Chapter 6

Introduction to Vectors

Scalars are quantities with **magnitude only** whereas vectors are those quantities having **both a magnitude and a direction.** Vectors are used to model a variety of fundamental processes occurring in engineering, physics and the sciences. The material presented in the pages that follow investigates both scalar and vectors quantities and operations associated with their use in solving applied problems. In particular, differentiation and integration techniques associated with both scalar and vector quantities will be investigated.

Vectors and Scalars

A vector is any quantity which possesses both magnitude and direction.

A scalar is a quantity which possesses a magnitude but does not possess a direction.

Examples of vector quantities are force, velocity, acceleration, momentum, weight, torque, angular velocity, angular acceleration, angular momentum.

Examples of scalar quantities are time, temperature, size of an angle, energy, mass, length, speed, density

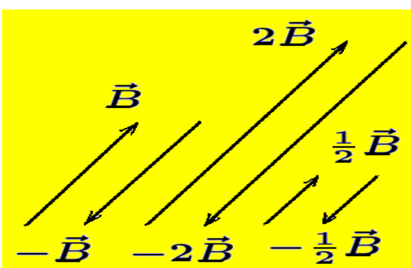

Figure 6-1.
Scalar multiplication.

A vector can be represented by an arrow. The **orientation of the arrow determines the direction of the vector**, and the **length of the arrow is associated with the magnitude of the vector.** The magnitude of a vector $\vec{B}$ is denoted $|\,\vec{B}\,|$ or B and represents the length of the vector. The **tail end of the arrow is called the origin**, and **the arrowhead is called the terminus**. Vectors are usually denoted by letters in bold face type. When a bold face type is inconvenient to use, then a letter with an arrow over it is employed, such as, $\vec{A}$, $\vec{B}$, $\vec{C}$. Throughout this text the arrow notation is used in all discussions of vectors.

Properties of Vectors

Some important properties of vectors are

1. Two vectors $\vec{A}$ and $\vec{B}$ are equal if they have the **same magnitude (length) and direction**. Equality is denoted by $\vec{A} = \vec{B}$.

2. The magnitude of a vector is **a nonnegative scalar quantity**. The magnitude of a vector $\vec{B}$ is denoted by the symbols B or $|\vec{B}|$.
3. A vector $\vec{B}$ is equal to zero only if its magnitude is zero. A vector whose magnitude is zero is called the **zero or null vector** and denoted by the symbol $\vec{0}$.
4. Multiplication of a nonzero vector $\vec{B}$ by a positive scalar m is denoted by $m\vec{B}$ and produces a new vector whose direction is the same as $\vec{B}$ but whose magnitude is m times the magnitude of $\vec{B}$. Symbolically, $|m\vec{B}| = m|\vec{B}|$. If m is a negative scalar the direction of $m\vec{B}$ is opposite to that of the direction of $\vec{B}$. In figure 6-1 several vectors obtained from $\vec{B}$ by scalar multiplication are exhibited.
5. Vectors are considered as **"free vectors".** The term "free vector" is used to mean the following. Any vector may be moved to a new position in space provided that in the new position it is **parallel to and has the same direction as its original position.** In many of the examples that follow, there are times when a given vector is moved to a convenient point in space in order to emphasize a special geometrical or physical concept. See for example figure 6-1.

Vector Addition and Subtraction

Let $\vec{C} = \vec{A} + \vec{B}$ denote **the sum of two vectors** $\vec{A}$ and $\vec{B}$. To find the vector sum $\vec{A}+\vec{B}$, slide the origin of the vector $\vec{B}$ to the terminus point of the vector $\vec{A}$, then draw the line from the origin of $\vec{A}$ to the terminus of $\vec{B}$ to represent $\vec{C}$. Alternatively, start with the vector $\vec{B}$ and place the origin of the vector $\vec{A}$ at the terminus point of $\vec{B}$ to construct the vector $\vec{B}+\vec{A}$. Adding vectors in this way employs the **parallelogram law for vector addition** which is illustrated in the figure 6-2. Note that vector addition is commutative. That is, using the shifted vectors $\vec{A}$ and $\vec{B}$, as illustrated in the figure 6-2, the commutative law for vector addition $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, is illustrated using the parallelogram illustrated. The addition of vectors can be thought of as connecting the origin and terminus of directed line segments.

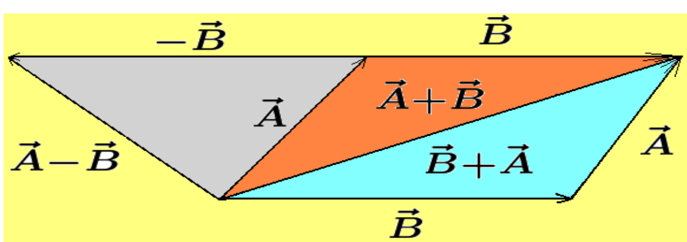

Figure 6-2. Parallelogram law for vector addition

If $\vec{F} = \vec{A} - \vec{B}$ denotes **the difference of two vectors** $\vec{A}$ and $\vec{B}$, then $\vec{F}$ is determined by the above rule for vector addition by writing $\vec{F} = \vec{A} + (-\vec{B})$. Thus, subtraction of the vector $\vec{B}$ from the vector $\vec{A}$ is represented by the addition of the vector $-\vec{B}$ to $\vec{A}$. In figure 6-2 observe that the vectors $\vec{A}$ and $\vec{B}$ are free vectors and have been translated to appropriate positions to illustrate the concepts of addition and subtraction. The sum of two or more force vectors is sometimes referred to as **the resultant force**. In general, the **resultant force** acting on an object is calculated by using **a vector addition** of all the forces acting on the object.

Vectors **constitute a group under the operation of addition**. That is, the following four properties are satisfied.

1. **Closure property** If $\vec{A}$ and $\vec{B}$ belong to a set of vectors, then their sum $\vec{A} + \vec{B}$ must also belong to the same set.
2. **Associative property** The insertion of parentheses or grouping of terms in vector summation is immaterial. That is,

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \tag{6.50}$$

3. **Identity element** The zero or null vector when added to a vector does not produce a new vector. In symbols, $\vec{A} + \vec{0} = \vec{A}$. The null vector is called the identity element under addition.
4. **Inverse element** If to each vector $\vec{A}$, there is associated a vector $\vec{E}$ such that under addition these two vectors produce the identity element, and $\vec{A} + \vec{E} = \vec{0}$, then the vector $\vec{E}$ is called the inverse of $\vec{A}$ under vector addition and is denoted by $\vec{E} = -\vec{A}$.

   Additional properties satisfied by vectors include

5. **Commutative law** If in addition all vectors of the group satisfy $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, then the set of vectors is said to form a commutative group under vector addition.
6. **Distributive law** The distributive law with respect to scalar multiplication is

$$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}, \qquad \text{where } m \text{ is a scalar.} \tag{6.51}$$

Definition (Linear combination)

If there exists constants $c_1, c_2, \ldots, c_n$, not all zero, together with a set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$, such that

$$\boldsymbol{\vec{A} = c_1\vec{A}_1 + c_2\vec{A}_2 + \cdots + c_n\vec{A}_n,}$$

then the vector $\vec{A}$ is said to be a linear combination of the vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$.

Definition (Linear dependence and independence of vectors)
Two nonzero vectors $\vec{A}$ and $\vec{B}$ are said to be* linearly dependent *if it is possible to find scalars k_1, k_2 not both zero, such that the equation

$$k_1\vec{A} + k_2\vec{B} = \vec{0} \tag{6.3}$$

***is satisfied. If $k_1 = 0$ and $k_2 = 0$ are the only scalars for which the above equation is satisfied, then the vectors $\vec{A}$ and $\vec{B}$ are said to be* linearly independent.**

This definition can be interpreted geometrically. If $k_1 \neq 0$, then equation (6.3) implies that $\vec{A} = -\frac{k_2}{k_1}\vec{B} = m\vec{B}$ showing that $\vec{A}$ is a scalar multiple of $\vec{B}$. That is, $\vec{A}$ and $\vec{B}$ have the same direction and therefore, they are called **colinear vectors**. If $\vec{A}$ and $\vec{B}$ are not colinear, then they are linearly independent **(noncolinear)**. If two nonzero vectors $\vec{A}$ and $\vec{B}$ are linearly independent, then any vector $\vec{C}$ lying in the plane of $\vec{A}$ and $\vec{B}$ can be expressed as a linear combination of the these vectors. Construct as in figure 6-3 a parallelogram with diagonal $\vec{C}$ and sides parallel to the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

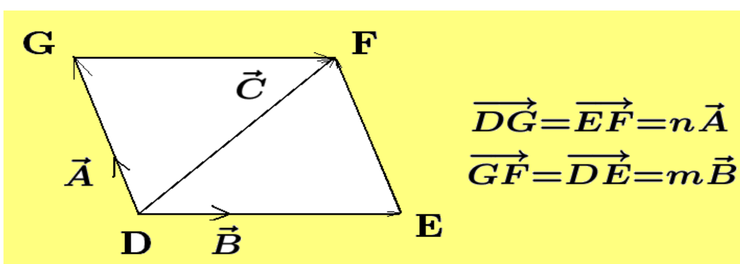

Figure 6-3. Vector $\vec{C}$ is a linear combination of vectors $\vec{A}$ and $\vec{B}$.

Since the vector side $\overrightarrow{DE}$ is parallel to $\vec{B}$ and the vector side $\overrightarrow{EF}$ is parallel to $\vec{A}$, then there exists scalars m and n such that $\overrightarrow{DE} = m\vec{B}$ and $\overrightarrow{EF} = n\vec{A}$. With vector addition,

$$\vec{C} = \overrightarrow{DE} + \overrightarrow{EF} = m\vec{B} + n\vec{A} \tag{6.54}$$

which shows that $\vec{C}$ is a linear combination of the vectors $\vec{A}$ and $\vec{B}$.

Example 6-1. Show that the medians of a triangle meet at a trisection point.

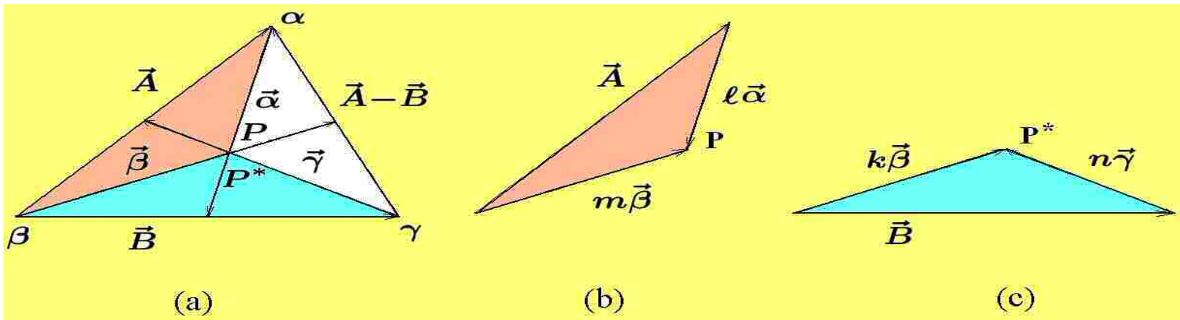

Figure 6-4. Constructing medians of a triangle

Solution: Let the sides of a triangle with vertices α, β, γ be denoted by the vectors $\vec{A}$, $\vec{B}$, and $\vec{A}-\vec{B}$ as illustrated in the figure 6-4. Further, let $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ denote the vectors from the respective vertices of α, β, γ to the midpoints of the opposite sides. By using the above definitions one can construct the following vector equations

$$\vec{A}+\vec{\alpha}=\frac{1}{2}\vec{B} \qquad \vec{B}+\frac{1}{2}(\vec{A}-\vec{B})=\vec{\beta} \qquad \vec{B}+\vec{\gamma}=\frac{1}{2}\vec{A}. \tag{6.5}$$

Let the vectors $\vec{\alpha}$ and $\vec{\beta}$ intersect at a point designated by P, Similarly, let the vectors $\vec{\beta}$ and $\vec{\gamma}$ intersect at the point designated P^*. The problem is to show that the points P and P^* are the same. Figures 6-4(b) and 6-4(c) illustrate that for suitable scalars k, ℓ, m, n, the points P and P^* determine the vectors equations

$$\vec{A}+\ell\vec{\alpha}=m\vec{\beta} \qquad \text{and} \qquad \vec{B}+n\vec{\gamma}=k\vec{\beta}. \tag{6.6}$$

In these equations the scalars k, ℓ, m, n are unknowns to be determined. Use the set of equations (6.5), to solve for the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ in terms of the vectors $\vec{A}$ and $\vec{B}$ and show

$$\vec{\alpha}=\frac{1}{2}\vec{B}-\vec{A} \qquad \vec{\beta}=\frac{1}{2}(\vec{A}+\vec{B}) \qquad \vec{\gamma}=\frac{1}{2}\vec{A}-\vec{B}. \tag{6.7}$$

These equations can now be substituted into the equations (6.6) to yield, after some simplification, the equations

$$(1-\ell-\frac{m}{2})\vec{A}=(\frac{m}{2}-\frac{\ell}{2})\vec{B} \quad \text{and} \quad (\frac{k}{2}-\frac{n}{2})\vec{A}=(1-n-\frac{k}{2})\vec{B}.$$

Since the vectors $\vec{A}$ and $\vec{B}$ are linearly independent (noncolinear), the scalar coefficients in the above equation must equal zero, because if these scalar coefficients were not zero, then the vectors $\vec{A}$ and $\vec{B}$ would be linearly dependent (colinear) and a triangle would not exist. By equating to zero the scalar coefficients in these equations, there results the simultaneous scalar equations

$$(1 - \ell - \frac{m}{2}) = 0, \qquad (\frac{m}{2} - \frac{\ell}{2}) = 0, \qquad (\frac{k}{2} - \frac{n}{2}) = 0, \qquad (1 - n - \frac{k}{2}) = 0$$

The solution of these equations produces the fact that $k = \ell = m = n = \frac{2}{3}$ and hence the conclusion $P = P^*$ is a trisection point.

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Unit Vectors

A vector having length or magnitude of one is called **a unit vector**. If $\vec{A}$ is a nonzero vector of length $|\vec{A}|$, a unit vector in the direction of $\vec{A}$ is obtained by multiplying the vector $\vec{A}$ by the scalar $m = \frac{1}{|\vec{A}|}$. The unit vector so constructed is denoted

$$\hat{\mathbf{e}}_A = \frac{\vec{A}}{|\vec{A}|} \qquad \text{and satisfies} \qquad |\,\hat{\mathbf{e}}_A| = 1.$$

The symbol $\hat{\mathbf{e}}$ is reserved for unit vectors and the notation $\hat{\mathbf{e}}_A$ is to be read **"a unit vector in the direction of $\vec{A}$."** The hat or carat ($\hat{}$) notation is used to represent a **unit vector or normalized vector.**

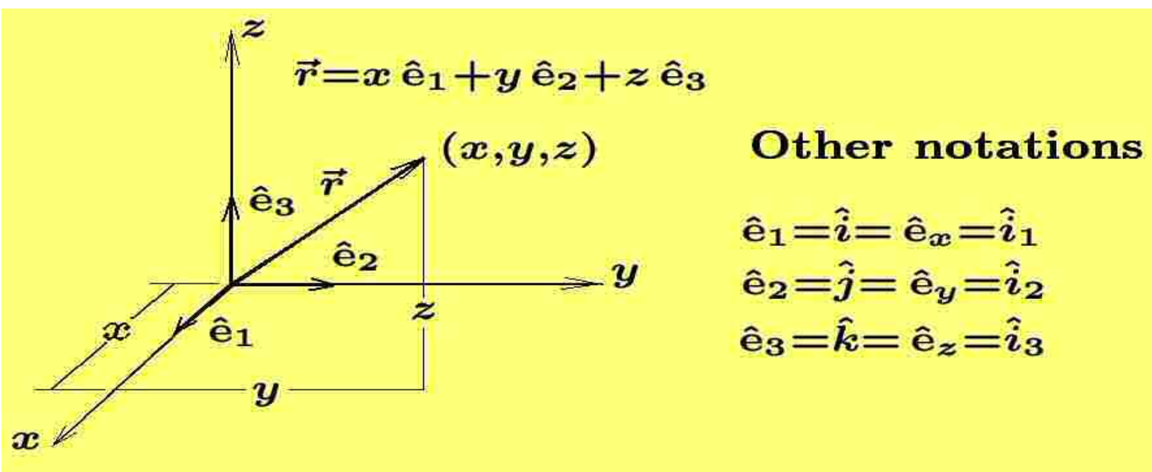


Figure 6-5. Cartesian axes.

The figure 6-5 illustrates unit base vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ in the directions of the positive x, y, z-coordinate axes in a rectangular three dimensional cartesian coordinate system. These unit base vectors in the direction of the x, y, z axes have historically been represented by a variety of notations. Some of the more common notations employed in various textbooks to denote **rectangular unit base vectors** are

$$\hat{\imath},\ \hat{\jmath},\ \hat{k}, \quad \hat{\mathbf{e}}_x,\ \hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_z, \quad \hat{\imath}_1,\ \hat{\imath}_2,\ \hat{\imath}_3, \quad \overline{1}_x,\ \overline{1}_y,\ \overline{1}_z, \quad \hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$$

The notation $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ to represent the unit base vectors in the direction of the x, y, z axes will be used in the discussions that follow as this notation makes it easier to generalize vector concepts to n-dimensional spaces.

Scalar or Dot Product (inner product)

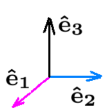
The **scalar or dot product of two vectors** is sometimes referred to as **an inner product of vectors**.

Definition (Dot product) ***The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted***

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\,\cos\theta, \tag{6.8}$$

and represents the magnitude of $\vec{A}$ times the magnitude $\vec{B}$ times the cosine of θ, where θ is the angle between the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

The angle between any two of the orthogonal unit base vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in cartesian coordinates is 90° or $\frac{\pi}{2}$ radians. Using the results $\cos\frac{\pi}{2} = 0$ and $\cos 0 = 1$, there results the following dot product relations for these unit vectors



$$\begin{array}{lll} \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1 & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 = 0 & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 = 0 \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 = 0 & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 = 1 & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_2 = 0 \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3 = 1 \end{array} \tag{6.9}$$

Using **an index notation** the above dot products can be expressed $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \delta_{ij}$ where the subscripts i and j can take on any of the integer values $1, 2, 3$. Here δ_{ij} is the **Kronecker delta symbol**[1] defined by $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

The dot product satisfies the following properties

Commutative law $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Distributive law $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

Magnitude squared $\vec{A}\cdot\vec{A} = A^2 = |\vec{A}|^2$

which are proved using the definition of a dot product.

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[1] Leopold Kronecker (1823-1891) A German mathematician.

The physical interpretation of **projection** can be assigned to the dot product as is illustrated in figure 6-6. In this figure $\vec{A}$ and $\vec{B}$ are nonzero vectors with $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_B$ **unit vectors in the directions of** $\vec{A}$ **and** $\vec{B}$, respectively. The figure 6-6 illustrates the physical interpretation of the following equations:

$$\hat{\mathbf{e}}_B \cdot \vec{A} = |\vec{A}|\,\cos\theta = \text{Projection of } \vec{A} \text{ onto direction of } \hat{\mathbf{e}}_B$$

$$\hat{\mathbf{e}}_A \cdot \vec{B} = |\vec{B}|\,\cos\theta = \text{Projection of } \vec{B} \text{ onto direction of } \hat{\mathbf{e}}_A.$$

In general, the dot product of a nonzero vector $\vec{A}$ with a unit vector $\hat{\mathbf{e}}$ is given by $\vec{A}\cdot\hat{\mathbf{e}} = \hat{\mathbf{e}}\cdot\vec{A} = |\vec{A}||\,\hat{\mathbf{e}}|\cos\theta$ and represents the projection of the given vector onto the direction of the unit vector. The **dot product of a vector with a unit vector** is a **basic fundamental concept** which arises in a variety of science and engineering applications.

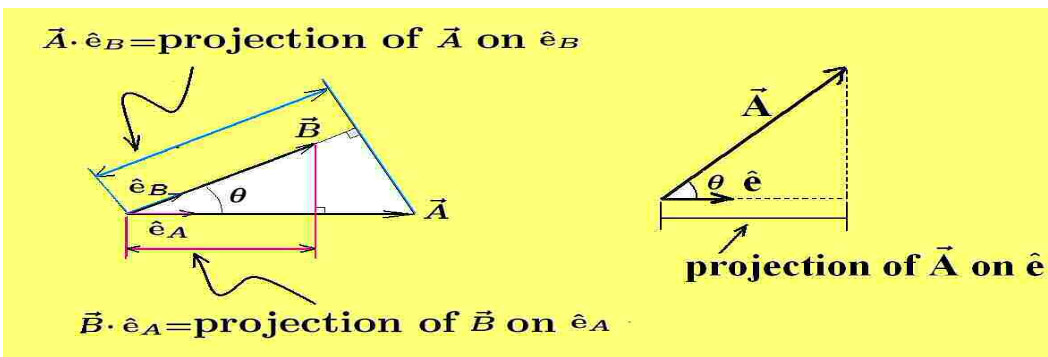


Figure 6-6. Projection of one vector onto another.

Observe that if the dot product of two vectors is zero, $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = 0$, then this implies that either $\vec{A} = \vec{0}$, $\vec{B} = \vec{0}$, or $\theta = \frac{\pi}{2}$. If $\vec{A}$ and $\vec{B}$ are **both nonzero vectors** and their **dot product is zero**, then the angle between these vectors, when their origins coincide, must be $\theta = \frac{\pi}{2}$. One can then say **the vector** $\vec{A}$ **is perpendicular to the vector** $\vec{B}$ or one can state that **the projection of** $\vec{B}$ **on** $\vec{A}$ **is zero.** If $\vec{A}$ and $\vec{B}$ are nonzero vectors and $\vec{A}\cdot\vec{B} = 0$, then the vectors $\vec{A}$ and $\vec{B}$ are said to be **orthogonal vectors**.

Direction Cosines Associated With Vectors

Let $\vec{A}$ be a nonzero vector having its origin at the origin of a rectangular cartesian coordinate system. The dot products

$$\vec{A} \cdot \hat{\mathbf{e}}_1 = A_1 \qquad \vec{A} \cdot \hat{\mathbf{e}}_2 = A_2 \qquad \vec{A} \cdot \hat{\mathbf{e}}_3 = A_3 \tag{6.10}$$

represent respectively, **the components or projections of the vector** $\vec{A}$ onto the x, y and z-axes. The projections $A_1,\ A_2,\ A_3$ of the vector $\vec{A}$ onto the coordinate axes are scalars which are called **the components of the vector** $\vec{A}$. From the definition of the dot product of two vectors, the scalar components of the vector $\vec{A}$ satisfy the equations

$$A_1 = \vec{A} \cdot \hat{\mathbf{e}}_1 = |\vec{A}| \cos\alpha, \qquad A_2 = \vec{A} \cdot \hat{\mathbf{e}}_2 = |\vec{A}| \cos\beta, \qquad A_3 = \vec{A} \cdot \hat{\mathbf{e}}_3 = |\vec{A}| \cos\gamma, \tag{6.11}$$

where $\alpha,\ \beta,\ \gamma$ are respectively, the smaller angles between the vector $\vec{A}$ and the $x,\ y,\ z$ coordinate axes. The cosine of these angles are referred to as the direction cosines of the vector $\vec{A}$. These angles are illustrated in figure 6-7.

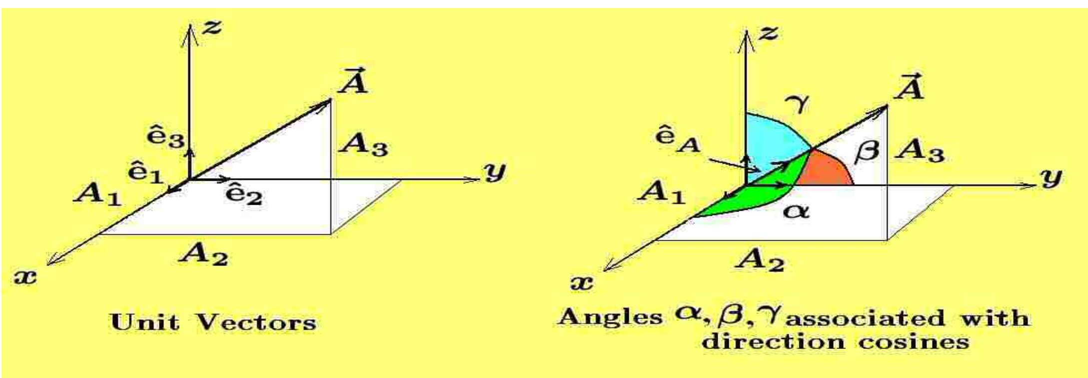


Figure 6-7. Unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_A = \cos\alpha\ \hat{\mathbf{e}}_1 + \cos\beta\ \hat{\mathbf{e}}_2 + \cos\gamma\ \hat{\mathbf{e}}_3$

The vector quantities

$$\vec{A}_1 = A_1\,\hat{\mathbf{e}}_1, \qquad \vec{A}_2 = A_2\,\hat{\mathbf{e}}_2, \qquad \vec{A}_3 = A_3\,\hat{\mathbf{e}}_3 \tag{6.12}$$

are called **the vector components of the vector** $\vec{A}$. From the **addition property of vectors,** the vector components of $\vec{A}$ may be added to obtain

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3 = |\vec{A}|(\cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3) = |\vec{A}|\ \hat{\mathbf{e}}_A \tag{6.13}$$

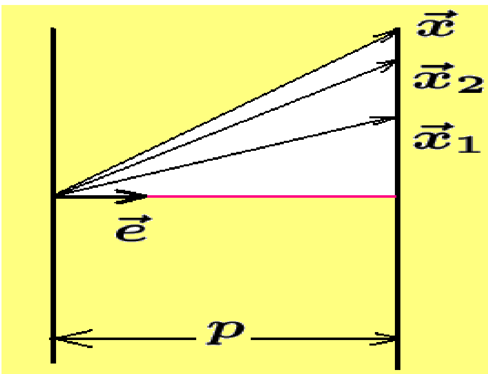
This vector representation $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is called the **component form of the vector** $\vec{A}$ and the unit vector $\hat{\mathbf{e}}_A = \cos\alpha\ \hat{\mathbf{e}}_1 + \cos\beta\ \hat{\mathbf{e}}_2 + \cos\gamma\ \hat{\mathbf{e}}_3$ is **a unit vector in the direction of** $\vec{A}$**.**

Any numbers proportional to the direction cosines of a line are called **the direction numbers of the line.** Show for $a : b : c$ the direction numbers of a line which are not all zero, then the direction cosines are given by

$$\cos\alpha = \frac{a}{r} \qquad \cos\beta = \frac{b}{r} \qquad \cos\gamma = \frac{c}{r},$$

where $r = \sqrt{a^2 + b^2 + c^2}$.

Example 6-2.



Sketch a large version of the letter H. Consider the sides of the letter H as parallel lines a distance of p units apart. Place a unit vector $\hat{\mathbf{e}}$ perpendicular to the left side of H and pointing toward the right side of H. Construct a vector $\vec{x}_1$ which runs from the origin of $\hat{\mathbf{e}}$ to a point on the right side of the H. Observe that $\hat{\mathbf{e}} \cdot \vec{x}_1 = p$ is a projection of $\vec{x}_1$ on $\hat{\mathbf{e}}$. Now construct another vector $\vec{x}_2$, different from $\vec{x}_1$, again from the origin of $\hat{e}$ to the right side of the H. Note also that $\hat{e} \cdot \vec{x}_2 = p$ is a projection of $\vec{x}_2$ on the vector $\hat{\mathbf{e}}$. Draw still another vector $\vec{x}$, from the origin of $\hat{\mathbf{e}}$ to the right side of H which is different from $\vec{x}_1$ and $\vec{x}_2$. Observe that the dot product $\hat{\mathbf{e}} \cdot \vec{x} = p$ representing the projection of $\vec{x}$ on $\hat{\mathbf{e}}$ still produces the value p.

Assume you are given $\hat{\mathbf{e}}$ and p and are asked to solve the vector equation $\hat{\mathbf{e}} \cdot \vec{x} = p$ for the unknown quantity $\vec{x}$. You might think that there is some operation like vector division, for example $\vec{x} = p/\hat{\mathbf{e}}$, whereby $\vec{x}$ can be determined. However, if you look at the equation $\hat{\mathbf{e}} \cdot \vec{x} = p$ as a projection, one can observe that there would be an infinite number of solutions to this equation and for this reason there is **no division of vector quantities.**

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Component Form for Dot Product

Let $\vec{A}$, $\vec{B}$ be two nonzero vectors represented in the component form

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3, \qquad \vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$$

The **dot product** of these two vectors is

$$\vec{A} \cdot \vec{B} = (A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3) \cdot (B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3) \tag{6.14}$$

and this product can be expanded utilizing the distributive and commutative laws to obtain

$$\begin{aligned}\vec{A}\cdot\vec{B} =\ & A_1B_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 + A_1B_2\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 + A_1B_3\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3\\ &+ A_2B_1\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + A_2B_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 + A_2B_3\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3\\ &+ A_3B_1\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 + A_3B_2\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_2 + A_3B_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3.\end{aligned} \tag{6.15}$$

From the previous properties of the dot product of unit vectors, given by equations (6.9), the dot product reduces to the form

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3. \tag{6.16}$$

Thus, the dot product of two vectors produces a scalar quantity which is **the sum of the products of like components.**

From the definition of the dot product the following useful relationship results:

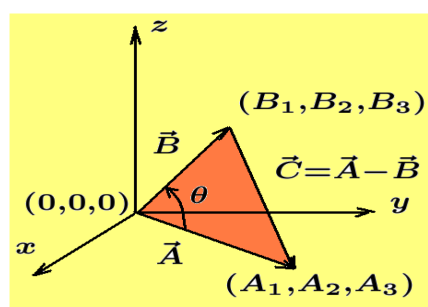
$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\theta. \tag{6.17}$$

This relation may be used to find the angle between two vectors when their origins are made to coincide and their components are known. If in equation (6.17) one makes the substitution $\vec{A}=\vec{B}$, there results the special formula

$$\vec{A}\cdot\vec{A} = A_1^2 + A_2^2 + A_3^2 = A\cdot A\cos 0 = A^2 = |\vec{A}|^2. \tag{6.18}$$

Consequently, the magnitude of a vector $\vec{A}$ is given by the square root of the sum of the squares of its components or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$

The previous dot product definition is motivated by the **law of cosines** as the following arguments demonstrate. Consider three points having the coordinates $(0,0,0)$, (A_1,A_2,A_3), and (B_1,B_2,B_3) and plot these points in a cartesian coordinate system as illustrated. Denote by $\vec{A}$ the directed line segment from $(0,0,0)$ to (A_1,A_2,A_3) and denote by $\vec{B}$ the directed straight-line segment from $(0,0,0)$ to (B_1,B_2,B_3).



One can now apply the distance formula from analytic geometry to represent the lengths of these line segments. We find these lengths can be represented by

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad\text{and}\quad |\vec{B}| = \sqrt{B_1^2 + B_2^2 + B_3^2}.$$

Let $\vec{C} = \vec{A} - \vec{B}$ denote the directed line segment from (B_1, B_2, B_3) to (A_1, A_2, A_3). The length of this vector is found to be

$$|\vec{C}| = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2}.$$

If θ is the angle between the vectors $\vec{A}$ and $\vec{B}$, the law of cosines is employed to write

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta.$$

Substitute into this relation the distances of the directed line segments for the magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Expanding the resulting equation shows that the law of cosines takes on the form

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - 2|\vec{A}||\vec{B}|\cos\theta.$$

With elementary algebra, this relation simplifies to the form

$$A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\theta$$

which suggests the definition of a dot product as $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$.

Example 6-3. If $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is a given vector in component form, then

$$\vec{A} \cdot \vec{A} = A_1^2 + A_2^2 + A_3^3 \quad \text{and} \quad |\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

The vector

$$\widehat{e}_A = \frac{1}{|\vec{A}|}\vec{A} = \frac{A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}} = \cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3$$

is **a unit vector in the direction of $\vec{A}$**, where

$$\cos\alpha = \frac{A_1}{|\vec{A}|}, \quad \cos\beta = \frac{A_2}{|\vec{A}|}, \quad \cos\gamma = \frac{A_3}{|\vec{A}|}$$

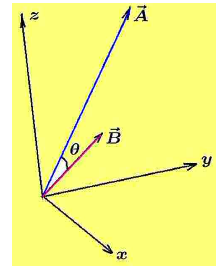
are the direction cosines of the vector $\vec{A}$. The dot product

$$\widehat{e}_A \cdot \widehat{e}_A = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

shows that **the sum of squares of the direction cosines is unity.**

■

Example 6-4. Given the vectors



$$\vec{A} = 2\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3 \quad \text{and} \quad \vec{B} = \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$$

Find:

(a) $|\vec{A}|,\ |\vec{B}|,\ \vec{A}\cdot\vec{B},\ |\vec{A}+\vec{B}|$

(b) The angle between the vectors $\vec{A}$ and $\vec{B}$

(c) The direction cosines of $\vec{A}$ and $\vec{B}$

(d) A unit vector in the direction $\vec{C} = \vec{A} - \vec{B}$.

Solution

$$(a) \quad |\vec{A}| = \sqrt{(2)^2+(3)^2+(6)^2} = \sqrt{49} = 7$$
$$|\vec{B}| = \sqrt{(1)^2+(2)^2+(2)^2} = \sqrt{9} = 3$$
$$\vec{A}\cdot\vec{B} = (2)(1)+(3)(2)+(6)(2) = 20$$

$$\vec{A}+\vec{B} = 3\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_2 + 8\,\hat{\mathbf{e}}_3$$
$$|\vec{A}+\vec{B}| = \sqrt{(3)^2+(5)^2+(8)^2} = \sqrt{98}$$

$$(b) \qquad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta \quad \Longrightarrow \quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{20}{7\cdot 3} = \frac{20}{21}$$

$$\theta = \arccos\left(\frac{20}{21}\right) = 0.3098446 \text{ radians } = 17.753 \text{ degrees}$$

or one can determine that

$$\tan\theta = \frac{\sqrt{(21)^2-(20)^2}}{20} = \frac{\sqrt{41}}{20} \quad \Longrightarrow \theta = 0.3098446 \quad \text{radians}$$

(c) A unit vector in the direction of the vector $\vec{A}$ is obtained by multiplying $\vec{A}$ by the scalar $\frac{1}{|\vec{A}|}$ to obtain

$$\hat{\mathbf{e}}_A = \frac{\vec{A}}{|\vec{A}|} = \cos\alpha_1\,\hat{\mathbf{e}}_1 + \cos\beta_1\,\hat{\mathbf{e}}_2 + \cos\gamma_1\,\hat{\mathbf{e}}_3 = \frac{2}{7}\,\hat{\mathbf{e}}_1 + \frac{3}{7}\,\hat{\mathbf{e}}_2 + \frac{6}{7}\,\hat{\mathbf{e}}_3$$

which implies the direction cosines are $\cos\alpha_1 = \frac{2}{7},\ \cos\beta_1 = \frac{3}{7},\ \cos\gamma_1 = \frac{6}{7}$ In a similar fashion one can show $\hat{\mathbf{e}}_B = \frac{\vec{B}}{|\vec{B}|} = \cos\alpha_2\,\hat{\mathbf{e}}_1 + \cos\beta_2\,\hat{\mathbf{e}}_2 + \cos\gamma_2\,\hat{\mathbf{e}}_3 = \frac{1}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{2}{3}\,\hat{\mathbf{e}}_3$ which implies the direction cosines are $\cos\alpha_2 = \frac{1}{3},\ \cos\beta_2 = \frac{2}{3},\ \cos\gamma_2 = \frac{2}{3}$

(d) $\vec{C} = \vec{A} - \vec{B} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3$ and $|\vec{C}| = |\vec{A}-\vec{B}| = \sqrt{(1)^2+(1)^2+(4)^2} = \sqrt{18} = 3\sqrt{2}$ Unit vector in direction of $\vec{C}$ is $\hat{\mathbf{e}}_C = \frac{\vec{C}}{|\vec{C}|} = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3}{3\sqrt{2}}$. Make note of the fact that the sum of the squares of the direction cosines equals unity.

■

Example 6-5. (The Schwarz inequality)

Show that for any two vectors $\vec{A}$ and $\vec{B}$ one can write the Schwarz inequality $|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$ the equality holding if $\vec{A}$ and $\vec{B}$ are colinear.

Solution If $\vec{A}$ and $\vec{B}$ are nonzero quantities, then $|\vec{A}\cdot\vec{B}|$ must be a positive quantity. Consider a graph of the function

$$\begin{aligned} y = y(x) = &| \vec{A} + x\vec{B} |^2 = (\vec{A}+x\vec{B})\cdot(\vec{A}+x\vec{B}) \\ y(x) = &\vec{A}\cdot\vec{A} + x\vec{A}\cdot\vec{B} + x\vec{B}\cdot\vec{A} + x^2\vec{B}\cdot\vec{B} \\ y(x) = &| \vec{B} |^2\ x^2 + 2(\vec{A}\cdot\vec{B})x + | \vec{A} |^2 = ax^2 + bx + c \end{aligned}$$

Note that if $y(x) > 0$ for all values of x, then this would imply the graph of $y(x)$ must not cross the $x-$axis. If $y(x)$ did cross the $x-$axis, then the equation $y(x) = 0$ would have the two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

in which case the discriminant $b^2 - 4ac$ would be positive. If $y(x)$ does not cross the $x-$axis, then the discriminant would satisfy $b^2 - 4ac \le 0$. Here $b = 2(\vec{A}\cdot\vec{B})$, $a = | \vec{B} |^2$ and $c = | \vec{A} |^2$ and the condition that the discriminant be less than or equal zero can be expressed

$$b^2 - 4ac = 4(\vec{A}\cdot\vec{B})^2 - 4\,| \vec{B} |^2 | \vec{A} |^2 \le 0$$

or
$$| \vec{A}\cdot\vec{B} | \le | \vec{A} || \vec{B} |$$

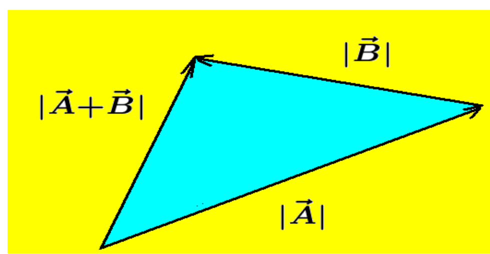
an inequality known as the **Schwarz inequality**.

■

Example 6-6. The triangle inequality

Show that for two vectors $\vec{A}$ and $\vec{B}$ the inequality $| \vec{A}+\vec{B} | \le | \vec{A} | + | \vec{B} |$ must hold.

This inequality is known as **the triangle inequality** and indicates that the length of one side of a triangle is always less than the sum of the lengths of the other two sides.



Solution To prove the triangle inequality one can use the Schwarz inequality from the previous example. Observe that

$$| \vec{A}+\vec{B} |^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

or

$$|\vec{A}+\vec{B}|^2 = |\vec{A}|^2 + 2(\vec{A}\cdot\vec{B}) + |\vec{B}|^2 \le |\vec{A}|^2 + 2|\vec{A}\cdot\vec{B}| + |\vec{B}|^2 \tag{6.19}$$

Using the Schwarz inequality $|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$ the equation (6.19) can be expressed

$$|\vec{A}+\vec{B}|^2 \le |\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|^2 = (|\vec{A}| + |\vec{B}|)^2 \tag{6.20}$$

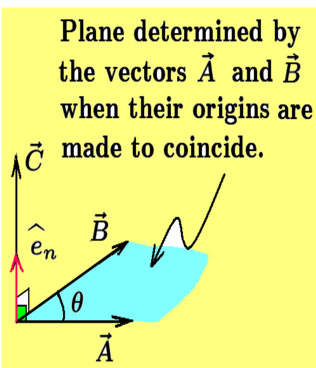
Taking the square root of both sides of the equation (6.20) gives **the triangle inequality** $|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}|$.

■

The Cross Product or Outer Product

The **cross or outer product of two nonzero vectors** $\vec{A}$ and $\vec{B}$ is denoted using the notation $\vec{A}\times\vec{B}$ and represents the construction of a new vector $\vec{C}$ defined as

$$\vec{C} = \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{\mathbf{e}}_n, \tag{6.21}$$

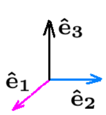


where θ is the smaller angle between the two nonzero vectors $\vec{A}$ and $\vec{B}$ when their origins coincide, and $\hat{\mathbf{e}}_n$ is a unit vector perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide. The direction of $\hat{\mathbf{e}}_n$ is determined by the **right-hand rule**. Place the fingers of your right-hand in the direction of $\vec{A}$ and rotate the fingers toward the vector $\vec{B}$, then the thumb of the right-hand points in the direction $\vec{C}$.

The vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ then form a **right-handed system.**[2] Note that the cross product $\vec{A}\times\vec{B}$ is a vector which will always be perpendicular to the vectors $\vec{A}$ and $\vec{B}$, whenever $\vec{A}$ and $\vec{B}$ are linearly independent.

A special case of the above definition occurs when $\vec{A}\times\vec{B} = \vec{0}$ and in this case one can state that either $\theta = 0$, which implies the vectors $\vec{A}$ and $\vec{B}$ **are parallel** or $\vec{A} = \vec{0}$ or $\vec{B} = \vec{0}$.

Use the above definition of a cross product and show that the orthogonal unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ satisfy the relations



$$\begin{array}{lll}
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 = \vec{0} & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \\
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_2 = \vec{0} & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_1 \\
\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_3 = \vec{0}
\end{array} \tag{6.22}$$

[2] Note many European technical books use left-handed coordinate systems which produces results different from using a right-handed coordinate system.

Properties of the Cross Product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{(noncommutative)}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad \text{(distributive law)}$$

$$m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) \quad m \text{ a scalar}$$

$$\vec{A} \times \vec{A} = \vec{0} \quad \text{since } \vec{A} \text{ is parallel to itself.}$$

Let $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ and $\vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$ be two nonzero vectors in component form and form the cross product $\vec{A} \times \vec{B}$ to obtain

$$\vec{A} \times \vec{B} = (A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3) \times (B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3). \tag{6.23}$$

The cross product can be expanded by using the distributive law to obtain

$$\begin{aligned}\vec{A} \times \vec{B} = \; & A_1B_1\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 + A_1B_2\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 + A_1B_3\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 \\ & + A_2B_1\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 + A_2B_2\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2 + A_2B_3\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 \\ & + A_3B_1\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 + A_3B_2\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 + A_3B_3\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_3.\end{aligned} \tag{6.24}$$

Simplification by using the previous results from equation (6.22) produces the important cross product formula

$$\vec{A} \times \vec{B} = (A_2B_3 - A_3B_2)\,\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\,\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\,\hat{\mathbf{e}}_3, \tag{6.25}$$

This result that can be expressed in **the determinant form**[3]

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \hat{\mathbf{e}}_1 - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \hat{\mathbf{e}}_2 + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \hat{\mathbf{e}}_3. \tag{6.26}$$

In summary, the cross product of two vectors $\vec{A}$ and $\vec{B}$ is a new vector $\vec{C}$, where

$$\vec{C} = \vec{A} \times \vec{B} = C_1\,\hat{\mathbf{e}}_1 + C_2\,\hat{\mathbf{e}}_2 + C_3\,\hat{\mathbf{e}}_3 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

with components

$$C_1 = A_2B_3 - A_3B_2, \qquad C_2 = A_3B_1 - A_1B_3, \qquad C_3 = A_1B_2 - A_2B_1 \tag{6.27}$$

[3] For more information on determinants see chapter 10.

Geometric Interpretation

A geometric interpretation that can be assigned to the magnitude of the cross product of two vectors is illustrated in figure 6-8.

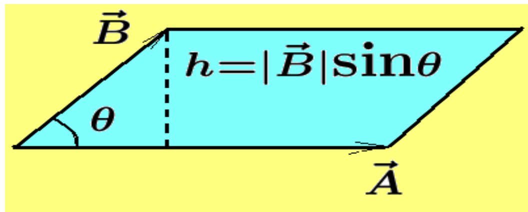


Figure 6-8. Parallelogram with sides $\vec{A}$ and $\vec{B}$.

The area of the parallelogram having the vectors $\vec{A}$ and $\vec{B}$ for its sides is given by

$$\text{Area} = |\vec{A}| \cdot h = |\vec{A}||\vec{B}| \sin\theta = |\vec{A} \times \vec{B}|. \tag{6.28}$$

Therefore, the magnitude of the cross product of two vectors represents the **area of the parallelogram formed from these vectors when their origins are made to coincide.**

Vector Identities

The following vector identities are often needed to simplify various equations in science and engineering.

1. $$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{6.29}$$
2. $$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \tag{6.30}$$
   An identity known as the **triple scalar product**.
3. $$\begin{aligned}(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) &= \vec{C}\left[\vec{D} \cdot (\vec{A} \times \vec{B})\right] - \vec{D}\left[\vec{C} \cdot (\vec{A} \times \vec{B})\right] \\ &= \vec{B}\left[\vec{A} \cdot (\vec{C} \times \vec{D})\right] - \vec{A}\left[\vec{B} \cdot (\vec{C} \times \vec{D})\right] \end{aligned} \tag{6.31}$$
4. $$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{6.32}$$
   The quantity $\vec{A} \times (\vec{B} \times \vec{C})$ is called a **triple vector product**.
5. $$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{6.33}$$
6. The triple vector product satisfies
   $$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0} \tag{6.34}$$

Note that in the triple scalar product $\vec{A} \cdot (\vec{B} \times \vec{C})$ the parenthesis is sometimes omitted because $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is meaningless and so $\vec{A} \cdot \vec{B} \times \vec{C}$ can have only one meaning. The parenthesis just emphasizes this one meaning.

A physical interpretation can be assigned to the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is that **its absolute value represents the volume of the parallelepiped formed by the three noncoplaner vectors $\vec{A},\vec{B},\vec{C}$ when their origins are made to coincide.** The absolute value is needed because sometimes the triple scalar product is negative. This physical interpretation can be obtained from the following analysis.

In figure 6-9 note the following.

(a) The magnitude $|\vec{B}\times\vec{C}|$ represents the area of the parallelogram $PQRS$.

(b) The unit vector $\hat{\mathbf{e}}_n = \dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$ is normal to the plane containing the vectors $\vec{B}$ and $\vec{C}$.

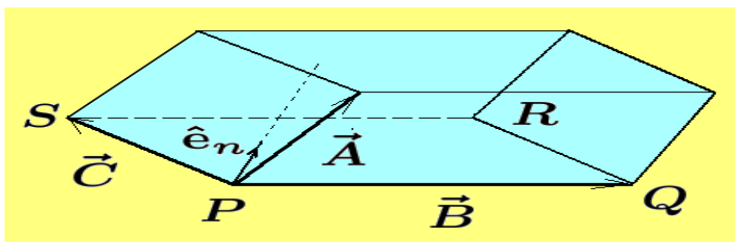


Figure 6-9. Triple scalar product and volume.

(c) The dot product $\vec{A}\cdot\hat{\mathbf{e}}_n = \vec{A}\cdot\dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|} = h$ represents the projection of $\vec{A}$ on $\hat{\mathbf{e}}_n$ and produces the height of the parallelepiped. These results demonstrate that

$$\left|\vec{A}\cdot(\vec{B}\times\vec{C})\right| = |\vec{B}\times\vec{C}|\,h = (\text{Area of base})(\text{Height}) = \text{Volume}.$$

so that the magnitude of the triple scalar product is the volume of the parallelepiped formed when the origins of the three vectors are made to coincide.

Example 6-7. Show that the triple scalar product satisfies the relations

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Note **the cyclic rotation of the symbols** in the above relations where the first symbol is moved to the last position and the second and third symbols are each moved to the left. This is called **a cyclic permutation** of the symbols.

Solution Use the determinant form for the cross product and express the triple scalar product as a determinant as follows.

$$\begin{aligned}
\vec{A}\cdot(\vec{B}\times\vec{C}) =&(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot[(B_2C_3 - B_3C_2)\,\hat{\mathbf{e}}_1 - (B_1C_3 - B_3C_1)\,\hat{\mathbf{e}}_2 + (B_1C_2 - B_2C_1)\,\hat{\mathbf{e}}_3]\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&A_1\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}
\end{aligned}$$

Determinants have the property[4] that the interchange of two rows of a determinant changes its sign. One can then show

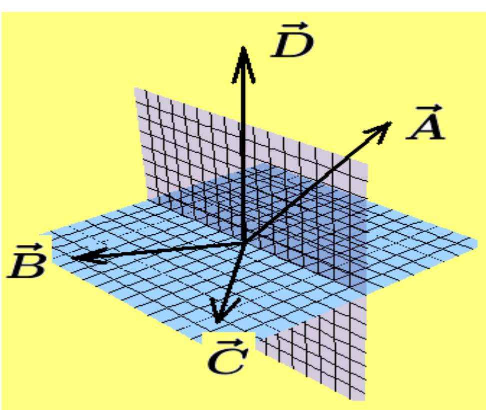
$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

or

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

■

Example 6-8. For nonzero vectors $\vec{A}, \vec{B}, \vec{C}$ show that the triple vector product satisfies $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$
That is, the triple vector product is not associative and the order of execution of the cross product is important.



Solution Let $\vec{B}\times\vec{C} = \vec{D}$ denote the vector perpendicular to the plane determined by the vectors $\vec{B}$ and $\vec{C}$. The vector $\vec{A}\times\vec{D} = \vec{E}$ is a vector perpendicular to the plane determined by the vectors $\vec{A}$ and $\vec{D}$ and therefore must lie in the plane of the vectors $\vec{B}$ and $\vec{C}$. One can then say the vectors $\vec{B}, \vec{C}$ and $\vec{A}\times(\vec{B}\times\vec{C})$ are coplanar and consequently there must exist scalars α and β such that

$$\vec{A}\times(\vec{B}\times\vec{C}) = \alpha\vec{B} + \beta\vec{C} \tag{6.35}$$

[4] See chapter 10 for properties of determinants.

In a similar fashion one can show that the vectors $(\vec{A}\times\vec{B})\times\vec{C},\ \vec{A}$ and $\vec{B}$ are coplanar so that there exists constants γ and δ such that

$$(\vec{A}\times\vec{B})\times\vec{C}=\gamma\vec{A}+\delta\vec{B} \tag{6.36}$$

The equations (6.35) and (6.36) show that in general

$$\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C}$$

■

Example 6-9. Show that the triple vector product satisfies

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\,\vec{B}-(\vec{A}\cdot\vec{B})\,\vec{C}$$

Solution Use the results from the previous example showing there exists scalars α and β such that

$$\vec{A}\times(\vec{B}\times\vec{C})=\alpha\vec{B}+\beta\vec{C} \tag{6.37}$$

Let $\vec{B}\times\vec{C}=\vec{D}$ and write

$$\vec{A}\times\vec{D}=\alpha\vec{B}+\beta\vec{C} \tag{6.38}$$

Take the dot product of both sides of equation (6.38) with the vector $\vec{A}$ to obtain the triple scalar product

$$\vec{A}\cdot(\vec{A}\times\vec{D})=\alpha(\vec{A}\cdot\vec{B})+\beta(\vec{A}\cdot\vec{C})$$

By the permutation properties of the triple scalar product one can write

$$\vec{A}\cdot(\vec{A}\times\vec{D})=\vec{A}\cdot(\vec{D}\times\vec{A})=\vec{D}\cdot(\vec{A}\times\vec{A})=0=\alpha(\vec{A}\cdot\vec{B})+\beta(\vec{A}\cdot\vec{C}) \tag{6.39}$$

The above result holds because $\vec{A}\times\vec{A}=\vec{0}$ and implies

$$\alpha(\vec{A}\cdot\vec{B})=-\beta(\vec{A}\cdot\vec{C})\quad\text{or}\quad\frac{\alpha}{\vec{A}\cdot\vec{C}}=\frac{-\beta}{\vec{A}\cdot\vec{B}}=\lambda$$

where λ is a scalar. This shows that the equation (6.37) can be expressed in the form

$$\vec{A}\times(\vec{B}\times\vec{C})=\lambda(\vec{A}\cdot\vec{C})\vec{B}-\lambda(\vec{A}\cdot\vec{B})\vec{C} \tag{6.40}$$

which shows that the vectors $\vec{A}\times(\vec{B}\times\vec{C})$ and $(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$ are colinear. The equation (6.40) must hold for all vectors $\vec{A},\vec{B},\vec{C}$ and so it must be true in the special case $\vec{A}=\hat{\mathbf{e}}_2,\ \vec{B}=\hat{\mathbf{e}}_1,\ \vec{C}=\hat{\mathbf{e}}_2$ where equation (6.40) reduces to

$$\hat{\mathbf{e}}_2\times(\,\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2)=\hat{\mathbf{e}}_1=\lambda\,\hat{\mathbf{e}}_1\quad\text{which implies}\quad\lambda=1$$

■

Example 6-10.

Derive the **law of sines** for the triangle illustrated in the figure 6-10.

Solution The sides of the given triangle are formed from the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and since these vectors are free vectors they can be moved to the positions illustrated in figure 6-10. Also sketch the vector $-\vec{C}$ as illustrated. The new positions for the vectors $\vec{A}, \vec{B}, \vec{C}$ and $-\vec{C}$ are constructed to better visualize certain vector cross products associated with the law of sines.

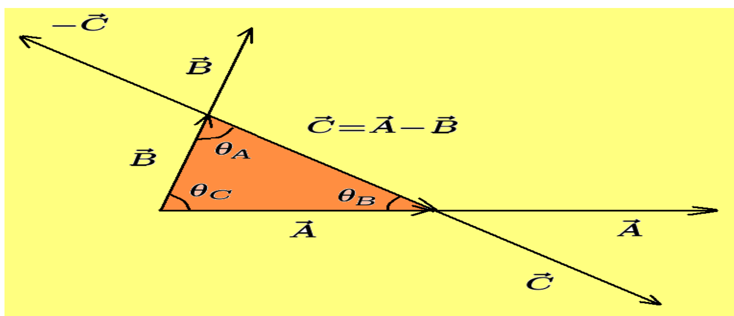


Figure 6-10. Triangle for law of sines.

Examine figure 6-10 and note the following cross products

$$\vec{C} \times \vec{A} = (\vec{A} - \vec{B}) \times \vec{A} = \vec{A} \times \vec{A} - \vec{B} \times \vec{A} = -\vec{B} \times \vec{A} = \vec{A} \times \vec{B}$$

$$\text{and} \quad \vec{B} \times (-\vec{C}) = \vec{B} \times (-\vec{A} + \vec{B}) = \vec{B} \times (-\vec{A}) + \vec{B} \times \vec{B} = \vec{A} \times \vec{B}.$$

Taking the magnitude of the above cross products gives

$$|\vec{C} \times \vec{A}| = |\vec{A} \times \vec{B}| = |\vec{B} \times (-\vec{C})|$$

or

$$AC \sin\theta_B = AB \sin\theta_C = BC \sin\theta_A.$$

Dividing by the product of the vector magnitudes ABC produces the law of sines

$$\frac{\sin\theta_A}{A} = \frac{\sin\theta_B}{B} = \frac{\sin\theta_C}{C}.$$

■

Example 6-11. Derive the **law of cosines** for the triangle illustrated.

