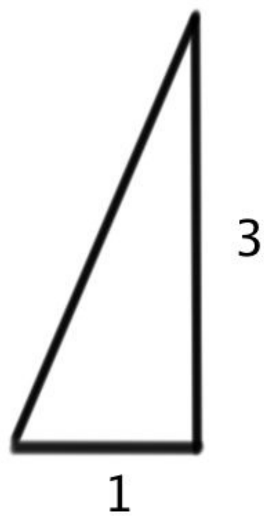
For one step up



For two steps



For three steps



So as you walk one along, the amount you have to go up is its gradient.

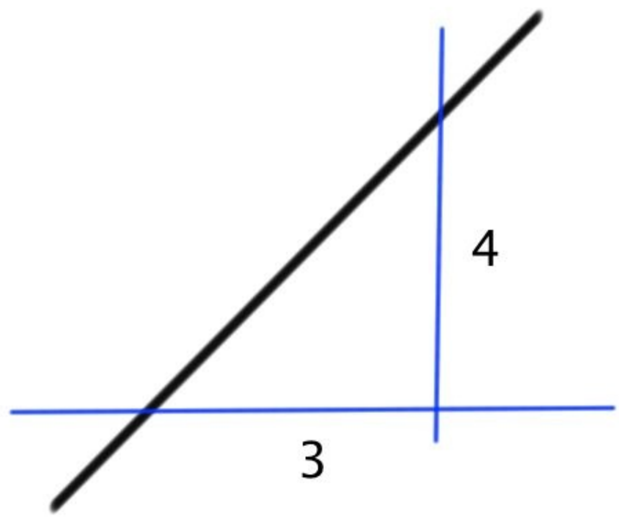
If you measure it over more than one horizontal step, you have to divide the vertical measurement by the horizontal
So

$$gradient = \frac{up}{across}$$

The reason we do this is to normalise the gradient. Since it is a measure per horizontal step, we need to reduce it in size so that the base of the triangle is only

1. We have seen normalisation before, in Percentages - In A Minute.

To calculate ANY gradient, we simply drop a vertical line anywhere. We then put a horizontal line across anywhere. Measuring the sides of this new triangle, and dividing (as we want the base to be 1), we find its gradient.



Here we need to do

$$gradient = \frac{4}{3}$$

To see what it would be for just one step, not 3. So we get

$$1\frac{1}{3}$$

Although sometimes we would leave it as

$$\frac{4}{3}$$

Since gradient is calculated using a division (unless the base is 1), how many types of gradient do you think there will be?

Three.

Since we have three types of division, from Division - In A Minute, we will have three types of gradient, which will be
> 1

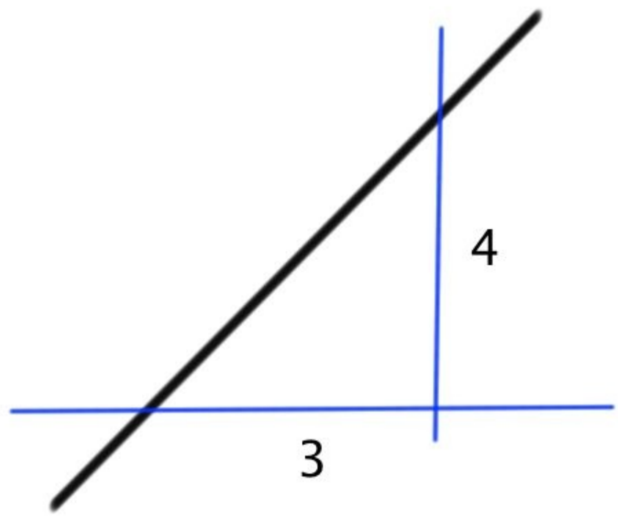
= 1

&

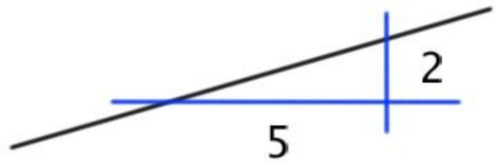
< 1

An example of > 1

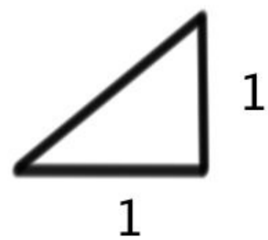
Is the one we’ve just looked at:



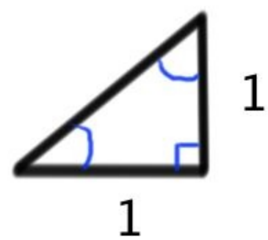
An example of < 1



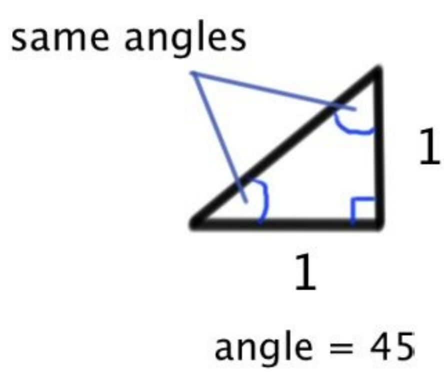
An example of = 1



For = 1, we might notice that the sides of the triangle will be the same. As a result this is an isosceles triangle. The triangles we create for calculating gradient are all ‘right-angled triangles’, in other words, they have an angle of 90 degrees. What would the other two angles be in this example?



Since the angles must add up to 180 degrees, as in any triangle (why?), the other two angles must add up to 90. Since it is isosceles, they will be the same, so a gradient of 1 means it has an angle of 45 degrees. **This is very important and is extremely useful to remember.**



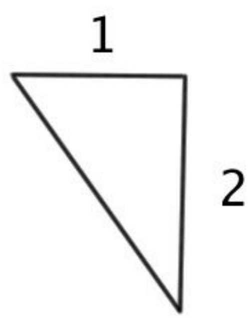
The letter we use to denote gradient is *m*, which is apparently short for *monter*, the French for ‘to climb’. You might like to think of it as being short for mountain.

So we’ve looked at all three types of gradient. However, here comes the third rule. We can always do the reverse! So we also have another type of gradient. Hills go up, but they also go down. So we also have negative gradients.

These are measured in exactly the same way, except we say for every step we go along, it is how many DOWN we have to go.

For example, if m = - 2

For every one we go along, we go down 2.



Just as for positive gradients, there are three types.

These will be... what?

Of course,

< - 1

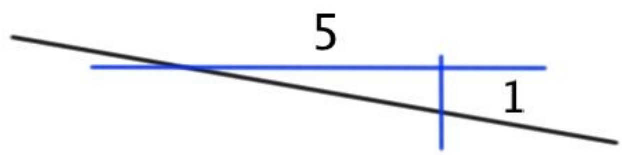
= - 1

&

> - 1

Bear in mind that the ‘more than’ type means a fraction here, and the ‘less-than’ means a whole number greater than 1 (or a mixed number), but negative. So this is also the reverse!

An example



For every 5 we go along, we go down one. So this will be negative. Dividing by 5 to make it only one horizontal step…

$$m = \frac{1}{5}$$

$$m = -\frac{1}{5}$$

As it’s negative. This is > - 1.

So they are all our types of gradients.

The Equation of A Straight Line

Where We Use Gradients

In ‘Sequences - In A Minute’, we saw how times tables were generalised into sequences. Instead of saying ‘the two times table’, mathematicians say ‘2n’. This had a variation, where we could have something like

$2n+1$

where this was the two times table, with 1 added to each term.

Can you remember what these sequences were called?

These were

Linear sequences

as they increased (or decreased) by a set amount. In this case, two.

We then saw that the development of algebra led to n being replaced by

x

where x is any number.

This led to a new equation for the above

$y=2x+1$

where

$2x+1$

is our original sequence, but including any value for x.

This meant that we had an infinity of possible answers for an infinite amount of possible inputs. The result of this was to have a constant, continuous set of inputs and outputs that led to a straight line which was connected by dots rather than huge leaps.

y

in

$y = 2x + 1$

is all the possible values.

For example, if $x = -2$

$y = 2(-2) + 1$

$y = -3$

So y is just the answer we get.

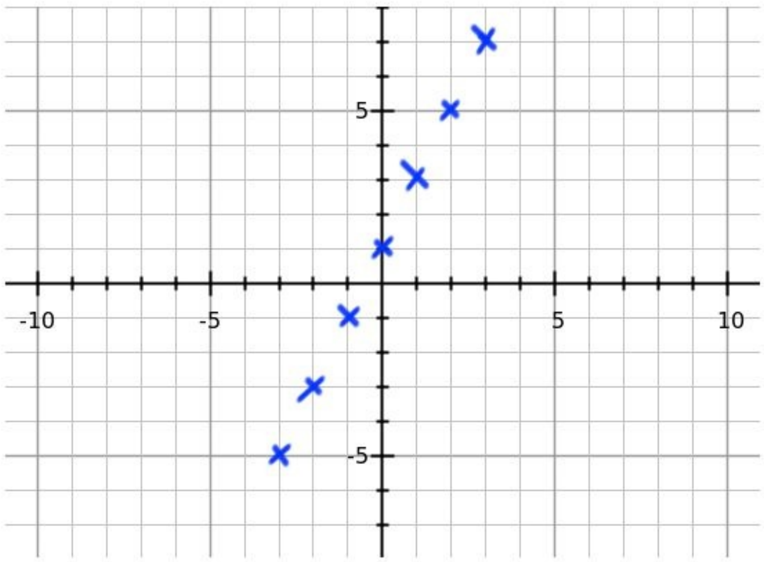
If we draw a table of this

x	- 3	- 2	- 1	**0**	**1**	**2**	**3**
$y = 2x + 1$	- 5	- 3	- 1	**1**	**3**	**5**	**7**

We can choose any values for x we like and calculate what the y values would be.

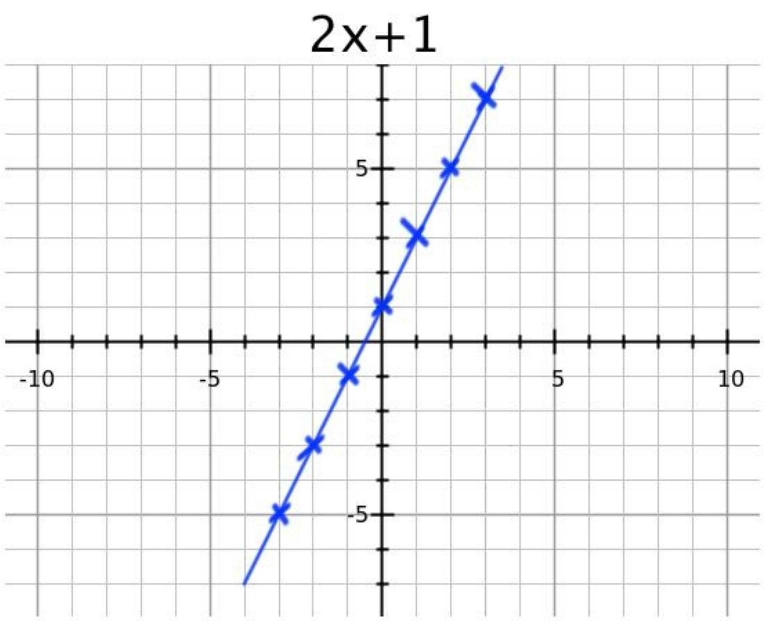
If we use round numbers we'll see that they follow the two times table, plus 1, from 2n+ 1, of course!

Once we have all these values, what we can do is plot them on a graph. This graph is technically known as the 'Cartesian Axis'. In other words it is two number lines. One for x and one for y. They cross each other at 90 degrees and at 0 for each.



Once we plot these points we can do a ‘dot-to-dot’, you may remember from your primary school days, and join them up!

This gives a straight line.

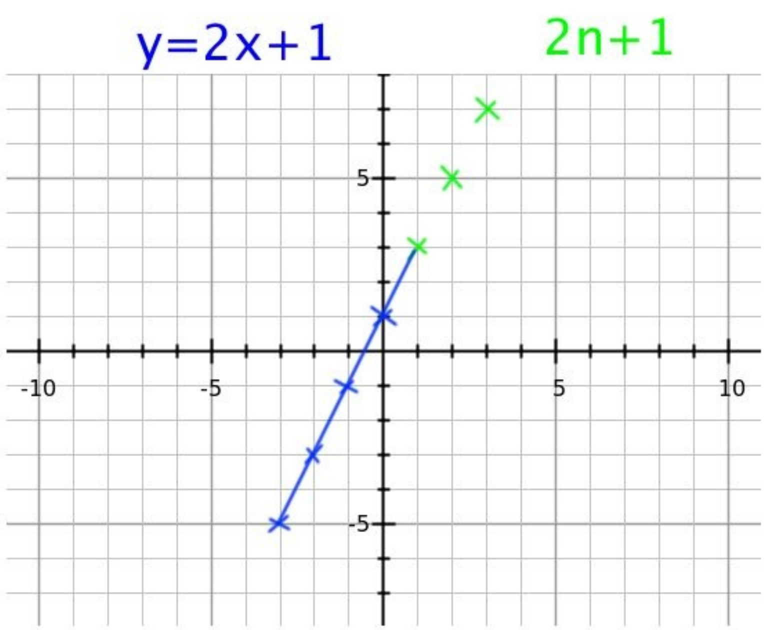


Not surprisingly then, this

$y = 2x + 1$

is called an *'Equation of A Straight Line'*.

This line actually represents all the possible values that x and y can take. Remember we said that now we are using x there are an infinite number of possibilities for x and y. So this is a true line. The previous graph only demonstrated this for 2n + 1 really. Here is a comparison.



What I would like you to do now is try a couple yourself.

Using graph paper, draw an x and y axis.

Then do three tables like these

tables

$y = 2x + 1$

$y = 3x + 2$

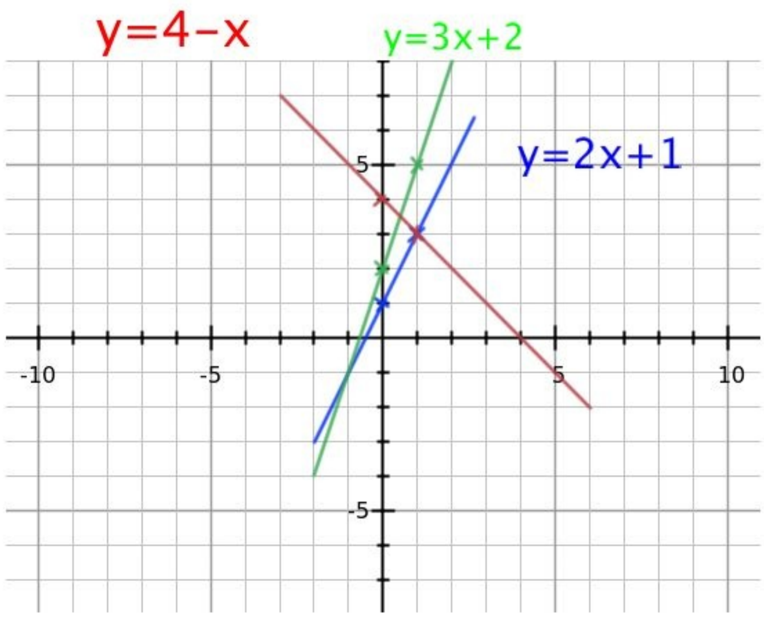
$y = 4 - x$

And calculate the y values

Having done that, plot the points. Then read on further to see if you are correct.

Make sure you do this exercise, as you need this sheet to make further calculations later in this book!

Did you get these?



Often people get the last one wrong.

Calculating the Gradient and ‘Cut

Calculating the gradient and ‘cut’.

Straight lines such as these have 2 characteristics, the gradient, which we’ve already met, and something I call the ‘cut’. This is the place where the line cuts the y-axis. Please calculate the gradients and cuts for all three of your straight lines.

Remember to calculate the gradient, just draw a horizontal and vertical line, and

$$gradient = \frac{up}{across}$$

Fill in this table, and see if you can spot any pattern?

	m	c
$y = 2x + 1$	**2**	**1**
$y = 3x + 2$	**3**	**2**
$y = 4 - x$	- 1	**4**

You might be able to see that the numbers in the equations and the columns m and c match up!

The numbers multiplied by x are the gradients, the numbers NOT multiplied by x are the cuts.

For the negative one, it’s the reverse. Of course we can always do the reverse...

This means that the information about the straight line’s characteristics is in the equation.

As a result, we don’t really need to make tables for straight lines. Since we know its gradient and cut, we know a starting point, its cut, and from there we can go along one, and up or down whatever the gradient is.

This will give a second point. Since two points is all we need for a straight line, we can just join them and extend either side.

If you look again at the graphs above, you’ll notice that I only made 2 points for each line. That was because I knew the gradient and cut of each line.

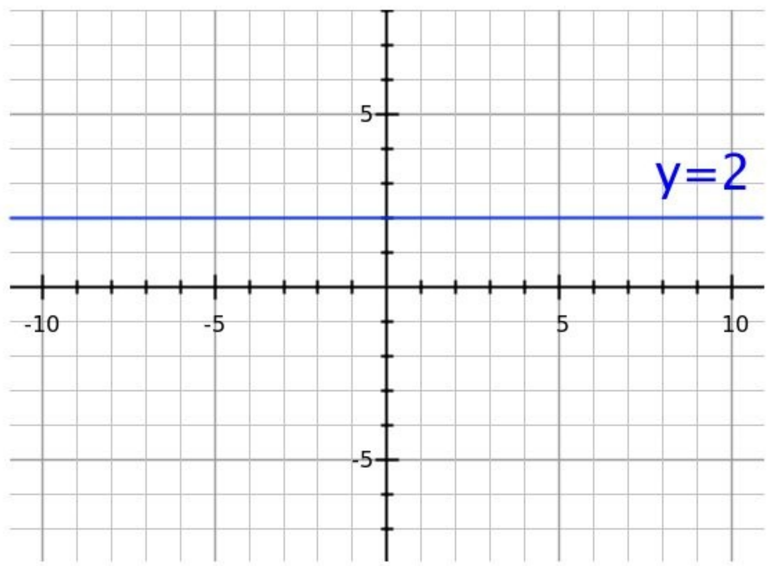
In practice, this is what mathematicians do. In an exam, you’d be expected to fill out a table.

Other Straight Line Graphs

Other Straight Line Graphs

$y = 2$

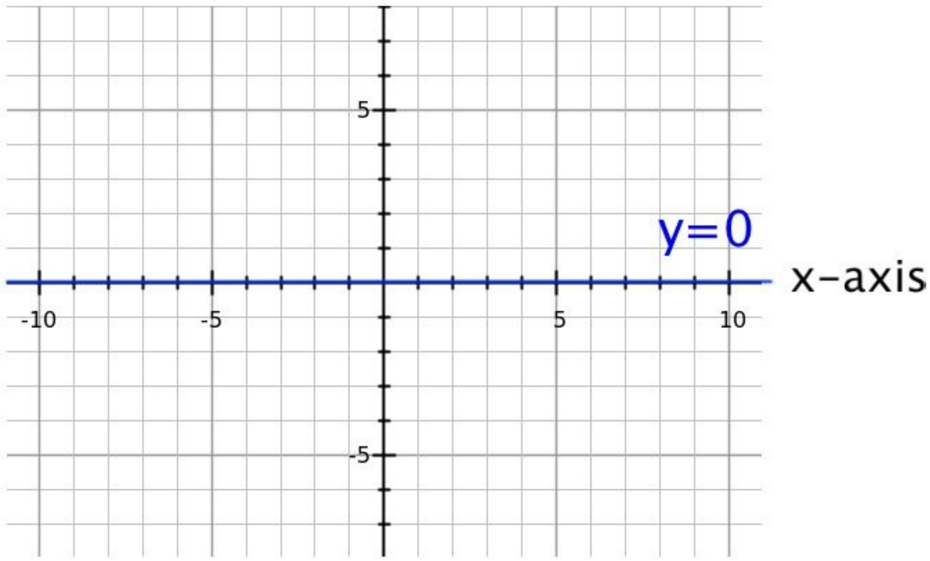
If m = 0, and we have a cut of 2, we'll have the equation $y = 2$ and this looks like this.



Note there is no gradient, so it is horizontal.

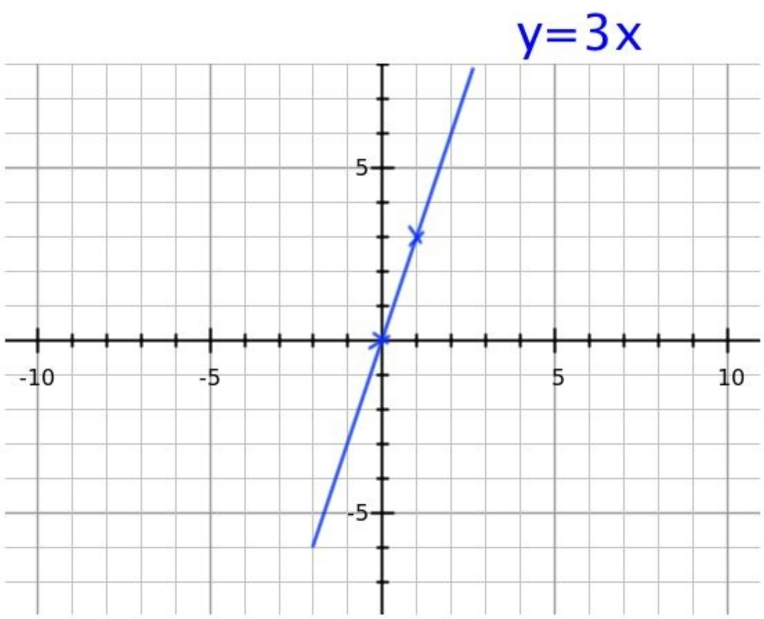
$y = 0$

This has no gradient or cut, so in fact it is the equation of the x-axis!



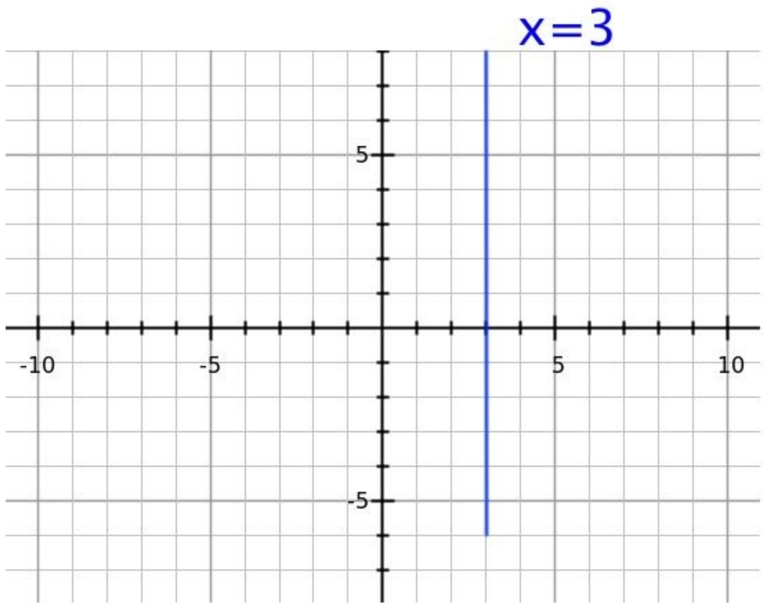
$y = 3x$

This has no cut, but a gradient of 3.



$x = 3$

Here, the equation does not begin y = ... so it is all the places that x = 3, which is a vertical line of points.



Read on to

Simultaneous Equations - In A Minute

to see what we will do with all of these lines.

Introduction to Simultaneous Equations

Introduction

Having looked at sequences and the equation of a straight line, we're now ready to do something with these which is of a practical or abstract use. These form the building blocks for solving all sorts of maths problems, and once we master these, we can learn how to do the same with quadratics, cubics or higher, as well as solve problems in calculus, which is especially useful.

In this book you will learn that it is quite straight-forward, not entirely abstract, and will leave you wondering why they made it so difficult at school!

Simultaneous Equations

Simultaneous Equations

Before we look at what these are, let's look at

The Three Types of Algebra

As you can see from the title, the good news is that there are only 3 types of algebra. The even better news is that we've come across one already, in Book 2, Division - In A Minute.

We saw a question like

$$\frac{6}{2}$$

and asked how to do it.

The first, intuitive instinct was to ask
'How many times does 2 go into 6?' or in other words,
'What do I have to multiply 2 by to get 6?'

and we called that the Algebra Trick, as it was asking
2 x ? = 6

or

$$2x = 6$$

what's x?

This is also the first type of algebra.

So Type 1

$$4x = 24$$

What's x?

x must be 6 as

4 x 6 = 24

Remember, when we see letters next to each other, like $4x$ or ab, this means $4\times x$ or $a\times b$.

Another one

$3x = 12$

$x = 4$

Because 3 x 4 = 12

and finally

$2x = 4$

$x = 2$

Because 2 x 2 = 4

All very simple.

So these are actually doing the reverse of divisions. We're just saying what we would multiply.

Another way to do these would be, of course, to do the reverse, and actually DO THE DIVISION, although this just leads us to asking what we would multiply it by, which is absolutely crazy! You turn it into a division, just to ask what would multiply it! Not surprisingly then, this is the method favoured in schools.

For example,

$4x = 12$

You could turn that into

$$x = \frac{12}{4}$$

By dividing both sides by 4.

So now you ask...

4 x ? = 12?

So to answer this you need to think of what multiplies to get 12! And of course it is 3.

But if we had just done that from the start, where we had

$$4x = 12$$

and asked, 4 x ? = 12, we could have got to 3 sooner.

So there’s ABSOLUTELY no need to do this division exercise. It’s a complete waste of time.

Unless.

Unless unless unless.

What if it was

$$4x = 13$$

then what?

We can still do it this way.

4 x ? = 13

leads to think, at least 3, with 1 remainder.

In other words

3 r 1

And so that becomes

$$3\frac{1}{4}$$

That's a small jump to doing these questions 'on the fly' instead of going through the rigmarole of writing

$$x = \frac{13}{4}$$

and then going through exactly the same thought process.

Okay. What about

$$8x = 2$$

Then what!?!

Here we see we may struggle to think of

8 x ? = 2

as it is clearly going to be less than 1.

So here, and only here, do we do that division. Why? Because we can clearly see that it's a less than 1 type, and these are not solved by doing a multiplication or the Algebra Trick.

As a result, we have to make it in a division so that we can simplify it.

This gives

$$x = \frac{2}{8}$$

simplifying,

$x = \frac{1}{4}$

And that makes sense. Since
$8 \times \frac{1}{4} = 2$

The great advantage of these **Three Types of Algebra** is that it is IMPOSSIBLE to get things wrong, as we can just place our answer back into the original question, and see if it works or not.

$8x = 2$

When

$x = \frac{1}{4}$

If it works it's correct.

If it doesn't, we've gone wrong somewhere.

That's all there is to Type 1 Algebra!

Type 2 Algebra

Type 2 Algebra

This is just Type 1 algebra with an extra bit. That’s all.

So for example

$2x + 7 = 15$

At school we're taught a complicated method for this now, but let's just look at what this says.

It says

‘Something plus 7 equals 15.’

Ok.

So we need to figure out what the ‘something’ is.

What must that something be equal to?

Since we know that to make 15, you need to add 8 to 7, that ‘something’ must be 8, right?

So

8 + 7 = 15

Therefore that something equals 8

So

$2x = 8$

What’s happened now is that from Type 2 algebra, we’ve got a Type 1.

And that’s our goal. If we have Type 2, make it Type 1, as these are simple.
This will always be the goal.

So

$2x = 8$

We say

‘2 x ? = 8’

and that must be

$x = 4$

Another example

$3x - 9 = 30$

First thought must be

What minus 9 equals 30?

Of course it is 39.

So we write

$$3x = 39$$

as this must logically be true.

So we have our Type 1

$$x = 13$$

We can test both our answers too.

$$2x + 7 = 15, x = 4$$

2(4) + 7 = 8 + 7 = 15 - yes that's correct!

$$3x - 9 = 30, x = 13$$

3(13) - 9 = 39 - 9 = 30 - yes that's correct!

Simple again!

So that's Type 2 algebra.

Type 3 Algebra

Type 3 Algebra

This is just Type 2 with an extra bit!

So we need to cascade down the Types. Change

Type 3 into Type 2
Type 2 into Type 1.

Let's look at an example.

$3x+1=x+6$

- Type 3 has inserted an extra x term on the right hand side here. Remember though that

1, we need to change it to Type 2

2, both sides are equal

To make it Type 2, we need to get rid of that extra bit on the right hand side. It's really ruined things. To do that we can think of picking it up or rubbing it off that side. However, since both sides are equal, we must do the same thing to the other side.

For example if we had

$4+5+2=3+6+2$

This is true.

If we remove the 2 from the right hand side, we need to do that to the other, or it won't be true anymore..

$4+5+2 \neq 3+6$

but

4 + 5 = 3 + 6

So back to our example

$3x+1=x+6$

If we remove an x from the right, we must from the left.

Since there are $3x's$ on the left, when we take one away, we are left with 2.

$2x+1=6$

This is now Type 2.

So we ask

‘What plus 1 equals 6?’

Obviously, it is 5

So therefore

$2x=5$

and now it is Type 1

$x=2\frac{1}{2}$

To check:

$3\left(2\frac{1}{2}\right)+1=2\frac{1}{2}+6$

$8\frac{1}{2}=8\frac{1}{2}$

which is true!

Another example

$4x + 3 = -3x + 5$

Again, this is a Type 3. We want to get rid of that extra x on the right hand side, to make it into a Type 2.

However, this is slightly different. We notice that there's a minus sign. So what do we do here?

Just the reverse!

Before, with a positive x, we subtracted. We took the x we didn't want away. Here, we just add! Essentially, we are trying to get a zero. Before we took away so there was none there. Here we have to add to make $-3x$ become zero. So we need to add... $3x$ of course!

If we add $3x$ to the right hand side to achieve this, we must also do the same to the left to equalise things.

So

$4x + 3 = -3x + 5$

becomes

$7x + 3 = 5$

We have Type 2

$7x = 2$

We have Type 1

$x = \frac{2}{7}$

Finished.

To check

$$4\left(\frac{2}{7}\right)+3=-3\left(\frac{2}{7}\right)+5$$

$$4\frac{1}{7}=4\frac{1}{7}$$

And that’s Type 3 algebra!

What We’re Doing This For

What We’re Doing This For

So why bother trying to find x? All we seem to be doing is unpicking an unnecessarily complicated puzzle for no reason.

Well, there is a reason!

Here’s what it’s for.

If we look at the first example, our Type 1,
$2x = 6$

what does this mean apart from
‘2 times something equals 6’?

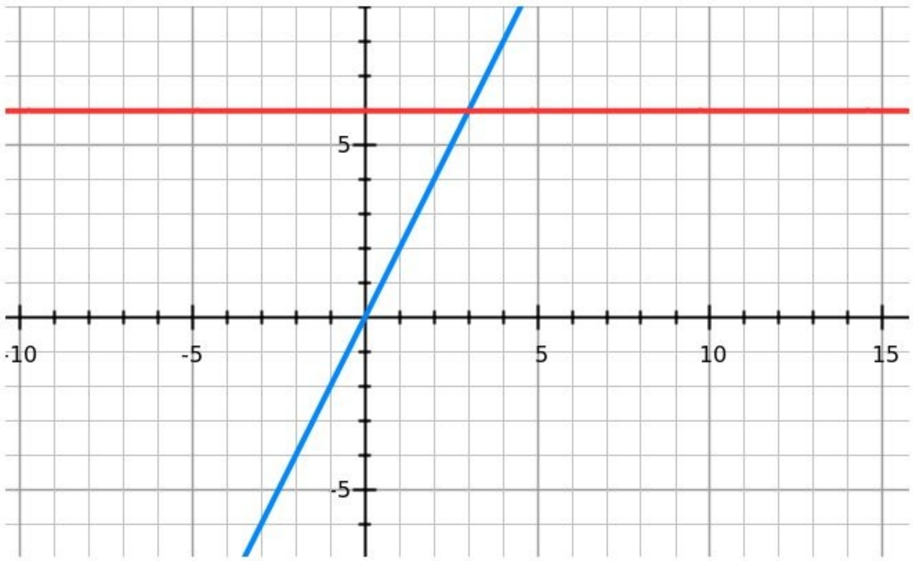
If we think back to the Equation of A Straight Line, in the previous book, we could sketch (or plot) two straight lines
$y = 2x$

Blue

and

$y = 6$

Red



If we do this we can see that they cross each other. The question that interests mathematicians is ‘where do they cross?’.

To find this, we could look at the graph, and read off the co-ordinates. However this isn’t always accurate, especially if you draw as badly as me!

So a better, exact, way to do it would be to notice something about

$y = 2x$ and $y = 6$

What do you notice about them?

Do they have anything in common?

Of course they both equal y.

What can we do with this? Let’s look at these two examples
6 + 4 = 10

2 x 5 = 10

Notice we’ve written two ways to describe ten. So we could write

6 + 4 = 2 x 5

and not even mention ten!

So we can do the same with our equations
$y = 2x$ and $y = 6$

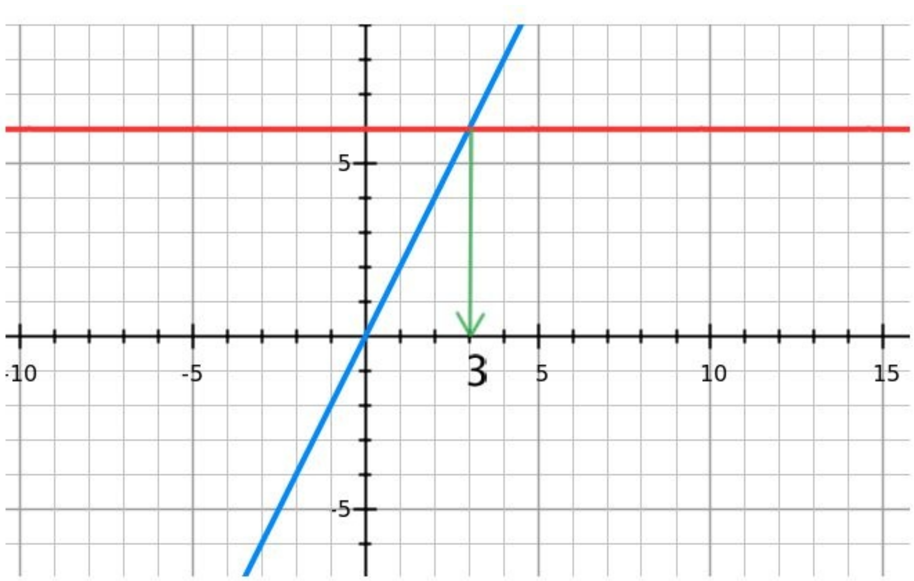
and write that as they both equal y, they must equal each other.

So

$2x = 6$

This is Type 1 algebra, and ‘solving’ it, will tell us where they two lines cross
$x = 3$

If we look on the graph, we can see this is true.



Because we know y = 6 the co-ordinates where they cross will be
(3, 6).

And we’ve found where they cross.

That’s just one thing we can do with the Three Types of Algebra.