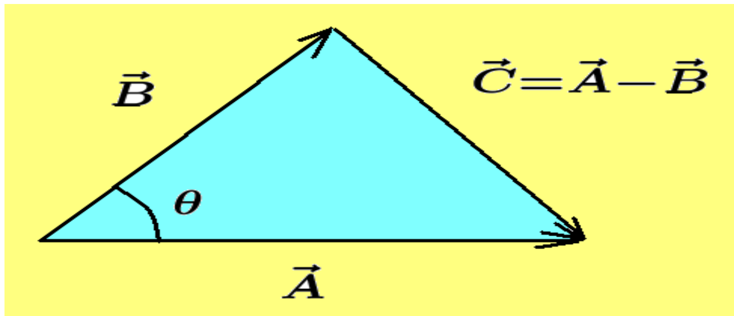


Figure 6-11. Triangle for law of cosines.

Solution Let $\vec{C} = \vec{A} - \vec{B}$ so that the dot product of $\vec{C}$ with itself gives

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B}$$

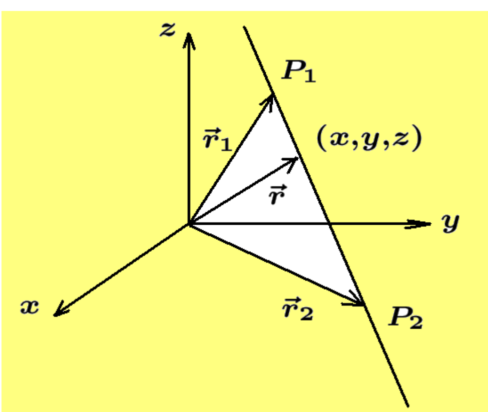
or

$$C^2 = A^2 + B^2 - 2AB\cos\theta,$$

where $A = |\vec{A}|, \quad B = |\vec{B}|, \quad C = |\vec{C}|$ represent the magnitudes of the vector sides.

■

Example 6-12. Find the **vector equation of the line** which passes through the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.



Solution Let

$$\begin{aligned} \vec{r}_1 &= x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3 \\ \text{and} \quad \vec{r}_2 &= x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3 \end{aligned}$$

denote position vectors to the points P_1 and P_2 respectively and let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of any other variable point on the line. Observe that the vector $\vec{r}_2 - \vec{r}_1$ is parallel to the line through the points P_1 and P_2. By vector addition the (x, y, z) position on the line is given by

$$\vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1) \qquad -\infty < \lambda < \infty \tag{6.41}$$

where λ is a scalar parameter. Note that as λ varies from 0 to 1 the position vector $\vec{r}$ moves from $\vec{r}_1$ to $\vec{r}_2$. An alternative form for the equation of the line is given by

$$\vec{r} = \vec{r}_2 + \lambda^*(\vec{r}_1 - \vec{r}_2) \qquad -\infty < \lambda^* < \infty$$

where λ^* is some other scalar parameter. This second form for the line has the position vector $\vec{r}$ moving from $\vec{r}_2$ to $\vec{r}_1$ as λ^* varies from 0 to 1. The vector $\pm(\vec{r}_2 - \vec{r}_1)$ is called **the direction vector of the line**. Equating the coefficients of the unit vectors in the equation (6.41) there results the scalar parametric equations representing the line. These parametric equations have the form

$$x = x_1 + \lambda(x_2 - x_1), \qquad y = y_1 + \lambda(y_2 - y_1), \qquad z = z_1 + \lambda(z_1 - z_2) \qquad -\infty < \lambda < \infty$$

If the quantities $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are different from zero, then the equation for the line can be represented in **the symmetric form**

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \tag{6.42}$$

Note that the equation of a line can also be represented as **the intersection of two planes**

$$\begin{aligned} N_1 x + N_2 y + N_3 z + D_1 &= 0 \\ M_1 x + M_2 y + M_3 z + D_2 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} \vec{N} \cdot (\vec{r} - \vec{r}_0) &= 0 \\ \vec{M} \cdot (\vec{r} - \vec{r}_1) &= 0 \end{aligned}$$

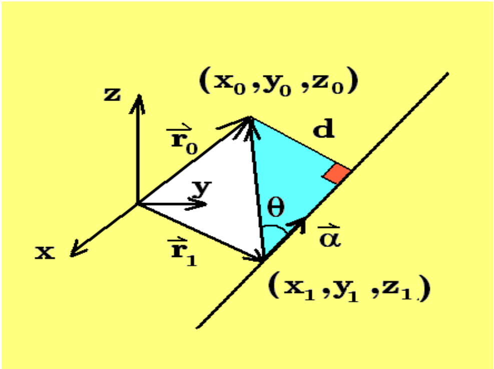
provided the planes are not parallel or $\vec{N} \neq k\vec{M}$, for k a nonzero constant.

■

Example 6-13. Show the **perpendicular distance from a point** (x_0, y_0, z_0) **to a given line** defined by $x = x_1 + \alpha_1 t, \quad y = y_1 + \alpha_2 t, \quad z = z_1 + \alpha_3 t$ is given by

$$d = \left| (\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|} \right| \quad \text{where} \quad \vec{\alpha} = \alpha_1 \,\hat{\mathbf{e}}_1 + \alpha_2 \,\hat{\mathbf{e}}_2 + \alpha_3 \,\hat{\mathbf{e}}_3$$

Solution The **vector equation of the line** is $\vec{r} = \vec{r}_1 + \vec{\alpha}\, t$, where (x_1, y_1, z_1) is a point on the line described by the position vector $\vec{r}_1$ and $\vec{\alpha}$ is **the direction vector of the line**. The vector $\vec{r}_0 - \vec{r}_1$ is a vector pointing from (x_1, y_1, z_1) to the point (x_0, y_0, z_0). These vectors are illustrated in the accompanying figure.

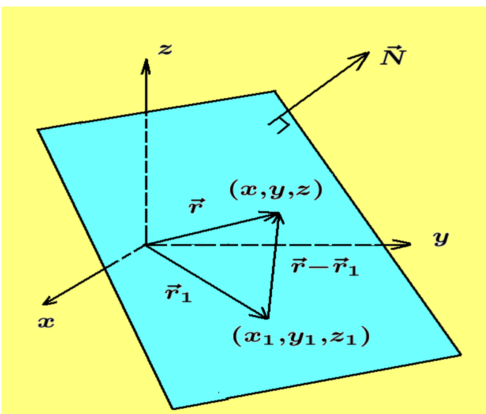


Define the unit vector $\hat{\mathbf{e}}_\alpha = \frac{1}{|\vec{\alpha}|}\vec{\alpha}$ and construct the line from (x_0, y_0, z_0) which is perpendicular to the given line and label this distance d. Our problem is to find the distance d. From the geometry of the right triangle with sides $\vec{r}_0 - \vec{r}_1$ and d one can write $\sin\theta = \dfrac{d}{|\vec{r}_0 - \vec{r}_1|}$. Use the fact that by definition of a cross product one can write

$$|(\vec{r}_0 - \vec{r}_1) \times \hat{\mathbf{e}}_\alpha| = |\vec{r}_0 - \vec{r}_1||\hat{\mathbf{e}}_\alpha|\sin\theta = d = \left|(\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|}\right|$$

■

Example 6-14. Find the **equation of the plane which passes through the point $P_1(x_1, y_1, z_1)$ and is perpendicular to the given vector $\vec{N} = N_1\hat{\mathbf{e}}_1 + N_2\hat{\mathbf{e}}_2 + N_3\hat{\mathbf{e}}_3$.**



Solution Let $\vec{r}_1 = x_1\hat{\mathbf{e}}_1 + y_1\hat{\mathbf{e}}_2 + z_1\hat{\mathbf{e}}_3$ denote the position vector to the point P_1 and let the vector $\vec{r} = x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3$ denote the position vector to any variable point (x, y, z) in the plane. If the vector $\vec{r} - \vec{r}_1$ lies in the plane, then it must be perpendicular to the given vector $\vec{N}$ and consequently the dot product of $(\vec{r} - \vec{r}_1)$ with $\vec{N}$ must be zero and so one can write

$$(\vec{r} - \vec{r}_1) \cdot \vec{N} = 0 \tag{6.43}$$

as the equation representing the plane. In scalar form, the equation of the plane is given as

$$(x - x_1)N_1 + (y - y_1)N_2 + (z - z_1)N_3 = 0 \tag{6.44}$$

■

Example 6-15. Find **the perpendicular distance d from a given plane**

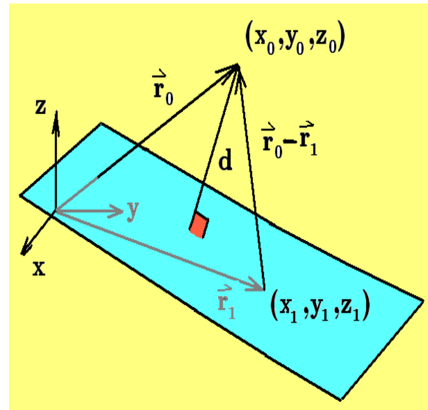
$$(x - x_1)N_1 + (y - y_1)N_2 + (z - z_1)N_3 = 0$$

to a given point (x_0, y_0, z_0).

Solution Let the vector $\vec{r}_0 = x_0\hat{\mathbf{e}}_1 + y_0\hat{\mathbf{e}}_2 + z_0\hat{\mathbf{e}}_3$ point to the given point (x_0, y_0, z_0) and the vector $\vec{r}_1 = x_1\hat{\mathbf{e}}_1 + y_1\hat{\mathbf{e}}_2 + z_1\hat{\mathbf{e}}_3$ point the point (x_1, y_1, z_1) lying in the plane.

Construct the vector $\vec{r}_0 - \vec{r}_1$ which points from the terminus of $\vec{r}_1$ to the terminus of $\vec{r}_0$ and construct the unit normal to the plane which is given by

$$\hat{\mathbf{e}}_N = \frac{N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}}$$



Observe that the dot product $\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)$ equals the projection of $\vec{r}_0 - \vec{r}_1$ onto $\hat{\mathbf{e}}_N$. This gives the distance

$$d = |\,\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)| = \left| \frac{(x_0 - x_1)N_1 + (y_0 - y_1)N_2 + (z_0 - z_{})N_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}} \right|$$

where the absolute value signs guarantee that the sign of d is always positive and does not depend upon the direction selected for the unit vector $\hat{\mathbf{e}}_N$.

■

Moment Produced by a Force

The **moment of a force with respect to a line is a measure of the forces tendency to produce a rotation about the line**. Let a force $\vec{F}$ acting at the point (x_1, y_1, z_1) be resolved into components parallel to the coordinate axes by expressing $\vec{F}$ in the component form

$$\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$$

That component of the force which is parallel to an axis has no tendency to produce a rotation about that axis. For example, the F_1 component is parallel to the $x-$axis and does not produce a rotation about this axis. For a chosen axis, the moment about that axis is the **product of the force component times the perpendicular distance of the force from the axis. By using the right-hand screw rule, one can assign a negative sign to the moment if it acts clockwise and a positive sign to the moment if it acts counterclockwise.** The moment of a force is a vector quantity which produces a definite sense of rotation about an axis.

With the use of figure 6-12 let us calculate the moment of a force $\vec{F}$, acting at the point (x_1, y_1, z_1), about the x-, y- and z-axes.

(a) For the moment about the x-axis produces

$$\begin{aligned} &F_1 \text{ component parallel to } x\text{-axis does not produce moment} \\ &(\text{Force})(\perp \text{ distance}) = +F_3 y_1 \ (\text{Counterclockwise rotation}) \\ &(\text{Force})(\perp \text{ distance}) = -F_2 z_1 \ \ (\text{Clockwise rotation}) \end{aligned}$$

The total moment about the x-axis is therefore the sum of these moments and given by

$$M_1 = F_3 y_1 - F_2 z_1.$$

(b) For the moment about the y-axis, one finds

(Force)($\perp$ distance) $= +F_1 z_1$ (Counterclockwise rotation)

F_2 component parallel to the y-axis does not produce a moment

(Force)($\perp$ distance) $= -F_3 x_1$ (Clockwise rotation)

The total moment about the y-axis is therefore

$$M_2 = F_1 z_1 - F_3 x_1.$$

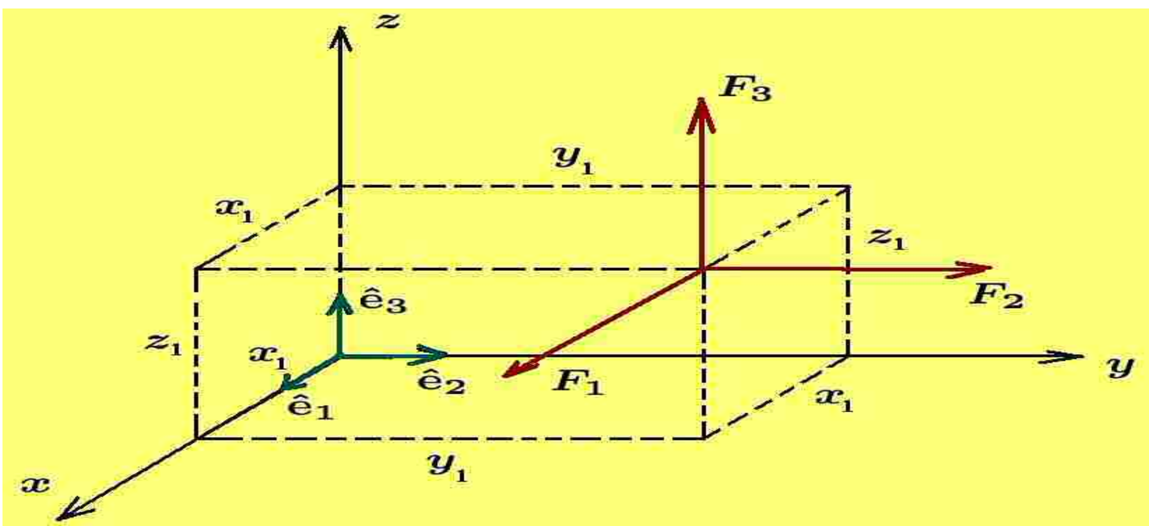


Figure 6-12. Moments produced by a force $\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$

(c) For the moment about the z-axis show that

(Force)($\perp$ distance) $= -F_1 y_1$ (Clockwise rotation)

(Force)($\perp$ distance) $= +F_2 x_1$ (Counterclockwise rotation)

F_3 component parallel to the z-axis does not produce a moment

The total moment about the z-axis is given by

$$M_3 = F_2 x_1 - F_1 y_1.$$

The total moment about the origin is a vector quantity represented as the vector sum of the above moments in the form

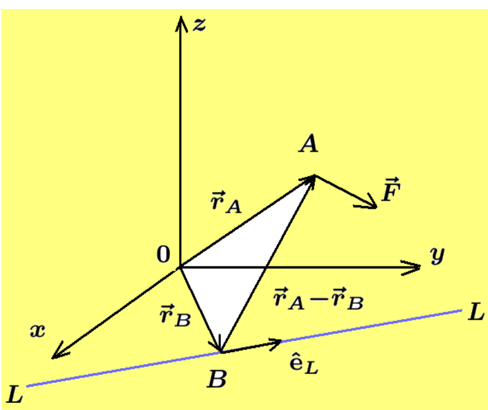
$$\begin{aligned}\vec{M}_0 &= M_1\,\hat{\mathbf{e}}_1 + M_2\,\hat{\mathbf{e}}_2 + M_3\,\hat{\mathbf{e}}_3 \\ &= (F_3y_1 - F_2z_1)\,\hat{\mathbf{e}}_1 + (F_1z_1 - F_3x_1)\,\hat{\mathbf{e}}_2 + (F_2x_1 - F_1y_1)\,\hat{\mathbf{e}}_3.\end{aligned} \tag{6.45}$$

If $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ is the position vector from the origin to the point (x_1, y_1, z_1), then the moment about the origin produced by the force $\vec{F}$ can be expressed as a cross product of the vectors $\vec{r}_1$ and $\vec{F}$ and written as

$$\vec{M}_0 = \vec{r}_1 \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ x_1 & y_1 & z_1 \\ F_1 & F_2 & F_3 \end{vmatrix}. \tag{6.46}$$

This is readily verified by expanding the equation (6.46) and showing the result is given by equation (6.45).

Moment About Arbitrary Line



Assume one has a force $\vec{F}$ acting through a given point A and $\vec{r}_A$ is the position from the origin to the point A. The moment about the origin is given by $\vec{M}_0 = \vec{r}_A \times \vec{F}$. The moment of the given force $\vec{F}$ about the lines representing the x, y and z axes are given by the projections of $\vec{M}_0$ on each of these axes. One finds these moments

$$\vec{M}_0 \cdot \hat{\mathbf{e}}_1 = M_1, \quad \vec{M}_0 \cdot \hat{\mathbf{e}}_2 = M_2, \quad \vec{M}_0 \cdot \hat{\mathbf{e}}_3 = M_3$$

To find the moment about a given line L, choose any point B on the line L and construct the position vector $\vec{r}_B$ from the origin to the point B. The vector $\vec{r}_A - \vec{r}_B$ then points from point B to the force $\vec{F}$ acting at point A as illustrated in the previous figure.

The moment of the force $\vec{F}$ about the point B is given by

$$\vec{M}_B = (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

Observe that this equation for $\vec{M}_B$ represents a position vector from point B to the force $\vec{F}$ crossed with $\vec{F}$ and has the exact same form as equation (6.46). The only difference being where the position vector to the force $\vec{F}$ is constructed. The moment about the line L is then the projection of the vector moment $\vec{M}_B$ on this line. If $\hat{\mathbf{e}}_L$ is a unit vector along the line, then $\vec{M}_B \cdot \hat{\mathbf{e}}_L$ represents the projection of $\vec{M}_B$ on L. The direction of the unit vector $\hat{\mathbf{e}}_L$ on the line L can point in one of two directions (i.e. $\hat{\mathbf{e}}_L$ or $-\hat{\mathbf{e}}_L$). However, once the direction of $\hat{\mathbf{e}}_L$ has been chosen one must be careful to analyze the dot product $\vec{M}_B \cdot \hat{\mathbf{e}}_L$ as its algebraic sign determines the rotation sense produced by the moment (i.e., clockwise or counterclockwise).

A **resultant force** is the algebraic sum of the forces associated with a system. The moment of a resultant force with respect to some axis is equal to the algebraic sum of the moments of the system forces with respect to the same axis.

Example 6-16. If $\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$ is a force acting at the end of the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ then the moment of the force about the origin is $\vec{M} = \vec{r} \times \vec{F} = (yF_3 - zF_2)\,\hat{\mathbf{e}}_1 + (zF_1 - xF_3)\,\hat{\mathbf{e}}_2 + (xF_2 - yF_1)\,\hat{\mathbf{e}}_3$. Make note that the moments of the force components are

$$\begin{aligned}
\vec{M}_1 =& \vec{r} \times (F_1\,\hat{\mathbf{e}}_1) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_1\,\hat{\mathbf{e}}_1) = -yF_1\,\hat{\mathbf{e}}_3 + zF_1\,\hat{\mathbf{e}}_2 \\
\vec{M}_2 =& \vec{r} \times (F_2\,\hat{\mathbf{e}}_2) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_2\,\hat{\mathbf{e}}_2) = xF_2\,\hat{\mathbf{e}}_3 - zF_2\,\hat{\mathbf{e}}_1 \\
\vec{M}_3 =& \vec{r} \times (F_3\,\hat{\mathbf{e}}_3) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_3\,\hat{\mathbf{e}}_3) = -xF_3\,\hat{\mathbf{e}}_2 + yF_3\,\hat{\mathbf{e}}_1
\end{aligned}$$

so that $\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3$

■

Differentiation of Vectors

Let us define what is meant by a derivative associated with a vector and consider some applications of these derivatives. Again notation plays an important part in the representation of the derivatives and therefore many examples are given to help clarify concepts as they arise.

The equation of a space curve can be described in terms of a position vector from the origin of a chosen coordinate system. For example, in cartesian coordinates the position vector of a space curve can have the form

$$\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3, \tag{6.47}$$

where the space curve is defined by the parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t). \tag{6.48}$$

where t represents some convenient parameter, say time. The derivative of the position vector $\vec{r}$ with respect to the parameter t is defined as

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}\end{aligned} \tag{6.49}$$

In component form the derivative is represented in a form where one can recognize the previous definition of a derivative of a scalar function. One finds

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \lim_{\Delta t \to 0} \frac{[x(t+\Delta t)\,\hat{\mathbf{e}}_1 + y(t+\Delta t)\,\hat{\mathbf{e}}_2 + z(t+\Delta t)\,\hat{\mathbf{e}}_3] - [x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3]}{\Delta t} \\ \frac{d\vec{r}}{dt} &= \lim_{\Delta t \to 0} \left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\,\hat{\mathbf{e}}_1 + \frac{y(t+\Delta t) - y(t)}{\Delta t}\,\hat{\mathbf{e}}_2 + \frac{z(t+\Delta t) - z(t)}{\Delta t}\,\hat{\mathbf{e}}_3\right] \\ \frac{d\vec{r}}{dt} &= \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = x'(t)\,\hat{\mathbf{e}}_1 + y'(t)\,\hat{\mathbf{e}}_2 + z'(t)\,\hat{\mathbf{e}}_3\end{aligned}$$

This shows that the derivative of the position vector (6.47) is obtained by differentiating each component of the vector. It will be shown that this derivative represents a vector tangent to the space curve at the point $(x(t), y(t), z(t))$ for any fixed value of the parameter t. Second-order and higher order derivatives are defined as derivatives of derivatives.

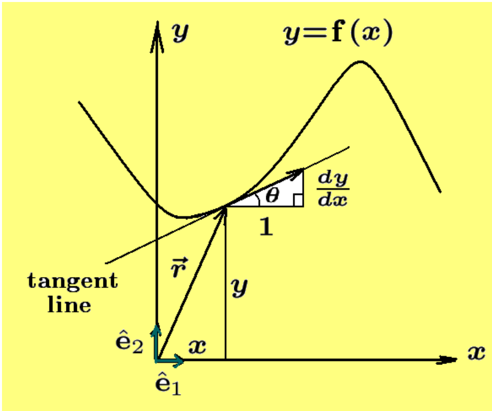
Example 6-17.

The two dimensional curve $y = f(x)$ can be represented by the position vector

$$\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$$

with the derivative

$$\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + \frac{df}{dx}\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 + \frac{dy}{dx}\,\hat{\mathbf{e}}_2$$

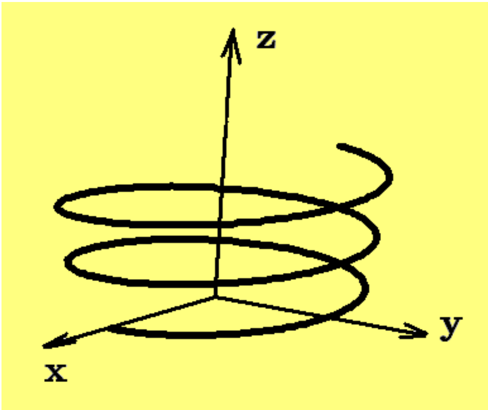


Note that at the point $(x, f(x))$ on the curve one can draw the derivative vector and show that it lies along the tangent line to the curve at the point $(x, f(x))$. This shows that the derivative $\frac{d\vec{r}}{dx}$ is a tangent vector to the curve $y = f(x)$.

In general, if $\vec{r} = \vec{r}(t)$ is the position vector of a three dimensional curve, then the vector $\frac{d\vec{r}}{dt}$ will be a tangent vector to the curve. This can be illustrated by drawing the secant line through the points $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ and showing the secant line then approaches the tangent line as Δt approaches zero.

■

Example 6-18.



Consider the space curve defined by the position vector

$$\vec{r} = \vec{r}(t) = \cos t\,\hat{\mathbf{e}}_1 + \sin t\,\hat{\mathbf{e}}_2 + t\,\hat{\mathbf{e}}_3.$$

This curve sweeps out a spiral called a helix[5]. The projection of the position vector $\vec{r}$ on the plane $z = 0$ generates a circle with unit radius about the origin.

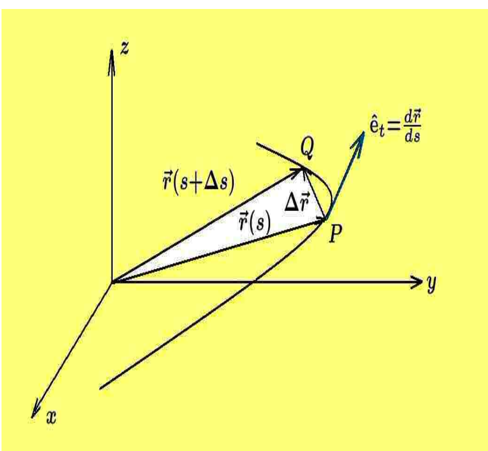
The first and second derivatives of the position vector with respect to the parameter t are

$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$
$$\frac{d^2\vec{r}}{dt^2} = -\cos t\,\hat{\mathbf{e}}_1 - \sin t\,\hat{\mathbf{e}}_2.$$

The vector $\frac{d\vec{r}}{dt}$ is tangent to the curve at the point $(\cos t, \sin t, t)$ for any fixed value of the parameter t.

■

Tangent Vector to Curve



Let s denote the distance along a curve measured from some fixed point on the curve and let the position vector of a point P on the curve be represented as a function of this distance. If the position vector is given by

$$\vec{r} = \vec{r}(s) = x(s)\,\hat{\mathbf{e}}_1 + y(s)\,\hat{\mathbf{e}}_2 + z(s)\,\hat{\mathbf{e}}_3$$

then the derivative with respect to arc length s is defined

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \vec{r}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\vec{r}(s+\Delta s) - \vec{r}(s)}{\Delta s}$$

This limiting statement can be interpreted by the illustration above with the vector $\vec{r}(s)$ pointing to some point P and the vector $\vec{r}(s+\Delta s)$ pointing to some near point

[5] The given equation sweeps out a right-handed helix. Can you determine the equation for a left-handed helix?

Q and the vector $\Delta\vec{r}$ representing the direction of the secant line through the points P and Q.

Letting the point Q approach the point P one finds the direction of the secant line vector $\Delta\vec{r}$ approaches the direction of the tangent to the curve at the point P. In this limiting process one can write

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\vec{r}}{\Delta s} = \frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 + \frac{dz}{ds}\,\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_t$$

where $\hat{\mathbf{e}}_t$ represents a unit tangent vector to the curve. Note that this tangent vector is a **unit vector** since the magnitude of this derivative is

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}} = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$$

since an element of arc length is given by $ds^2 = dx^2 + dy^2 + dz^2$. This shows the vector $\frac{d\vec{r}}{ds}$ is a **unit vector** which is tangent to the space curve $\vec{r} = \vec{r}(s)$.

By using chain rule differentiation one can assign a geometric interpretation to the derivative of a space curve $\vec{r} = \vec{r}(t)$ which is expressed in terms of a time parameter t. Using the chain rule one finds

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\frac{d\vec{r}}{ds} = v\,\hat{\mathbf{e}}_t = \vec{v}$$

Here $v = \frac{ds}{dt}$ is a scalar called speed and represents the change in distance with respect to time. The above equation shows the velocity vector is also tangent to the curve at any instant of time.

Differentiation Formulas

The derivative of any vector $\vec{v} = \vec{v}(t)$ is defined $\lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}$. Note the derivative of a constant vector is zero. Using the property that the limit of a sum is the sum of the limits, the above differentiation formula indicates that **each of the components of a vector must be differentiated**. Here it is assumed the unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_3$ are fixed constants and so their derivatives are zero.

For vector functions of the parameter t

$$\begin{aligned}
\vec{u} &= \vec{u}(t) = u_1(t)\,\hat{\mathbf{e}}_1 + u_2(t)\,\hat{\mathbf{e}}_2 + u_3(t)\,\hat{\mathbf{e}}_3 \\
\vec{v} &= \vec{v}(t) = v_1(t)\,\hat{\mathbf{e}}_1 + v_2(t)\,\hat{\mathbf{e}}_2 + v_3(t)\,\hat{\mathbf{e}}_3, \\
\vec{w} &= \vec{w}(t) = w_1(t)\,\hat{\mathbf{e}}_1 + w_2(t)\,\hat{\mathbf{e}}_2 + w_3(t)\,\hat{\mathbf{e}}_3
\end{aligned}$$

where the components $u_i(t)$,$v_i(t)$ and $w_i(t)$, $i = 1, 2, 3$ are continuous and differentiable, the following differentiation rules can be verified using the definition of a derivative as given by equation (6.49).

The derivative of a sum is the sum of the derivatives and $\frac{d}{dt}(\vec{u}+\vec{v}) = \frac{d\vec{u}}{dt} + \frac{d\vec{u}}{dt}$

The derivative of a dot product of two vectors is the first vector dotted with the derivative of the second vector plus the derivative of the first vector dotted with the second vector and one can write

$$\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{v}$$

The derivative of a cross product of two vectors gives a similar result

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$$

The derivative of a scalar function times a vector is similar to the product rule and one finds

$$\frac{d}{dt}(f(t)\vec{u}) = f(t)\frac{d\vec{u}}{dt} + \frac{df}{dt}\vec{u}$$

where $f = f(t)$ is a scalar function. In the special case $f = c$ is a constant one finds $\frac{d}{dt}(c\vec{u}) = c\frac{d\vec{u}}{dt}$.

If $\vec{u} = \vec{u}(s)$ and $s = s(t)$, then the **chain rule for differentiating vector functions** is given by

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{ds}\frac{ds}{dt}$$

The derivative of a triple scalar product is found to be

$$\frac{d}{dt}(\vec{u} \cdot \vec{v} \times \vec{w}) = \vec{u} \cdot \vec{v} \times \frac{d\vec{w}}{dt} + \vec{u} \cdot \frac{d\vec{v}}{dt} \times \vec{w} + \frac{d\vec{u}}{dt} \cdot \vec{v} \times \vec{w}$$

Each of the above derivative relations can be derived using the definition of a derivative.

Kinematics of Linear Motion

In the study of dynamics or physics one encounters Newton's three laws of motion. These three laws are sometimes expressed in the following form.

1. *A body at rest remains at rest and a body in motion remains in motion, unless acted upon by an external force.*
2. *The time rate of change of the linear momentum of a body is proportional to the force acting on the body, with the body moving in the direction of the applied force.*
3. *For every action there is an equal and opposite reaction.*

If $\vec{r}$ represents the length and direction of a line drawn to the center of mass of a body, then $\frac{d\vec{r}}{dt} = \vec{v}$ represents the instantaneous velocity of the body and $|\vec{v}| = v = |\frac{d\vec{r}}{dt}|$ represents the speed of the body. Let m denote the scalar mass of the body and let $\vec{w}$ denote the vector weight of the body. Here weight is a force given by $\vec{w} = m\vec{g}$, where $\vec{g}$ is the acceleration of gravity[6] Denote by $m\vec{v}$ the linear momentum of the body and let $\vec{F}$ denote the force acting on a body. Using these symbols Newton's second law can be expressed in the form

$$\frac{d}{dt}(m\vec{v}) = k\vec{F}$$

and if the mass m is a constant, then $m\frac{d\vec{v}}{dt} = k\vec{F}$ or $m\vec{a} = k\vec{F}$ where k is a proportionality constant and $\vec{a} = \frac{d\vec{v}}{dt}$ denotes the acceleration of the body. The value of the constant k depends upon the units used to measure distance, time and force.

The following is a set of units for force, mass, distance and time which allow for the proportionality constant to have the value $k = 1$. The notation of brackets around a quantity is used to denote "the dimensions of" the quantity. For example, the notation, $[y] =$ meters, is read, "The dimension of y is meters.[7]

(fps) System

In the foot (ft), pound (lb), second (sec) system of measurements, one uses

$$[distance] = \text{ft}, \qquad [mass] = \text{lb}, \qquad [time] = \text{sec}, \qquad [Force] = \text{slugs}$$

where $1\ slug \cdot ft/sec^2 = 1\ lb\ force$

(cgs) System

In the centimeter (cm), gram (g), second (sec) system of measurements, one uses

$$[distance] = \text{cm}, \qquad [mass] = \text{g}, \qquad [time] = \text{sec}, \qquad [Force] = \text{dynes}$$

where $1\ dyne = 1\ g \cdot m/sec^2$

(mks) System

In the meter (m), kilogram (kg), second (sec) system of measurements, one uses

$$[distance] = \text{m}, \qquad [mass] = \text{kg}, \qquad [time] = \text{sec}, \qquad [Force] = \text{N}$$

where $1\ N = 1\ Newton = 1\ kg \cdot m/sec^2$

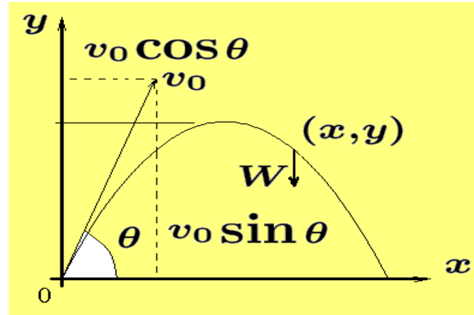
---

[6] The magnitude of the acceleration of gravity g varies between 9.78 $\frac{m}{sec^2}$ and 9.82 $\frac{m}{sec^2}$ and depends upon the position of latitude of the body. In this introduction, all particles and bodies are assumed to accelerate in a gravitational field at the same rate with a value of g=32 $\frac{ft}{sec^2}$ or g=980 $\frac{cm}{sec^2}$ or g=9.8 $\frac{m}{sec^2}$.

[7] Bracket notation for dimensions of a quantity was introduced by J.B.J. Fourier, theorie analytique de la chaleur, Paris, 1822.

Example 6-19.

A cannon ball of mass m is fired from a cannon with an initial velocity v_0 inclined at an angle θ with the horizontal as illustrated. Neglect air resistance and find the equations of motion, maximum height, and range of the cannon ball.



Solution: Let $y = y(t)$ denote the vertical height at any time t and let $x = x(t)$ denote the horizontal distance at any time t. Consider the cannon ball at a position (x, y) and examine the forces acting on it. In the y-direction the force due to the weight of the cannon ball is $W = mg$, $(g = 32\,\text{ft/sec}^2)$. The equation of motion in the y-direction is represented as

$$m\frac{d^2y}{dt^2} = -W = -mg. \tag{6.50}$$

Forces in the x-direction like air resistance are neglected. Newton's second law can then be expressed

$$m\frac{d^2x}{dt^2} = 0. \tag{6.51}$$

Make note of the fact that whenever time t is the independent variable, the dot notation

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \qquad \dot{y} = \frac{dy}{dt}, \qquad \ddot{y} = \frac{d^2y}{dt^2} \tag{6.52}$$

is often employed to denote derivatives. Using the dot notation the equations (6.50) and (6.51) would be represented

$$\ddot{y} = -g \qquad \text{and} \qquad \ddot{x} = 0 \tag{6.53}$$

Calculating the x and y-components of the initial velocity, the equations (6.50) and (6.51) are solved subject to the initial conditions:

$$\begin{aligned} x(0) &= 0, & y(0) &= 0 \\ \dot{x}(0) &= v_0\cos\theta, & \dot{y}(0) &= v_0\sin\theta, \end{aligned}$$

where v_0 is the initial speed and θ is the angle of inclination of the cannon. Solving the differential equations (6.50) and (6.51) by successive integrations gives

$$\begin{aligned} \dot{y} &= -gt + c_1, & \dot{x} &= c_3 \\ y &= -g\frac{t^2}{2} + c_1 t + c_2, & x &= c_3 t + c_4 \end{aligned}$$

where c_1, c_2, c_3, c_4 are constants of integration. The solution satisfying the initial conditions can be expressed as

$$\begin{aligned} y = y(t) = & -\frac{g}{2}t^2 + (v_0 \sin\theta)\, t \\ x = x(t) = & (v_0 \cos\theta)\, t. \end{aligned} \tag{6.54}$$

These are parametric equations describing the position of the cannon ball. The position vector describing the path of the cannon ball is given by

$$\vec{r} = \vec{r}(t) = (v_0 \cos\theta)\, t\, \hat{\mathbf{e}}_1 + (-\frac{g}{2}t^2 + (v_0 \sin\theta)\, t)\, \hat{\mathbf{e}}_2$$

The maximum height occurs where the derivative $\frac{dy}{dt}$ is zero, and the maximum range occurs when the height y returns to zero at some time $t > 0$. The derivative $\frac{dy}{dt}$ is zero when t has the value $t_1 = v_0 \sin\theta / g$, and at this time,

$$y_{max} = y(t_1) = \frac{v_0^2 \sin^2\theta}{2g}, \qquad x = x(t_1) = \frac{v_0^2 \sin 2\theta}{2g} \tag{6.55}$$

The maximum range occurs when $y = 0$ at time $t_2 = 2\dfrac{v_0 \sin\theta}{g}$, and at this time,

$$x_{max} = x(t_2) = \frac{v_0^2 \sin 2\theta}{g}.$$

Eliminating t from the parametric equations (6.54), demonstrates that the trajectory of the cannon ball is a parabola.

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Example 6-20. (Circular motion)

Consider a particle moving on a circle of radius r with **a constant angular velocity** $\omega = \frac{d\theta}{dt}$. Construct a cartesian set of axes with origin at the center of the circle. Assume the position of the particle at any given time t is given by the position vector

$$\vec{r} = \vec{r}(t) = r \cos\omega t\ \hat{\mathbf{e}}_1 + r \sin\omega t\ \hat{\mathbf{e}}_2 \qquad r \text{ and } \omega \text{ are constants.}$$

The **displacement of the particle** as it moves around the circle is given by $s = r\theta$ and the speed of the particle is $\dfrac{ds}{dt} = v = r\dfrac{d\theta}{dt} = r\omega$. The velocity of the particle is a vector quantity given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -r\omega \sin\omega t\ \hat{\mathbf{e}}_1 + r\omega \cos\omega t\ \hat{\mathbf{e}}_2 \tag{6.56}$$

The velocity vector is perpendicular to the position vector $\vec{r}$ since $\vec{v} \cdot \vec{r} = 0$ as can be readily verified. The velocity vector is a free vector and can be moved anywhere and so it is placed at the end of the position vector, as illustrated in the figure 6-13 to show that the velocity is tangent to the circle. The magnitude of the velocity $\vec{v}$ is the speed v given by

$$|\vec{v}| = v = \sqrt{r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t} = r\omega$$

One can define an **angular velocity vector** $\vec{\omega}$ as follows. Use the right-hand rule and point the fingers of your right-hand in the direction of the position vector $\vec{r}$ and then rotate your fingers in the direction of motion of the particle. Your thumb then points in the direction of the **angular velocity vector**. For circular motion counterclockwise in the x, y-plane, one can define the angular velocity vector $\vec{\omega} = \omega\ \hat{\mathbf{e}}_3$. By defining an angular velocity vector one can express the velocity vector of a rotating particle by

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & \omega \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix} = \hat{\mathbf{e}}_1 \left(-\omega r \sin\theta\right) - \hat{\mathbf{e}}_2 \left(-\omega r \cos\theta\right), \qquad \theta = \omega t \tag{6.57}$$

and this equation can be compared with equation (6.56).

The **acceleration of the rotating particle** is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -r\omega^2 \cos\omega t\ \hat{\mathbf{e}}_1 - r\omega^2 \sin\omega t\ \hat{\mathbf{e}}_2 = -\omega^2\vec{r}$$

This shows the acceleration is directed toward the origin. It is therefore called **a centripetal acceleration**.[8] The magnitude of the centripetal acceleration is

$$|\vec{a}| = \omega^2 r = \frac{v^2}{r} = v\omega$$

The acceleration can also be obtained by differentiating the vector velocity given by equation (6.57) to obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \omega \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

and since ω is a constant, then $\frac{d\vec{\omega}}{dt} = 0$ so that the above reduces to

$$\vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2\vec{r}$$

[8] Centripetal means “center-seeking”.

where the last simplification was obtained using **the vector identity** given by equation (6.32) and the result $\omega \cdot \vec{r} = 0$. The above results are derived under the assumption that the angular velocity $\omega = \frac{d\theta}{dt}$ was a constant.

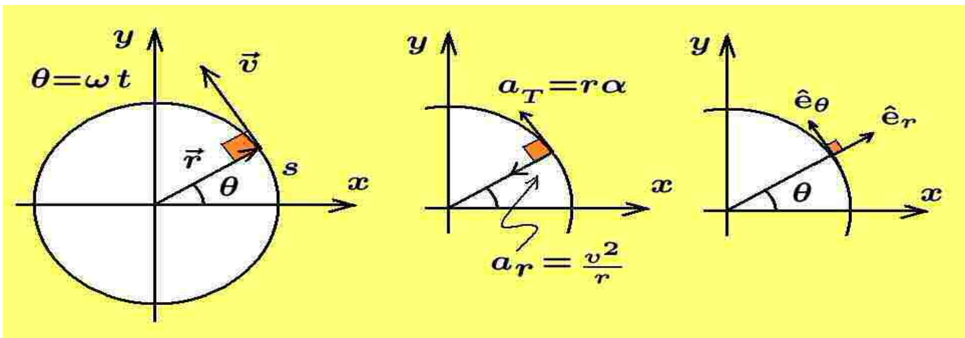

Figure 6-13. Particle moving in circular motion.

In contrast, let us examine what happens if **the angular velocity is not a constant**. The position vector to a particle undergoing circular motion is given by

$$\vec{r} = r\cos\theta\ \hat{\mathbf{e}}_1 + r\sin\theta\ \hat{\mathbf{e}}_2$$

where $\theta = \theta(t)$ is the angular displacement as a function of time. The velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -r\sin\theta\,\frac{d\theta}{dt}\ \hat{\mathbf{e}}_1 + r\cos\theta\,\frac{d\theta}{dt}\ \hat{\mathbf{e}}_2$$

Let $\dfrac{d\theta}{dt} = \omega(t)$ denote the angular speed which is a function of time t and express the velocity as

$$\vec{v} = -r\omega(t)\sin\theta\ \hat{\mathbf{e}}_1 + r\omega(t)\cos\theta\ \hat{\mathbf{e}}_2$$

The acceleration is obtained by taking the derivative of the velocity to obtain

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} =& -r\left[\omega\cos\theta\,\frac{d\theta}{dt} + \frac{d\omega}{dt}\sin\theta\right]\ \hat{\mathbf{e}}_1 + r\left[-\omega(t)\sin\theta\,\frac{d\theta}{dt} + \frac{d\omega}{dt}\cos\theta\right]\ \hat{\mathbf{e}}_2\\ \vec{a} =& -r\omega^2\cos\theta\ \hat{\mathbf{e}}_1 - r\alpha\sin\theta\ \hat{\mathbf{e}}_1 - r\omega^2\sin\theta\ \hat{\mathbf{e}}_2 + r\alpha\cos\theta\ \ \hat{\mathbf{e}}_2\end{aligned}$$

where $\alpha = \alpha(t) = \dfrac{d\omega}{dt}$ is the angular acceleration. The acceleration vector can be broken up into two components by writing

$$\vec{a} = -r\omega^2\left[\cos\theta\ \hat{\mathbf{e}}_1 + \sin\theta\ \hat{\mathbf{e}}_2\right] + r\alpha\left[-\sin\theta\ \hat{\mathbf{e}}_1 + \cos\theta\ \hat{\mathbf{e}}_2\right]$$

The physical interpretation applied to the acceleration vector is as follows. Observe that the vectors

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \qquad \text{and} \qquad \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$

are unit vectors and that these vectors are perpendicular to one another since they satisfy the dot product relation $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$. The vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ represent unit vectors in polar coordinates and are illustrated in the figure 6-13. The acceleration vector can then be expressed in the form

$$\vec{a} = -r\omega^2\,\hat{\mathbf{e}}_r + r\alpha\,\hat{\mathbf{e}}_\theta = \vec{a}_r + \vec{a}_t$$

where $\vec{a}_r = -r\omega^2\,\hat{\mathbf{e}}_r$ is called **the radial component of the acceleration** or **centripetal acceleration** and $\vec{a}_t = r\alpha\,\hat{\mathbf{e}}_\theta$ is called **the tangential component of the acceleration**. These components have the magnitudes

$$|\vec{a}_r| = r\omega^2 \qquad \text{and} \qquad |\vec{a}_t| = r\alpha$$

Note that if ω is a constant, then $\alpha = 0$ and consequently the tangential component of the acceleration will always be zero leaving only the radial component of acceleration.

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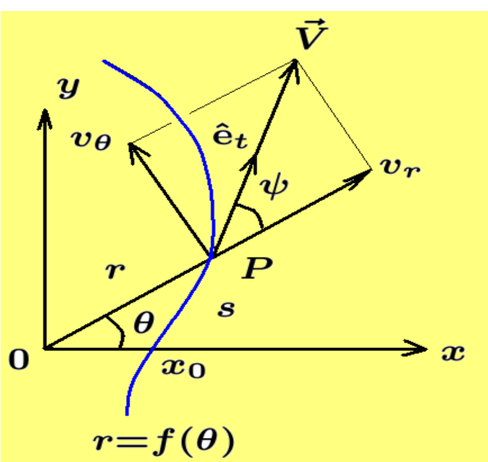
Example 6-21. Transverse and Radial Components of Velocity



Consider the motion of a particle which is described in polar coordinates by an equation of the form $r = f(\theta)$, where θ is measured in radians. Select a point P with coordinates (r, θ) on the curve and construct the radius vector $\vec{r}$ from the origin to the point P. Construct the tangent to the curve at the point P and define the angle ψ between the radius vector and the tangent. Label a fixed point on the curve, say the fixed point x_0 where the curve intersects the x-axis. Let s denote the arc length along the curve measured from x_0 to the point P. The velocity of the particle P as it moves along the curve is given by the change in distance with respect to time t and can be written $v = \dfrac{ds}{dt}$.

The velocity vector is in the direction of the tangent to the curve and **the component of the velocity along the direction** $0P$ is called the **radial component of the velocity** and denoted v_r. At the point P construct a line perpendicular to the line segment $0P$, then the component of the velocity projected onto this perpendicular line segment is called the **transverse component of the velocity** and denoted by v_θ. These projections of the velocity vector give the **radial and transverse components**

$$v_r = v\cos\psi \qquad \text{and} \qquad v_\theta = v\sin\psi$$

where $v = \dfrac{ds}{dt} = \sqrt{v_r^2 + v_\theta^2}$ is the magnitude of the velocity called the **speed of the particle**. The unit tangent vector to the curve is given by

$$\hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the radial direction and $\hat{\mathbf{e}}_\theta$ is a unit vector in the transverse direction. The derivative of the position vector with respect to the time t can be written as

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\;\hat{\mathbf{e}}_t = v\cos\psi\,\hat{\mathbf{e}}_r + v\sin\psi\,\hat{\mathbf{e}}_\theta = v_r\,\hat{\mathbf{e}}_r + v_\theta\,\hat{\mathbf{e}}_\theta$$

Therefore, when the position vector to the point P is written in the form

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \quad \text{or} \quad \vec{r} = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 = r\,\hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is the unit vector in the radial direction given by $\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$, one finds the derivative of this unit vector with respect to θ produces the vector

$$\frac{d\,\hat{\mathbf{e}}_r}{d\theta} = \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$

The derivative of the position vector with respect to arc length is a unit vector so that

$$\begin{aligned} \frac{d\vec{r}}{ds} =& \frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 = r\frac{d\,\hat{\mathbf{e}}_r}{ds} + \frac{dr}{ds}\,\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta \\ \text{where} \quad & \frac{d\,\hat{\mathbf{e}}_r}{ds} = -\sin\theta\frac{d\theta}{ds}\,\hat{\mathbf{e}}_1 + \cos\theta\frac{d\theta}{ds}\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta\,\frac{d\theta}{ds} \end{aligned}$$

Therefore, $\quad \dfrac{d\vec{r}}{ds} = r\dfrac{d\theta}{ds}\,\hat{\mathbf{e}}_\theta + \dfrac{dr}{ds}\,\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta$

Equating like components produces the result that

$$r\frac{d\theta}{ds} = \sin\psi \quad \text{and} \quad \frac{dr}{ds} = \cos\psi$$

The derivative of the position vector $\vec{r} = r\,\hat{\mathbf{e}}_r$ with respect to time t takes on the form

$$\frac{d\vec{r}}{dt} = r\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = r\frac{d\,\hat{\mathbf{e}}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = r\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = v_r\,\hat{\mathbf{e}}_r + v_\theta\,\hat{\mathbf{e}}_\theta = \vec{v}$$

where

$$\begin{aligned}
v_r =& \frac{dr}{dt} = v\cos\psi \quad \text{is the radial component of the velocity}\\
v_\theta =& r\frac{d\theta}{dt} = v\sin\psi \quad \text{is the transverse component of the velocity}\\
\hat{\mathbf{e}}_r =& \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \quad \text{is a unit vector in the radial direction}\\
\hat{\mathbf{e}}_\theta =& -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2 \quad \text{is a unit vector in the transverse direction}\\
\frac{d\vec{r}}{dt} =& \vec{v} = v\cos\psi\,\hat{\mathbf{e}}_r + v\sin\psi\,\hat{\mathbf{e}}_\theta \quad \text{is alternative form for the velocity vector}
\end{aligned}$$

Note also that if $\frac{d\theta}{dt} = \omega$ is the angular velocity, then one can write $v_\theta = r\omega$.

■

Example 6-22. **Angular Momentum**

Recall that a moment causes a rotational motion. Let us investigate what happens when Newton's second law is applied to rotational motion. The **angular momentum of a particle is defined as the moment of the linear momentum**. Let $\vec{H}$ denote the angular momentum; $m\vec{v}$, the linear momentum; and $\vec{r}$, the position vector of the particle, then by definition the moment of the linear momentum is expressed

$$\vec{H} = \vec{r} \times (m\vec{v}) = \vec{r} \times \left(m\frac{d\vec{r}}{dt}\right). \tag{6.58}$$

Differentiating this relation produces

$$\frac{d\vec{H}}{dt} = \vec{r} \times \left(m\frac{d^2\vec{r}}{dt^2}\right) + \frac{d\vec{r}}{dt} \times \left(m\frac{d\vec{r}}{dt}\right).$$

Observe that the second cross product term is zero because the vectors are parallel. Also note that by using Newton's second law, involving a constant mass, one can write

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}.$$

Comparing these last two equations it is found that the time rate of change of angular momentum is expressible in terms of the force $\vec{F}$ acting upon the particle. In particular, one can write

$$\frac{d\vec{H}}{dt} = \vec{r} \times \vec{F} = \vec{M}.$$

One of the many marvelous things introduced by the early Greek mathematicians was that symbols represent ideas and concepts. The symbols in our last equation tell us about a fundamental principal in Newtonian dynamics, that *the time rate of change of angular momentum equals the moment of the force acting on the particle.*

■

Angular Velocity

A **rigid body** is one where **any two distinct points remain a constant distance apart for all time**. A rigid body in motion can be studied by considering **both translational and rotational motion of the points within the body**. Assume there is no translational motion but only rotational motion of the rigid body. A simple rotation of every point in the rigid body, about a line through the body, can be described by (a) an axis of rotation L and (b) an angular velocity vector $\vec{\omega}$. If the axis of rotation remains fixed in space, then all points in the rigid body must move in circular arcs about the line L. Consider a point P revolving about L in a circular path of radius a as illustrated in figure 6-14.

The **average angular speed of the point** P is given by $\frac{\Delta\phi}{\Delta t}$, where $\Delta\phi$ is the angle swept out by P in a time interval Δt. The instantaneous angular speed is a scalar quantity ω determined by

$$\omega = \frac{d\phi}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}.$$

There is **a direction associated with the angular motion of** P **about the line** L and thus **an angular velocity vector** $\vec{\omega}$ **is introduced and defined so that**

(i) $\vec{\omega}$ **has a magnitude or length equal to the angular speed** ω,

(ii) $\vec{\omega}$ **is perpendicular to the plane of the circular path.**

(iii) The direction of $\vec{\omega}$ **is in the direction of advance of a right-hand screw when turned in the direction of rotation.**

Choose any point O on the line L and construct the position vector $\vec{r}$ from O to an arbitrary point P inside the rigid body. The arc length s swept out as P moves through the angle ϕ is given by $s = a\phi$. The magnitude of the linear speed v, of the point P, is given by

$$v = \frac{ds}{dt} = a\frac{d\phi}{dt} = a\omega = |\vec{v}|.$$

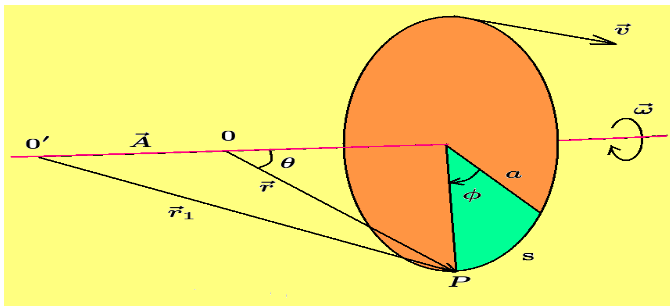


Figure 6-14. Rotation of a rigid body about a line.

The geometry in figure 6-14, is investigated and indicates that $a = |\vec{r}|\sin\theta$, and hence the magnitude of the velocity can be represented as

$$\frac{ds}{dt} = |\vec{v}| = |\vec{\omega}||\vec{r}|\sin\theta.$$

The velocity vector is always normal to the plane containing the position vector and the angular velocity vector. Therefore the velocity vector can be expressed as

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r} = |\vec{\omega}||\vec{r}|\sin\theta\,\hat{\mathbf{e}}_n,$$

where $\hat{\mathbf{e}}_n$ is a unit vector perpendicular to the plane containing the vectors $\vec{\omega}$ and $\vec{r}$. The above arguments demonstrate that the expression for the velocity of a rotating vector is independent of the orientation of the cartesian x-,y-,z-axes as long as the origin of the coordinate system lies on the axis of rotation. To prove this result let O' denote the origin of some new x', y', z' cartesian reference frame with its origin on the axis of rotation. If $\vec{r}_1$ is the position vector from this origin to the same point P considered earlier, one finds that

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 = \vec{\omega} \times \vec{r}_1.$$

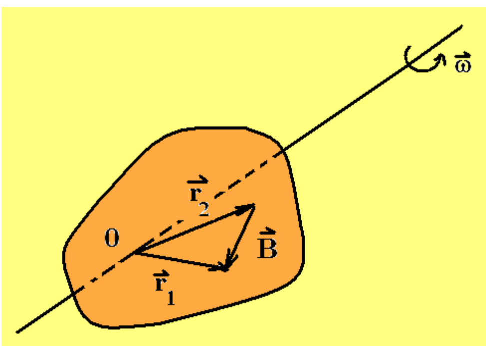
It therefore remains to show that $\vec{v}_1 = \vec{v}$. The geometry of figure 6-14, provides an aid in demonstrating that the vectors $\vec{r}_1$ and $\vec{r}$ are related by the vector equation

$$\vec{r}_1 = \vec{A} + \vec{r},$$

where $\vec{A}$ is a vector from the origin of one system to the origin of the other system and lying along the axis of rotation and in the same direction as $\vec{\omega}$. These results demonstrate that $\omega \times \vec{A} = \vec{0}$ and

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 = \vec{\omega} \times \vec{r}_1 = \vec{\omega} \times (\vec{A} + \vec{r}) = \vec{\omega} \times \vec{A} + \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{r} = \vec{v} = \frac{d\vec{r}}{dt},$$

Here the distributive law for cross products has been employed and the fact that both $\vec{\omega}$ and $\vec{A}$ have the same direction produced a cross product of zero.



Let $\vec{B}$ denote any vector connecting two fixed points within a rigid body which is rotating about a line with angular velocity $\vec{\omega}$. Let $\vec{r}_1$ denote a vector to the terminus of $\vec{B}$ and let $\vec{r}_2$ denote a vector to the origin of $\vec{B}$, as measured from **some origin on the axis of rotation**. One can then write

$$\frac{d\vec{r}_1}{dt} = \omega \times \vec{r}_1 \qquad \text{and} \qquad \frac{d\vec{r}_2}{dt} = \omega \times \vec{r}_2$$

Observe that by vector addition $\vec{r}_2 + \vec{B} = \vec{r}_1$ so that

$$\frac{d\vec{B}}{dt} = \frac{d\vec{r}_1}{dt} - \frac{d\vec{r}_2}{dt} = \omega \times \vec{r}_1 - \omega \times \vec{r}_2 = \omega \times (\vec{r}_1 - \vec{r}_2) = \omega \times \vec{B}$$

Therefore one can state that in general, if $\vec{B}$ **is any fixed vector lying within a rigid body which is rotating, then with respect to any origin on the axis of rotation, one can state that**

$$\frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B} \tag{6.59}$$

This is an important result used in the study of rotating bodies.

Two-Dimensional Curves

The graphical representation of a function $y = f(x)$ in a rectangular cartesian coordinate system can also be presented in a vector language. A graph of the function $y = f(x)$ can be represented by a position vector $\vec{r}$, measured from the origin, which sweeps out the curve as the parameter x varies. In figure 6-15, the position vector $\vec{r}$ is illustrated. This position vector has the representation

$$\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2. \tag{6.60}$$

As the parameter x varies, the position vector $\vec{r}$ represents the distance and direction of the point $(x, f(x))$ with respect to the origin. The derivative

$$\frac{d\vec{r}}{dx} = \vec{r}\,'(x) = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2 \tag{6.61}$$

is also illustrated in figure 6-15. Observe that the derivative represents the tangent vector to the curve at the point $(x, f(x))$. There can be two tangent vectors to the curve at $(x, f(x))$, namely $\dfrac{d\vec{r}}{dx} = \vec{r}\,'(x)$ and $-\dfrac{d\vec{r}}{dx} = -\vec{r}\,'(x)$. Unless otherwise stated, the tangent vector in the sense of increasing parameter x is to be understood.

The cross product of the unit vector $\hat{\mathbf{e}}_3$, out of the plane of the curve, and the tangent vector $\frac{d\vec{r}}{dx} = \vec{r}\,'(x)$ to the curve, gives a normal vector $\vec{N}$ to the curve at the point $(x, f(x))$. One calculates this normal vector using the cross product

$$\vec{N} = \hat{\mathbf{e}}_3 \times \frac{d\vec{r}}{dx} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & 1 \\ 1 & f'(x) & 0 \end{vmatrix} = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \tag{6.62}$$

Note **there can be two normal vectors to the curve at the point** $(x, f(x))$, namely $\vec{N}$ and $-\vec{N}$. To verify that $\vec{N}$ is normal to the tangent vector at a general point $(x, f(x))$ one can examine the dot product of the normal and tangent vector $\vec{N} \cdot \dfrac{d\vec{r}}{dx}$ and show

$$\vec{N} \cdot \frac{d\vec{r}}{dx} = [-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2] \cdot [\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2] = 0$$

which demonstrates these two vectors are perpendicular to one another. Further, the magnitudes of the normal vector $\vec{N}$ and the tangent vector $\frac{d\vec{r}}{dx}$ are equal and can be represented

$$|\vec{N}| = \left|\frac{d\vec{r}}{dx}\right| = \sqrt{1 + [f'(x)]^2}. \tag{6.63}$$

One can use the magnitudes of the tangent and normal vectors to construct unit vectors in the tangent and normal directions at each point $(x, f(x))$ on the plane curve. One finds these unit vectors have the form

$$\hat{\mathbf{e}}_t = \frac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}} \qquad \text{and} \qquad \hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}. \tag{6.64}$$

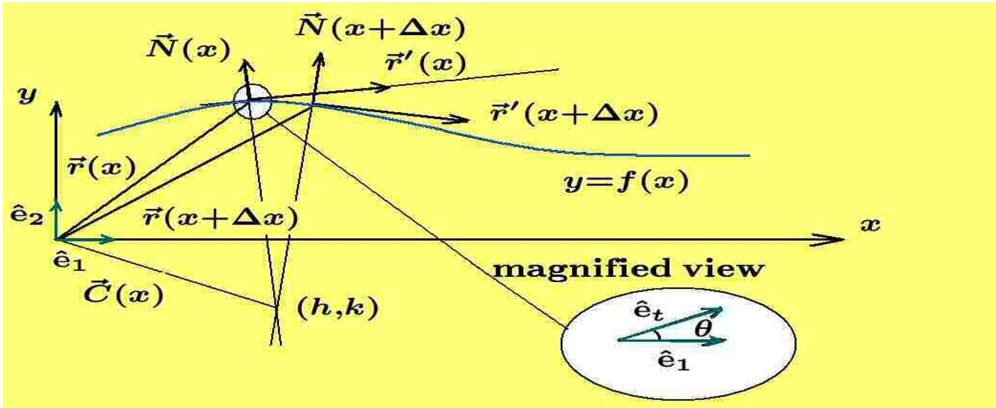


Figure 6-15.
Tangent line and normal line to plane curve change with position.

Recall from our earlier study of calculus that the arc length s measured along a curve from some fixed point $(x_0, f(x_0))$ is given by

$$s = \int_{x_0}^{x} \sqrt{1 + [f'(x)]^2}\, dx \tag{6.65}$$

and the derivative of this arc length with respect to the parameter x is

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}. \tag{6.66}$$

Using chain rule differentiation one finds

$$\frac{d\vec{r}}{ds}\frac{ds}{dx} = \frac{d\vec{r}}{dx} = \frac{d\vec{r}}{ds}\sqrt{1 + [f'(x)]^2}$$

or

$$\hat{\mathbf{e}}_t = \frac{d\vec{r}}{ds} = \frac{1}{\sqrt{1 + [f'(x)]^2}}\frac{d\vec{r}}{dx}$$

which shows the unit tangent vector to the curve is the derivative of the position vector with respect to arc length. The choice of the sign on the square root determines the direction of the unit tangent vector.

At each point on the plane curve the unit tangent vector $\hat{\mathbf{e}}_t$ makes an angle θ with the constant unit vector $\hat{\mathbf{e}}_1$. The **absolute value of the rate of change of this angle with respect to arc length is called the curvature** and is denoted by the Greek letter κ. The curvature is thus represented by

$$\kappa = \left|\frac{d\theta}{ds}\right|.$$

By using the results $\tan\theta = \dfrac{dy}{dx}$ and $ds^2 = dx^2 + dy^2$, one can calculate the derivatives

$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \qquad \text{and} \qquad \frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The chain rule for differentiation can be employed to calculate the curvature

$$\kappa = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\theta}{dx}\frac{dx}{ds}\right| = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{6.67}$$

The unit tangent vector $\hat{\mathbf{e}}_t$ satisfies $\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t = 1$. Differentiating this relation with respect to arc length s and simplifying produces

$$\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \cdot \hat{\mathbf{e}}_t = 2\,\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} = 0. \tag{6.68}$$

When the dot product of two nonzero vectors is zero, the two vectors are perpendicular to one another. Hence, the vector $\dfrac{d\,\hat{\mathbf{e}}_t}{ds}$ is perpendicular to the tangent vector $\hat{\mathbf{e}}_t$ when evaluated at a common point on the curve. It is known that the vector $\hat{\mathbf{e}}_n$ is perpendicular to the tangent vector. The vectors $\hat{\mathbf{e}}_n$ and $\dfrac{d\,\hat{\mathbf{e}}_t}{ds}$ are therefore colinear. Consequently there exists a suitable constant c such that

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = c\,\hat{\mathbf{e}}_n.$$

It is now demonstrated that $c = \kappa$, the curvature associated with the curve. To solve for the constant c differentiate $\hat{\mathbf{e}}_t$ with respect to the arc length s. From the expression

$$\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dx} = \frac{d\,\hat{\mathbf{e}}_t}{dx} = \frac{\sqrt{1 + [f'(x)]^2}f''(x)\,\hat{\mathbf{e}}_2 - [\,\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2][1 + [f'(x)]^2]^{-\frac{1}{2}}f'(x)f''(x)}{1 + [f'(x)]^2},$$

the derivative of the unit tangent vector with respect to arc length is given by

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{f''(x)}{[1 + [f'(x)]^2]^{\frac{3}{2}}}\left[\frac{-f'(x)\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}\right] = \frac{f''(x)}{[1 + [f'(x)]^2]^{\frac{3}{2}}}\,\hat{\mathbf{e}}_n.$$

Taking the absolute value of both sides of this equation shows that the scalar curvature κ is a function of position and is given by

$$\kappa = \frac{|f''(x)|}{[1 + [f'(x)]^2]^{\frac{3}{2}}}.$$

The **reciprocal of the curvature κ is called the radius of curvature** ρ. Note that **straight lines have a constant angle θ between a unit tangent vector and the x-axis and hence the curvature of straight lines is zero** since the curvature is a measure of how fast the tangent vector is changing with respect to arc length.

To understand the meaning of the radius of curvature, consider the vectors $\vec{N}(x)$ and $\vec{N}(x+\Delta x)$ which are normal to the curve $y = f(x)$ at the points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$. These vectors are illustrated in figure 6-15. For appropriate scalars α and β, the vector equations

$$\vec{C}(x) = \vec{r}(x) + \alpha\vec{N}(x) \quad \text{and} \quad \vec{C}(x) = \vec{r}(x+\Delta x) + \beta\vec{N}(x+\Delta x)$$

depict the common point of intersection (h,k) of these normal lines to the plane curve, provided these normal lines are not parallel. The scalars α and β are related by the vector equation

$$\vec{r}(x) + \alpha\vec{N}(x) = \vec{r}(x+\Delta x) + \beta\vec{N}(x+\Delta x). \tag{6.69}$$

If in the limit as $\Delta x \to 0$, the point of intersection (h,k) approaches a specific value, this limit point is called **the center of curvature.** To find the center of curvature (h,k), the scalar α (or β) must be determined. This is accomplished by expanding the above equations relating α and β. When the vector equation (6.69) is expanded, one finds have

$$x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2 - \alpha f'(x)\,\hat{\mathbf{e}}_1 + \alpha\,\hat{\mathbf{e}}_2 = (x+\Delta x)\,\hat{\mathbf{e}}_1 + f(x+\Delta x)\,\hat{\mathbf{e}}_2 - \beta f'(x+\Delta x)\,\hat{\mathbf{e}}_1 + \beta\,\hat{\mathbf{e}}_2.$$

Equate like components the two scalar equations and show

$$\begin{gathered} x - \alpha f'(x) - (x+\Delta x) + \beta f'(x+\Delta x) = 0 \qquad \text{and} \\ f(x) + \alpha - f(x+\Delta x) - \beta = 0. \end{gathered} \tag{6.70}$$

By eliminating β from these two equations one finds

$$\alpha\left[\frac{f'(x+\Delta x) - f'(x)}{\Delta x}\right] = 1 + \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] f'(x+\Delta x). \tag{6.71}$$

In this equation let $\Delta x \to 0$ and solve for α and find

$$\alpha = \frac{1 + [f'(x)]^2}{f''(x)}, \qquad f''(x) \neq 0. \tag{6.72}$$

From this result the center of curvature is found to have the position vector

$$\vec{C}(x) = \vec{r}(x) + \alpha \vec{N}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2 + \frac{1+[f'(x)]^2}{f''(x)}\left[-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\right]. \tag{6.73}$$

Note the position vector can also be expressed in the form

$$\vec{C}(x) = \vec{r}(x) + \rho\,\hat{\mathbf{e}}_n, \qquad \text{where} \qquad \rho = \frac{1}{\kappa}, \tag{6.74}$$

and consequently the coordinates of the center of curvature can be determined. These coordinates are given by

$$h = x - \frac{f'(x)}{f''(x)}(1+[f'(x)]^2) \quad \text{and} \quad k = f(x) + \frac{1}{f''(x)}(1+[f'(x)]^2), \tag{6.75}$$

provided that $f''(x) \neq 0$. For $f''(x) = 0$, there is a point of inflection, and the circle of curvature degenerates into a straight line which is the tangent line to the point of inflection of the curve. Consider the set of all circles which have their centers on the normal line to the curve and which pass through the point where the normal line intersects the curve (i.e., circles are tangent to the tangent vectors). Of all the circles, **there is only one which has a contact of the second order** and this circle has its center at the center of curvature (h, k). A contact of second order means that not only does the circle and curve have a common point of intersection and a common first derivative but also that they have a common second derivative. A proof of these statements is now offered. Let the equation of the circle be denoted by

$$(\xi - h)^2 + (\eta - k)^2 = \rho^2, \tag{6.76}$$

where the (ξ, η) axes coincide with the (x, y) axes and h, k, ρ are the functions of x derived above. If one considers x as being held constant and treats η as a function of ξ, then by differentiating the equation of the circle (6.76) twice one produces the derivatives

$$(\xi - h) + (\eta - k)\frac{d\eta}{d\xi} = 0 \qquad \text{and} \qquad 1 + \left(\frac{d\eta}{d\xi}\right)^2 + (\eta - k)\frac{d^2\eta}{d\xi^2} = 0 \tag{6.77}$$

At the common point of intersection where $(\xi, \eta) = (x, f(x))$ one finds

$$\xi - h = \frac{f'(x)}{f''(x)}(1+[f'(x)]^2) \qquad \text{and} \qquad \eta - k = -\frac{1}{f''(x)}(1+[f'(x)]^2)$$

so that

$$\frac{d\eta}{d\xi} = -\frac{\xi - h}{\eta - k} = f'(x) \qquad \text{and} \qquad \frac{d^2\eta}{d\xi^2} = -\frac{1 + \left(\frac{d\eta}{d\xi}\right)^2}{\eta - k} = f''(x)$$

This shows that the first and second derivatives at the common point of intersection of the curve and circle are the same and so this intersection is called **a contact of order two**.

Scalar and Vector Fields

Of extreme importance in science and engineering are the concepts of a **scalar field** and a **vector field**.

Scalar and vector fields

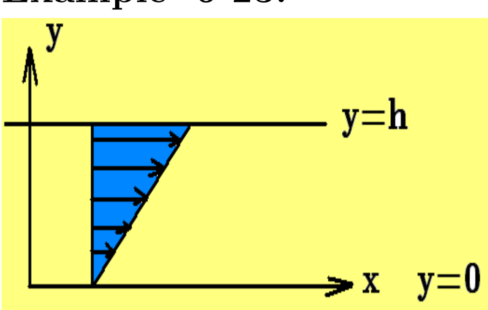
Let R denote a region of space in a cartesian coordinate system. If corresponding to each point (x, y, z) of the region R there corresponds a scalar function $\phi = \phi(x, y, z)$, then a scalar field is said to exist over the region R. If to each point (x, y, z) of a region R there corresponds a vector function

$$\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{e}_1 + F_2(x, y, z)\,\hat{e}_2 + F_3(x, y, z)\,\hat{e}_3,$$

then a vector field is said to exist in the region R.

That is, a scalar field is a one-to-one correspondence between points in space and scalar quantities and a vector field is a one-to-one correspondence between points in space and vector quantities. The functions which occur in the representation of a vector or scalar fields are assumed to be single valued, continuous, and differentiable everywhere within their region of definition.

Example 6-23.



An example of a vector field is the velocity of a fluid. In such a velocity field, at each point in some specified region a velocity vector exists which describes the fluid velocity. The velocity vector is a function of position within the specified region. Consider water flowing in a channel having a depth h as illustrated. Construct a set of x, y axes with $y = 0$ representing the bottom of the channel and $y = h$ representing the top of the channel. If the velocity of the fluid in the channel is given by the one-dimensional vector field $\vec{v} = \alpha y\,\hat{\mathbf{e}}_1$, for $0 \le y \le h$ and α is some proportionality constant, then the vector field associated with this flow can be graphically illustrated by sketching the vectors $\vec{v}$ at various depths in the channel. The resulting images represent one way of illustrating a vector field. The resulting sketch is called a vector field plot.

■

Example 6-24.

Consider the two-dimensional vector field $\vec{v} = \vec{v}(x, y) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2$. There are computer programs that can graphically illustrate this vector field by plotting vectors at selected points within a specified region. The resulting images of all the vectors illustrated at a finite set of points is called a vector field plot. The figure 6-16 illustrates a vector field plot for the above vector $\vec{v}$ sketched at selected points over the region $R = \{(x, y) \mid -5 \le x \le 5,\ -5 \le y \le 5\ \}$.

An alternative method of illustrating a vector field is to define a set of curves, called field lines, where each curve has the property that at each point (x, y) on any curve, the tangent to the curve at (x, y) has **the same direction as the vector field at that point**. If $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ is a position vector to a point (x, y) on a field line, then $d\vec{r}$ gives the direction of the tangent line and if this direction is to have the same direction as $\vec{v}$, then the two directions must be proportional and requires that

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 = k\vec{v}(x, y) = k\,[y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2] = ky\,\hat{\mathbf{e}}_1 + kx\,\hat{\mathbf{e}}_2 \tag{6.78}$$

where k is some proportionality constant.

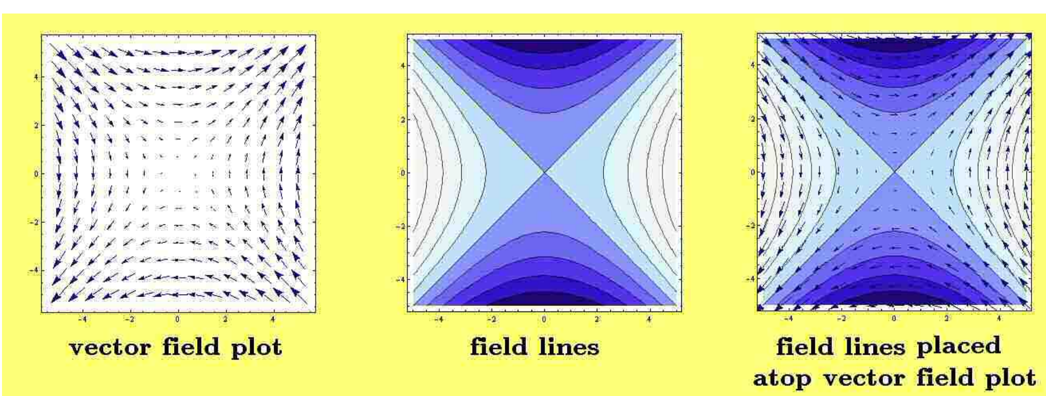


Figure 6-16. Vector field plot for $\vec{v} = \vec{v}(x, y) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2$

If these direction are the same, then by equating like components one must have

$$dx = ky \quad \text{and} \quad dy = kx \qquad \text{or} \qquad \frac{dx}{y} = \frac{dy}{x} = k \tag{6.79}$$

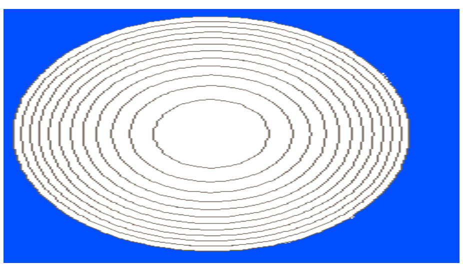
The equation (6.79) requires that $x\,dx = y\,dy$ and if one integrates both sides of this equation one obtains the family of field lines

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{C}{2} \qquad \text{or} \qquad x^2 - y^2 = C \tag{6.80}$$

where $C/2$ is selected as the constant of integration to make all terms have a factor of 2 in the denominator. Plotting these curves over the region R for various values of the constant C gives the field lines illustrated in the figure 6-16. The final figure in figure 6-16 illustrates the field lines atop the vector field plot in order that you can get a comparison of the two techniques.

■

Example 6-25.



An example of a two-dimensional scalar field is a scalar function $\phi = \phi(x,y)$ representing the temperature at each point (x,y) inside some specified region. The scalar field can be visualized by plotting the family of curves $\phi(x,y) = C$ for various values of the constant C.

The resulting family of curves are called level curves and represent curves where the temperature has a constant value. If the scalar field $\phi = \phi(x,y)$ represented height of the water above some reference point, then one can think of say an island where at different times the level of the water makes a contour of the island shape. In this case the family of curves $\phi(x,y) = C$, for various values of the constant C, are called level curves or contour plots since at various heights C the contour of the island is given. Example contour plots are illustrated in figures 6-16 and 6-17.

Note that there are many computer programs capable of drawing contour plots or level curves associated with a given scalar function. The figure 6-17 illustrates contour plots or level curves for several different two-dimensional scalar functions as the level C changes.

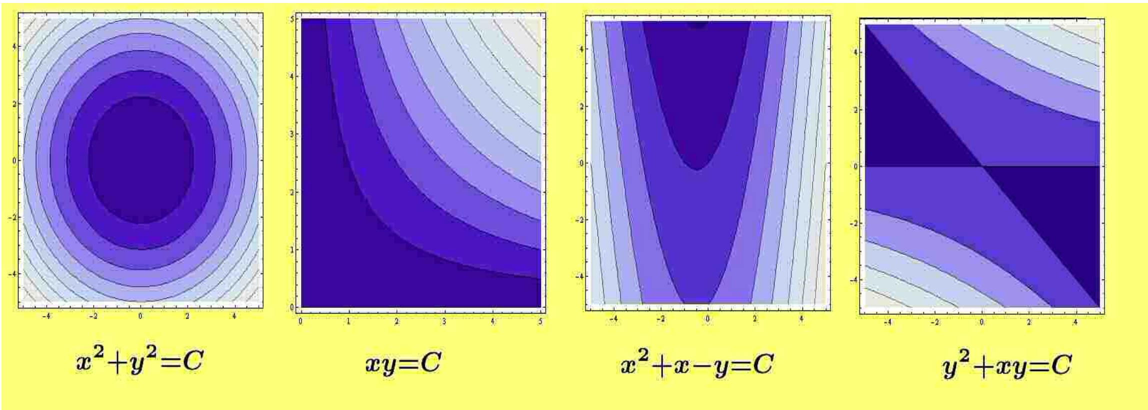


Figure 6-17. Contour plots of selected two-dimensional scalar functions.

■

A vector field is a one–to–one correspondence between points in space and vector quantities, whereas a scalar field is a one–to–one correspondence between points in space and scalar quantities. The concept of scalar and vector fields has many generalizations. A scalar field assigns a single number $\phi(x,y,z)$ to each point of space. A two-dimensional vector field would assign two numbers $(F_1(x,y,z),\ F_2(x,y,z))$ to each point of space, and a three-dimensional vector field would assign three numbers $(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ to each point of space. An immediate generalization would be that an n-dimensional vector field would assign an n-tuple of numbers $(F_1, F_2, \dots, F_n)$ to each point of space. Here each component F_i is a function of position, and one can write

$$F_i = F_i(x,y,z),\ \ i = 1,\dots,n.$$

Other immediate ideas that come to mind are the concepts of assigning n^2 numbers to each point in space or n^3 numbers to each point in space. These higher dimensional correspondences lead to the study of matrices and tensor fields which are functions of position. In science and engineering, there is great interest in how such scalar and vector fields change with position and time.

Partial Derivatives

If a vector field $\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ is referenced with respect to a fixed set of cartesian axes, then the partial derivatives of this vector field are given by:

$$\begin{aligned}
\frac{\partial \vec{F}}{\partial x} &= \frac{\partial F_1}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial x}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial x}\,\hat{\mathbf{e}}_3 \\
\frac{\partial \vec{F}}{\partial y} &= \frac{\partial F_1}{\partial y}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial y}\,\hat{\mathbf{e}}_3 \\
\frac{\partial \vec{F}}{\partial z} &= \frac{\partial F_1}{\partial z}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial z}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial z}\,\hat{\mathbf{e}}_3.
\end{aligned} \tag{6.81}$$

Observe that each component of the vector field $\vec{F}$ must be differentiated.

The higher partial derivatives are defined as derivatives of derivatives. For example, the second order partial derivatives are given by the expressions

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial x}\right), \quad \frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial y}\right), \quad \frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial y}\right), \tag{6.82}$$

where each component of the vectors are differentiated. This is analogous to the definitions of higher derivatives previously considered.

Total Derivative

The total differential of a vector field $\vec{F} = \vec{F}(x, y, z)$ is given by

$$d\vec{F} = \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial y}dy + \frac{\partial \vec{F}}{\partial z}dz$$

or

$$\begin{aligned}
d\vec{F} = &\left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right)\hat{\mathbf{e}}_1 \\
&+ \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right)\hat{\mathbf{e}}_2 \\
&+ \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right)\hat{\mathbf{e}}_3.
\end{aligned} \tag{6.83}$$

Example 6-26.

For the vector field

$$\vec{F} = \vec{F}(x, y, z) = (x^2 y - z)\,\hat{\mathbf{e}}_1 + (yz^2 - x)\,\hat{\mathbf{e}}_2 + xyz\,\hat{\mathbf{e}}_3$$

calculate the partial derivatives

$$\frac{\partial \vec{F}}{\partial x}, \quad \frac{\partial \vec{F}}{\partial y}, \quad \frac{\partial \vec{F}}{\partial z}, \qquad \frac{\partial^2 \vec{F}}{\partial x \partial y}$$

Solution: Using the above definitions produces the results

$$\begin{aligned}
\frac{\partial \vec{F}}{\partial x} &= 2xy\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + yz\,\hat{\mathbf{e}}_3, & \frac{\partial \vec{F}}{\partial z} &= -\hat{\mathbf{e}}_1 + 2yz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3 \\
\frac{\partial \vec{F}}{\partial y} &= x^2\,\hat{\mathbf{e}}_1 + z^2\,\hat{\mathbf{e}}_2 + xz\,\hat{\mathbf{e}}_3, & \frac{\partial^2 \vec{F}}{\partial x \partial y} &= \frac{\partial^2 \vec{F}}{\partial y \partial x} = 2x\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_3.
\end{aligned}$$

■

Notation

The position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ is sometimes represented in matrix notation as a row vector $\vec{r} = (x, y, z)$ or a column vector $\vec{r} = \text{col}(x, y, z)$. Sometimes the substitution $x = x_1,\ y = x_2$ and $z = x_3$ is made and these vectors are represented as $\vec{x} = (x_1, x_2, x_3)$ or $\vec{x} = \text{col}(x_1, x_2, x_3)$ and a vector function

$$\vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$$

is represented in the form of either a row vector or column vector

$$\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x})) \quad \text{or} \quad \vec{F}(\vec{x}) = \text{col}(F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x}))$$

where the representation of the basis vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ is to be understood and col is used to denote a column vector.

This change in notation is made in order that scalar and vector concepts can be extended to represent scalars and vectors in higher dimensions. For example, the representation $\vec{x} = (x_1, x_2, x_2, \ldots, x_n)$ would represent an $n-$dimensional vector, The scalar function $\phi = \phi(\vec{x}) = \phi(x_1, x_2, \ldots, x_n)$ would represent a scalar function of n-variables and the vector $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x}))$ would represent an n-dimensional vector function of position.

Gradient, Divergence and Curl

The **gradient of a scalar function** $\phi = \phi(x, y, z)$ is the vector function

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

If the scalar function is represented in the form $\phi = \phi(x_1, x_2, x_3)$, then the gradient vector is sometimes expressed in the form

$$\text{grad}\,\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right)$$

where it is to be understood that the partial derivatives are to be evaluated at the point $(x_1, x_2, x_3) = (x, y, z)$. The vector operator

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3$$

called the "del operator"or "nabla", is sometimes used to represent the gradient as an operator operating upon a scalar function to produce

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

The **divergence of a vector function** $\vec{F}(x,y,z)$ is **a scalar function** defined by

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

If one uses the notation $\vec{x} = (x_1, x_2, x_3)$ the divergence is expressed

$$\operatorname{div}\vec{F}(\vec{x}) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

The del operator can be used to represent the divergence using the dot product operation

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot\left(\vec{F}_1\,\hat{\mathbf{e}}_1 + \vec{F}_2\,\hat{\mathbf{e}}_2 + \vec{F}_3\,\hat{\mathbf{e}}_3\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The **curl of a vector function** $\vec{F}(x,y,z)$ is defined by the determinant operation[9]

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{e}}_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{e}}_3$$

If the notation $\vec{F} = \vec{F}(x_1, x_2, x_3)$ is used, then the curl is sometimes represented in the form

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)$$

where the unit base vectors are to be understood. The operations of gradient, divergence and curl will be investigated in more detail in the next chapter.

Taylor Series for Vector Functions

Consider a vector function

$$\vec{F} = \vec{F}(\vec{x}) = \vec{F}(x_1, x_2) = F_1(x_1, x_2)\,\hat{\mathbf{e}}_1 + F_2(x_1, x_2)\,\hat{\mathbf{e}}_2$$

which is continuous and possesses $(n+1)$ partial derivatives. The Taylor series expansion for this function is just applying the Taylor series expansion to each of the scalar functions F_1, F_2. Associated with the vector $\vec{h} = (h_1, h_2)$ is the vector operator

$$\vec{h}\cdot\nabla = h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2}$$

[9] See chapter 10 for properties of determinants.

so that if ϕ represents either of the components F_1 or F_2 one can write

$$(\vec{h}\cdot\nabla)\phi = h_1\frac{\partial\phi}{\partial x_1} + h_2\frac{\partial\phi}{\partial x_2}$$

Observe that the operator

$$\begin{aligned}(\vec{h}\cdot\nabla)^2\phi =&(\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla)\phi\\ =&h_1\frac{\partial}{\partial x_1}\left(h_1\frac{\partial\phi}{\partial x_1}+h_2\frac{\partial\phi}{\partial x_2}\right)+h_2\frac{\partial}{\partial y}\left(h_1\frac{\partial\phi}{\partial x_1}+h_2\frac{\partial\phi}{\partial x_2}\right)\\ =&h_1^2\frac{\partial^2\phi}{\partial {x_1}^2}+2h_1h_2\frac{\partial^2\phi}{\partial x_1\,\partial x_2}+h_2^2\frac{\partial^2\phi}{\partial {x_2}^2}\end{aligned}$$

In a similar fashion one can show

$$\begin{aligned}(\vec{h}\cdot\nabla)^3\phi =&(\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla)^2\phi\\ =&h_1^3\frac{\partial^3\phi}{\partial x_1^3}+3h_1^2h_2\frac{\partial^3\phi}{\partial x_1^2\partial x_2}+3h_1h_2^2\frac{\partial^3\phi}{\partial x_1\partial x_2^2}+h_2^3\frac{\partial^3\phi}{\partial x_2^3}\end{aligned}$$

and in general for any positive integer n one can use the binomial expansion to calculate the operator

$$(\vec{h}\cdot\nabla)^n\,\phi = \left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}\right)^n\phi$$

This operator can be used to represent the Taylor series expansion of a function $F = F(\vec{x})$ where $\vec{x} = (x_1, x_2)$. If $\vec{x}_0 = (x_1^0, x_2^0)$ is a constant and $\vec{h} = (h_1, h_2)$ denotes a small vector displacement from the point $\vec{x}_0$, then the Taylor series expansion can be written

$$F(\vec{x}_0+\vec{h}) = \sum_{m=1}^{n}\frac{1}{m!}(\vec{h}\cdot\nabla)^m F(\vec{x})\Big|_{\vec{x}=\vec{x}_0} + \frac{1}{(n+1)!}(\vec{h}\cdot\nabla)^{n+1}F(\vec{x})\Big|_{\vec{x}=\vec{x}_0} \tag{6.84}$$

where all derivatives are to be evaluated at the point $\vec{x}_0$.

In three dimensions vectors of the form

$$\vec{F} = \vec{F}(\vec{x}) = \vec{F}(x_1,x_2,x_3) = F_1(x_1,x_2,x_3)\,\hat{\mathbf{e}}_1 + F_2(x_1,x_2,x_3)\,\hat{\mathbf{e}}_2 + F_3(x_1,x_2,x_3)\,\hat{\mathbf{e}}_3$$

which have $(n+1)$ partial derivatives can be expanded in a Taylor series by expanding each of the components in a Taylor series. Associated with the vector displacement $\vec{h} = (h_1, h_2, h_3)$ one can define the operator

$$(\vec{h}\cdot\nabla) = h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + h_3\frac{\partial}{\partial x_3}$$

and find that the Taylor series expansion has the same form as equation (6.84)

Differentiation of Composite Functions

Let $\phi = \phi(x, y, z)$ define a scalar field and consider a curve passing through the region where the scalar field is defined. Express the curve through the scalar field in the parametric form

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

with parameter t. The value of the scalar ϕ, at the points (x, y, z) along the curve, is a function of the coordinates on the curve. By substituting into ϕ the position of a general point on the curve, one can write

$$\phi = \phi(x(t), y(t), z(t)).$$

By substituting the time-varying coordinates of the curve into the function ϕ, one creates a composite function. The time rate of change of this composite function ϕ, as one moves along the curve, is derived from chain rule differentiation and

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}. \tag{6.85}$$

The equation (6.85) gives us the general rule

$$\frac{d[\ ]}{dt} = \frac{\partial[\ ]}{\partial x}\frac{dx}{dt} + \frac{\partial[\ ]}{\partial y}\frac{dy}{dt} + \frac{\partial[\ ]}{\partial z}\frac{dz}{dt} \tag{6.86}$$

where the quantity inside the brackets can be any scalar function of x, y and z. The second derivative of ϕ can be calculated by using the product rule and

$$\begin{aligned} \frac{d^2\phi}{dt^2} = & \frac{\partial\phi}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\frac{d}{dt}\left[\frac{\partial\phi}{\partial x}\right] \\ & + \frac{\partial\phi}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left[\frac{\partial\phi}{\partial y}\right] \\ & + \frac{\partial\phi}{\partial z}\frac{d^2z}{dt^2} + \frac{dz}{dt}\frac{d}{dt}\left[\frac{\partial\phi}{\partial z}\right]. \end{aligned} \tag{6.87}$$

To evaluate the derivatives of the terms inside the brackets of equation (6.87) use the general differentiation rule given by equation (6.86). This produces a second derivative having the form

$$\begin{aligned} \frac{d^2\phi}{dt^2} = & \frac{\partial\phi}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left[\frac{\partial^2\phi}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2\phi}{\partial x\,\partial y}\frac{dy}{dt} + \frac{\partial^2\phi}{\partial x\,\partial z}\frac{dz}{dt}\right] \\ & + \frac{\partial\phi}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left[\frac{\partial^2\phi}{\partial y\,\partial x}\frac{dx}{dt} + \frac{\partial^2\phi}{\partial y^2}\frac{dy}{dt} + \frac{\partial^2\phi}{\partial y\,\partial z}\frac{dz}{dt}\right] \\ & + \frac{\partial\phi}{\partial z}\frac{d^2z}{dt^2} + \frac{dz}{dt}\left[\frac{\partial^2\phi}{\partial z\,\partial x}\frac{dx}{dt} + \frac{\partial^2\phi}{\partial z\,\partial y}\frac{dy}{dt} + \frac{\partial^2\phi}{\partial z^2}\frac{dz}{dt}\right]. \end{aligned} \tag{6.88}$$

Higher derivatives can be calculated by using the product rule for differentiation together with the rule for differentiating a composite function.

Integration of Vectors

Let $\vec{u}(s) = u_1(s)\,\hat{\mathbf{e}}_1 + u_2(s)\,\hat{\mathbf{e}}_2 + u_3(s)\,\hat{\mathbf{e}}_3$ denote a vector function of arc length, where the components $u_i(s)$, $i = 1, 2, 3$ are continuous functions. The indefinite integral of $\vec{u}(s)$ is defined as the indefinite integral of each component of the vector. This is expressed in the form

$$\begin{aligned}\int \vec{u}(s)\,ds =& \int u_1(s)\,ds\,\hat{\mathbf{e}}_1 + \int \vec{u}_2(s)\,ds\,\hat{\mathbf{e}}_2 + \int \vec{u}_3(s)\,ds\,\hat{\mathbf{e}}_3 + \vec{C},\\ =& \vec{U}(s) + \vec{C}.\end{aligned} \tag{6.89}$$

where $\vec{U}(s)$ is a vector such that $\frac{d\vec{U}}{ds} = \vec{u}(s)$ and $\vec{C}$ is a vector constant of integration.

The definite integral of $\vec{u}$ is defined as

$$\int_a^b \vec{u}(s)\,ds = \vec{U}(s)\Big|_a^b = \vec{U}(b) - \vec{U}(a), \quad \text{where} \quad \frac{d\vec{U}(s)}{ds} = \vec{u}(s). \tag{6.90}$$

The following are some properties associated with the integration of vector functions. These properties are stated without proof.

1. For $\vec{c}$ a constant vector

$$\int \vec{c}\cdot\vec{u}(s)\,ds = \vec{c}\cdot\int \vec{u}(s)\,ds \quad \text{and} \quad \int \vec{c}\times\vec{u}(s)\,ds = \vec{c}\times\int \vec{u}(s)\,ds$$

2. For $\vec{c}_1$ and $\vec{c}_2$ constant vectors, the integral of a sum equals the sum of the integrals

$$\int \left[\vec{c}_1\cdot\vec{u}(s) + \vec{c}_2\cdot\vec{v}(s)\right]\,ds = \vec{c}_1\cdot\int \vec{u}(s)\,ds + \vec{c}_2\cdot\int \vec{v}(s)\,ds,$$

3. Integration by parts takes on the form

$$\int_a^b f(s)\vec{u}(s)\,ds = f(s)\vec{U}(s)\Big|_a^b - \int_a^b f'(s)\vec{U}(s)\,ds, \tag{6.91}$$

where $f(s)$ is a scalar function and $\dfrac{d\vec{U}(s)}{ds} = \vec{u}(s)$.

Example 6-27. The acceleration of a particle is given by

$$\vec{a} = \sin t\, \hat{\mathbf{e}}_1 + \cos t\, \hat{\mathbf{e}}_2.$$

If at time $t = 0$ the position and velocity of the particle are given by

$$\vec{r}(0) = 6\, \hat{\mathbf{e}}_1 - 3\, \hat{\mathbf{e}}_2 + 4\, \hat{\mathbf{e}}_3 \qquad \text{and} \qquad \vec{v}(0) = 7\, \hat{\mathbf{e}}_1 - 6\, \hat{\mathbf{e}}_2 - 5\, \hat{\mathbf{e}}_3,$$

find the position and velocity as a function of time.

Solution: An integration of the acceleration with respect to time produces the velocity and

$$\int \vec{a}(t)\, dt = \vec{v} = \vec{v}(t) = -\cos t\, \hat{\mathbf{e}}_1 + \sin t\, \hat{\mathbf{e}}_2 + \vec{c}_1,$$

where $\vec{c}_1$ is a vector constant of integration. From the above initial condition for the velocity, the constant $\vec{c}_1$ can be determined. One finds

$$\vec{v}(0) = -\, \hat{\mathbf{e}}_1 + \vec{c}_1 = 7\, \hat{\mathbf{e}}_1 - 6\, \hat{\mathbf{e}}_2 - 5\, \hat{\mathbf{e}}_3 \qquad \text{or} \qquad \vec{c}_1 = 8\, \hat{\mathbf{e}}_1 - 6\, \hat{\mathbf{e}}_2 - 5\, \hat{\mathbf{e}}_3.$$

Consequently, the velocity can be expressed as a function of time in the form

$$\vec{v} = \vec{v}(t) = \frac{d\vec{r}}{dt} = (-\cos t + 8)\, \hat{\mathbf{e}}_1 + (\sin t - 6)\, \hat{\mathbf{e}}_2 - 5\, \hat{\mathbf{e}}_3.$$

An integration of the velocity with respect to time produces the position vector as a function of time and

$$\begin{aligned} \int \vec{v}(t)\, dt = \int \frac{d\vec{r}}{dt}\, dt = \int (-\cos t + 8)\, dt\ \hat{\mathbf{e}}_1 + \int (\sin t - 6)\, dt\ \hat{\mathbf{e}}_2 - 5 \int dt\ \hat{\mathbf{e}}_3 + \vec{c}_2 \\ \vec{r}(t) = (-\sin t + 8t)\, \hat{\mathbf{e}}_1 + (-\cos t - 6t)\, \hat{\mathbf{e}}_2 - 5t\, \hat{\mathbf{e}}_3 + \vec{c}_2, \end{aligned}$$

where $\vec{c}_2$ is a vector constant of integration. From the above initial conditions, at time $t = 0$, one can determine this vector constant of integration and

$$\vec{r}(0) = -\, \hat{\mathbf{e}}_2 + \vec{c}_2 = 6\, \hat{\mathbf{e}}_1 - 3\, \hat{\mathbf{e}}_2 + 4\, \hat{\mathbf{e}}_3 \qquad \text{or} \qquad \vec{c}_2 = 6\, \hat{\mathbf{e}}_1 - 2\, \hat{\mathbf{e}}_2 + 4\, \hat{\mathbf{e}}_3.$$

The position vector as a function of time can be expressed as

$$\vec{r} = \vec{r}(t) = (-\sin t + 8t + 6)\, \hat{\mathbf{e}}_1 + (-\cos t - 6t - 2)\, \hat{\mathbf{e}}_2 + (-5t + 4)\, \hat{\mathbf{e}}_3.$$

■

Example 6-28. A particle in a force field $\vec{F} = \vec{F}(x, y, z)$ having a position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ moves according to Newton's second law such that

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \qquad \text{or} \qquad \vec{F}\,dt = m\,d\vec{v}.$$

An integration over the time interval t_1 to t_2 produces

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\,\vec{v}(t_2) - m\,\vec{v}(t_1).$$

The quantity $\int_{t_1}^{t_2} \vec{F}\,dt$ is called the linear impulse on the particle over the time interval (t_1, t_2). The quantity $m\vec{v}$ is called the linear momentum of the particle. The above equation tells us that the linear impulse equals the change in linear momentum.

■

Example 6-29. In 10 seconds a particle with a mass of 1 gram changes velocity from

$$\vec{v}_1 = 6\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 7\,\hat{\mathbf{e}}_3 \text{ cm/s} \qquad \text{to} \qquad \vec{v}_2 = -2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3 \text{ cm/s}.$$

What average force produces this change?

Solution: The average force over a time interval (t_1, t_2) is given by

$$\vec{F}\text{avg} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \vec{F}\,dt.$$

But the integral $\int_{t_1}^{t_2} \vec{F}\,dt$ is the linear impulse and equals the change in linear momentum given by $m\vec{v}_2 - m\vec{v}_1$. The average force is therefore

$$\begin{aligned}\vec{F}\text{avg} &= \frac{1}{10}\left[(-2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3) - (6\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 7\,\hat{\mathbf{e}}_3)\right] \\ &= \frac{1}{5}\left[-4\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3\right] \text{ dynes.}\end{aligned}$$

■

Line Integrals of Scalar and Vector Functions.

An important type of vector integration is **integration by line integrals.** Let C be a curve defined by a position vector

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3,$$

where x, y, z define some parametric representation of the curve C. The element of arc length along the curve, when squared, is given by

$$ds^2 = d\vec{r} \cdot d\vec{r} = dx^2 + dy^2 + dz^2.$$

An integration (summation) produces the following formulas for the arc length s.

1. If $y = y(x)$ and $z = z(x)$ are known in terms of the parameter x, the arc length between two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ on the curve can be represented in the form
$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx. \tag{6.92}$$
2. If the parametric equations of the curve are given by $x = x(t)$, $y = y(t)$ and $z = z(t)$, the arc length between two points P_0 and P_1 on the curve is given by
$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt, \tag{6.93}$$
where the parametric values $t = t_0$ and $t = t_1$ correspond to the points P_0 and P_1 and
$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0$$
$$x(t_1) = x_1, \quad y(t_1) = y_1, \quad z(t_1) = z_1.$$

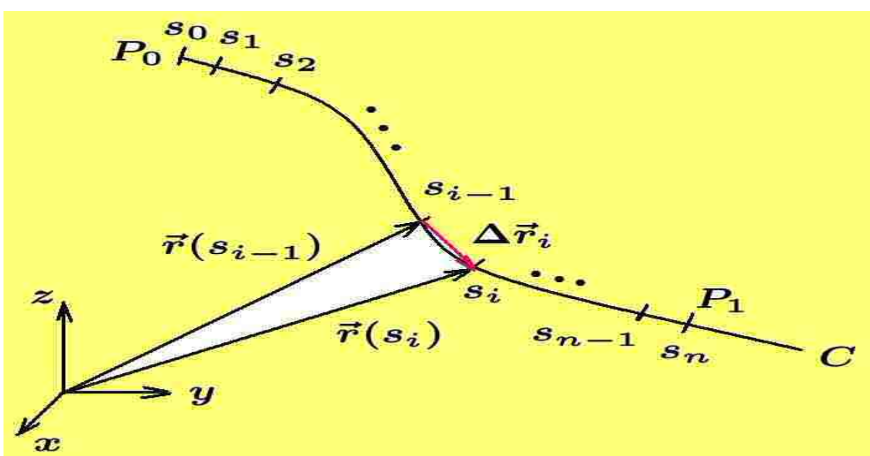


Figure 6-18. Curve C partitioned into $n-$segments between P_0 and P_1.

The above formulas result indirectly from the following limiting process. On that part of the curve between the given points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, the arc length along the curve is divided into n segments by a set of numbers

$$s_0 < s_1 < \ldots < s_n,$$

where corresponding to each value of the arc length parameter s_i there is a position vector $\vec{r}(s_i) = x(s_i)\,\hat{\mathbf{e}}_1 + y(s_i)\,\hat{\mathbf{e}}_2 + z(s_i)\,\hat{\mathbf{e}}_3$, for $i = 1, \ldots, n$, as illustrated in figure 6-18.

A change in the element of arc length from $\vec{r}(s_{i-1})$ to $\vec{r}(s_i)$ is defined as

$$\Delta s_i = |\vec{r}(s_i) - \vec{r}(s_{i-1})| = |\Delta \vec{r}_i|\,.$$

The total arc length is obtained from the sum of these elements of arc length as the number of these lengths increase without bound and the partition gets finer and finer. In symbols, this limit is denoted as

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \int_{s_0}^{s_n} ds.$$

The above definition for arc length along the curve suggests how values of a scalar field can be summed as one moves through the scalar field along a curve C.

Definition (Line integral of a scalar function along a curve C.) ***Let $f = f(x,y,z)$ denote a scalar function of position. The line integral of f along a curve C is defined as***

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i, \tag{6.94}$$

where (x_i^*, y_i^*, z_i^*) is a point on the curve in the ith subinterval Δs_i and where the symbol $\int_C$ denotes an integral taken along the given curve C. This type of integral is called a line integral along the curve.

Similarly, define the summation of a vector field as one moves through the field along a curve C. This produces the following definition of a line integral of a vector function along a curve C.

Definition (Line integral along a curve C involving a dot product.) ***Let***

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathrm{e}}_1 + F_2(x,y,z)\,\hat{\mathrm{e}}_2 + F_3(x,y,z)\,\hat{\mathrm{e}}_3$$

denote a vector function of position. The line integral of $\vec{F}$ along a given curve C, defined by a position vector $\vec{r} = \vec{r}(s) = x(s)\,\hat{\mathrm{e}}_1 + y(s)\,\hat{\mathrm{e}}_2 + z(s)\,\hat{\mathrm{e}}_3$, is defined as

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(x_i^*,y_i^*,z_i^*)\cdot\frac{\vec{\Delta} r_i}{\Delta s_i}\Delta s_i$$
$$=\int_C\left(F_1\frac{dx}{ds}+F_2\frac{dy}{ds}+F_3\frac{dz}{ds}\right)ds, \qquad (6.95)$$
$$=\int_C \vec{F}\cdot d\vec{r}=\int_C\vec{F}\cdot\frac{d\vec{r}}{ds}\,ds$$

where (x_i^*,y_i^*,z_i^*) is a point inside the ith subinterval of the arc length Δs_i.

In the above definition the dot product $\vec{F}\cdot\frac{\vec{\Delta} r_i}{\Delta s_i}$ represents the projection of the vector $\vec{F}$ or component of $\vec{F}$ in the direction of the tangent vector to the curve C. The line integral of the vector function may be thought of as representing a summation of the tangential components of the vector $\vec{F}$ along the curve C between the points P_0 and P_1. Line integrals of this type arise in the calculation of the work done in moving through a force field along a curve. Here the work is given by a summation of force times distance traveled.

In particular, the above line integral can be expressed in the form

$$\int_C \vec{F}\cdot d\vec{r}=\int_C\vec{F}\cdot\frac{d\vec{r}}{ds}\,ds=\int_C\vec{F}\cdot\hat{\mathbf{e}}_t\,ds=\int_C F_1dx+F_2dy+F_3dz, \qquad (6.96)$$

where at each point on the curve C, the dot product $\vec{F}\cdot\hat{\mathbf{e}}_t$ is a scalar function of position and represents the projection of $\vec{F}$ on the unit tangent vector to the curve.

Summations of cross products along a curve produce another type of line integral.

Definition (Line integral along a curve C involving cross products.)
The line integral

$$\int_C \vec{F}\times d\vec{r}$$

is defined by the limiting process

$$\int_C \vec{F}\times d\vec{r}=\lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(x_i^*,y_i^*,z_i^*)\times\Delta\vec{r}_i, \qquad (6.97)$$

where $\vec{F}=\vec{F}(x_i^*,y_i^*,z_i^*)$ is the value of $\vec{F}$ at a point (x_i^*,y_i^*,z_i^*) in the ith subinterval of arc length on the curve C.

Integrals of this type arise in the calculation of magnetic dipole moments associated with current loops.

Note that each of the line integrals requires knowing the values of x, y and z along a given curve C and these values must be substituted into the integrand and after this substitution the summation process reduces to an ordinary integration.

Work Done.

Consider a particle moving from a point P_O to a point P_1 along a curve C which lies in a force field $\vec{F} = \vec{F}(x, y, z)$. At each point (x, y, z) on the curve there are force vectors acting on the particle as illustrated in figure 6-19.

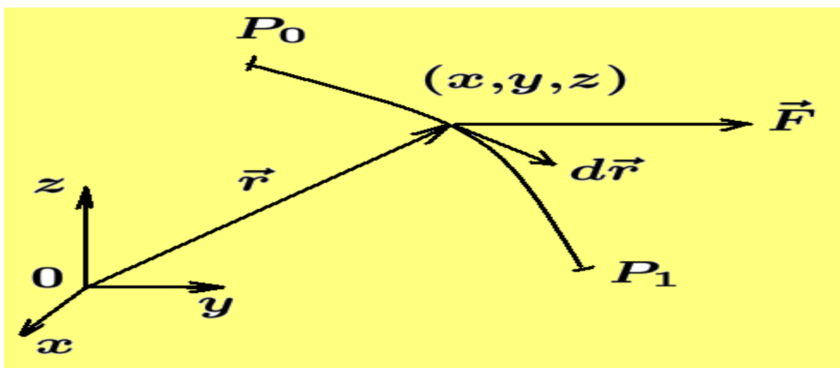


Figure 6-19. Moving along a curve C in a force field $\vec{F}$.

Examine the particle at a general point (x, y, z) on the given curve C. Construct the position vector $\vec{r}$, the force vector $\vec{F}$, and the tangent vector $d\vec{r}$ acting at this general point on the curve. The line integral

$$W_{P_0 P_1} = \int_C \vec{F} \cdot d\vec{r} = \int_{P_0}^{P_1} \vec{F} \cdot \frac{d\vec{r}}{ds} ds = \int_{P_1}^{P_1} \vec{F} \cdot \hat{\mathbf{e}}_t \, ds$$

is a summation of the tangential component of the force times distance traveled along the curve C. Consequently, the above integral represents the work done in moving through the force field from point P_0 to P_1 along the curve C.

Example 6-30. Let a particle with constant mass m move along a curve C which lies in a vector force field $\vec{F} = \vec{F}(x, y, z)$. Also, let $\vec{r}$ denote the position vector of the particle in the force field and on the curve C. As the particle moves along the curve, at each point (x, y, z) of the curve, the particle experiences a force $\vec{F}(x, y, z)$

which is determined by the vector force field. Newton's second law of motion is expressed

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt}.$$

The work done in moving along the curve C between two points A and B can then be expressed as

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_A^B \vec{F}\cdot\vec{v}\,dt = \int_A^B m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = \int_A^B m\vec{v}\cdot\frac{d\vec{v}}{dt}\,dt.$$

Now utilize the vector identity

$$\frac{1}{2}\frac{d}{dt}\left(v^2\right) = \frac{1}{2}\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right) = \vec{v}\cdot\frac{d\vec{v}}{dt},$$

so that the above line integral can be expressed in the form

$$\int_A^B \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_A^B \vec{F}\cdot\vec{v}\,dt = \int_A^B \frac{m}{2}\frac{d}{dt}\left(v^2\right)\,dt,$$

which is easily integrated. One finds

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r} = \frac{m}{2}v^2\Big|_A^B = \frac{m}{2}\left(v_B^2 - v_A^2\right) = E_k(v_B) - E_k(v_A).$$

In this equation the line integral $W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$ is called the work done in moving the particle from A to B through the force field $\vec{F}$. The quantity $E_k(v) = \frac{m}{2}v^2$ is called the kinetic energy of the particle. The above equation tells us that the work done in moving a particle from A to B in a force field $\vec{F}$ must equal the change in the kinetic energy of the particle between the points A and B.

■

Representation of Line Integrals

The line integral $\int \vec{F}\cdot d\vec{r}$ can be expressed in many different forms:

1.
$$\int_A^B \vec{F}\cdot d\vec{r} = \int_{t_A}^{t_B} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_{t_A}^{t_B} \vec{F}\cdot\vec{v}\,dt$$

Integrals of this form are used if $\vec{F} = \vec{F}(t)$ and $\vec{v} = \vec{V}(t)$ are known functions of the parameter t.

2.
$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds = \int_{s_A}^{s_B} \vec{F}\cdot\hat{\mathbf{e}}_t\,ds$$

Here $\vec{F}\cdot\hat{\mathbf{e}}_t$ is the tangential component of the force $\vec{F}$ along the given curve C. This form of the line integral is used if $\vec{F}=\vec{F}(s)$ and $\hat{\mathbf{e}}_t$ are known functions of the arc length s.