Another Example

$2x+1=6$

If we solve this we find

$2x=5$

and

$x=2\frac{1}{2}$

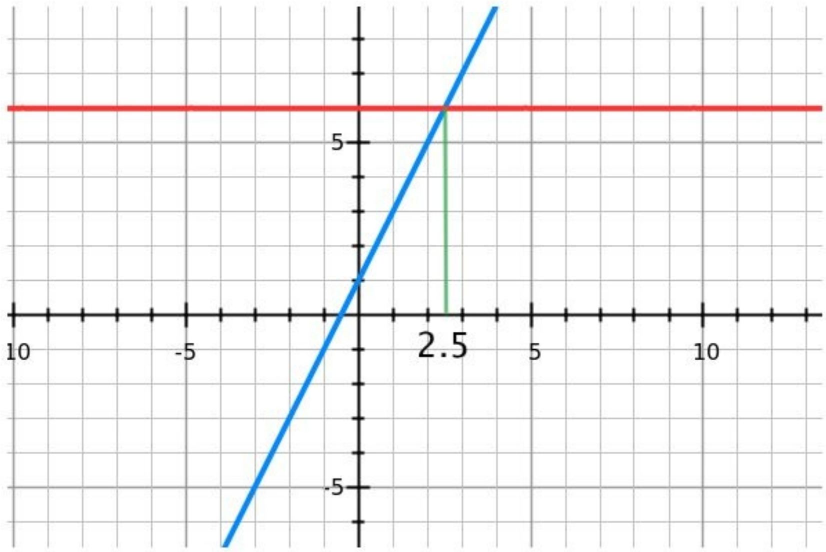
If we sketch the lines

$$y = 2x + 1$$

and

$$y = 6$$

We will have



And we can see that they cross at

$x = 2\frac{1}{2}$

Finally

$4x + 3 = -3x + 5$

Solving, as above, gives

$7x = 2$

$x = \frac{2}{7}$

Again, sketching

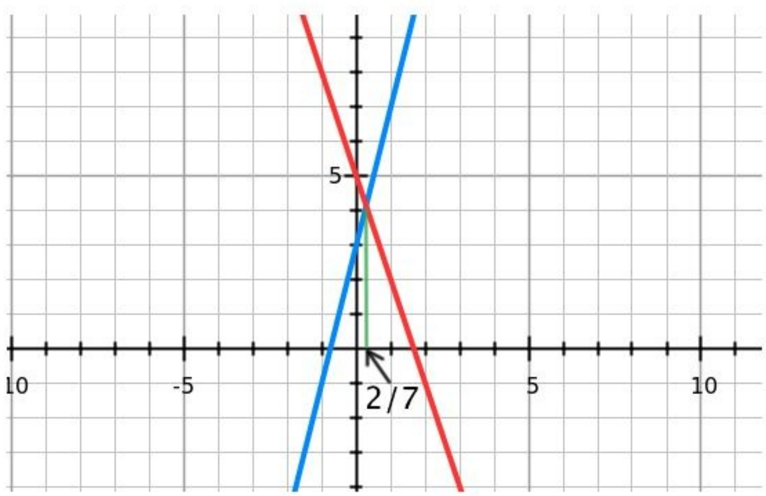
$y = 4x + 3$

&

$y = -3x + 5$

we see that they cross at

$x = \frac{2}{7}$



Real-Life Example of Simultaneous Equations

Real-Life Example of Simultaneous Equations

So what's the point of drawing graphs to find where they intersect? Why do we care?

Often in real life, we don't get to use algebra very much. However there have been a couple of occasions in my life where I have.

Here is one.

Years ago, when the internet began, there was something called the 'Internet Cafe'. Here you could surf the internet while downing a coffee.

I went to one while visiting a city. They had two options,

1. Pay as you go, for £3 an hour
2. Become a member for £5 and get £1 an hour for ever.

Which is the best option?

The question really is, how much time do you want to use the computers for? If you want to use it for an hour, we can see it'd be cheaper to pay as you go. But at what point would it be cheaper to become a member?

If we describe each situation as a straight line equation, we find
Option 1

Cost = 3t

The cost will be 3 pounds per hour, and if t=1, or 1 hour, the cost will be £3. In equation of a line terms, the gradient is 3, since this is essentially the three times table.

Option 2

Cost = 1t + 5

Here we have an initial cost (or cut) of £5. Then for every hour after that we pay £1.

If we make these slightly more mathematical

$$C = 3t$$
$$C = t + 5$$

Since they both equal C, we can make them equal to each other
$$3t = t + 5$$

And solve for t to find at what point it would be cheaper to become a member (or the break-even point).

This is Type 3 algebra, since t is on both sides
$$2t = 5$$

$$t = 2\frac{1}{2}$$

So after $2\,{}^{1}\!/_{2}$ hours, it would be cheaper to be a member. If you want to use the internet for less than that, go Pay as You Go, but otherwise, being a member is better.

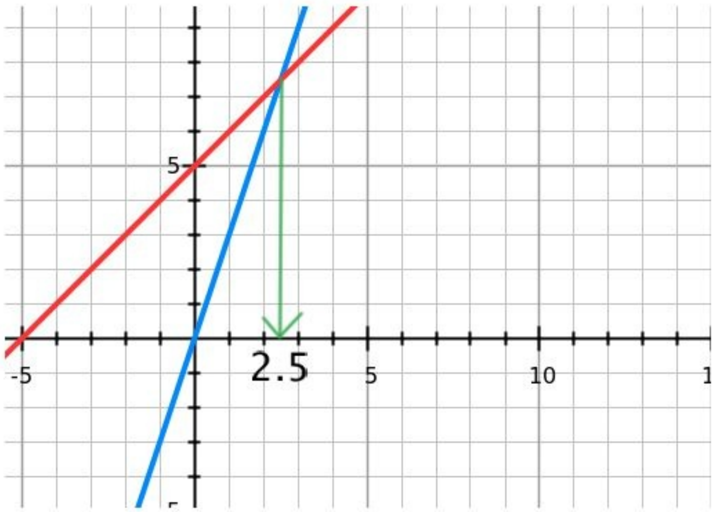
We can see this on a graph also.

$$C = 3t$$

Blue line

$C = t + 5$

Red line



Simultaneous Equations - School Style

Simultaneous Equations - School Style

If you’re reading this for exam purposes, you will need to know a different technique for finding where two lines cross each other.

For more complicated situations, far in advance of A-level maths, this method forms the basis to solve them. However, for some reason, this method is still required at GCSE.

What you will be given is the equations of two straight lines, as above, but in a different format, such as
$2x + y = 6$

all this really is

$y = -2x + 6$

that is to say, it is a straight line with gradient of - 2 and cut of 6.

If we have another straight line, such as
$3x - y = 14$

which is really

$y = 3x - 14$

we could find out where they cross by making them equal to each other as we did above.

However, since you get them in the format
$2x + y = 6$
$3x - y = 14$

We can do them in a different way.

Instead of the above method, we could simply add the equations.

This would give

$$2x + y = 6$$

$$+$$

$$3x - y = 14$$

$$\overline{5x + 0 = 20}$$

Or

$$5x = 20$$

Then we have Type 1 algebra, to give

$$x = 4$$

Since we now know that $x = 4$,

$$2(4) + y = 6$$

$$8 + y = 6$$

$$y = -2$$

Co-ordinates

$$(4, -2)$$

That’s it!

If the original signs had been the opposite, we could have just done the reverse, and subtracted.

So if it was

$$2x + y = 6$$

$3x + y = 14$

We could have subtracted instead, and this would lead to

$$\begin{array}{r} 2x + y = 6 \\ - \quad 3x + y = 14 \\ \hline -x + 0 = -8 \end{array}$$

$x = 8$

There is one other variation.

It was just lucky that the last example added nicely. But what if they don't?

In this example, we can't use that technique.

$2x + 3y = 8$

$5x + 2y = 9$

In this case we need to make either the x's or the y's the same number and then we can subtract the two equations.

Here the lowest numbers are the y's, (3 and 2). Because of this we can use a technique we learnt in Fractions - In A Minute, where we find the lowest common number of the two times tables.

2 4 6 8 10 12

3 6

which turns out to be **6.**

So we can make the y's equal to 6 by multiplication (just like with Fractions).

$$2x+3y=8...\times 2$$
$$4x+6y=16$$

and

$$5x+2y=9...\times 3$$

$$15x+6y=27$$

Now they both have equal y terms.

$$4x+6y=16$$
$$15x+6y=27$$

If we subtract these, but first change the order for convenience

$$15x+6y=27$$

-

$$4x+6y=16$$

$$11x+0=11$$

$$x=1$$

to find y

$$2(1)+3y=8$$

$$3y=6$$

$$y=2$$

Introduction to Quadratic Equations

To start this book, I want to ask you a *couple of unusual questions.*

One, is there gravity in space?

Yes or No?

When the space shuttle or rocket takes off, what path does it take to get into space?

SPACE

Bear in mind your answers to these questions, as we begin to learn about...

Multiplying Brackets

Multiplying Brackets

What’s

5(7) =

35

Because ‘brackets mean multiply’!

What about if we split the 7 up into
5(5 + 2) =

what then?

The best thing to do here is to multiply straight out, to get
5(5 + 2) = 25 + 10 = 35

Why not just add the 5 + 2? Because I want you to see that you should never use that method. Brackets mean multiply, so multiply!

For example, it doesn’t take much to show that adding doesn’t always work.

$5(x+1)=5x+5$

It is impossible to add the $x+1$. Since this will be the case for the majority of the time, never add.

Multiplying a bracket like this is also known as ‘expanding’ a bracket.
Sometimes you’ll see a question like
Expand

$4(2x-3)$

and the answer will be

$8x-12$

Because you simply multiply.

Try these

$3(x-2)$

$4(4x-9)$

$3(a+b)$

Ans.

$3x-6$

$16x-36$

$3a+3b$

Factorising

Doing the Reverse

Okay, so we can multiply (or expand) a bracket. Of course, according to the Third Rule, there must be a reverse to this.

Since we are multiplying, the reverse of it is clearly dividing. However this is known as ‘factorising’. But factorising and dividing are the same word! For instance,
If we divide

$$\frac{12}{4}$$

we are factorising it, because we want to know
$4 \times ? = 12$

In other words, what is the other factor (apart from 4) that goes into 12. This is of course 3. So we have factorised 12, to find 3, using another factor. This is an important idea. Please hold on to it.

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

As above, we factorised (divided) 12, using a factor (4). From that we found another factor (3). Both 4 and 3 divide into 12.

To do the same with

$$3x - 6$$

we need to do the same. A factor of both $3x$ and -6 is needed to be able to divide or factorise this expression.

We already know it is 3 because we have done the original question(!), but we can see that both $3x$ and -6 can be divided by 3.

So we take 3 to one side, and write a bracket...

$3($

We then fill in the bracket using the idea of

What times by 3 to get $3x$?

x

So we now have

$3(x$

And what times by 3 to get - 6?

-2

So we have

$3(x-2)$

and we close the bracket as there are no more terms.

And we end up with what we started with.

For

$3a+3b$

Again we can see that it will be 3 as a ‘common factor’ as it is called. The number or term that will divide into both.

Placing 3 in front of a bracket

$3($

and asking

What times 3 gives $3a$

a

$3(a$

And what times 3 gives $3b$

b

So we have

$3(a+b)$

Interesting note.

If we only have letters to factorise, this is actually EASIER than with numbers, as we don't have to find highest common numbers or anything like that.

For instance

$ab + ac + ad$

The common factor is a

$a($

And running through the same questions will give
$a(b + c + d)$

Much easier.

Multiplying - In A Minute

Multiplying 2 Brackets

Going back to

$5(5+2)$

where all of this started from, we've learnt to multiply that out (or expand) and to do the reverse, divide (or factorise).

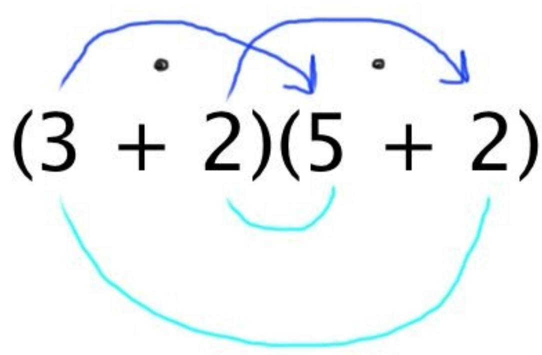
If we take this a little further by writing the 5 as
$(3+2)(5+2)$

What should we do here?

I've already asked you not to add what's inside the brackets, so don't do that! We know the answer will be 35, but how do we do this?

The school method is to use the smiley face or FOIL technique. By now you'll have realised that I never use the school method, as I've invented something better! So how?

For now, let's look at the smiley face method.



This will give

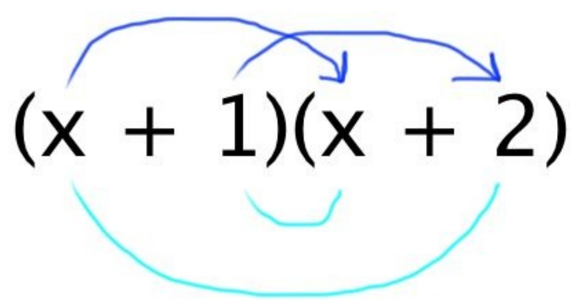
$15+4+10+6=35$

That’s fine.

What about for

$(x+1)(x+2)$

Same process



$x^2+2+1x+2x$

$=x^2+3x+2$

Notice I’ve added these up afterwards in order. The order we use in mathematics is to do with powers. So we have the highest power first (x^2), then the next ($3x$, which is x^1) and then no x at all. Actually this is $2x^0$, but what is x^0 equal to?

$x^0=1$

So it is 2 x 1 = 2.

So a disadvantage with the smiley face method is having to put these in order. This is known as ‘degree order’.

My method

This should be very familiar to you by now, especially if you have read ‘Multiplication - In A Minute’.

Instead of writing the brackets in a row,
$(x+1)(x+2)$

Write them in a column

$$\begin{array}{r} (x+1) \\ \times(x+2) \end{array}$$

And using exactly the same multiplication technique you’ve used countless times since Book 1, multiply the brackets!

$$\begin{array}{r} (x+1) \\ \Downarrow \\ \times(x+2) \\ \text{------------} \\ +2 \end{array}$$

then...

$$\begin{array}{r} (x+1) \\ \times \\ \times(x+2) \end{array}$$

$3x+2$

The cross here will be

$2x+1x=3x$

And finally the left column

$$\begin{array}{r} (x+1) \\ \Downarrow \\ \times(x+2) \end{array}$$

----

x^2+3x+2

Much easier, and note that the terms drop out in degree order (every time).

Try a few yourself now.

$(x+3)(x+4)$

$(x+3)(x-4)$

$(x-2)(x-3)$

Ans.

$x^2+7x+12$

x^2-x-12

x^2-5x+6

Each time you should find they easily drop out.

Here is the last one as a worked example to check your working (initially placed in height order to show you which to do first. Don't do it like this… just the last line is all that is necessary).

$$\begin{array}{r} (x-2) \\ \times(x-3) \end{array}$$

$$\begin{array}{rr} & +6 \\ -5x & \end{array}$$

x^2

$x^2 - 5x + 6$

What We’re Doing This For

What We’re Doing This For

Okay, so we can multiply brackets now, great. What is it for?

Now the new expressions you have got are called quadratics. What I want you to do is to sketch our first one,

$x^2 + 3x + 2$

to do this, we again - as in Equation of A Straight Line - make y equal to it and use a table of values, such as this one

$y = x^2 + 3x + 2$

x	- 3	- 2	- 1	**0**	**1**	**2**	**3**	**4**
$y = x^2 + 3x + 2$							**20**	

We calculate the y values by inserting the x values into the equation, just like we did for the Equation of A Straight Line

For example

When $x = 3$

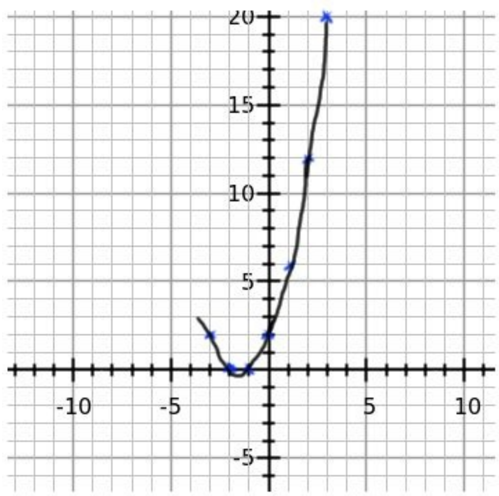
$y = (3)^2 + 3(3) + 2 = 20$

Fill in the rest of the table and plot the graph.

When you draw the lines between the dots, make them curvy!

x	- 3	- 2						

			- 1	**0**	**1**	**2**	**3**	**4**
$y = x^2 + 3x + 2$	**2**	**0**	**0**	**2**	**6**	**12**	**20**	**30**



(Note I didn’t put the final (4, 30) co-ordinate in also.

We go back now to the questions I asked you at the start of the book.

Is there gravity in space?

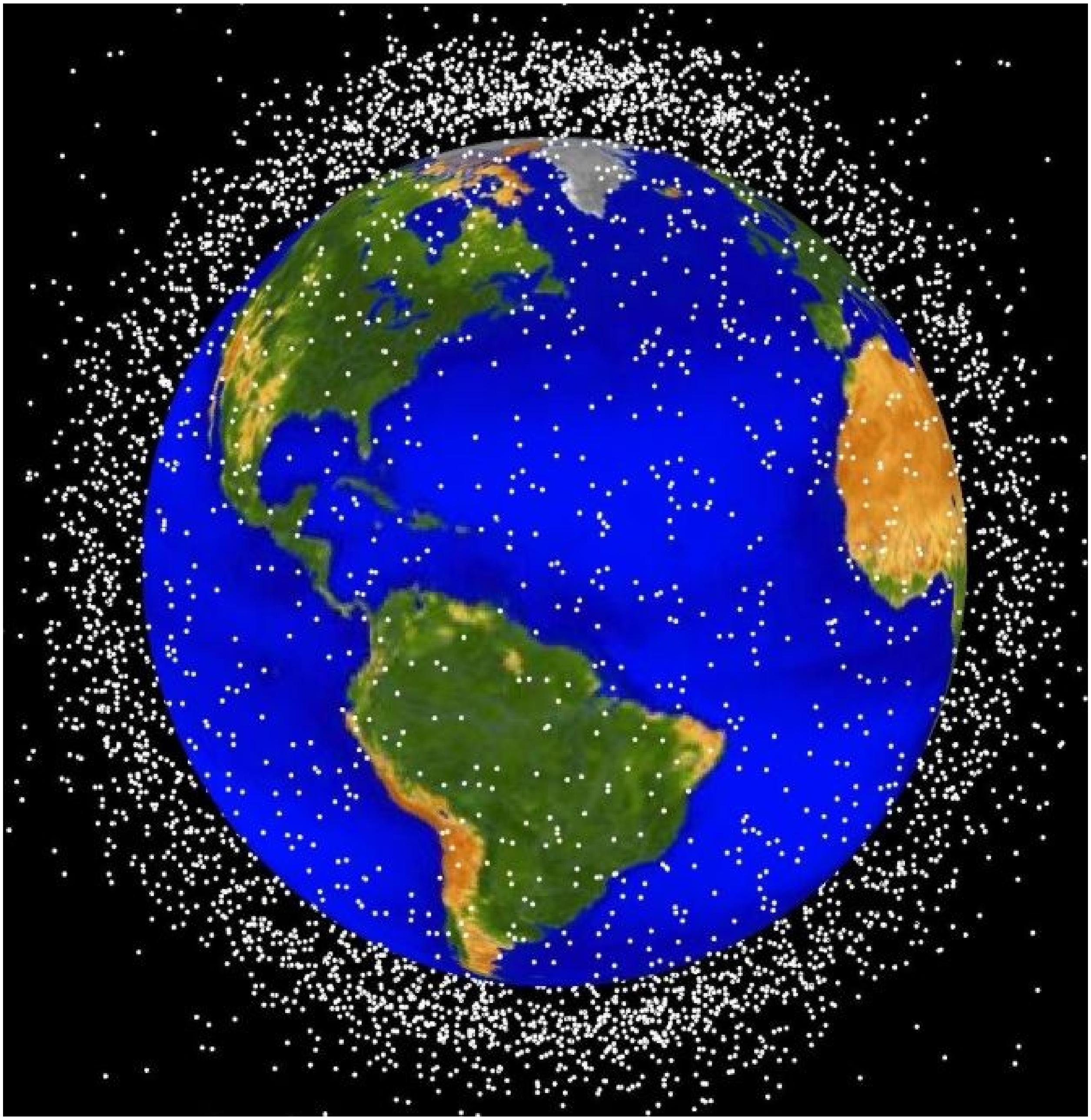
Most people say no. Why?

Because astronauts float and their teachers have told them there is no gravity in space. So is this correct?

Er, no.

There is gravity everywhere!

It is gravity keeping the moon in its orbit, gravity keeping the Earth in its orbit around the Sun, and gravity keeping the Sun in its orbit around the centre of the galaxy! Plus there are hundreds of satellites in orbit, how are they staying up there..?

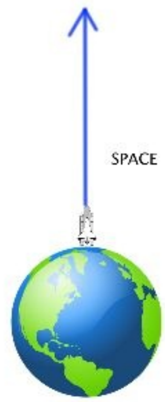


So how come astronauts float?

Let’s look at the second question first.

What path does the space shuttle take to get into space?

Again, people usually say, ‘it goes straight up’.



Is this what you said?

In fact, this would be the worst way for that to happen. If it went straight up, it would eventually run out of fuel, slow down (due to gravity pulling it back down again, as we’ve just established THERE IS gravity in space), come to a stop, and a bit like Wily Coyote, pause, and then plummet downwards.

What happens in fact is that it gets thrown.

If you throw a ball, maybe it will go 20-30 metres. That is because gravity pulls it down before it has a chance to get anywhere. If you threw it faster, it would go further before gravity pulled it down.

Imagine you threw it so fast, that by the time it fell down, it had reached the horizon. In fact you have to throw it at 5 miles per second/8 kilometres per second for that to happen.

What would happen is that as the Earth is round, the ground curves away. So the ball would fall, but as the Earth curves away, it doesn’t hit it.

You can find this explained in my blog, here

You can see this demonstrated in this time lapse picture. This is probably my favourite picture. You can see the curve that is so beautiful. And notice it goes down, completely contrary to what you would think. It falls off the Earth.

The shape of this curve is the same as a quadratic. If you turn your graph upside down, you should see the same shape.

It is called a Parabola.

And it is described by a quadratic equation. And this is just ONE of the many reasons why we study quadratics - although this is my favourite.

Why are astronauts 'weightless'?

They're not. They, like the space shuttle or space station they are in, are also falling towards the Earth, because of gravity. They are also traveling at 8 kilometres per second.

Because they are falling at the same rate as the shuttle, effectively they are skydiving but inside a lift that is also skydiving. So to them it appears they can just float around. The real issue is that to feel weight you need to have the floor pushing up against you. If that’s falling away from you, there’s nothing to push on you.

So, as the lift is falling, there is nothing to push back on their feet where they are standing. Relatively to them though, everything is still, and they are able to 'float'. In fact, if you were stood still by the shuttle, it would rush past you at around 17,500 mph as it falls towards Earth. To them, they are falling at the same rate as the shuttle so they are able to move around. It's a like a very long skydive without any air resistance in a lift that is doing the same thing at the same time.

You feel the same thing in a lift yourself. When it initially drops, to go down, you feel weightless for a millisecond. When it decelerates to a stop, it makes you feel slightly heavier.

Notice on the graph you have drawn there is a cut. Where is it?

At $y = 2$

So $c = 2$

If we look at the original equation again, we see that the number not times by x is 2. This will be the cut. This is always the case, and is the same as we saw for the Equation of A Straight Line.

$y = x^2 + 3x + 2$

2 is the cut.

We can now employ the Third Rule of maths and do the reverse. To find this quadratic, we multiplied out those brackets.

We’re now going to reverse that process to change our quadratic
$x^2 + 3x + 2$

back to the brackets.

How will we do this?

The problem we have is that there is no common factor. x only appears in 2 of the terms. As a result, this can seem like a tricky problem. However, since we know the origin of this expression (from multiplying), we can figure out how to get back to those brackets.

Remember they came from the column multiplication technique. As a result, we know that the 2 was due to a multiplication, and the $3x$ came from an addition.

In other words, we need to find 2 numbers that multiply to make 2 and add to make 3.

The x^2 just comes from the left column, $x \times x$

2 numbers that multiply to make 2 can be

1 x 2
Or - 1 x - 2

Remember to get in the habit of always writing the negative numbers as well, since it could well be them.

Adding these numbers

1 + 2 = 3

- Or -1 - 2 = - 3

So it must be both positive 1 & 2.

This means we have

$x^2 + 3x + 2$
$(x+1)(x+2)$

and the expression is factorised.

How could we check this was correct?

Of course, multiply them again!

$$\begin{array}{r} (x+1) \\ \times(x+2) \end{array}$$

----

x^2+3x+2

which is correct.

So now we can go in both directions.

If you're given brackets you can multiply them to get a quadratic.

If you're given a quadratic expression, you can factorise (divide) it to get the original brackets.

Try these

$x^2+7x+12$

$x^2-7x+12$

x^2+5x+6

x^2-x-2

Ans.

$(x+3)(x+4)$

$(x-3)(x-4)$

$(x+3)(x+2)$

$(x+1)(x-2)$

Brackets don't have to be in any particular order. (Why?)

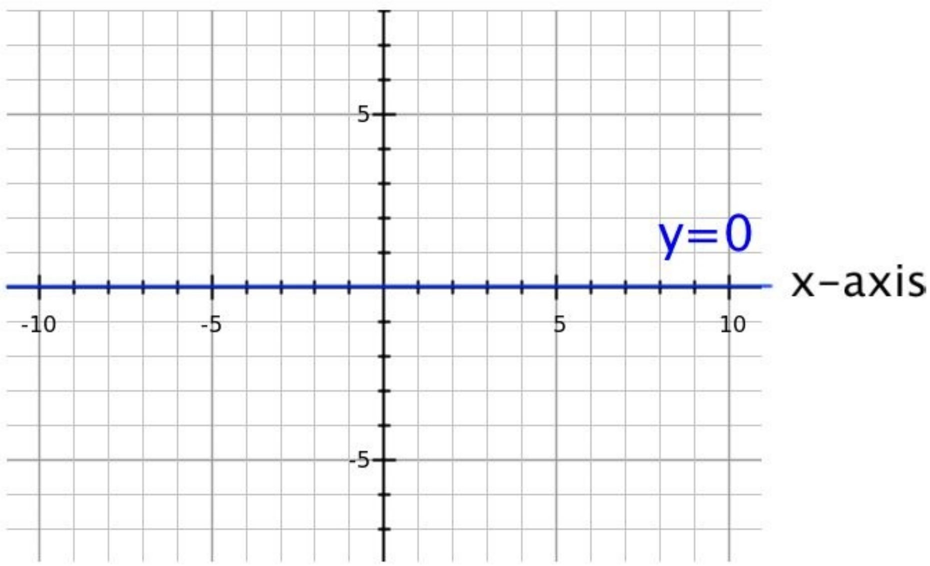
Solving A Quadratic

Now we can factorise a quadratic, there is something else we can do.

From Equation of A Straight Line, can you remember what the equation of the x-axis is?

You might recall it is

$y = 0$



In ‘Simultaneous Equations’ we saw that if we have two equations like
$y = 2x$

and

$y = 6$

we can make them equal to each other to find out where they cross. Or, if we’re asked to solve algebra like this, this is one of the things we’re doing it for.

$2x = 6$

If we want to find out where the quadratic crosses the x-axis, we can use the two equations of each for this.

$$y = x^2 + 3x + 2$$

and

$y = 0$

and make them equal to each other.

$x^2 + 3x + 2 = 0$

since they both equal y

Fine so far.

But if we want to know what the value of x will be to make the expression on the left equal to zero, we're going to struggle a bit. We could guess what it might be... say, using $x = 1$

$1^2 + 3(1) + 2 = 6 \neq 0$

and keep guessing. But guesswork is unmathematical. **NEVER GUESS.**

(In this case we could refer to your original table, and you'll find it there. But we want a method that doesn't rely on this.)
Now we know how to factorise the quadratic, we could do that.

So

$x^2 + 3x + 2 = 0$

becomes

$(x+1)(x+2)=0$

which is still true.

Remembering what brackets mean, this says that these two things multiplied together equals zero.

So what does each one equal?

Each bracket must equal zero too.

It’s a neat trick to tell us what x will be.

So

$x+1=0$

and

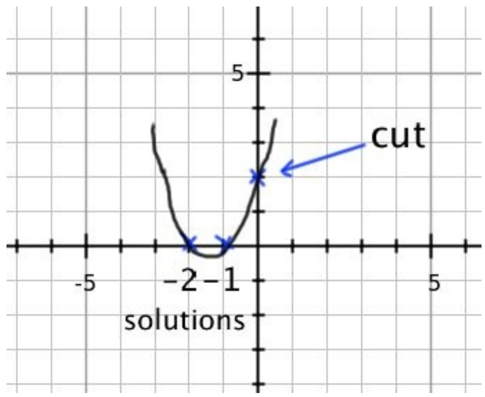
$$x + 2 = 0$$

this means that

$$x = -1$$

and
$x = -2$

So now we know where our quadratic crosses the x-axis, and the y-axis. We know the shape (parabolic) too, so it's easy for us to sketch a quadratic without having to calculate a table of values - because all we have to do is solve for x to find out where it crosses the x-axis!



This was our goal. To be able to create, factorise, solve and sketch a quadratic.

Now we can do that.

Try this one yourself.

$y = x^2 + 4x + 3$

Factorise

Solve
Sketch.

Also, try to find the ‘*minimum value*’. This is the location of the lowest point on the curve.

Answer below.

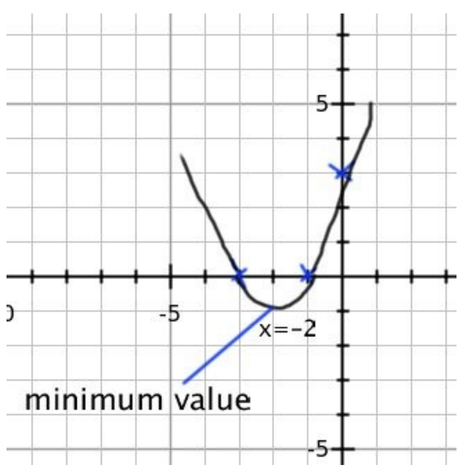
Did you find the minimum value?

It is

- $x = -2$

There are actually four different ways to find this value. One of the ways is to use our solutions. Because the quadratic is symmetrical, the lowest point is always halfway in between the two solutions.

Once we’ve solved the quadratic, it is then easy to find the minimum value as it will be halfway between the solutions.



How could we find the y-value of this point?

One way to do it would be to use the equation of the quadratic.

$y = x^2 + 4x + 3$

since it begins ‘y equals...’

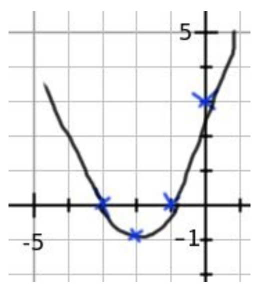
all we have to do is substitute in the x-value.

$y = (-2)^2 + 4(-2) + 3$

$= -1$

So the co-ordinate for the minimum value is
(- 2, - 1)

Now we know four points where the quadratic goes through. As a result, just like for the equation of a straight line, we don’t need to actually do a table of values in order to sketch one. If we have enough points, and we know the shape (always a parabola) we can just sketch through the four points, as below.



Do the same with these questions

$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 - x - 2$

Non-factorisable Quadratics

Non-factorisable Quadratics

Sometimes you'll be asked to solve a quadratic that isn't factorisable. However it will cross through the x-axis, so you'll be asked to find out where.

To be able to do this we use a quadratic formula, which comes from the general expression of a quadratic.

This is

$ax^2 + bx + c$

for example,

$x^2 + 3x + 2$

would have

$a = 1$
$b = 3$
$c = 2$

The c here is of course the cut!

Another example

$9x^2 + 29x - 28$

$a = 9$
$b = 29$
$c = -28$

If we equate the general expression to zero, to find where it crosses the x-axis, we will have
$ax^2 + bx + c = 0$

We want to know what x will be in this to make the equation equal to zero.

What we can do is to 're-arrange' the equation so we have x in terms of everything else. The derivation of this is left to the book ‘Changing The Subject’ as this is what we’re effectively doing...

You may have come across this formula already. In any case, the format I use is different to the school one (this may not come as a surprise by now!) If we use my format, we actually get a bonus piece of information, plus it tells you something very interesting about quadratics and squares in general.

So here it is

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

I appreciate it probably looks a bit scary! But we’ll look at it bit by bit.

Firstly, why is there a plus and a minus in front of the square root?

To answer this, what is the square root of 9, or $\sqrt{9}$

Everyone usually says 3. But no, this isn’t the answer.

It is 3... or - 3!

Because $-3 \times -3 = 9$, then - 3 is the square root also.

As a result, any square root has two possible answers. So that’s why that’s in the formula. Use this from now on for your square roots in general!

Where this formula differs from the one you’ll usually find in textbooks is to have the

$$\frac{-b}{2a}$$

The reason I write it like this is because it will give us the minimum value!

For example, for

$y = x^2 + 3x + 2$

We found that the minimum was

$-\frac{3}{2} = -1\frac{1}{2}$

If we had used the formula the

$\frac{-b}{2a}$

would give us

- $-\frac{3}{2} = -1\frac{1}{2}$

That is, it gives you the minimum too!

In fact, from now on you can easily find the minimum of a quadratic in this way.

Another example would be

$y = x^2 + 4x + 3$

What is the minimum here?

$\frac{-b}{2a}$

$-\frac{4}{2} = -2$

as we saw for yourself when you did it.

If we do an example we can see how the formula works.

$x^2 + 3x + 2$

would have

$$a = 1$$
$$b = 3$$
$$c = 2$$

substituting into the formula gives

$$x = \frac{-3}{2(1)} \pm \frac{\sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-3}{2} \pm \frac{\sqrt{9-8}}{2}$$

$$x = \frac{-3}{2} \pm \frac{\sqrt{1}}{2} = x = \frac{-3}{2} \pm \frac{1}{2}$$

$$x = -\frac{3}{2} - \frac{1}{2} = -2$$

OR

$$x = -\frac{3}{2} + \frac{1}{2} = -1$$

we have the minimum value

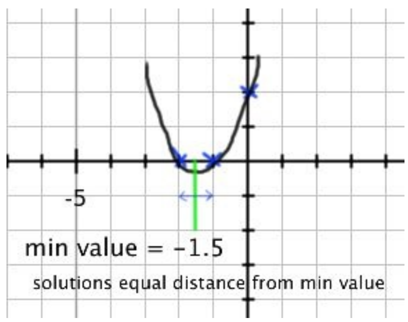
- min = $-\frac{3}{2} = -1\frac{1}{2}$

- and then either side of that the same distance lie the solutions. Because we use the minimum value as our starting point, we see the symmetry of the parobola, quadratic and square numbers all in one.

Here we see that each solution is the same distance

$$\left(\frac{1}{2}\right)$$

away from the minimum value. This will always be the case for any solution for any quadratic. Once we find the minimum, which is easy to do, we can then go right and left the same distance to find the solutions. If we use the formula in this way, we get 3 pieces of information instead of just 2 (as in the school method). That's why I prefer you to use my version of the formula.



Try the formula on these questions

$x^2+7x+12$

$x^2-7x+12$

x^2+5x+6

x^2-x-2

Ans.

Now if we try

x^2+5x+4

Working this out, we see that in the square root, we have a square number. In fact, this is because so far I've only asked you to use the formula on factorisable quadratics. If the number in the square root is a square number, they end up cancelling each other out, and we always get a round number. So we have a minimum value, and a 'nice' number right and left of it.

$$x=-\frac{5}{2}\pm\frac{\sqrt{25-16}}{2}$$
$$=-\frac{5}{2}\pm\frac{\sqrt{9}}{2}$$
$$=-\frac{5}{2}\pm\frac{3}{2}$$

In reality, we use the formula to solve non-factorisable quadratics. We should always try to factorise it first, and only if that fails do we try the formula.

For instance,

$$x^2+7x+1=0$$

This quadratic isn't factorisable, because there isn't two numbers that multiply to get 1 but add to get 7. As a result, the solutions will be surds. Let's see what that means.

$a=1$

$b=7$

$c=1$

$$x=-\frac{7}{2}\pm\frac{\sqrt{7^2-4(1)(1)}}{2}$$

We find that the minimum is

$$-\frac{7}{2}$$

and that the solutions will be

$\pm\frac{\sqrt{45}}{2}$

right and left of it.

$\sqrt{45}$

is a surd (from Indices - In A Minute) and we can simplify this to get
$\frac{3\sqrt{5}}{2}$

This is the exact answer. This also tells us that it wasn't factorisable.

If we use a calculator to find what this will be we'll see that we have
$x = -\frac{7}{2} \pm 3.35$

-

$x = -3.5 \pm 3.35$

Gives that

-

$x = -3.5 + 3.35 = -0.15$

-
- Or that,
-

$x = -3.5 - 3.35 = -6.85$

-

There would be no way we'd think of these numbers to factorise it! This is where the formula comes in.

Try these yourself

Solve and find the minimum value of these:

$x^2 + 4x + 2$

$x^2 - 9x + 4$

$x^2 - 9x - 14$

Ans.

$x = -2 \pm \sqrt{2}$

$x = \dfrac{9}{2} \pm \dfrac{\sqrt{73}}{2}$

$x = \dfrac{9}{2} \pm \dfrac{\sqrt{137}}{2}$

Note we can keep these last two they way they are as the numbers in the square roots are primes, and will not simplify.

Any Type of Quadratic

When a is not equal to 1

Now we know how to tackle all quadratic of every kind, except one.

So far we've looked at quadratics where a is always 1, *i.e.* ones like
$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 - x - 2$

We can have any type here though, such as
$2x^2 + 7x + 3$

Again, we try to factorise these first. To do this we follow the same method of finding two numbers to multiply to make 3 but add to make 7.

Of course that won't work!

However, this is where the $2x^2$ part comes in.

Because we have this instead of just the usual x^2, we also have to find 2 numbers that multiply to get 2.

So this adds an extra complication, but all we have to do is match up the right numbers.

To get 3, we would need

3 x 1
Or - 3 x - 1

and to get 2, we would need

2 x 1
Or - 2 x - 1

Since it is entirely positive, we can take it (to begin with) that to get 2, we would use 2 x 1.

So we have

$$2x^2 + 7x + 3$$
$$(2x + ?)(x + ?)$$

To fill in the rest, let's try 3 in the first bracket and 1 in the second.

$$(2x + 3)(x + 1)$$

If we multiply this we get

$$\begin{array}{r} (2x + 3) \\ \times(x + 1) \end{array}$$

$$2x^2 + 5x + 3$$

Which is almost correct!

However, we want $7x$ in the middle.

If we try 1 in the first bracket and 3 in the second, we get

$$\begin{array}{l} (2x + 1) \\ \times(x + 3) \end{array}$$

$$2x^2 + 7x + 3$$

Which is what we're looking for, so we know they're the correct brackets.

$$2x^2 + 7x + 3$$
$$(2x + 1)(x + 3)$$

With practice and familiarity, you will get to a stage where you'll begin to do this process in your head, but for now trying different possibilities is fine.

To go ahead and solve this - that is, find out where the quadratic crosses the x-axis, we just equate it to ... what?

$y = 0$

So we have

$$\begin{aligned}2x^2 + 7x + 3 &= 0\\(2x+1)(x+3) &= 0\end{aligned}$$

So

$$2x + 1 = 0$$

And

$$x + 3 = 0$$

$$x = -\frac{1}{2}$$

$$x = -3$$

Another Example

Another example

$3x^2 + 13x + 4$

To get four, there are four possibilities.

4 x 1

- 4 x - 1

2 x 2

- 2 x - 2

To get 3, there are two

3 x 1

- 3 x - 1

Again, since they're all positive terms in the quadratic, we can start with positive values.

Try

$(3x + 2)(x + 2)$

This would give

$3x^2 + 8x + 4$

which isn't right.

We need to get $13x$.

Since 3 x 4 = 12, which is near to 13, let's try that combination.

$(3x+1)(x+4)$

gives

$3x^2+13x+4$

which is correct.

It is useful to see what we are aiming for and try to 'make our way there' by using two of our numbers to do it. It may give us a pointer in the right direction.

Here's one to try that came up in an actual GCSE exam. (I know, because I was taking it!)
It's quite hard, so take your time over it. In fact it's one of the hardest I have seen in an exam.

$9x^2+29x-28$

Ans.

$(9x-7)(x+4)$

(In the exam itself I didn't actually spot this and went via the 'back door' by using the quadratic formula. If you try this you'll see why it's not an easy option, but do-able. Remember I didn't have a calculator. Notice that the number in the square root is a square number. What does that mean?)

Finally we can look at this type using the *formula* but one that isn't factorisable, such as
$2x^2 + 4x - 4$

$a = 2$
$b = 4$
$c = -4$

This would give

$$x = \frac{-4}{2(2)} \pm \frac{\sqrt{16 - 4(2)(-4)}}{2(2)}$$

$$x = -1 \pm \frac{\sqrt{16 + 32}}{4}$$

$$x = -1 \pm \frac{4\sqrt{3}}{4}$$

$$x = -1 \pm \sqrt{3}$$

The minimum value would be at

$x = -1$

and the two solutions are

$\sqrt{3}$

right and left of that.

Completing the Square

Completing the Square

When we factorise and solve a quadratic, we get 2 pieces of information about it. The two solutions and the minimum value, as it is half-way between the solutions. When we use the quadratic formula, we get the same information (if we use it my way).

There's a third way to get this information, but it also gives us a couple of bits extra. As a result, this is the method I tend to use if I want to know everything about a quadratic I'm using.

So what do we do? Let's say we want to find out all about this quadratic
$y = x^2 + 3x + 2$

Instead of factorising or using a formula, we almost take half of it and write
$(x+1.5)^2$

It's like half the x^2, and half the 3.

What we do now is multiply out the brackets and see what we would get.

$(x+1.5)$
$\times(x+1.5)$

$x^2 + 3x + 2.25$

(this is where knowing how to square numbers that end in 5 comes in so useful. Do you remember this from Squaring - In A Minute?)
Note that

$(x+1.5)^2$

isn't

$x^2 + 1.5^2$

which is a common mistake.

Our answer of

$x^2 + 3x + 2.25$

is very similar to the original quadratic!

The first two terms are the same, but the number is slightly out.

So if we subtract 0.25 from these brackets, it would be the same.

Giving

$(x+1.5)^2 - 0.25$

So this is the square completed.

I'll talk a bit more about why it is called this later on in the book.

If we were to multiply this back out again, it would look like

$y = x^2 + 3x + 2$

So this is just another way of saying the same thing.

What information does this give us?

Firstly, it gives us the minimum value. If we take the opposite sign of the number in the brackets, we instantly know what it is.

$x = -1.5$

which is true, from what we saw earlier.

Here is the bonus.

The number outside the brackets, - 0.25, is the y-value of the minimum value. So we get the whole thing in one. The minimum value coordinate is then
(- 1.5, - 0.25)

note we don't use the opposite here, just exactly what it is. There's a reason the y value is negative too - do you know why?

So immediately, we have the minimum value - and its y-coordinate.

To find the solutions, we just equate this all to zero.

We do that because we want to know where the line

$$y = x^2 + 3x + 2$$

and

$$y = 0$$

cross each other.

$$y = x^2 + 3x + 2 = 0$$

which is

$$(x + 1.5)^2 - 0.25 = 0$$

We now shuffle this equation around to find x

$$(x + 1.5)^2 = 0.25$$

Square root both sides

$$x + 1.5 = \pm\sqrt{0.25}$$

$$x = -1.5 \pm \sqrt{0.25}$$

which is

$x = -1.5 \pm \frac{1}{2}$

which is

$x = -2$
$x = -1$

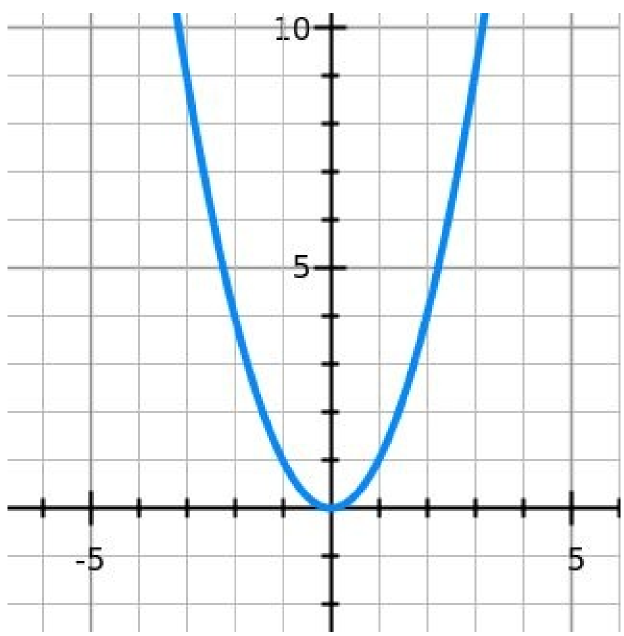
as we saw earlier.

We've got the solutions, both coordinates of the minimum value and now we can find something else!

We've looked at lots of quadratics, but actually they all form part of the same template. They use the same model and every quadratic we've looked at is just a variation of that.

The fundamental quadratic they all come from is

$y = x^2$



If you were to place a mouse on this and move it around the graph, it would still be $y = x^2$ in shape, but it would have different solutions and minimum values.

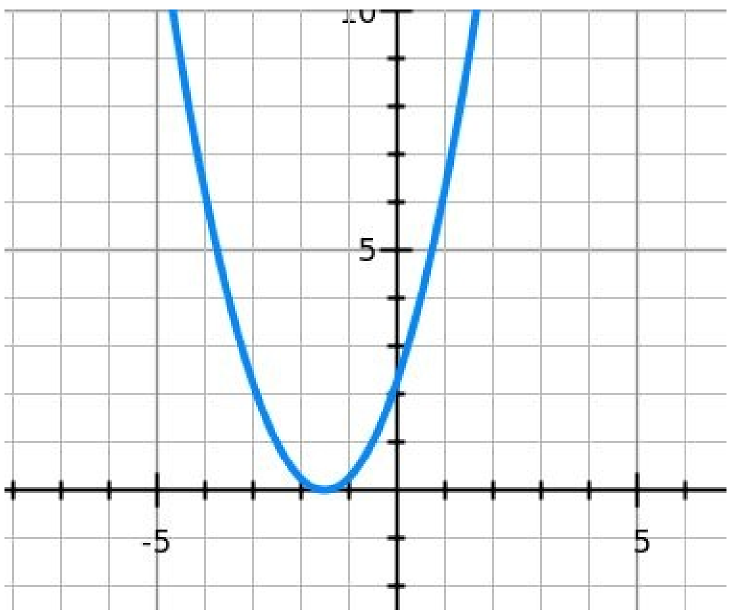
All quadratics are in this exact situation.

Looking at our completed square

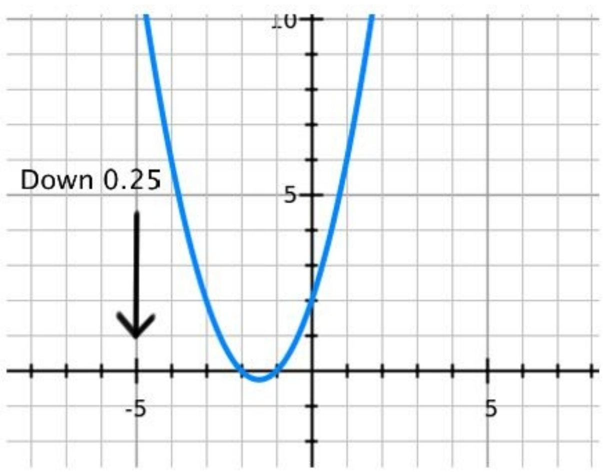
$(x+1.5)^2 - 0.25$

What this is says is that our $y = x^2$ has had two moves.

The first is a move to the left of 1.5



and the second is a move down of 0.25



So we end up with our original quadratic
$y = x^2 + 3x + 2$

which has a cut of 2 and solutions at - 1 and - 2, as well as that minimum at
$x = -1.5, y = -0.25$

What our completed square has told us is the TRANSFORMATION that x^2 has made to look like
$y = x^2 + 3x + 2$

which can be of a lot of use.

Not surprisingly this is the same as the minimum value because we are actually charting the movement of the quadratic from the origin (0, 0) to where it finishes. This will always be the minimum.

We use this concept of 'transformation' in other areas of maths - this is the first time we come across it. I will refer to it again when we look at transformations.

This is written as
[- 1.5, - 0.25]

Try this one yourself.

$y = x^2 + 5x + 6$

Answer.

Working through

$(x + 2.5)^2 - 0.25$

It is just a coincidence this is the same y-value!

From this we know the minimum value

(- 2.5, - 0.25)

and the transformation

[- 2.5, - 0.25]

and to find the solutions

Equate to zero

$(x+2.5)^2 - 0.25 = 0$

$(x+2.5)^2 = 0.25$

$x + 2.5 = \pm\sqrt{0.25}$

$x = -2.5 \pm \sqrt{0.25}$

Which gives

$x = -3$

Or

$x = -2$

We start to see a pattern developing here.

We can see that the solutions will always be the x - minimum value plus or minus the square root of the (positive) y - minimum value. So in practice we can just jump straight to that. In an exam you have to show each line of working.

For example, we could have

$y = x^2 + 7x + 6$

The Completed Square

$(x+3.5)^2 - 6.25$

Therefore the solutions will be the x - minimum value, plus or minus the square root of the (positive) y - minimum value.

$$x = -3.5 \pm \sqrt{6.25}$$

$$x = -1, x = -6$$

Easy!

An Even Easier Way To Find the Y-Minimum Value

So far to find the y-value we've just squared the brackets to see what that would give us. Then we just need to make an adjustment to match with the original equation.

In the previous example, the completed square multiplied out would give $x^2 + 7x + 12.25$

So we need to subtract 6.25 from that to get back down to 6.

That's fine.

A quicker way is just to square that number only. Because the first two terms are always the same (that's the idea), the only difference will ever be the number. So if we just squared 3.5

$$3.5^2 = 12.25$$

We then have the number we need.

There's a second reason for this. The actual location of the minimum value is x = −3.5

If we put this into the original equation, like we did when we used the solutions to find the minimum value, we find that the middle term is always double the first term. So it is working backwards on us for no reason - *I.e.* It's a waste of time to do it that way.

$$(-3.5)^2 + 7(-3.5) + 6$$

$12.25 - 24.5 + 6$

$= -6.25$

The second term is exactly double the first, meaning we just make our value negative, just to add positive 6. We can avoid all this working out by just squaring the number in the original completed square bracket, then decided what we need to get down to our number, as above.

Solve Quadratics In Your Head!

Now we know the two concepts above, we can apply both to solve a quadratic in our head.

Using completing the square, let’s say we want to solve
$x^2 + 7x + 10 = 0$

Immediately we complete the square, giving
$(x + 3.5)^2$

$x = -3.5$

For the minimum value.

And to calculate the y-value, as we just saw, we just square this

12.25

And we have to subtract −2.25 to get to 10, so that’s our y-value.

So now we can just jump straight to our solutions
$x = -3.5 \pm \sqrt{2.25}$

The number in the square root is a square number itself (15^2), so it is easily square rooted in your mind to 1.5

Our solutions will be

$x = -3.5 \pm 1.5$

$x = -5, -2$

Practicing a few of these will make it easy for you to do in your head.

When *a* is Not Equal to 1

Again, as with factorising or using the formula, we may want to complete the square on a quadratic where *a* isn't 1.

For instance,

$2x^2 + 8x + 5$

Here, just factorise the first two terms first, to give
$2(x^2 + 4x) + 5$

Then we just complete the square on what is in the brackets
$2(x + 2)^2$

This will give

$2(x^2 + 4x + 4)$

$2x^2 + 8x + 8$

So almost the same.

We just need to minus 3 to get to 5, so we have
$2(x + 2)^2 - 3$

And the square is completed.

Our minimum value will be

(- 2, - 3)

And the solutions will be

$$2(x+2)^2 = 3$$
$$(x+2)^2 = \frac{3}{2}$$
$$x = -2 \pm \sqrt{\frac{3}{2}}$$

The only difference we have to make here is to our y-value. Because this is $2x^2$ instead of our usual x^2, the quadratic stretches upwards. As a result, our solutions will be closer to the minimum value. Just remember to divide by *a* before square rooting, if *a* is not equal to 1!

A Closer Look At The Quadratic Formula

A Closer Look At the Formula

Looking again at the quadratic formula, there is another piece of information we can get from it.

(Amazing, eh, how much information is in quadratics?)

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

In the square root part, there is a

$$b^2 - 4ac$$

As we saw in our examples, if this number is positive, we had two solutions.

(If the number was positive and a square number itself, the quadratic was factorisable too).

What if the number isn't positive?

What if it is zero or negative? Then what?

If the number is zero, as in this example

$$x^2 + 4x + 4$$

$$a = 1$$
$$b = 4$$
$$c = 4$$

$$b^2 - 4ac$$

$$(4)^2 - 4(1)(4) = 0$$

We end up with only 1 solution.

We can see this from the formula

$$x = -\frac{4}{2} \pm \frac{\sqrt{0}}{2}$$

The right hand side

$$\frac{\sqrt{0}}{2}$$

is zero

So we only have

$$x = -\frac{4}{2}$$

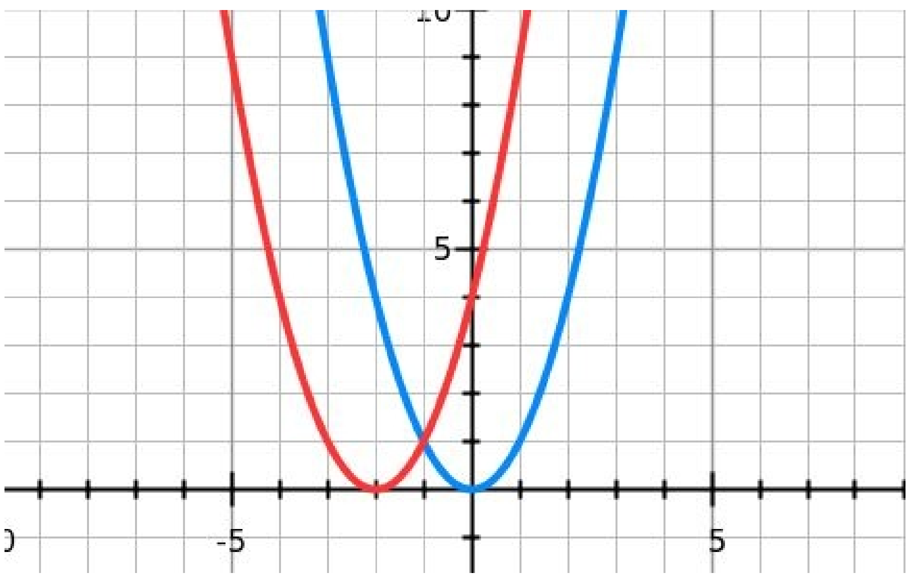
$$x = -2$$

What does this mean?

It means that the quadratic sits on the x-axis at this point, only touching at one co-ordinate.

To show this, if we complete the square on it we find,
$(x+2)^2$

There is no y-value because we don't need to make any adjustment up or down. So the transformation of this quadratic is from $y = x^2$ and two to the left.



We can see that it has only one solution.

*It is interesting to note that y is also zero here.

Finally, what if the square root is negative, as in this example
$y = x^2 + 2x + 10$

$b^2 - 4ac$

$(2)^2 - 4(1)(10)$

$= -36$

Since it is impossible to square root a negative number*, there is no answer to this. So there are no solutions, so the quadratic doesn't cross the x-axis at all!

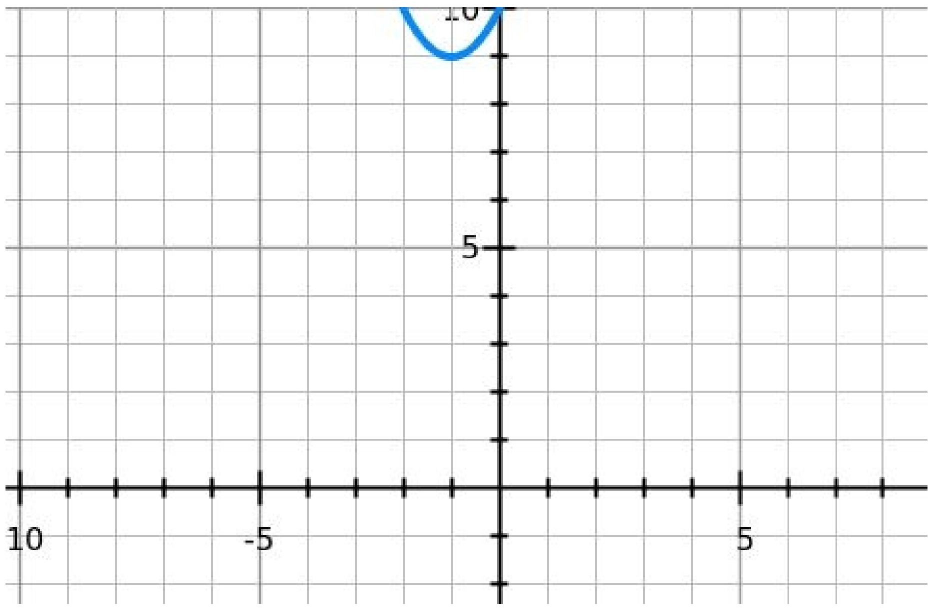
Using

$b^2 - 4ac$

in this way is called using the ***discriminant.***

This can check for us - quite quickly - whether the quadratic has any solutions, and if so, how many.

We can see this is true from its graph.



However, we can check whether there are solutions much more quickly in a much simpler way.

Instead of using the *discriminant*, we could just use the y-value in the completed square.

If it is negative, there are 2 solutions.
If it is zero, there is 1 solution (it sits on the x-axis) If it is positive, there are no solutions.

This is because the y-value is part of the minimum value. If it sits below the x-axis, there must be 2 solutions. If it sits on the x-axis, at $y=0$, that must be 1 solution, and if it is above the x-axis, there can't be any solutions.

It is easier to find y, as all we have to do is
Square half the x-coefficient, and see how far that is from c, as we saw in an earlier chapter.

For example, for

$y=x^2+2x+10$

$(1)^2 = 1$

And we need positive 9 to get to 10. Therefore the minimum is 9 whole units above the x-axis. There is no way this quadratic has solutions!

Try this technique for the following

$x^2 + 7x + 12$

$x^2 - 7x + 12$

$x^2 + 5x + 6$

$x^2 + 6x + 9$

$x^2 + 8x + 9$

Answers

Yes, two

Yes, two

Yes, two

Yes, one

No

From now on you can very easily and very quickly calculate whether a quadratic has solutions. *Complete the square* on it and it will tell you

- the x and y co-ordinates of the minimum value
- the transformation it has undergone
- whether it has any solutions, and if so how many

- what the solutions are

This is why completing the square is SO useful.

Final example

Doing a Quadratic - In A Minute!

$x^2 + 7x + 12$

Completing the Square

$(x + 3.5)^2 - 0.25$

Minimum - (- 3.5, - 0.25)
Transformation - [- 3.5, - 0.25]
Solutions, YES, two
They are
$x = -3.5 \pm \sqrt{0.25}$

(From the earlier chapter - minimum value plus or minus the square root of the positive y-value)
$x = -3, -4$

What Multiplying Two Expressions Together Will Give

What Multiplying Two Expressions Together Will Give

When we looked at multiplying brackets such as
$(x+1)(x+2)$

or

$(x+6)(x-1)$

we of course, got a quadratic.

But what shape does that give?

Your first thought might be a parabola.

However, it also gives another shape.

We are multiplying two expressions together. They actually represent numbers - it's just that we don't know what the numbers are.

Thinking back to Multiplication - In A Minute, when we multiply two numbers together, what shape do we get?

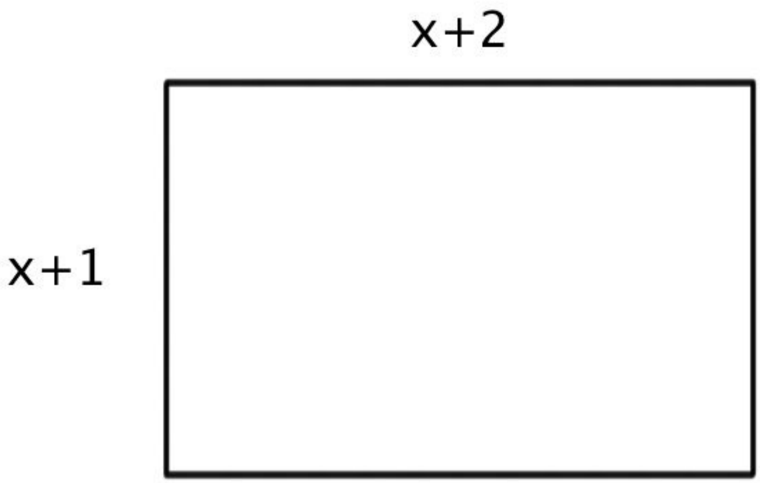
A rectangle.

So our expressions above, when multiplied will give a rectangle.

For example

$(x+1)(x+2)$

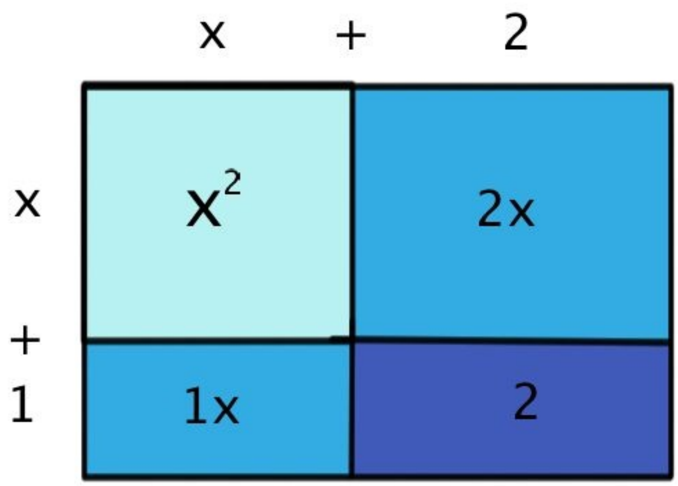
Looks like this



And when multiplied out gives

$x^2 + 3x + 2$

and this will look like this in the rectangle



Since the light blue shape has sides of x and x
The blue shapes have sides of 1 and x and 2 and x respectively, and The purple shape has sides of 1 and 2.

These 'shapes' are all rectangles and one square (light blue).

The information we get about the rectangle is the area of it.

So remember, when we are solving for x, not only are we trying to find where the curve crosses the x-axis, but we're trying to find what the lengths would be, given the area. When we do this, we set the quadratic equal to zero. This means we're saying the area is zero, so what would the lengths be? Of course, each length must be zero for that to happen, so that's why when we have
$(x+1)(x+2)=0$

$(x+2)=0$

x must be - 2 to make zero.

If the area of the rectangle isn't zero, we can also find out what the area of the sides will be.

Let's say we have

$x^2+5x+4=10$

So the rectangle has an area of 10.

$x^2+5x+4=10$
$(x+1)(x+4)=10$

These are the lengths of the sides that give 10.

What would x be to make this work? (We can probably see this quite easily in this example, but just to see how we would do it if it was more complicated.) To find this, we solve the quadratic as normal.

However, to do this, we need to have the situation of there being no area or having the equation equal to zero.

As a result, we minus 10 from both sides.

$x^2+5x-6=0$

and factorise and solve as usual

$(x-1)(x+6)=0$

$x=1, x=-6$

Looking again at our original quadratic,

$$x^2+5x+4=10$$
$$(x+1)(x+4)=10$$

We can’t use -6 here, as this wouldn’t make sense, as both lengths would be negative. So we take the value $x=1$

This tells us that the two sides of the rectangle will be
(1 + 1)(1 + 4)

2 x 5

This of course, multiplies to make 10!

Why Is it Called ‘Completing The Square'?

Why Is It Called Completing the Square?

As we have just seen, when we multiply brackets together, we get the shape of a rectangle.

Completing the Square is turning that rectangle into a square.

However, even though it is possible to turn the area of a rectangle into a square (via square rooting), this doesn’t happen when we complete the square because of the way we do it.

As we just sort of half the expression, this isn’t the same as square rooting it.
For example, if we complete the square on
$x^2 + 3x + 2$

this is

$$(x+1.5)^2 - 0.25$$

This isn't

$$\sqrt{x^2+3x+2}$$

Actually when we complete the square, we do get a square of course, since that is what

$$(x+1.5)^2$$

gives.

But since it has the extra term, it either takes away or adds to a square, so our shape is 'a square and a bit' or 'a square with a bit taken away'.

However, if we complete the square on this

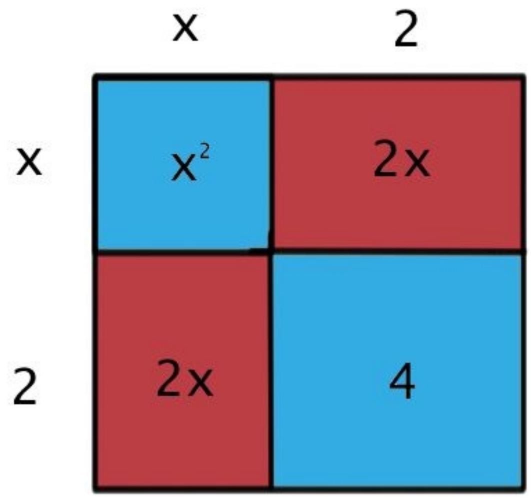
$$x^2+4x+4$$

we get

$(x+2)^2$

which is a square.

This is because the original quadratic was a square itself



This is technically known as a ‘binomial square’.

This is related to the transformation of the curve $y=x^2$ too. If the original quadratic was a square, the transformation will just be left or right.

If it was a rectangle, it can be both left or right, and up or down.

Simultaneous Equations - Extended

Simultaneous Equations - Extended

To solve a quadratic, we are trying to find where it crosses the x-axis. To do this we equate it to the equation of the x-axis,

$$y = 0$$

As we have done many times already.

However, there is nothing to say we can't find where the quadratic crosses any line. Here are three examples.

Example 1

Another horizontal line on the graph.

Let's say we want to know where

$$y = x^2 + 5x + 6$$

crosses the line

$$y = 2$$

Equating

$$x^2 + 5x + 6 = 2$$

At this stage we need to make it equal to zero, like the area situation in the previous chapter.

This gives

$x^2 + 5x + 4 = 0$

by taking 2 from both sides.

We then factorise and solve as usual

$x = -4$

$x = -1$

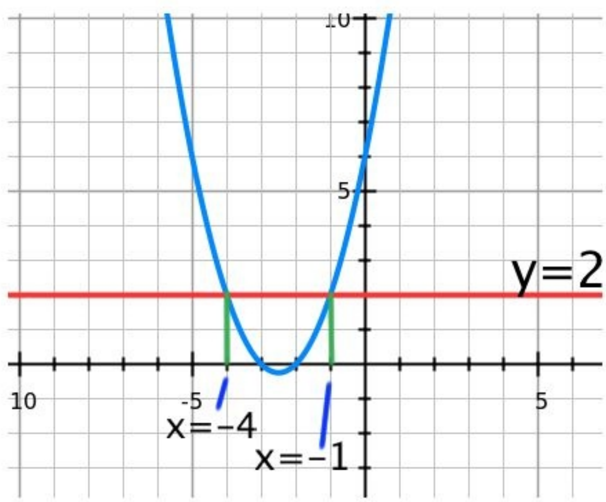
So we find that the curve crosses the line

$y = 2$

At the points,

$x = -4$

$x = -1$



And we can see this is true.

Example 2

Let’s say we we want to find where a quadratic crosses a straight line which isn’t horizontal.

For example let’s say we want to find where
$y = x^2 + 6x + 4$

crosses

$y = 2x + 1$

Again, because we want to know where they cross, we equate them
x$x^2 + 6x + 4 = 2x + 1$

And again, we want to have zero on the right hand side, since this is our trick.

This gives

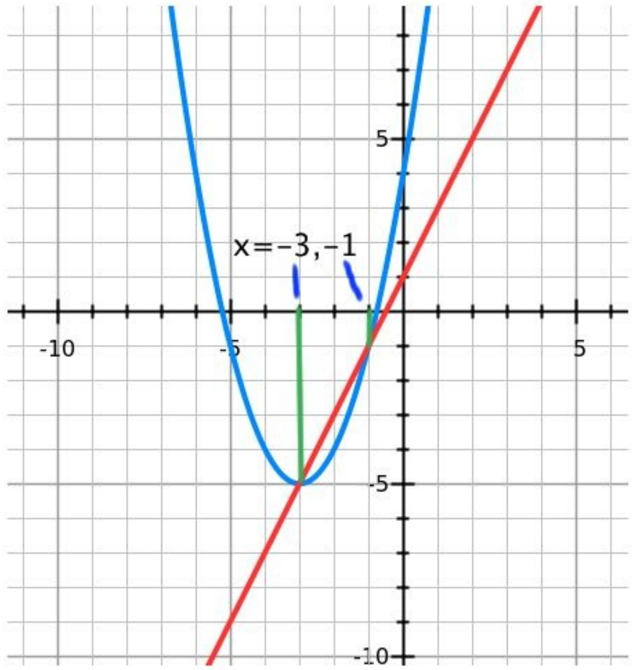
$x^2 + 4x + 3 = 0$

We factorise and solve as usual

$(x+1)(x+3) = 0$

$x = -3$
$x = -1$

Looking at the graph



We see this is true.

Example 3

Finally, we can even see where a quadratic crosses another quadratic!

If we have

$y = x^2 + 3x + 2$

and

$y = x^2 + 5x + 6$

Equating these

$x^2 + 5x + 6 = x^2 + 3x + 2$

Because they both have x^2 terms, we can cancel these to give
$5x+6=3x+2$

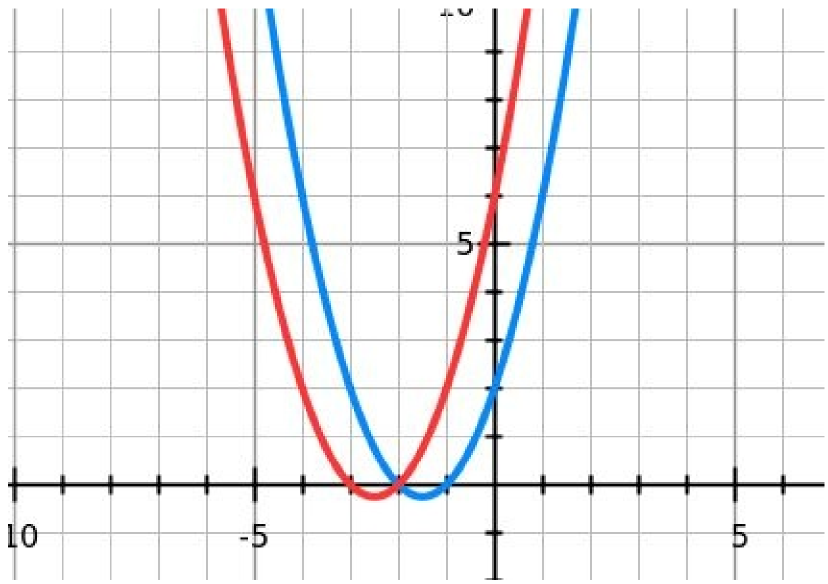
This is Type 3 algebra, and to solve we then have
$2x+6=2$

$2x=-4$

$x=-2$

So they cross at

$x=-2$



Epilogue

Epilogue

Quadratics are of a lot of importance in mathematics. Master the skills in this book. Now you can do a Quadratic - In A Minute. It will give you the tools you need for now and for more advanced maths. You have to deal with quadratics over and over again, not just to solve for x to find where it crosses the x-axis, but in the following areas.