3. For a force field given by

$$\vec{F}=\vec{F}(x,y,z)=F_1(x,y,z)\,\hat{\mathbf{e}}_1+F_2(x,y,z)\,\hat{\mathbf{e}}_2+F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

and the position vector of a point (x,y,z) on a curve C given by

$$\vec{r}=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3 \quad \text{with} \quad d\vec{r}=dx\,\hat{\mathbf{e}}_1+dy\,\hat{\mathbf{e}}_2+dz\,\hat{\mathbf{e}}_3,$$

Here the work done is represented in the form

$$\int_A^B \vec{F}\cdot d\vec{r}=\int_A^B F_1dx+F_2dy+F_3dz.$$

Line integrals are written in this form when a parametric representation of the curve is known. In the special case where $\vec{r}=x\,\hat{\mathbf{e}}_1+0\,\hat{\mathbf{e}}_2+0\,\hat{\mathbf{e}}_3$, the above line integral reduces to an ordinary integral.

4. The line integral $\int_C \vec{F}\cdot d\vec{r}$ may be broken up into a sum of line integrals along different portions of the curve C. If the curve C is comprised of n separate curves $C_1, C_2, \ldots, C_n$, one can write

$$\int_C \vec{F}\cdot d\vec{r}=\int_{C_1}\vec{F}\cdot d\vec{r}+\int_{C_2}\vec{F}\cdot d\vec{r}+\cdots+\int_{C_n}\vec{F}\cdot d\vec{r}.$$

5. When the curve C is a simple closed curve (i.e., the curve does not intersect itself), the line integral is represented by

$$\circlearrowleft\!\!\!\!\!\!\oint_{C} \vec{F}\cdot d\vec{r} \qquad \text{or} \qquad \circlearrowright\!\!\!\!\!\!\oint_{C} \vec{F}\cdot d\vec{r} \tag{6.98}$$

where the direction of integration is either in the counterclockwise sense or clockwise sense. Whenever the line integral is represented in the form $\oint_C \vec{F}\cdot d\vec{r}$ then it is to be understood that the integration direction is in the counterclockwise sense which is known as the positive sense. Note that when the curve is a simple closed curve, there is no need to specify a beginning and end point for the integration. One need only specify a direction to the integration. The integration is said to be in the positive sense if the integration is in a counterclockwise direction or it is said to be in the negative sense if the direction of integration is clockwise. The sense of integration is the same as that for angular measure. The situation is illustrated in figure 6-20.

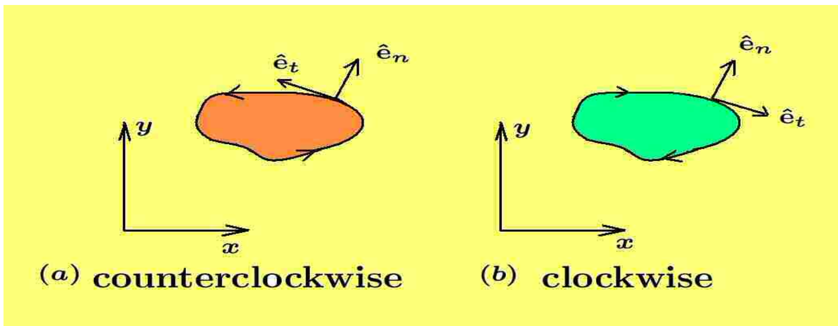


Figure 6-20. Direction of integration for line integrals.

The direction of integration around a simple closed curve can be referenced with respect to the unit outward normal $\hat{\mathbf{e}}_n$ and to the unit tangent vector $\hat{\mathbf{e}}_t$ to the simple close curve as the direction of the unit tangent produces an oriented simple closed curve.

6. If the direction of integration is reversed, then the sign of the line integral changes so that one can write

$$\oint_C \vec{F} \cdot d\vec{r} = -\oint_C \vec{F} \cdot d\vec{r}$$

Example 6-31. Consider a particle moving in a two-dimensional force field, where at any point (x, y) the force in pounds acting on the particle is given by

$$\vec{F} = \vec{F}(x,y) = (x^2 + y)\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2$$

Find the work done in moving the particle from the origin to the point B along the path illustrated in figure 6-21, where distance traveled is measured in units of feet.

Solution: Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ denote the position vector of a point on the path OAB illustrated in figure 21. The work done is obtained by evaluating the line integral

$$W = \int_0^B \vec{F} \cdot d\vec{r}$$

Using the property that line integrals may be broken up into integration along separate curves, one can write $W = \int_O^B \vec{F} \cdot d\vec{r} = \int_O^A \vec{F} \cdot d\vec{r} + \int_A^B \vec{F} \cdot d\vec{r}$ where $\vec{F} \cdot d\vec{r} = (x^2 + y)\,dx + xy\,dy$.

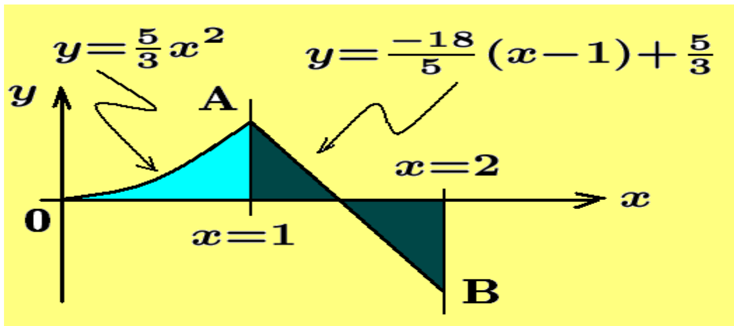


Figure 6-21. Find the work done in moving particle from origin to point B.

The portion of the work done in moving along the parabola from 0 to A, where $y = \frac{5}{3}x^2$ and $dy = \frac{5}{3}(2x\,dx)$, is

$$\int_O^A \vec{F} \cdot d\vec{r} = \int_0^1 [x^2 + (\frac{5}{3}x^2)]\,dx + x(\frac{5}{3}x^2)\,\frac{5}{3}(2x\,dx) = 2$$

The portion of the work done in moving along the straight-line from A to B, where $y = \frac{-18}{5}(x-1) + \frac{5}{3}$ and $dy = \frac{-18}{5}dx$, is expressed as

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_1^2 [x^2 + (\frac{-18}{5}(x-1) + \frac{5}{3}]\,dx + x(\frac{-18}{5}(x-1) + \frac{5}{3})(\frac{-18}{5}dx) = 4$$

The total work done is therefore given by the sum $W = 2 + 4 = 6$ ft-lbs. Here the unit of work is the unit of force times unit of distance traveled.

∎

Example 6-32. Compute the value of the line integral

$$\oint_C \vec{F} \cdot d\vec{r},$$

where $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ and C is the circle $x^2 + y^2 = 1$.

Solution: Let the circular path be represented in the parametric form

$$x = \cos t \qquad y = \sin t,$$

then the above line integral can be written

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_C (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2)\cdot(dx\;\hat{\mathbf{e}}_1 + dy\;\hat{\mathbf{e}}_2)\\ &= \int_C x\,dx + y\,dy\\ &= \int_0^{2\pi} (\cos t)(-\sin t)\,dt + (\sin t)(\cos t)\,dt = 0.\end{aligned}$$

Here the direction of integration is in the positive sense as the parameter t varies from 0 to 2π.

■

Example 6-33. Compute the value of the line integral $\oint_C \vec{F}\times d\vec{r}$, where $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ and C is the circle $x^2 + y^2 = 1$

Solution: Write

$$\vec{F}\times d\vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ x & y & 0\\ dx & dy & 0\end{vmatrix} = \hat{\mathbf{e}}_3(x\,dy - y\,dx)$$

and therefore

$$\oint_C \vec{F}\times d\vec{r} = \oint_C \hat{\mathbf{e}}_3(x\,dy - y\,dx).$$

If the circular path of integration is represented in the parametric form

$$x = \cos t \qquad y = \sin t$$

one finds

$$\oint_C \vec{F}\times d\vec{r} = \hat{\mathbf{e}}_3 \int_0^{2\pi} (\cos t)(\cos t)\,dt - (\sin t)(-\sin t)\,dt = \hat{\mathbf{e}}_3 \int_0^{2\pi} dt = 2\pi\,\hat{\mathbf{e}}_3.$$

■

Example 6-34.

Examine the work done in moving a particle through the force field

$$\vec{F} = (x+z)\,\hat{\mathbf{e}}_1 + (y+z)\,\hat{\mathbf{e}}_2 + 2z\,\hat{\mathbf{e}}_3$$

as the particle moves along the curve C described by the position vector

$$\vec{r} = t\,\hat{\mathbf{e}}_1 + t^2\,\hat{\mathbf{e}}_2 + (-3t+1)\,\hat{\mathbf{e}}_3$$

as the parameter t ranges from 0 to 2.

Solution: The work done is determined by evaluating the line integral $\int_C \vec{F}\cdot d\vec{r}$ where

$$\vec{F}\cdot d\vec{r} = (x+z)\,dx + (y+z)\,dy + 2z\,dz$$

On the given curve C use $x=t$, $y=t^2$ and $z=-3t+1$ with $dx=dt$, $dy=2t\,dt$ and $dz=-3\,dt$ and substitute these values into the line integral describing the work done. This produces the result

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 \big[(t-3t+1)(dt) + (t^2-3t+1)(2t\,dt) + 2(-3t+1)(-3\,dt)\big] = 18$$

where work has the units of force times units of distance traveled.

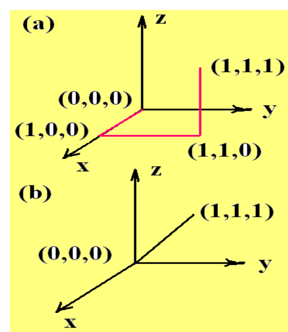
■

Example 6-35.

For $\vec{F} = x(y+1)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x+1)\,\hat{\mathbf{e}}_3$ evaluate the line integral $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r}$

(a) Along the line segments illustrated.

(b) Along the straight line path from $(0,0,0)$ to $(1,1,1)$.



Solution $\vec{F}\cdot d\vec{r} = x(y+1)\,dx + y\,dy + z(x+1)\,dz$

(a) Along $(0,0,0)$ to $(1,0,0)$, $x=1,\ z=0,\ 0\le x\le 1$

$$\int_0^1 x\,dx = \frac{1}{2}$$

Along $(1,0,0)$ to $(1,1,0)$, $x=1,\ z=0,\ 0\le y\le 1$

$$\int_0^1 y\,dy = \frac{1}{2}$$

Along $(1,1,0)$ to $(1,1,1)$, $x=1,\ y=1,\ 0\le z\le 1$

$$\int_0^1 2z\,dz = 1$$

therefore $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = \frac{1}{2}+\frac{1}{2}+1 = 2$

(b) The straight line path from $(0,0,0)$ to $(1,1,1)$ is represented by the parametric equation

$$x=t,\quad y=t,\quad z=t$$

for $0\le t\le 1$. Therefore

$$\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 [t(t+1)+t+t(t+1)]\,dt = \frac{13}{6}$$

The work done in moving from $(0,0,0)$ to $(1,1,1)$ is path dependent.

■

Exercises

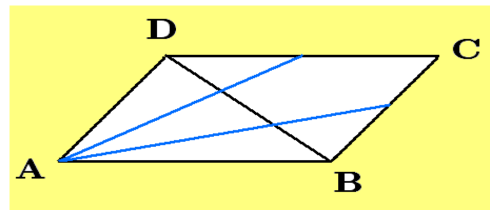
▶ **6-1.** For the vectors $\vec{A} = 3\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\vec{B} = 6\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ calculate

$$(a)\quad \vec{A}+\vec{B} \qquad (b)\quad 6\vec{A}-3\vec{B} \qquad (c)\quad \vec{A}+2\vec{B}$$

▶ **6-2.** Use vectors to show that the diagonals of a parallelogram bisect one another.

▶ **6-3.** Use vectors to show that the line segment connecting the midpoints of two sides of a triangle is parallel to the third side and has one half the magnitude of the third side.

▶ **6-4.** In the parallelogram $ABCD$ illustrated, construct lines from the vertex A to the midpoints of the sides DC and BC. Show that these lines trisect the diagonal BD.



▶ **6-5.** Are the given vectors linearly dependent or linearly independent?

$$\begin{array}{lll}
(a)\ \vec{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\,\hat{\mathbf{e}}_3 & (b)\ \vec{A} = 2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3 & (c)\ \vec{A} = 3\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 \\
\quad\ \vec{B} = -4\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2 & \quad\ \vec{B} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 & \quad\ \vec{B} = -\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3 \\
\quad\ \vec{C} = 7\,\hat{\mathbf{e}}_1 + 6\,\hat{\mathbf{e}}_2 - 6\,\hat{\mathbf{e}}_3 & \quad\ \vec{C} = 3\,\hat{\mathbf{e}}_3 & \quad\ \vec{C} = 14\,\hat{\mathbf{e}}_1 - 4\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3.
\end{array}$$

▶ **6-6.** If $\vec{A}, \vec{B}, \vec{C}$ are nonzero vectors and $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$, then determine if the following statements are true or false.

(i) The vectors $\vec{A}, \vec{B}, \vec{C}$ are linearly independent.

(ii) The vectors $\vec{A}, \vec{B}, \vec{C}$ are linearly dependent.

Justify your answers.

▶ **6-7.** Let $\vec{A} = \vec{A}(t)$ denote a vector which has a constant length C for all values of the parameter t.

(a) Show that $\vec{A}\cdot\vec{A} = C^2$

(b) Show that the derivative vector $\dfrac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

▶ **6-8.** Show that for $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ and $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ the distance d of an arbitrary point (x_0, y_0, z_0) from the line $\vec{r} = \vec{r}_1 + t\vec{A}$, is given by

$$d = |(\vec{r}_0 - \vec{r}_1) \times \hat{\mathbf{e}}_A|$$

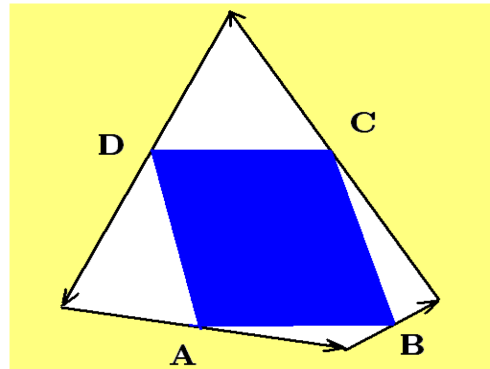
where $\hat{\mathbf{e}}_A$ is a unit vector in the direction of $\vec{A}$ and $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ is a position vector to the arbitrary point.

▶ **6-9.** Consider the triangle defined by the three vertices $(6,0,0)$, $(0,6,0)$ and $(0,0,12)$. Use vector methods to find the area of the triangle.

▶ **6-10.** Let the sides of a quadrilateral be denoted by the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ such that

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{0}.$$

Use vectors to show that the lines joining the midpoints of the sides of this quadrilateral form a parallelogram.



▶ **6-11.** Let $\vec{r}_0$ represent the position vector of the center of a sphere of radius ρ and let $\vec{r}$ represent the position vector of a variable point on the surface of the sphere. Find the equation of the sphere in a vector form. Simplify your result to a scalar form.

▶ **6-12.** For $\vec{A}=\hat{\mathbf{e}}_1+2\,\hat{\mathbf{e}}_2+2\,\hat{\mathbf{e}}_3$ and $\vec{B}=7\,\hat{\mathbf{e}}_1+4\,\hat{\mathbf{e}}_2+4\,\hat{\mathbf{e}}_3$

(a) Find a unit vector in the direction of $\vec{B}$.

(b) Find a unit vector in the direction of $\vec{A}$.

(c) Find the projection of $\vec{A}$ on $\vec{B}$.

(d) Find the projection of $\vec{B}$ on $\vec{A}$.

▶ **6-13.** (a) Find a unit vector perpendicular to the vectors

$$\vec{A}=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3 \quad \text{and} \quad \vec{B}=\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3$$

(b) Find the projection of $\vec{B}$ on $\vec{A}$.

▶ **6-14.** For $\vec{A}=-\,\hat{\mathbf{e}}_1+\sqrt{3}\,\hat{\mathbf{e}}_2+\sqrt{5}\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_\alpha=\cos\alpha\,\hat{\mathbf{e}}_1+\sin\alpha\,\hat{\mathbf{e}}_2$

(a) Verify that $\hat{\mathbf{e}}_\alpha$ is a unit vector for all α.

(b) Find the projection of $\vec{A}$ on $\hat{\mathbf{e}}_\alpha$.

(c) For what angle α is the projection equal to zero?

(d) For what angle α is the projection a maximum?

▶ **6-15.** Assume $\vec{A}(t)$ has derivatives of all orders. Find the constant vectors $\vec{A}_0,\vec{A}_1,\ldots,\vec{A}_n,\ldots$ if

$$\vec{A}(t)=\vec{A}_0+\vec{A}_1\frac{(t-t_0)}{1!}+\vec{A}_2\frac{(t-t_0)^2}{2!}+\cdots+\vec{A}_n\frac{(t-t_0)^n}{n!}+\cdots$$

Hint: Evaluate $\vec{A}(t)$ at $t=t_0$, then differentiate $\vec{A}(t)$ and evaluate result at $t=t_0$.

▶ **6-16.** Given the vectors $\vec{A} = \hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ and $\vec{B} = 3\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3$
Evaluate the following quantities:

(a)	$\vec{A}\times\vec{B}$	(d)	$(\vec{A}+\vec{B})\times\vec{A}$	(g)	$(\vec{A}+3\vec{B})\times\vec{B}$
(b)	$\vec{B}\times\vec{A}$	(e)	The angle between $\vec{A}$ and $\vec{B}$	(h)	$(\vec{B}-\vec{A})\times(\vec{B}+\vec{A})$
(c)	$\vec{A}\cdot\vec{B}$	(f)	$3\vec{A}\times 2\vec{B}$	(i)	$\vec{A}\cdot(\vec{A}+\vec{B})$

▶ **6-17.** The sides of a parallelogram are $\vec{A} = \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ and $\vec{B} = 2\,\hat{\mathbf{e}}_1 + 9\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$.
(a) Find the vectors which represent the diagonals of this parallelogram.
(b) Find the area of the parallelogram.

▶ **6-18.** Determine the direction cosines of the vector $\vec{\rho} = \sqrt{2}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$.

▶ **6-19.** Explain why two vectors are said to be linearly dependent if their vector cross product is the zero vector.

▶ **6-20.** Three noncolinear points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$ determine a plane. Let $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ denote the position vectors from the origin to each of these points, respectively, and let $\vec{r}$ denote the position vector of any variable point (x,y,z) in the plane.
(a) Describe and illustrate the vector $\vec{r}_3 - \vec{r}_1$.
(b) Describe and illustrate the vector $\vec{r}_2 - \vec{r}_1$.
(c) Describe and illustrate the vector $(\vec{r}_2 - \vec{r}_1)\times(\vec{r}_3 - \vec{r}_1)$.
(d) Explain the geometrical significance $(\vec{r}-\vec{r}_1)\cdot[(\vec{r}_2 - \vec{r}_1)\times(\vec{r}_3 - \vec{r}_1)] = 0$.

▶ **6-21.** Find the parametric equations of the given line. Also find the tangent vector to the given line $\vec{r} = 3\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 + \lambda(\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2)$.

▶ **6-22.** (a) Find the area of the triangle having vertices at the points

$$P_1(0,0,0) \quad P_2(0,3,4) \quad P_3(4,3,0).$$

(b) Find a unit normal vector to the plane passing through the above three points.
(c) Find the equation of the plane in part (b).

▶ **6-23.** **Distance between two skew lines** Let line ℓ_1 pass through points $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$. Let line ℓ_2 pass through the points $P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$. (a) Show $\vec{N} = \overline{P_0P_1}\times\overline{P_2P_3}$ is perpendicular to both lines. (b) Show the projection of $\overline{P_2P_1}$ onto $\vec{N}$ gives the distance between the lines.

▶ **6-24.** Is the point $(6, 13, 12)$ on the line which passes through the points $P_1(1, 0, 1)$ and $P_2(3, 5, 2)$? Find the equation of the line.

▶ **6-25.**

(a) Derive the vector equations of a line in the following forms.

$$(\vec{r} - \vec{r}_1) \times (\vec{r}_2 - \vec{r}_1) = \vec{0} \qquad \text{and} \qquad \vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1)$$

for a line passing through the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

(b) Show these vector equations produce the same scalar equations for determining points on the line.

▶ **6-26.** Sketch the vectors $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_\beta = \cos\beta\,\hat{\mathbf{e}}_1 + \sin\beta\,\hat{\mathbf{e}}_2$ assuming α and β are acute constant angles.

(a) Show $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$ are unit vectors.

(b) From the dot product $\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta$ derive the addition formula for $\cos(\beta \pm \alpha)$

(c) From the cross product $\hat{\mathbf{e}}_\alpha \times \hat{\mathbf{e}}_\beta$, derive the addition formula for $\sin(\beta \pm \alpha)$

▶ **6-27.** Verify that $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \pm\,\hat{\mathbf{e}}_k$ where the + sign is used if (ijk) is an even permutation of (123) and the – sign is used if (ijk) is an odd permutation of (123).

(a) Verify the above by taking three consecutive numbers from the set $\{1, 2, 3, 1, 2, 3\}$ for the values of i, j, k. These are called the even permutations of the numbers (123).

(b) Verify the above by taking three consecutive numbers from the set $\{3, 2, 1, 3, 2, 1\}$ for the values of i, j, k. These are called the odd permutations of the numbers (123).

▶ **6-28.** Let $\alpha_1,\ \beta_1,\ \gamma_1$ and $\alpha_2,\ \beta_2,\ \gamma_2$ be the direction angles of two lines. Move each line parallel to itself until it passes through the origin. The angle between two lines is defined as the angle between the shifted lines, which pass through the origin.

(a) Show that the angle θ between two lines can be expressed in terms of the direction cosine of the lines and

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2.$$

(b) Find the angle between the lines defined by the equations

$$\vec{r} = (1 + 2t)\,\hat{\mathbf{e}}_1 + (1 + t)\,\hat{\mathbf{e}}_2 + (1 + 2t)\,\hat{\mathbf{e}}_3 \quad \text{and}$$

$$\vec{r} = (1 + 2t)\,\hat{\mathbf{e}}_1 + (2 + 2t)\,\hat{\mathbf{e}}_2 + (6 + t)\,\hat{\mathbf{e}}_3.$$

▶ **6-29.** Find the shortest distance from the point $(-1, 17, 7)$ to the line which passes through the points $P_1(2, 5, 4)$ and $P_2(3, 7, 6)$. Hint: See problem 6-8.

► **6-30.** If $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ and $\vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$ show that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

► **6-31.** If $\vec{A}\times\vec{B} = \vec{0}$ and $\vec{B}\times\vec{C} = \vec{0}$, then calculate $\vec{A}\times\vec{C}$. Justify your answer.

► **6-32.** (a) Find the equation of the plane which passes through the points

$$P_1(3,10,13) \qquad P_2(0,11,12) \qquad P_3(5,12,14).$$

(b) Find the perpendicular distance from the origin to this plane.
(c) Find the perpendicular distance from the point $(6,3,18)$ to this plane.

► **6-33.** Show that the rules for calculating the moment of a force about a line L can be altered as follows: If $\vec{r}$ is the position vector from a point P on the line L to any point on the line of action of the force $\vec{F}$, then $\vec{M} = \vec{r}\times\vec{F}$ is the moment about point P on the line L and $\vec{M}\cdot\hat{\mathbf{e}}_L$ is the moment about the line L, where $\hat{\mathbf{e}}_L$ is a unit vector in the direction of L.

► **6-34.** A force $\vec{F} = 100(\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 - 2\,\hat{\mathbf{e}}_3)$ lbs acts at the point $P_1(2,2,4)$.
(a) Find the moment of $\vec{F}$ about the origin.
(b) Find the moment of $\vec{F}$ about the point $P_2(-1,3,-4)$.
(c) Find the moment about the line passing through the origin and the point P_2.

► **6-35.** Find the indefinite integral of the following vector functions

$$(a)\quad \vec{u}(t) = t\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - t^2\,\hat{\mathbf{e}}_3 \qquad (b)\quad \vec{u}(t) = t\,\hat{\mathbf{e}}_1 + \sin t\;\hat{\mathbf{e}}_2 + \cos t\;\hat{\mathbf{e}}_3$$

► **6-36.** Find the position vector and velocity of a particle which has an acceleration given by $\vec{a} = \cos t\;\hat{\mathbf{e}}_1 + \sin t\;\hat{\mathbf{e}}_2$ if at time $t=0$ the position and velocity are given by $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = 2\,\hat{\mathbf{e}}_3$.

► **6-37.** The acceleration of a particle is given by $\vec{a} = \hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2$. If at time $t=0$ the velocity is $\vec{v} = \vec{v}(0) = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3$ and its position vector is $\vec{r} = \vec{r}(0) = \hat{\mathbf{e}}_2$, then find the velocity and position as a function of time.

► **6-38.** **Distance between parallel planes** If $(\vec{r}-\vec{r}_0)\cdot\vec{N} = 0$ and $(\vec{r}-\vec{r}_1)\cdot\vec{N} = 0$ are the equations of parallel planes, then show the distance between the planes is given by the projection of $\vec{r}_1 - \vec{r}_0$ onto the normal vector $\vec{N}$.

▶ **6-39.** In a rectangular coordinate system a particle moves around a unit circle in the plane $z = 0$ with a constant angular velocity of $\omega = 5\,\text{rad/sec}$

(a) What is the angular velocity vector for this system?

(b) What is the velocity of the particle at any time t if the position of the particle is

$$\vec{r} = \cos 5t\ \hat{\mathbf{e}}_1 + \sin 5t\ \hat{\mathbf{e}}_2?$$

▶ **6-40.** A particle moves along a curve having the parametric equations

$$x = e^t, \qquad y = \cos t, \qquad z = \sin t.$$

(a) Find the velocity and acceleration vectors at any time t.

(b) Find the magnitude of the velocity and acceleration when $t = 0$.

▶ **6-41.** Let $x = x(t)$, $y = y(t)$ denote the parametric representation of a curve in two-dimensions. Using chain rule differentiation, show that the center of curvature vector, at any parameter value t, can be represented by

$$\vec{c}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\,(-\dot{y}\,\hat{\mathbf{e}}_1 + \dot{x}\,\hat{\mathbf{e}}_2)$$

provided $\dot{x}\ddot{y} - \dot{y}\ddot{x}$ is different from zero. Here the notation $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$ has been employed.

▶ **6-42.** Find the center and radius of curvature as a function of x for the given curves.

$$(a) \quad (x-2)^2 + (y-3)^2 = 16 \qquad\qquad (b) \quad y = e^x$$

▶ **6-43.** Let $\vec{e}$ denote a unit vector and let $\vec{A}$ denote a nonzero vector. In what direction $\vec{e}$ will the projection $\vec{A} \cdot \vec{e}$ be a maximum?

▶ **6-44.** Assume $\vec{A} = \vec{A}(t)$ and $\vec{B} = \vec{B}(t)$.

(a) Show that $\dfrac{d}{dt}\left(\vec{A} \cdot \vec{B}\right) = \vec{A} \cdot \dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt} \cdot \vec{B}$

(b) Show that $\dfrac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \vec{A} \times \dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt} \times \vec{B}$

▶ **6-45.** Given $\vec{A} = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + t^3\,\hat{\mathbf{e}}_3$ and $\vec{B} = \sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$.

Find $\quad (a)\ \dfrac{d}{dt}\left(\vec{A} \cdot \vec{B}\right) \qquad (b)\ \dfrac{d}{dt}\left(\vec{A} \times \vec{B}\right) \qquad (c)\ \dfrac{d}{dt}\left(\vec{B} \cdot \vec{B}\right)$

▶ **6-46.** For $\vec{u} = \vec{u}(t) = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + 2t\,\hat{\mathbf{e}}_3$ and $\vec{v} = \vec{v}(t) = t^3\,\hat{\mathbf{e}}_1 + t^2\,\hat{\mathbf{e}}_2 + t^6\,\hat{\mathbf{e}}_3$ find the derivatives

$$(a) \quad \frac{d}{dt}(\vec{u} \cdot \vec{v}) \qquad\qquad (b) \quad \frac{d}{dt}(\vec{u} \times \vec{v})$$

▶ **6-47.** If $\vec{U} = \vec{U}(x,y) = (2x^2y + y^2x)\,\hat{\mathbf{e}}_1 + (xy + 3x^2y)\,\hat{\mathbf{e}}_2$,

then find $\dfrac{\partial \vec{U}}{\partial x}, \dfrac{\partial \vec{U}}{\partial y}, \dfrac{\partial^2 \vec{U}}{\partial x^2}, \dfrac{\partial^2 \vec{U}}{\partial y^2}, \dfrac{\partial^2 \vec{U}}{\partial x\,\partial y}$

▶ **6-48.** Consider a rigid body in pure rotation with angular velocity given by $\vec{\omega} = \omega_1\,\hat{\mathbf{e}}_1 + \omega_2\,\hat{\mathbf{e}}_2 + \omega_3\,\hat{\mathbf{e}}_3$.. For 0 an origin on the axis of rotation and the vector $\vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ denoting the position vector of a particle P in the rigid body, show that the components x, y, z must satisfy the differential equations

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y, \qquad \frac{dy}{dt} = \omega_3 x - \omega_1 z, \qquad \frac{dz}{dt} = \omega_1 y - \omega_2 x$$

▶ **6-49.** For the space curve $\vec{r} = \vec{r}(t) = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + t^2\,\hat{\mathbf{e}}_3$ find

(a) $\dfrac{d\vec{r}}{dt}$ and $\dfrac{ds}{dt} = \left|\dfrac{d\vec{r}}{dt}\right|$

(b) The unit tangent vector to the curve at any time t.

▶ **6-50.** For $\vec{A},\ \vec{B},\ \vec{C}$ functions of time t show

$$\frac{d}{dt}\left[\vec{A} \times (\vec{B} \times \vec{C})\right] = \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{dt}\right) + \vec{A} \times \left(\frac{d\vec{B}}{dt} \times \vec{C}\right) + \frac{d\vec{A}}{dt} \times \left(\vec{B} \times \vec{C}\right)$$

▶ **6-51.** Letting $x = r\cos\theta, \quad y = r\sin\theta$ the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ becomes a function of r and θ which can be denoted $\vec{r} = \vec{r}(r,\theta)$.

(a) Show that $\frac{\partial \vec{r}}{\partial r}$ is perpendicular to the vector $\frac{\partial \vec{r}}{\partial \theta}$ and assign a physical interpretation to your results.

(b) Find unit vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ in the directions $\frac{\partial \vec{r}}{\partial r}$ and $\frac{\partial \vec{r}}{\partial \theta}$ and sketch these unit vectors.

▶ **6-52.** Evaluate the given line integrals along the curve $y = 3x$ from $(1,3)$ to $(2,6)$ using $\vec{F} = \vec{F}(x,y) = xy\,\hat{\mathbf{e}}_1 + (y - x)\,\hat{\mathbf{e}}_2$.

$$(a) \quad \int_C \vec{F} \cdot d\vec{r} \qquad (b) \quad \int_C \vec{F} \times d\vec{r}$$

▶ **6-53.** For $\vec{F} = (xy+1)\,\hat{\mathbf{e}}_1 + (x+z+1)\,\hat{\mathbf{e}}_2 + (z+1)\,\hat{\mathbf{e}}_3$, evaluate the line integral $I = \int_C \vec{F} \cdot d\vec{r}$, where C is the curve consisting of the straight-line segments $\overline{OA} + \overline{AB} + \overline{BC}$, where O is the origin $(0,0,0)$, and A, B, C are, respectively, the points $(1,0,0), \quad (1,1,0), \quad (1,1,1)$.

► **6-54.** Evaluate the line integral $I = \int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = 3(x+y)\,\hat{\mathbf{e}}_1 + 5xy\,\hat{\mathbf{e}}_2$ and C is the curve $y = x^2$ between the points $(0,0)$ and $(2,4)$.

► **6-55.** For $\vec{F} = x\,\hat{\mathbf{e}}_1 + 2xy\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve consisting of the straight-line segments $\overline{OA} + \overline{AB}$, where O is the origin and $A,\ B$ are respectively the points $(1,1,0),\quad (1,1,2)$.

► **6-56.** For $P_1 = (1,1,1)$ and $P_2 = (2,3,5)$, evaluate the line integral

$$I = \int_{P_1}^{P_2} \vec{A}\cdot d\vec{r}, \qquad \text{where} \qquad \vec{A} = yz\,\hat{\mathbf{e}}_1 + xz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$$

and the integration is

(a) Along the straight-line joining P_1 and P_2.

(b) Along any other path joining P_1 to P_2.

► **6-57.** Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = yz\,\hat{\mathbf{e}}_1 + 2x\,\hat{\mathbf{e}}_2 + y\,\hat{\mathbf{e}}_3$ and C is the unit circle $x^2 + y^2 = 1$ lying in the plane $z = 2$.

► **6-58.** Find the work done in moving a particle in the force field $\vec{F} = x\,\hat{\mathbf{e}}_1 - z\,\hat{\mathbf{e}}_2 + 2y\,\hat{\mathbf{e}}_3$ along the parabola $y = x^2,\ z = 2$ between the points $(0,0,2)$ and $(1,2,2)$.

► **6-59.** Find the work done in moving a particle in the force field $\vec{F} = y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ along the straight-line path joining the points $(1,1,1)$ and $(2,3,5)$.

► **6-60.** Sketch some level curves $\phi(x,y) = k$ for the values of k indicated.

$$(a)\quad \phi = 4x - 2y, \quad k = -2,-1,0,1,2 \qquad (c)\quad \phi = x^2 + y^2, \quad k = 0,1,9,25$$

$$(b)\quad \phi = xy, \quad k = -2,-1,0,1,2 \qquad (d)\quad \phi = 9x^2 + 4y^2, \quad k = 16,36,64$$

Give a physical interpretation to your results.

► **6-61.** Sketch the two-dimensional vector fields or their associated field lines.

$$(a)\quad \vec{F} = x\,\hat{\mathbf{e}}_1 - y\,\hat{\mathbf{e}}_2 \qquad (b)\quad \vec{F} = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 \qquad (c)\quad \vec{2}y\,\hat{\mathbf{e}}_1 + 2x\,\hat{\mathbf{e}}_2$$

► **6-62.**

(a) Show line through (x_0,y_0,z_0) and parallel to vector $\vec{A}$ is $(\vec{r} - \vec{r}_0)\times\vec{A} = \vec{0}$

(b) Show line through (x_0,y_0,z_0) and (x_1,y_1,z_1) is given by $(\vec{r} - \vec{r}_0)\times(\vec{r} - \vec{r}_1) = \vec{0}$

(c) Show line through (x_0,y_0,z_0) and perpendicular to the vectors $\vec{A}$ and $\vec{B}$ is given by $(\vec{r} - \vec{r}_0)\times(\vec{A}\times\vec{B}) = \vec{0}$

(d) Find equation of line through (x_0,y_0,z_0) and perpendicular to plane through the noncolinear points P_1,P_2 and P_3.

▶ **6-63.** For the curves defined by the given parametric equations, find the position vector, velocity vector and acceleration vector at the given time.

$$(a)\quad x=t,\quad y=2t,\quad z=3t,\qquad t_0=1$$
$$(b)\quad x=\cos 2t,\quad y=\sin 2t,\quad z=0,\qquad t_0=0$$
$$(c)\quad x=\cos 2t,\quad y=\sin 2t,\quad z=3t,\qquad t_0=\pi$$

▶ **6-64.** Show for $\vec{A}=\vec{A}(t)$, $\vec{B}=\vec{B}(t)$, and $\vec{C}=\vec{C}(t)$ that

$$\frac{d}{dt}\left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]=\vec{A}\cdot\vec{B}\times\frac{d\vec{C}}{dt}+\vec{A}\cdot\frac{d\vec{B}}{dt}\times\vec{C}+\frac{d\vec{A}}{dt}\cdot\vec{B}\times\vec{C}$$

▶ **6-65.** If $\vec{F}=(x^2+z)\,\hat{\mathbf{e}}_1+xyz\,\hat{\mathbf{e}}_2+x^2y^2z^2\,\hat{\mathbf{e}}_3$ find the partial derivatives

$$(a)\ \frac{\partial\vec{F}}{\partial x},\quad (b)\ \frac{\partial\vec{F}}{\partial y},\quad (c)\ \frac{\partial\vec{F}}{\partial z},\quad (d)\ \frac{\partial^2\vec{F}}{\partial x^2},\quad (e)\ \frac{\partial^2\vec{F}}{\partial y^2},\quad (f)\ \frac{\partial^2\vec{F}}{\partial z^2}$$

▶ **6-66.** Find the partial derivatives

$$(a)\ \frac{\partial\Phi}{\partial x},\quad (b)\ \frac{\partial\Phi}{\partial y},\quad (c)\ \frac{\partial^2\Phi}{\partial x^2},\quad (d)\ \frac{\partial^2\Phi}{\partial y^2},\quad (e)\ \frac{\partial^2\Phi}{\partial x\,\partial y}$$

in each of the following cases.

$$(i)\quad \Phi=u^2+v^2\quad \text{with } u=xy \text{ and } v=x+y$$
$$(ii)\quad \Phi=uv \text{ with } u=xy \text{ and } v=x+y$$
$$(iii)\quad \Phi=v^2+2v \text{ with } v=x+y$$

▶ **6-67.** Let $\Phi=\Phi(r,\theta)$ denote a scalar function of position in polar coordinates. If the coordinates are changed to cartesian, where $x=r\cos\theta \qquad y=r\sin\theta$,

(a) Show that

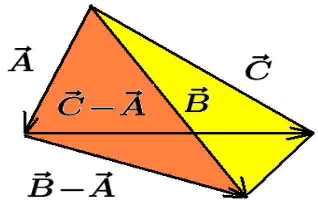
$$\frac{\partial\Phi}{\partial y}=\frac{\partial\Phi}{\partial r}\sin\theta+\frac{\partial\Phi}{\partial\theta}\frac{\cos\theta}{r}$$
$$\frac{\partial^2\Phi}{\partial y^2}=\frac{\partial\Phi}{\partial r}\frac{\cos^2\theta}{r}+\frac{\partial^2\Phi}{\partial r^2}\sin^2\theta+2\frac{\partial^2\Phi}{\partial r\,\partial\theta}\frac{\sin\theta\cos\theta}{r}-2\frac{\partial\Phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2\Phi}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

(b) Show that $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}=\dfrac{\partial^2\Phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\Phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2}{}^2$

▶ **6-68.** Show the equation of the tangent plane to point (x_1,y_1,z_1) on the surface of sphere centered at (x_0,y_0,z_0), having radius a, is given by $(\vec{r}-\vec{r}_1)\cdot(\vec{r}_1-\vec{r}_0)=0$ Sketch a diagram illustrating these vectors.

▶ **6-69.** For the scalar function of position $F=F(u,v)$, where $u=u(x,y),\ v=v(x,y)$ calculate the quantities $\dfrac{\partial F}{\partial x},\quad \dfrac{\partial F}{\partial y},\quad \dfrac{\partial^2 F}{\partial x^2},\quad \dfrac{\partial^2 F}{\partial x\,\partial y},\quad \dfrac{\partial^2 F}{\partial y^2}$

▶ **6-70.** Consider the tetrahedron defined by the vectors $\vec{A}, \vec{B}, \vec{C}$ illustrated.



(a) Show the vectors $\vec{n}_1 = \frac{1}{2}\vec{A}\times\vec{B}$, $\vec{n}_2 = \frac{1}{2}\vec{B}\times\vec{C}$, $\vec{n}_3 = \frac{1}{2}\vec{C}\times\vec{A}$, $\vec{n}_4 = \frac{1}{2}(\vec{C}-\vec{A})\times(\vec{B}-\vec{A})$ are normal to the faces of the tetrahedron with magnitudes equal to the area of the faces. (b) Show $\vec{n}_1+\vec{n}_2+\vec{n}_3+\vec{n}_4 = \vec{0}$

▶ **6-71.** Find the work done in moving a particle in a counterclockwise direction around a unit circle in the $z=0$ plane if the particle moves in the force field

$$\vec{F} = \vec{F}(x,y,z) = (x+y+z)\,\hat{\mathbf{e}}_1 + (2x-y+3z)\,\hat{\mathbf{e}}_2 + (3x-y-z)\,\hat{\mathbf{e}}_3.$$

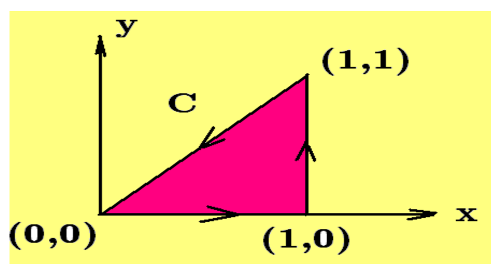
▶ **6-72.** The straight-line defined by the parametric equations

$$x = 2+\lambda, \qquad y = 3+2\lambda, \qquad z = 4-2\lambda$$

with parameter λ, is drawn through the force field $\vec{F} = \vec{F}(x,y,z) = xy\,\hat{\mathbf{e}}_1 + yz\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$. Evaluate the given line integrals along this line from the point $P_1(2,3,4)$ to the point $P_2(4,7,0)$

$$(a) \quad \int_{P_1}^{P_2} (x^2+y^2)ds \qquad (b) \quad \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} \qquad (c) \quad \int_{P_1}^{P_2} \vec{F}\times d\vec{r}$$

▶ **6-73.** A particle moves around the closed curve C illustrated in figure. It moves in a vector field $\vec{F}$ defined by



$$\vec{F} = \vec{F}(x,y) = 6\,(y^2-x)\,\hat{\mathbf{e}}_1 + 6\,x\,\hat{\mathbf{e}}_2.$$

Evaluate the line integrals in parts (a) and (b).

(a) $\oint \vec{F}\cdot d\vec{r}$ $\quad (b)$ $\oint \vec{F}\times d\vec{r}$ $\quad (c)$ Show that $\circlearrowleft\!\!\oint \vec{F}\cdot d\vec{r} = -\circlearrowright\!\!\oint \vec{F}\cdot d\vec{r}$

▶ **6-74.** Evaluate the given line integrals along the path $C = \{\,(x,y)\,|\, x=2t,\ y=1+t+t^2\,\}$ from $t=0$ to $t=3$.

$$(a) \quad \int_C y\,dx + (x+y)\,dy \qquad (b) \quad \int_C y\,dx - x\,dy \qquad (c) \quad \int_C 2xy\,dx + x^2\,dy$$

▶ **6-75.** Evaluate the given line integrals around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, both clockwise and counterclockwise.

$$(a) \quad \oint_C x(y+1)\,dx + (x+1)y\,dy \qquad (b) \quad \oint_C (x^2-y^2)\,dx + (x^2+y^2)\,dy \qquad (c) \quad \oint_C y\,dx + x\,dy$$

Chapter 7

Vector Calculus I

One aspect of vector calculus can be described as taking many of the concepts from scalar calculus, generalizing these concepts and representing them in a vector format. These alternative vector representations have many applications in representing two-dimensional and three-dimensional physical problems. Let us begin by examining the representation of curves using vectors.

Curves

A **two-dimensional curve** can be defined

(i) Explicitly $\quad y = f(x)$

(ii) Implicitly $\quad F(x,y) = 0$

(iii) Parametrically $\quad x = x(t), \quad y = y(t)$

(iv) As a vector $\quad \vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2$ or $\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$

A **three-dimensional curve** can be defined

(i) Parametrically $\quad x = x(t), \quad y = y(t), \quad z = z(t)$

(ii) As a vector $\quad \vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$

(iii) A curve in space is sometimes defined as the intersection of two surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ and in this special case the curve is defined by a set of (x,y,z) values which are common to both surfaces.

It is assumed that the functions used to define these curves are continuous single-valued functions which are everywhere differentiable. Also note that **the parametric and vector representations of a curve are not unique.**

In two-dimensions a parametric curve $\{x(t), y(t)\}$, for $a \le t \le b$ has end points $(x(a), y(a))$ and $(x(b), y(b))$. A curve is called a **closed curve** if its **end points** coincide and $x(a) = x(b)$ and $y(a) = y(b)$. If (x_0, y_0) is a point on the given curve, which is not an end point, such that there exists more than one value of the parameter t such that $(x(t), y(t)) = (x_0, y_0)$, then the point (x_0, y_0) is called **a multiple point** or **a point where the curve crosses itself**. A curve is called a **simple closed curve** if it has no multiple points and the end points coincide. Simple closed curves are defined by one-to-one mappings. The above definitions of end points, closed curve, simple closed curve and multiple points apply to parametric curves $\{x(t), y(t), z(t)\}$ in three-dimensions and to n-dimensional parametric curves defined by $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ as the parameter t ranges from a to b.

A curve is called an **oriented curve** if

(i) The curve is piecewise smooth.

(ii) The position vector $\vec{r} = \vec{r}(t)$, when expressed in terms of a parameter t, determines the direction of the tangent vector to each point on the curve.

(iii) The direction of the tangent vector is said to determine the orientation of the curve.

(iv) A plane curve which is a simple closed curve which does not cross itself is said to have either a clockwise or counterclockwise orientation which depends upon the directions of the tangent vector at each point on the closed curve.

Tangents to Space Curve

In three-dimensions the derivative vector $\frac{d\vec{r}}{dt} = x'(t)\,\hat{\mathbf{e}}_1 + y'(t)\,\hat{\mathbf{e}}_2 + z'(t)\,\hat{\mathbf{e}}_3$ is tangent to the point $(x(t), y(t), z(t))$ on the curve $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ for any fixed value of the parameter t. The tangent line to the curve $\vec{r} = \vec{r}(t)$ at the point where the parameter has the value $t = t^*$ is given by

$$\vec{R} = \vec{R}(\lambda) = \vec{r}(t^*) + \lambda \frac{d\vec{r}}{dt}\bigg|_{t=t^*} \qquad -\infty < \lambda < \infty$$

where λ is a parameter. The tangent line defined by the vector $\vec{R}$ can also be expressed in the expanded form

$$\vec{R} = \vec{R}(\lambda) = (x(t^*) + \lambda x'(t*))\,\hat{\mathbf{e}}_1 + (y(t^*) + \lambda y'(t^*))\,\hat{\mathbf{e}}_2 + (z(t^*) + \lambda z'(t^*))\,\hat{\mathbf{e}}_3$$

where t^* represents some fixed value of the parameter t. The element of arc length ds along the curve $\vec{r} = \vec{r}(t)$ is obtained from the relation

$$ds^2 = d\vec{r}\cdot d\vec{r} = (dx)^2 + (dy)^2 + (dz)^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt)^2$$

$$\text{and} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{7.1}$$

$$\text{so that one can write} \quad \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The total arc length for the curve $\vec{r} = \vec{r}(t)$ for $t_0 \le t \le t_1$ is given by

$$\text{arc length of curve} = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The unit tangent vector to the curve can therefore be expressed by

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{1}{\frac{ds}{dt}}\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \tag{7.2}$$

which shows that the derivative of the position vector with respect to arc length s produces a **unit tangent vector to the curve.**

Example 7-1. **Reflection property for the parabola.**

The parabola $y^2 = 4px$ with focus F having coordinates $(p, 0)$ can be represented parametrically. One parametric representation for the position vector is

$$\vec{r} = \vec{r}(t) = \frac{t^2}{4p}\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2, \qquad -\infty < t < \infty \tag{7.3}$$

and the resulting parabola is illustrated in the figure 7-1. In this figure assume the surface of the parabola is a mirrored surface.

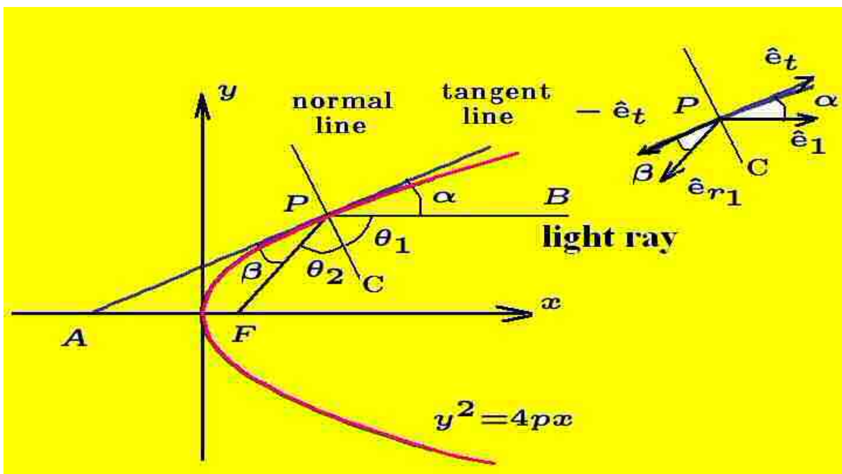


Figure 7-1. Light ray PB gets reflected to ray through focus.

The derivative vector

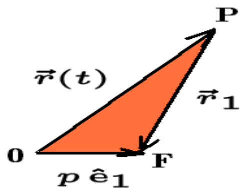
$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{t}{2p}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$$

produces a tangent vector to the curve and the vector

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{\frac{t}{2p}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + t^2/4p^2}} = \frac{t\,\hat{\mathbf{e}}_1 + 2p\,\hat{\mathbf{e}}_2}{\sqrt{4p^2 + t^2}}$$

is a unit tangent vector to the curve.

Consider a general point P on the parabola where a light ray PB parallel to the $x-$axis hits the parabola. Construct the normal to the parabola and label the angle $\angle BPC$ the angle θ_1 and then label the angle $\angle FPC$ the angle θ_2. The angle θ_1 is called the **angle of incidence** and the angle θ_2 is called the **angle of reflection.** Also in figure 7-1 are the complementary angles to θ_1 and θ_2. These angles are labeled as α and β.



Construct the vector $\vec{r}_1$ from point P to the focus F and by using vector addition show with the aid of equation (7.3) that

$$\vec{r}(t) + \vec{r}_1 = p\,\hat{\mathbf{e}}_1 \quad \text{or} \quad \vec{r}_1 = (p - t^2/4p)\,\hat{\mathbf{e}}_1 - t\,\hat{\mathbf{e}}_2$$

A unit vector in the direction of $\vec{r}_1$ is

$$\hat{\mathbf{e}}_{r_1} = \frac{(p - t^2/4p)\,\hat{\mathbf{e}}_1 - t\,\hat{\mathbf{e}}_2}{\sqrt{(p - t^2/4p)^2 + t^2}} = \frac{(4p^2 - t^2)\,\hat{\mathbf{e}}_1 - 4pt\,\hat{\mathbf{e}}_2}{\sqrt{(4p^2 - t^2)^2 + 16p^2t^2}}$$

Using the definition of the dot product one can show

$$\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_1 = \cos\alpha = \frac{t}{\sqrt{4p^2 + t^2}}$$

$$(-\,\hat{\mathbf{e}}_t) \cdot \hat{\mathbf{e}}_{r_1} = \cos\beta = \frac{-t(4p^2 - t^2) + 8p^2t}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}} = \frac{t(t^2 + 4p^2)}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}}$$

If $\cos\alpha = \cos\beta$ for all values of the parameter t, then one must show that

$$\frac{t}{\sqrt{4p^2 + t^2}} = \frac{t(t^2 + 4p^2)}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}} \tag{7.4}$$

Using algebra one can establish that equation (7.4) is indeed true and so the angles α and β are equal. Simplify the equation (7.4) to the form

$$\sqrt{(4p^2 + t^2)^2 + 16p^2t^2} = t^2 + 4p^2$$

and then square both sides to show

$$16p^4 - 8p^2t^2 + t^4 + 16p^2t^2 = (t^2 + 4p^2)^2$$

which reduces to the identity $(t^2 + 4p^2)^2 = (t^2 + 4p^2)^2$. The equality of the angles α and β implies $\theta_1 = \theta_2$ or the angle of incidence is equal to the angle of reflection. One can also show that the distances AF=FP which shows the triangle PFA is an isosceles triangle with angle $\angle FAP$ equal to angle $\angle APF$ implying the complementary angles θ_1 and θ_2 are equal. These results show that all light coming in parallel to the $x-$axis will be reflected by the mirrored parabolic surface and pass through the focus. Conversely, if a light source is placed at the focus, than rays of light from the focus are reflected parallel to the $x-$axis.

■

Example 7-2. **Reflection property of the ellipse.**

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having eccentricity $e < 1$ and foci at the points F_1, F_2 with coordinates $(c, 0)$ and$(-c, 0)$ respectively. For this ellipse $b^2 = a^2 - c^2$ and $c = ae$. If P represents an arbitrary point (x_0, y_0) on the ellipse, then one can construct the vector $\vec{r}_1$ from P to F_1 and also construct the vector $\vec{r}_2$ from point P to F_2. The magnitude of these vectors when summed gives

$$|\vec{r}_1| + |\vec{r}_2| = 2a \tag{7.5}$$

The vectors $\vec{r}_1$, $\vec{r}_2$ and the ellipse are illustrated in the figure 7-2. If the ellipse is mirrored, then a ray of light from the focus F_1 will reflect from an arbitrary point P on the ellipse to the focus at F_2.

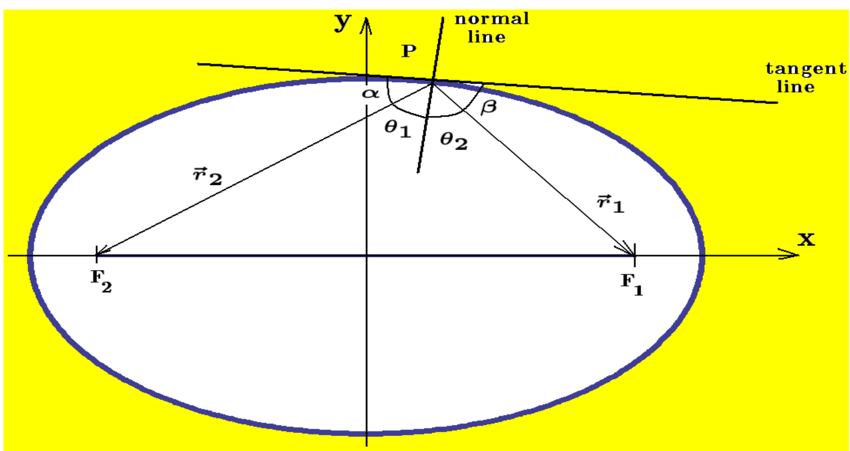


Figure 7-2. Light ray from one focus passes through other focus.

The position vector of a general point on the ellipse can be represented in the parametric form

$$\vec{r} = \vec{r}(t) = a\cos t\,\hat{\mathbf{e}}_1 + b\sin t\,\hat{\mathbf{e}}_2, \qquad 0 \le t \le 2\pi \tag{7.6}$$

A point P on the ellipse with coordinates (x_0, y_0) is described by equation (7.6) by assigning the proper value for the parameter t. The proper value for the parameter t, call it t_0, is determined by solving the equations

$$x_0 = a\cos t \qquad \text{and} \qquad y_0 = b\sin t$$

simultaneously, to obtain $t_0 = \tan^{-1}\left(\frac{ay_0}{bx_0}\right)$. The derivative vector

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{\mathbf{e}}_1 + b\cos t\,\hat{\mathbf{e}}_2$$

evaluated at the value t_0, represents a tangent vector to the ellipse at the point P. A unit vector in the direction of the tangent line at the point P is then given by

$$\hat{\mathbf{e}}_t = \frac{-a\sin t\,\hat{\mathbf{e}}_1 + b\cos t\,\hat{\mathbf{e}}_2}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

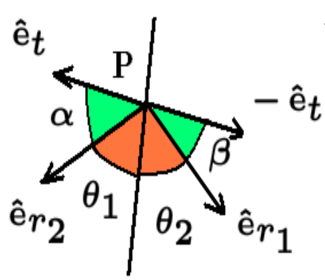
where everything is understood to be evaluated at $t = t_0$. Using vector addition one can show the vectors $\vec{r}_1$ and $\vec{r}_2$ must satisfy

$$\vec{r}(t) + \vec{r}_1 = c\,\hat{\mathbf{e}}_1 \qquad \text{and} \qquad \vec{r}(t) + \vec{r}_2 + c\,\hat{\mathbf{e}}_1 = \vec{0}$$

These equations allow one to express the vectors $\vec{r}_1$ and $\vec{r}_2$ in the form

$$\begin{aligned}\vec{r}_1 =& (c - a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2\\ \vec{r}_2 =& (-c - a\cos t)\,\hat{\mathbf{e}}_2 - b\sin t\,\hat{\mathbf{e}}_2\end{aligned}$$

where again, these vectors are to be evaluated at the parameter value t_0.