Trigonometry

Matrices
Differential Equations Differentiation

and many others.

Try to achieve a fluency with them so you can do them with ease. Now you understand how they work, and the different ways of solving and finding out information about them, you’re well on your way to being able to tackle all of maths.

Introduction to Inequalities

Introduction

Inequalities are the reverse situation of equations. Up to now, we've tried to solve equations to find out the value of x, and it always gives one (or more) specific values. However, we can also look at the reverse scenario, where, using the Third Rule, we see what happens when one side *doesn't* equal the other.

In this book we will look at the Three Types of Algebra to see how this works. We'll also see how it works with quadratics.

Three Types of Algebra for Inequalities

Three Types of Algebra for Inequalities

In Simultaneous Equations - In A Minute, we first met the Three Types of Algebra. For example, something like

$2x = 6$

With inequalities we look at the same situation, but when they DON'T equal each other. We could have

$2x < 6$

$2x \leq 6$

$2x > 6$

or

$2x \geq 6$

What do these all mean?

Let’s tackle one at a time.

$2x < 6$

Means ‘when is 2x less than 6?’

We solve this as usual, giving

$x < 3$

The answer then is when x is less than 3.

For

$2x \leq 6$

Same again, but this time when x is less than or equal to 6

So for this it would be

$x \leq 3$

Or in other words when x is less than or equal to 3.

For the reverse situations,

$2x > 6$

Here we are asking when 2x is more than 6.

Solving as usual

$x > 3$

The answer is when x is more than 3.

So when x is more than 3, 2x is not surprisingly, more than 6.

Simple stuff!

Finally

When

$2x \geq 6$

$x \geq 3$

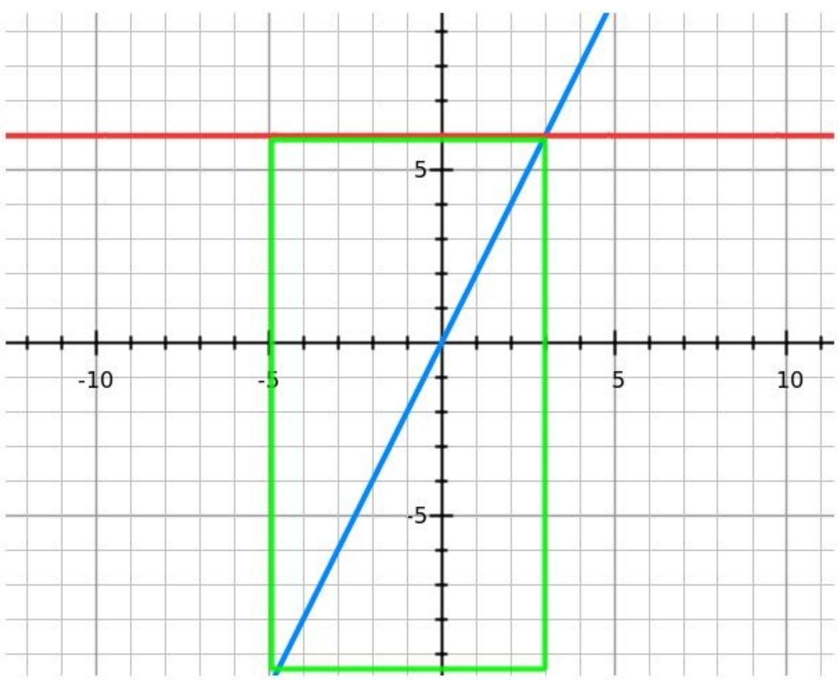
So when x is equal to or more than 3, 2x is equal or more than 6.

They are all the types of inequalities there are.

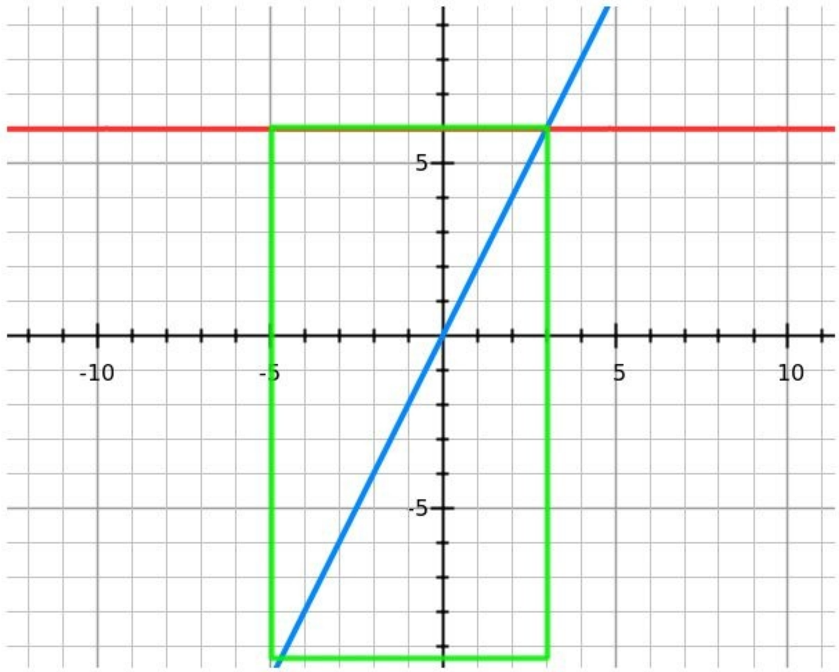
To see what this looks like on a graph, here's a graph of each situation.

As with simultaneous equations, these all represent straight lines. Instead of trying to find out where they cross, we can see where, in fact, they do not cross! We are going to find regions instead of specific points.

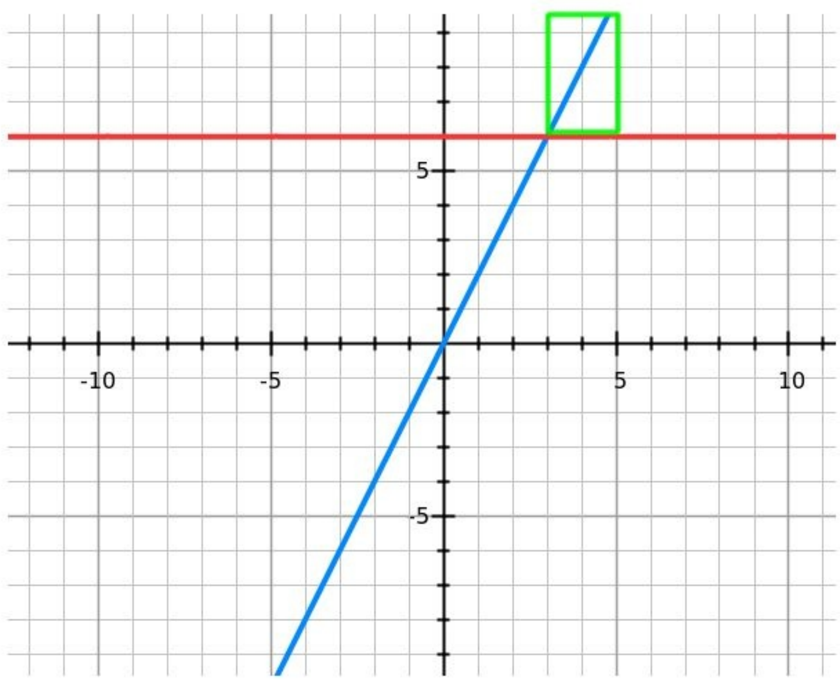
$2x < 6$



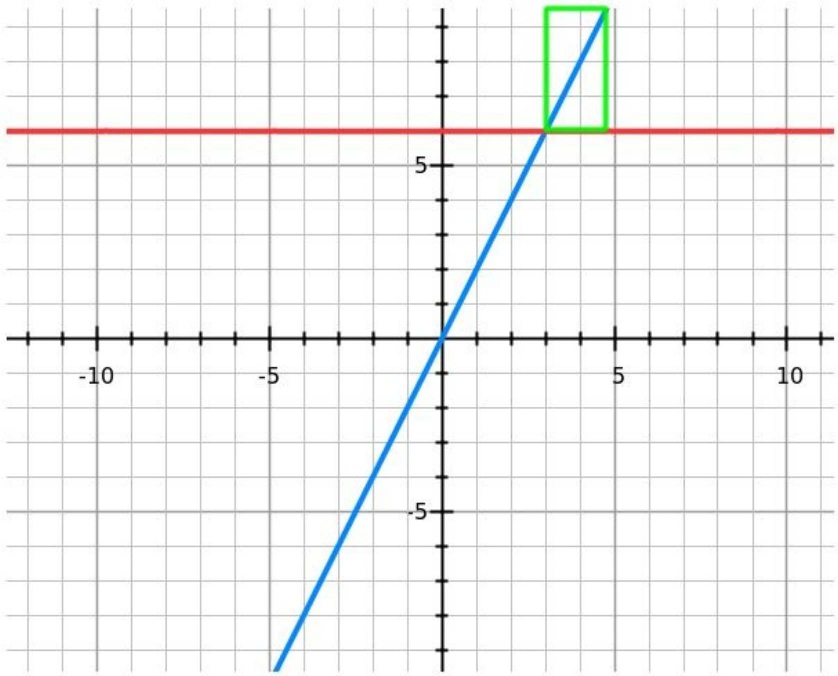
$2x \leq 6$



$2x > 6$



$2x \geq 6$



5
-5
-10
-5
5
10

Type 2 Algebra

Type 2 Algebra

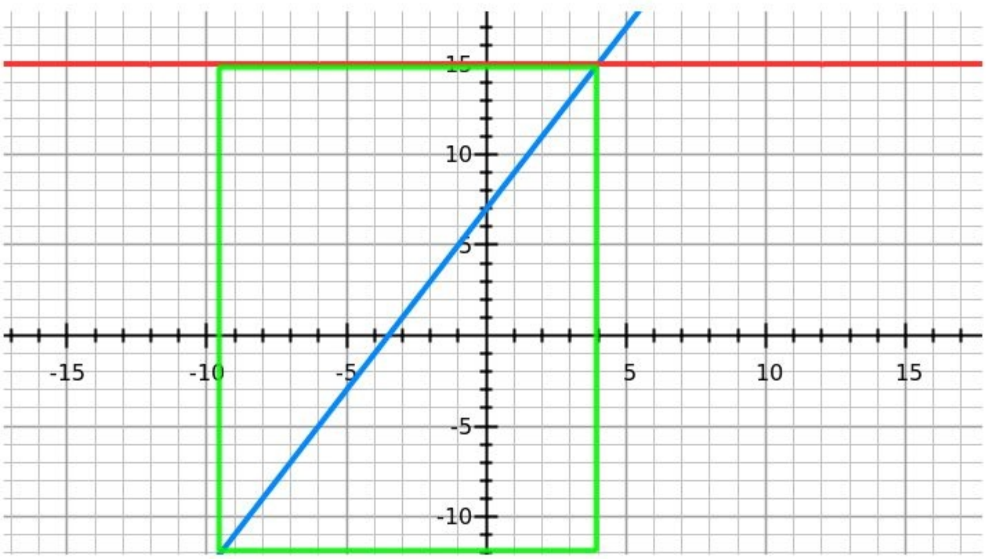
$2x + 7 < 15$

This is exactly the same treatment as we had in Simultaneous Equations, giving

$2x < 8$

$x < 4$

We can see this on a graph



Type 3 Algebra

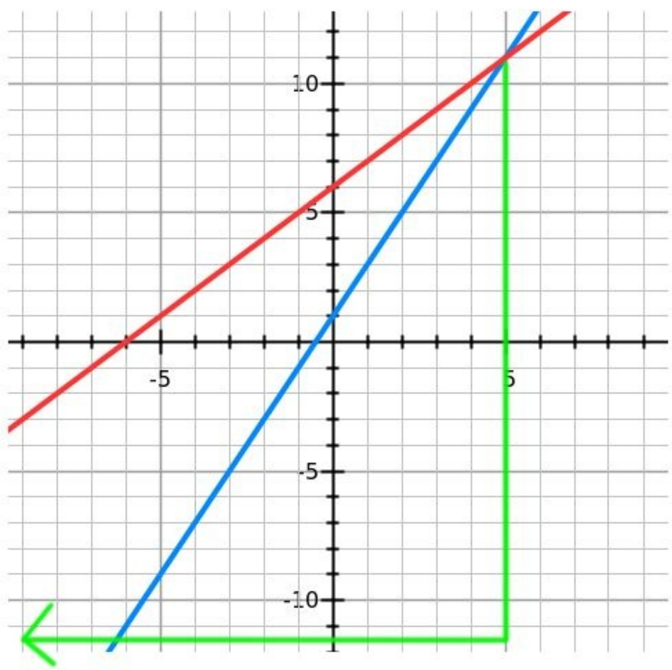
Type 3 Algebra

$2x+1 < x+6$

Again, exactly the same as with Simultaneous Equations!

$x+1 < 6$

$x < 5$



When the Inequality Sign Flips

When the Inequality Sign Flips

If we think back to Gradient, we discovered that there were three types of negative gradient, and these were

$m > -1$

$m = -1$

and

$m < -1$

Remember I pointed out that 'fractions' were in the > -1 region, which is the opposite to the positive gradient situation. For normal division and positive gradients, we saw fractions when the division was < 1.

This direction flip is something we need to bear in mind when we are solving inequalities. For example

$\frac{2}{3} < 1$

But if I multiply the left hand side by a negative (minus 1), this gives

$-\frac{2}{3} > -1$

The sign has to flip for this to make sense.

When solving, if we had

$7 - x < 5$

We can see the answer will have to be when

$x > 2$

How do we get to this?

If we subtract 7 from both sides

- $-x < -2$

Multiplying both sides by minus 1

$x < 2$

wouldn't make sense!

If x was less than 2, we would have
$7-1<5$

which isn't true.

If the sign flips in our answer to
$x>2$

Now we have

$$7-3<5$$

Which makes sense.

The treatment of inequalities is exactly the same as with simultaneous equations, except for this one detail. Watch out for it!

Inequalities of Quadratics

Inequalities of Quadratics

The two simplest situations we can find with quadratics is where the quadratic is either negative or positive.

For each, we just have to bear in mind a different way of writing for each.

For example,
Let’s say we have our usual quadratics of

$x^2 + 3x + 2$

Where is it positive?

If we set it to

$$x^2 + 3x + 2 > 0$$

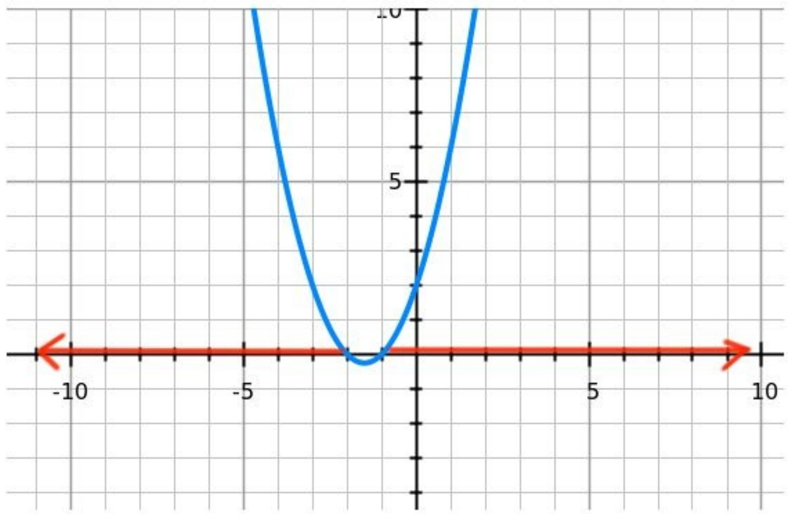
That's what this means.

From here, we solve as usual, by any means we like (factorising, formula, or completing the square).

This gives solutions of

$x = -1$
&
$x = -2$

So we know that the quadratic is positive to the left and right of these values.



We'd write this as

$x < -2$

&

$x > -1$

(be careful to write the lowest number first!)
If we wanted to find the same thing for when the quadratic was negative, we would have

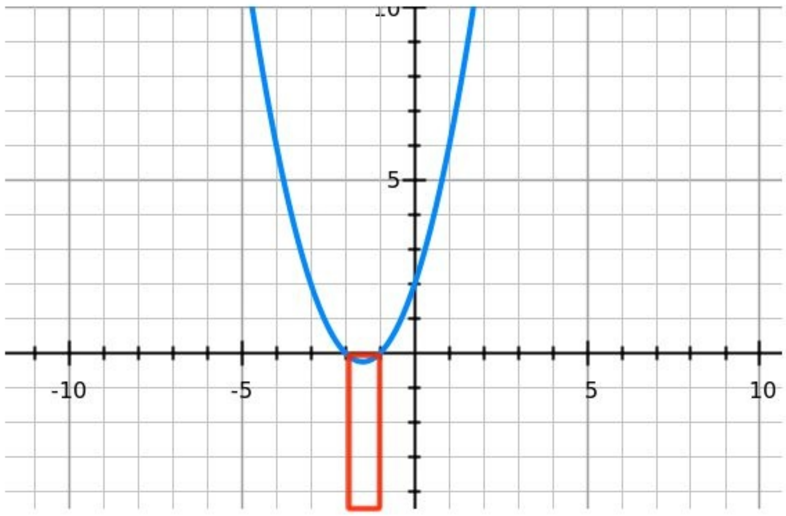
$x^2 + 3x + 2 < 0$

And we'd follow the same method.

Then we'd have a different way of writing as the answers wouldn't be two different regions. In fact, we are just describing one.

This becomes

- $-2 < x < -1$



We can see the region here where the quadratic is negative. This has two borders, so we just write x to be inside them.

Quadratics and Straight Lines

Quadratics and Straight Lines

Of course the x-axis is a straight line. The previous example is really our template for all of these questions.

If we want to know where a straight line is less than a quadratic, again, we can see this on a graph.

However, for accuracy, and speed, we can do this algebraically.

Let's say we want to know where

$y = 2x + 1$

is less than

$$y = x^2 + 9x + 11$$

Again this would be written

$$2x+1<x^2+9x+11$$

or we can write this in reverse

$$x^2+9x+11>2x+1$$

To find these regions, we just treat it the same as in Quadratics - In A Minute, where we wanted to know where they crossed each other.

Firstly we move

$$2x+1$$

over to the left-hand side to give

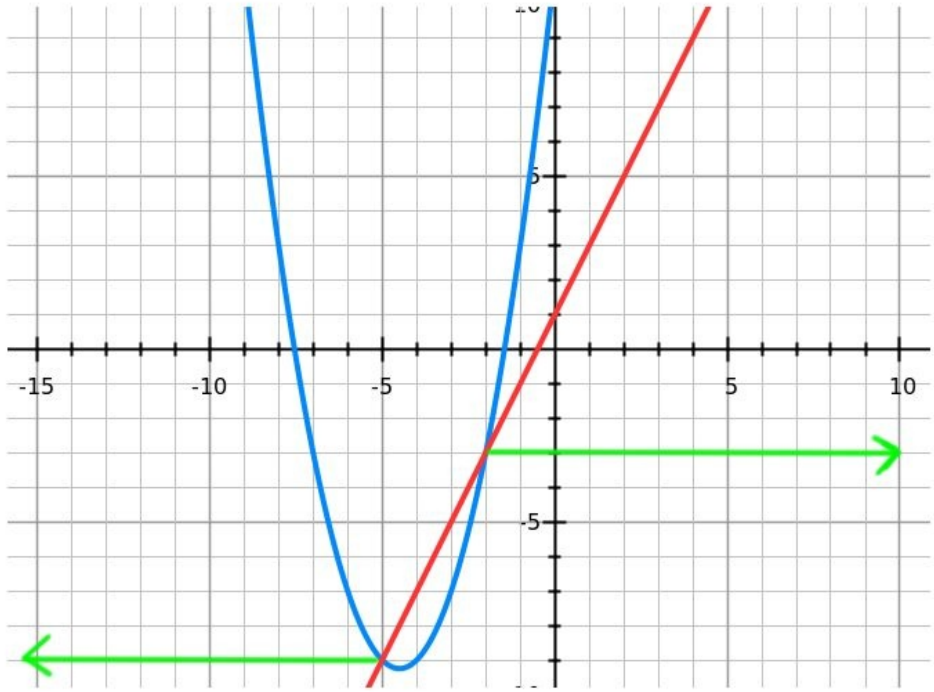
$$x^2+7x+10>0$$

And solve as normal

$$x = -2$$
$$x = -1$$

This then gives

$$x < -5$$
$$x > -2$$



In between - 5 and - 2, we can see that

$$y = 2x + 1$$

is below the quadratic.

Introduction to Changing The Subject

Introduction

Changing the subject…

This doesn't mean let's stop doing maths and talk about something else!

This is sometimes called 'Rearranging Formulae' in schools. I don't call it that because it's a title that is not all-encompassing. In other words, if you learn to change the subject, you can do it with anything, it doesn't particularly have to be a formula. In fact we'll see that we've come across this concept already in the Three Types of Algebra.

In this book we're going to examine changing the subject from simple equations to more complicated expressions. To do this we'll see how we can use BIDMAS - you'll see that this can be used in an unexpected way!

After that we'll look at how solving our 3 types of algebra is essentially changing the subject. We will then look at complicated situations where it's not clear what our first step should be.

Finally, we'll look at some famous formulae in science and re-arrange them.

Using BIDMAS to Change the Subject

Using BIDMAS to Change the Subject

First of all, what is the subject?

If we look at these few equations, the subject is the letter that everything else is equal to.

So for

$E = mc^2$

E is the subject

$v = u + at$

v

is the subject

$T = 2\pi\sqrt{\frac{l}{g}}$

T

is the subject.

To change the subject, this means putting a different letter from the equation in the place of the original. To do this, we must follow some mathematical rules.

You may have come across

BIDMAS

before.

I have not mentioned it as yet in my course. This is because I don’t believe it’s entirely true. The letters stand for
BRACKETS INDICES DIVISION MULTIPLICATION ADDITION SUBTRACTION

and denote the order you should do things if you’re not sure.

For example

3 + 5 x 8

Should be

3 + 40 = 43

as you should do the multiplication before the addition.

So far you’ll have noticed we’ve survived not using this acronym. That’s because I’ve explained certain things in a different way. Another reason I’ve not referred to it is that it has B at the start. This is usually taken to mean that you should ‘do brackets first’ which as we’ve demonstrated in an earlier book, isn’t true.

So to use BIDMAS

I want you to drop the B!

So now only IDMAS.

This is now correct, as we’ll see again.

So IDMAS tells us the normal order of things to do in a question in arithmetic or algebra. That’s fine.

When we change the subject, we follow the Third Rule, and do IDMAS in reverse, in other words
SAMDI

And follow it along.

Examples

Let’s look at a basic example
$a = b + c$

What’s the subject?

a

Let’s say we want to make b the subject.

What we do is run through SAMDI and ask

Is there anything Subtracting from b, the subject?

No.

Anything Adding?

Yes. What? c.

So to get rid of c, we do the complete opposite of adding, and subtract. It then goes on the OTHER side.

This gives

$a - c = b$

We subtract c, and put it with a on the other side.

In this case, this leaves b on its own, so we now have b as the subject, which was our goal.

By convention, we place the subject first, so we write
$b = a - c$

Finished!

Let's look at another example

$a = b - c$

Again, what is the subject?

a

Let's make b the subject.

Running through SAMDI

Is there anything being subtracted from b?

Yes. c

So let’s do the opposite and add c to the other side.

$a + c = b$

Again, we have b on its own so it is now the subject.

By convention

$b = a + c$

As you can see, we just run through IDMAS backwards, in reverse, and this will change the subject for us.

Let's try

$a = bc$

Again, a is the current subject, let's make it b

Is anything subtracted from b?
No.

Is there anything added to b?

No.

Is there anything multiplied to b?

Yes. c

So we do the reverse, and DIVIDE by c on the other side.

This gives

$$\frac{a}{c} = b$$

Again, by convention

$$b = \frac{a}{c}$$

Finally

Let’s say we have

$$a = \frac{b}{c}$$

and we try to make b the subject.

Running through IDMAS

Anything subtracted from b? No.
Anything added? No.
Anything multiplied? No.
Anything divided? Yes! c

So we do the reverse, and MULTIPLY by c.

This gives

$$ac = b$$

and by convention

$b = ac$

The last letter, indices, means that there is a power on the letter we want to make as the subject, so we must reverse this.

For a famous equation like

$E = mc^2$

and we want to make c the subject, we have a problem as it has a square on it!

But we follow the process as normal.

Anything subtracted from c?
Anything added?
Anything multiplied? Yes. m.

So we divide on the other side giving

$\frac{E}{m} = c^2$

Anything divided? No.

Any indices? Yes. The square.

So the opposite is square rooting.

This gives

$$\sqrt{\frac{E}{m}} = c$$

and by convention

$$c = \sqrt{\frac{E}{m}}$$

How We've Used This Before

We have actually seen everything we’ve done so far in earlier books, I just didn’t explicitly say ‘this is changing the subject’.

For example

$2x = 6$

If we say we want to make x the subject, that is, solve this equation, and we follow IDMAS in reverse we could do it that way. Because they are numbers though, and I've asked you to tackle this kind of question intuitively, this hasn't come up.

We could say
Is anything subtracting from x? No.
Anything added? No.
Anything multiplied? Yes. 2.

So divide by 2 on the other side and we have
$x = 3$

So our Three Types of Algebra have all been an exercise in changing the subject.

Exceptions To The Rule

Exceptions To The Rule

There are 3 main exceptions to following IDMAS backwards, and they are little hurdles that need to be recognised and jumped over before using IDMAS at all.

Exception No.1

There is a division in the equation.

For example

$$a = \frac{b+c}{t}$$

and we want to make c the subject.

Because of that division of t, we can't do anything yet. You may remember from Division - In A Minute that division isn't very accommodating. This is why!

So the first step is to GET RID OF THE DIVISION

And do the reverse to give

$$at = b + c$$

We now can follow IDMAS/SAMDI.

$$c = at - b$$

Exception No. 2

Brackets are in the equation.

(this is something we saw in ‘Completing the Square’, as I will describe later).

$a(b+c)=d$

Let’s say we want to make c the subject.

The problem we have is that c is inside brackets, so we have to get it out of there.

So, before using IDMAS/SAMDI, we must multiply the brackets!

This gives

$ab+ac=d$

We now follow IDMAS/SAMDI

To give

$ac=d-ab$
$c=\dfrac{d-ab}{a}$

In the previous example, if we wanted to make a the subject, instead of multiplying the brackets, we could be a little cleverer and just divide. This solves our question in one go.

$a(b+c)=d$

$a=\dfrac{d}{b+c}$

Exception No. 3

The *‘repeated factor’*.

What this means is that the subject we want to change our equation to appears twice.

For example

$ab + bc = d$

And we want to make b the subject.

What would we do here?

Really, we only want one b. To get this, all we have to do is factorise, to give
$b(a + c) = d$

and then, as in the previous example

$$b = \frac{d}{a + c}$$

We could then have combinations of these exceptions. So something like
$$\frac{a}{a + c} = b + c$$

and we want to make a the subject.

GETTING RID OF DIVISION first,

$a = (a + c)(b + c) = ab + ac + bc + cc$

We now have three a's!

Bringing them all together

$a - ab - ac = bc + cc$

Factorising

$a(1 - b - c) = bc + cc$

and dividing by the bracket

$$a = \frac{bc + cc}{1 - b - c}$$

We could also factorise the top line as it contains c twice.

$$a = \frac{c(b + c)}{1 - b - c}$$

Here I've factorised the top line - but why? Why bother? This is answered in the book 'Algebraic Fractions - In A Minute'.

Another example would be

$$(a + c)^2 = b$$

Let's say we want to make a the subject. Again, this is trapped in a bracket, but worse, this bracket is also squared. Before we can apply IDMAS/SAMDI, we need to square root just to make it a bracket at all!

$$a + c = \sqrt{b}$$

$$a = -c + \sqrt{b}$$

This is what we saw in 'Completing the Square' which led to a fast way to solve a quadratic, by square rooting the left-hand side to make x the subject.

$$y = x^2 + 3x + 2 = 0$$

which is

$$(x + 1.5)^2 - 0.25 = 0$$

We now shuffle this equation around to find x

$$(x + 1.5)^2 = 0.25$$

Square root both sides

$$x + 1.5 = \pm\sqrt{0.25}$$

$$x = -1.5 \pm \sqrt{0.25}$$

which is

$$x = -1.5 \pm \frac{1}{2}$$

Here we are making x the subject, although it is locked up in a bracket and squared. You can see we follow IDMAS in reverse to make it the subject, as soon as we get rid of that squared bracket.

Famous Science Formulae Rearranged

Famous Science Formulae Rearranged

$F = ma$

This is Newton’s 2nd law of motion, which tells us how much force you get from an accelerating mass.

Making a the subject

$$\frac{F}{m} = a$$
$$a = \frac{F}{m}$$

$$F = \frac{Gm_1m_2}{r^2}$$

This is Newton's law of Gravitation, which calculates the gravitation force between two bodies (things) that have mass (almost everything!).

Making G the subject…

$$G = \frac{Fr^2}{m_1m_2}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

This is the formula which describes simple harmonic motion. In other words, if a pendulum oscillates regularly, we can calculate how long it will take based on its length.

Making g the subject

$$T^2 = 4\pi^2 \frac{l}{g}$$

$$g = 4\pi^2 \frac{l}{T^2}$$

Introduction to Cubics

Introduction

In Quadratics - In A Minute we saw what happened when we multiplied two brackets. This gave a rectangle, and a parabola, and a new expression, which began
$x^2 + ...$

In Cubics, we will look at what happens when we take this one step further, and multiply by another bracket. We will see a new and simple way to do this, which was taken from Multiplying - In A Minute.

From then on we'll see what shape we get this time, what information we get about a cubic, how to factorise and solve them, and whether we can find their minimum (and maximum!) values.

This book is suitable for A-level in UK and will demonstrate advanced techniques to achieve things that take much longer using traditional school methods.

Firstly, let's look at multiplying 3 brackets!

Multiplying Three Brackets

Multiplying Three Brackets

Here, let's say we choose to multiply
$(x+1)(x-2)(x+3)$

For the first two brackets, we could multiply them using the column method from Quadratics, viz:
$(x+1)$

$(x-2)$

Giving

$x^2 - x - 2$

This would then need to be multiplied by

$(x+3)$

Instead of multiplying each term and collecting like terms, we can do this by the 3 x 3 method of the 'Union Jack Situation' we saw in Multiplying, viz:
$x^2 - x - 2$

$\times (x+3)$

giving

$x^3 + 2x^2 - 5x - 6$

If you're not sure how I did this, here's a reminder of that method for numbers.

…So for this method there are 5 steps. 4 of them will be exactly the same as you have already seen, the 5th but, middle, step is the 'Union Jack Situation'.

Let’s take a look at the multiplication above:

123 x 421

Place in a column

  123
x 421

Step 1:

Exactly the same: right hand column.
  123
    ⇩
x 421

-----

3 x 1, gives

  123
    ⇩
x 421

    3

Next step, exactly the same: cross
  123

⨯
x 421

3

So ignore the 1 and 4 on the left column, we pretend they're not there and then this gives, 2 x 1 + 3 x 2 = 8

So

123
x 421

83

So far, this has been exactly the same. Here, we see the new 'Union Jack Situation' come in.

We now place our union jack in between the numbers, viz:

123
✱
x 421

83

So we have 3 multiplications to do and remember 3 answers! After achieving fluency with the 2 x 2 method, this becomes easy.

So we have, multiplying along the lines,
1 x 1 + 3 x 4 + 2 x 2
= 1 + 12 + 4

= 17

So we now have

  123
x 421

$_{1}$783

So that’s 3 out of 5 steps!

Steps 4 and 5, as I mentioned above, now follow the symmetry of mathematics.

Step 4

  123
  X
x 421

$_{1}$783

Will not ignore the right hand column completely.

So we have 1 x 2 + 2 x 4 = 10

Adding the carry of 1, gives us 11.

  123
x 421

$_{1}$1$_{1}$783

Step 5 is simply multiplying the left column.

123
⇩
x 421

$_1 1 _1 783$

1 x 4 = 4
Plus the carried 1, = 5.

Giving us

123
x 421

$5_1 1_1 783$

Therefore

123 x 421 = 51 783.

I have gone through each step very slowly and carefully as it will be the first time you've seen it. With practice, you will be able to do this without writing any crosses or union jacks and just write down numbers, to lead up to the answer!

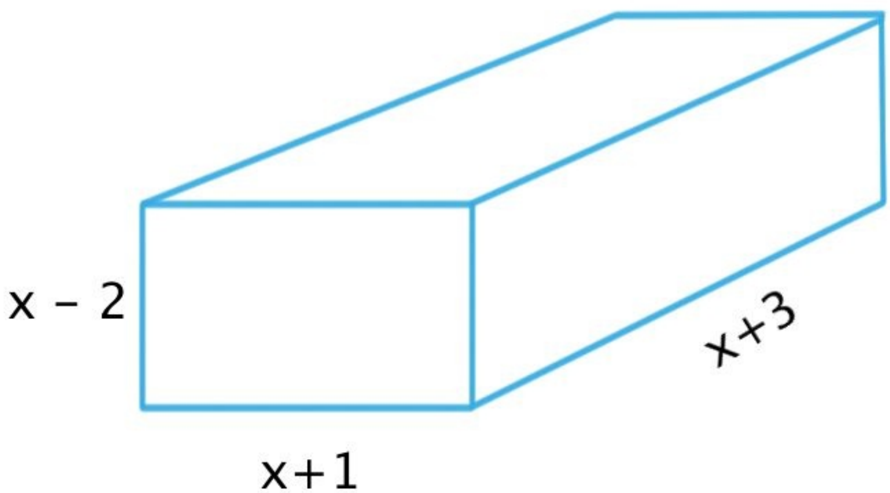
You can watch this live video of me doing a 3 x 3 digit multiplication of 124 x 132 at this YouTube link.

This is a cubic. So called as its highest power is x^3.

We saw in **Quadratics** that two brackets gave both a rectangle and a parabola. What shape do you think this will give?

This will give a

cuboid



And what information will that give us about the cuboid?

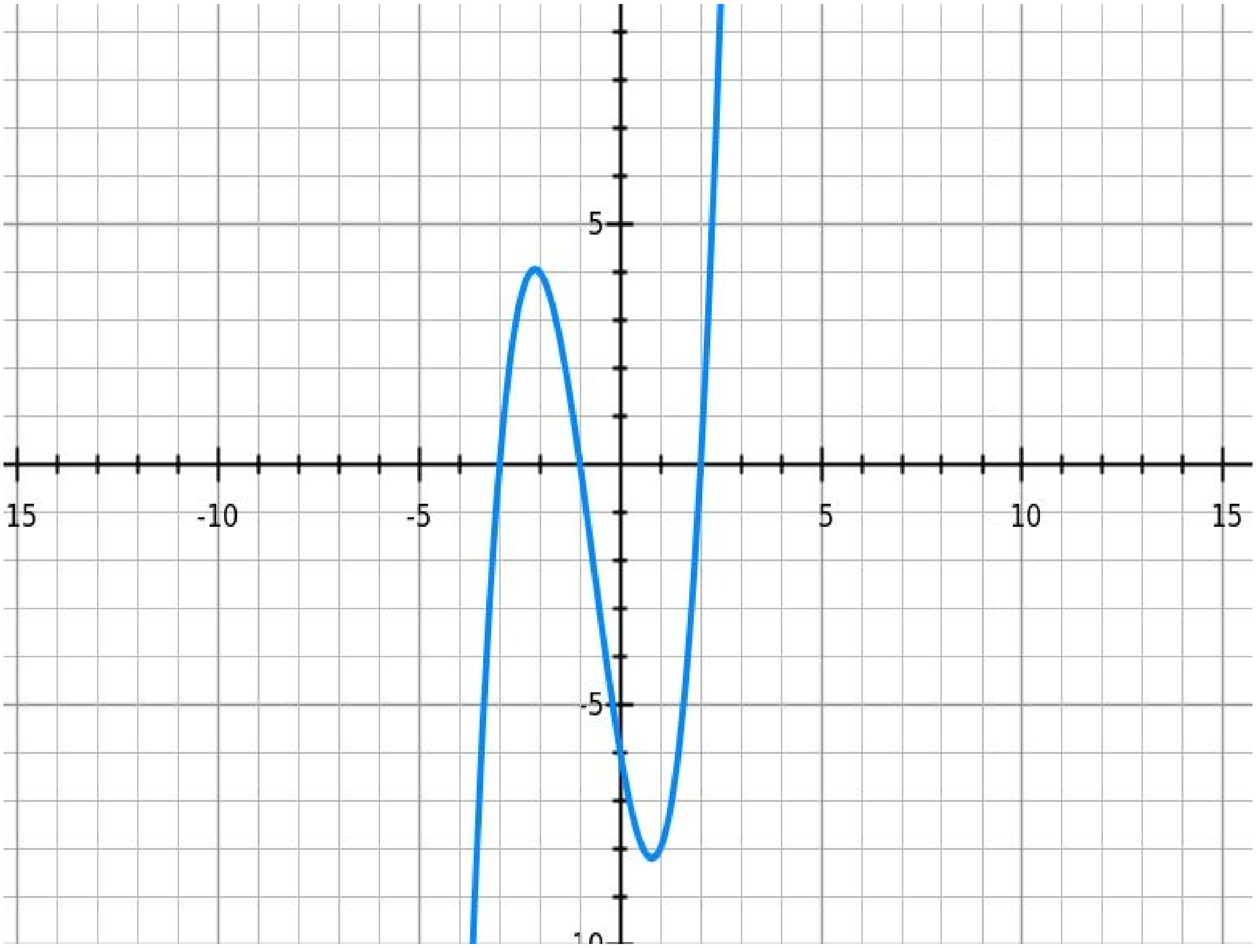
The volume.

The image on the graph we can find by plotting points using a table.

$y = x^3 + 2x^2 - 5x - 6$

x	- 3	- 2	- 1	0	1	2
$y = x^3 + 2x^2 - 5x - 6$	0	4	0	- 6	-8	0

If we plot these points, we’ll see it gives an unusual looking graph, which isn’t symmetrical.



This is because it is an ‘odd’ function. A quadratic, being symmetrical, is an even function.   This will be examined in more detail in ‘Functions - In A Minute’.

We can see from this table where the solutions of the cubic are here. y = 0 when x equals - 3, - 1 and 2.

Once we know how to multiply these brackets out, using the Third Rule, we will want to know how to return to the brackets, or factorise, this cubic.

Factorising a Cubic

Factorising a Cubic

As we saw in Quadratics,

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

So to factorise this cubic, we need to find one factor, and using that find the others.

If we demonstrate this with the same cubic we've just found, we can see how it works.

We know in this cubic that the solutions are

$x = -3$

$x = -1$

$x = 2$

This means that

$x + 3 = 0$

$x + 1 = 0$

$x - 2 = 0$

As x has these values.

In other words, these are the factors of the cubic. You'll note that these are exactly the same as the brackets we started off from in the previous chapter. *These brackets multiply to form the cubic*, so they are its factors.

But what if you don't have the solutions to start off with? What if you want to find the solutions without drawing the graph?

Again,

To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.

So our first step will be to find at least one of the factors of the cubic. To do this, since we know that they are linked to the solutions, we could just 'try' solutions and see if they give zero.

Here we should be a little clever. Don't just try ANY solution. Use numbers that are factors of the 'cut'. The number on the end of the cubic, not multiplied by x . Our number for c is

- c = - 6

So try
1, -1
2, -2
3, -3
and
6, -6

So let's start...

Let

$x = 1$

$y = (1)^3 + 2(1)^2 - 5(1) - 6$

we see that this doesn't give zero. In fact, it gives - 8 as we saw for our table.

Let's try

$x = -1$

This gives

$y = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

This does give zero, so we know that

$x = -1$

is a solution and

$x + 1 = 0$

$(x + 1)$

is a factor.

Now we have one factor, we can use it to find the others.

Just like the algebra trick for division, we can ask, what times by $(x+1)$ to give the cubic?

So if we write

$(x + 1)(.........)$

We can fill the second bracket with a quadratic which will multiply to give that cubic.

The first term will be x^2, in order to give x^3 when they multiply.

$(x + 1)(x^2 ....)$

$$(x + 1)(x^2 \qquad)$$

The final term, the number, will have to multiply by 1 to get - 6.

Therefore it must be - 6

$$(x + 1)(x^2 \quad -6)$$

$(x+1)(x^2 ......-6)$

The middle term of the quadratic we can find in two different ways.

Way 1.

We need to see how many x^2's we need to make the same number in the cubic.

In the cubic we have

$2x^2$

In the multiplication we already have

$1\times x^2 = x^2$

$$(x + 1)(x^2 \quad -6)$$

So we'll need another x^2 to make it to

$2x^2$

In the middle then we'll have

$(x+1)(x^2+x-6)$

$$(x + 1)(x^2 + x - 6)$$

That’s it!

We’ve found the other factor that multiplies by
$(x+1)$

to get

x^3+2x^2-5x-6

or

Way 2.

We can see how many $x's$ we need to get the same in the cubic.

In the cubic we have

- $-5x$

To get this, we can look at our - 6 and note that when it multiplies by x, we get

- $-6x$

So we need just a single positive x to get it to

- $-5x$

Hence we just need x in the centre.

Factorising a Cubic 2

Now to find the other two solutions/factors, we factorise/solve the quadratic we now have.

$(x^2 + x - 6)$

This gives

$(x+3)(x-2)$

From **Quadratics - In A Minute**

From here, we have factorised the cubic entirely. This gives
$(x+1)(x+3)(x-2)$

Equating the cubic to zero will tell us where it intersects the x-axis, as again, we saw with Quadratics.

Therefore

$x^3+2x^2-5x-6=0$

and we can write

$$(x+1)(x+3)(x-2)=0$$

since this is like a ‘pre-multiplied’ version of that same cubic.

If these terms all multiply to equal zero, as in Quadratics, we can say that each bracket must equal zero.

Or, if these brackets multiply to form a cubic, which as volume, then if the volume of the shape is zero, then its dimensions must be zero.

So

$$x+3=0$$

$$x+1=0$$

$$x-2=0$$

As we saw already.

Then for these to make sense, x must be equal to

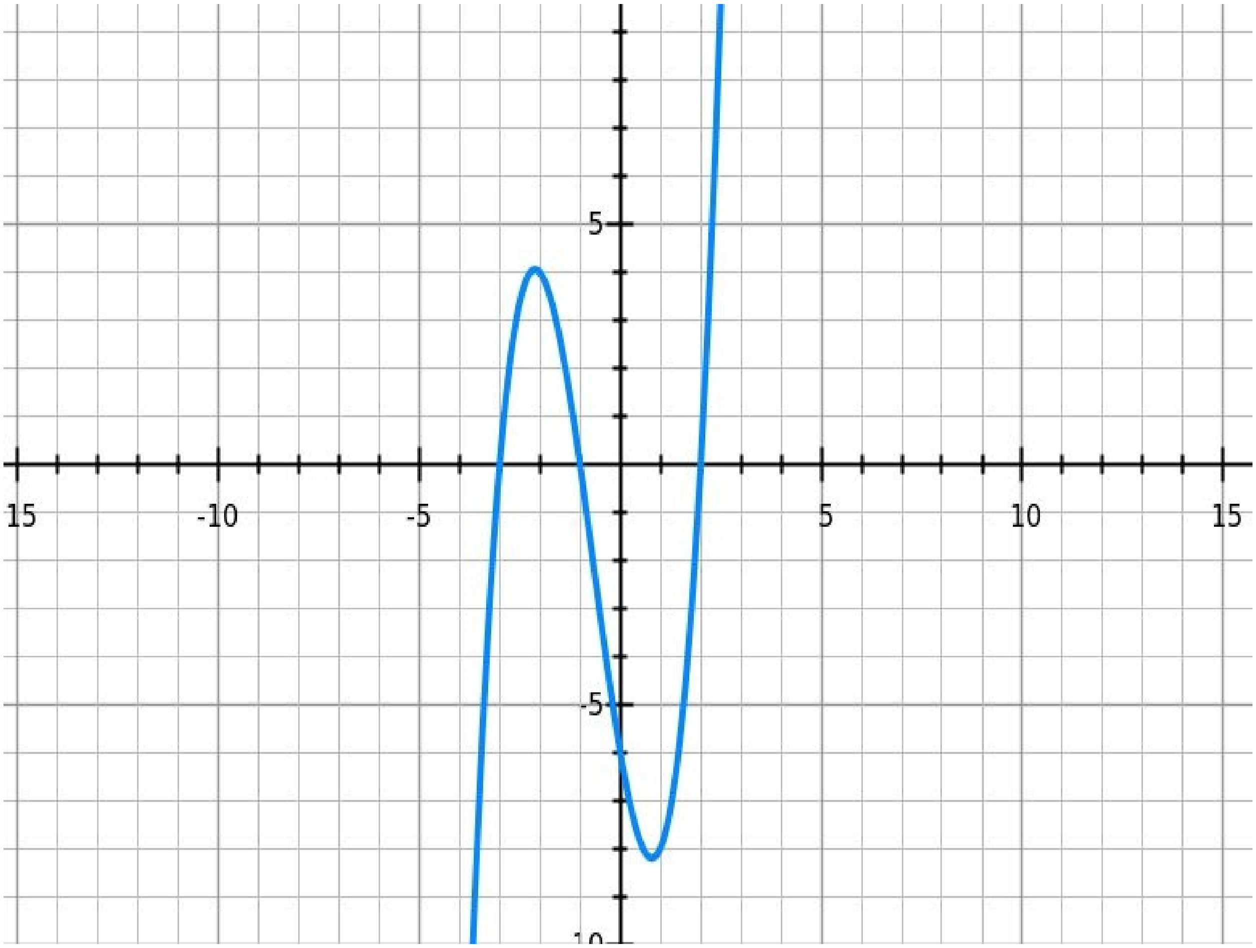
$$x=-3$$

$$x=-1$$

$$x=2$$

which is where the cubic crosses the x-axis.

We can see this on the graph too.



This idea that when you multiply factors together to get an answer is technically known as the ‘Factor Theorem’ which is just a complicated way of saying that *To factorise, we use one factor to find the others. A factor is something that divides in to the original expression or number.*

That is, it is possible to factorise a cubic... with its factors. That’s all.

Remains of The Day

Remains of The Day

So of course, we have the reverse of this situation.

Either an expression like

$(x+1)$

will be a factor of a cubic, or it won't.

If it's not, all that happens is that it won't divide in exactly.

This is just like

$$\frac{15}{4}$$

Four is not a factor of 15, as it doesn't go in exactly. We see this because when we divide them there is a remainder.

Similarly for algebra, if we were to divide our cubic
$x^3 + 2x^2 - 5x - 6$

by

$(x + 4)$

because it's not a factor (as we know now) it won't divide in exactly. As a result there will be a remainder.

The imaginatively titled **'Remainder Theorem'** is the name for this situation. To find the remainder, we just use the same method for when we wanted to check whether an expression was a factor, by writing it as
$x + 4 = 0$

So we try

$x = -4$

in the cubic.

If we try this we get

$y = (-4)^3 + 2(-4)^2 - 5(-4) - 6$

this gives

$y = -18$

And since this isn't zero, we know that

$(x+4)$

isn't a factor.

If we had done a division where the cubic was divided by
$(x+4)$

we would have seen that this was our remainder.

Hence the name 'Remainder Theorem'.

In other words, *if our remainder is zero,*

it is a factor

If our remainder is non-zero,

it is NOT a factor.

You've been doing this with numbers for years, so this is the same concept.

As yet, I haven't described how to actually do this.

How to divide a cubic by an expression such as

$(x+4)$

The reason I haven't explained it is because I don't want you to!

There is a way of finding the answer without having to divide at all.

Again, we just need to imagine what we need to multiply
$(x+4)$

by.

First of all, we need to find the remainder.

We do this, as we've said, by substituting

$x = -4$

This gave

$$y = -18$$

as we saw.

If we think back to our division example above with numbers

$$\frac{15}{4}$$

We know the remainder is 3.

If we subtract that from 15 we get 12.

THEN we can figure out what multiplies 4 to get 12 (also 3).

Since we subtracted the remainder, we were able to perform the division.

We do the same for the cubic.

We subtract the remainder from it, giving

$$x^3 + 2x^2 - 5x - 6 - (-18)$$

giving

$$x^3 + 2x^2 - 5x + 12$$

(Since that minus times a minus is a plus, from Negative Numbers - In A Minute)
We can then figure out what multiplies

$$(x + 4)$$

to get this.

We use the same bracket technique

$$(x + 4)(..........)$$

and fill the second bracket with terms.

Again, to start with

$$(x + 4)(x^2 ....)$$

As they multiply to get x^3

To get 12, we'll need + 3

So we have

$(x+4)(x^2....+3)$

And finally to get the x term in the centre, we will need

- $-2x$

To reduce our x^2 terms to $2x^2$.

Giving

$(x+4)(x^2-2x+3)$

If we now add on our remainder, the answer to the division of the original cubic by

$(x+4)$

will be

$$(x^2 - 2x + 3) - 18$$

Which we can check by multiplying out again

$$(x+4)(x^2 - 2x + 3)$$

$$(x^2 - 2x + 3)$$

X $\quad (x+4)$

which gives

$x^3 + 2x^2 - 5x + 12$

if we then add the remainder of - 18

we get

the original cubic

$x^3 + 2x^2 - 5x + 12$

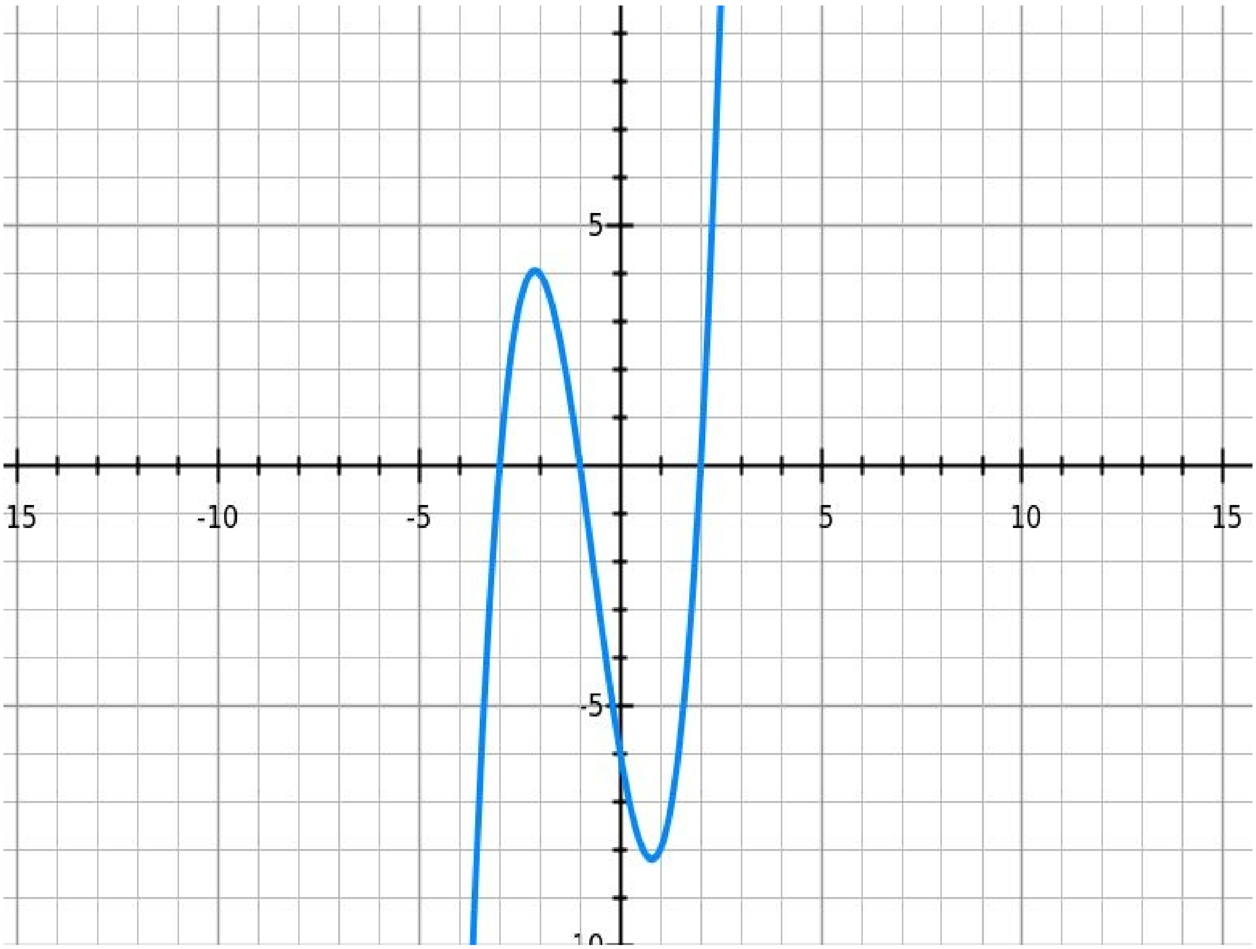
We can divide a cubic like this for any expression.

We DO NOT need to use long division. EVER!

Finding the maximum or minimum values

Finding the maximum or minimum values

You can see from the graph of the cubic that we have a maximum and minimum value.



In Quadratics, we spent a lot of time finding different and easy ways to find the minimum value (or maximum, depending on whether it was a negative quadratic). For cubics, there's no magic algebraic way of doing it. You can't do it in your head like you can for Quadratics!

This is because a cubic is an 'odd' function. By that we mean the reverse of 'even'. An even function is symmetrical. An odd function is not. As a result, we have no algebraic way of finding these points.

But it IS possible to find them.

To find out how, read Book 25 - Gradient/Differentiation 1.

Introduction to Advanced Mental Multiplication

Introduction to Advanced Multiplication Techniques

In this book we're going to examine some ultra-fast ways to multiply numbers. In the original book, Multiplication - In A Minute, we saw a technique to apply for every situation for the times tables, and then for numbers that are 2 digits in size, as well as 3, and 4 digits.

We then saw in further books this method being applied for decimals, percentages, standard form and quadratics. Later in the book, I will explain the algebra behind how this works. This is what I call 'Algebraic Arithmetic'. However I don't call it this at the start as it's somewhat of a scary name. However, that is why the book has this subtitle.

We shall now 'move on' from this technique a little, by seeing if sometimes we can calculate mentally (in our heads) instead of relying on pencil and paper.

For this we're going to rely on one method for this for both single digit and double digit numbers.

It is a method taught in 'Squaring - In A Minute', but I will outline it again here. We'll then see it applied to everything.

Once we've looked at examples such as

7 x 9

and

19 x 23

we’ll look at squaring larger numbers manually.

We’ll also then look at ‘doing the reverse’ of this, square rooting these back again. Once we understand how this works, we’re away.

After that, we’ll look at a way of becoming more familiar with square numbers, by using them to find other square numbers.

We’ll examine how we can check our answers are correct, in seconds, and without having to divide.

We’ll then use this technique to perform a couple of magic tricks - useful for parties! This will have your friends scratching their heads and thinking you are some kind of wizard.

To finish, we’ll look at the algebra behind these systems, and see how they work. We will find them very simple, and you’ll likely be amazed they are not more widely known.

The Times Tables

The Times Tables

In Book 1, we saw how to find the answers to times tables - by which I mean single digit numbers, such as 7 x 9 - using 3 possible methods, whatever was your preference.

In this more advanced book we're going to use square numbers to find them all.

In other words, the numbers from this table in red. When the same numbers are multiplied, instead of getting a rectangle, we get a square.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Firstly, you may not know the square numbers up to ten. That's fine. We can use my squaring system for this also. Here's a brief reminder. For example: For

7^2

We go to the nearest ten, which for numbers we're dealing with at the moment, is always 10.

This is 3 away.

We then go in the opposite direction for the same distance.

This takes us to 4.

We multiply together, giving

4 x 10 = 40

The distance we had to move to the nearest ten, in this case 3, we square and add on.

$3^2 = 9$

40 + 9 = 49.

So that's

7^2

We can use this for all the squares from 6 onwards.

The first 5 you just need to learn. We had to know 3-squared for this one! You can do this by adding repeatedly every time, since Multiplication Is Just Addition. There is also a way of doing it using our above method. However this doesn't use the nearest ten (why not?).
What we could is always move a distance of 2, so we know we always add on 4.

For 4 x 4

We could do 2 x 6 + 4 = 16

For 3 x 3

1 x 5 + 4 = 9

For 2 x 2 of course we know it is 4, since that is our usual square to use!

And 1 x 1... hopefully you know that one by now, since we've seen it in AREA, as a definition for a metre square.

Using Squares For The Times Tables

Let's say we want to find

5 x 7

All we do is look for the number exactly in between 5 and 7.

This is 6.

We *square* this.

$6^2 = 36$

We then *square and subtract* the difference we had to go to get from 5 or 7 to 6. This is 1.

$1^2 = 1$

36 - 1 = 35

Another example

5 x 9

Here our middle number is 7.

We square this.

$7^2 = 49$

We then square the difference and subtract.

$2^2 = 4$

49 - 4 = 45

How about

8 x 12

We square 10.

$10^2 = 100$

Square and subtract the difference, 4,
100 - 4 = 96

As you can see we can use this method quite easily, especially if we know the square numbers. As a result, out of all of those numbers in the table, you only have to know the squares!

Another one

3 x 5

$4^2 - 1 = 15$

3 x 7

$5^2 - 4 = 21$

7 x 11

$9^2 - 4 = 77$

We can do any multiplication in this way. And in our heads!

Odd Number Differences

So far, you may have noticed that the differences are always a symmetrical distance apart, so that when we find the number in the middle, it is always a round number, rather than something like 7.5.

We can tackle this problem in one of two ways. The first way is recommended, but if you want to have some fun, use the second method.

Let's say we want to do

7 x 8

Probably the best way would be to square either number and then add or subtract as necessary.

So

$7^2 = 49$

So 7 x 8 = 56

Since we add on an extra 7.

Or we could do that with 8^2

Giving

64 - 8 = 56

Another example

5 x 8

Here our middle number is 6.5. If instead we make it easier by looking at

6 x 8

We could then do

$7^2 - 1 = 48$

6 x 8 = 48

So

5 x 8 = 40

This is all done in your head, and comes with practice and fluency.

Another way to do it would be to use the fact that we know how to square numbers ending in 5. Again, looking at Squaring - In A Minute
We can do

75^2

by again looking at the nearest ten.

This is 70 AND 80.

Then we just do

7 x 8 + 25

= 5625

6.5^2

would be

42.25

For our example above

5 x 8

We could do our middle number squared and then subtract the square of the difference.

Again this difference will end in 5, so this is just another simple one to do.

For 5 x 8

We have

$6.5^2 - 1.5^2$

42.25 - 2.25 = 40

This will always work out neatly like this.

4 x 7

$5.5^2 - 1.5^2$

30.25 - 2.25 = 28

4 x 9

$6.5^2 - 2.5^2$

42.25 - 6.25 = 36

Once we master this squaring technique, we can easily see how to calculate all our times tables, just based on that diagonal in that grid, plus knowing squares that end in 5.

In practice, you might just want to use the first method outlined for odd differences as it is the simplest.

But this second method can also be used for two-digit numbers also.

How to Multiply Two Digit Numbers In Your Head

How to Multiply Two Digit Numbers In Your Head

The previous chapter took care of all single digit multiplications. We saw we could tackle these in a couple of ways, and it is quite surprising, I think, that you can take these different routes, but end up at the same destination. That is the beauty of arithmetic, algebra and mathematics.

To do two digit numbers, such as

17 x 23

We can use exactly the same method. Again, we need to rely on our squaring method from Squaring - In A Minute.

Make sure you achieve fluency with this.

For example for

17 x 23

This is quite easy.

Our middle number is 20.

Squaring this

$20^2 = 400$

We then subtract the square of the difference

3^2

400 - 9 = 391

Finished!

How about

29 x 33

Middle number is 31.

Square this.

$31^2 = 961$

Subtract the square of the difference (4)
961 - 4 = 957

The real advantage of this method is being able to do this mentally. You can just quickly think of the square and you're nearly at the answer.

Another example

41 x 49

Easy...

$45^2 - 16$

2025 - 16

= 2009

76 x 82

We can think of as $79^2 - 9$

6241-9

= 6 232

And so on.

For numbers that are far apart, such as
78 x 28

We can again use the same method, but with a larger difference between them.

In this case we have a difference of 50

So that’s

$53^2 - 25^2$

2809 - 625

= 2 184

Again, familiarity with squares and my squaring system is vital here.

Another...

94 x 38

Difference = 56

So half-way point is 38 + 28 = 66

Squaring...

$66^2 - 28^2$

4356 - 784

= 3 572

37 x 27

$32^2 - 5^2$

1024 - 25

= 999

107 x 85

$96^2 - 11^2$

9216 - 121

= 9 095

Odd Number Differences

Let's say like the last example we have
107 x 86

In that case we can do

107 x 85 + 107

Calculating

= 9 095 + 107

= 9 202

Or

23 x 28

We can do 24 x 28 - 28

$26^2 - 4 - 28$

676 - 32

= 644

Finally

93 x 38

We can do

94 x 38 - 38

$66^2 - 28^2 - 38$

4356 - 784 - 38

= 3 534

Once you become familiar with the squares, these multiplications become easier and easier.

We start to read multiplications in terms of the squares needed instead of the standard way.

For example

35 x 43 can now read

$38^2 - 9$

and

57 x 61

$59^2 - 4$

Squaring Large Numbers

Squaring Large Numbers

In Squaring - In A Minute, we looked at squaring up to around 105, or thereabouts. I also hinted it was possible to square larger numbers.

We can do something like
218^2

By doing

$2 \times 236(00) + 18^2$

47200 + 324

47 524

Simple!

Or

397^2

4 x 394(00) + 9

160000 - 2400 + 9 = 157 609

Here I did 4 x 400(00) and subtracted 6 400s, or you could just do
394
$\times 4$

$15_3 7_1 600$

- +9

----

157 609

So we can do any square by just using a different ‘nearest ten’, that is, using anything as our focal point to square around.

One final example

825^2

8 x 850(00) + 625

= 680 625

It is amazing how quickly this method works, with some basic times tables and a few squares as our tools!

Discovering More Square Numbers

One good way to find more square numbers is to multiply two square numbers together.

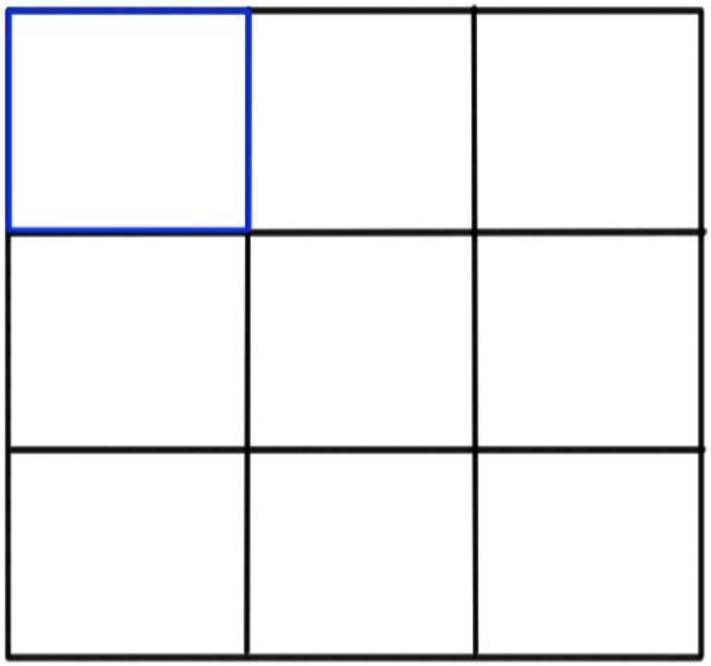
For example,

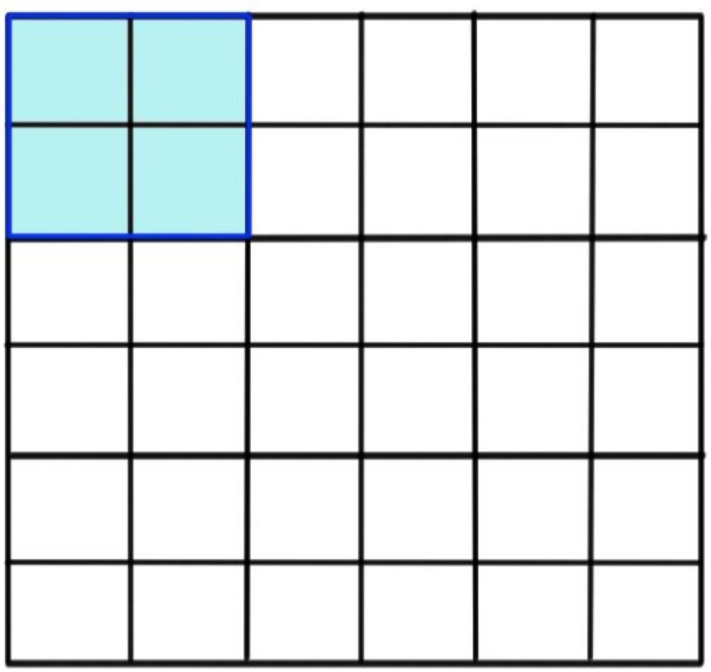
4 and 9

are both square numbers. When we multiply them, we get another square number.

4 x 9 = 36

This is because we can fit more squares into the original square.





If we want to have a general ‘look around’ we can keep doing this to see what square numbers there are out there.

For example,

9 x 49 = 441

25 x 81 = 2025

And so on.

All these answers will be squares.

And to find out what squares they are is very simple.

All we have to do is square root each number and multiply them.

For example

25 x 81

5 x 9 = 45

So

$$45^2 = 25 \times 81 = 2025$$

Or

9 x 49

3 x 7 = 21

$$21^2 = 9 \times 49 = 441$$

We can discover whatever we like, and it’s a very simple way to have a look for ourselves, out of interest.

Finding Square Roots from Square Numbers

Finding Square Roots from Square Numbers

Once we get some large square numbers like
2025

How can we reverse this process, like the Third Rule says we can, to find out the original square root?

As we've just seen, we can do it if we know the origin of the number from multiplying two other squares.

But what if you know it is a square number, but you don't know its origin?

For example

4225

is square. What is its square root?

To find this, we look at the ending.

We see it ends in 25. This immediately tells us that the square will end in 5.

Looking at 42, we note that

$6^2 = 36$

and

$7^2 = 49$

49 is too large, so it must be 6.

So the square root is

65!

Easy.

What about this square number?

1156?

The number ends in 6, which tells us that it must have been squared from a number ending in 4, since

$$4^2 = 16$$

Again, we know that

$$3^2 = 9$$

and

$$4^2 = 16$$

So

$$30^2 = 900$$

$$40^2 = 1600$$

Therefore it must be

34.

However, couldn't it be

36?

This would also generate a number ending in 6.

To test this, we just square it back to see if it gives us 1156.

34^2

$34^2 = 3 \times 380 + 16$

1140 + 16.

1156

So it was 34^2

For all squares that don't end in 5, there will be two possibilities, so we must try and confirm it is one or the other.

Now try

1849

This ends in 9, implying our square will end in 3, or 7.

We know that

$40^2 = 1600$

$50^2 = 2500$

Since it lies in between, it must be 43 or 47.

Trying

43^2

4 x 46 + 9

1849

So it is 43.

Trying

8464

This ends in 4 so it must be a square ending in 2 or 8.

$90^2 = 8100$

So it must be

92 or 98

Trying

92^2

9 x 94 + 4

= 8464

It is 92

Finally

9801

This ends in 1, so we know our square ends in 1 or 9.

$90^2 = 8100$

and

$100^2 = 10000$

It must be 91 or 99.

Trying

91^2

9 x 92 + 1

8281

(this seemed unlikely also, since the number is so close to 100^2)
Trying

99^2

98 x 10 + 1

9801

Therefore it is 99.

Numbers that aren't square numbers are harder to square root, although it is possible to do it arithmetically. It is usually done algebraically and this is for a later book.

Arithmetically if we wanted to square root
200

We would think of the nearest square number
This is

$14^2 = 196$

So we know it is around 14.

Trying

14.1^2

This gives

  141
x 141

19881

Which, when adjusted for decimal places (see Decimals - In A Minute)
Becomes

198.81

Trying

   142
x 142

20164

Is over 200.

So we know it's around 14.1 to 14.2

From here we could try

14.15

and so on, to try to get closer to 100.

This is fairly laborious, but do-able.

I leave this for you to try.

Four-digit multiplication is in 'Multiplication - In A Minute'!

We can also use this technique to find

$\sqrt{2}$

To find this we can multiply by 100

and find

$\sqrt{200}$

this we have just done, and found it was

14.1

So

$\sqrt{2} = 1.41$

To find

$\sqrt{3}$

we can find the square root of 300 in a similar fashion.

Again, I leave this as an exercise...

Start at

$17^2 = 289$

The Christmas Party Where I Was Called A Wizard

The Christmas Party Where I Was Called A Wizard

One Christmas, all of my friends and family had a party trick prepared for entertainment. My one was a maths trick of course!

I asked them to choose any four-digit number. They said
3214

I wrote it down and I asked for another one.

2312

I wrote this down.

Giving them a calculator, I said
'Now I want you to multiply these on this calculator - but DON'T tell me the answer.'

So they did it.

'Now, read out the answer, in any order, and leave one number out. I will tell you what number is missing!'

They read out

8 6 7 0 4 7

I announced the missing number was...

3!

They were amazed!

Then I said

‘That was nothing! Here’s a better trick. This time, let’s choose a number together.’

Asking one person at a time for a single digit, they gave me

3 2 7

and I said,

‘And I’ll add 6’

giving

3276

Then I said

‘Now multiply this by ANY 4-digit number, but DON’T tell me this either! Again, read out the answer in any order, leave a number out, and I will tell you the missing number’.

My friends were flabbergasted. Their faces looking doubtful, but amused, they wondered how I could possibly do this.

They read out the numbers...

6 5 3 1 1 9 1...

I looked thoughtful and said..

‘1!’

They were thoroughly astounded!

I will show you how to perform this trick. But first we need to know...

How To Check Multiplications Are Correct - In Seconds

How To Check Multiplications Are Correct - In Seconds

When we carry out a multiplication, how can we check it is correct? Apart from using a calculator, which seems to defeat the object of the exercise.

We could do a division, and that would tell us.

So, as a simple example,
4 x 3 = 12

If we divided

$$\frac{12}{4}$$

we should get 3.