Unit vectors in the directions of $\vec{r}_1$ and $\vec{r}_2$ can be expressed

$$\begin{aligned}\hat{\mathbf{e}}_{r_1} =& \frac{(c - a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2}{\sqrt{(c - a\cos t)^2 + b^2\sin^2 t}}\\ \hat{\mathbf{e}}_{r_2} =& \frac{-(c + a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2}{\sqrt{(c + a\cos t)^2 + b^2\sin^2 t}}\end{aligned}$$

By employing the definition of the dot product of unit vectors one can verify that

$$\begin{aligned}\hat{\mathbf{e}}_{r_1}\cdot(-\,\hat{\mathbf{e}}_t) = \cos\beta =& \frac{a\sin t(c - a\cos t) + b^2\sin t\cos t}{r_1\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\\ \hat{\mathbf{e}}_{r_2}\cdot(\,\hat{\mathbf{e}}_t) = \cos\alpha =& \frac{a\sin t(c + a\cos t) - b^2\sin t\cos t}{(2a - r_1)\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\end{aligned}$$

where $r_1 = |r_1| = \sqrt{(c - a\cos t)^2 + b^2\sin^2 t}$. If $\cos\alpha = \cos\beta$ for all values of the parameter t, then one must show that

$$\frac{a\sin t(c - a\cos t) + b^2\sin t\cos t}{r_1\sqrt{a^2\sin^2 t + b^2\cos^2 t}} = \frac{a\sin t(c + a\cos t) - b^2\sin t\cos t}{(2a - r_1)\sqrt{a^2\sin^2 t + b^2\cos^2 t}} \tag{7.7}$$

Using algebra one can verify that the equation (7.7) reduces down to an identity so that the angles α and β are equal. This in turn implies $\theta_1 = \theta_2$ which states that the angle of incidence equals the angle of reflection.

Sound waves also are reflected in the same way. Elliptically shaped rooms or domes have the property that someone whispering at one focus of the ellipse can easily be heard at the other focus of the ellipse. This gives rise to the phrase "whispering galleries". Constructions which make use of this reflection property of the ellipse can be found at Statuary Hall in the United States capital, St Paul's Cathedral in London, the Grand Central Terminal in New York City and in certain museums throughout the world.

■

Example 7-3. **Reflection property of the hyperbola.**

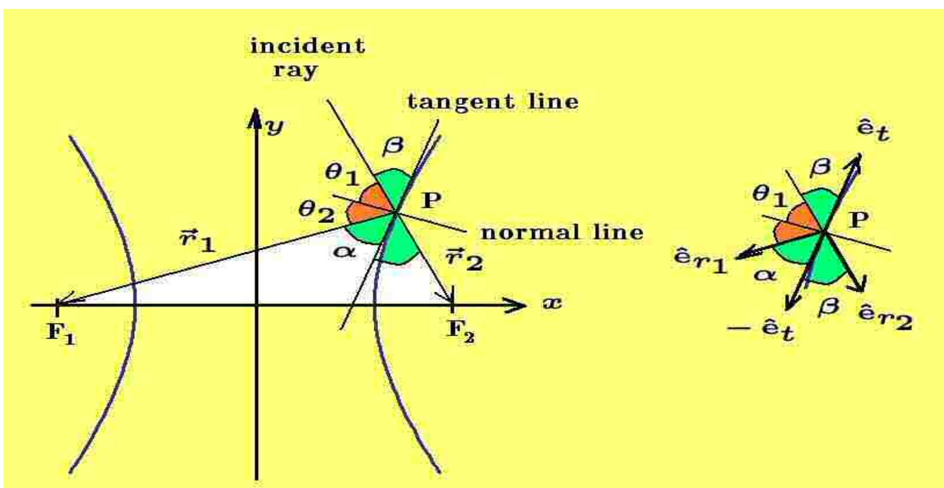


Figure 7-3.
Light ray directed toward one focus reflects and goes through other focus.

Sketch the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci F_1, F_2 having coordinates $(-c, 0)$ and $(c, 0)$ respectively, where $c = ae$ and $e > 1$ is the eccentricity of the hyperbola. Construct an incident ray aimed at the focus F_2 which passes through a known point (x_0, y_0) on the hyperbola. Label the point where this ray intersects the hyperbola as point P. Sketch the tangent line and normal line to the hyperbola at the point P and label the angle of incidence as θ_1 and the angle of reflection as θ_2. The complementary angles associated with these angles are labeled α and β respectively. Next construct the vector $\vec{r}_1$ running from the point P on the hyperbola to the focus F_1 and then construct the vector $\vec{r}_2$ running from the P to the other focus F_2 as illustrated in the figure 7-3.

The position vector to a general point on the right branch of the hyperbola can be expressed

$$\vec{r} = \vec{r}(t) = a\cosh t\,\hat{\mathbf{e}}_1 + b\sinh t\,\hat{\mathbf{e}}_2$$

where t is a parameter. The value of the parameter t, call it t_0, corresponding to the point P having coordinates (x_0, y_0) is obtained by solving the equations

$$x_0 = a\cosh t \qquad \text{and} \qquad y_0 = b\sinh t$$

simultaneously to obtain $t_0 = \tanh^{-1}\left(\dfrac{ay_0}{bx_0}\right)$. The derivative of the position vector is

$$\frac{d\vec{r}}{dt} = a\sinh t\,\hat{\mathbf{e}}_1 + b\cosh t\,\hat{\mathbf{e}}_2$$

and this vector, when evaluated using the parameter value t_0 is a tangent vector to the hyperbola at the point P. The vector

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{a\sinh t\,\hat{\mathbf{e}}_1 + b\cosh t\,\hat{\mathbf{e}}_2}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}$$

also evaluated at t_0, is a unit tangent vector to the hyperbola at the point P. Using vector addition one can show that the vectors $\vec{r}_1$ and $\vec{r}_2$ are given by

$$\vec{r}_1 = -\left(a\cosh t + c\right)\hat{\mathbf{e}}_1 - b\sinh t$$
$$\vec{r}_2 = -\left(a\cosh t - c\right)\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2$$

all to be evaluated at the parameter value t_0. Unit vectors in the directions of $\vec{r}_1$ and $\vec{r}_2$ are

$$\hat{\mathbf{e}}_{r_1} = \frac{-(a\cosh t + c)\,\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2}{\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t}}$$
$$\hat{\mathbf{e}}_{r_2} = \frac{-(a\cosh t - c)\,\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t}}$$

also to be evaluated at $t = t_0$. The given hyperbola satisfies the properties that $a^2 + b^2 = c^2$ and $|\vec{r}_1| - |\vec{r}_2| = 2a$ or

$$\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t} - \sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} = 2a$$

for all values of the parameter t. The angles α and β constructed at the point P can be calculated from the dot products

$$
\begin{aligned}
(-\hat{\mathbf{e}}_t)\cdot\hat{\mathbf{e}}_{r_2} = \cos\alpha =& \frac{a\sinh t\,(a\cosh t + c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t})}\\
=& \frac{a\sinh t\,(a\cosh t + c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a)} \qquad (7.8)\\
(-\hat{\mathbf{e}}_t)\cdot\hat{\mathbf{e}}_{r_1} = \cos\beta =& \frac{a\sinh t\,(a\cosh t - c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t})} \qquad (7.9)
\end{aligned}
$$

If $\cos\alpha = \cos\beta$ then one must show that the right-hand sides of equations (7.8) and (7.9) are equal. Setting the right-hand sides equal to one another and simplifying produces

$$
\frac{a^2\sinh t\cosh t + ac\sinh t + b^2\sinh t\cosh t}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a} = \frac{a^2\sinh t\cosh t - ac\sinh t + b^2\sinh t\cosh t}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t}} \qquad (7.10)
$$

To show equation (7.10) reduces to an identity, first show equation (7.10) can be written

$$
|\vec{r}_2| = \frac{c\cosh t - a}{c\cosh t + a}\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a
$$

and then use the fact that $c^2 = a^2 + b^2$ and $|\vec{r}_2| = c\cosh t - a$ to simplify the above equation to the form

$$
c\cosh t + a = \sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a \qquad (7.11)
$$

It is now an easy exercise to show equation (7.11) reduces to an identity.

All this algebra shows that the angles α and β are equal and consequently the complementary angles θ_1 and θ_2 are also equal, showing the hyperbola has the property that the angle of incidence equals the angle of reflection. The above results imply that a ray of light aimed at the focus F_2 will be reflected and pass through the other focus. This reflection property of the hyperbola is **one of the basic principles used in the construction of a reflecting telescope.**

■

Normal and Binormal to Space Curve

Recall that the **unit tangent vector to a space curve** $\vec{r} = \vec{r}(t)$, for any value of the parameter t, is given by the equation

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \tag{7.12}$$

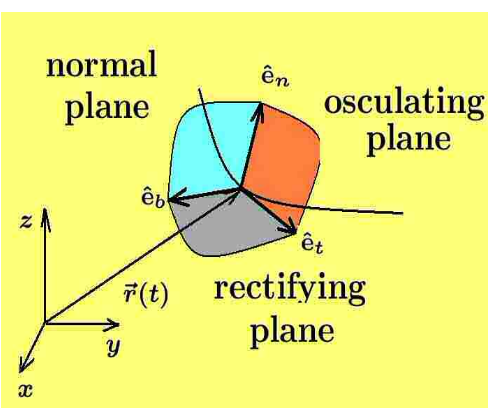
and satisfies $\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t = 1$. Differentiating this relation with respect to the **arc length parameter** s one finds

$$\frac{d}{ds}\left[\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t\right] = \hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \cdot \hat{\mathbf{e}}_t = 0 \quad \text{or} \qquad 2\,\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} = 0 \tag{7.13}$$

The zero dot product in equation (7.13) demonstrates that the vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is perpendicular to the unit tangent vector $\hat{\mathbf{e}}_t$. Note that this vector can be calculated using the chain rule for differentiation $\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{dt}$ where $\frac{ds}{dt}$ is calculated using the equation (7.1). Observe that there are an infinite number of vectors which are perpendicular to the unit tangent vector $\hat{\mathbf{e}}_t$. The **unit vector with the same direction as the vector** $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is called **the principal unit normal vector to the curve** $\vec{r}(t)$ for each value of the parameter t. The **principal unit normal vector** in the direction of the derivative vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is given the label $\hat{\mathbf{e}}_n$. The vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ has the same direction as $\hat{\mathbf{e}}_n$ and so one can write

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n \tag{7.14}$$

where κ is a scaling constant called **the curvature of the curve** $\vec{r}(t)$. The curvature κ will vary as the parameter t changes. The quantity $\rho = \frac{1}{\kappa}$ is called the **radius of curvature at the point associated with the parameter value of** t. The unit vector $\hat{\mathbf{e}}_b$ calculated from the **cross product of** $\hat{\mathbf{e}}_t$ **and** $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$, is perpendicular to both the unit tangent $\hat{\mathbf{e}}_t$ and unit normal $\hat{\mathbf{e}}_n$ and is called **the unit binormal vector to the curve** as the parameter t changes.



The vectors $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b$ are called **a moving triad along the curve** $\vec{r}(t)$ because the unit vectors $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b$ generated a localized **right-handed coordinate system** which changes as the parameter t changes. The plane which contains the unit tangent $\hat{\mathbf{e}}_t$ and principal normal $\hat{\mathbf{e}}_n$ is called the **osculating plane**. The plane containing the unit binormal $\hat{\mathbf{e}}_b$ and unit normal $\hat{\mathbf{e}}_n$ is called **the normal plane.** The plane which is perpendicular to the principal normal $\hat{\mathbf{e}}_n$ is called **the rectifying plane.** Let $\vec{r}(t^*)$ denote the position vector to a fixed point on the given curve and let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a variable point in one of these planes. One can then show

$$\begin{aligned}&\text{The osculating plane can be written} \quad (\vec{r} - \vec{r}(t^*)) \cdot \hat{\mathbf{e}}_b = 0\\ &\text{The normal plane can be written} \quad (\vec{r} - r(t^*)) \cdot \hat{\mathbf{e}}_t = 0\\ &\text{The rectifying plane can be written} \quad (\vec{r} - \vec{r}(t^*)) \cdot \hat{\mathbf{e}}_n = 0\end{aligned} \tag{7.15}$$

The equations of the straight lines through the fixed point $\vec{r}(t^*)$ and having the directions of $\hat{\mathbf{e}}_t, \hat{\mathbf{e}}_n$ or $\hat{\mathbf{e}}_b$ are given by

$$\begin{aligned}(\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_t =& \vec{0} \qquad \text{Tangent line}\\ (\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_n =& \vec{0} \qquad \text{Line normal to curve}\\ (\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_b =& \vec{0} \qquad \text{Line in binormal direction}\end{aligned} \tag{7.16}$$

Let us examine the three unit vectors $\hat{\mathbf{e}}_t, \hat{\mathbf{e}}_b$ and $\hat{\mathbf{e}}_n$ and their derivatives with respect to the arc length parameter s. One can calculate the derivatives $\frac{d\,\hat{\mathbf{e}}_t}{ds}$, $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ and $\frac{d\,\hat{\mathbf{e}}_n}{ds}$ with the aid of the triple scalar product relations

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

It has been demonstrated that $\frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$ and $\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$, consequently one finds that

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \times \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} + \kappa\,\hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} \tag{7.17}$$

Take the dot product of both sides of equation (7.17) with the vector $\hat{\mathbf{e}}_t$ and use the above triple scalar product result to show

$$\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} = \hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} = 0$$

This result shows that the vector $\hat{\mathbf{e}}_t$ is perpendicular to the vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$. By differentiating the relation $\hat{\mathbf{e}}_b \cdot \hat{\mathbf{e}}_b = 1$ one finds that

$$\hat{\mathbf{e}}_b \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} + \frac{d\,\hat{\mathbf{e}}_b}{ds} \cdot \hat{\mathbf{e}}_b = 2\,\hat{\mathbf{e}}_b \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} = 0$$

which implies that the vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ is also perpendicular to the vector $\hat{\mathbf{e}}_b$. These two results show that the derivative vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ must be in the direction of the normal vector $\hat{\mathbf{e}}_n$. Hence, there exists a constant K such that

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = K\,\hat{\mathbf{e}}_n \tag{7.18}$$

where K is a constant. By convention, the constant K is selected as $-\tau$, where τ is called **the torsion** and the reciprocal $\sigma = \frac{1}{\tau}$ is called **the radius of torsion.** Taking the dot product of both sides of equation (7.18) with the unit vector $\hat{\mathbf{e}}_n$ gives

$$\tau = \tau(s) = -\,\hat{\mathbf{e}}_n \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} \tag{7.19}$$

The torsion is a measure of the **twisting of a curve out of a plane** and is a measure of how the osculating plane changes with respect to arc length. The torsion can be positive or negative and if the torsion is zero, then the curve must be **a plane curve**. The three vectors $\hat{\mathbf{e}}_t,\ \hat{\mathbf{e}}_b,\ \hat{\mathbf{e}}_n$ form a right-handed system of unit vectors and so one can write $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t$. Differentiating this relation with respect to arc length gives

$$\frac{d\,\hat{\mathbf{e}}_n}{ds} = \hat{\mathbf{e}}_b \times \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_b}{ds} \times \hat{\mathbf{e}}_t = \hat{\mathbf{e}}_b \times \kappa\,\hat{\mathbf{e}}_n - \tau\,\hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_t = -\kappa\,\hat{\mathbf{e}}_t + \tau\,\hat{\mathbf{e}}_b \tag{7.20}$$

These results give the Frenet[1]-Serret[2] formulas

$$\begin{aligned} \frac{d\,\hat{\mathbf{e}}_t}{ds} =& \kappa\,\hat{\mathbf{e}}_n \\ \frac{d\,\hat{\mathbf{e}}_b}{ds} =& -\tau\,\hat{\mathbf{e}}_n \\ \frac{d\,\hat{\mathbf{e}}_n}{ds} =& \tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t \end{aligned} \tag{7.21}$$

Using matrix notation[3], the Frenet-Serret formulas can be written as

$$\begin{bmatrix} \frac{d\,\hat{\mathbf{e}}_t}{ds} \\ \frac{d\,\hat{\mathbf{e}}_b}{ds} \\ \frac{d\,\hat{\mathbf{e}}_n}{ds} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & -\tau \\ -\kappa & \tau & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_t \\ \hat{\mathbf{e}}_b \\ \hat{\mathbf{e}}_n \end{bmatrix} \tag{7.22}$$

Recall that if $\vec{B}$ is a vector which rotates about a line with angular velocity $\vec{\omega}$, then $\frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B}$. One can use this result to give a physical interpretation to the Frenet-Serret formulas. One can write

$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \kappa\,\hat{\mathbf{e}}_n \frac{ds}{dt} = \kappa \frac{ds}{dt}\,\hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t = \vec{\omega} \times \hat{\mathbf{e}}_t \qquad \text{where} \quad \vec{\omega} = \kappa \frac{ds}{dt}\,\hat{\mathbf{e}}_b$$

---

1 Jean Frédéric Frenet (1816-1900) A French mathematician.
2 Joseph Alfred Serret (1819-1885) A French mathematician.
3 See chapter 10 for a description of the matrix notation.

This is interpreted as showing the vector $\hat{\mathbf{e}}_t$ rotates about a line through $\hat{\mathbf{e}}_b$ with angular velocity $\vec{\omega}$. If the curvature $\kappa = 0$, then $\vec{\omega}$ is also zero and so the tangent vector $\hat{\mathbf{e}}_t$ is not rotating and consequently **the curve is a straight line.** Similarly, one can write

$$\frac{d\,\hat{\mathbf{e}}_b}{dt} = \frac{d\,\hat{\mathbf{e}}_b}{ds}\frac{ds}{dt} = -\tau\,\hat{\mathbf{e}}_n\frac{ds}{dt} = -\tau\frac{ds}{dt}\,\hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t = \tau\frac{ds}{dt}\,\hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_b = \vec{\omega} \times \hat{\mathbf{e}}_b \quad \text{where} \quad \vec{\omega} = \tau\frac{ds}{dt}\,\hat{\mathbf{e}}_t$$

and this result is interpreted as meaning the vector $\hat{\mathbf{e}}_b$ is rotating about the $\hat{\mathbf{e}}_t$ direction with angular velocity $\vec{\omega}$. If the torsion $\tau = 0$, then there is no rotation of the binormal vector and so the curve **remains a plane curve** in the plane of the normal and tangent vectors.

It is left as an exercise to give a physical interpretation to the derivative $\dfrac{d\,\hat{\mathbf{e}}_n}{dt}$.

Example 7-4. Determine how to calculate the curvature κ of a space curve.

Solution Use the fact $\dfrac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$ so that $|\frac{d\,\hat{\mathbf{e}}_t}{ds}| = \kappa$, since $\hat{\mathbf{e}}_n$ is a unit vector. Let $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ denote the position vector to a point on the space curve where t represents some parameter. If the arc length parameter s is used, then

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{1}{|\frac{d\vec{r}}{dt}|}\frac{d\vec{r}}{dt} = \hat{\mathbf{e}}_t$$

is a unit tangent vector to the curve and the derivative of this vector with respect to arc length s gives

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$$

Taking the dot product of this vector with itself gives

$$\frac{d^2\vec{r}}{ds^2}\cdot\frac{d^2\vec{r}}{ds^2} = (\kappa\,\hat{\mathbf{e}}_n\,)\cdot(\kappa\,\hat{\mathbf{e}}_n\,) = \kappa^2 = [x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2$$

where

$$\begin{aligned} x'(t) =& \frac{dx}{ds}\frac{ds}{dt} = x'(s)\frac{ds}{dt} \qquad \Longrightarrow \qquad x'(s) = \frac{x'(t)}{\frac{ds}{dt}} \\ x''(t) =& x'(s)\frac{d^2s}{dt^2} + \frac{d}{dt}x'(s)\frac{ds}{dt} \\ x''(t) =& x'(s)\frac{d^2s}{dt^2} + x''(s)\left(\frac{ds}{dt}\right)^2 \end{aligned}$$

and solving for $x''(s)$ one finds $x''(s) = \dfrac{x''(t) - x'(s)\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^2}$ The derivatives for $y''(s)$ and $z''(s)$ are calculate in a similar fashion.

■

Example 7-5. Determine how to calculate the torsion τ of a space curve.

Solution The derivative $\dfrac{d\vec{r}}{ds} = \hat{\mathbf{e}}_t$ is a unit tangent vector to the space curve and $\dfrac{d^2\vec{r}}{ds^2} = \dfrac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$. Calculating the third derivative of the position vector with respect to the arc length parameter gives

$$\frac{d^3\vec{r}}{ds^3} = k\frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa(\tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t) + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$

Use the properties of the triple scalar product with

$$\frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right) = \hat{\mathbf{e}}_t\cdot\left(\kappa\,\hat{\mathbf{e}}_n\times[\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n]\right) \tag{7.23}$$

together with the cross products

$$\hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t, \qquad \hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_t = -\,\hat{\mathbf{e}}_b, \qquad \hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_n = \vec{0}$$

and show equation (7.23) simplifies to $\dfrac{d\vec{r}}{ds}\cdot\left(\dfrac{d^2\vec{r}}{ds^2}\times\dfrac{d^3\vec{r}}{ds^3}\right) = \hat{\mathbf{e}}_t\cdot[k^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b] = \kappa^2\tau$
Using the result from the previous example that $\kappa^2 = [x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2$ one can write

$$\tau = \frac{\dfrac{d\vec{r}}{ds}\cdot\left(\dfrac{d^2\vec{r}}{ds^2}\times\dfrac{d^3\vec{r}}{ds^3}\right)}{[x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2}$$

which can also be expressed as the determinant

$$\tau = \frac{1}{[x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2}\begin{vmatrix} x'(s) & y'(s) & z'(s) \\ x''(s) & y''(s) & z''(s) \\ x'''(s) & y'''(s) & z'''(s) \end{vmatrix}$$

■

Example 7-6. **Velocity and Acceleration.**

A physical example illustrating the use of the unit tangent and normal vectors is found in determining the normal and tangential components of the velocity and acceleration vectors as a particle moves along a curve. If $\vec{r}$ denotes the position vector of the particle, $\vec{v}$ its velocity, and $\vec{a}$ its acceleration, then

$$\vec{v} = \frac{d\vec{r}}{dt}, \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}, \qquad v^2 = \vec{v}\cdot\vec{v} = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \left(\frac{ds}{dt}\right)^2, \tag{7.24}$$

where s is the arc length along the curve. Using chain rule differentiation gives

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{\mathbf{e}}_t v.$$

Analysis of this equation demonstrates that the velocity vector $\vec{v}$ is directed along the tangent vector at any time t and has the magnitude given by $v = \frac{ds}{dt}$ which represents the speed of the particle.

The derivative of the velocity vector with respect to time t is the acceleration and
$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{\mathbf{e}}_t \frac{dv}{dt} + \frac{d\,\hat{\mathbf{e}}_t}{dt} v.$$
From the Frenet-Serret formula and using chain rule differentiation, it can be shown that the time rate of change of the unit tangent vector is
$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \frac{v}{\rho}\,\hat{\mathbf{e}}_n.$$
Substituting this result into the acceleration vector gives
$$\vec{a} = \frac{dv}{dt}\,\hat{\mathbf{e}}_t + \frac{v^2}{\rho}\,\hat{\mathbf{e}}_n.$$
The resulting acceleration vector lies in the osculating plane. The tangential component of the acceleration is given by $\dfrac{dv}{dt}$, and the normal component of the acceleration is given by $\dfrac{v^2}{\rho}$. ■

Surfaces

A surface can be defined

(*i*) Explicitly $\quad z = f(x,y)$

(*ii*) Implicitly $\quad F(x,y,z) = 0$

(*iii*) Parametrically $\quad x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$

(*iv*) As a vector $\quad \vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3$

or $\quad \vec{r} = \vec{r}(x,y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + f(x,y)\,\hat{\mathbf{e}}_3$

(*v*) By rotating a curve about a line.

Here again it should be noted that **the parametric representation of a surface is not unique.**

If the functions used to define the above surfaces are continuous and differentiable functions and are such that the functions defining the surface and their partial derivatives are all well defined at points on the surface, then the surfaces are called **smooth surfaces.** If the surface is defined implicitly by an equation of the form $F(x,y,z) = 0$, then those points on the surface where at least one of the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ is different from zero are called **regular points on the surface.** If all of these partial derivatives are zero at a point on the surface, then that point is called **a singular point of the surface.**

To represent **a curve on a given surface** defined in terms of two parameters u and v, one can specify how these parameters change. For example, if

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3 \tag{7.25}$$

defines a given surface, then

(i) One can specify that the parameters u and v change as as a function of time t and write $u = u(t)$ and $v = v(t)$, then the position vector $\vec{r} = \vec{r}(u,v)$ becomes a function of a single variable t

$$\vec{r} = \vec{r}(t) = x(u(t),v(t))\,\hat{\mathbf{e}}_1 + y(u(t),v(t))\,\hat{\mathbf{e}}_2 + z(u(t),v(t))\,\hat{\mathbf{e}}_3, \qquad a \le t \le b$$

which sweeps out a curve lying on the surface.

(ii) If one specifies that v is a function of u, say $v = f(u)$, this reduces the vector $\vec{r}(u,v)$ to a function of a single variable which defines the curve on the surface. This surface curve is given by

$$\vec{r} = \vec{r}(u) = x(u,f(u))\,\hat{\mathbf{e}}_1 + y(u,f(u))\,\hat{\mathbf{e}}_2 + z(u,f(u))\,\hat{\mathbf{e}}_3$$

(iii) An equation of the form $g(u,v) = 0$ implicitly defines u as a function of v or v as a function of u and can be used to define a curve on the surface. The equation $g(u,v) = 0$, together with the equation (7.25), is said to define the surface curve implicitly.

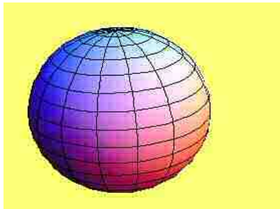
(iv) Consider the special curves

$$\begin{aligned} \vec{r} =& \vec{r}(u,v_0) \qquad v_0 \text{ constant} \\ \vec{r} =& \vec{r}(u_0,v) \qquad u_0 \text{ constant} \end{aligned}$$

sketched on the surface for the values

$$\begin{aligned} u_0 &\in \{\ \alpha, \alpha+h, \alpha+2h, \alpha+3h, \dots\ \} \\ v_0 &\in \{\ \beta, \beta+k, \beta+2k, \beta+3k, \dots\ \} \end{aligned}$$

where α, β, h and k have fixed constant values. These special curves are called **coordinate curves on the surface.** The partial derivatives $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ evaluated at a common point (u_0,v_0) are **tangent vectors to the coordinate curves** and the cross product $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ produces **a normal to the surface.**



Coordinate curves on sphere.

For example, consider the unit sphere

$$\vec{r}(u,v) = \cos u \sin v\, \hat{\mathbf{e}}_1 + \sin u \sin v\, \hat{\mathbf{e}}_2 + \cos v\, \hat{\mathbf{e}}_3$$

where $0 \le u \le 2\pi$ and $0 \le v \le \pi$. The curves $\vec{r}(u_0, v)$ for equi-spaced constants u_0 gives the coordinate curves called lines of longitude on the sphere. The curves $\vec{r}(u, v_0)$ for equi-spaced constants v_0 give the curves called lines of latitude on the sphere.

A surface is called **an oriented surface** if

(i) each nonboundary point on the surface has two unit normals $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$. By selecting one of these unit normals one is said to give an **orientation to the surface.** Thus, an oriented surface will always have two orientations.

(ii) The unit normal selected defines a surface orientation and this unit normal must vary continuously over the surface.

(iii) Each nonboundary point on the oriented surface has a tangent plane.

(iv) If the surface is that of a solid, then the unit normal at each point on the surface which is directed outward from the surface is usually selected as the **preferred orientation for the closed surface.**

A surface S is said to be a **simple closed surface** if the surface divides all of three dimensional space into three regions defined by

(i) points interior to S, where the distance between any two points inside S is finite.

(ii) points on the surface S.

(iii) points exterior to the surface S.

A **smooth surface** is one where a **normal vector can be constructed at each point of the surface**.

The sphere

The general equation of a sphere is

$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0$$

where α, β, γ and δ are constants. This is a simple closed surface with outward normal defining its orientation. It is customary to complete the square on the x, y and z terms and express this equation in the form

$$\left(x + \frac{\alpha}{2}\right)^2 + \left(y + \frac{\beta}{2}\right)^2 + \left(z + \frac{\gamma}{2}\right)^2 = \frac{\alpha^2 + \beta^2 + \gamma^2}{4} - \delta \tag{7.26}$$

After completing the square on the x, y and z terms, the following cases can arise.

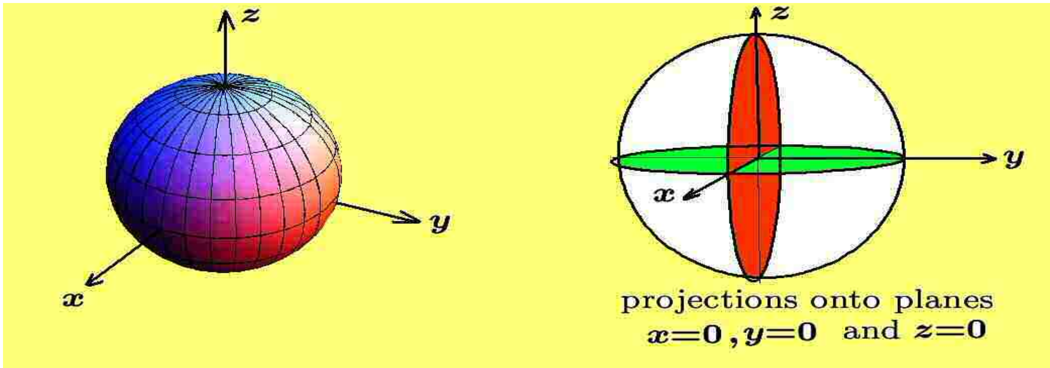


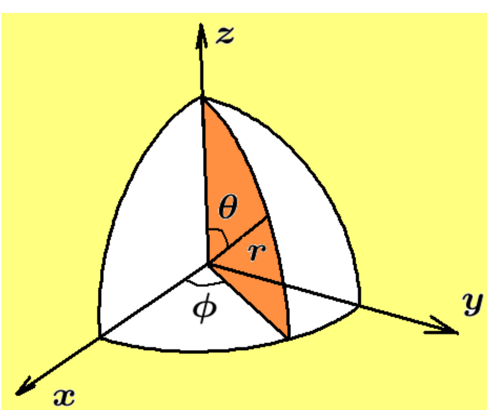
Figure 7-4.
Sphere centered at origin and projections onto planes $x = 0$, $y = 0$ and $z = 0$.

$$\text{If} \quad \frac{\alpha^2+\beta^2+\gamma^2}{4} - \delta = \begin{cases} r^2 > 0, & \text{then } r \text{ is radius of sphere centered at } \left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right) \\ 0, & \text{then } 0 \text{ is radius of sphere centered at } \left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right) \\ -r^2 < 0, & \text{then no real sphere exists} \end{cases}$$

In the case the right-hand side of equation (7.26) is negative, then a virtual sphere is said to exist. A sphere centered at the point (x_0, y_0, z_0) with radius $r > 0$ has the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{7.27}$$

The figure 7-4 illustrates a sphere and projections of the sphere onto the $x = 0$, $y = 0$ and $z = 0$ planes.



A sphere with constant radius $r > 0$ and centered at the origin can also be represented in the parametric form

$$\begin{aligned} x &= x(\phi, \theta) = r \sin\theta \cos\phi, \\ y &= y(\phi, \theta) = r \sin\theta \sin\phi, \\ z &= z(\phi, \theta) = r \cos\theta \end{aligned} \tag{7.28}$$

where $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. These parameters are illustrated in the accompanying figure. Note that when θ is held constant, one obtains a coordinate curve representing a line of latitude on the sphere given by $\lambda = \frac{\pi}{2} - \theta$ and when ϕ is held constant, one obtains a coordinate curve representing some line of longitude on the sphere.

The representation $\vec{r} = \vec{r}(\phi, \theta) = r \sin\theta \cos\phi \,\hat{\mathbf{e}}_1 + r \sin\theta \sin\phi \,\hat{\mathbf{e}}_2 + r \cos\theta \,\hat{\mathbf{e}}_3$ is a vector representation for points on the sphere of radius r with $\vec{r}(\phi_0, \theta)$ a curve of longitude and $\vec{r}(\phi, \theta_0)$ a line of latitude and these curves are called **coordinate curves** on the surface of the sphere. The vectors $\frac{\partial \vec{r}}{\partial \phi}$ and $\frac{\partial \vec{r}}{\partial \theta}$ are tangent vectors to the coordinate curves. The cross product $\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}$ produces **a normal vector to the surface of the sphere**.

The Ellipsoid

The ellipsoid centered at the point (x_0, y_0, z_0) is represented by the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1 \tag{7.29}$$

and if

$a = b > c$ it is called **an oblate spheroid**.

$a = b < c$ it is called **a prolate spheroid**.

$a = b = c$ it is called **a sphere of radius** a.

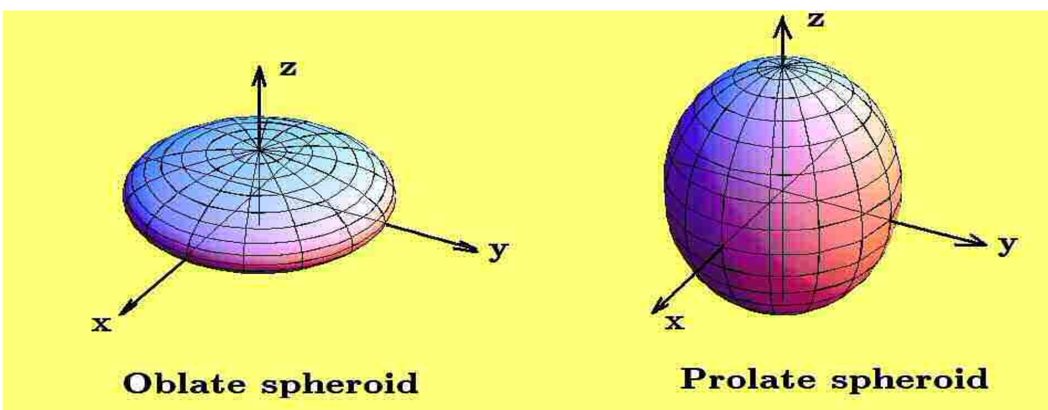


Figure 7-5. Oblate and prolate spheroids.

The ellipsoid can also be represented by the parametric equations

$$x - x_0 = a \cos\theta \cos\phi, \qquad y - y_0 = b \cos\theta \sin\phi, \qquad z - z_0 = c \sin\theta \tag{7.30}$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $-\pi \le \phi \le \pi$. The figure 7-5 illustrates the oblate and prolate spheroids centered at the origin.

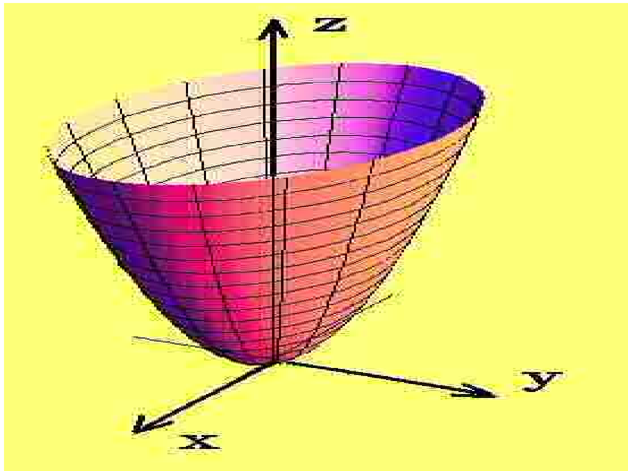


Figure 7-6. Elliptic paraboloid

The Elliptic Paraboloid

The elliptic paraboloid centered at the point (x_0, y_0, z_0) is described by the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = \frac{z-z_0}{c} \tag{7.31}$$

It can also be represented by the parametric equations

$$x - x_0 = a\sqrt{u}\cos v, \qquad y - y_0 = b\sqrt{u}\sin v, \qquad z - z_0 = cu \tag{7.32}$$

where $0 \le v \le 2\pi$ and $0 \le u \le h$. The elliptic paraboloid centered at the origin is illustrated in the figure 7-6.

The Elliptic Cone

The elliptic cone centered at the point (x_0, y_0, z_0) is represented by an equation having the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = \frac{(z-z_0)^2}{c^2} \tag{7.33}$$

A parametric representation for the elliptic cone is given by

$$x - x_0 = au\cos v, \qquad y - y_0 = bu\sin v, \qquad z - z_0 = cu$$

for $0 \le v \le 2\pi$ and $-h \le u \le h$. The elliptic cone centered at the origin is illustrated in the figure 7-7.

The position vector $\vec{r} = \vec{r}(u,v) = au\cos v\,\hat{\mathbf{e}}_1 + bu\sin v\,\hat{\mathbf{e}}_2 + cu\,\hat{\mathbf{e}}_3$ describes a point on the surface centered at the origin and the curves $\vec{r}(u_0,v)$, $\vec{r}(u,v_0)$ define the coordinate curves. The partial derivatives of $\vec{r}$ with respect to u and v are tangent vectors to the coordinate curves and these vectors can be used to construct a normal vector to the surface.

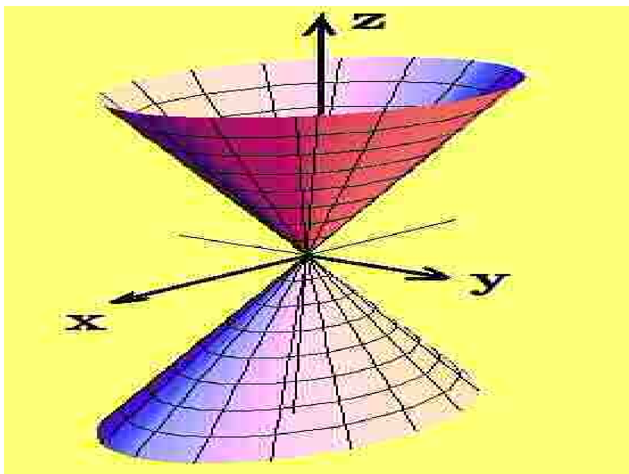


Figure 7-7. Elliptic cone

The Hyperboloid of One Sheet

The hyperboloid of one sheet centered at the point (x_0,y_0,z_0) and symmetric about the $z-$axis is given by the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = 1 \tag{7.34}$$

It can also be represented using the parametric equations

$$x - x_0 = a\cos u\cosh v, \qquad y - y_0 = b\sin u\cosh v, \qquad z - z_0 = c\sinh v \tag{7.35}$$

where $0 \le u \le 2\pi$ and $-h < v < h$. Here h is usually selected as a small number, say $h = 1$ as the selection of h as a large number gives a scaling difference between the parameters and distorts the final image.

The Hyperboloid of Two Sheets

The hyperboloid of two sheets centered at the point (x_0,y_0,z_0) and symmetric about the $z-$axis is describe by the equation

$$-\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1 \tag{7.36}$$

It can also be represented by the parametric equations

$$x - x_0 = a\cos v \sinh u, \qquad y - y_0 = b\sin v \sinh u, \qquad z - z_0 = c\cosh u \tag{7.37}$$

where $0 \le v \le 2\pi$ and $0 \le u \le h$, with both $c > 0$ and $c < 0$ producing a surface with two parts. Here again the selection of h should be of the same magnitude or less than v or else the final image gets distorted. The hyperboloid of one sheet and some hyperboloids of two sheets are illustrated in the figure 7-8. Note in this figure that the axes x, y and z have undergone various permutations. These permutations show that the axis of symmetry for the hyperboloid of two sheets is always associated with the term which has the positive sign. In a similar fashion one can do a permutation of the symbols x, y and z in the equation describing the hyperboloid of one sheet to obtain different axes of symmetry.

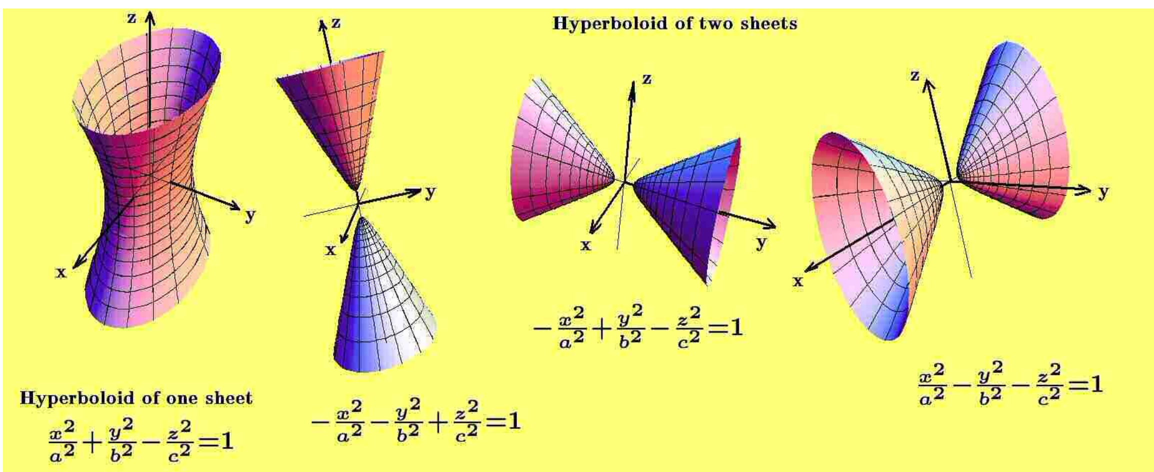


Figure 7-8. Hyperboloid of one sheet and several hyperboloids of two sheets.

In a similar fashion one can perform a permutation of the symbols x, y and z to give alternative representations of any of the surfaces previously defined.

One can use the parametric equations to define a position vector $\vec{r} = \vec{r}(u, v)$ from which the coordinate curves $\vec{r}(u_0, v)$ and $\vec{r}(u, v_0)$ can be constructed. The partial derivatives of $\vec{r}(u, v)$ with respect to u and v produce tangent vectors to the coordinate curves and these tangent vectors can be used to construct a normal vector to each point on the surface.

The Hyperbolic Paraboloid

The hyperbolic paraboloid centered at the the point (x_0, y_0, z_0) is described by the equation

$$\frac{z - z_0}{c} = -\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}, \tag{7.38}$$

This surface is saddle shaped and can also be described using the parametric equations

$$x - x_0 = u, \qquad y - y_0 = v, \qquad z - z_0 = c\left(-\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) \tag{7.39}$$

where $-h \le u \le h$ and $-k \le v \le k$ for selected constants h and k. These parametric equations can be used to construct the two-parameter surface $\vec{r}(u, v)$ from which the coordinate curves and normal vector can be constructed.

It is left as an exercise to show that under a rotation of axes and scaling using the equations

$$\frac{x - x_0}{a} = \bar{x}\cos\theta - \bar{y}\sin\theta, \qquad \frac{y - y_0}{b} = \bar{x}\sin\theta + \bar{y}\cos\theta, \qquad \frac{z - z_0}{c} = \bar{z}$$

with $\theta = \pi/4$, the hyperbolic paraboloid can be represented $\bar{z} = \bar{x}\bar{y}$.

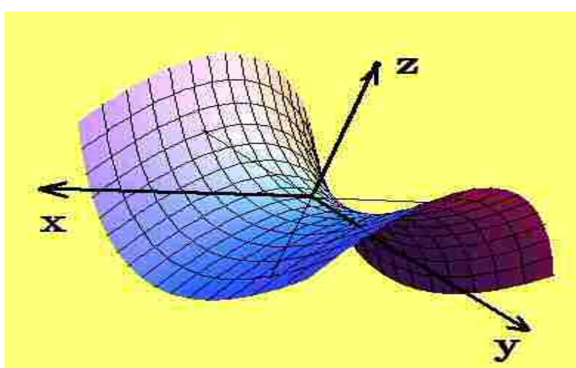


Figure 7-9. Hyperbolic paraboloid.

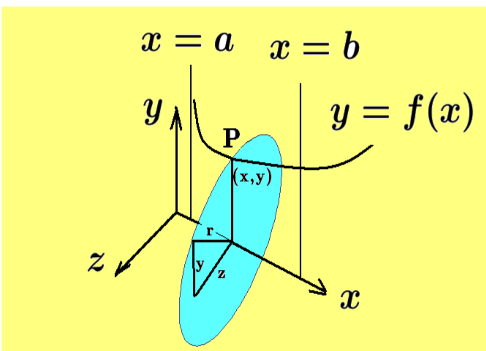
Surfaces of Revolution

Any surface which can be created by rotating a curve about a fixed line is called a surface of revolution. The fixed line about which the curve is rotated is called the axis of revolution. Some examples of surfaces of revolution are the sphere which is created by rotating the semi-circle $x^2 + y^2 = r^2$, $-r \le x \le r$ and $y \ge 0$ about the $y = 0$

axis. A paraboloid is obtained by rotating the parabola $y = x^2$, $0 \le x \le x_0$ about the $x = 0$ axis.

The general procedure for determining the equation for representing a surface of revolution is as follows. First select a general point P on the given curve and then rotate the point P about the axis of revolution to form a circle. This usually involves some parameter used to describe the general point. One can then determine the equation of the surface by eliminating the parameter from the resulting equations.

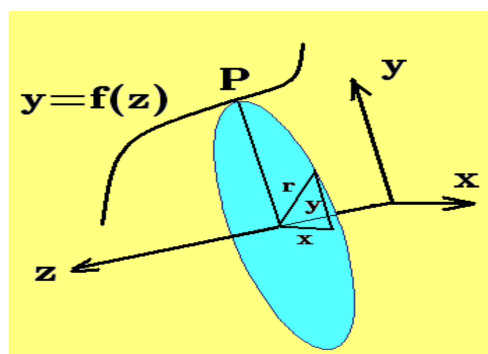
Example 7-7. A curve $y = f(x)$ for $a \le x \le b$ is rotated about the x–axis. Find the equation describing the surface of revolution.



Solution A general point P on the given curve, when rotated about the x–axis produces the circle $y^2 + z^2 = r^2$, where $r = f(x)$ is the radius of the circle. Eliminating the parameter r gives the equation of the surface of revolution as $y^2 + z^2 = [f(x)]^2$ ■

Example 7-8. The curve $y = f(z)$ for $a \le z \le b$ is rotated about the z–axis. Find the equation describing the surface of revolution.

Solution A general point P on the given curve is rotated about the z–axis to form the circle $x^2 + y^2 = r^2$ where $r = f(z)$ is the radius of the circle. Eliminating r between these two equations gives the equation for the surface of revolution as $x^2 + y^2 = [f(z)]^2$



■

Example 7-9. A curve described by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $t_0 \le t \le t_1$, is rotated about the line

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}$$

where $\vec{b} = b_1\,\hat{\mathbf{e}}_1 + b_2\,\hat{\mathbf{e}}_2 + b_3\,\hat{\mathbf{e}}_3$ is the direction vector of the line. Find the equation of the surface of revolution.

Solution In the figure 7-10 the point P represents a general point on the space curve. Let the coordinates of this point be denoted by $(x(t^*), y(t^*), z(t^*))$ where $t_0 \le t^* \le t_1$ and t^* is held constant. Construct the vector $\vec{r}_P$ from the origin to the point P and construct the position vector $\vec{r}_0$ from the origin to the fixed point (x_0, y_0, z_0) on the axis of rotation. A **unit vector** in the direction of the axis or rotation is described by

$$\hat{\mathbf{e}}_B = \frac{1}{|\vec{b}|}\vec{b} = \frac{b_1\,\hat{\mathbf{e}}_1 + b_2\,\hat{\mathbf{e}}_2 + b_3\,\hat{\mathbf{e}}_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3 \tag{7.40}$$

where $\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_B = B_1^2 + B_2^2 + B_3^2 = 1$. Also construct the vector $\vec{r}_P - \vec{r}_0$ from the point (x_0, y_0, z_0) to the point P as illustrated in the figure 7-10.

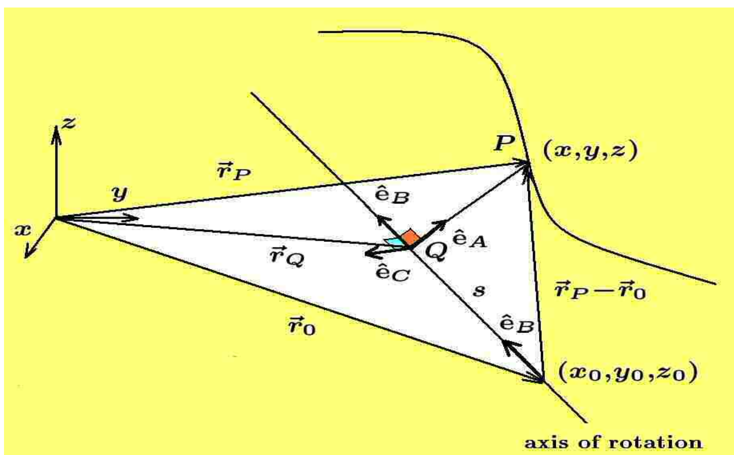


Figure 7-10. Space curve revolved about line to form surface of revolution.

Consider a line perpendicular to the axis of rotation and passing through the point P. Denote by Q the point where this line intersects the axis of rotation. The distance s from the point (x_0, y_0, z_0) to the point Q is given by the projection of the vector $\vec{r}_P - \vec{r}_0$ onto the unit vector $\hat{\mathbf{e}}_B$. This projection gives the distance

$$s = \hat{\mathbf{e}}_B \cdot (\vec{r}_P - \vec{r}_0) \tag{7.41}$$

The point Q can be described by the position vector

$$\vec{r}_Q = \vec{r}_0 + s\,\hat{\mathbf{e}}_B \tag{7.42}$$

The distance from P to Q represents the radius of the circle of revolution when the point P is revolved about the axis of rotation. This distance, call it R, is given by the magnitude of the vector $\vec{r}_P - \vec{r}_Q$ and one can determine this distance from the dot product relation

$$R^2 = (\vec{r}_P - \vec{r}_Q) \cdot (\vec{r}_P - \vec{r}_Q) \tag{7.43}$$

Construct the unit vector $\hat{\mathbf{e}}_A$ pointing from the point Q to the point P by expanding the equation

$$\hat{\mathbf{e}}_A = \frac{\vec{r}_P - \vec{r}_Q}{|\vec{r}_P - \vec{r}_Q|} = \frac{\vec{r}_P - \vec{r}_Q}{R} \tag{7.44}$$

The unit vector $\hat{\mathbf{e}}_C$ which is perpendicular the unit vectors $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_B$ can be constructed using the cross product

$$\hat{\mathbf{e}}_C = \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B \tag{7.45}$$

Note that when the point P is revolved about the axis of rotation, the circle generated lies in the plane of the vectors $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_C$ and a point on this circle can be described using the equation

$$\vec{r} = \vec{r}_Q + R\cos\theta\,\hat{\mathbf{e}}_A + R\sin\theta\,\hat{\mathbf{e}}_C, \qquad 0 \le \theta \le 2\pi \tag{7.46}$$

Recall that the point P represents a general point on the space curve and so the vector in equation (7.46) is really a function of the two variables t and θ and one can express the equation (7.46) as the two parameter surface described by

$$\vec{r} = \vec{r}(t,\theta) = \vec{r}_Q + R\cos\theta\,\hat{\mathbf{e}}_A + R\sin\theta\,\hat{\mathbf{e}}_C, \qquad 0 \le \theta \le 2\pi \tag{7.47}$$

where the vectors $\vec{r}_Q$, $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_C$ are all calculated in terms of the parameter t and can be constructed using the equations (7.40), (7.41), (7.42), (7.43), (7.44), (7.45). That is, to construct the surface of revolution, one can construct a circle for each value of the space parameter t varying between two fixed values, say $t_0 \le t \le t_1$.

An alternative method for constructing the circles of revolution of each point P on the space curve for $t_0 \le t \le t_1$ is as follows. First, assume P is fixed and construct the sphere centered at the point (x_0, y_0, z_0) which passes through the point P. This sphere has a radius given by $r = |\vec{r}_P - \vec{r}_0|$ and this radius is a function of the parameter t^* used to describe the point P. The equation of the sphere centered at (x_0, y_0, z_0) and passing through the point P is given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{7.48}$$

Next construct the plane which passes through the point P and is perpendicular to the axis of rotation. The equation of this plane is given by

$$(\vec{r} - \vec{r}_P) \cdot \hat{\mathbf{e}}_B = 0 \qquad \text{or} \qquad (x - x(t^*))B_1 + (y - y(t^*))B_2 + (z - z(t^*))B_3 = 0 \tag{7.49}$$

This plane is the plane of rotation of the point P and it intersects the sphere in the circle described by the point P as it moves around the axis of rotation. To obtain the equation for the surface of revolution one must eliminated the parameter t^* from the equations (7.48) and (7.49). This elimination is not always an easy task to perform.

■

Ruled Surfaces

A surface $\vec{r} = \vec{r}(u,v)$ or $z = f(x,y)$ or $F(x,y,z) = 0$ is called a ruled surface if it has the following property. *Through each point on the surface it is possible to draw a straight line which lies entirely on the surface.* For example, consider the set of all straight lines which pass through a fixed point V and which intersect a fixed curve C, which is not a straight line through V. The surface generated is called a general cone with the point V called the vertex of the cone, the curve C being called the directrix of the cone and the lines on the surface of the cone are called the generating lines. Some example cones are illustrated in the figure 7-11.

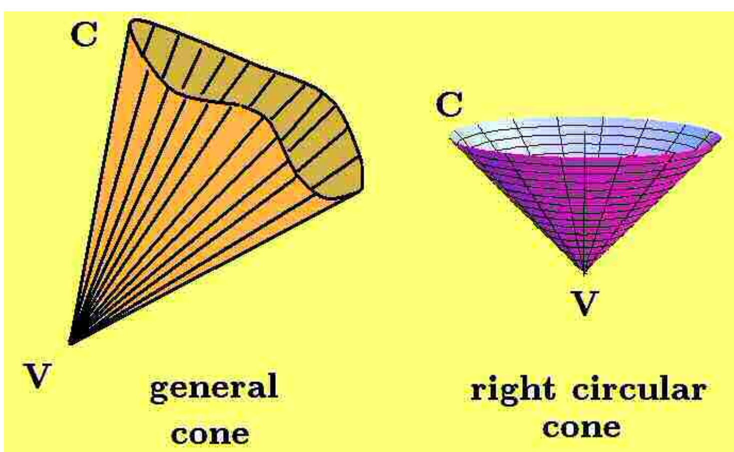


Figure 7-11. A cone is an example of a ruled surface.