However the problem is to get this answer we do a multiplication, what I call the Algebra Trick, so we're just doing the same question again effectively.

What we need is an easy, independent method to confirm our answer.

The one we can use is called the Digit Sum Method.

For example,

Let's say we're doing

14 x 21

and we get

294

as our answer

To check this is correct, all we have to do is sum the digits (and a couple of steps extra).

$$\begin{array}{rl} & 14 = 5 \\ \times & 21 = 3 \end{array}$$

Adding 1 and 4 gives 5 and 2 and 1 gives 3.

Since we are multiplying, we multiply these numbers.

We get

5 x 3 = 15.

Because this is called the DIGIT SUM method, we keep adding until we get a single digit.

This gives

1 + 5 = **6**

And this is our Magic Number for the Question.

All we have to do is see if this matches our answer.

If we add up our answer’s digits, we get
2 + 9 + 4 = 15

1 + 5 = **6**

That’s correct!

The two numbers match up!

Let’s try

32 x 38

Using the squaring technique, this gives
$35^2 - 9$

$1225 - 9$

= 1216

Checking

3 + 2 = 5
3 + 8 = 11 = 1 + 1 = 2

Multiplying

5 x 2 = 10 = 1 + 0 = **1**

Does the answer add to 1?

1 + 2 + 1 + 6 = 10 = 1 + 0 = **1**

Yes!

It is correct.

Throwing OUT 9s

One neat thing in this system is that if we come across a 9, we can ignore it. This is because 9 = 0. It has no value.

Looking at the above example, we got
1216.

The

2, 1, and 6, add up to 9. If we had ignored them, we would have just been left with 1.

So, ignoring 9s works.

This is called ‘*Throwing out 9s*’.

Now run back through all the multiplications in this book and check whether they are correct using this method.

Use it yourself from now on for a super-fast check to see if your working is correct!

The Christmas Party Magic Trick

The Christmas Party Magic Trick

Once you know this method, you can demonstrate an excellent trick.

At a dinner-party, or such like, ask for someone to provide a calculator, pencil and paper. The calculator is for them.

Ask them to choose a random 3 or 4 digit number - it doesn't matter!

Let's say they randomly choose

2401

then ask for another - from someone else!

Let's say they go for

9832

Ok. At this stage, quickly add these numbers up and multiply their digits. So that would be
2401 = 7
9832 = 22 = 4

You can even write 7 and 4 here on the page, as you add. Away from or opposite to the number you've added. Smart friends will figure out it's an addition. Otherwise, it won't mean anything to them. Or if you want to be more subtle, I hold my 7th finger and my 4th finger with my thumbs.

Do 7 x 4 = 28 = 10 = 1.

Your magic number is 1.

What we do as they say these numbers is write them down and pretend to be deep in thought.

You now tell them you want them to calculate the answer on their calculator. They will tell you every number in the answer in any order, except one of them. Whatever is missing - you will tell them!

They then begin to read out the numbers.

Let's say they give you

6 6 6 2 3 2 0

What number is missing?

As they are reading these numbers out, add them up.

6 + 6 + 6 + 2 + 3 + 2 + 0 = 25

This equals 7

However, we want 1.

So the number that must be missing will get us from 7 to 1.

To get from 7 to 1 is only 3. Why?

7 + 3 = 10

1 + 0 = 1

So we just need 3.

Say, "I think the number that is missing is... [pause]...3!"

And they will be amazed.

It doesn't matter what numbers they give you. As long as one is left out.

Let's say they give you

6 6 6 2 3 2 3

And you add these up.

These will give

28

2 + 8 = 10

1 + 0 = 1

Aha! So here we have a problem! We've reached 1, even though that's what we want.

So this means that the number that is missing must be 0. Or... 9. Remember in this system, 9=0.

So here you say... "Well, this is a hard one. You've chosen well there. The number missing could be one of two!! I'm going to go for..."

and then choose one.

If you're lucky, it will be correct.

If it is wrong... play along as if it is very funny how wrong you are.

Then turn around and say... "Ok...I'm going to go for [the other one]".

They'll still be amazed.

If they're not so impressed by this, and you've just been genuinely unlucky, then promise them that you will show them a greater trick.

THIS TIME, you only want to know one four-digit number. They can multiply it by ANYTHING and you will perform the same trick EVEN THOUGH YOU DON'T KNOW WHAT THEY CHOSE!

So here's how to do it.

For this to work, it must be a multiple of 9.

As we saw in the Digit Sum Method, if a number contains a 9, we can ignore it and ‘throw it out’.

This is true even if it’s in the multiplication.

For example

9 x 5

The answer is

45

this sums to 9.

18 x 5

The answer

90

This sums to 9.

18 sums to 9.

(1 + 8)

So if there is a nine involved in the question, the answer must sum to 9 too.

So let each person choose a single number this time, and say 3 people choose

4 2 1

Here, you have to make it 9, so you say
“Ok, and I’ll add a 2”

This will give

4212

Which sums to 9.

NO MATTER WHAT THEY MULTIPLY IT BY NOW, the answer will sum to 9 also.

So instruct them to multiply it by any other 4 digit number they choose, but NOT to tell you.

Let’s say they choose 1234

(It doesn’t matter what they choose, but to show you how it works.)
The answer they will get will be

5 197 608

They read out...

‘8 0 6 9 1 5...what’s missing?’

Adding these as they go, you get
8 + 0 + 6 + 9 + 1 + 5 = 29

2 + 9 = 11 = 2

So you are seven short!

So you have a deep think.

At this stage, they don't believe you know the answer to this. How can you? You don't even know what was in the question! How could you know the answer?

Then say
'Hmm. Tough. I reckon... it's 7!'

And they will be amazed again!

Again, it may happen that they choose all numbers except 9 or 0. Again, you'll have to do the 'one of two' remark. You'd be unlucky for this to happen twice. But don't worry. They will be amazed at how you did it!

Remember, never explain a magic trick. Leave it at that.

Algebra Behind the Multiplication Method

Algebra Behind the Multiplication Method

In Book 1 and throughout the series, we used my refined method for multiplication which goes as follows.

The method takes 3 steps, so I'm going to explain each one. That means writing it out 3 times. In practice, we would only write out the question once.

Let's look at 14 x 21.

First, place it in a column.

```
  14
x 21
----
```

As above, start on the right hand column

```
  14
   ⇩
x 21
----
   4
```

which is 4 x 1, giving us 4.

Step 1 complete.

Step 2 is where you'll see the revolutionary difference that you won't have seen before.

14
✕
x 21

4

Here, draw a small cross between the numbers and multiply along the lines of the cross.

So that gives:
1 x 1 = 1
4 x 2 = 8

Remember that multiplication is just addition, ADD these 2 answers in your mind. Really the goal is to have no working, so we need to store these in our memory for a moment.

So we have 8 + 1 = 9.

Giving

```
  14
x 21
----
  94
```

Step 3 we just multiply along the left column, just as we multiplied along the right.

```
  14
  ⇩
x 21
----
 294
```

Giving, of course, 1 x 2 = 2.

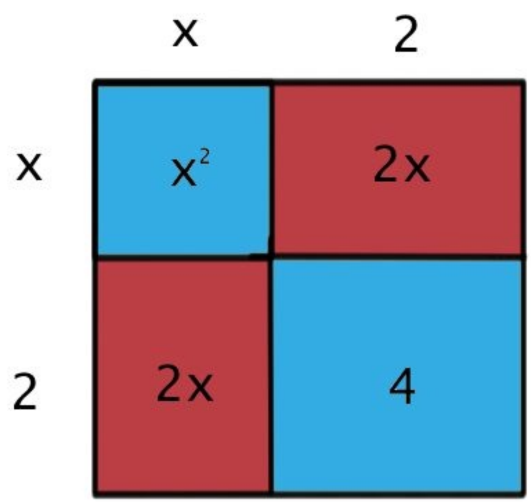
So 14 x 21 = 294.

No working was involved. We had some simple multiplications to make, one addition to store in our memory as we multiplied the 2 in the centre, and the answer appeared as if by magic.

How does this work?

We’re now going to look at the algebra behind how this works, to see why I call it ‘Algebraic Arithmetic’.

First of all, you may recognise this from Quadratics - In A Min



This was the Binomial Square which came from multiplying brackets
$(x+2)^2$

$=(x+2)(x+2)$

Once I saw this, I realised it could be applied to multiplication of numbers also.

If we think of 14 x 21 as tens and units, we can change our $(x+2)$ to

$(t+u)^2$

In other words

$(t+u)$
$\times(t+u)$

----

where the tens unit is being replaced by t and the unit is replaced by u. In practice, these numbers will be different, most likely, but the beauty of it is that it doesn't matter whether they are different are not, the algebra holds true.

We saw above that these multiplied by doing the right column, then the cross, then the left column.

If we do that here, we get

u^2

for the right column

$$tu + tu = 2tu$$

for the cross

and

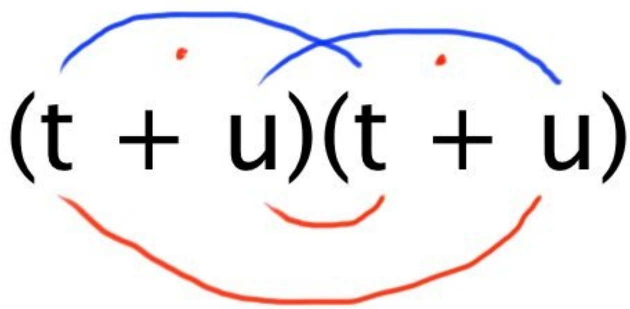
t^2

for the left column

$t^2 + 2tu + u^2$

Which is a binomial square, just like the image above.

This is in fact the expansion of two brackets given by FOIL or the smiley face method also.



What is clever is that we can call t and u anything and it will still work. We still get a binomial square algebraically. However because those values are different we get a rectangle in practice.

We can see then that we are using the concept of a binomial square for ALL multiplications, and it doesn’t matter that the numbers are different.

We have unit times unit.

The cross of u times t added to u times t is the same as multiplying it by 2 if they are the same. If not, we add them.

And finally we have ten times ten.

I first came across the binomial square when I was researching Maria Montessori. Montessori was an educator from Italy who had a different way of going about education (sound familiar?). I was curious to see how she taught maths. I spoke to a teacher who showed me a curious cube.

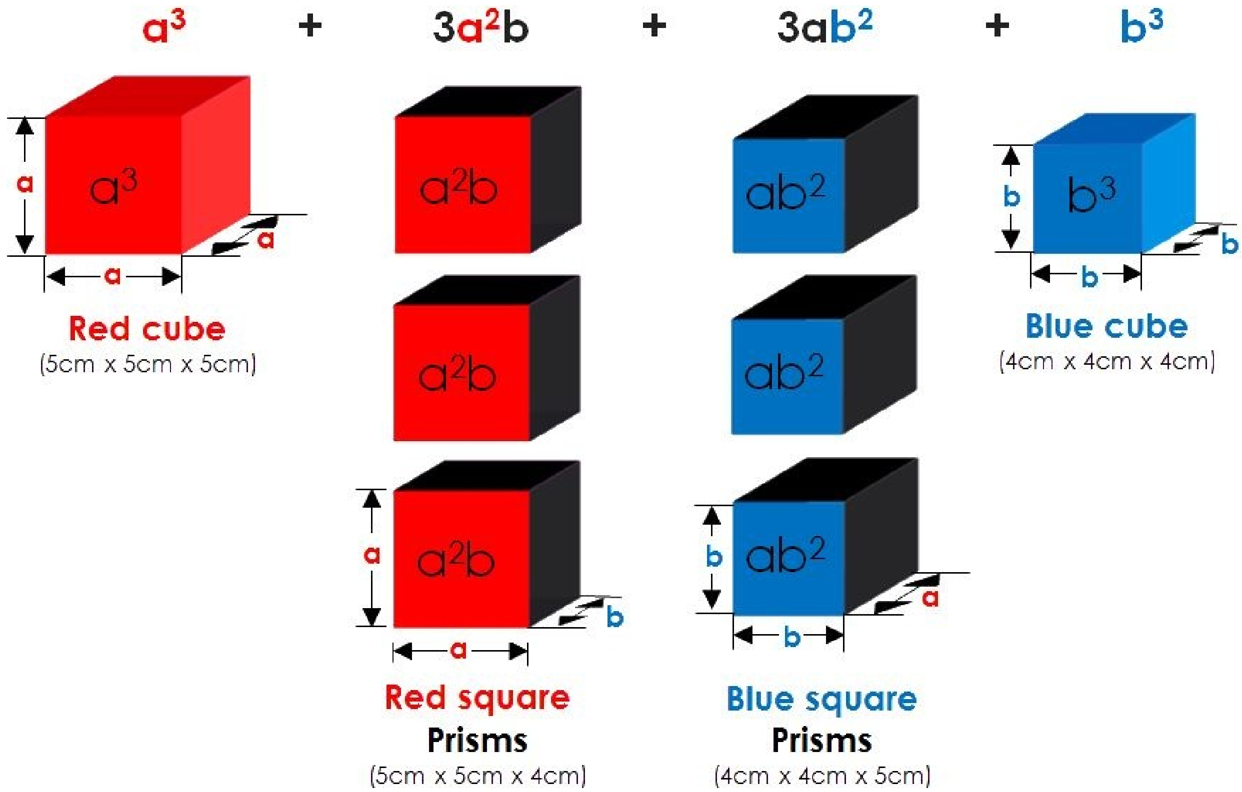
This was something that 5-year-olds put together. After I played around with it I realised it was a binomial cube.

This is a 3-D realisation of this expression.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Which comes from just multiplying brackets!

And how it works…all these terms add up to make a larger cube. Just like our normal multiplication of two numbers form a square.



Montessori was a genius to think of this. This is A-level material, being done by 5 year-olds! I was amazed. The teacher herself didn't understand it, but children can put it together with ease.

This led me to think of the square (which features on the top of the box) and then how to multiply numbers using this.

Algebra Behind The Squaring System

Algebra Behind The Squaring System

When I was 17, I was idly thinking of

$14^2 = 196$

(I don't know why!)

I thought that $4^2 = 16$, and that if you subtract that from 196 you have 180.

I thought that was funny. That is

10 x 18.

Which are both equidistant from 14, the number I was squaring.

I stumbled on my squaring system.

I then applied the concept to other squares I knew, such as

$12^2 = 144$

And realised that worked too.

10 x 14 + 2 x 2

I had discovered something! I showed my maths teacher at school. She wasn't impressed. 'Now prove it', she said.

Here's the algebra behind it.

When we use the method, the first step is to go to the nearest ten.

If we choose 21 to be our number to square, our nearest ten is 20. For that we need to go 1.

So let's call $x = 21$ and $y = 1$

What we do is subtract 1 to get 20 and add 1 to get 22. Then we add the number we've moved squared onto our answer.

Algebraically this is

$$(x-y)(x+y)+y^2$$
$$20\times 22+1^2$$

Multiplying these brackets using the column method, as well as adding y^2

$$\begin{array}{r} (x-y) \\ \times(x+y) \end{array}$$

$$x^2+(xy-xy)-y^2+y^2$$

The $xy's$ cancel.

The $y^2's$ cancel!

This just leaves x^2.

The number we were trying to square!

And that's the algebra.

Has this been discovered before? Yes, it has. I understand it was discovered by Vedic maths, but information is sketchy.

One thing I was very proud of was that, by coming up with this method, I had outdone Richard Feynman. Feynman is a big hero and idol of mine, and his books taught me a lot about physics and how to learn (especially). In one of his parables, he mentions that a colleague of his could 'square numbers around 50'. His method for this was the same as mine. Feynman never seemed to realise this could be extended further for all squares. I only discovered this story a year or so after I'd discovered the method myself.

If you want to read the story yourself, I strongly recommend 'Surely You're Joking, Mr Feynman', a book that influenced me greatly.

Algebra Behind Advanced Multiplication Using Squares

Algebra Behind Advanced Multiplication Using Squares

Once we know the squaring system, we can combine that with a concept in maths known as the **Difference of Two Squares**.

This is what we've used for our multiplications earlier in this book.

It comes from

$(x-y)(x+y)$

As we saw from my squaring system, this is the first part of it. What is special about it is that it always simplifies to

$x^2 - y^2$

which is why it is called the Difference of Two Squares!

This concept was what we used for a multiplication like

27 x 37

Choosing $x = 32$, gives that $y = 5$, so we have
(32 - 5)(32 + 5)

This will multiply out as

$32^2 - 5^2$

from the Difference of Two Squares.

(this is why you should always multiply brackets!)
This gives

1024 - 25

= 999

Introduction to Gradient/Tangent

Introduction

In the book Gradient/Equation of A Straight Line, we saw that it was a short step from using the times tables to become straight lines. These straight lines had gradients, which, it turned out, happened to be the very times tables that formed the sequences for the values of the table for the equation of a straight line!

We then saw that we didn't need these tables at all, as the information contained within the equation gave us everything we needed to know, the gradient and the cut.

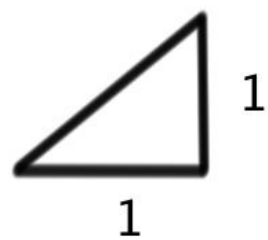
The gradient is something we're going to expand on here and see what else can be done with it other than measuring the steepness of straight lines. In fact, we are going to do something similar, but use a different measure of gradient. It is just like switching from using kilometres to miles (or from degrees to radians - Book 21!) or vice versa. The thing we're measuring will be the same (steepness), but we'll just use a different unit.

Concept Connected To Gradient

Concept Connected To Gradient

So let’s look at this different way to express steepness.

As we saw in earlier books, the definition of gradient was that for every one you went along, the amount you go up (or down) is the gradient. So if you took one step along, and one step up, the gradient would be one.



Do you remember what this concept of using one as a measure was called? It was called...

Normalisation.

We saw in percentages this use of one (100%) as a benchmark measure, even if we’re not measuring something out of 100. In units of distance, area and volume, we saw they were based on one (1m, 1m x 1m, and 1m x 1m x 1m). We saw it again in gradient with the idea being for every one you go along...the amount you go up (or down) is how steep your line or hill is. This comparison to 1 is extremely important for the topic of trigonometry which we are beginning now.

Here is a real life example of normalisation courtesy of a film called Spinal Tap, called, It Goes Up to 11.

The interviewer naturally and intuitively tries to normalise the amplifier by making the maximum 10, but the musician doesn’t get what he’s on about.

However, going back to gradient, what concept, do you think, is similar or connected to gradient?

What other way could we measure steepness?

If you’re not sure, think of this

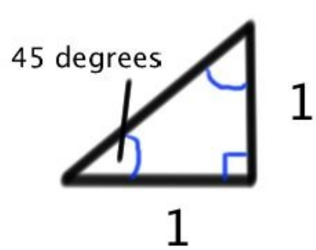
When we see a plane taking off, it doesn’t take off at a gradient, but at an...

ANGLE.

So another way of measuring could be using angles and degrees.

So angle is another measure.

So then, gradient and angle are connected. Whatever a gradient is, there must be some angle connected to it. In fact, I alluded to this in the book Gradient/Eqn of A Straight Line when I pointed out that a gradient of 1 has an angle of 45 degrees, as it is an isosceles triangle.



So the natural question is does every gradient have an associated angle? (Did you wonder this?)

To get a visual idea of what this is like, we could draw a graph from a table of values that we can calculate easily ourselves.

Let’s look at the following angles, and then figure out their gradients.

0°

45°

90°

135°

180°

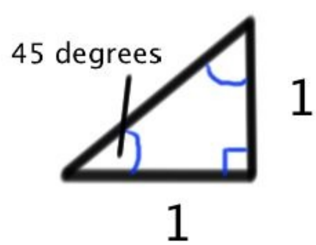
225°

and then make a graph of gradient versus angle and see what that would look like. We could also use the graph to find other values of either angle or gradient, depending on which we’re trying to convert to.

Okay, so zero degrees would be flat.

What will be its gradient? Zero! For every one we go along, we go up zero.

45 degrees we have already examined, the gradient is 1.



90 degrees? What will this be?

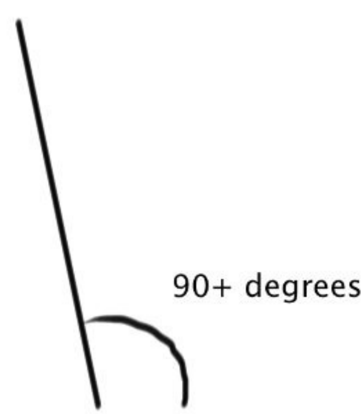
As we can see it just goes straight up! What value will that take? If we think of gradient as being the difficulty of climbing a hill, we saw with 45 degrees that it wasn't too bad. For every one we went along, we went one up. For 90 degrees, how difficult is this to climb? Impossible? And what number would that be?

Infinity - which has the symbol

∞

Which is a strange graph! It goes off towards the edge of the universe (to infinity and beyond, ha ha)! However, we know this is true via calculation - 89.99999 degrees is a very large number too.

After 90 degrees, what happens?



As we saw with straight lines, we know have a negative gradient. It is like going down hill.   For every one you go along right, it goes down (negative). So just after 90 degrees the graph goes from plus infinity to minus infinity! Very unusual graph.

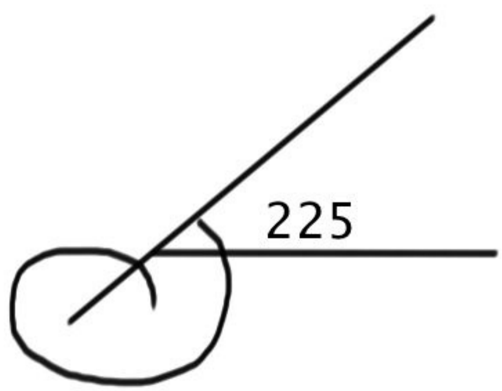
Then we have our next value. 135 degrees.

Just like the third rule, we see this is the reverse of 45 degrees

and for every one we go along, we go down one, so the gradient is - 1.

At 180 degrees, again we see that the gradient is zero.

At 225, we see that this is just 45 degrees again.

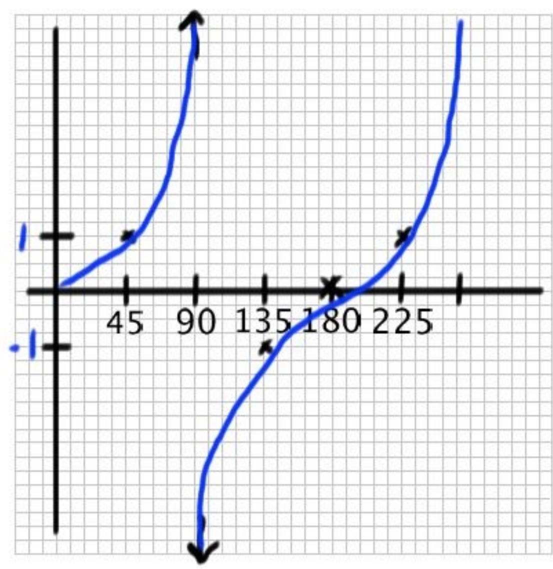


and the cycle repeats.

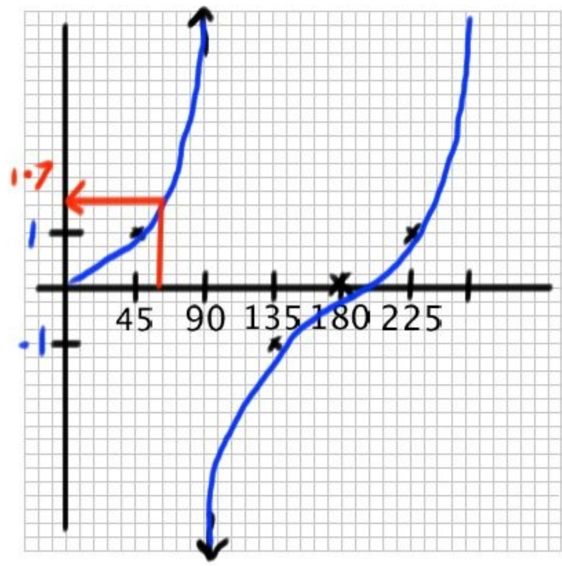
We can now create a table of values

Angle	0	45	90	135	180	225
Gradient	0	1	∞	-1	0	1

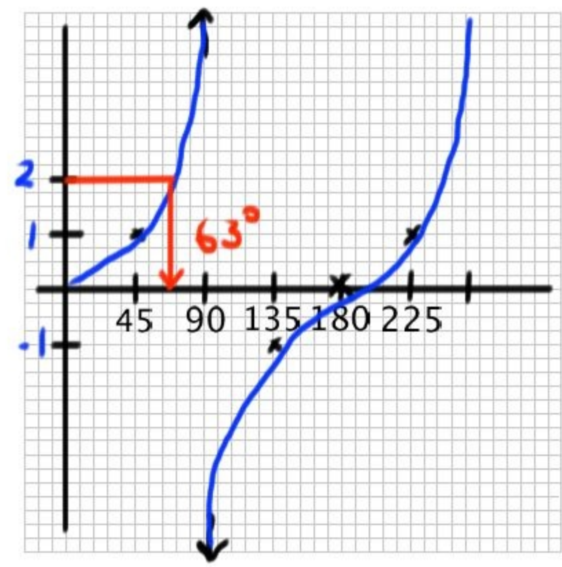
and then draw a 'Gradient Graph' which will show equivalent values of gradients and angles up to 360 degrees, or just 180 since it repeats.



From this graph we could find any gradient for any angle. So let's say you had an angle of 60 degrees and wanted to know what its gradient was. All you do is read up and across and you get about 1.7.



And of course, we can do the reverse. If we have a gradient, we can use it to find an angle. For example, a gradient of 2 would give an angle of around 63 degrees.



So we have a handy-ish device for conversion between the two. Obviously this is not very practical (or accurate) and a table of complete values is available in this list.

Angle	Gradient	Angle	Gradient	Angle	Gradient
0	0.0000				
1	0.0174	31	0.6008	61	1.8040
2	0.0349	32	0.6248	62	1.8807
3	0.0524	33	0.6494	63	1.9626
4	0.0699	34	0.6745	64	2.0603
5	0.0874	35	0.7002	65	2.1445
6	0.1051	36	0.7265	66	2.2460
7	0.1227	37	0.7535	67	2.3558
8	0.1405	38	0.7812	68	2.4750
9	0.1583	39	0.8097	69	2.6050
10	0.1763	40	0.8390	70	2.7474
11	0.1943	41	0.8692	71	2.9042
12	0.2125	42	0.9004	72	3.0776
13	0.2308	43	0.9325	73	3.2708
14	0.2493	44	0.9656	74	3.4874
15	0.2679	45	1.0000	75	3.7320
16	0.2867	46	1.0355	76	4.0107
17	0.3057	47	1.0723	77	4.3314
18	0.3249	48	1.1106	78	4.7046
19	0.3443	49	1.1503	79	5.1445
20	0.3639	50	1.1917	80	5.6712
21	0.3838	51	1.2348	81	6.3137
22	0.4040	52	1.2799	82	7.1153
23	0.4244	53	1.3270	83	8.1443
24	0.4452	54	1.3763	84	9.5143
25	0.4663	55	1.4281	85	11.4300
26	0.4877	56	1.4825	86	14.3006
27	0.5095	57	1.5398	87	19.0811
28	0.5317	58	1.6003	88	28.6362
29	0.5543	59	1.6642	89	57.2899
30	0.5773	60	1.7320	90	------------

We can see some examples in this list of our calculations above, which I've ringed in red.

Note that as the angle increases towards 90 the values get higher in bigger leaps and that there is no value for 90 degrees.

In fact, in truth, we don't need this table either. Of course, this information is saved on your calculator. You just have to know where it is.

An important idea here is that the word 'Gradient' is very closely linked to the word 'Tangent' in maths (we'll see why later) and in fact, I've been lying. This is not called a Gradient Graph, but a **Tangent** Graph. They are effectively the same.

So... if you press the TAN button on your calculator, imagine it says GRADIENT because it will give you the gradient of any angle you put in.

Pressing

TAN 45

Will give you an answer of 1. Why? Because 45 degrees is a gradient of 1! So you're effectively saying Gradient 45?

TAN 60

will give you

$\sqrt{3}$

which is 1.73

Try

TAN 90

You'll see it doesn't like that.

To do the reverse (third rule!) we need to tell the calculator what we're doing, so we just need to use the 2nd function or shift button to reverse the process. You can see this in the picture above.

The symbol for that is

$\tan^{-1}$

Which means, of course the reverse (or more correctly the inverse) of Tangent.

In this case, we have the gradient to start with, and we want to know the angle.

So if we do

$\tan^{-1} 1$

we get

45 degrees of course.

$\tan^{-1} 2$

Means what angle is a gradient of 2?

And we get

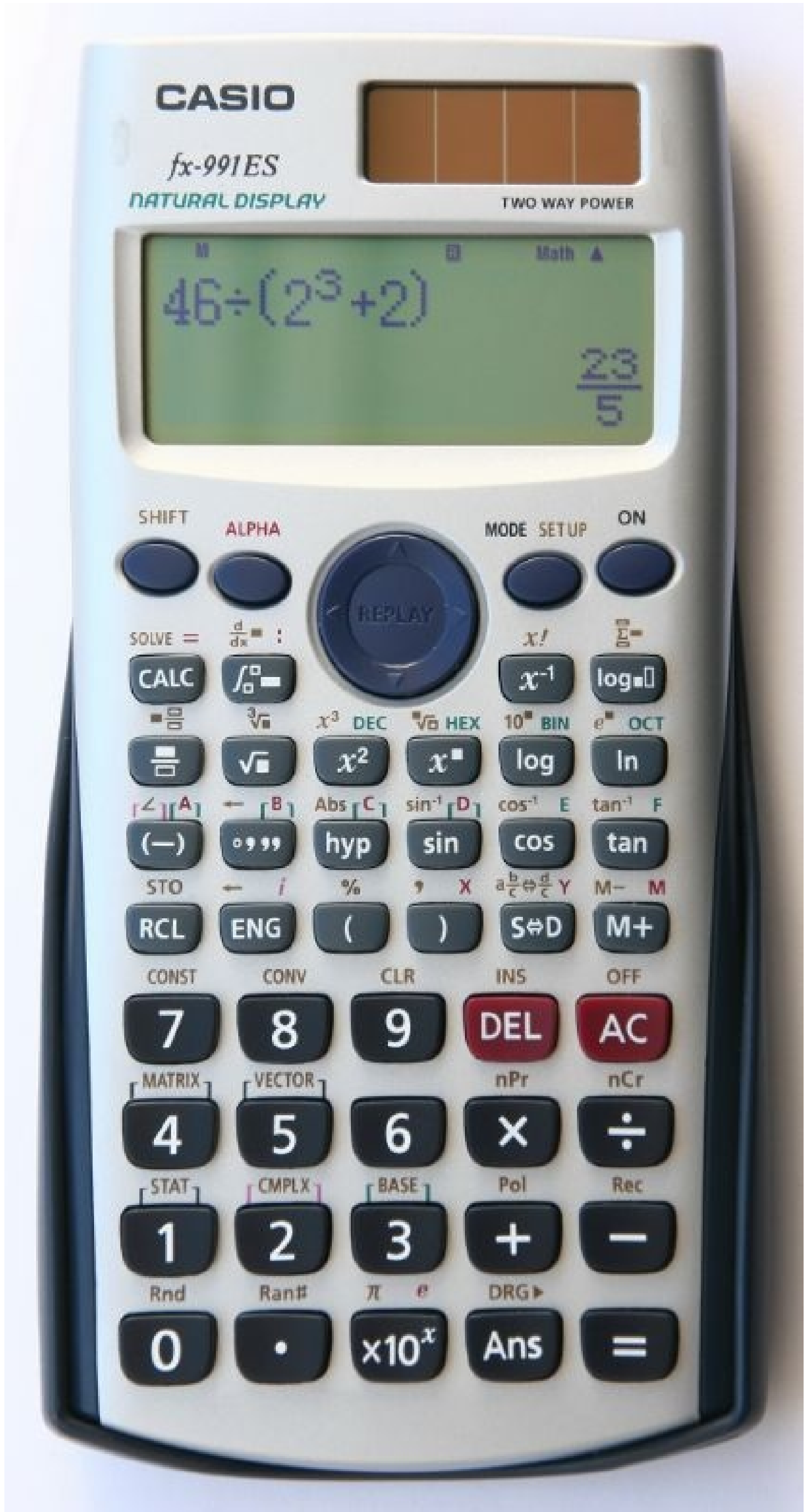
63.4 degrees.

And so on.

So your calculator has all these values stored in its memory. (It doesn't calculate them.)
A quick note about a calculator to buy.

I recommend the

Casio Fx - 991 ES or ES Plus



Because it’s full of functions that are very convenient for use in mathematics. If you’re not using this calculator, you need to get it, as it is permitted for use in school exams like GCSE and A-Level, and can do many functions the average school calculator can’t. This makes it great for CHECKING your answer in an exam (not doing it for you!). In the exam you can check if you are correct, but you’ll not receive full marks if you just write an answer. But if you made a silly mistake in your working, and got a different answer, the calculator will tell you!

I write more about this in a chapter of my book - Guarantee An A*… in the exam technique section.

Putting Our Values To Use

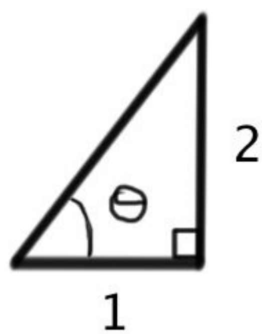
Putting Our Values To Use

So what's the point of all this? Why is it useful to be able to switch between an angle and a gradient and vice versa?

If we look at our gradient 'triangle' which is formed when we try to calculate a gradient, we see we get a... right angled triangle!

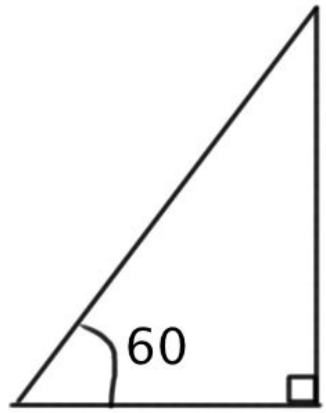
That means that if we have just a triangle, and not a line, slope or hill, we can use the same tool to find out information about this triangle as with the line.

For example, if we had a triangle with a gradient of 2, we would also be able to tell that its angle would be 63.4 degrees.



Similarly, if we had its angle, we could find its gradient. If the angle was 60 degrees, we would know that it would have sides that divide to give an answer of

$\sqrt{3}$

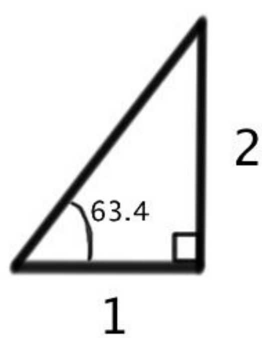


From this we can have 3 possible scenarios.

1 We can find an angle given the sides of the triangle (the gradient) 2 We can find one of the sides of the triangle given the angle and one side 3 We can find one of the other sides of the triangle given the angle and one side
An example follows of each.

Angle

Given the gradient, or sides of a triangle, we can find the angle using our graph/table/calculator.



Here we have a gradient of 2.

Therefore, if we call the angle theta θ (rhymes with eater), we can find it just by saying that

$\tan\theta = 2$

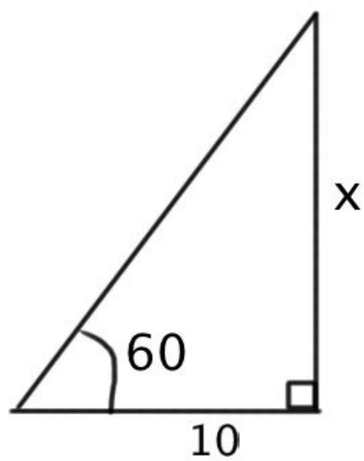
$\tan^{-1} 2 = \theta$

$\theta = 63.4°$

Note that the values of θ and 2 interchange when we use the inverse version of tangent.

Sides

In this scenario, we have the angle, but we don't know one of the sides.



So if the angle is 60 degrees, we know we can find the gradient, since it is

TAN 60 =

$\sqrt{3}$

which is 1.73

However, since we also know that

$$\frac{x}{10} = gradient$$

we have two expressions which we can put equal to each other.

Tan 60 = gradient = 1.73

$$\frac{x}{10} = gradient$$

$$\frac{x}{10} = \sqrt{3} = 1.73$$

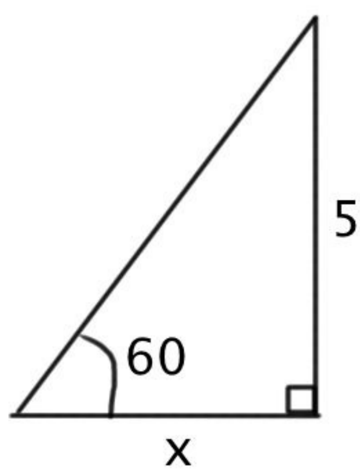
Therefore,

x = 17.3

Finished!

In the opposite scenario, we don't know the bottom of the triangle (the width) but we know the height.

For the same problem with the same angle,



We will have

Tan 60 = 1.73

$$\frac{5}{x} = \sqrt{3}$$

Therefore we need to use the reciprocal to solve this giving

$$\frac{x}{5} = \frac{1}{\sqrt{3}}$$

and multiplying by 5

Gives

$$x = \frac{5}{\sqrt{3}}$$

Or

$$x = 2.89$$

These are the three types of problems we could have.

This is entering a new world. **Trigonometry.** This is Greek for three-tri, line-gon, measure-metry.

In other words, trying to find the lengths of the lines in triangles. We can also find angles of course!

For gradient/tangent, we divide the height by the width. We need to do that to 'normalise' the gradient, so that the base is 1. Naturally, there are other ways to divide two sides of a triangle. The first of which would be to divide the width by the height, which of course, is the complete reverse of gradient.

This is known as the

Co-tangent

Which kind of makes sense!

All that means is that it is the RECIPROCAL of tangent, or gradient. So

$$\cot\theta = \frac{1}{\tan\theta}$$

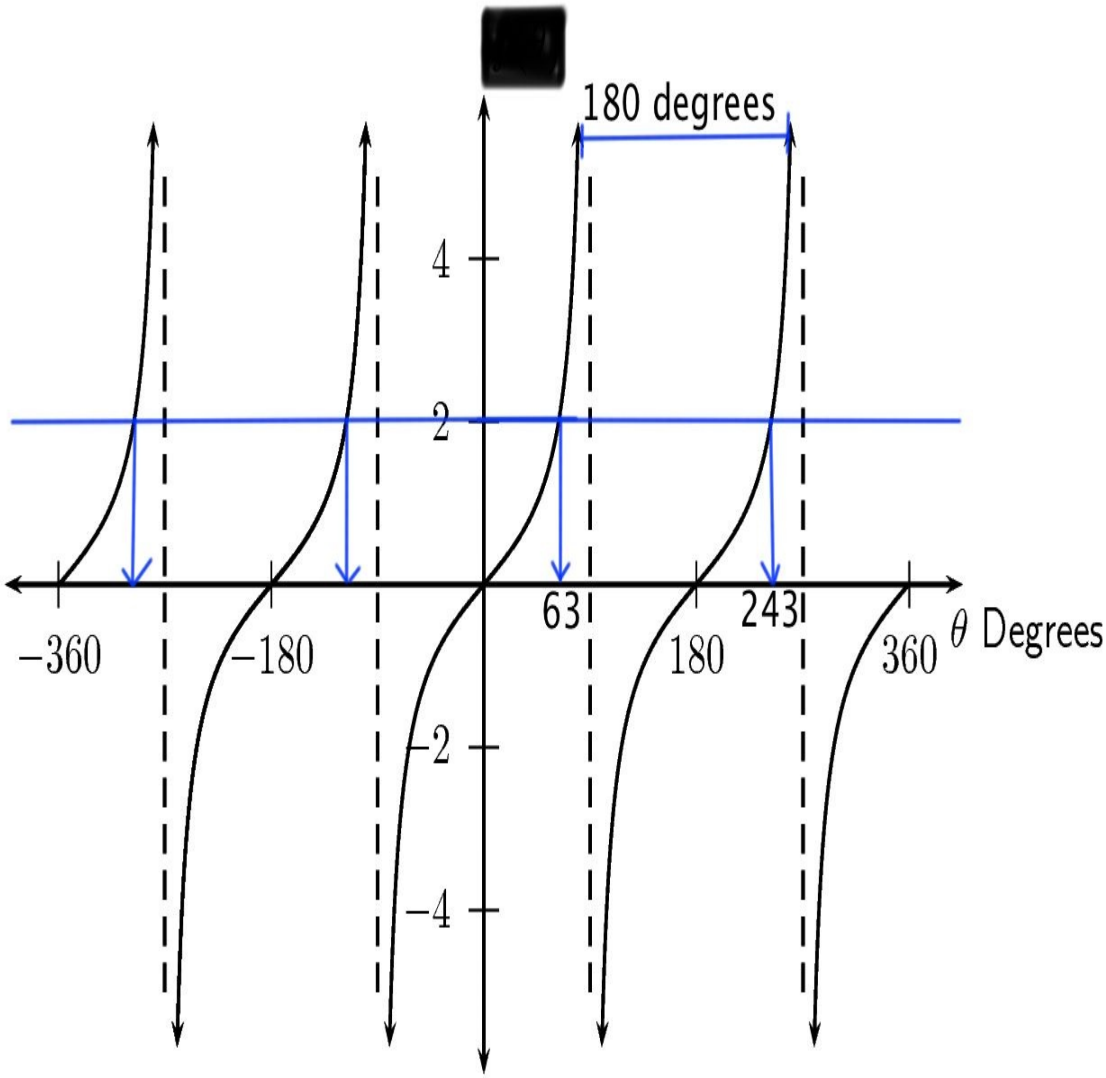
There are four other possible ways to divide the sides of a triangle, which we'll look at in the next book.

Secondary Solutions

Secondary Solutions

If we look again at the graph of gradient, SORRY, tangent, we see that we can read off our values for gradient or angle, depending on which we want. However, being maths, we can generalise and realise that for any value of a gradient, there'll actually be more than one answer. Because our gradient goes around in a circle, this means that every 180 degrees it's going to repeat. We saw this with our first manual graph, where we saw that 0 and 180 degrees had a gradient of zero, and 45 and 225 degrees had a gradient of 1. As a result, we are just going round and round in a circle as we move to the right (or left) on the graph. Consequently, the values will repeat over and over again ad infinitum.

Therefore, our answers so far have been incorrect.



If we want to solve

$\tan\theta = 2$

(i.e. find the angle when the gradient is 2)

there are an INFINITE number of possible answers, since only the first is the one we'd use in reality, doesn't mean that is all of them, and we have seen that if we rotate 180 and 360 degrees we'll get the same answer again. Since there's nothing stopping us rotating repeatedly, the answer here would actually be

$63.4 \pm 180n°$

where n, as we saw in Sequences - In A Minute, was the counting numbers, 1, 2, 3,...etc.

So our possible (positive) answers are, to begin with
63.4, 243.4, 423.4, 603.4,....

and the same for negative, just subtracting 180 degrees each time.

We can see this on the graph too.

So we've now generalised beyond a right-angled triangle. We often do this in mathematics!

Use of Radians

Just like switching from miles to kilometres, we can do everything so far in Radians instead of degrees. In fact, mathematically, this is preferable.

All this means doing is switching your calculator to Radians and using a tan graph with radians as a measure instead.

Introduction to Sine & Cosine

Introduction

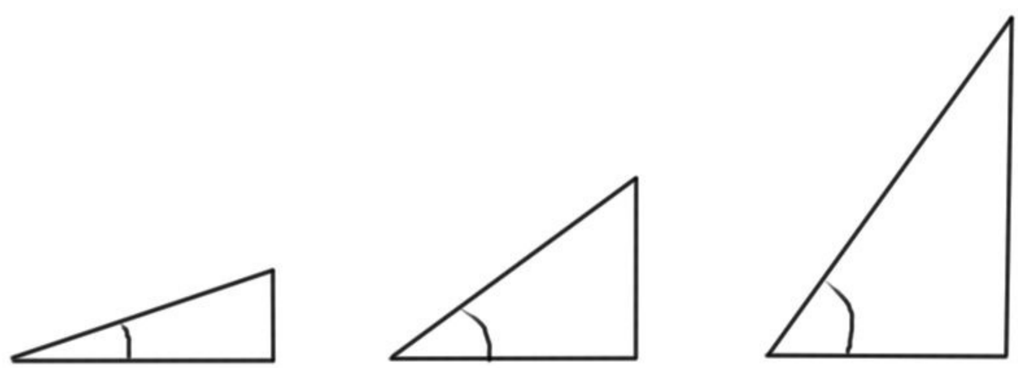
In Gradient/Tangent In A Minute, we saw that we used the concept of gradient to find angles of triangles, and of course, the reverse, finding sides if we had the angle.

In Sine and Cosine, we will see this concept extended to the other sides of a triangle. We noted that there are 6 possible ways to divide the three sides of a triangle, with the previous book looking at the first two. We will look at the other four in this book.

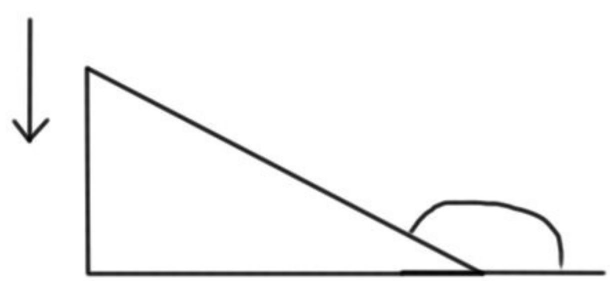
Normalising the Hypotenuse

Normalising the Hypotenuse

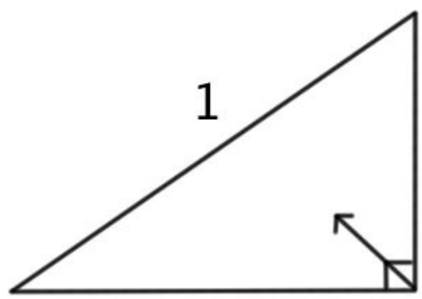
To start with, we are going to look at the height of the triangle. If we look at a variety of possibilities of angle, we will see that as the angle of a triangle gets larger, the height gets larger too.



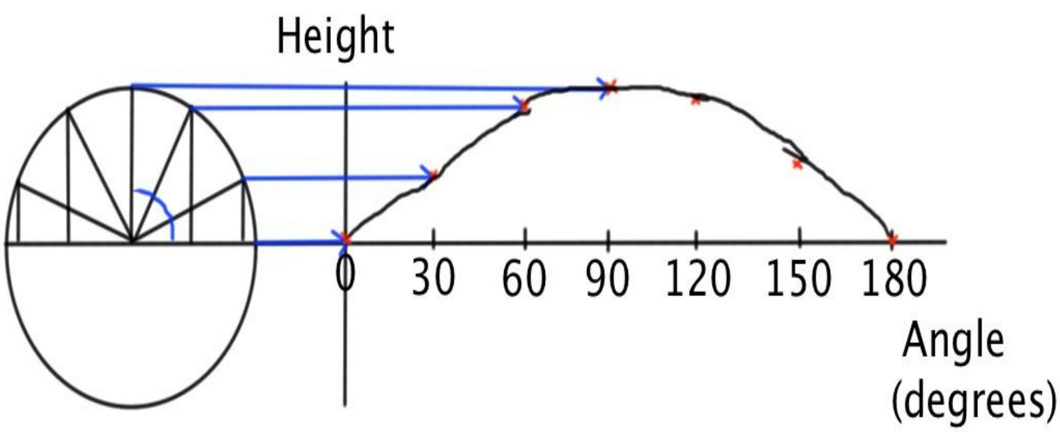
If we extend this for more than 90 degrees, we will see this then comes back down again. (The reverse, of course.)



In Gradient/Tangent, our definition was that for every one we went along horizontally, how many we went up (or down) was the gradient. In other words, the base of the triangle was ‘normalised’. This meant it had to be one - even if we had to divide to make it one. Looking at height, we see that it varies as angle increases. This time we will normalise (make one) the *hypotenuse* of the triangle. This is the name of the longest side of the triangle directly opposite the right angle.

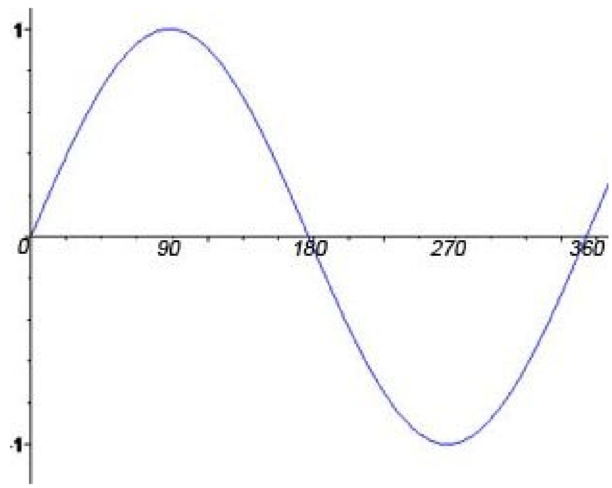


What we can now do is develop a graph where we look at the height of the triangle as it goes from 0 degrees to 180 degrees, in 30 degree intervals, and with a hypotenuse of 1 throughout. You will see that this is basically a circle with a radius of 1, but we are concerned with the triangles within!



You can see how I did this from a video, here.

If we were to keep going around the full 360, the bottom half of the circle would just be the reverse (yet again) and negative, so the full graph would give this.



From this graph we can find a relationship between the angle of a triangle and its height. If we know the angle of the triangle, we can find its height. If we know the height of the triangle we can find its angle.

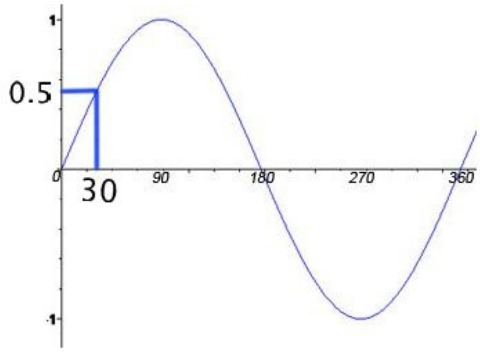
This is known as a Sine Graph. You can think of sine meaning height.

Again, like Tangent, we could use this graph or a table of values, but our calculator has these values stored.

We can see the graph oscillates up and down. This is due to the fact that the angle is going around in a circle.

There's an excellent video of this here.

For example, as we can see from our graph, an angle of 30 degrees gives a height of 0.5.



So Height at 30 degrees is 0.5. Or we would write

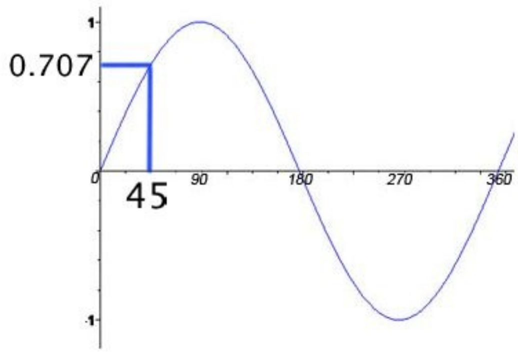
SIN 30 = 0.5

This is only if the length of the hypotenuse of the triangle is one. If the hypotenuse was double that (2), the height would be doubled too. In fact, we can also think of this 'height' value as being the fraction of the size of the hypotenuse. So if the triangle had an angle of 30 degrees, and a hypotenuse of 10, its height would be 5.

In reverse, a height of

$$\frac{\sqrt{2}}{2}$$

gives an angle of 45 degrees.



We tell the calculator we are doing the reverse by using 2nd function -

$\sin^{-1}$

$\sin^{-1}\frac{\sqrt{2}}{2} = 45$

Which we write

$45°$

Putting Our Values To Use 1

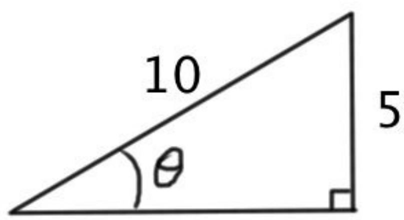
Putting Our Values To Use 1

From Gradient/Tangent, we saw that there were three possibilities. We could either find the angle given the sides, or sides given the angle! In Sine we have exactly the same concept, but different sides.

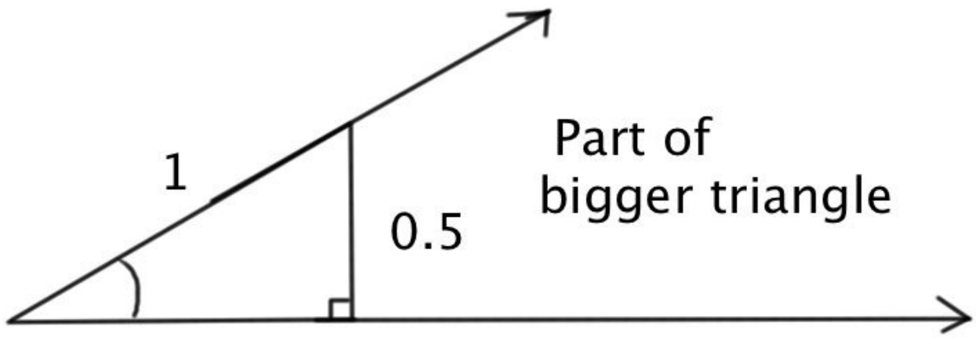
The two sides are of course the height of the triangle and its hypotenuse.

ANGLE

Let's say we have a triangle with a height of 5 and a hypotenuse of length 10.



First of all, we have to divide that hypotenuse by 10 to make it 1. We therefore have to do the same to the height, to make it a similar triangle. This gives a height of 0.5.



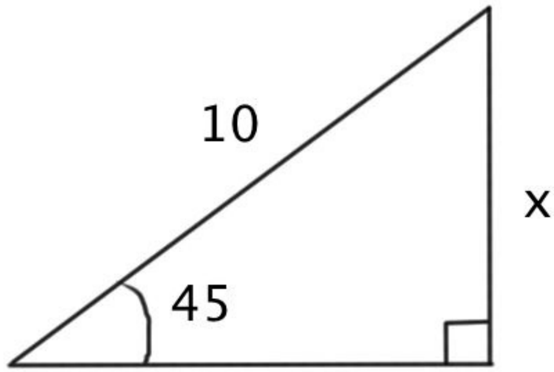
Now we know the actual 'height' of the triangle since we have normalised the hypotenuse, we can now use this to find the angle.

Since we are going backwards from the height to the angle, we use the 2nd function;
$\sin^{-1} 0.5 = 30°$

And so we have found our angle.

SIDE 1

Let's say we know our angle, and we know the length of the hypotenuse this time.



Here then we can find the height of the triangle straight away.

Height at 45 degrees:

Type

$\sin 45$

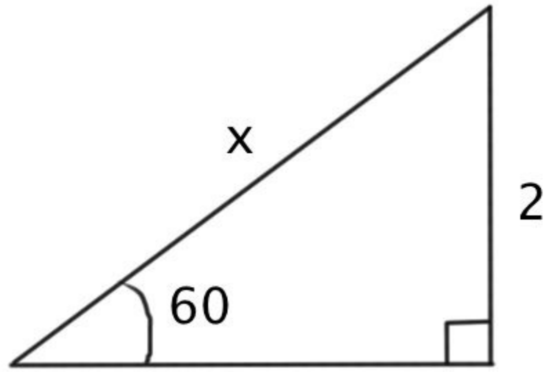
$= \frac{\sqrt{2}}{2} = 0.707$

If our hypotenuse was 1, this would be our height. But the hypotenuse is 10. So we need to make the height ten times bigger, giving
7.07

SIDE 2

For the other side, the situation is reversed, where again we know the angle and the height, but we don't know the length of the hypotenuse.

If we have that the angle is 60 degrees, and the height is 2, obviously we know the height in two different ways here!



However, our height measurement is if the hypotenuse is 1. So there's no way that the hypotenuse is 1 because our height is more than 1 too. Therefore we need to see what the height would be if the hypotenuse is 1.

To find this, again we just type

$\sin 60$

You can think of this as 'height at 60'

And this gives

$= \frac{\sqrt{3}}{2} = 0.866$

However, our height is 2 so that means we need to normalise this hypotenuse.

To do that, we need to divide the height by our hypotenuse, which will be

$\frac{2}{x}$

As we know that this will equal the height too, we set them equal to each other.

$\frac{2}{x} = \frac{\sqrt{3}}{2}$

As with Tangent, we then just use the reciprocal

$$\frac{x}{2} = \frac{2}{\sqrt{3}}$$

And multiply by 2

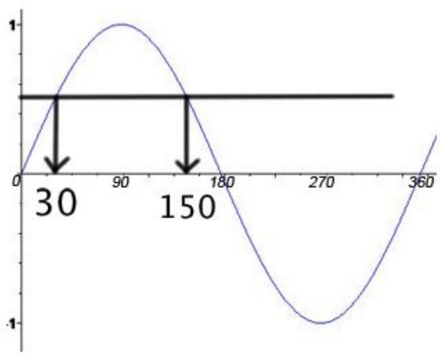
$$x = \frac{4}{\sqrt{3}} = 2.31$$

Secondary Solutions

Secondary Solutions

As we saw with Tangent, there are other solutions other than solving for the angle in a triangle.

If we look at the graph, we can see that if we extend the line out further both sides of the y-axis, the line will cross the curve again (and again).



Looking at the next solution to the right, how can we figure out what it would be?

Because it is a symmetrical curve, we can see that the second solution will exactly mirror our first.

So if we have that

$\sin\theta = 0.5$

This will give 30 degrees as a primary solution. This is what your calculator will give, or what you would expect for a normal triangle problem. Therefore, the second solution will be 180 degrees minus 30 degrees.

This will give 150 degrees.

In fact, like tangent, there are an infinite number of solutions. As we can see they will cycle around every 360 degrees, so our solutions will actually be $30 \pm 360n°$

$150 \pm 360n°$

Normally we limit the range in order to just get between one or two cycles of the sine wave, viz:
$0 \le \theta \le 360$

would give just

$30°, 150°$

as answers.

CALCULATOR NOTE!

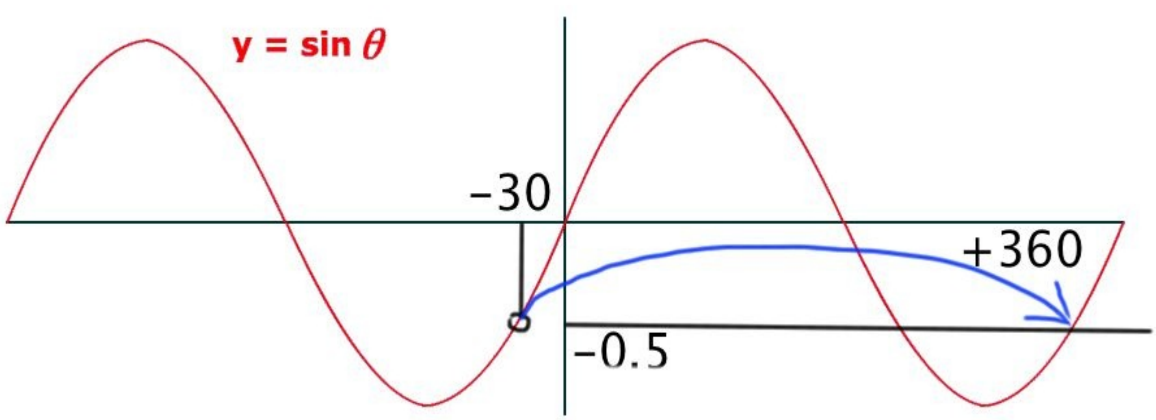
If you type in a value for

$\sin\theta = -0.5$

The calculator gives you a negative answer of - 30 degrees! If you're looking between a range of

$0 \le \theta \le 360$

then obviously this lies outside the range. To find your values, you need to add 360 degrees and work out the second one from there using symmetry.



This will give

330.

The reason the calculator gives you the ‘wrong’ value is because it’s programmed to give values between

$-180 \leq \theta \leq 180$

it doesn’t know that you only want positive values!

So watch out for that.

Using Sine To Find the Area of A Triangle

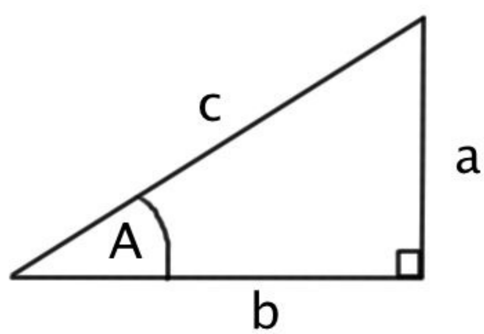
Using Sine To Find the Area of A Triangle

Because Sine means ‘Height’ of a triangle, we can also use it to find the area of a triangle. As we have seen in Squaring & Area - In A Minute, a triangle’s area is calculated just by halving a rectangle. One of the sides of a rectangle is its height.

We have seen that multiplying 2 numbers gives a rectangle. Halving that will give a triangle. The ‘formula’ will be

$$A_{Triangle} = \frac{1}{2}bh$$

Since we can now replace height with Sine, we generate a new formula, which means that we don’t even need the height any more! We can use the angle instead.



In this triangle, Angle A is opposite side a (the height) and b is the base (makes sense!) and c is the hypotenuse.

As we said, is c is 1, then we can use Sine.

If c isn’t 1, we have to normalise by dividing by c.

So we find that

$$\sin A = \frac{a}{c}$$

The height of this triangle is then

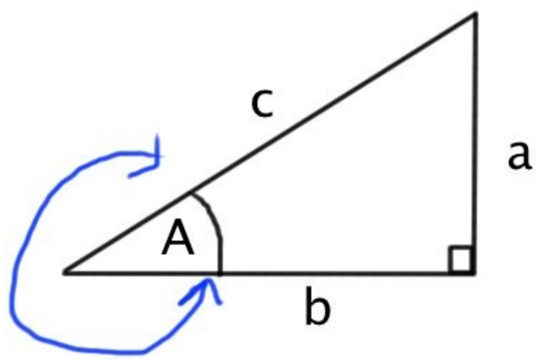
$$a = c\sin A$$

If we replace the h in the area formula with this we get

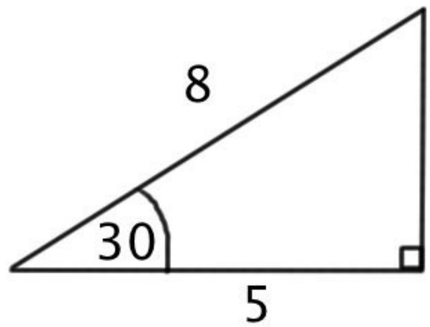
$$A_{Triangle} = \frac{1}{2}bc\sin A$$

Which given the lengths b and c, and the angle A, we can use to find the Area of the triangle.

In fact, the information we've got is all around one corner. If we have this info, we can find the area.



Example



To find the area of this triangle, we just substitute values into our formula.

$$A_{Triangle} = \frac{1}{2} bc \sin A$$

since

b = 5
c = 8
A = 30
Sin A = 0.5

We get

A = 10 square units

The Other Two Sides of The Triangle

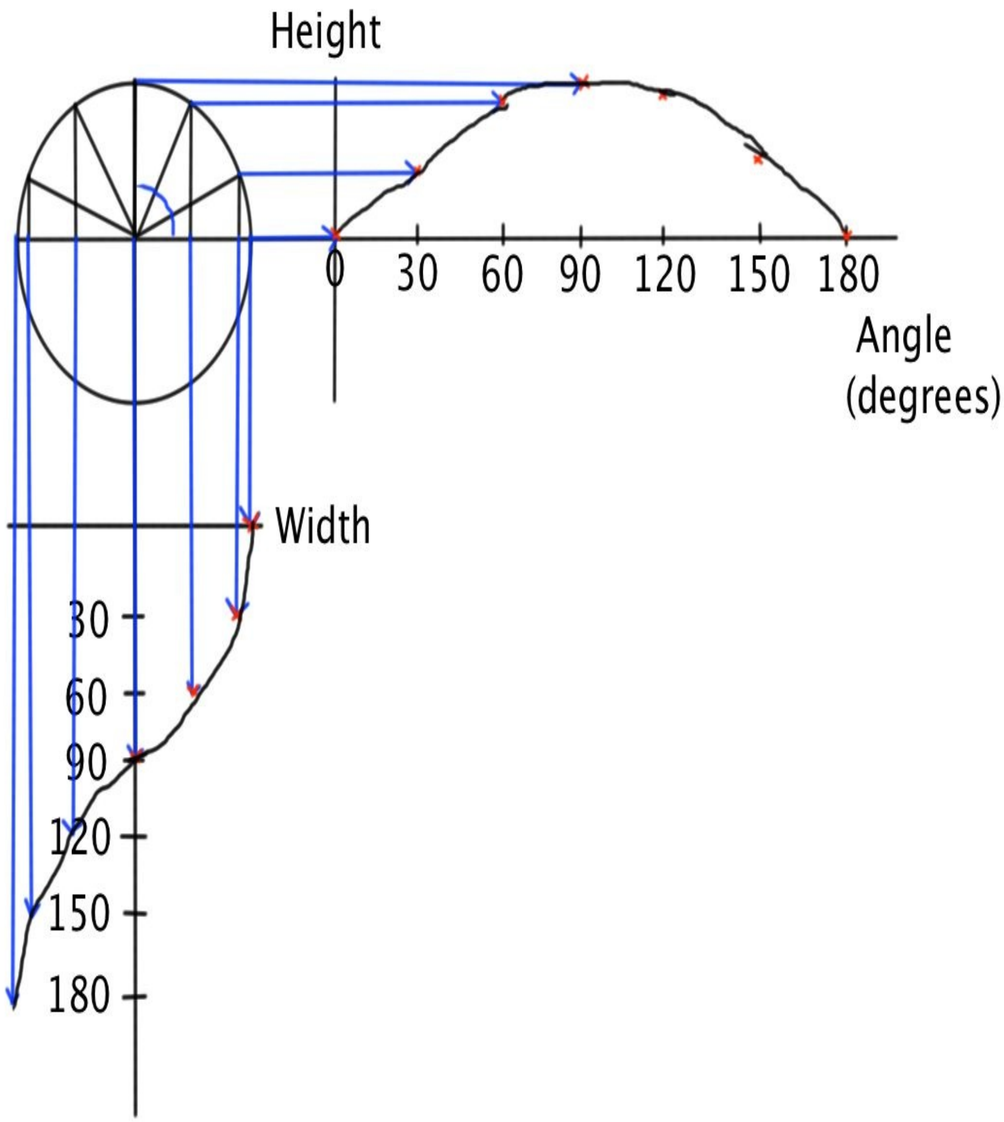
The Other Two Sides of The Triangle

Up to now we've looked at the gradient, and how this is led us to being able to find the angle.

We've also looked at the height of the triangle, and again, this has led us to find the angle.

We're now going to look at what is at ninety degrees to the height, the width, and again, use this to find the angle.

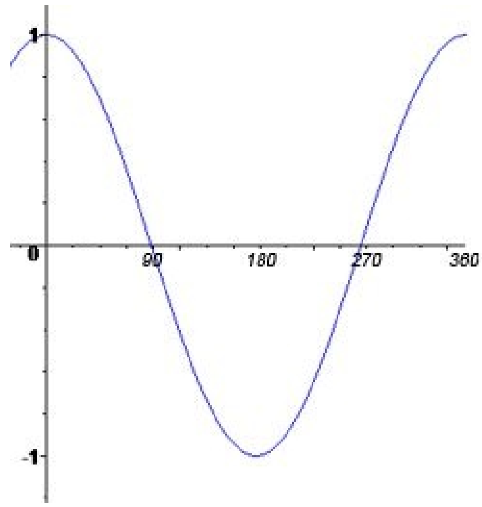
Since the width is at ninety degrees, we can look at the same circle as for the height, but work downwards. In other words, as if it was rotated ninety degrees! We move along in 30 degree increments again and see what graph this gives us.



We can see that at an angle of zero, the width is at a maximum and this works its way down to zero as the angle rotates around to 90.

It then continues as we get to 180, and at that point, does the exact reverse.

If we now rotate this graph 90 degrees we get this graph in a more familiar orientation.



As we can see, this is actually the sine graph, but advanced 90 degrees onwards. As a result, this is called the

Co-sine graph

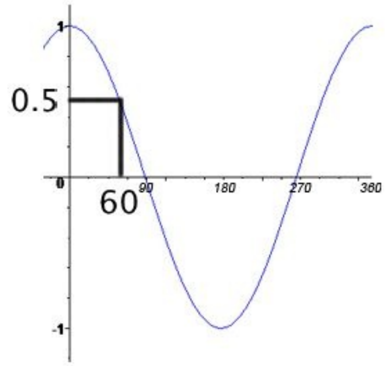
As it is just sine but moved along 90. Perhaps not surprising considering we were looking for the width, which is 90 degrees from the height!

Just with the sine graph, we can find values now of the angle or the width, depending on what information we have to start with.

For example, if we have the angle of a triangle, we can use that to find the width of the triangle.

Again, we are normalising the hypotenuse, so it must be one. And again the value for the width we get is a *fraction* of the hypotenuse, so if it isn't one, it will be that fraction of whatever it is!

So let's say we get that for angle of 60 degrees, we find the width is 0.5.



If the hypotenuse is 1, the width will indeed by one-half. But if the hypotenuse is 3, it will be half of *that*, so it will be 1.5.

So we now have a graph for the width of the triangle, and we can see that this varies with angle, with a maximum at 0 degrees and zero at 90 degrees, the complete REVERSE of Sine.

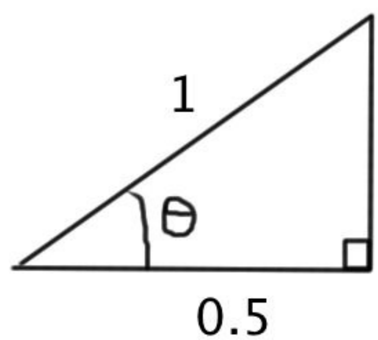
Putting Our Values To Use 2

Putting Our Values To Use 2

Just as with sine, we can now use this graph to find the angle or width in the same 3 possible scenarios.

1 We have the angle and we can find a side (width) 2 We have the angle and we can find the other side (hypotenuse) 3 We have both sides so we can find the angle.

We will now see this in action.



In this triangle, we have the width of the triangle, and the hypotenuse is normalised, *i.e.* it is one.

To find the angle, we just type

$\cos\theta = 0.5$

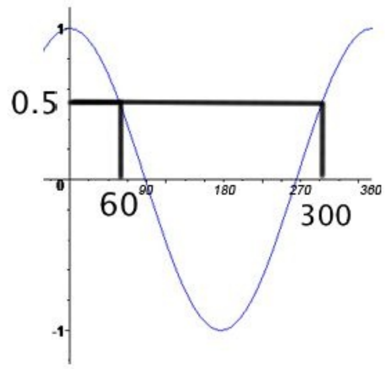
$\cos^{-1} 0.5 = \theta$

and this will give

$\theta = 60°$

So now we know the angle for that triangle.