Another example of a ruled surface are general cylindrical surfaces which can be described as a collection of straight lines all parallel to a given direction.

In general, a ruled surface can be thought of as set of points created by moving a straight line. One way of creating the equation of a ruled surface is to consider two curves where both curves are defined in terms of a parameter t and represented by the position vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$ as illustrated in the figure 7-12.

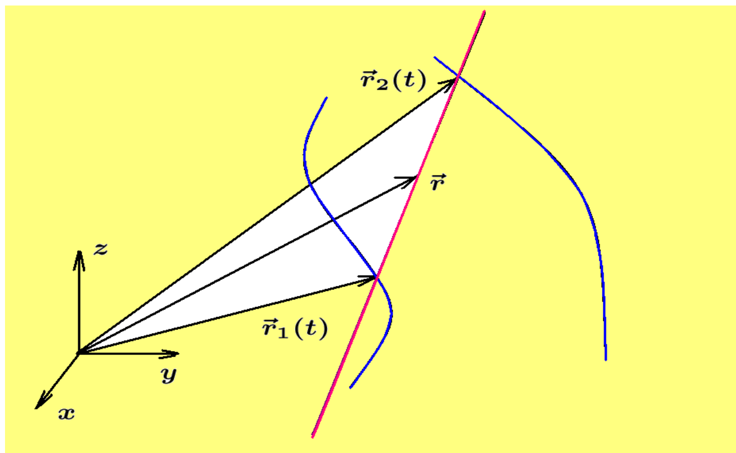


Figure 7-12. Generating a ruled surface using two curves.

For a fixed value of the parameter t, one can draw a straight line between the two points $\vec{r}_1(t)$ and $\vec{r}_2(t)$ as illustrated in the figure 7-12. If $\vec{r}$ is the position vector to a general point on this line it can be represented by the equations

$$\vec{r} = \vec{r}(t,u) = (1-u)\vec{r}_1(t) + u\vec{r}_2(t) \qquad -u_0 \le u \le u_0 \tag{7.50}$$

where u is a parameter and u_0 is some specified constant . Note that when $u = 0$, then $\vec{r} = \vec{r}_1$ and when $u = 1$, $\vec{r} = \vec{r}_2$. As the parameter t changes the line sweeps out a surface.

Ruled surfaces can be observed on cylinders, cones, hyperboloids of one sheet, as well as elliptic and hyperbolic paraboloids. Ruled surfaces have been studied since the time of the early Greeks and many architectural structures can be described as ruled surfaces.

Surface Area

The position vector

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3 \qquad \alpha \le u \le \beta, \quad \gamma \le v \le \delta \tag{7.51}$$

defines a surface in terms of two parameters u and v. The family of curves $\vec{r}(u,v_0)$, with v_0 taking on selected constant values, defines **a set of coordinate curves on the surface**. Similarly, the family of curves $\vec{r}(u_0,v)$, with u_0 taking on selected constant values, defines another set of coordinate curves. The vector $\frac{\partial \vec{r}}{\partial u}$ is **a tangent vector to the coordinate curve** $\vec{r}(u,v_0)$ and the vector $\frac{\partial \vec{r}}{\partial v}$ is **a tangent vector to the coordinate curve** $\vec{r}(u_0,v)$. If at every common point of intersection of the coordinate curves $\vec{r}(u_0,v)$ and $\vec{r}(u,v_0)$ one finds that $\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\Big|_{(u_0,v_0)} = 0$, then the coordinate curves are said to form **an orthogonal net on the surface**.

The vector

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv \tag{7.52}$$

lies in the tangent plane to the point $\vec{r}(u,v)$ on the surface and one can say that the vector element $d\vec{r}$ **defines a parallelogram with vector sides** $\frac{\partial \vec{r}}{\partial u}\,du$ **and** $\frac{\partial \vec{r}}{\partial v}\,dv$ as illustrated in the figure 7-13.

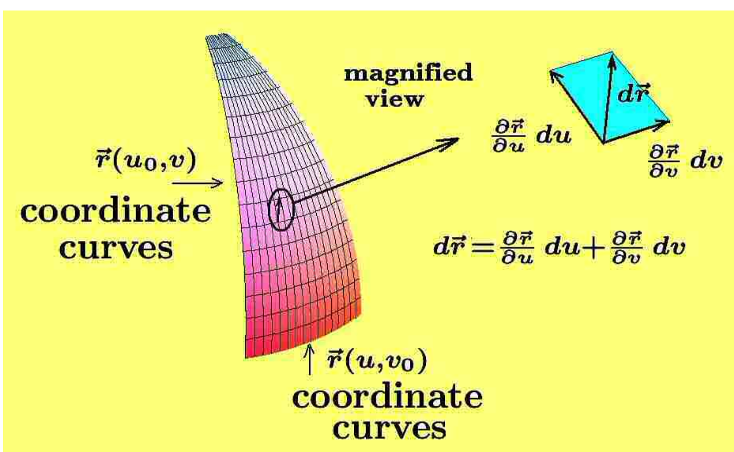


Figure 7-13. Defining an element of area on a surface.

Define **the element of surface area** dS on a given surface as the area of the elemental parallelogram formed using the vector components of $d\vec{r}$. Recall that the magnitude of the cross product of the sides of a parallelogram gives the area of the parallelogram and consequently one can express the element of surface area as

$$dS = \left| \frac{\partial \vec{r}}{\partial u}\, du \times \frac{\partial \vec{r}}{\partial v}\, dv \right| = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \, du dv \tag{7.53}$$

Using the vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

with $\vec{A} = \vec{C} = \frac{\partial \vec{r}}{\partial u}$ and $\vec{B} = \vec{D} = \frac{\partial \vec{r}}{\partial v}$ one finds

$$\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) = \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}\right)\left(\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\right) - \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\right) \cdot \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\right)$$

Define the quantities

$$\begin{aligned}
E =& \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\
F =& \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\
G =& \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2
\end{aligned} \tag{7.54}$$

then one can write

$$dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \, du dv = \sqrt{EG - F^2}\, du dv \tag{7.55}$$

Alternatively one can write

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} =& \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \\
=& \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\hat{\mathbf{e}}_1 - \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\hat{\mathbf{e}}_3
\end{aligned}$$

and the magnitude of this cross product is given by

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(-\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2} \tag{7.56}$$

Expanding the equation (7.56) one finds that the element of surface can be represented by the equation (7.55). To find the area of the surface one need only evaluate the double integral

$$\text{Surface Area} = \int_\alpha^\beta \int_\gamma^\delta \sqrt{EG - F^2}\, dv\, du \tag{7.57}$$

which represents a summation of the elements of surface area over the surface between appropriate limits assigned to the parameters u and v.

Note that the vectors $\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ and $-\vec{N} = \frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}$ are both normal vectors to the surface $\vec{r} = \vec{r}(u, v)$ and

$$\hat{\mathbf{e}}_n = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\sqrt{EG - F^2}}$$

are unit normals to the surface.

In the special case the surface is defined by $\vec{r} = \vec{r}(x, y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x, y)\,\hat{\mathbf{e}}_3$ one can show the element of surface area is given by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \frac{dx\,dy}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

Here the surface element dS is projected onto the xy-plane to determine the limits of integration.

In the special case the surface is defined by $\vec{r} = \vec{r}(x, z) = x\,\hat{\mathbf{e}}_1 + y(x, z)\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ one can show the element of surface area is given by

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1|} = \frac{dx\,dz}{\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}}$$

Here the surface element dS is projected onto the xz-plane to determine the limits of integration.

In the special case the surface is defined by $\vec{r} = \vec{r}(y, z) = x(y, z)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is found to be given by

$$dS = \frac{dy\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|} = \frac{dy\,dz}{\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}}$$

In this case the surface element dS is projected onto the yz-plane to determine the limits of integration.

Arc Length

Consider a curve $u = u(t)$, $v = v(t)$ on a surface $\vec{r} = \vec{r}(u, v)$ for $t_0 \le t \le t_1$. The **element of arc length** ds associated with this curve can be determined from the vector element

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv$$

using

$$\begin{aligned} ds^2 =& d\vec{r} \cdot d\vec{r} = \left(\frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv\right) \cdot \left(\frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv\right) \\ ds^2 =& E\, du^2 + 2F\, du\, dv + G\, du^2 \end{aligned} \tag{7.58}$$

where E, F, G are given by the equations (7.54). The length of the curve is then given by the integral

$$s = \text{arc length} = \int_{t_0}^{t_1} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt \tag{7.59}$$

where the limits of integration t_0 and t_1 correspond to the endpoints associated with the curve as determined by the parameter t.

The Gradient, Divergence and Curl

The **gradient** is a field characteristic that describes **the spatial rate of change of a scalar field**. Let $\phi = \phi(x, y, z)$ represent a scalar field, then the gradient of ϕ is a vector and is written

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial x}\, \hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\, \hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\, \hat{\mathbf{e}}_3. \tag{7.60}$$

Here it is assumed that the scalar field $\phi = \phi(x, y, z)$ possesses first partial derivatives throughout some region R of space in order that the gradient vector exists. The operator

$$\nabla = \frac{\partial}{\partial x}\, \hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\, \hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\, \hat{\mathbf{e}}_3 \tag{7.61}$$

is called the "del "operator or nabla operator and can be used to express the gradient in the operator form

$$\text{grad}\ \phi = \nabla \phi = \left(\frac{\partial}{\partial x}\, \hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\, \hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\, \hat{\mathbf{e}}_3\right) \phi. \tag{7.62}$$

Note that the operator is not commutative and $\nabla \phi \neq \phi \nabla$.

If $\vec{v} = \vec{v}(x, y, z) = v_1(x, y, z)\, \hat{\mathbf{e}}_1 + v_2(x, y, z)\, \hat{\mathbf{e}}_2 + v_3(x, y, z)\, \hat{\mathbf{e}}_3$ denotes a vector field with components which are well defined, continuous and everywhere differential, then **the divergence of** $\vec{v}$ is defined

$$\text{div}\, \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{7.63}$$

Using the del operator ∇ the divergence can be represented

$$
\begin{aligned}
\operatorname{div}\vec{v} = \nabla\cdot\vec{v} =& (\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3)\cdot(v_1\,\hat{\mathbf{e}}_1 + v_2\,\hat{\mathbf{e}}_2 + v_3\,\hat{\mathbf{e}}_3)\\
\operatorname{div}\vec{v} = \nabla\cdot\vec{v} =& \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}
\end{aligned}
\tag{7.64}
$$

Again make note of the fact that the del operator is not commutative and $\nabla\cdot\vec{v} \neq \vec{v}\cdot\nabla$. **If the divergence of a vector field is zero, $\nabla\cdot\vec{v} = 0$, then the vector field is called solenoidal.**

If $\vec{v} = \vec{v}(x,y,z) = v_1(x,y,z)\,\hat{\mathbf{e}}_1 + v_2(x,y,z)\,\hat{\mathbf{e}}_2 + v_3(x,y,z)\,\hat{\mathbf{e}}_3$ denotes a vector field with components which are well defined, continuous and everywhere differential, then **the curl or rotation of $\vec{v}$** is written[4] $\operatorname{curl}\vec{v} = \nabla\times\vec{v} = \operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$ which can be expressed in the expanded determinant form[5]

$$
\begin{aligned}
\operatorname{curl}\vec{v} = \nabla\times\vec{v} =& \operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}\\
=& \,\hat{\mathbf{e}}_1 \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix}\\
\operatorname{curl}\vec{v} = \nabla\times\vec{v} =& \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\,\hat{\mathbf{e}}_1 - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\,\hat{\mathbf{e}}_2 + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\,\hat{\mathbf{e}}_3
\end{aligned}
\tag{7.65}
$$

If the curl of a vector field is zero, curl $\vec{v} = \nabla\times\vec{v} = \vec{0}$, then the vector field is said to be irrotational.

Example 7-10. Find the gradient of the scalar field $\phi = x^2y + zxy^2$ at the point $(1,1,2)$.

Solution Using the above definition show that

$$
\operatorname{grad}\,\phi = \nabla\phi = (2xy + zy^2)\,\hat{\mathbf{e}}_1 + (x^2 + 2xyz)\,\hat{\mathbf{e}}_2 + xy^2\,\hat{\mathbf{e}}_3
$$

and $\quad \operatorname{grad}\,\phi\Big|_{(1,1,2)} = 4\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$

■

[4] The curl of $\vec{v}$ is sometimes referred to as the rotation of $\vec{v}$ and written $\operatorname{rot}\vec{v}$.

[5] See chapter 10 for properties of determinants.

Example 7-11. Find the divergence of the vector field given by
$$\vec{v} = xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3$$

Solution By definition

$$\begin{aligned}\operatorname{div}\vec{v} = \nabla\cdot\vec{v} &= \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot\left(xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3\right)\\ \operatorname{div}\vec{v} = \nabla\cdot\vec{v} &= yz + z^2 + xy^2\end{aligned}$$

■

Example 7-12. Find the curl of the vector field $\vec{v} = xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3$

Solution By definition

$$\operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ xyz & yz^2 & zxy^2\end{vmatrix} = \hat{\mathbf{e}}_1\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz^2 & zxy^2\end{vmatrix} - \hat{\mathbf{e}}_2\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ xyz & zxy^2\end{vmatrix} + \hat{\mathbf{e}}_3\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ xyz & yz^2\end{vmatrix}$$
$$\operatorname{curl}\vec{v} = \nabla\times\vec{v} = (2xyz - 2yz)\,\hat{\mathbf{e}}_1 - (zy^2 - xy)\,\hat{\mathbf{e}}_2 - xz\,\hat{\mathbf{e}}_3$$

■

Properties of the Gradient, Divergence and Curl

Let $u = u(x,y,z)$ and $v = v(x,y,z)$ denote scalar functions which are continuous and differentiable everywhere and let $\vec{A} = \vec{A}(x,y,z)$ and $\vec{B} = \vec{B}(x,y,z)$ denote vector functions which are continuous and differentiable everywhere. One can then verify that the del or nabla operator has the following properties.

(i) $\operatorname{grad}(u+v) = \operatorname{grad}u + \operatorname{grad}v$ or $\nabla(u+v) = \nabla u + \nabla v$

(ii) $\operatorname{grad}(uv) = u\operatorname{grad}v + v\operatorname{grad}u$ or $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(iii) $\operatorname{grad}f(u) = f'(u)\operatorname{grad}u$ or $\nabla(f(u)) = f'(u)\,\nabla u$

(iv) $|\operatorname{grad}u| = |\nabla u| = \sqrt{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2}$

(v) If a vector field is irrotational $\operatorname{curl}\vec{F} = \vec{0}$, then it is **derivable from a scalar function by taking the gradient**, then one can write $\vec{F} = \vec{F}(x,y,z) = \operatorname{grad}u(x,y,z)$, or $\vec{F} = \nabla u$. The vector field $\vec{F}$ is called **a conservative vector field**. The function u from which the vector field is derivable is called the **scalar potential.**

(vi) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$ or $\operatorname{div}(\vec{A}+\vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$

(vii) $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$ or $\operatorname{curl}(\vec{A}+\vec{B}) = \operatorname{curl}\vec{A} + \operatorname{curl}\vec{B}$

(viii) $\nabla(u\vec{A}) = (\nabla u)\cdot\vec{A} + u(\nabla\cdot\vec{A})$

(ix) $\nabla\times(u\vec{A}) = (\nabla u)\times\vec{A} + u(\nabla\times\vec{A})$

(x) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(xi) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - \vec{B}(\nabla\cdot\vec{A}) - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B})$

(xii) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$

(xiii) $\nabla\cdot(\nabla u) = \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

The operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the **Laplacian operator**.

(xiv) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

(xv) $\nabla\times(\nabla u) = \text{curl}\,(\text{grad}\,u) = \vec{0}$ **The curl of the gradient of u is the zero vector.**

(xvi) $\nabla\cdot(\nabla\times\vec{A}) = \text{div}\,(\text{curl}\,\vec{A}) = 0$ **The divergence of the curl of $\vec{A}$ is the scalar zero.**

(xvii) If a vector field $\vec{F}(x,y,z)$ is solenoidal, then it is **derivable from a vector function $\vec{A} = \vec{A}(x,y,z)$ by taking the curl**. One can then write $\vec{F} = \text{curl}\,\vec{A}$ and hence $\text{div}\,\vec{F} = 0$. The vector function $\vec{A}$ is called **the vector potential** from which $\vec{F}$ is derivable.

(xviii) If f is a function of $u_1, u_2, \ldots, u_n$ where $u_i = u_i(x,y,z)$ for $i = 1,2,\ldots,n$, then

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \cdots + \frac{\partial f}{\partial u_n}\nabla u_n$$

Many properties and physical interpretations associated with the operations of gradient, divergence and curl are given in the next chapter.

Example 7-13. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x,y,z). Show that

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{1}{r}\vec{r} = \hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the direction of $\vec{r}$.

Solution Let $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, then

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2x = \frac{x}{r}\\
\frac{\partial r}{\partial y} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2y = \frac{y}{r}\\
\frac{\partial r}{\partial z} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2z = \frac{z}{r}
\end{aligned}$$

Substituting for the partial derivatives in the gradient gives

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{1}{r}\vec{r} = \hat{\mathbf{e}}_r$$

■

Example 7-14. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x, y, z). Show that

$$\text{grad}\,(\frac{1}{r}) = \text{grad}\,\frac{1}{|\vec{r}|} = -\frac{1}{r^2}\text{grad}\,(r) = -\frac{1}{r^3}\vec{r} = -\frac{1}{r^2}\,\hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the direction of $\vec{r}$.

Solution Let $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ so that $\frac{1}{r} = (x^2+y^2+z^2)^{-1/2}$. By definition

$$\text{grad}\,(\frac{1}{r}) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = \frac{-x}{r^3} \\
\frac{\partial}{\partial y}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y) = \frac{-y}{r^3} \\
\frac{\partial}{\partial z}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = \frac{-z}{r^3}
\end{aligned}$$

Substituting for the partial derivatives in the gradient gives

$$\text{grad}\,(\frac{1}{r}) = \text{grad}\,\frac{1}{|\vec{r}|} = -\frac{\vec{r}}{r^3} = -\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right) == \frac{1}{r^2}\text{grad}\,r = -\frac{1}{r^2}\,\hat{\mathbf{e}}_r$$

■

Example 7-15. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x, y, z) and let $r = |\vec{r}|$. Find $\text{grad}\,(r^n)$.

Solution By definition

$$\text{grad}\,(r^n) = \nabla(r^n) = \frac{\partial r^n}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r^n}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r^n}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial r^n}{\partial x} &= nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = r^{n-2}x \\
\frac{\partial r^n}{\partial y} &= nr^{n-1}\frac{\partial r}{\partial y} = nr^{n-1}\frac{y}{r} = nr^{n-2}y \\
\frac{\partial r^n}{\partial z} &= r^{n-1}\frac{\partial r}{\partial z} = nr^{n-1}\frac{z}{r} = nr^{n-2}z
\end{aligned}$$

so that

$$\text{grad}\,(r^n) = nr^{n-2}\vec{r} = nr^{n-1}\,\hat{\mathbf{e}}_r$$

■

Example 7-16. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x,y,z) and let $r = |\vec{r}|$. Find $\text{grad}\, f(r)$ where $f = f(r)$ is any continuous differentiable function of r.

Solution By definition

$$\text{grad}\, f(r) = \frac{\partial f}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{df}{dr}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}\\
\frac{\partial f}{\partial y} &= \frac{df}{dr}\frac{\partial r}{\partial y} = f'(r)\frac{y}{r}\\
\frac{\partial f}{\partial z} &= \frac{df}{dr}\frac{\partial r}{\partial z} = f'(r)\frac{z}{r}
\end{aligned}$$

so that

$$\text{grad}\, f(r) = f'(r)\frac{1}{r}\vec{r} = f'(r)\,\hat{\mathbf{e}}_r$$

Compare this result with the result from the previous example.

■

Example 7-17. If $\phi = \phi(x,y,z)$ is continuous and possess derivatives which are also continuous, show that the curl of the gradient of ϕ produces the zero vector. That is, show

$$\text{curl}(\text{grad}\,\phi) = \nabla \times (\nabla\phi) = \vec{0}$$

Solution The function ϕ is differentiable so that

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

and the curl of this vector is represented

$$\begin{aligned}
\text{curl}(\text{grad}\,\phi) =& \nabla \times (\nabla\phi) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}\\
\text{curl}(\text{grad}\,\phi) =& \hat{\mathbf{e}}_1\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right) - \hat{\mathbf{e}}_2\left(\frac{\partial^2\phi}{\partial x\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial x}\right) + \hat{\mathbf{e}}_3\left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right) = \vec{0}
\end{aligned}$$

because the mixed partial derivatives inside the parenthesis are equal to one another.

■

Example 7-18. Show that $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Solution Calculate $\nabla \times \vec{A}$ using determinants to obtain

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{\mathbf{e}}_1 - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{\mathbf{e}}_3$$

One can then calculate the curl of the curl as

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \end{vmatrix} \tag{7.66}$$

The $\hat{\mathbf{e}}_1$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_1 \left[\frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \right] = \hat{\mathbf{e}}_1 \left[\frac{\partial^2 A_2}{\partial x \, \partial y} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial x \, \partial z} \right]$$

By adding and subtracting the term $\dfrac{\partial^2 A_1}{\partial x^2}$ to the above result one finds the $\hat{\mathbf{e}}_1$ component can be expressed in the form

$$\hat{\mathbf{e}}_1 \left\{ \left[-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right] + \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \right\} \tag{7.67}$$

In a similar fashion it can be verified that the $\hat{\mathbf{e}}_2$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_2 \left\{ \left[-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} \right] + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \right\} \tag{7.68}$$

and the $\hat{\mathbf{e}}_3$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_3 \left\{ \left[-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2} \right] + \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \right\} \tag{7.69}$$

Adding the results from the equations (7.67), (7.68), (7.69) one obtains the result

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{7.70}$$

■

Directional Derivatives

Let $\vec{r} = \vec{r}(s)$ denote **an arbitrary space curve which passes through the point** $P(x, y, z)$ **of the region** R, where the scalar function $\phi = \phi(x, y, z)$ exists and has all first-order partial derivatives which are continuous. Here the space curve is expressed in terms of the arc length parameter s, where s is measured from some fixed point on the curve. In general, the scalar field $\phi = \phi(x, y, z)$ varies with position and has different values when evaluated at different points in space. Let us evaluate ϕ at points along the curve $\vec{r}$ to **determine how ϕ changes with position along the curve**. The rate of change of ϕ with respect to arc length along the curve is given by

$$
\begin{aligned}
\frac{d\phi}{ds} &= \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \\
\frac{d\phi}{ds} &= \left(\frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3\right) \cdot \left(\frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 + \frac{dz}{ds}\,\hat{\mathbf{e}}_3\right) \\
\frac{d\phi}{ds} &= \text{grad }\phi \cdot \frac{d\vec{r}}{ds} = \nabla\,\phi \cdot\, \hat{\mathbf{e}}_t,
\end{aligned}
$$

where the right-hand side is to be evaluated at a point P on the arbitrary curve $\vec{r}(s)$ in R. The right-hand side of this equation is the dot product of the gradient vector with the unit tangent vector to the curve at the point P and physically represents **the projection of the vector** grad ϕ **in the direction of this tangent vector.** Note that the curve $\vec{r}(s)$ represents **an arbitrary curve through the point** P, and hence, the unit tangent vector represents **an arbitrary direction.** Therefore, one may interpret the derivative $\frac{d\phi}{ds} = \text{grad }\phi \cdot \vec{e}$ as representing **the rate of change of ϕ as one moves in the direction $\vec{e}$.** Here the derivative equals **the projection of the vector** grad ϕ **in the direction** $\vec{e}$. Such derivatives are called **directional derivatives.**

(Directional derivative) ***The component of the gradient $\phi = \phi(x, y, z)$ in the direction of a unit vector $\hat{\mathrm{e}} = \cos\alpha\,\hat{\mathrm{e}}_1 + \cos\beta\,\hat{\mathrm{e}}_2 + \cos\gamma\,\hat{\mathrm{e}}_3$ is equal to the projection $\nabla\phi\cdot\hat{\mathrm{e}}$ and is called the directional derivative of ϕ in the direction $\hat{\mathrm{e}}$. The directional derivative is written as***

$$
\begin{aligned}
\frac{d\phi}{ds} =& \text{grad }\phi \cdot \vec{e} = \nabla\phi\cdot\hat{\mathrm{e}} \\
=& \left(\frac{\partial \phi}{\partial x}\,\hat{\mathrm{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathrm{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathrm{e}}_3\right) \cdot (\cos\alpha\,\hat{\mathrm{e}}_1 + \cos\beta\,\hat{\mathrm{e}}_2 + \cos\gamma\,\hat{\mathrm{e}}_3)
\end{aligned}
\tag{7.71}
$$

where s denotes distance in the direction $\vec{e}$. If $\hat{\mathrm{e}} = \hat{\mathrm{e}}_n$ is a unit normal vector to a surface, the notation $\dfrac{\partial\phi}{\partial n} = \text{grad }\phi\cdot\hat{\mathrm{e}}_n$ is used to denote **a normal derivative to the surface.**

The directional derivative is a measure of how the scalar field ϕ changes as you move in a certain direction. Since **the maximum projection of a vector is the magnitude of the vector itself**, the gradient of ϕ is **a vector which points in the direction of**

the greatest rate of change of ϕ. The length of the gradient vector is $|\text{grad}\ \phi|$ and represents the magnitude of this greatest rate of change.

In other words, the gradient of a scalar field is a vector field which represents the direction and magnitude of the greatest rate of change of the scalar field.

Example 7-19. Show **the gradient of ϕ is a normal vector to the surface $\phi = \phi(x, y, z) = c =$ constant.**

Solution: Let $\vec{r}(s)$, where s is arc length, represent any curve lying in the surface $\phi(x, y, z) = c$. Along this curve the scalar field has the value $\phi = \phi(x(s), y(s), z(s)) = c$ and the rate of change of ϕ along this curve is given by

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} = \frac{dc}{ds} = 0$$

$$\text{or} \qquad \frac{d\phi}{ds} = \text{grad}\ \phi \cdot \frac{d\vec{r}}{ds} = \text{grad}\ \phi \cdot \hat{\mathbf{e}}_t = 0.$$

The resulting equation tells us that the vector grad ϕ is perpendicular to the unit tangent vector to the curve on the surface. But this unit tangent vector lies in the tangent plane to the surface at the point of evaluation for the gradient. **Thus, grad ϕ is normal to the surface $\phi(x, y, z) = c$.** The family of surfaces $\phi = \phi(x, y, z) = c$, for various values of c, are called **level surfaces**. In two-dimensions, the family of curves $\phi = \phi(x, y) = c$, for various values of c, are called **level curves**. **The gradient of ϕ is a vector perpendicular to these level surfaces or level curves.** ■

Example 7-20. Find the unit tangent vector at a point on the curve defined by the intersection of the two surfaces

$$F(x, y, z) = c_1 \quad \text{and} \quad G(x, y, z) = c_2,$$

where c_1 and c_2 are constants.

Solution: If two surfaces $F = c_1$ and $G = c_2$ intersect in a curve, then at a point (x_0, y_0, z_0) common to both surfaces and on the curve one can calculate the normal vectors to both surfaces. These normal vectors are

$$\nabla F = \text{grad}\ F \quad \text{and} \quad \nabla G = \text{grad}\ G$$

which are evaluated at the point (x_0, y_0, z_0) common to both surfaces and on the curve of intersection of the surfaces. The cross product

$$(\nabla F) \times (\nabla G)$$

is a vector tangent to the curve of intersection and perpendicular to both of the normal vectors ∇F and ∇G. A unit tangent vector to the curve of intersection is constructed having the form form

$$\hat{\mathbf{e}}_t = \frac{\nabla F \times \nabla G}{|\nabla F \times \nabla G|}.$$

■

Example 7-21. In two-dimensions a curve $y = f(x)$ or $\vec{r} = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$ can be represented in the implicit form $\phi = \phi(x, y) = y - f(x) = 0$ so that

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 = \vec{N}$$

is a vector normal[6] to the curve at the point $(x, f(x))$. A unit normal to this curve is given by

$$\hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}$$

The vector $\vec{T} = \dfrac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2$ and unit vector $\hat{\mathbf{e}}_t = \dfrac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}$ are tangent to the curve and one can verify that $\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_t = 0$ showing these vectors are orthogonal.

■

Applications for the Gradient

In two-dimensions, let $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ denote a unit vector in an arbitrary, but constant, direction α and let $\phi = \phi(x, y)$ denote any scalar function of position. At a point (x_0, y_0), the directional derivative of ϕ in the direction α becomes

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial \phi}{\partial x}\cos\alpha + \frac{\partial \phi}{\partial y}\sin\alpha$$

and the magnitude of this directional derivative changes as the angle α changes. As the angle α varies, the maximum and minimum directional derivatives, at the point (x_0, y_0), occur in those directions α which satisfy

$$\frac{d}{d\alpha}\left[\frac{d\phi}{ds}\right] = -\frac{\partial \phi}{\partial x}\sin\alpha + \frac{\partial \phi}{\partial y}\cos\alpha = 0. \tag{7.72}$$

Note there exists two angles α lying in the region between 0 and 2π radians which satisfy the above equation. These directions must be tested to see which corresponds to a maximum and which corresponds to a minimum directional derivative. These

---

[6] Always remember that there are two normals to a curve, namely $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$,

angles specify the directions one should travel in order to achieve the maximum (or minimum) rate of change of the scalar ϕ.

A physical example illustrating this idea is heat flow. Heat always flows from regions of higher temperature to regions of lower temperature. Let $T(x,y)$ denote a scalar field which represents the temperature T at any point (x,y) in some region R within a material medium. The level curves $T(x,y)=T_0$ are called isothermal curves and represent the constant "levels" of temperature. The vector $\operatorname{grad} T$, evaluated at a point on an isothermal curve, points in the direction of greatest temperature change. The vector is also normal to the isothermal curve. Fourier's law of heat conduction states that the heat flow $\vec{q}$ [joules/cm^2 sec] is in a direction opposite to this greatest rate of change and

$$\vec{q} = -k \operatorname{grad} T,$$

where k [joules/cm$-$sec$-$deg C] is the thermal conductivity of the material in which the heat is flowing.

Example 7-22. In two-dimensions a curve $y=f(x)$ can be represented in the implicit form $\phi=\phi(x,y)=y-f(x)=0$ so that

$$\operatorname{grad} \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 = \vec{N}$$

is a vector normal to the curve at the point $(x, f(x))$. A unit normal vector to the curve is given by

$$\hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}}$$

Another way to construct this normal vector is as follows. The position vector $\vec{r}$ describing the curve $y=f(x)$ is given by $\vec{r} = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$ with tangent $\dfrac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2$. The unit tangent vector to the curve is given by $\hat{\mathbf{e}}_t = \dfrac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}}$. The vector $\hat{\mathbf{e}}_3$ is perpendicular to the planar surface containing the curve and consequently the vector $\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_t$ is normal to the curve. This cross product is given by

$$\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_t = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}} = \hat{\mathbf{e}}_n$$

and produces a unit normal vector to the curve. Note that there are always two normals to every curve or surface. It is important to observe that if $\vec{N}$ is normal to a point on the surface, then the vector $-\vec{N}$ is also a normal to the same point on the surface. If the surface is a closed surface, then one normal is called an **inward normal** and the other an **outward normal**.

■

Maximum and Minimum Values

The directional derivative of a scalar field ϕ in the direction of a unit vector $\vec{e}$ has been defined by the projection

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \vec{e}.$$

Define a second directional derivative of ϕ in the direction $\vec{e}$ as the directional derivative of a directional derivative. The second directional derivative is written

$$\frac{d^2\phi}{ds^2} = \text{grad } \left[\frac{d\phi}{ds}\right] \cdot \vec{e} = \text{grad } \left[\text{grad } \phi \cdot \vec{e}\right] \cdot \vec{e}. \tag{7.73}$$

Higher directional derivatives are defined in a similar manner.

Example 7-23. Let $\phi(x, y)$ define a two-dimensional scalar field and let

$$\hat{\mathbf{e}}_\alpha = \cos\alpha\, \hat{\mathbf{e}}_1 + \sin\alpha\, \hat{\mathbf{e}}_2$$

represent a unit vector in an arbitrary direction α. The directional derivative at a point (x_0, y_0) in the direction $\hat{\mathbf{e}}_\alpha$ is given by

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha,$$

where it is to be understood that the derivatives are evaluated at the point (x_0, y_0). The second directional derivative is given by

$$\begin{aligned}
\frac{d^2\phi}{ds^2} &= \text{grad } \left(\frac{d\phi}{ds}\right) \cdot \hat{\mathbf{e}}_\alpha \\
\frac{d^2\phi}{ds^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha\right)\cos\alpha + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha\right)\sin\alpha \\
\frac{d^2\phi}{ds^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\alpha\cos\alpha + \frac{\partial^2\phi}{\partial y^2}\sin^2\alpha.
\end{aligned}$$

■

Directional derivatives can be used to determine the maximum and minimum values of functions of several variables. Recall from calculus a function of a single variable $y = f(x)$ has a relative maximum (or relative minimum) at a point x_0 if for any x in a neighborhood of x_0 and different from x_0, the inequality $f(x) < f(x_0)$ (or

$f(x) > f(x_0)$) holds. The determination of relative maximum and minimum values of a differential function $y = f(x)$ over an interval (a, b) consists of

1. Determining the critical points where $f'(x) = 0$ and then testing these critical points.
2. Testing the boundary points $x = a$ and $x = b$.

The second derivative test for relative maximum and minimum values states that if x_0 is a critical point, then

1. $f(x)$ has the maximum value $f(x_0)$ if $f''(x_0) < 0$ (i.e., curve is concave downward if the second derivative is negative).
2. $f(x)$ has a minimum value $f(x_0)$ if $f''(x_0) > 0$ (i.e., the curve is concave upward if the second derivative is positive).

The above concepts for the relative maximum and minimum values of a function of one variable can be extended to higher dimensions when one must deal with functions of more than one variable. The extension of these concepts can be accomplished by utilizing the gradient and directional derivatives.

In the following discussion, it is assumed that the given surface is in an explicit form. If the surface is given in the implicit form $F(x, y, z) = 0$, then it is assumed that one can solve for z in terms of x and y to obtain $z = z(x, y)$. By a **delta neighborhood of a point** (x_0, y_0) in two-dimensions is meant the set of all points inside the circular disk

$$N_0(\delta) = \{x, y \mid (x - x_0)^2 + (y - y_0)^2 < \delta^2\}.$$

The function $z(x, y)$, which is continuous and whose derivatives exist, has a relative maximum at a point (x_0, y_0) if $z(x, y) < z(x_0, y_0)$ for all x, y in a some δ neighborhood of (x_0, y_0). Similarly, the function $z(x, y)$ has a relative minimum at a point (x_0, y_0) if $z(x, y) > z(x_0, y_0)$ for all x, y in some δ neighborhood of the point (x_0, y_0). Points where the surface $z = z(x, y)$ has a relative maximum or minimum are called critical points and at these points one must have

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0$$

simultaneously. Critical points are those points where the tangent plane to the surface $z = z(x, y)$ is parallel to the x, y plane. If the points (x, y) are restricted to a region R of the plane $z = 0$, then the boundary points of R must be tested separately for the determination of any local maximum or minimum values on the surface.

The problem of determining the relative maximum and minimum values of a function of two variables is now considered. In the discussions that follow, note that the problem of determining the maximum and minimum for a function of two variables is reduced to the simpler problem of finding the maximum and minimum of a function of a single variable.

If (x_0, y_0) is a critical point associated with the surface $z = z(x, y)$, then one can slide the free vector given by $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ to the critical point and construct a plane normal to the plane $z = 0$, such that this plane contains the vector $\hat{\mathbf{e}}_\alpha$. This plane intersects the surface in a curve. The situation is depicted graphically in the figure 7-14

At a critical point where $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, the directional derivative satisfies

$$\frac{dz}{ds} = \text{grad } z \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha = 0$$

for all directions α.

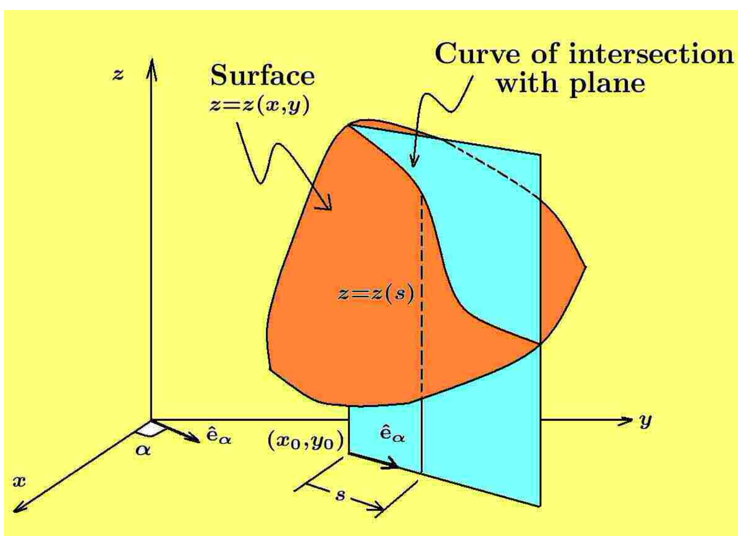


Figure 7-14. Curve of intersection with plane containing $\hat{\mathbf{e}}_\alpha$.

Here the directional derivative represents the variation of the surface height z with respect to a distance s in the $\hat{\mathbf{e}}_\alpha$ direction. (i.e., measure the rate of change of the scalar field z which represents the height of the curve.) To picture what the above equations are describing, let

$$x = x_0 + s\cos\alpha \quad \text{and} \quad y = y_0 + s\sin\alpha$$

represent the equation of the line of intersection of the plane $z = 0$ with the plane normal to $z = 0$ containing $\hat{\mathbf{e}}_\alpha$. The plane containing $\hat{\mathbf{e}}_\alpha$ and the normal to the plane $z = 0$ intersects the surface $z(x, y)$ in a curve given by

$$z = z(x, y) = z(x_0 + s\cos\alpha, y_0 + s\sin\alpha) = z(s).$$

The directional derivative of the scalar field $z(x, y)$ in the direction α is then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

Observe that the curve of intersection $z = z(s)$ is a two-dimensional curve, and the methods of calculus may be applied to determine the relative maximum and minimum values along this curve. However, one must test this curve of intersection corresponding to all directions α.

One can conclude that at a critical point (x_0, y_0) one must have $\frac{dz}{ds} = 0$ for all α. If in addition $\frac{d^2z}{ds^2} > 0$ for all directions α, then $z_0 = z(x_0, y_0)$ corresponds to a relative minimum. If the second derivative $\frac{d^2z}{ds^2} < 0$ for all directions α, then $z_0 = z(x_0, y_0)$ corresponds to a relative maximum.

Calculate the second directional derivative and show

$$\frac{d^2z}{ds^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 z}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha.$$

The sign of the second directional derivative determines whether a maximum or minimum value for z exists, and hence one must be able to analyze this derivative for all directions α. Let

$$A = \frac{\partial^2 z}{\partial x^2} \qquad B = \frac{\partial^2 z}{\partial x\,\partial y} \qquad C = \frac{\partial^2 z}{\partial y^2}$$

represent the values of the second partial derivatives evaluated at a critical point (x_0, y_0). One can then express the second directional derivative in a form which is more tractable for analysis. Factor out the leading term and then complete the square on the first two terms to obtain

$$\begin{aligned}\frac{d^2z}{ds^2} &= A\cos^2\alpha + 2B\cos\alpha\sin\alpha + C\sin^2\alpha \\ &= A\left[\cos^2\alpha + 2\frac{B}{A}\cos\alpha\sin\alpha + \frac{C}{A}\sin^2\alpha\right] \\ &= A\left[\left(\cos\alpha + \frac{B}{A}\sin\alpha\right)^2 + \frac{(AC-B^2)}{A^2}\sin^2\alpha\right].\end{aligned} \tag{7.74}$$

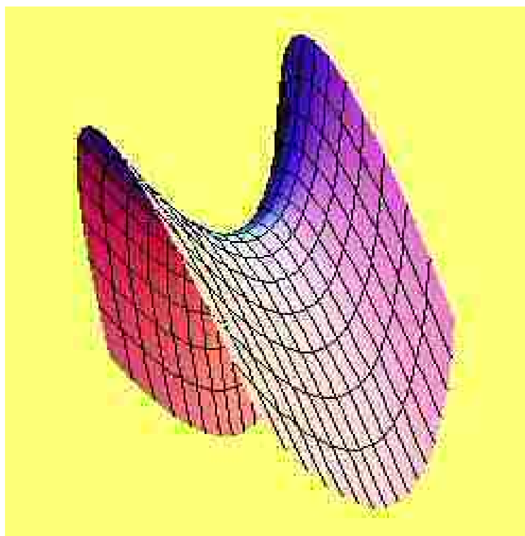


Figure 7-15. Saddle point for $z(x_0, y_0)$

One can now make the following observations:

1. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 = 0$, then in those directions α which satisfy $\cos\alpha + \frac{B}{A}\sin\alpha = 0$, the second derivative vanishes. For all other values of α, the second derivative is of constant sign, which is the same sign as A. If the above conditions are satisfied, then the second derivative test for a maximum or minimum fails.
2. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 < 0$, then the second derivative is not of constant sign, but assumes different signs in different directions α. In particular, for the special case $\alpha = 0$ one finds $\frac{d^2z}{ds^2} = A$ and for α satisfying $\cos\alpha + \frac{B}{A}\sin\alpha = 0$ there results

$$\frac{d^2z}{ds^2} = \frac{A(AC-B^2)}{A^2}\sin^2\alpha.$$

Hence, if $A > 0$, then $A(AC - B^2)$ is negative and alternatively if $A < 0$, then $A(AC - B^2)$ is positive. In either case, the second derivative has a nonconstant sign value and in this situation the critical point (x_0, y_0) is said to correspond to a saddle point. Such a critical point is illustrated in figure 7-15.

3. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 > 0$, the second derivative is of constant sign, which is the sign of A.
   (a) If $A > 0$, $\frac{d^2z}{ds^2} > 0$, the curve $z = z(s)$ is concave upward for all α, and hence the critical point corresponds to a relative minimum.
   (b) If $A < 0$, $\frac{d^2z}{ds^2} < 0$, the curve $z = z(s)$ is concave downward for all α, and therefore the critical point corresponds to a relative maximum.

Example 7-24. Find the maximum and minimum values of

$$z = z(x, y) = x^2 + y^2 - 2x + 4y$$

Solution: The first and second partial derivatives of z are

$$\frac{\partial z}{\partial x} = 2x - 2, \quad \frac{\partial z}{\partial y} = 2y + 4, \quad A = \frac{\partial^2 z}{\partial x^2} = 2, \quad C = \frac{\partial^2 z}{\partial y^2} = 2, \quad B = \frac{\partial^2 z}{\partial x\, \partial y} = 0$$

Setting the first partial derivatives equal to zero and solving for x and y gives the critical points. For this example there is only one critical point which occurs at $(x_0, y_0) = (1, -2)$. From the second derivatives of z one finds $A = 2$, $B = 0$, $C = 2$ and $AC - B^2 = 4 > 0$, and consequently the critical point $(1, -2)$ corresponds to a relative minimum of the function.

The use of level curves to analyze complicated surfaces is sometimes helpful. For example, the level curves of the above function can be expressed in the form

$$z = z(x, y) = (x - 1)^2 + (y + 2)^2 - 5 = k = \text{constant}.$$

By assigning values to the constant k one can determine the general character of the surface. It is left as an exercise to show these level curves are circles which are cross section of the surface known as a paraboloid.

■

Lagrange Multipliers

Consider the problem of finding stationary values associated with a function $f = f(x, y)$ subject to a constraint condition that $g = g(x, y) = 0$. Recall that a necessary condition for $f = f(x, y)$ to have an extremum value at a point (a, b) requires that the differential $df = 0$ or

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{7.75}$$

Whenever the small changes dx and dy are independent, one obtains the necessary conditions that

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

at a critical point. Whenever a constraint condition is required to be satisfied, then the small changes dx and dy are no longer independent and one must find the relationship between the small changes dx and dy as the point (x, y) moves along the constraint curve. From the differential relation $dg = 0$ one finds that

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy = 0$$

must be satisfied. Assume that $\frac{\partial g}{\partial y} \neq 0$, then one can obtain

$$dy = \frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}\,dx \tag{7.76}$$

as the dependent relationship between the small changes dx and dy.

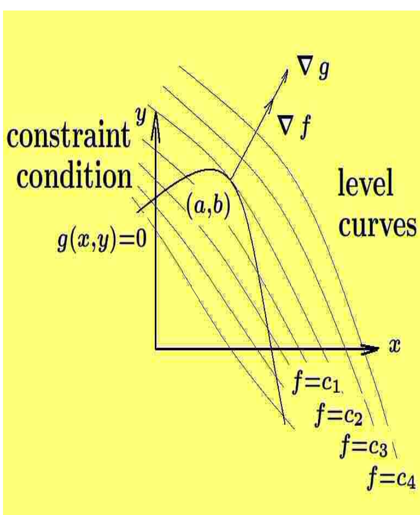


Figure 7-17.
Maximum-minimum problem with constraint.

Substitute the dy from equation (7.76) into the equation (7.75) to produce the result

$$df = \frac{1}{\frac{\partial g}{\partial y}}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) dx = 0 \tag{7.77}$$

that must hold for an arbitrary change dx. This gives the following necessary condition. The critical points (x, y) of the function f, subject to the constraint equation $g(x, y) = 0$, must satisfy the equations

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} &= 0 \\ g(x, y) &= 0 \end{aligned} \tag{7.78}$$

simultaneously.

The equations (7.78) can be interpreted that when a member of the family of curves $f(x,y)=c=constant$ is tangent to the constraint curve $g(x,y)=0$, there results the common values of

$$\frac{dy}{dx}=\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=\frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} \qquad \Rightarrow \qquad \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=0.$$

One can give a physical picture of the problem. Think of the constraint condition given by $g=g(x,y)=0$ as defining a curve in the x,y-plane and then consider the family of level curves $f=f(x,y)=c$, where c is some constant. A representative sketch of the curve $g(x,y)=0$, together with several level curves from the family, $f=c$ are illustrated in the figure 7-17. Among all the level curves that intersect the constraint condition curve $g(x,y)=0$ select that curve for which c has the largest or smallest value. Here it is assumed that the constraint curve $g(x,y)=0$ is a smooth curve without singular points.

If (a,b) denotes a point of tangency between a curve of the family $f=c$ and the constraint curve $g(x,y)=0$, then at this point both curves will have gradient vectors that are collinear and so one can write $\nabla f+\lambda\nabla g=\vec{0}$ for some constant λ called a Lagrange multiplier. This relationship together with the constraint equation produces the three scalar equations

$$\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=&0\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=&0\\ g(x,y)=&0.\end{aligned} \qquad \Rightarrow \qquad \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=0 \tag{7.79}$$

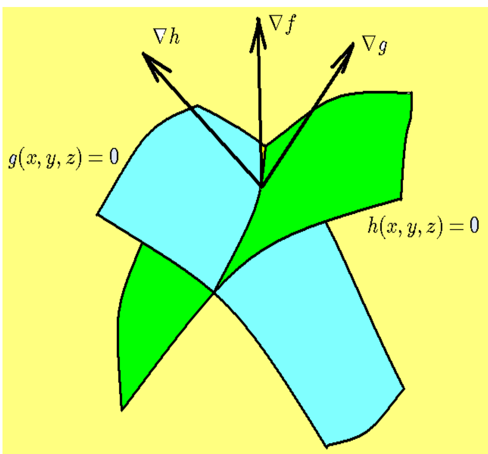
Lagrange viewed the above problem in the following way. Define the function

$$F(x,y,\lambda)=f(x,y)+\lambda g(x,y) \tag{7.80}$$

where $f(x,y)$ is called an objective function and represents the function to be maximized or minimized. The parameter λ is called a Lagrange multiplier and the function $g(x,y)$ is obtained from the constraint condition. Lagrange observed that a stationary value of the function F, without constraints, is equivalent to the problem of stationary values of f with a constraint condition because one would have at a stationary value of F the conditions

$$\begin{aligned} \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} =&0 \\ \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} =&0 \\ \frac{\partial F}{\partial \lambda} = g(x,y) =&0 \qquad \text{The constraint condition.} \end{aligned} \tag{7.81}$$

These represent three equations in the three unknowns x, y, λ that must be solved. The equations (7.80) and (7.81) are known as **the Lagrange rule** for the method of Lagrange multipliers.



The method of Lagrange multipliers can be applied in higher dimensions. For example, consider the problem of finding maximum and minimum values associated with a function $f = f(x,y,z)$ subject to the constraint conditions $g(x,y,z) = 0$ and $h(x,y,z) = 0$. Here the equations $g(x,y,z) = 0$ and $h(x,y,z) = 0$ describe two surfaces that may or may not intersect. Assume the surfaces intersect to give a space curve.

The problem is to find an extremal value of $f = f(x,y,z)$ as (x,y,z) varies along the curve of intersection of surfaces $g = 0$ and $h = 0$. At a critical point where a stationary value exists, the directional derivative of f along this curve must be zero. Here the directional derivative is given by $\frac{df}{ds} = \nabla f \cdot \hat{\mathbf{e}}_t$, where $\hat{\mathbf{e}}_t$ is a unit tangent vector to the space curve and $\nabla f = \operatorname{grad} f$ denotes the gradient of f. Note that if the directional derivative is zero, then ∇f must lie in a plane normal to the curve of intersection.

Another way to view the problem, and also suggest that the concepts can be extended to higher dimensional spaces, is to introduce the notation $\bar{x} = (x_1, x_2, x_3) = (x,y,z)$ to denote a vector to a point on the curve of intersection of the two surfaces $g(x_1,x_2,x_3) = 0$ and $h(x_1,x_2,x_3) = 0$. At a stationary value of f one must have

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = \operatorname{grad} f \cdot d\bar{x} = 0$$

This implies that grad f is normal to the curve of intersection since it is perpendicular to the tangent vector $d\bar{x}$ to the curve of intersection. At a stationary point, the normal plane containing the vector grad f also contains the vectors ∇g and ∇h since $dg = \text{grad}\, g \cdot d\bar{x} = 0$ and $dh = \text{grad}\, h \cdot d\vec{x} = 0$ at the stationary point. Hence, if these three vectors are noncollinear, then there will exist scalars λ_1 and λ_2 such that

$$\nabla f + \lambda_1 \nabla g + \lambda_2 \nabla h = 0 \tag{7.82}$$

at a stationary point. The equation (7.82) is a vector equation and is equivalent to the three scalar equations

$$\begin{aligned}
\frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} =&0 \\
\frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} =&0 \\
\frac{\partial f}{\partial z} + \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z} =&0.
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g}{\partial x_1} + \lambda_2 \frac{\partial h}{\partial x_1} =&0 \\
\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g}{\partial x_2} + \lambda_2 \frac{\partial h}{\partial x_2} =&0 \\
\frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial g}{\partial x_3} + \lambda_2 \frac{\partial h}{\partial x_3} =&0.
\end{aligned}$$

depending upon the notation you are using. These three equations together with the constraint equations $g = 0$ and $h = 0$ gives us five equations in the five unknowns $x, y, z, \lambda_1, \lambda_2$ that must be satisfied at a stationary point.

By the Lagrangian rule one can form the function

$$F = F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 g(x, y, z) + \lambda_2 h(x, y, z)$$

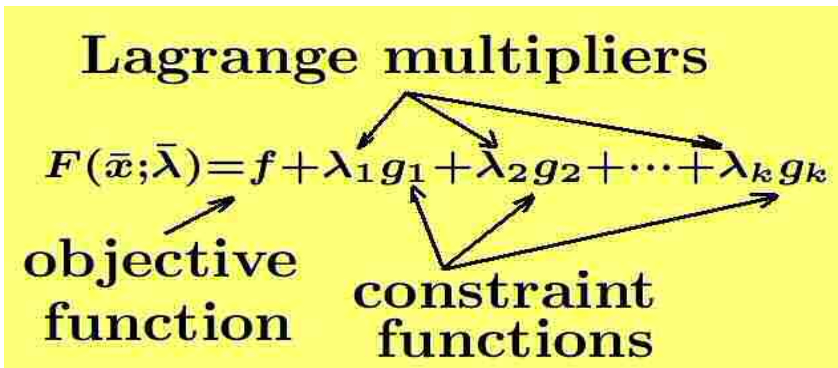
where $f(x, y, z)$ is the objective function, $g(x, y, z)$ and $h(x, y, z)$ are the constraint functions and λ_1, λ_2 are the Lagrange multipliers. Observe that F has a stationary value where

$$\begin{aligned}
\frac{\partial F}{\partial x} =&\frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} = 0 \\
\frac{\partial F}{\partial y} =&\frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} = 0 \\
\frac{\partial F}{\partial z} =&\frac{\partial f}{\partial z} + \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z} = 0 \\
\frac{\partial F}{\partial \lambda_1} =&g(x, y, z) = 0 \\
\frac{\partial F}{\partial \lambda_2} =&h(x, y, z) = 0
\end{aligned} \tag{7.83}$$

These are the same five equations, with unknowns $x, y, z, \lambda_1, \lambda_2$, for determining the stationary points as previously noted.

Generalization of Lagrange Multipliers

In general, to find an extremal value associated with a n-dimensional function given by $f = f(\bar{x}) = f(x_1, x_2, \ldots, x_n)$ subject to k constraint conditions that can be written in the form $g_i(\bar{x}) = g_i(x_1, x_2, \ldots, x_n) = 0$, for $i = 1, 2, \ldots, k$, where k is less than n. It is required that the gradient vectors $\nabla g_1, \nabla g_2, \ldots, \nabla g_k$ be linearly independent vectors, then one can employ the method of Lagrange multipliers as follows. The Lagrangian rule requires that the function $F = f + \sum_{i=1}^{k} \lambda_i g_i$ can be written in the expanded form



(7.84)

which contains the objective function f, summed with each of the constraint functions g_i, multiplied by a Lagrange multiplier λ_i, for the index i having the values $i = 1, \ldots, k$. Here the function F and consequently the function f has stationary values at those points where the following equations are satisfied

$$\begin{aligned} \frac{\partial F}{\partial x_i} =& 0, \quad \text{for } i = 1, \ldots, n \\ \frac{\partial F}{\partial \lambda_j} =& 0, \quad \text{for } j = 1, \ldots, k \end{aligned} \tag{7.85}$$

The equations (7.85) represent a system of $(n+k)$ equations in the $(n+k)$ unknowns $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k$ for determining the stationary points. In general, the stationary points will be found in terms of the λ_i values. The vector $(\bar{x}_0, \bar{\lambda}_0)$ where $\bar{x}_0$ and $\bar{\lambda}_0$ are solutions of the system of equations (7.85) can be thought of as critical points associated with the Lagrangian function $F(\bar{x}, \bar{\lambda})$ given by equation (7.84). The resulting stationary points must then be tested to determine whether they correspond to a relative maximum value, minimum value or saddle point. One can form the Hessian[7] matrix associated with the function $F(\bar{x}; \bar{\lambda})$ and analyze this matrix at the

[7] See page 318 for definition of Hessian matrix.

critical points. Whenever the determinant of the Hessian matrix is zero at a critical point, then the critical point $(\bar{x}_0, \bar{\lambda}_0)$ is said to be degenerate and one must seek an alternative method to test for an extremum.

Vector Field and Field Lines

A vector field is a vector-valued function representing a mapping from R^n to a vector $\vec{V}$. Any vector which varies as a function of position in space is said to represent a vector field. The vector field $\vec{V} = \vec{V}(x,y,z)$ is a one-to-one correspondence between points in space (x,y,z) and a vector quantity $\vec{V}$. This correspondence is assumed to be continuous and differentiable within some region R. Examples of vector fields are velocity, electric force, mechanical force, etc. Vector fields can be represented graphically by plotting vectors at selected points within a region. These kind of graphical representations are called vector field plots. Alternative to plotting many vectors at selected points to visualize a vector field, it is sometimes easier to use the concept of field lines associated with a vector field. A field line is a curve where at each point (x,y,z) of the curve, the tangent vector to the curve has the same direction as the vector field at that point. If $\vec{r} = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ is the position vector describing a field line, then by definition of a field line the tangent vector $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3$ evaluated at a point t_0 must be in the same direction as the vector $\vec{V}_0 = \vec{V}(x(t_0), y(t_0), z(t_0))$. If this relation is true for all values of the parameter t, then one can state that the vectors $\frac{d\vec{r}}{dt}$ and $\vec{V}$ must be colinear at each point on the curve representing the field line. This requires

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = k\left[V_1(x,y,z)\,\hat{\mathbf{e}}_1 + V_2(x,y,z)\,\hat{\mathbf{e}}_2 + V_3(x,y,z)\,\hat{\mathbf{e}}_3\right]$$

where k is some proportionality constant. Equating like components in the above equation one obtains the system of differential equations

$$\frac{dx}{dt} = kV_1(x,y,z), \qquad \frac{dy}{dt} = kV_2(x,y,z), \qquad \frac{dz}{dt} = kV_3(x,y,z)$$

which must be solved to obtain the equations of the field lines.

Surface Integrals

In this section various types of surface integrals are introduced. In particular, surface integrals of the form

$$\iint\limits_R f(x,y,z)\,d\vec{S}, \qquad \iint\limits_R \vec{F}\cdot d\vec{S}, \qquad \iiint\limits_R \vec{F}\times d\vec{S},$$

are defined and illustrated. Throughout the following discussion all surfaces are considered to be oriented (two-sided) surfaces.

Consider a surface in space with an element of surface area dS constructed at some general point on the surface as is illustrated in figure 7-16.

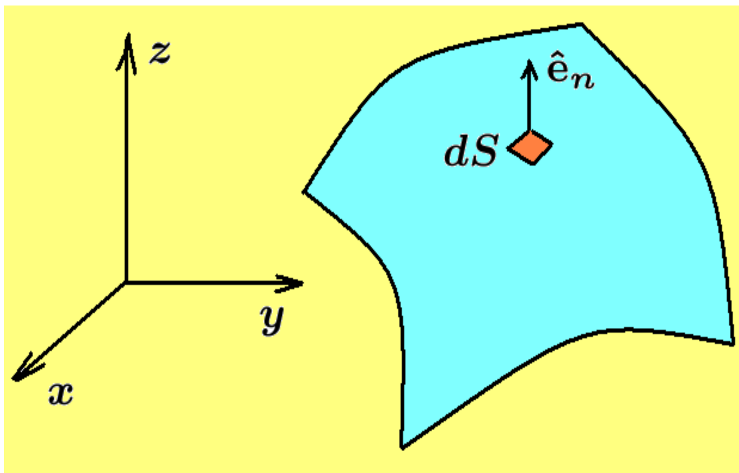


Figure 7-16. Element of surface area.

In the representation of various vector integrals, it is convenient to **define vector elements of surface area $d\vec{S}$ whose magnitude is dS and whose direction is the same as the unit outward normal $\hat{\mathbf{e}}_n$ to the surface.** Define this vector element of surface area as $d\vec{S} = \hat{\mathbf{e}}_n \, dS$ which can be considered as the limit associated with the area $\vec{\Delta} S = \hat{\mathbf{e}}_n \, \Delta S$.

Normal to a Surface

If $\hat{\mathbf{e}}_n$ is a normal to a smooth surface, then $-\hat{\mathbf{e}}_n$ is also normal to the surface. That is, all smooth orientated surfaces possess two normals. If the surface is a closed surface, there is an inside surface and an outside surface. The outside surface is called the positive side of the surface. The unit normal to the positive side of a surface is called the positive normal or outward normal. If the surface is not closed, then one can arbitrarily select one side of the surface and call it the positive side, therefore, the normal drawn to this positive side is also called the outward normal.

If the surface is expressed in an implicit form $F(x, y, z) = 0$, then a unit normal to the surface can be obtained from the relation:

$$\hat{\mathbf{e}}_n = \frac{\text{grad } F}{|\text{grad } F|}.$$

If the surface is expressed in the explicit form $z = z(x, y)$, then a unit normal to the surface can be found from the relation

$$\hat{\mathbf{e}}_n = \frac{\text{grad } [z(x,y) - z]}{|\text{ grad } [z(x,y) - z]|} = \frac{\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 - \,\hat{\mathbf{e}}_3}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{7.86}$$

Surfaces can also be expressed in the parametric form

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

where u and v are parameters. The functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ must be such that one and only one point (u, v) maps to any given point on the surface. These functions are also assumed to be continuous and differentiable. In this case, the position vector to a point on the surface can be represented as

$$\vec{r} = \vec{r}(u, v) = x(u, v)\,\hat{\mathbf{e}}_1 + y(u, v)\,\hat{\mathbf{e}}_2 + z(u, v)\,\hat{\mathbf{e}}_3.$$

The curves

$$\vec{r}(u, v)\Big|_{v=\text{Constant}} \qquad \text{and} \qquad \vec{r}(u, v)\Big|_{u=\text{Constant}}$$

sweep out coordinate curves on the surface and the vectors

$$\frac{\partial \vec{r}}{\partial u}, \qquad \frac{\partial \vec{r}}{\partial v}$$

are tangent vectors to these coordinate curves. A unit normal to the surface at a point P on the surface can then be calculated from the cross product of the tangent vectors tangent vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ evaluated at the point P. One can calculate the unit normal

$$\hat{\mathbf{e}}_n = \frac{\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}.$$

It should be noted that if the cross product

$$\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u} \neq \vec{0},$$

the surface is called a smooth surface. If at a point with surface coordinates (u_0, v_0) this cross product equals the zero vector, the point on the surface is called a singular point of the surface.

Example 7-25. The parametric equations

$$x = a\cos u \sin v, \qquad y = a\sin u \sin v, \qquad z = a\cos v$$

with $0 \le u \le 2\pi$ and $0 \le v \le \pi$, represent the surface of a sphere of radius a. These parametric equations were obtained from the geometry of figure 7-18.

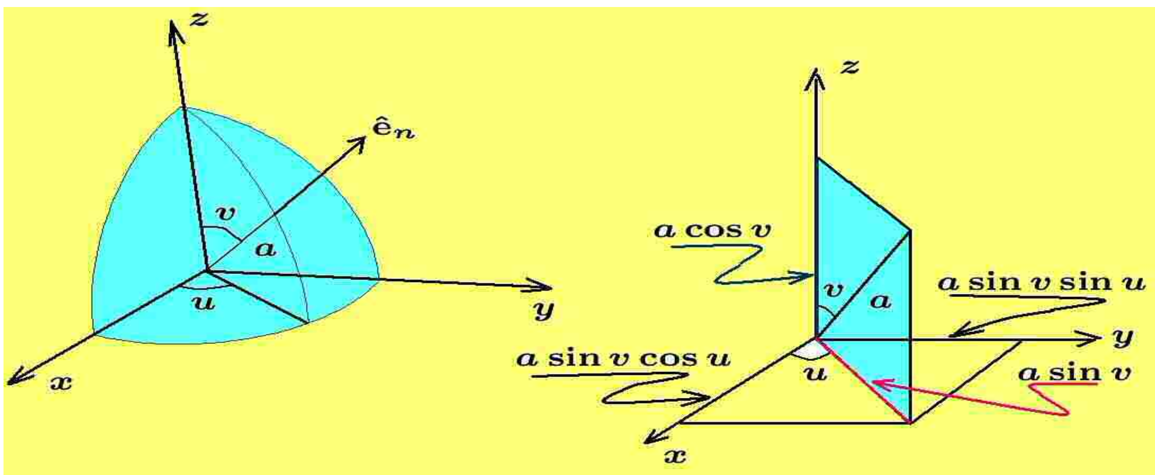


Figure 7-18. Surface of a sphere of radius a.

The position vector of a point on the surface of this sphere can be represented by the vector

$$\vec{r}(u,v) = a\cos u \sin v\,\hat{\mathbf{e}}_1 + a\sin u \sin v\,\hat{\mathbf{e}}_2 + a\cos v\,\hat{\mathbf{e}}_3.$$

For u_0 and v_0 constants, the curves $\vec{r}(u_0,v)$, $\quad 0 \le v \le \pi$, are meridian lines on the sphere while the curves $\vec{r}(u,v_0)$, $\quad 0 \le u \le 2\pi$, are circles of constant latitude. The tangent vectors to these curves are found by taking the derivatives

$$\begin{aligned}\frac{\partial \vec{r}}{\partial u} &= -a\sin u \sin v\,\hat{\mathbf{e}}_1 + a\cos u \sin v\,\hat{\mathbf{e}}_2\\ \frac{\partial \vec{r}}{\partial v} &= a\cos u \cos v\,\hat{\mathbf{e}}_1 + a\sin u \cos v\,\hat{\mathbf{e}}_2 - a\sin v\,\hat{\mathbf{e}}_3.\end{aligned}$$

From these tangent vectors, a normal vector to the surface is constructed by taking a cross product and

$$\vec{N} = \frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}.$$

It can be verified that a unit normal to this surface is

$$\hat{\mathbf{e}}_n = \frac{\vec{N}}{|\vec{N}|} = \frac{1}{a}\vec{r}.$$

That is, the unit outer normal to a point P on the surface of the sphere has the same direction as the position vector $\vec{r}$ to the point P.

■

Example 7-26. If the surface is described in the explicit form $z = z(x, y)$ the position vector to a point on the surface can be represented

$$\vec{r} = \vec{r}(x, y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x, y)\,\hat{\mathbf{e}}_3$$

This vector has the partial derivatives

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

so that a normal to the surface can be calculated from the cross product

$$\vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

A unit normal to the surface is

$$\hat{\mathbf{e}}_n = \frac{\vec{N}}{|\vec{N}|} = \frac{-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} = n_x\,\hat{\mathbf{e}}_1 + n_y\,\hat{\mathbf{e}}_2 + n_z\,\hat{\mathbf{e}}_3$$

where n_x, n_y, n_z are the direction cosines of the unit normal. Note also that the vector

$$\hat{\mathbf{e}}_{n^*} = \frac{\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

is also normal to the surface.

Example 7-27. Consider a surface described by the implicit form $F(x,y,z)=0$. Recall that equations of this form define z as a function of x and y and the derivatives of z with respect to x and y are given by

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Substituting these derivatives into the equation (7.86) and simplifying one finds that the direction cosines of the unit normal to the surface are given by

$$n_x = \frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad n_y = \frac{\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad n_z = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

Tangent Plane to Surface

Consider a smooth surface defined by the equation

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3$$

In order to construct a tangent plane to a regular point $\vec{r} = \vec{r}(u_0, v_0)$ where the surface coordinates have the values (u_0, v_0), one must first construct the normal to the surface at this point. One such normal is

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3$$

The point on the surface is

$$x_0 = x(u_0, v_0), \qquad y_0 = y(u_0, v_0), \qquad z_0 = z(u_0, v_0)$$

which can be described by the position vector $\vec{r}_0 = \vec{r}(u_0, v_0)$. If $\vec{r}$ represents the variable point

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

which varies over the plane through the point (x_0, y_0, z_0), then the vector $\vec{r} - \vec{r}_0$ must lie in the tangent plane and consequently is perpendicular to the normal vector $\vec{N}$. One can then write

$$(\vec{r} - \vec{r}_0) \cdot \vec{N} = 0 \tag{7.87}$$

as the equation of the plane through the point (x_0, y_0, z_0) which is perpendicular to $\vec{N}$ and consequently tangent to the surface. In equation (7.87) one can substitute any of the normal vectors calculated in the previous examples.

The equation of the line through the point (x_0, y_0, z_0) which is perpendicular to the tangent plane is given by

$$\vec{r} = \vec{r}_0 + \lambda \vec{N} \tag{7.88}$$

where λ is a scalar. The equation of the line can also be expressed by the parametric equations

$$x = x_0 + \lambda N_1, \qquad y = y_0 + \lambda N_2, \qquad z = z_0 + \lambda N_3 \tag{7.89}$$

where again, the normal vector $\vec{N}$ can be replaced by any of the normals previously calculated.

■

Element of Surface Area

Consider the case where the surface is given in the explicit form $z = z(x, y)$. In this case, the position vector of a point on the surface is given by

$$\vec{r} = \vec{r}(x, y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x, y)\,\hat{\mathbf{e}}_3. \tag{7.90}$$

The curves

$$\vec{r}(x, y)\Big|_{y=\text{ Constant}} \qquad \text{and} \qquad \vec{r}(x, y)\Big|_{x=\text{Constant}}$$

are coordinate curves lying in the surface which intersect at a common point (x, y, z). The vectors

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

are tangent to these coordinate curves, and consequently the differential of the position vector

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x}\,dx + \frac{\partial \vec{r}}{\partial y}\,dy$$

lies in the tangent plane to the surface at the common point of intersection of the coordinate curves. This differential is illustrated in figure 7-19.

Consider an element of area $\Delta A = \Delta x\,\Delta y$ in the xy plane of figure 7-19. When this element of area is projected onto the surface $z = z(x, y)$, it intersects the surface in an element of surface area ΔS. When projected onto the tangent plane to the surface it intersects the tangent plane in an element of surface area ΔR. These projections are illustrated in figure 7-19(c).

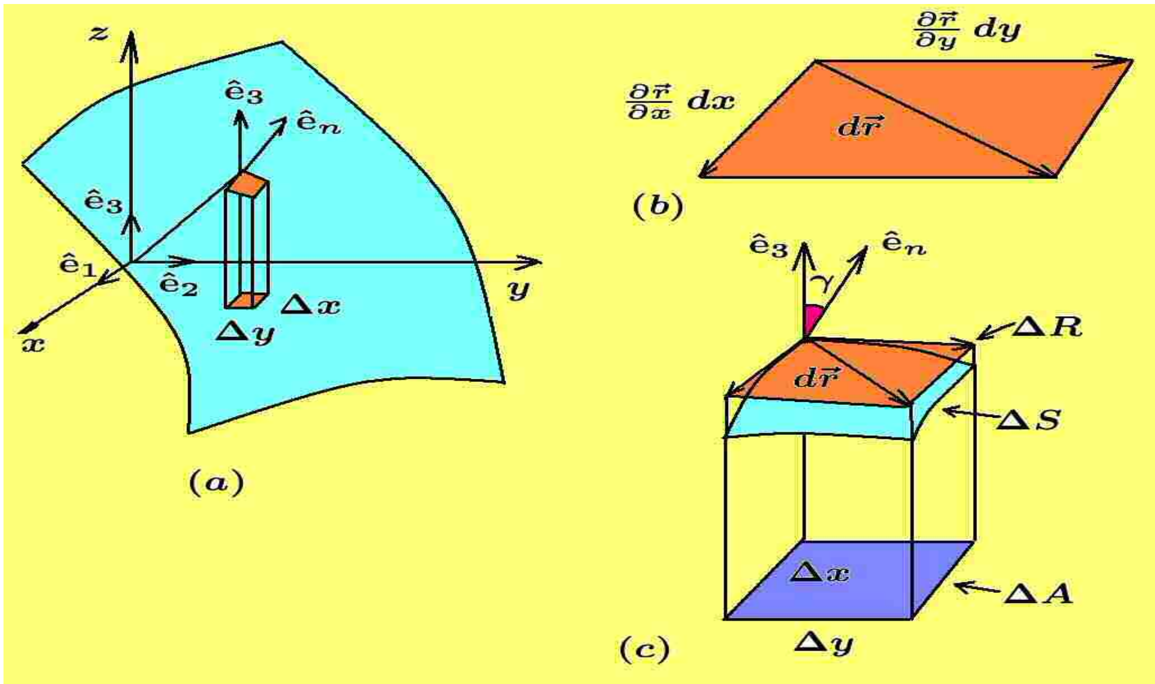


Figure 7-19. Element of surface area and element parallelogram.

In the limit as Δx and Δy tend toward zero, ΔR approaches ΔS and one can define $d\vec{R} = d\vec{S}$, where the element of area dR lies in the tangent plane to the surface at the point (x, y, z). **In the limit as Δx and Δy approach zero, the element of area is defined as the area of the elemental parallelogram defined by the vector $d\vec{r}$ and illustrated in figure 7-19(b). The area of this elemental parallelogram can be calculated from the cross product relation**

$$\left(\frac{\partial \vec{r}}{\partial x}\, dx\right) \times \left(\frac{\partial \vec{r}}{\partial y}\, dy\right) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ dx & 0 & \frac{\partial z}{\partial x}\, dx \\ 0 & dy & \frac{\partial z}{\partial y}\, dy \end{vmatrix} = \left(-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right) dx\, dy. \tag{7.91}$$

The area of the elemental parallelogram is the magnitude of the above cross product, and can be expressed

$$dS = dR = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy. \tag{7.92}$$

Given a surface in the explicit form $z = z(x, y)$, define the outward normal to the surface $\phi(x, y, z) = z - z(x, y) = 0$ by

$$\hat{\mathbf{e}}_n = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}}. \tag{7.93}$$

From equations (7.92) and (7.93), obtain the vector element of surface area

$$d\vec{S} = \hat{\mathbf{e}}_n\, dS = \left(-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right) dx\, dy.$$

Taking the dot product of both sides of the above equation with the unit vector $\hat{\mathbf{e}}_3$ gives

$$|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|\, dS = dx\, dy \qquad \text{or} \qquad dS = \frac{dx\, dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = \frac{dx\, dy}{\cos\gamma} \tag{7.94}$$

with an absolute value placed upon the dot product to ensure that the surface area is positive (i.e., recall that there are two normals to the surface which differ in sign). In equation (7.94), the element of surface area has been expressed in terms of its projection onto the xy plane. The angle $\gamma = \gamma(x, y)$ is the angle between the outward normal to the surface and the unit vector $\hat{\mathbf{e}}_3$. This representation of the element of surface area is valid provided that $\cos\gamma \neq 0$; That is, it is assumed that the surface is such that the normal to the surface is nowhere parallel to the xy plane.

We have previously shown that for surfaces which have a normal parallel to the xy plane, the element of surface area can be projected onto either of the planes $x = 0$ or $y = 0$. If the surface element is projected onto the plane $x = 0$, then the element of surface area takes the form

$$dS = \frac{dy\, dz}{|\,\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n|} \tag{7.95}$$

and if projected onto the plane $y = 0$ it has the form

$$dS = \frac{dx\, dz}{|\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n|}. \tag{7.96}$$

If the element of surface area dS is projected onto the $z = 0$ plane, the total surface area is then

$$S = \iint_R dS = \iint_R \frac{dx\, dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|},$$

where the integration extends over the region R, where the surface is projected onto the $z = 0$ plane. Similar integrals result for the other representations of surface area.

Example 7-28. Find the surface area of that part of the plane

$$\phi(x,y,z) = 2x + 2y + z - 12 = 0$$

which lies in the first octant.

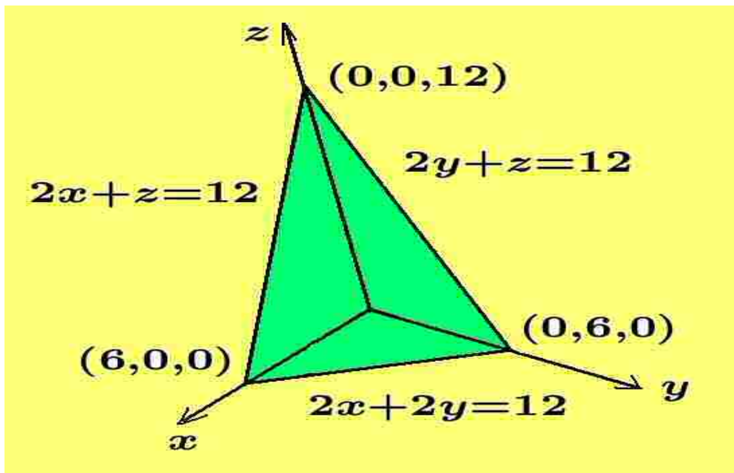


Figure 7-20. Surface area of plane in first octant.

Solution The given plane is sketched as in figure 7-20.

The unit normal to the plane is

$$\hat{\mathbf{e}}_n = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3.$$

The projection of the surface element dS onto the $z = 0$ plane produces

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = 3dx\,dy$$

By summing $dx\,dy$ over the region where $x > 0,\ y > 0$, and $x + y \le 6$, one obtains the limits of integration for the surface area. The surface area is determined from the integral

$$S = \int_{x=0}^{x=6} \int_{y=0}^{y=6-x} 3dx\,dy = \int_0^6 3(6-x)\,dx = -\frac{3}{2}(6-x)^2 \Big|_0^6 = 54.$$

If the element of surface area is projected onto the plane $y = 0$, there results

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n|} = \frac{3}{2}dx\,dz$$

and the limits of summation are determined as dx and dz range over the region $x > 0,\ z > 0$, and $2x + z \le 12$. This produces the surface integral

$$S = \int_{x=0}^{x=6} \int_{z=0}^{z=12-2x} \frac{3}{2}\, dx\, dz = \int_0^6 \frac{3}{2}(2)(6-x)\, dx = 54.$$

Similarly, if the element dS is projected onto the plane $x = 0$, it can be verified that

$$dS = \frac{dy\, dz}{|\,\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n|} = \frac{3}{2} dydz$$

and the surface area is given by

$$S = \int_{y=0}^{y=6} \int_{z=0}^{12-2y} \frac{3}{2} dz\, dy = 54.$$

■

Element of Volume

In a general (u, v, w) curvilinear coordinate system the (x, y, z) rectangular coordinates of a point are given as functions of (u, v, w) and written

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w)$$

so that the position vector to a point P can be written

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\, \hat{\mathbf{e}}_1 + y(u, v, w)\, \hat{\mathbf{e}}_2 + z(u, v, w)\, \hat{\mathbf{e}}_3$$

The vector $\frac{\partial \vec{r}}{\partial u}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u, v_0, w_0)$, the vector $\frac{\partial \vec{r}}{\partial v}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u_0, v, w_0)$ and the vector $\frac{\partial \vec{r}}{\partial w}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u_0, v_0, w)$. Unit vectors to the coordinates curves are

$$\hat{\mathbf{e}}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{|\frac{\partial \vec{r}}{\partial u}|}, \qquad \hat{\mathbf{e}}_v = \frac{\frac{\partial \vec{r}}{\partial v}}{|\frac{\partial \vec{r}}{\partial v}|}, \qquad \hat{\mathbf{e}}_w = \frac{\frac{\partial \vec{r}}{\partial w}}{|\frac{\partial \vec{r}}{\partial w}}$$

The magnitudes h_u, h_v, h_w defined by

$$h_u = |\frac{\partial \vec{r}}{\partial u}|, \qquad h_v = |\frac{\partial \vec{r}}{\partial v}|, \qquad h_w = |\frac{\partial \vec{r}}{\partial w}|$$

are called scaled factors. The vector change

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw = h_u\, du\ \hat{\mathbf{e}}_1 + h_v\, dv\ \hat{\mathbf{e}}_2 + h_w\, dw\ \hat{\mathbf{e}}_3$$

can be thought of as defining **an element of volume** dV in the shape of a parallelepiped with vector sides $\vec{A} = h_u\,du\;\hat{\mathbf{e}}_1,\ \vec{B} = h_v\,dv\;\hat{\mathbf{e}}_2$ and $\vec{C} = h_w\,dw\;\hat{\mathbf{e}}_3$. The volume of this parallelepiped is given by

$$dV = |\vec{A}\cdot(\vec{B}\times\vec{C})| = |(h_u\,du\;\hat{\mathbf{e}}_1)\cdot((h_v\,dv\;\hat{\mathbf{e}}_2)\times(h_w\,dw\;\hat{\mathbf{e}}_3)| = h_u h_v h_w\,du\,dv\,dw$$

In rectangular coordinates (x, y, z) one finds $h_x = 1,\ h_y = 1$, and $h_z = 1$ and the element of volume is $dV = dxdydz$.

In cylindrical coordinates (r, θ, z) where $x = r\cos\theta, y = r\sin\theta, z = z$ one finds $h_r = 1,\ h_\theta = r$ and $h_z = 1$ and the element of volume is $dV = r\,dr\,d\theta\,dz$

In spherical coordinates (ρ, θ, ϕ) where $x = \rho\sin\theta\cos\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\theta$, one finds $h_\rho = 1,\ h_\theta = \rho,\ h_\phi = \rho\sin\theta$ and the element of volume is $dV = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi$

These elements of volume must be summed over appropriate regions of space in order to calculate volume integrals of the form

$$\iiint f(x,y,z)\,dxdydz, \qquad \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz, \qquad \iiint f(\rho,\theta,\phi)\,\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$$

Surface Placed in a Scalar Field

If a surface is placed in a region of a scalar field $f(x, y, z)$, one can divide the surface into n small areas

$$\Delta S_1,\ \Delta S_2, \ldots, \Delta S_n.$$

For n large, define $f_i = f_i(x_i, y_i, z_i)$ as the value of the scalar field over the surface element $\vec{\Delta} S_i$ as i ranges from 1 to n. The summation of the elements $f_i\Delta\vec{S}_i$ over all i as n increases without bound defines the surface integral

$$\iint\limits_R f(x,y,z)\,\hat{\mathbf{e}}_n dS = \iint\limits_R f(x,y,z)d\vec{S} = \lim_{n\to\infty}\sum_{i=1}^{n} f_i(x_i,y_i,z_i)\Delta\vec{S}_i, \tag{7.97}$$

where the integration is determined by the way one represents the element of surface area $d\vec{S}$. The integral can be represented in different forms depending upon how the given surface is specified.

Surface Placed in a Vector Field

For a surface S in a region of a vector field $\vec{F} = \vec{F}(x, y, z)$ the integral

$$\iint\limits_R \vec{F}\cdot d\vec{S} = \iint\limits_R \vec{F}\cdot\,\hat{\mathbf{e}}_n dS \tag{7.98}$$

represents a scalar which is the sum of the projections of $\vec{F}$ onto the normals to the surface elements. If the surface is divided into n small surface elements ΔS_i, where $i = 1, \ldots, n$. Let $\vec{F}_i = \vec{F}(x_i, y_i, z_i)$ represent the value of the vector field over the ith surface element. The summation of the elements

$$\vec{F}_i \cdot \Delta\vec{S}_i = \vec{F}_i \cdot \hat{\mathbf{e}}_{n_i} \Delta S_i$$

over all surface elements represents the sum of the normal components of $\vec{F}_i$ multiplied by ΔS_i as i varies from 1 to n. A summation gives the surface integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{S}_i = \iint_R \vec{F} \cdot d\vec{S}. \tag{7.99}$$

Again, the form of this integral depends upon how the given surface is represented. Integrals of this type arise when calculating the volume rate of change associated with velocity fields. It is called a flux integral and represents the amount of a substance moving across an imaginary surface placed within the vector field.

The vector integral

$$\iint_R \vec{F} \times d\vec{S}$$

represents a vector which is obtained by summing the vector elements $\vec{F}_i \times \Delta\vec{S}_i$ over the given surface. The fundamental theorem of integral calculus enables such sums to be expressed as integrals and one can write

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_i \times \Delta\vec{S}_i = \iint_R \vec{F} \times d\vec{S}. \tag{7.100}$$

Integrals of this type arise as special cases of some integral theorems that are developed in the next chapter.

Each of the above surface integrals can be represented in different forms depending upon how the element of surface area is represented. The form in which the given surface is represented usually dictates the method used to calculate the surface area element. Sometimes the representation of a surface in a different form is helpful in determining the limits of integration to certain surface integrals.

Example 7-29. Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where S is the surface of the cube bounded by the planes

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad z = 0, \quad z = 1$$

and $\vec{F}$ is the vector field $\vec{F} = (x^2 + z)\,\hat{\mathbf{e}}_1 + (xy - z)\,\hat{\mathbf{e}}_2 + (x + y)\,\hat{\mathbf{e}}_3$

Solution The given surface is illustrated in figure 7-21.

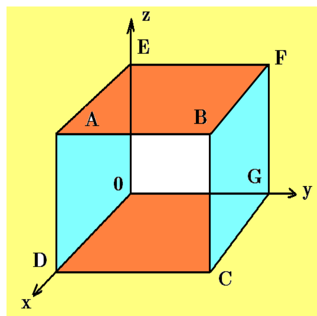


Figure 7-21. Surface of a cube.

The given surface is piecewise continuous and thus the surface integral can be broken up and written as the sum of the surface integrals over each face of the cube. The following calculations illustrates the mechanics involved in evaluating this type of surface integral.

(i) On face ABCD the unit normal to the surface is the vector $\vec{n} = \hat{\mathbf{e}}_1$ and x has the value 1 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 (1 + z)\, dydz = \int_0^1 (1 + z)\, dz = \frac{3}{2}$$

(ii) On face EFG0 the unit normal to the surface is the vector $\vec{n} = -\hat{\mathbf{e}}_1$ and x has the value 0 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 -z\, dydz = \int_0^1 -z\, dz = -\frac{1}{2}$$

(iii) On face BFGC the unit normal to the surface is the vector $\vec{n} = \hat{\mathbf{e}}_2$ and y has the value 1 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 (x - z)\, dxdz = \int_0^1 \frac{1}{2}\, dz - \int_0^1 z\, dz = 0$$

(iv) On face AE0D the unit normal to the surface is the vector $\vec{n} = -\hat{\mathbf{e}}_2$ and y has the value 0 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 z\, dzdx = \int_0^1 z\, dz = \frac{1}{2}$$

(v) On face ABFE the unit normal to the surface is the vector $\vec{n} = \hat{\mathbf{e}}_3$and z has the value 1 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 (x + y)\, dxdy = \int_0^1 \frac{1}{2}\, dy + \int_0^1 y\, dy = 1$$

(vi) On face DCG0 the unit normal to the surface is the vector $\vec{n} = -\hat{\mathbf{e}}_3$ and z has the value 0 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 -(x + y)\, dxdy = -1$$

A summation of the surface integrals over each face gives

$$\iint_R \vec{F} \cdot d\vec{S} = \frac{3}{2} - \frac{1}{2} + 0 + \frac{1}{2} + 1 - 1 = \frac{3}{2}.$$

■

Example 7-30. Evaluate the surface integral $\iint_R f(x, y, z)\, d\vec{S}$, where S is the surface of the plane

$$G(x, y, z) = 2x + 2y + z - 1 = 0$$

which lies in the first octant and $f = f(x, y, z)$ is the scalar field given by $f = xyz$.

Solution The given surface is sketched in figure 7-22. The unit normal at any point on the surface is

$$\hat{\mathbf{e}}_n = \frac{\text{grad } G}{|\text{grad } G|} = \frac{2}{3}\, \hat{\mathbf{e}}_1 + \frac{2}{3}\, \hat{\mathbf{e}}_2 + \frac{1}{3}\, \hat{\mathbf{e}}_3.$$

The element of surface area dS is projected upon the xy plane giving

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = 3\,dx\,dy$$

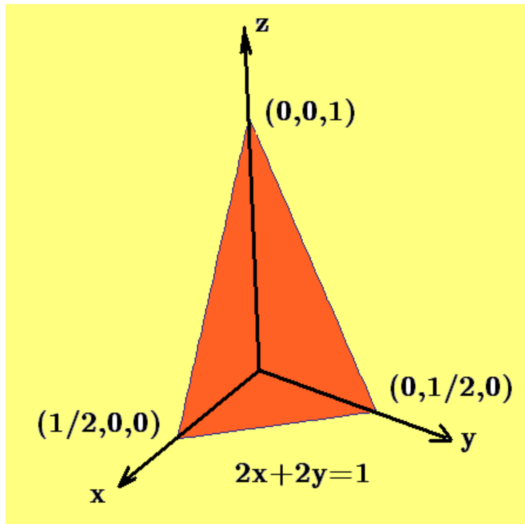


Figure 7-22. Plane $2x + 2y + z = 1$ in first octant.

On the surface $z = 1-2x-2y$, and therefore the surface integral can be represented in terms of only x and y. One finds

$$\begin{aligned}
\iint_R f(x,y,z)\,d\vec{S} &= \iint_R xyz\ \hat{\mathbf{e}}_n dS \\
&= \int_{x=0}^{x=\frac{1}{2}} \int_{y=0}^{y=\frac{1}{2}-x} xy(1-2x-2y)\left[\frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3\right] 3dx\,dy \\
&= (2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (xy - 2x^2y - 2xy^2)\,dx\,dy
\end{aligned}$$

Integrate with respect to y and show

$$\iint_R f(x,y,z)\,d\vec{S} = (2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3) \int_0^{\frac{1}{2}} \left[\frac{1}{2}x(\frac{1}{2}-x)^2 - x^2(\frac{1}{2}-x)^2 - \frac{2}{3}x(\frac{1}{2}-x)^3\right] dx$$

Now integrate with respect to x and simplify the result to obtain

$$\iint_R f(x,y,z)\,d\vec{S} = \frac{1}{1920}\,(2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3)$$

■

Example 7-31. Evaluate the surface integral $\iint\limits_R \vec{F}\times d\vec{S}$, where S is the plane $2x+2y+z-1=0$ in the first octant and $\vec{F}=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$.

Solution Here

$$\iint\limits_R \vec{F}\times d\vec{S}=\iint\limits_R \vec{F}\times\,\hat{\mathbf{e}}_n\,dS$$

and from the previous example

$$\hat{\mathbf{e}}_n=\frac{2}{3}\,\hat{\mathbf{e}}_1+\frac{2}{3}\,\hat{\mathbf{e}}_2+\frac{1}{3}\,\hat{\mathbf{e}}_3.$$

As in the previous example, the element of surface area is projected upon the xy plane to obtain $dS=3dx\,dy$. Therefore,

$$\vec{F}\times\,\hat{\mathbf{e}}_n=\begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ x & y & z\\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3}\end{vmatrix}=\frac{1}{3}(y-2z)\,\hat{\mathbf{e}}_1-\frac{1}{3}(x-2z)\,\hat{\mathbf{e}}_2+\frac{2}{3}(x-y)\,\hat{\mathbf{e}}_3$$

and the surface integral is

$$\iint\limits_R \vec{F}\times d\vec{S}=\,\hat{\mathbf{e}}_1\iint\limits_R(y-2z)\,dx\,dy-\,\hat{\mathbf{e}}_2\iint\limits_R(x-2z)\,dx\,dy+2\,\hat{\mathbf{e}}_3\iint\limits_R(x-y)\,dx\,dy.$$

Here the element of surface area has been projected upon the xy plane and all integrations are with respect to x and y. Consequently, one must express z in terms of x and y. From the equation of the plane, the value of z on the surface is given by $z=1-2x-2y$ and the surface integral becomes

$$\begin{aligned}\iint\limits_R \vec{F}\times d\vec{S}=&\;\;\hat{\mathbf{e}}_1\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-x}(5y+4x-2)\,dy\,dx\\ &-\,\hat{\mathbf{e}}_2\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-x}(5x+4y-2)\,dy\,dx\\ &+2\,\hat{\mathbf{e}}_3\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-x}(x-y)\,dy\,dx.\end{aligned}$$

These integrals are easily evaluated and the final result is

$$\iint\limits_R \vec{F}\times d\vec{S}=-\frac{1}{16}\,\hat{\mathbf{e}}_1+\frac{1}{16}\,\hat{\mathbf{e}}_2+\frac{1}{48}\,\hat{\mathbf{e}}_3.$$

■

Summary

When a surface is represented in parametric form, the position vector of a point on the surface can be represented as

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3$$

where

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v)$$

is the parametric representation of the surface. The differential of the position vector $\vec{r} = \vec{r}(u,v)$ is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv = \vec{S}_1 + \vec{S}_2 \tag{7.101}$$

and this differential can be thought of as a vector addition of the component vectors $\vec{S}_1 = \frac{\partial \vec{r}}{\partial u}\,du$ and $\vec{S}_2 = \frac{\partial \vec{r}}{\partial v}\,dv$ which make up the sides on an elemental parallelogram having area dS lying on the surface. The vectors $\vec{S}_1$ and $\vec{S}_2$ are tangent vectors to the coordinate curves $\vec{r}(u, v_2)$ and $\vec{r}(u_1, v)$ where u_1 and v_2 are constants. A representation of coordinate curves on a surface and an element of surface area are illustrated in the figure 7-23.

The unit normal to the surface at a point having the surface coordinates (u,v), can be found from either of the cross product relations

$$\hat{\mathbf{e}}_n = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|} \quad \text{or} \quad \hat{\mathbf{e}}_n = \mp \frac{\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{u}}{\partial v}\right|} \tag{7.102}$$

The above results differing in sign. That is, $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$ are both normals to the surface and selecting one of these vectors gives an orientation to the surface.

Example 7-32. Find the unit normal to the sphere defined by

$$x = r\cos\phi\sin\theta,\;\; y = r\sin\phi\sin\theta,\;\; z = r\cos\theta$$

Solution Here

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \theta} =& r\cos\phi\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\phi\cos\theta\,\hat{\mathbf{e}}_2 - r\sin\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \phi} =& - r\sin\phi\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\phi\sin\theta\,\hat{\mathbf{e}}_2\end{aligned}$$

and the cross product is

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \end{vmatrix} = \hat{\mathbf{e}}_1(r^2\sin^2\theta\cos\phi) + \hat{\mathbf{e}}_2(r^2\sin^2\theta\sin\phi) + \hat{\mathbf{e}}_3(r^2\sin\theta\cos\theta)$$

One finds $\left|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\phi}\right| = r^2\sin\theta$ so that a unit vector to the surface of the sphere is

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

■

For $\vec{r}$ a position vector to a point on the surface, the element $d\vec{r}$ lies in the tangent plane to the surface at the point determined by the parameters u and v. The element of surface area dS is also determined from the differential element $d\vec{r}$ and is given by the magnitude of the cross products of the vectors $\vec{S}_1$ and $\vec{S}_2$ representing the sides of the elemental parallelogram which defines the element of surface area. This element of surface area is calculated from the cross product

$$dS = \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|\,dudv$$

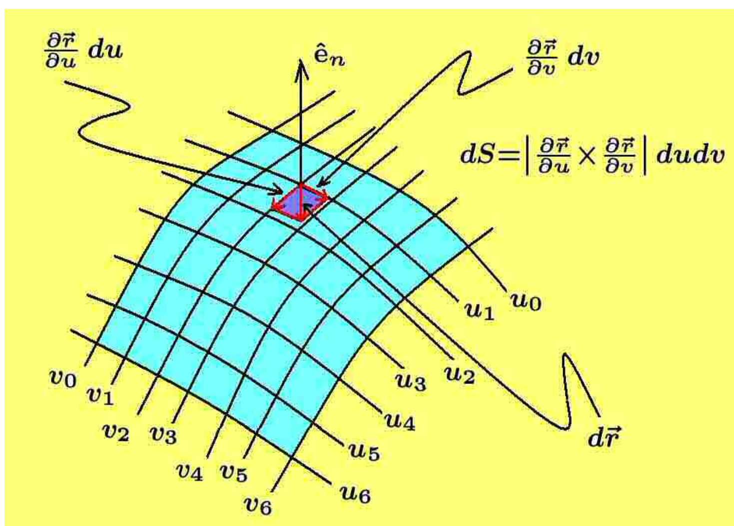


Figure 7-23.
Coordinate curves on surface and elemental parallelogram.

Using the dot product relation

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

one can readily verify that

$$\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| = \sqrt{EG-F^2}, \tag{7.103}$$

where

$$E = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$
$$F = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$
$$G = \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Then the surface area can be represented in the form

$$S = \int_{R_{uv}} \sqrt{EG - F^2}\, du\, dv, \tag{7.104}$$

where the integration is over those parameter values u and v which define the surface.

The various surface integrals can also be represented in terms of the parameters u and v. These integrals have the forms

$$\iint_R f(x,y,z)\, d\vec{S} = \iint_{R_{uv}} f(x(u,v), y(u,v), z(u,v)) \sqrt{EG - F^2}\, \hat{\mathbf{e}}_n\, du\, dv$$

and

$$\iint_R \vec{F}(x,y,z) \cdot d\vec{S} = \iint_{R_{uv}} \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \hat{\mathbf{e}}_n \sqrt{EG - F^2}\, du\, dv. \tag{7.105}$$

Example 7-33. A cylinder of radius a and height h has the parametric representation $x = x(\theta, z) = a\cos\theta$, $y = y(\theta, z) = a\sin\theta$, $z = z(\theta, z) = z$, where the parameters θ and z, are illustrated in figure 7-24, and satisfy $0 \le \theta \le 2\pi$ and $0 \le z \le h$.

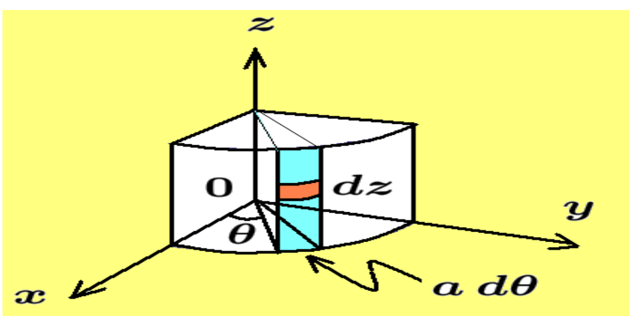


Figure 7-24.
Surface area in cylindrical coordinates.

A point on the surface of the cylinder can be represented by the position vector

$$\vec{r} = \vec{r}(\theta, z) = a\cos\theta\, \hat{\mathbf{e}}_1 + a\sin\theta\, \hat{\mathbf{e}}_2 + z\, \hat{\mathbf{e}}_3.$$

The coordinate curves are

$$\text{The straight-lines,} \quad \vec{r}(\theta_0, z),\ 0 \le z \le h$$
$$\text{and the circles,} \quad \vec{r}(\theta, z_0),\ 0 \le \theta \le 2\pi,$$

where θ_0 and z_0 are constants. The tangent vectors to the coordinate curves are given by

$$\frac{\partial \vec{r}}{\partial \theta} = -a\sin\theta\,\hat{\mathbf{e}}_1 + a\cos\theta\,\hat{\mathbf{e}}_2 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$$

Consequently, we have $E = a^2$, $F = 0$, and $G = 1$. The element of surface area is then $dS = \sqrt{EG - F^2}\,d\theta\,dz = a\,d\theta\,dz$. The surface area of the cylinder of height h is therefore

$$S = \int_0^h \int_0^{2\pi} a\,d\theta\,dz = 2\pi a h.$$

■

Volume Integrals

The summation of scalar and vector fields over a region of space can be expressed by volume integrals having the form

$$\iiint_V f(x,y,z)\,dV \qquad \text{and} \qquad \iiint_V \vec{F}(x,y,z)\,dV,$$

where $dV = dx\,dy\,dz$ is an element of volume and V is the region over which the integrations are to extend.

The integral of the scalar field is an ordinary triple integral. The triple integral of the vector function $\vec{F} = \vec{F}(x,y,z)$ can be expressed as

$$\iiint_V \vec{F}\,dV = \hat{\mathbf{e}}_1 \iiint_V F_1(x,y,z)\,dV + \hat{\mathbf{e}}_2 \iiint_V F_2(x,y,z)\,dV + \hat{\mathbf{e}}_3 \iiint_V F_3(x,y,z)\,dV, \tag{7.106}$$

where each component is a scalar triple integral.

Whenever appropriate, the above integrals are sometimes expressed

(i) in cylindrical coordinates (r,θ,z), where $x = r\cos\theta, y = r\sin\theta, z = z$ and the element of volume is represented $dV = r\,dr\,d\theta\,dz$

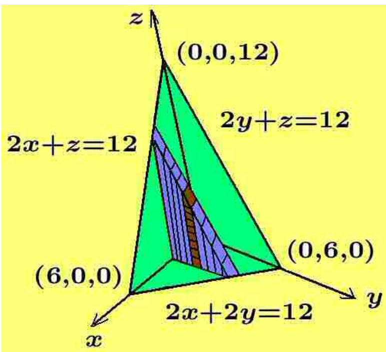
(ii) in spherical coordinates (ρ,θ,ϕ) where $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\theta$ and the element of volume is $dV = \rho^2\sin\theta\,d\rho\,d\phi\,d\theta$.

(iii) in curvilinear coordinates (u,v,w) where $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$ and the element of volume is given by[8] $dV = \left|\frac{\partial \vec{r}}{\partial u} \cdot \left(\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w}\right)\right| du\,dv\,dw$ where $\vec{r} = x(u,v,w)\,\hat{\mathbf{e}}_1 + y(u,v,w)\,\hat{\mathbf{e}}_2 + z(u,v,w)\,\hat{\mathbf{e}}_3$

[8] See pages 143 and 156 for details.

Example 7-34. Evaluate the integral $\iiint_V f(x,y,z)\,dV$, where $f(x,y,z) = 6(x+y)$, $dV = dxdydz$ is an element of volume and V represents the volume enclosed by the planes $2x+2y+z=12$, $x=0$, $y=0$, and $z=0$. The integration is to be performed over this volume.

Solution



The figure on the left is a copy of the figure 7-20 with summations of the element dV illustrated. From this figure the limits of integration can be determined. The volume element is $dV = dxdydz$ is placed at the general point (x,y,z) within the volume. This volume element can be visualized as a cube inside the volume. Summation of these cubic elements aids in determining the limits of integration for the integral to be calculated. If this cube is summed in the z-direction, a parallelepiped is produced. This parallelepiped has lower limit $z=0$ and upper limit $z = 12-2x-2y$. If the parallelepiped is summed in the y-direction, then a triangular slab is formed with lower limit $y=0$ and upper limit $y=6-x$. Summing the triangular slabs in the x-direction from $x=0$ to $x=6$ gives the limits of integration in the x-direction. At each stage of the summation process, the volume element is weighted by the scalar function $f(x,y,z)$ giving the integral $\iiint_V f(x,y,z)\,dV$. From all this summation one can verify the above integral can be expressed.

$$
\begin{aligned}
\iiint_V f(x,y,z)\,dV &= \int_{x=0}^{x=6}\int_{y=0}^{y=6-x}\int_{z=0}^{z=12-2x-2y} 6(x+y)\,dz\,dy\,dx \\
&= \int_0^6 \left[\int_0^{6-x} 6(x+y)(12-2x-2y)\,dy\right] dx \\
&= \int_0^6 (432 - 36x^2 + 4x^3)\,dx = 1296
\end{aligned}
$$

This integral is calculated by first integrating in the z-direction holding the other variables constant. This is followed by an integration in the y-direction holding x constant. The last integration is then in the x-direction.

■

Example 7-35. Evaluate the integral $\iiint\limits_V \vec{F}(x,y,z)\,dV$, where

$$\vec{F} = x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

and $dV = dxdydz$ is a volume element. The limits of integration are determined from the volume bounded by the surfaces $y = x^2$, $y = 4$, $z = 0$, and $z = 4$.

Solution From figure 7-25 the limits of integration can be determined by sketching an element of volume $dV = dx\,dy,\,dz$ and then summing these elements in the $x-$direction from $x = 0$ to $x = \sqrt{y}$ to form a parallelepiped. Next sum the parallelepiped in the $z-$direction from $z = 0$ to $z = 4$ to form a slab. Finally, the slab can be summed in $y-$direction from $y = 0$ to $y = 4$ to fill up the volume.

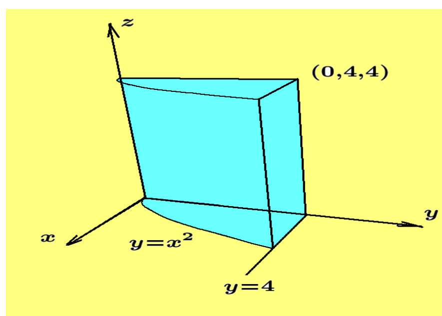


Figure 7-25.
Volume bounded by $y = x^2$, and the planes $y = 4$, $z = 0$ and $z = 4$

One then has

$$\iiint\limits_V \vec{F}\cdot dV = \int_{y=0}^{y=4}\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{y}} (x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)\,dx\,dz\,dy$$
$$= \int_0^4\int_0^4\int_0^{\sqrt{y}} [x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3]\,dxdzdy$$

Perform the integrations over each vector component and show that

$$\iint\limits_V \vec{F}\cdot dV = 16\,\hat{\mathbf{e}}_1 + \frac{128}{3}\,\hat{\mathbf{e}}_2 + \frac{64}{3}\,\hat{\mathbf{e}}_3$$

■

Volume Elements Revisited

Consider **the volume element** $dV = dxdydz$ from cartesian coordinates and introduce a change of variables

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

from an x, y, z rectangular coordinate system to a u, v, w curvilinear coordinate system. One finds the vector

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\,\hat{\mathbf{e}}_1 + y(u, v, w)\,\hat{\mathbf{e}}_2 + z(u, v, w)\,\hat{\mathbf{e}}_3$$

is the position vector of a general point within a region determined by the restrictions placed upon the u, v, w variables. The surfaces

$$\vec{r}(u, v, w_0), \quad \vec{r}(u, v_0, w), \quad \vec{r}(u_0, v, w)$$

are called **coordinates surfaces** and the curves

$$\vec{r}(u_0, v_0, w), \quad \vec{r}(u_0, v, w_0), \quad \vec{r}(u, v_0, w_0)$$

are called **coordinate curves.** The coordinate curves represent intersections of the coordinate surfaces. The partial derivatives

$$\frac{\partial \vec{r}}{\partial u}, \qquad \frac{\partial \vec{r}}{\partial v}, \qquad \frac{\partial \vec{r}}{\partial w}$$

represent tangent vectors to the coordinate curves and the quantities

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

are called scale factors associated with the tangents to the coordinate curves. These scale factors are used to calculate unit vectors

$$\hat{\mathbf{e}}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u}, \qquad \hat{\mathbf{e}}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}, \qquad \hat{\mathbf{e}}_w = \frac{1}{h_w}\frac{\partial \vec{r}}{\partial w}$$

to the coordinate curves. If these unit vectors are all perpendicular to one another the coordinate system is called **an orthogonal coordinate system.** The differential

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv + \frac{\partial \vec{r}}{\partial w}\,dw$$

represents a small change in $\vec{r}$. One can think of the differential $d\vec{r}$ as the diagonal of a parallelepiped having the vector sides

$$\frac{\partial \vec{r}}{\partial u}\,du, \qquad \frac{\partial \vec{r}}{\partial v}\,dv, \quad \text{and} \quad \frac{\partial \vec{r}}{\partial w}\,dw.$$

The volume of this parallelepiped produces the volume element dV of the curvilinear coordinate system and this volume element is given by the formula

$$dV = \left| \frac{\partial \vec{r}}{\partial u} \cdot \left(\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w} \right) \right| du\,dv\,dw.$$

This result can be expressed in the alternate form

$$dV = \left| J\begin{pmatrix} x,y,z \\ u,v,w \end{pmatrix} \right| du\,dv\,dw,$$

where one can make use of the property of representing scalar triple products in terms of determinants to obtain

$$J\begin{pmatrix} x,y,z \\ u,v,w \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The quantity $J\begin{pmatrix} x,y,z \\ u,v,w \end{pmatrix}$ is called **the Jacobian of the transformation** from x,y,z coordinates to u,v,w coordinates. The absolute value signs are to insure the element of volume is positive.

As an example, the volume element $dV = dx\,dy\,dz$ under the change of variable to **cylindrical coordinates** (r,θ,z), with coordinate transformation

$$x = x(r,\theta,z) = r\cos\theta, \quad y = y(r,\theta,z) = r\sin\theta, \quad z = z(r,\theta,z) = z$$

has the Jacobian determinant $\left| J\begin{pmatrix} x,y,z \\ r,\theta,z \end{pmatrix} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$ which gives the new volume element $dV = r\,dr\,d\theta\,dz$.

As another example, the volume element $dV = dx\,dy\,dz$ under the change of variable to **spherical coordinates** (ρ,θ,ϕ), where

$$x = x(\rho,\theta,\phi) = \rho\sin\theta\cos\phi, \quad y = y(\rho,\theta,\phi) = \rho\sin\theta\sin\phi, \quad z = z(\rho,\theta,\phi) = \rho\cos\theta$$

one finds the Jacobian $\left| J\begin{pmatrix} x,y,z \\ \rho,\theta,\phi \end{pmatrix} \right| = \begin{vmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix} = \rho^2\sin\theta$ giving the new volume element $dV = \rho^2 sin\theta\,d\rho\,d\phi\,d\theta$.
Verification of the above results is left as an exercise.

Cylindrical Coordinates (r, θ, z)

The transformation from rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z) is given by

$$x = x(r, \theta, z) = r\cos\theta, \qquad y = y(r, \theta, z) = r\sin\theta, \qquad z = z(r, \theta, z) = z$$

so that a general position vector is given by $\vec{r} = \vec{r}(r, \theta, z) = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ In cylindrical coordinates the coordinate surfaces are

$$\begin{aligned}
\vec{r}(r_0, \theta, z) =& r_0\cos\theta\,\hat{\mathbf{e}}_1 + r_0\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \quad &\text{a cylinder}\\
\vec{r}(r, \theta_0, z) =& r\cos\theta_0\,\hat{\mathbf{e}}_1 + r\sin\theta_0\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \quad &\text{a plane perpendicular to } z\text{−axis}\\
\vec{r}(r, \theta, z_0) =& r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3 \quad &\text{a plane through the } z\text{−axis}
\end{aligned}$$

These surfaces are illustrated in the figure 7-26.