We could also expand this idea more generally and actually find secondary solutions. This would give, looking at the graph



$300°$

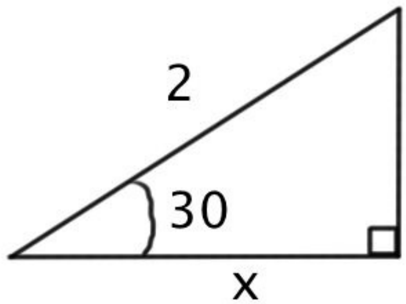
As cosine is symmetrical at 180 degrees.

In fact, as there are an infinite number of solutions, the values would be
$60° \pm 360n$
$300° \pm 360n$

But again these are often limited to a range.

SIDE 1

If we had the hypotenuse, which isn't 1, and an angle, such as this example



From the angle we could find the width, x.

So

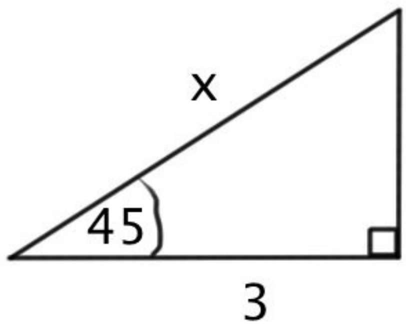
$$\cos 30 = \frac{\sqrt{3}}{2}$$

However, this is the width per 1 length of the hypotenuse. We know the hypotenuse is 2, so that means we have to double the width to get the right answer for this triangle.

$$\cos 30 = \frac{\sqrt{3}}{2} = \frac{x}{2}$$

$$x = \sqrt{3} = 1.73$$

SIDE 2



This is the reverse problem, where we have an angle and the width, but not the hypotenuse.

In this situation we know the width because it's given, but our value of

Cos 45 gives a different width.

That's because it's width per 1 length of the hypotenuse. So we know it isn't 1!

Therefore to find it we

$$\cos 45 = \frac{\sqrt{2}}{2} = \frac{3}{x}$$

Using the reciprocal,

$$\cos 45 = \frac{x}{3} = \frac{2}{\sqrt{2}}$$

And multiplying by 3

$$x = \frac{6}{\sqrt{2}}$$

$$x = 4.24$$

The Relationship Between Sine, Cosine, Tangent, Gradient

The Relationship Between Sine, Cosine, Tangent, Gradient, Height & Width

We have known for a long time that

$$\frac{height}{width} = gradient$$

But now, we know new definitions for these three terms
tangent = gradient
sine = height
cosine = width

Therefore, replacing these, we now know that

$$\frac{\sin}{\cos} = \tan$$

or more correctly

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

This is known as a true mathematical fact. This is absolutely and fundamentally true and can never be disproved. It is true on Earth and every part of the universe. It is part of the fabric of the universe and science. It is amazing that human beings have discovered it!

It is specifically known as a ‘trigonometric identity’.

So if we look back at the graphs of sine and cosine, if we were to somehow divide them, they would give a tangent graph.

Weird, huh?

Reverse of Sine

Reverse of Sine & Cosine

As with tangent, both sine and cosine have a reverse.

These are found by their reciprocals.

The reverse of

$\sin\theta$

is

Cosecant, or cosec for short,

as it is

$$\frac{1}{\sin\theta}$$

and the reverse of

$$\cos\theta$$

is

Secant, or sec

as it is

$$\frac{1}{\cos\theta}$$

These are just the final two ways to divide two sides of a triangle.

Cosecant

is just the

$$\frac{hypotenuse}{height}$$

and

secant

is just

$$\frac{hypotenuse}{width}$$

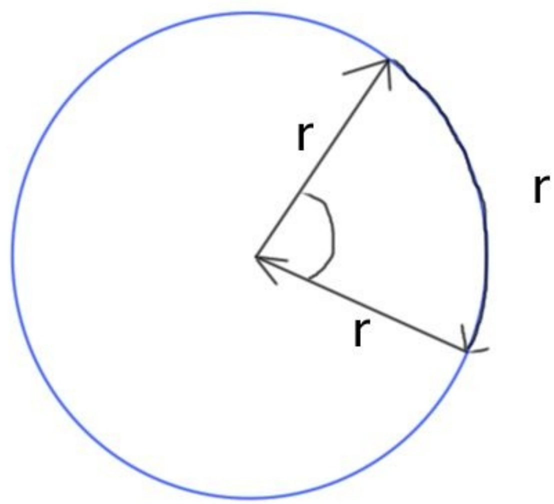
This is all the possible ways that two sides of a triangle can be divided!

A New Angle

A New Angle

If we look back to calculating the circumference, we can see something interesting.

When we drop our first radius around the circumference, as if we are on our way to finding its length, how about we draw a line back to the centre?



Then what we have is an angle created by the radius itself.

In fact, this is the angle that the universe prefers. The use of degrees is a human invention (once again), which is associated with things being a multiple of 6. (That's why we have 12 months a year, 12 inches in a foot, 60 seconds in a minute, 60 minutes an hour, 24 hours a day...etc). However the universe is unconcerned with human inventions, so in fact degrees don't fit very well into the scheme of things.

It turns out that this 'radius angle' is better.

In fact, it is called a 'radius angle' or RADIus ANgle...RADIAN. for short.

And because it is created by the radius on the circumference, can you guess/calculate how many radians are in a circle (or 360 degrees)? Of course, it is 2π also. Very neat!

So there are 2π radians in a circle.

If we want to know the value of 1 radian, we just divide 360 by 2π. This is around 57.3 degrees.

So that gives you an idea.

You'll notice on your calculator that it will be set to degrees as a default. However you can always tell a true mathematician. When you switch his calculator on (if they use one), it will be set to radians [R].

As a result, everything we have done in the proceeding two books can be done using Radians, and in fact, it is preferable to do so. For example, in the graphic above, although you may not recognise the squiggly long S on the left, the other terms,

$Sinx$

And

$\frac{\pi}{3}$

You will have come across by now.

$\frac{\pi}{3}$

Represents an angle of

$60°$

Since it is

$$\frac{2\pi}{6}$$

Which simplifies to

$$\frac{\pi}{3}$$

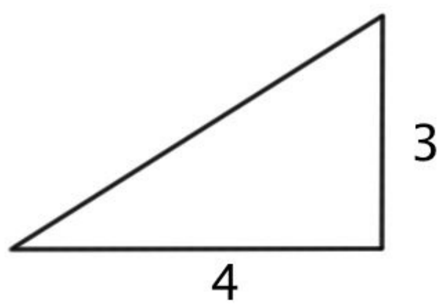
Pythagoras' theorem

Pythagoras' theorem

So far in right angled triangles we've been able to find other information about a triangle as long as we have 2 pieces of information about it.

Another variation of this is to have the gradient, but instead of using that to find the angle, we can use it to find the other side.

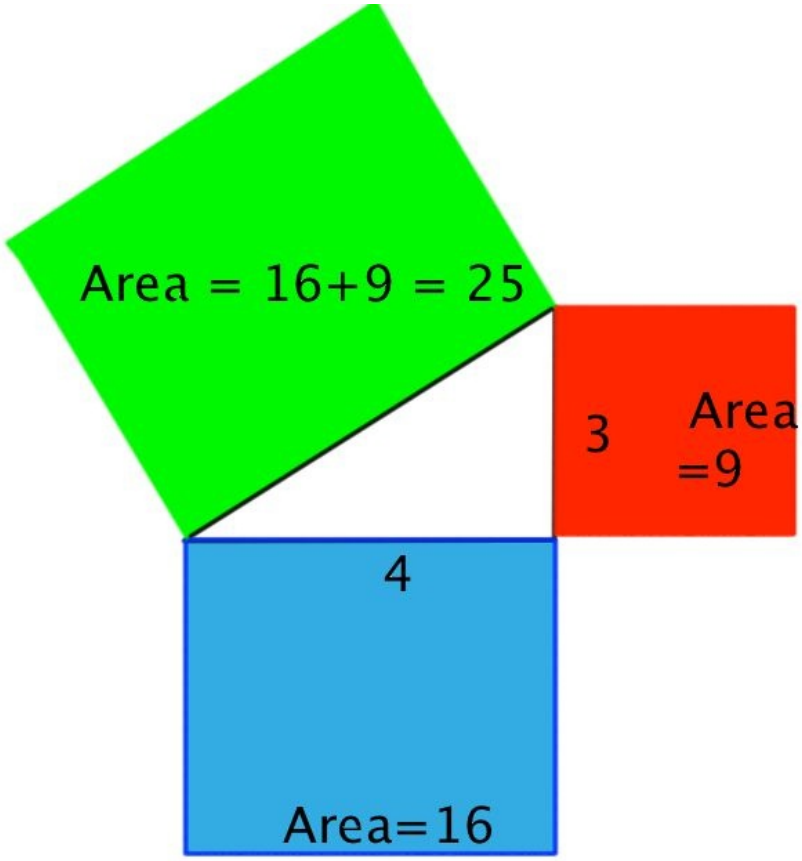
At the moment, if I asked you to find the length of the hypotenuse in this triangle, you could do it, but only by using the concepts of tangent and sine (or cosine). You would first find the angle then use that to find the hypotenuse, since you know the height (or width).



However, if we have BOTH measurements (height and width) of the gradient, we could just skip this and find the hypotenuse in one go.

How to do this was discovered by Pythagoras.

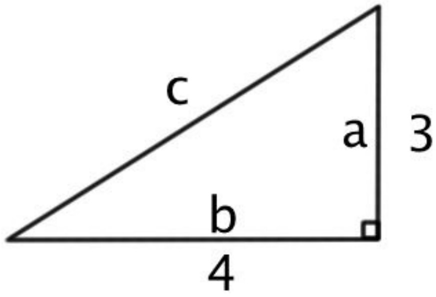
He realised that a right angled triangle had the property that the area of a square on one side, plus the area of another square on the other, would add together to form the area of a square on the hypotenuse.



Or in other words,

The square of the hypotenuse is equal to the sum of the squares of the other two sides.

Writing this a formula, and calling c the hypotenuse
$c^2 = a^2 + b^2$



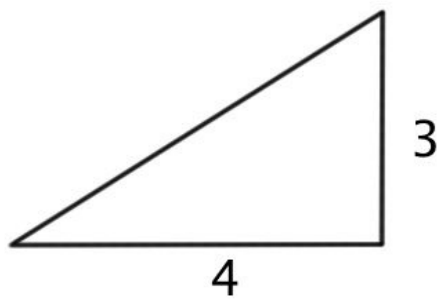
Or, making it as easy as ABC

$$a^2 + b^2 = c^2$$

From this we can find the hypotenuse from our two sides!

Example

a = 3 b = 4



To find c, we know that

$c^2 = a^2 + b^2$

Substituting

$c^2 = 3^2 + 4^2$

= 9 + 16
= 25

If

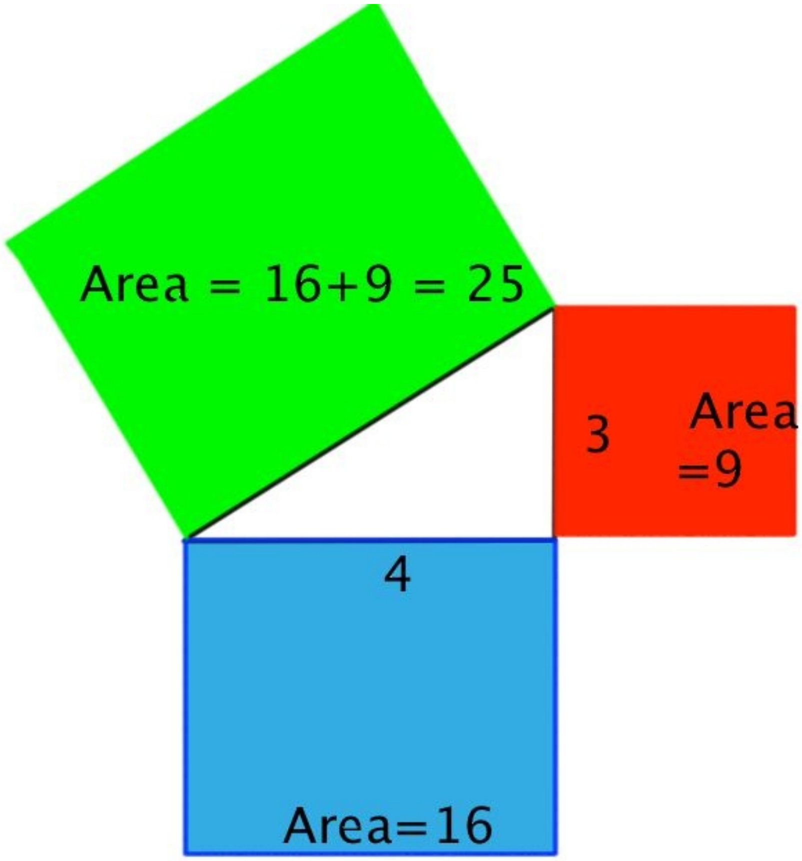
$c^2 = 25$

From Changing The Subject - In A Minute, we need to square root
$c = \sqrt{25} = 5$

So

c = 5

And that’s doing Pythagoras’ Theorem!



In fact, this is a famous triangle in mathematics, as it is common result. It's also a nice pattern... 3, 4, 5!

It just so happens that the squares of 3 and 4 add to form the square of 5.

Another example of this is 5, 12, 13.

This is because

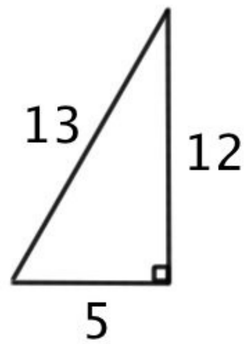
$$5^2 = 25$$
$$12^2 = 144$$

adding together

= 169

which happens to be

13^2



Most of the time, they will not be such round numbers.

If we had

a = 4 and b = 6

We get

16 + 36

= 52

$c^2 = 52$

$c = \sqrt{52}$

$= 2\sqrt{13}$

Which is not a round number! This will often be the case.

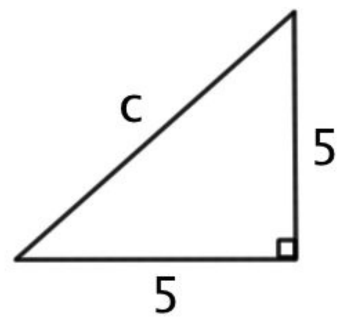
Special Situation

Special Situation

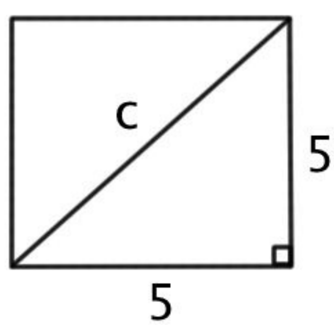
Let’s examine a situation where we have a triangle that is isosceles.

Let’s say we had a triangle where

a = 5 and b = 5



This could also be the diagonal length of a square.



To find c we use the same method

$c^2 = 25 + 25$

= 50

$c = \sqrt{50}$

$= 5\sqrt{2}$

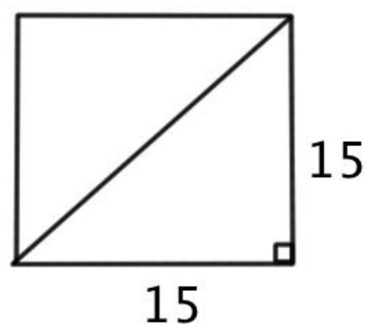
So we see the original length (5) is multiplied by root 2.

This will be true no matter what the length. As long as a and b are the same. In other words, the length of a diagonal of a square is just

$\sqrt{2}$

times by the side of the square.

What is the length of this diagonal?



Reverse Situation

Reverse Situation

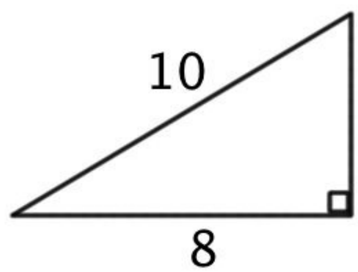
The reverse of this would be where we have the hypotenuse, but not one of the sides. From this can we find the side's length?

Of course, we can. We just reverse the formula, by changing the subject.

Let's say we're trying to find a.

And we have that

c = 10 and b = 8



Pythagoras' theorem is
$c^2 = a^2 + b^2$

Since we want a, we change the subject by following the rules set out in Changing The Subject - In A Minute

giving

$c^2 - b^2 = a^2$

$a^2 = c^2 - b^2$

$= 10^2 - 8^2$

= 100 - 64

= 36

$a^2 = 36$

a = 6

So we just do the exact reverse.

The Pythagorean Theorem Is Not Just For Triangles Once

The Pythagorean Theorem Is Not Just For Triangles

Once people learn Pythagoras' theorem they think it's only ever used in this way. In fact, the Pythagorean Theorem forms the final part of what I call the '**Trinity**' of A-Level, or advanced maths, because it is a concept that is used over and over again in a variety of surprising ways.

The Trinity of A-Level Maths is

Quadratics
Gradient/Tangent
Pythagoras' Theorem

Once they are understood and can be used fluently, these 3 concepts form a huge part of A-Level and are used repeatedly.

Now you're reading Pythagoras' Theorem - In A Minute you have now completed the third item in the Trinity and you are fully prepared for A-Level!

In this book, we will examine 3 other ways in which Pythagoras' Theorem is used.

No. 1

Combining Trigonometry with Pythagoras' Theorem Gives a New Equation

In a right-angled triangle, we now know that

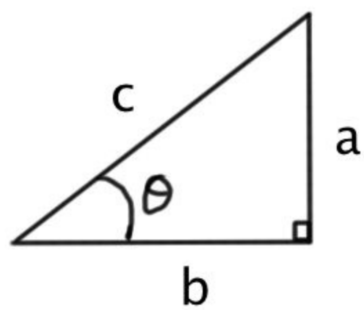
$c^2 = a^2 + b^2$

However, we also know that

$\sin\theta = \frac{a}{c}$

and

$\cos\theta = \frac{b}{c}$



So

$a = c\sin\theta$

and

$b = c\cos\theta$

If we substitute our values for a and b into Pythagoras, we get

$c^2 = (c\sin\theta)^2 + (c\cos\theta)^2$

$c^2 = c^2\sin^2\theta + c^2\cos^2\theta$

This is where

$\sin\theta \times \sin\theta = \sin^2\theta$

Which is how it is written.

If we factorise, since c is common on the RHS.

$$c^2 = c^2(\sin^2\theta + \cos^2\theta)$$

We see that

$$\sin^2\theta + \cos^2\theta = 1$$

As otherwise this expression wouldn't be true, since c-squared would be altered by multiplying it by any other value.

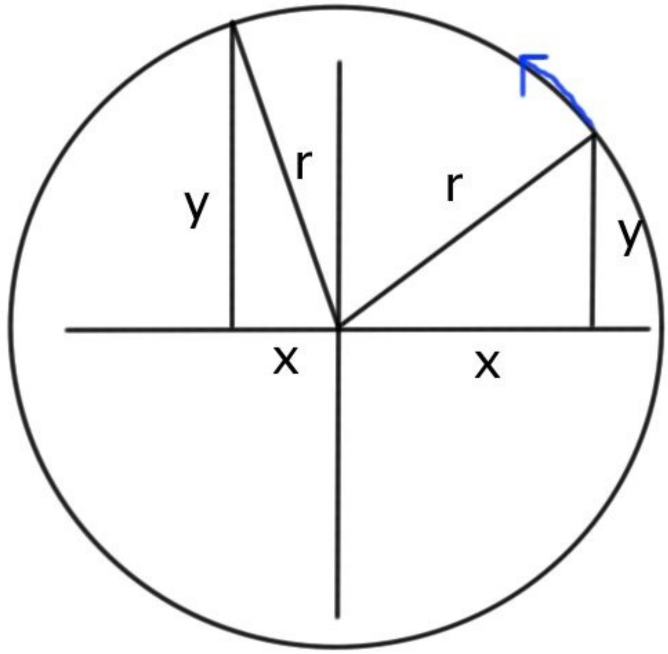
So this is yet another complicated way of writing 1!

This gives a new equation, but which is actually called an identity, which is a true mathematical fact. This identity helps us to solve problems in trigonometry.

No. 2

We Derive The Equation of A Circle

If we rename the sides of our triangle, x, y and r, we see that at any point as r rotates around the centre of the origin, Pythagoras' theorem holds true.



$x^2 + y^2 = r^2$

If we sweep around the whole 360, we get a path which gives a circle. So Pythagoras' theorem also gives the Equation of A Circle!

Where r is its radius.

No. 3

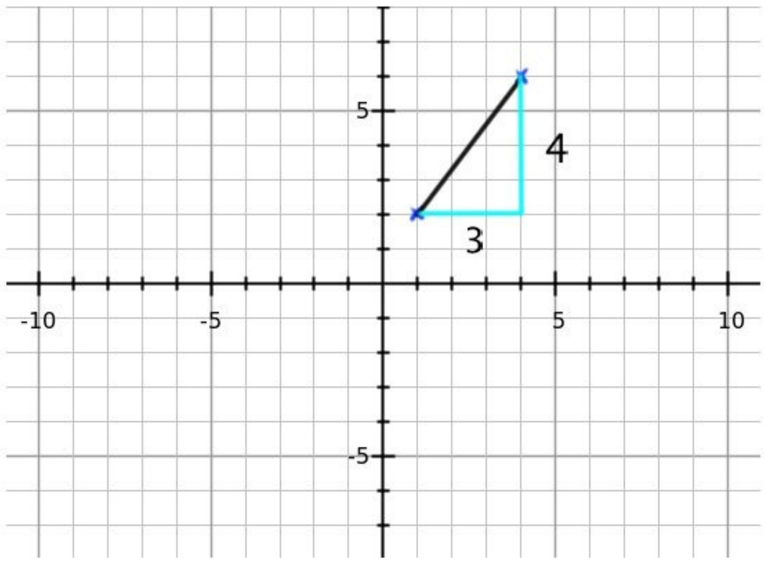
Finding the Length of A Line

If we have a line with co-ordinates on a graph, we can find its length, even though we have no way of measuring it.

Let's say we have the points

(1, 2) and (4, 6)

Between which is a straight line.



Because we have rulers that are at right angles, we can't measure anything that is diagonal. However, this is where Pythagoras comes in!

If we draw a triangle underneath this line, we can find the length of ITS sides very easily.

We can then use Pythagoras.

In this example, we can see it will be a 3, 4, 5 triangle.

So our length will be 5.

We will be using all of these ideas a lot in the future.

Introduction to Sine & Cosine Rules

Introduction

In the previous 3 books we looked at right-angled triangles exclusively, and we were able to find sides and angles, as long as we had at least 2 of each!

In this book we’ll look at what to do if we have to deal with non right-angled triangles. We’ll see that our first strategy, the Sine Rule, can deal with almost any situation. However, there is one exception, and this will be dealt with using the Cosine Rule.

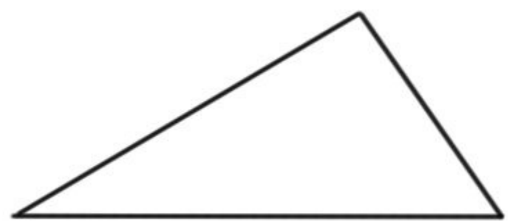
We’ll look at how each rule comes about and a few examples of how to apply them.

The Sine Rule

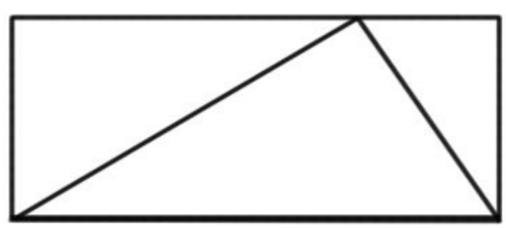
The Sine Rule

If we have a right-angled triangle, Tangent, Sine and Cosine can be used to handle all situations. This all stemmed from the concept of gradient. If we don't have an angle, we can use Pythagoras' theorem to find another length. We really have a few options.

However, if we have a non right-angled triangle, we can't use any of these strategies. So it is necessary to derive a method for this.



If we look again at the area of a triangle, we know, from the beginning, that multiplying two numbers together gives a rectangle. So halving this will give a triangle.



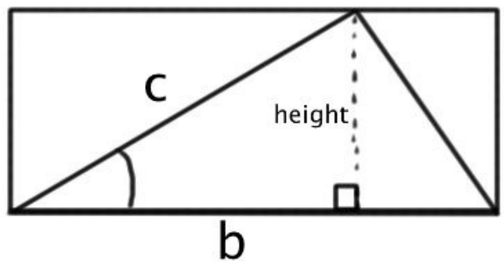
So our formula for Area is

$$A_{triangle} = \frac{1}{2}bh$$

and when we replaced the height with our new definition of 'height', which is a normalised length when the hypotenuse is 1, we get

$A_{triangle} = \frac{1}{2} bc \sin A$

So starting at angle A, and looking at the two lengths that form the angle, will give us the area.

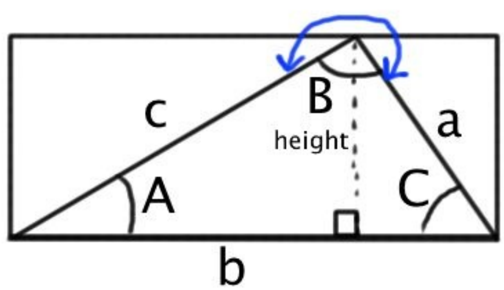


What's great about this formula is that it doesn't have to be a right-angled triangle.

Because the triangle has 3 vertices (sharp points!) we can work our way around each vertex and attach a formula for each.

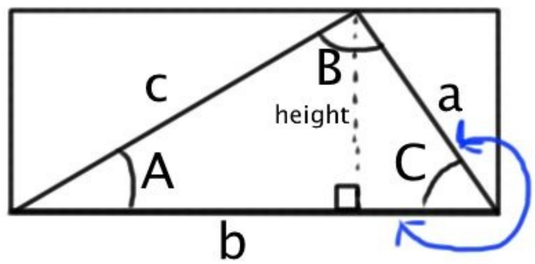
So we'd also have

$A_{triangle} = \frac{1}{2} ac \sin B$



and

$$A_{triangle} = \frac{1}{2}ab\sin C$$



And since they will all have the same area, we could put them equal to each other.

$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

So we have three alternate ways to write the Area. Note that each contains the 3 different ways to write abc in order.

If we manipulate this a little, we can develop a formula.

Multiplying through by

$$\frac{2}{abc}$$

We get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We now have relationships between an angle and its opposite side. This essentially tells us that the bigger the opposite side, the bigger the angle there must be.

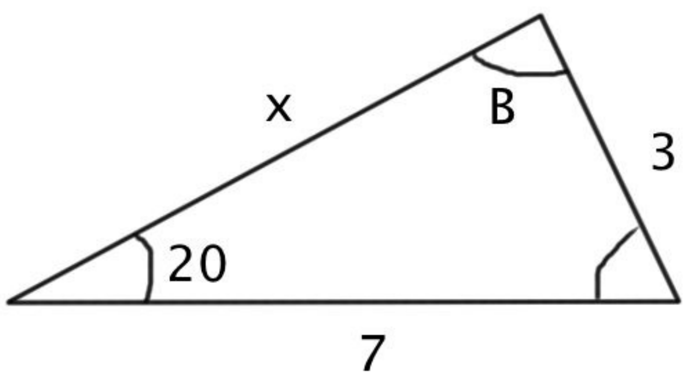
The three sides of a triangle are all equal to each other in this way. Very neat.

We can also write this in its reverse format (of course!) and using the reciprocal, gives

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Using this new formula, we can now use it in a similar way as using Tangent, Sine and Cosine, to find sides or angles, depending on what information we have to start with.

Example.



In this example we can find B.

We know that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Calling Angle A = 20, and its side will be a = 3

So Angle B faces side of length 7, so we'll say that b = 7.

This is all we need to use the Sine Rule. We don't need to know C or x.

So we have

$$\frac{\sin 20}{3} = \frac{\sin B}{7}$$

Re-arranging to make B the subject, gives

$$\sin B = \frac{7 \times \sin 20}{3}$$

This gives that

$$\sin B = 0.798$$

With right-angled triangles, this would tell us that the height of a triangle would be 0.798 times the hypotenuse, but this is the clever thing - we're not doing a right-angled triangle here, but it will still give us a value… as if it were.

$$\sin^{-1} 0.798 = 52.9°$$

$$B = 52.9°$$

Right-Angled Triangles

Right-Angled Triangles

We can even use this for right-angled triangles if we wanted. One of the angles will likely be

$$\sin 90 = 1$$

So there isn't much point.

We could just use Tangent, Sine or Cosine. The Sine Rule is a more general approach to all problems, whereas using the trig functions of Tangent, Sine & Cosine is for more particular cases - in particular when we have a right-angled triangle. A reductive case.

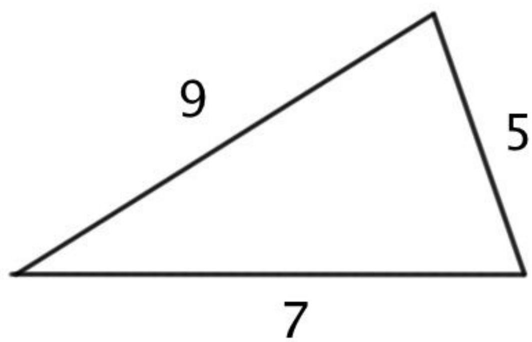
However, I did have a student once who only used the Sine Rule for everything. She never saw that she was repeating work unnecessarily by always finding Sin 90 - which is what just using the trig functions does for you - but she was happy. So I leave it to you the choice!

Having said that, it's important as we progress to realise that the trig functions are not just used for solving triangles - that is just one application of them. Knowing and understanding them will be extremely useful later. And that's why I gave you a glimpse of this by showing you 'Secondary Solutions' in the earlier books.

One Situation Remains - The Cosine Rule

One Situation Remains - The Cosine Rule

We are able to use the Sine Rule for virtually every possibility. As long as we don't have all angles or all sides! If we have a triangle and we only know what sides or angles it has, the Sine Rule won't help us at all.



As a result, we can use another rule for just this situation. As you may have guessed from the title of this book, it is the Cosine Rule.

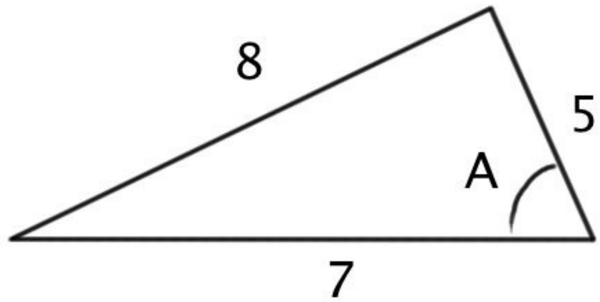
The Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos A$

I'll explain where it comes from at the end of the book.

You can probably see that it contains hints of Pythagoras' theorem and of course, a Cosine, and indeed, it is a mixture of trig functions and Pythagoras.

Let's see it being used.



In this triangle we know all the sides, so, normally, to find angle A would be impossible. But we can use the Cosine Rule here, by substituting values.

First, let's re-arrange the equation to put the angle up front.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{7^2 + 5^2 - 8^2}{2(7)(5)}$$

$$= \frac{10}{70} = \frac{1}{7}$$

$$\cos^{-1}\frac{1}{7} = A$$

$$A = 81.8°$$

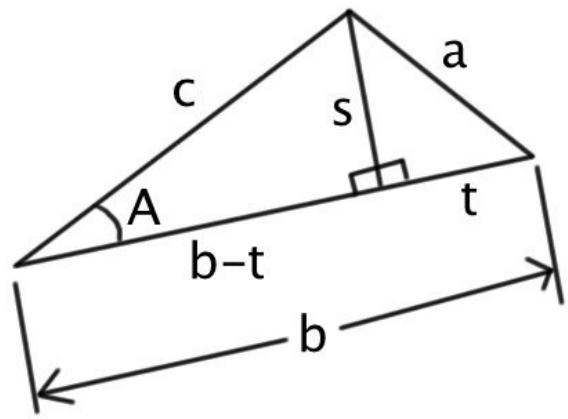
Another issue is when we have sides and angles, but they don't face each other. Again, this is what the Sine Rule depends on, and we could use the Cosine Rule instead.

So we now have a way of doing every type of triangle!

Derivation of Cosine Rule

Derivation of Cosine Rule (Not Essential)

Let’s say we have a triangle like this one.



We can see that there are a few relationships we can make.

We can see that

$$t^2 + s^2 = a^2$$

And

$$\sin A = \frac{s}{c}$$

$$\cos A = \frac{b-t}{c}$$

So that

$$c \cos A = b - t$$

And

$t = b - c\cos A$

And

$s = c\sin A$

If we now substitute s and t into the top formula, Pythagoras' theorem, we get
$a^2 = t^2 + s^2$

$a^2 = (b - c\cos A)^2 + (c\sin A)^2$

To do this we use the column method, viz:

$$\begin{array}{l} b - c\cos A \\ \times b - c\cos A \\ \hline b^2 - 2bc\cos A + c^2\cos^2 A \end{array}$$

Which gives

$$a^2 = b^2 - 2bc\cos A + c^2\cos^2 A + c^2\sin^2 A$$

Factorising all of this

$$a^2 = b^2 - 2bc\cos A + c^2(\cos^2 A + \sin^2 A)$$

We note that we know the value of

$$\cos^2 A + \sin^2 A$$

We saw in Pythagoras' Theorem - In A Minute, that this must be equal to one.

Therefore

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Where we have just switched round
$-2bc\cos A + c^2$

And that’s the Cosine Rule!

As you can see it is just a mixture of Pythagoras’ theorem and trigonometric functions, sine and cosine, blended together to find out something new!

Exam Technique

Exam technique

In schools, although the end result of all of your study of mathematics is an exam, they never seem to give you any strategies for taking one! As a result, most students end up just doing question 1, then question 2, and so on. This seems like the normal thing to do.

But there is a far superior strategy, which, although seems to waste time initially, actually saves you time and guarantees you will pass.

The problem with the strategy above is that if you come across a ‘hard’ question then you can end up spending a lot of time on something that you may not even get correct in the end!

Plus, as you are going along, it’s like you are walking through a jungle, hacking away at the foliage, desperate to find a clearing for a break or civilisation.

What would be more useful would be to have a plan view of the jungle so you’d know when it comes to an end and where the less dense foliage is!

To achieve that, what you do is before writing ANYTHING, read through the paper. This will immediately give you a view of how many questions there are, which are easy and which are hard, and most importantly, your brain will subconsciously begin working on them. Plus knowing what task is ahead of you is a great way of calming any nerves at the beginning. If you never know your task in full for the length of the whole exam, you’re never sure that it isn’t suddenly going to get worse. It’s like waiting. If it’s a countdown, you know how long it will be, that’s fine. When you don’t know how long you’re waiting for, it seems to stretch out forever!

Then, tick off the easy ones! Whichever you think “I’m glad that’s there, I can do it”, tick it off.

As you’re doing this, you’ll see and hear in your periphery everyone else scribbling furiously. Don't worry about that. Let them go ahead with the hacking. You've got a better plan.

When you've finished that, which takes 3-4 minutes, then DO THE EASY ONES FIRST. There's no law that says you have to do them in order. Go through the jungle the way you know better - through the less dense foliage! Do all those easy ones and get those marks in the bank! Isn't that how you're going to pass this thing?

When all those easy ones are done, you can now turn your attention to the harder questions.

And this is the brilliant thing. In the time you've been warming up, doing those easier questions, your brain has been hard at work in its subconscious on those harder questions! So I bet you that when you look at them again they'll be just that little bit easier! And that means you'll do better in this exam than ever before.