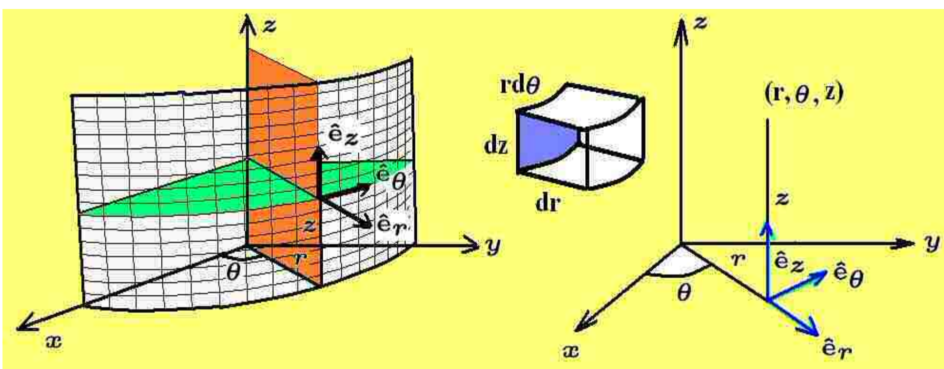


Figure 7-26.
Coordinate surfaces and coordinate curves for cylindrical coordinates (r, θ, z).

The coordinate curves are

$$\begin{aligned}
\vec{r}(r_0, \theta_0, z), &\quad \text{lines perpendicular to plane } z = 0\\
\vec{r}(\theta_0, z_0), &\quad \text{lines emanating from the origin}\\
\vec{r}(r_0, \theta, z_0), &\quad \text{circles of radius } r_0 \text{ in the plane } z = z_0
\end{aligned}$$

The vectors $\dfrac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$, $\dfrac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2$, $\dfrac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$ are tangent vectors to the coordinate curves and the vectors

$$\hat{\mathbf{e}}_r = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2, \qquad \hat{\mathbf{e}}_\theta = \frac{1}{r}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2, \qquad \hat{\mathbf{e}}_z = \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3 \quad (7.107)$$

are unit vectors tangent to the coordinate curves, where $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$, $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_z = 0$ and $\hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_z = 0$. The unit vector $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta$ produces the triad system $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$ so that the cylindrical coordinate system is a right-handed orthogonal coordinate system. The unit vectors are sometimes expressed in the matrix[9] form

$$\hat{\mathbf{e}}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In the cylindrical coordinate system the element of volume is given by $dV = r\,dr\,d\theta\,dz$ and the element of surface area is $dS = r\,d\theta\,dz$. The direction $\hat{\mathbf{e}}_r$ is called the radial direction, the direction $\hat{\mathbf{e}}_\theta$ is called the azimuthal direction and the direction $\hat{\mathbf{e}}_z$ is called the vertical direction.

Spherical Coordinates (ρ, θ, ϕ)

The transformation from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ) is given by the equations

$$x = x(\rho,\theta,\phi) = \rho\sin\theta\cos\phi, \qquad y = y(\rho,\theta,\phi) = \rho\sin\theta\sin\phi, \qquad z = z(\rho,\theta,\phi) = \rho\cos\theta$$

and the general position vector is given by

$$\vec{r} = \vec{r}(\rho,\theta,\phi) = \rho\sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \rho\sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \rho\cos\theta\,\hat{\mathbf{e}}_3$$

In this coordinate system the coordinate surfaces are

$$\begin{aligned} &\vec{r}(\rho_0,\theta,\phi), && \text{a sphere} \quad x^2+y^2+z^2=\rho_0^2 \\ &\vec{r}(\rho,\theta_0,\phi), && \text{a cone} \quad x^2+y^2=\tan^2\theta z^2 \\ &\vec{r}(\rho,\theta,\phi_0), && \text{a plane through the } z\text{−axis } y = x\tan\phi \end{aligned}$$

The coordinate curves in spherical coordinates are obtained from the intersection of the coordinate surfaces and can be represented by

$$\begin{aligned} &\vec{r}(\rho_0,\theta_0,\phi), && \text{circles of latitude} \\ &\vec{r}(\rho_0,\theta,\phi_0), && \text{meridian curve} \\ &\vec{r}(\rho,\theta_0,\phi_0), && \text{lines through the origin} \end{aligned}$$

These coordinate surfaces and coordinate lines are illustrated in the figure 7-27.

[9] See chapter 10 for a discussion of the matrix calculus.

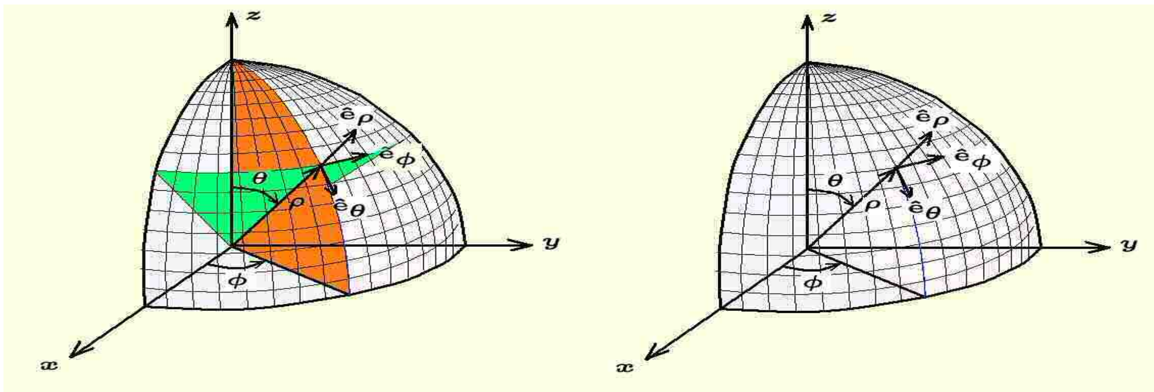


Figure 7-27.
Coordinate surfaces and coordinate curves for cylindrical coordinates.

The partial derivative vectors

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \rho} =& \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \theta} =& \rho\cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \rho\cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \rho\sin\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \phi} =& -\rho\sin\theta\sin\phi\,\hat{\mathbf{e}}_1 + \rho\sin\theta\cos\phi\,\hat{\mathbf{e}}_2\end{aligned}$$

are tangent vectors to the coordinate curves and the scaled vectors

$$\begin{aligned}\hat{\mathbf{e}}_\rho =& \frac{\partial \vec{r}}{\partial \rho} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_\theta =& \frac{1}{\rho}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_\phi =& \frac{1}{\rho\sin\theta} = -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2\end{aligned} \tag{7.108}$$

are unit vectors tangent to the coordinate curves. The spherical coordinate system is a right-handed orthogonal coordinate system because

$$\hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\theta = 0, \qquad \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\phi = 0, \qquad \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\phi = 0, \qquad \hat{\mathbf{e}}_\rho \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\phi$$

The above unit vectors are sometimes expressed in the matrix form[10] as the column vectors.

$$\hat{\mathbf{e}}_\rho = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \qquad \hat{\mathbf{e}}_\theta = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}, \qquad \hat{\mathbf{e}}_\phi = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

[10] See chapter 10 for a discussion of matrices.

The element of volume in spherical coordinates is given by $dV = \rho^2 \sin\theta\, d\theta\, d\phi\, d\rho$ and the element of surface area is $dS = \rho^2 \sin\theta\, d\theta\, d\phi$, with ρ constant. The direction $\hat{\mathbf{e}}_\rho$ is called the radial direction, the vector $\hat{\mathbf{e}}_\theta$ is called the polar direction[11] and the direction $\hat{\mathbf{e}}_\phi$ is called the azimuthal direction.

Example 7-36. For $\vec{F} = (x-z)\,\hat{\mathbf{e}}_1 + (y-x)\,\hat{\mathbf{e}}_2 + (z+x+y)\,\hat{\mathbf{e}}_3$, let S denote the surface enclosing the volume V bounded by the hemisphere $x^2+y^2+z^2=1,\ z \ge 0$, and the plane $z=0$. Calculate (i) $I_1 = \iiint_V \nabla \cdot \vec{F}\, dV$ (ii) $I_2 = \iint_S \vec{F} \cdot \hat{\mathbf{e}}_n\, dS$

Solution Show $\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = 3$ and use spherical coordinates with $dV = \rho^2 \sin\theta\, d\rho\, d\phi\, d\theta$, and show

$$I_1 = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{1} 3\rho^2 \sin\theta\, d\rho\, d\phi\, d\theta = 2\pi$$

Break the surface integral I_2 into an integration I_{upper} over the hemisphere and an integral I_{lower} surface integral over the plane $z=0$. On I_{upper} use $dS = \sin\theta\, d\theta\, d\phi$ and $\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ with $\vec{F} \cdot \hat{\mathbf{e}}_n = x^2+y^2+z^2 = 1$. One finds

$$I_{upper} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\, d\phi = 2\pi$$

On the plane $z=0$, use $dS = dxdy$ and $\hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_3$ with $\vec{F} \cdot \hat{\mathbf{e}}_n = -(z+x+y)\Big|_{z=0} = -(x+y)$ so that

$$I_{lower} = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -(x+y)\, dydx = 0$$

and consequently $I_2 = I_{upper} + I_{lower} = 2\pi$.

Example 7-37. Let S denote the surface of the hemisphere $x^2+y^2+z^2=1,\ z \ge 0$ and let C denote the curve $x^2+y^2=1$ lying on the surface S. Calculate the integrals

$$(i) \quad I_3 = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n\, dS \qquad (ii) \quad I_4 = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = y\,\hat{\mathbf{e}}_1 + (z^2+2x)\,\hat{\mathbf{e}}_2 + 2yz\,\hat{\mathbf{e}}_3$

Solution One finds $\operatorname{curl} \vec{F} = \hat{\mathbf{e}}_3$ and on the hemisphere $\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$, so that $\operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n = z$. Let $dS = \dfrac{dxdy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = \dfrac{dxdy}{z}$ and show

$$I_3 = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx = \pi$$

[11] The angle θ is called the polar angle or zenith angle and the angle ϕ is called the azimuthal angle.

To evaluate the line integral let

$$x = \cos\theta,\ \ y = \sin\theta \text{ with } dx = -\sin\theta\, d\theta,\ \ dy = \cos\theta\, d\theta$$

and show

$$I_4 = \int_C \vec{F}\cdot d\vec{r} = \int_C (y-z)\,dx + (2x-y)\,dy = \int_0^{2\pi} [-\sin^2\theta + 2\cos^2\theta - \sin\theta\cos\theta]\,d\theta = \pi$$

Note that $z=0$ on C.

■

Example 7-38. Evaluate the flux integral $I = \iint\limits_S \vec{F}\cdot d\vec{S}$ where the vector field is given by $\vec{F} = \vec{F}(x,y,z) = z\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + x\,\hat{\mathbf{e}}_3$ and S is the surface of the unit sphere $x^2+y^2+z^2=1$.

Solution Transform to spherical coordinates where the position vector to a point of the unit sphere is

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 = \sin\theta\cos\phi\ \hat{\mathbf{e}}_1 + \sin\theta\sin\phi\ \hat{\mathbf{e}}_2 + \cos\theta\ \hat{\mathbf{e}}_3$$

for $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. An element of surface area in spherical coordinates is $dS = \sin\theta\, d\theta d\phi$ and a unit normal $\hat{\mathbf{e}}_n$ to the surface of the unit sphere is in the same direction as the vector $\vec{r}$ above so that one can write

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\ \hat{\mathbf{e}}_1 + \sin\theta\sin\phi\ \hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

Substituting these values into the flux integral one obtains

$$I = \iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} [\cos\theta(\sin\theta\cos\phi) + (\sin\theta\sin\phi)(\sin\theta\sin\phi) + (\sin\theta\cos\phi)\cos\theta]\ \sin\theta\, d\theta d\phi$$

Evaluating the inner integral, holding ϕ constant gives

$$I = \int_0^{2\pi} \frac{4}{3}\sin^2\phi\, d\phi$$

which can be integrated to obtain the value $I = \dfrac{4\pi}{3}$.

■

Exercises

► **7-1.** Sketch the given surfaces (i) $\dfrac{y^2}{a^2}+\dfrac{z^2}{b^2}=\dfrac{x^2}{c^2}$ $\qquad (ii)$ $\dfrac{z^2}{a^2}+\dfrac{x^2}{b^2}=\dfrac{y^2}{c^2},\quad a>b>c$

► **7-2.** Sketch the given surfaces (i) $\dfrac{y^2}{a^2}+\dfrac{z^2}{b^2}=\dfrac{x}{c}$ $\qquad (ii)$ $\dfrac{z^2}{a^2}+\dfrac{x^2}{b^2}=\dfrac{y}{c},\ a>b>c$

► **7-3.** Sketch the given surfaces defined by the parametric equations

$$\begin{array}{llll}(i) & x-x_0=u, & y-y_0=v, & z-z_0=c\left(\dfrac{u^2}{a^2}+\dfrac{v^2}{b^2}\right)\\ (ii) & x-x_0=u, & y-y_0=v, & z-z_0=c\left(\dfrac{u^2}{a^2}-\dfrac{v^2}{b^2}\right)\end{array}$$

► **7-4.** The curve $\vec{r}=\vec{r}(t)=\alpha\cos\omega t\,\hat{\mathbf{e}}_1+\alpha\sin\omega t\,\hat{\mathbf{e}}_2+\beta t\,\hat{\mathbf{e}}_3$, where α,β and ω are constants, describes a circular helix of radius α. For this space curve calculate the following quantities.

(a) The unit tangent vector $\hat{\mathbf{e}}_t$

(b) The unit normal vector $\hat{\mathbf{e}}_n$

(c) The unit binormal vector $\hat{\mathbf{e}}_b$

(d) The curvature κ

(e) The torsion V

► **7-5.** If $\vec{r}=\vec{r}(t)$ denotes a space curve, show that the curvature is given by

$$\kappa=\frac{\sqrt{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'')-(\vec{r}\,'\cdot\vec{r}\,'')^2}}{(\vec{r}\,'\cdot\vec{r}\,')^{3/2}}$$

where $'=\frac{d}{dt}$ denotes differentiation with respect to the argument of the function.

► **7-6.** If $\vec{r}=\vec{r}(x)=x\,\hat{\mathbf{e}}_1+y(x)\,\hat{\mathbf{e}}_2$ is the position vector describing a curve in the $x,y-$plane, show that the curvature is given by

$$\kappa=\frac{|y''|}{(1+(y')^2)^{3/2}}$$

where $'=\frac{d}{dx}$ denotes differentiation with respect to the argument of the function.

► **7-7.**

(a) For $\vec{r}=\vec{r}(s)$ the position vector of a curve, show that $\dfrac{d\vec{r}}{ds}\times\dfrac{d^2\vec{r}}{ds^2}=\kappa\,\hat{\mathbf{e}}_b$.

(b) For $\vec{r}=\vec{r}(s)$ the position vector of a curve, show that $\left|\dfrac{d\vec{r}}{ds}\times\dfrac{d^2\vec{r}}{ds^2}\right|=\kappa$.

▶ **7-8.** If $\vec{r} = \vec{r}(t)$ denotes a space curve, show that the torsion can be calculate from the relation

$$V = \frac{\vec{r}\,' \cdot (\vec{r}\,'' \times \vec{r}\,''')}{(\vec{r}\,' \cdot \vec{r}\,')(\vec{r}\,'' \cdot \vec{r}\,'') - (\vec{r}\,' \cdot \vec{r}\,'')^2}$$

where prime $' = \frac{d}{dt}$ always denotes differentiation with respect to the argument of the function. Hint: Show $\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \right) = \kappa^2 V$

▶ **7-9.**

(a) Find the curvature of the straight line $\vec{r} = \vec{r}(t) = \,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + (7\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3)t$

(b) Find the torsion of the plane curve $\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2$

▶ **7-10.** For $\vec{r} = \vec{r}(t)$ the position vector of a curve, show that

$$|\vec{r}\,' \times \vec{r}\,''| = \kappa |\vec{r}\,'|^3, \quad \text{where} \quad ' = \frac{d}{dt}$$

▶ **7-11.** Find the directional derivative of ϕ in the specified direction, at the given point.

(i) $\phi = y^2x^2z + x^3z, \qquad P(1,1,1), \quad 3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3$

(ii) $\phi = xyz, \qquad P(2,1,-1) \quad 5\,\hat{\mathbf{e}}_1 - 4\,\hat{\mathbf{e}}_2 + 20\,\hat{\mathbf{e}}_3$

(iii) $\phi = xy^2 + yz^3, \qquad P(1,-1,0), \quad 2\,\hat{\mathbf{e}}_1 - 5\,\hat{\mathbf{e}}_2 - 14\,\hat{\mathbf{e}}_3$

(iv) $\phi = x^2y^2 + yz^3x, \qquad P(1,1,1), \quad \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$

▶ **7-12.**

(i) Let $\phi = x^2y$ define a two-dimensional scalar field. Find the directional derivative of ϕ at the point $(2, \sqrt{3})$ in the direction $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$

(ii) In what direction α is the directional derivative a maximum?

(iii) In what direction α is the directional derivative a minimum?

▶ **7-13.** Show that $\frac{d}{dt}\left[\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right) \right] = \vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3} \right)$

▶ **7-14.** Prove that $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$

▶ **7-15.** Discuss the critical points of the function

$$z = z(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{2}x^2 - \frac{3}{2}y^2 - 2x + 2y$$

▶ **7-16.** Show that the Frenet-Serret formulas may be expressed in the form

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \vec{\omega} \times \,\hat{\mathbf{e}}_t, \qquad \frac{d\,\hat{\mathbf{e}}_b}{ds} = \vec{\omega} \times \,\hat{\mathbf{e}}_b, \qquad \frac{d\,\hat{\mathbf{e}}_n}{ds} = \vec{\omega} \times \,\hat{\mathbf{e}}_n$$

by finding the vector $\vec{\omega}$. Hint: Let $\vec{\omega} = \alpha\,\hat{\mathbf{e}}_t + \beta\,\hat{\mathbf{e}}_n + \gamma\,\hat{\mathbf{e}}_b$ and examine the above cross products to solve for α, β, and γ.

▶ **7-17.** Let $\vec{r}(s)$ denote the position vector of a space curve which is defined in terms of the arc length s.

(a) Show that the equation of the rectifying plane can be written as

$$(\vec{r}(s) - \vec{r}(s_0)) \cdot \frac{d^2\vec{r}(s_0)}{dx^2} = 0$$

(b) Show that the equation of the osculating plane can be written as

$$[\vec{r}(s) - \vec{r}(s_0)] \cdot \left[\frac{d\vec{r}(s_0)}{ds} \times \frac{d^2\vec{r}(s_0)}{ds^2}\right] = 0$$

(c) Show that the equation of the normal plane can be written as

$$[\vec{r}(s) - \vec{r}(s_0)] \cdot \frac{d\vec{r}(s_0)}{ds} = 0$$

▶ **7-18.** Show that the direction cosines (ℓ_1, ℓ_2, ℓ_3) of the normal to the surface $\vec{r} = \vec{r}(u,v)$ are given by

$$\ell_1 = \frac{\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}}{D}, \quad \ell_2 = \frac{\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}}{D}, \quad \ell_3 = \frac{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}{D},$$

where

$$D = \sqrt{EG - F^2}.$$

▶ **7-19.** Show that the direction cosines (ℓ_1, ℓ_2, ℓ_3) of the normal to the surface $F(x,y,z) = 0$ are given by

$$\ell_1 = \frac{\frac{\partial F}{\partial x}}{H}, \quad \ell_2 = \frac{\frac{\partial F}{\partial y}}{H}, \quad \ell_3 = \frac{\frac{\partial F}{\partial z}}{H},$$

where

$$H^2 = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2.$$

▶ **7-20.** Show that the direction cosines (ℓ_1, ℓ_2, ℓ_3) of the normal to the surface $z = z(x,y)$ are given by

$$\ell_1 = \frac{-\frac{\partial z}{\partial x}}{H}, \quad \ell_2 = \frac{-\frac{\partial z}{\partial y}}{H}, \quad \ell_3 = \frac{1}{H},$$

where

$$H^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1$$

▶ **7-21.** Find a unit normal vector to the cylinder $x^2 + y^2 = 1$

▶ **7-22.** Find a unit normal vector to the sphere $x^2 + y^2 + z^2 = 1$

▶ **7-23.** Find a unit normal vector to the plane $ax + by + cz = d$

▶ **7-24.** Evaluate the surface integral $\iint_R x^2yz\,d\vec{S}$ over the cylinder $x^2 + y^2 = 1$ lying in the first octant between the planes $z = 0$ and $z = 2$.

▶ **7-25.** Evaluate the surface integral $\iint_R xyz\,d\vec{S}$ where integration is over the upper half of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

▶ **7-26.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = x\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2$ and S is the surface of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = 2$.

▶ **7-27.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = z\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ lying in the first octant.

▶ **7-28.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = \hat{\mathbf{e}}_3$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

▶ **7-29.** Evaluate the surface integral $\iint_R \vec{F} \times d\vec{S}$, where $\vec{F} = (z+y)\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2 - y\,\hat{\mathbf{e}}_3$ and S is the surface of the plane $z = 1$, where $0 \le x \le 1$ and $0 \le y \le 1$.

▶ **7-30.** Show that any curve on a surface defined by the parametric equations

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

has an element of arc length given by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where E, F, G are defined by the equations (7.54).

▶ **7-31.**

(a) Show that when the curve $z = f(x)$, $x_0 < x < x_1$, is rotated 360° about the z-axis, the surface formed has a surface area

$$S = \int_0^{2\pi} \int_{x_0}^{x_1} r\sqrt{1 + [f'(r)]^2}\,dr\,d\theta$$

(b) The curve $z = f(x) = \frac{H}{R}x$ for $0 \le x \le R$ is rotated 360° about the z axis. Find the surface area generated.

▶ **7-32.** Consider a circle of radius $\rho < a$ centered at $x = a > 0$ in the xz plane. The parametric equations of this circle are

$$x - a = \rho\cos\theta, \quad z = \rho\sin\theta, \qquad 0 \le \theta \le 2\pi, \quad a > \rho.$$

If the circle is rotated about the z-axis, a torus results.

(a) Show that the parametric equations of the torus are

$$x = (a + \rho\cos\theta)\cos\phi, \quad y = (a + \rho\cos\theta)\sin\phi, \quad z = \rho\sin\theta, \quad 0 \le \phi \le 2\pi$$

(b) Find the surface area of the torus.

(c) Find the volume of the torus.

▶ **7-33.** Calculate the arc length along the given curve between the points specified.

$$\begin{array}{ll} (a) & y = x, \qquad p_1(0,0), \qquad p_2(3,3) \\ (b) & x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi \\ (c) & x = t, \quad y = 2t, \quad z = 2t, \qquad p_1(0,0,0), \quad p_2(2,4,4) \\ (d) & y = x^2, \qquad p_1(0,0), \qquad p_2(2,4) \end{array}$$

▶ **7-34.**

(a) Describe the surface $\vec{r} = u\,\hat{\mathbf{e}}_1 + v\,\hat{\mathbf{e}}_2$ and sketch some coordinate curves on the surface.

(b) Describe the surface $\vec{r} = v\cos u\,\hat{\mathbf{e}}_1 + v\sin u\,\hat{\mathbf{e}}_2$ and sketch some coordinate curves on the surface.

(c) Describe the surface $\vec{r} = \sin u\cos v\,\hat{\mathbf{e}}_1 + \sin u\sin v\,\hat{\mathbf{e}}_2 + \cos u\,\hat{\mathbf{e}}_3$ and sketch some coordinate curves on the surface.

(d) Construct a unit normal vector to each of the above surfaces.

▶ **7-35.** Evaluate the surface integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = 4y\,\hat{\mathbf{e}}_1 + 4(x+z)\,\hat{\mathbf{e}}_2$ and S is the surface of the plane $x + y + z = 1$ lying in the first octant.

▶ **7-36.** Evaluate the surface integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = x^2\,\hat{\mathbf{e}}_1 + y^2\,\hat{\mathbf{e}}_2 + z^2\,\hat{\mathbf{e}}_3$ and S is the surface of the unit cube bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $x = 1,\ y = 1,\ z = 1$.

▶ **7-37.** Evaluate the surface integral $I = \iint\limits_S f(x,y,z)\,dS$, where $f(x,y,z) = 2(x+1)y$ and S is the surface of the cylinder $x^2+y^2=1,\ 0 \le z \le 3$, in the first octant.

▶ **7-38.** Evaluate the integral $I = \iint\limits_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = (x+z)\,\hat{\mathbf{e}}_1 + (y+z)\,\hat{\mathbf{e}}_2 - (x+y)\,\hat{\mathbf{e}}_3$ and S is the surface of the sphere $x^2+y^2+z^2=9$ where $z \ge 0$.

▶ **7-39.** Evaluate the surface integral $I = \iint\limits_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = 4y\,\hat{\mathbf{e}}_1 + 4(x+z)\,\hat{\mathbf{e}}_2$ and S is the surface of the plane $x+y+z=1$ which lies in the first octant.

▶ **7-40.** **(Lagrange multipliers)**

Lagrange multipliers are used to help find the maximum or minimum values associated with functions of several variables when the variables are subject to certain constraint conditions. The following is a two-dimensional example of finding the minimum value of a function when the variables in the problem are subject to constraints. Let D denote the distance from the origin $(0,0)$ to a point (x,y) which lies on the line $x+y+2=0$. Let $F(x,y) = D^2 = x^2+y^2$ denote the square of this distance. The mathematical problem is to find values for (x,y) which minimize $F(x,y)$ when (x,y) is constrained to move along the given line. Mathematically one writes

$$\begin{aligned} \text{Minimize} \quad & F(x,y) = x^2+y^2 \\ \text{subject to the constraint condition} \quad & G(x,y) = x+y-2 = 0. \end{aligned}$$

The point (x,y), where F has a minimum value, is called a critical point.

(a) Show that at a critical point ∇F is normal to the curve $F =$ constant and ∇G is normal to the line $G=0$.

(b) Show that at a critical point the vectors ∇F and ∇G are colinear. Consequently, one can write

$$\nabla F + \lambda \nabla G = \vec{0}$$

where λ is a scalar called a Lagrange multiplier.

(c) Show that at a critical point which minimizes F, the function $H = F + \lambda G$, satisfies the equations

$$\frac{\partial H}{\partial \lambda} = 0, \qquad \frac{\partial H}{\partial x} = 0, \qquad \frac{\partial H}{\partial y} = 0.$$

Calculate these equations and find the point (x,y) which minimizes F.

▶ **7-41.** **(Lagrange multipliers)**
Use Lagrange multipliers to

$$\text{Minimize} \qquad \omega = \omega(x,y,z) = x^2 + y^2 + z^2,$$

$$\text{subject to the constraint conditions:} \qquad \begin{aligned} g(x,y,z) &= x + y + z - 6 = 0 \\ h(x,y,z) &= 3x + 5y + 7z - 34 = 0 \end{aligned}$$

▶ **7-42.** Given the vector field $\vec{F} = (x^2 + y - 4)\,\hat{\mathbf{e}}_1 + 3xy\,\hat{\mathbf{e}}_2 + (2xz + z^2)\,\hat{\mathbf{e}}_3$. Evaluate the surface integral

$$I = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

over the upper half of the unit sphere centered at the origin.

▶ **7-43.** Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = x^2\,\hat{\mathbf{e}}_1 + (y+6)\,\hat{\mathbf{e}}_2 - z\,\hat{\mathbf{e}}_3$ and S is the surface of the unit cube bounded by the planes $x = 0, \; y = 0, \; z = 0$ and $x = 1, \; y = 1, \; z = 1.$

▶ **7-44.**

(a) Show in the special case the surface is defined by $\vec{r} = \vec{r}(x,y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x,y)\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy$$

(b) Show in the special case the surface is defined by $\vec{r} = \vec{r}(x,z) = x\,\hat{\mathbf{e}}_1 + y(x,z)\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\,dz$$

(c) Show in the special case the surface is defined by $\vec{r} = \vec{r}(y,z) = x(y,z)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dy\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1|} = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\,dz$$

▶ **7-45.** Given n particles having masses $m_1, m_2, \ldots, m_n$. Let $\vec{r}_i, \; i = 1, 2, \ldots, n$ denote the position vector describing the position of the ith particle. Find the vector describing the center of mass of the system of particles.

► **7-46.** Show that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \begin{vmatrix} \vec{A}\cdot\vec{C} & \vec{A}\cdot\vec{D} \\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{D} \end{vmatrix}$

► **7-47.** Let $F = F(x,y,z)$ and $G = G(x,y,z)$ denote continuous functions which are everywhere differentiable. Show that

$$(a) \quad \nabla(F+G) = \nabla F + \nabla G \qquad (b) \quad \nabla(FG) = F\nabla G + G\nabla F \qquad (c) \quad \nabla\left(\frac{F}{G}\right) = \frac{G\nabla F - F\nabla G}{G^2}$$

► **7-48.** **(Least-squares)**

(a) Assume that $(x_1,y_1), (x_2,y_2), \ldots, (x_i,y_i), \ldots, (x_N,y_N)$ are N known distinct data points that are plotted in the x,y plane along with a sketch of the straight line

$$y = \alpha x + \beta$$

where α and β are constants to be determined. The situation is illustrated in figure 7-28.

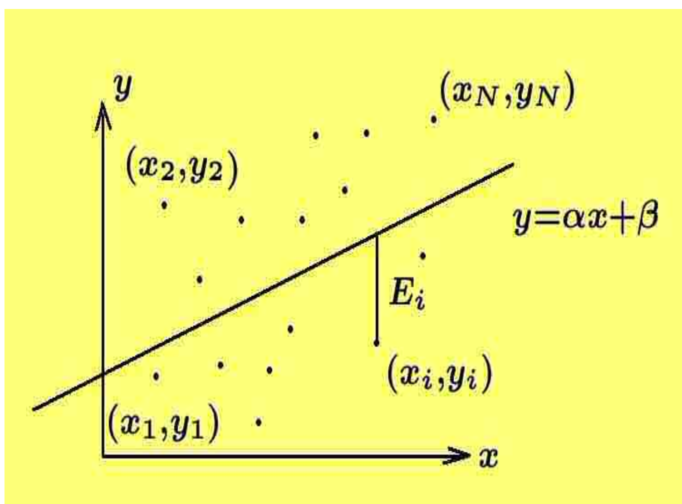


Figure 7-28. Linear least-squares fit.

Each data point (x_i,y_i) has associated with it an error E_i which is defined as the difference between the y value of the line and the y value of the data point. For example, the error associated with the data point (x_i,y_i) can be written

$$E_i = E_i(\alpha,\beta) = \{y \text{ of line}\} - \{y \text{ of data point}\}$$

$$E_i = E_i(\alpha,\beta) = (\alpha x_i + \beta) - y_i.$$

Find the error associated with each data point and then square these errors and sum them to obtain the quantity $\sum_{i=1}^{N} E_i^2$ called the sum of the errors squared. The "best "linear least squares fit to all the data points is defined as the line which minimizes the sum of the errors squared. This requires finding those values of α and β which minimize the sum of the errors squared given by

$$E(\alpha, \beta) = \sum_{i=1}^{N} E_i^2 = \sum_{i=1}^{N} (\alpha x_i + \beta - y_i)^2\,, \quad \text{to have a minimum value}$$

(a) Show that the best linear least-squares fit requires that the coefficients α and β be chosen to satisfy the equations

$$\alpha = \frac{N\sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{\Delta}$$
$$\beta = \frac{\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i\right) - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i y_i\right)}{\Delta},$$

where $$\Delta = N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2.$$

(b) Given the data points

$$(1, 10),\ (2, 4),\ (2.5, 6),\ (3, 12),\ (3.5, 5),\ (4, 10)$$

Find the best linear least-squares fit. Hint: Construct a table of values of the form

x_i	y_i	x_i^2	$x_i y_i$
$\sum_{i=1}^{4} x_i$	$\sum_{i=1}^{4} y_i$	$\sum_{i=1}^{4} x_i^2$	$\sum_{i=1}^{4} x_i y_i$

(c) Plot the least squares straight line and the data points.

Chapter 8

Vector Calculus II

In this chapter we examine in more detail the operations of gradient, divergence and curl as well as introducing other mathematical operators involving vectors. There are several important theorems dealing with the operations of divergence and curl which are extremely useful in modeling and representing physical problems. These theorems are developed along with some examples to illustrate how powerful these results are. Also considered is the representation of the many vector operations and their use when dealing with a general orthogonal coordinate system.

Vector Fields

Let $\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_3 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ denote a continuous vector field with continuous partial derivatives in some region R of space. A vector field is **a one-to-one correspondence between points in space and vector quantities** so by selecting a discrete set of points $\{(x_i, y_i, z_i)\}\,|\,i = 1,\ldots,n,\ (x_i,y_i,z_i) \in R\}$ one could sketch in tiny vectors each proportional to the given vector evaluated at the selected points. This would be one way of visualizing the vector field. Imagine a surface being placed in this vector field, then at each point (x,y,z) on the surface there is associated a vector $\vec{F}(x,y,z)$. This is another way of visualizing a vector field. One can think of the surface as being punctured by arrows of different lengths. These arrows then represent the direction and magnitude of the vectors in the vector field. The situation is illustrated in figure 8-1.

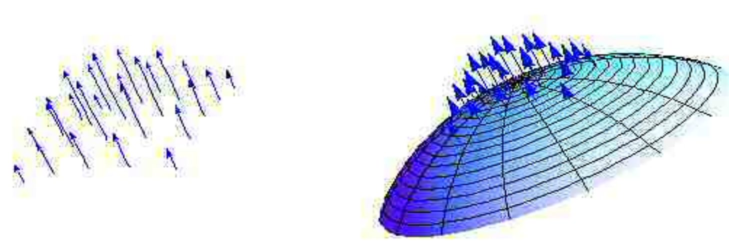


Figure 8-1. Representation of vector field $\vec{F}(x,y,z)$ at a select set of points.

Another way to visualize a vector field is to create a bundle of curves in space, where **each curve in the bundle** has the property that at every point (x,y,z) on any one curve, **the direction of the tangent vector to the curve is the same as the direction of the vector field $\vec{F}(x,y,z)$ at that point.** Curves with this property are called **field lines** associated with the vector field $\vec{F}(x,y,z)$. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ be a position vector to a point on a field line curve, then $d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3$ is in the direction of the tangent to the curve. If the curve is a field line, then one can write $d\vec{r} = \alpha\vec{F}$, where α is a proportionality constant. That is, if the curve is a field line, then the vectors $d\vec{r}$ and $\vec{F}$ are colinear at all points along the curve and one can write

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3 = \alpha F_1(x,y,z)\,\hat{\mathbf{e}}_1 + \alpha F_2(x,y,z)\,\hat{\mathbf{e}}_2 + \alpha F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

where α is some proportionality constant. By equating like components in the above equation, one obtains

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)} = \alpha \tag{8.1}$$

The equations (8.1) represent a system of differential equations to be solved to obtain the representation of the field lines.

Example 8-1. Find and sketch the field lines associated with the vector field

$$\vec{V} = \vec{V}(x,y) = (3-x)(4-y)\,\hat{\mathbf{e}}_1 + (6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$$

Solution If $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ describes a field line, then one can write

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 = \alpha\vec{V}(x,y) = \alpha(3-x)(4-y)\,\hat{\mathbf{e}}_1 + \alpha(6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$$

where α is a proportionality constant. Equating like components one can show that the field lines must satisfy

$$dx = \alpha(3-x)(4-y), \qquad dy = \alpha(6-x^2)(4+y^2)$$

or one could write

$$\frac{dx}{(3-x)(4-y)} = \frac{dy}{(6-x^2)(4+y^2)} = \alpha \tag{8.2}$$

Separate the variables in equation (8.2) to obtain

$$\frac{6-x^2}{3-x}\,dx = \frac{4-y}{4+y^2}\,dy$$

and then integrate both sides to obtain

$$\int \frac{6-x^2}{3-x}\,dx = \int \frac{4-y}{4+y^2}\,dy \tag{8.3}$$

Use a table of integrals and evaluate the integrals and then collect all the constants of integration and combine them into just one arbitrary constant C to obtain the result

$$3x + \frac{1}{2}x^2 + 3\ln|x-3| = 2\tan^{-1}(y/2) - \frac{1}{2}\ln(4+y^2) + C \tag{8.4}$$

The equation (8.4) represents **a one-parameter family of curves** which describe the field lines associated with the given vector field. Assign values to the constant C and sketch the corresponding field line. Place arrows on the curves to show the direction of the vector field.

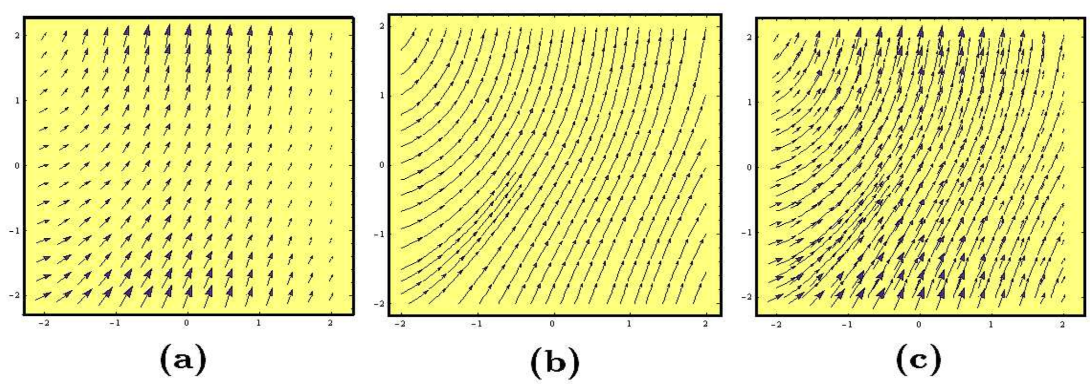


Figure 8-2.
Vector plot and field lines for $\vec{V}(x,y) = (3-x)(4-y)\,\hat{\mathbf{e}}_1 + (6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$

The figure 8-2 illustrates three graphs created by a computer. The figure 8-2(a) represents a vector field plot over the region $-2 \le x \le 2$ and $-2 \le y \le 2$. The figure 8-2(b) is a graph of the field lines associated with the vector field with arrows placed on the field lines. The figure 8-2(c) is the vectors of figure 8-2(a) placed on top of the field lines of figure 8-2(b) to compare the different representations.

■

Divergence of a Vector Field

The study of field lines leads to the concept of **intensity of a vector field** or **the density of the field lines in a region.** To visualize this, place an imaginary surface in a vector field and try to determine how the vector field punctures this surface. Let the surface be divided into n small areas ΔS_i and let $\vec{F}_i = \vec{F}(x_i, y_i, z_i)$ denote the value of the vector field associated with each surface element. The dot product $\vec{F}_i \cdot \Delta\vec{S}_i = \vec{F}_i \cdot \hat{\mathbf{e}}_n \Delta S_i$ represents the projection of the vector $\vec{F}_i$ onto the normal to the element ΔS_i multiplied by the area of the element. Such a product is a measure of **the number of field lines which pass through the area** ΔS_i and is called a **flux across the surface boundary**. The total flux across the surface is denoted by the **flux integral**

$$\varphi = \lim_{\substack{n\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{S}_i = \iint_R \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \hat{\mathbf{e}}_n \, dS \tag{8.5}$$

The surface area over which the integration is performed can be part of a surface or it can be over all points of a closed surface. The evaluation of a flux integral over a closed surface measures the total contribution of the normal component of the vector field over the surface.

The term flux can mean flow in some instances. For example, let an imaginary plane surface of area 1 square centimeter be placed perpendicular to a uniform velocity flow of magnitude V_0, such that the velocity is the same at all points over the surface. In 1 second there results a column of fluid V_0 units long which passes through the unit of surface area. The dimension of the flux integral is volume per unit of time and can be interpreted as the rate of flow or flux of the velocity across the surface.

In the above example, the flux was a quantity which is recognized as volume rate of flow. In many other problems **the flux is only a definition** and does not readily have any physical meaning. For example, the electric flux over the surface of a sphere due to a point charge at its center is given by the flux integral $\iint_R \vec{E} \cdot d\vec{S}$, where $\vec{E}$ is the electrostatic intensity. The flux cannot be interpreted as flow because nothing is flowing. In this case the flux is considered as **a measure of the density of the field lines that pass through the surface** of the sphere.

The value for the flux depends upon the size of the surface that is placed in the vector field under consideration and therefore cannot be used to describe a characteristic of the vector field. However, if an arbitrary closed surface is placed in a vector field and the flux integral over this surface is evaluated and the result is divided by the volume enclosed by the surface, one obtains the ratio of $\frac{\text{Flux}}{\text{Volume}}$. By letting the volume and surface area of the arbitrary closed surface approach zero, the ratio of $\frac{\text{Flux}}{\text{Volume}}$ turns out to measure **a point characteristic of the vector field** called **the divergence**. Symbolically, the divergence is a scalar quantity and is defined by the limiting process

$$\text{div}\,\vec{F} = \lim_{\substack{\Delta V\to 0\\ \Delta S\to 0}} \frac{\iint\limits_R \vec{F}\cdot d\vec{S}}{\Delta V} = \lim_{\substack{\Delta V\to 0\\ \Delta S\to 0}} \frac{\text{Flux}}{\text{Volume}} \tag{8.6}$$

Consider the evaluation of this limit in the special case where the closed surface is a sphere. Consider a sphere of radius $\epsilon > 0$ centered at a point $P_0(x_0, y_0, z_0)$ situated in a vector field

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3.$$

Express the sphere in the parametric form as

$$\begin{aligned} x &= x_0 + \epsilon\sin\theta\cos\phi, \qquad & 0 \le \phi \le 2\pi\\ y &= y_0 + \epsilon\sin\theta\sin\phi, \qquad & 0 \le \theta \le \pi\\ z &= z_0 + \epsilon\cos\theta \end{aligned}$$

then the position vector to a point on this sphere is given by

$$\vec{r} = \vec{r}(\theta,\phi) = (x_0 + \epsilon\sin\theta\cos\phi)\,\hat{\mathbf{e}}_1 + (y_0 + \epsilon\sin\theta\sin\phi)\,\hat{\mathbf{e}}_2 + (z_0 + \epsilon\cos\theta)\,\hat{\mathbf{e}}_3$$

The coordinate curves on the surface of this sphere are

$$\vec{r}(\theta_0,\phi) \quad \text{and} \quad \vec{r}(\theta,\phi_0) \quad \theta_0, \phi_0 \text{ constants}$$

and one can show the element of surface area on the sphere is given by

$$dS = \left|\, \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}\,\right| \, d\theta\, d\phi = \sqrt{EG - F^2}\, d\theta\, d\phi = \epsilon^2 \sin\theta\, d\theta\, d\phi$$

A **unit normal** to the surface of the sphere is given by

$$\hat{\mathbf{e}}_n = \frac{\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}\right|} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

The flux integral given by equation (8.6) and integrated over the surface of a sphere can then be expressed as

$$\begin{aligned} \varphi &= \iint\limits_R \vec{F}\cdot d\vec{S} = \iint\limits_R \vec{F}\cdot\hat{\mathbf{e}}_n\, dS\\ \varphi &= \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} \vec{F}(x_0 + \epsilon\sin\theta\cos\phi, y_0 + \epsilon\sin\theta\sin\phi, z_0 + \epsilon\cos\theta)\,\hat{\mathbf{e}}_n \epsilon^2 \sin\theta\, d\theta d\phi. \end{aligned}$$

Treating the vector function $\vec{F}$ as a function of ϵ, one can expand $\vec{F}$ in a Taylor's series about $\epsilon = 0$, to obtain

$$\vec{F} = \vec{F}(x_0, y_0, z_0) + \epsilon \frac{d\vec{F}}{d\epsilon} + \frac{\epsilon^2}{2!}\frac{d^2\vec{F}}{d\epsilon^2} + \frac{\epsilon^3}{3!}\frac{d^3\vec{F}}{d\epsilon^3} + \cdots \tag{8.7}$$

where all the derivatives are to be evaluated at $\epsilon = 0$. Substituting the expressions for the unit normal and the Taylor's series into the flux integral produces

$$\varphi = \epsilon^2\mu_0 + \epsilon^3\mu_1 + \epsilon^4\mu_2 + \cdots,$$

where

$$\begin{aligned}
\mu_0 &= \int_0^\pi \int_0^{2\pi} \vec{F}(x_0, y_0, z_0) \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta \\
\mu_1 &= \int_0^\pi \int_0^{2\pi} \frac{d\vec{F}}{d\epsilon}\bigg|_{\epsilon=0} \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta \\
\mu_2 &= \int_0^\pi \int_0^{2\pi} \frac{d^2\vec{F}}{d\epsilon^2}\bigg|_{\epsilon=0} \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta,
\end{aligned}$$

plus higher order terms in ϵ. The vector $\vec{F}(x_0, y_0, z_0)$ is a constant and an evaluation of the integral defining μ_0 produces $\mu_0 = 0$. To calculate the second integral defining μ_1 observe that the chain rule for functions of more than one variable produces the result

$$\begin{aligned}
\frac{d\vec{F}}{d\epsilon} &= \frac{\partial\vec{F}}{\partial x}\frac{\partial x}{\partial\epsilon} + \frac{\partial\vec{F}}{\partial y}\frac{\partial y}{\partial\epsilon} + \frac{\partial\vec{F}}{\partial z}\frac{\partial z}{\partial\epsilon} \\
\frac{d\vec{F}}{d\epsilon} &= \frac{\partial\vec{F}}{\partial x}\sin\theta\cos\phi + \frac{\partial\vec{F}}{\partial y}\sin\theta\sin\phi + \frac{\partial\vec{F}}{\partial z}\cos\theta
\end{aligned}$$

This result can be expressed in the component form

$$\begin{aligned}
\frac{d\vec{F}}{d\epsilon}\bigg|_{\epsilon=0} = & \left(\frac{\partial F_1}{\partial x}\sin\theta\cos\phi + \frac{\partial F_1}{\partial y}\sin\theta\sin\phi + \frac{\partial F_1}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_1 \\
& + \left(\frac{\partial F_2}{\partial x}\sin\theta\cos\phi + \frac{\partial F_2}{\partial y}\sin\theta\sin\phi + \frac{\partial F_2}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_2 \\
& + \left(\frac{\partial F_3}{\partial x}\sin\theta\cos\phi + \frac{\partial F_3}{\partial y}\sin\theta\sin\phi + \frac{\partial F_3}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_3 \bigg|_{\epsilon=0}
\end{aligned}$$

where all the derivatives are to be evaluated at $\epsilon = 0$. Evaluating the integrals defining μ_1, produces

$$\mu_1 = \frac{4}{3}\pi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\bigg|_{\epsilon=0}$$

The flux integral then has the form

$$\varphi = \frac{4}{3}\pi\epsilon^3\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \epsilon^4\mu_2 + \epsilon^5\mu_3 + \cdots$$

where the derivatives are to be evaluated at $\epsilon = 0$. The volume of the sphere of radius ϵ centered at the point (x_0, y_0, z_0) is given by $\frac{4}{3}\pi\epsilon^3$ and consequently the limit of the ratio of $\frac{\text{Flux}}{\text{Volume}}$ as ϵ tends toward zero produces the scalar relation

$$\operatorname{div}\vec{F} = \lim_{\substack{\Delta V\to 0\\ \Delta S\to 0}} \frac{\iint\limits_R \vec{F}\cdot d\vec{S}}{\Delta V} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big|_{\epsilon=0} \tag{8.8}$$

Recalling the definition of the operator ∇, the mathematical expression of the divergence may be represented

$$\begin{aligned}\operatorname{div}\,\vec{F} = \nabla\cdot\vec{F} =& \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot(F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3)\\ =& \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned} \tag{8.9}$$

Example 8-2. Find the divergence of the vector field

$$\vec{F}(x,y,z) = x^2y\,\hat{\mathbf{e}}_1 + (x^2 + yz^2)\,\hat{\mathbf{e}}_2 + xyz\,\hat{\mathbf{e}}_3$$

Solution: By using the result from equation (8.9), the divergence can be expressed

$$\operatorname{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(x^2+yz^2)}{\partial y} + \frac{\partial(xyz)}{\partial z} = 2xy + z^2 + xy = 3xy + z^2$$

■

The Gauss Divergence Theorem

A relation known as **the Gauss divergence theorem** exists between **the flux and divergence of a vector field**. Let $\vec{F}(x,y,z)$ denote a vector field which is continuous with continuous derivatives. For **an arbitrary closed sectionally continuous surface** S **which encloses a volume** V, the Gauss' divergence theorem states

$$\iiint\limits_V \operatorname{div}\,\vec{F}\,dV = \iiint \nabla\cdot\vec{F}\,dV = \iint\limits_S \vec{F}\cdot d\vec{S} = \iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS. \tag{8.10}$$

which states that the surface integral of the normal component of $\vec{F}$ summed over a closed surface equals the integral of the divergence of $\vec{F}$ summed over the volume enclosed by S. This theorem can also be represented in the expanded form as

$$\iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint\limits_S (F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_n\,dS, \tag{8.11}$$

where $\hat{\mathbf{e}}_n$ is the exterior or positive normal to the closed surface.

The proof of the Gauss divergence theorem begins by first verifying the integrals

$$\begin{aligned}\iiint_V \frac{\partial F_1}{\partial x}\, dV &= \iint_S F_1\, \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n\, dS\\ \iiint_V \frac{\partial F_2}{\partial y}\, dV &= \iint_S F_2\, \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n\, dS\\ \iiint_V \frac{\partial F_3}{\partial z}\, dV &= \iint_S F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\, dS.\end{aligned} \tag{8.12}$$

The addition of these integrals then produces the desired proof. Note that the arguments used in proving each of the above integrals are essentially the same for each integral. For this reason, only the last integral is verified.

Let the closed surface S be composed of an upper half S_2 defined by $z = z_2(x, y)$ and a lower half S_1 defined by $z = z_1(x, y)$ as illustrated in figure 8-3. An element of volume $dV = dx\, dy\, dz$, when summed in the $z-$direction from zero to the upper surface, forms a parallelepiped which intersects both the lower surface and upper surface as illustrated in figure 8-3. Denote the unit normal to the lower surface by $\hat{\mathbf{e}}_{n_1}$ and the unit normal to the upper surface by $\hat{\mathbf{e}}_{n_2}$. The parallelepiped intersects the upper surface in an element of area dS_2 and it intersects the lower surface in an element of area dS_1. The projection of S for both the upper surface and lower surface onto the xy-plane is denoted by the region R.

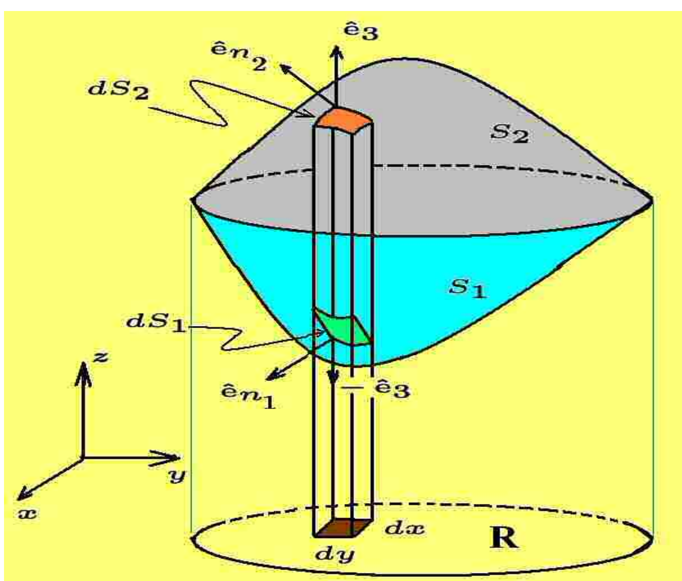


Figure 8-3. Integration over a simple closed surface.

An integration in the z-direction produces

$$\iiint_V \frac{\partial F_3}{\partial z}\, dz dx\, dy = \iint_R F_3(x,y,z)\Big|_{z_1(x,y)}^{z_2(x,y)} dx\, dy$$
$$= \iint_R F_3(x,y,z_2(x,y))\, dx\, dy - \iint_R F_3(x,y,z_1(x,y))\, dx\, dy.$$

The element of surface area on the upper and lower surfaces can be represented by

$$\text{On the surface } S_2, \qquad dS_2 = \frac{dx\, dy}{\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_2}}$$
$$\text{On the surface } S_1, \qquad dS_1 = \frac{dx\, dy}{-\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_1}}$$

so that the above integral can be expressed as

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_{S_2} F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_2}\, dS_2 + \iint_{S_1} F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_1}\, dS_1 = \iint_S F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\, dS,$$

which establishes the desired result.

Similarly by dividing the surface into appropriate sections and projecting the surface elements of these sections onto appropriated planes, the remaining integrals may be verified.

Example 8-3. Verify the divergence theorem for the vector field

$$\vec{F}(x,y,z) = 2x\, \hat{\mathbf{e}}_1 - 3y\, \hat{\mathbf{e}}_2 + 4z\, \hat{\mathbf{e}}_3$$

over the region in the first octant bounded by the surfaces

$$z = x^2 + y^2, \qquad z = 4, \quad x = 0, \quad y = 0$$

Solution The given surfaces define a closed region over which the integrations are to be performed. This region is illustrated in figure 8-4. The divergence of the given vector field is given by

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2 - 3 + 4 = 3,$$

and thus the volume integral part of the Gauss divergence theorem can be determined by summing the element of volume $dV = dx\, dy\, dz$ first in the $z-$direction from the surface $z = x^2 + y^2$ to the plane $z = 4$. The resulting parallelepiped is then summed in the $y-$direction from $y = 0$ to the circle $y = \sqrt{4 - x^2}$ to form a slab. The slab is then summed in the $x-$direction from $x = 0$ to $x = 2$.

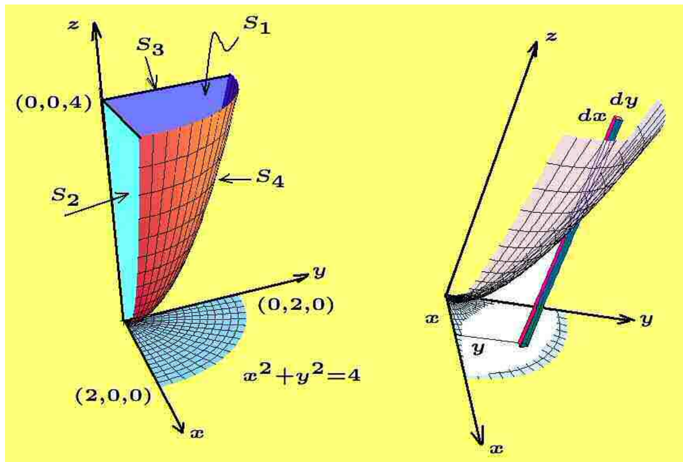


Figure 8-4. Integration over closed surface area defined by $S_1 \cup S_2 \cup S_3 \cup S_4$.

The resulting volume integral is then represented

$$\begin{aligned}\iiint\limits_V \operatorname{div} \vec{F}\, dV =& \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \int_{z=x^2+y^2}^{z=4} 3\, dz dy\, dx \\ =& \int_0^2 \int_0^{\sqrt{4-x^2}} 3[4-(x^2+y^2)]\, dy\, dx \\ =& \int_0^2 3(4y - x^2y - \frac{1}{3}y^3)\Big|_0^{\sqrt{4-x^2}} dx \\ =& \int_0^2 (8-2x^2)\sqrt{4-x^2}\, dx = 6\pi.\end{aligned}$$

For the surface integral part of Gauss' divergence theorem, observe that the surface enclosing the volume is composed of four sections which can be labeled S_1, S_2, S_3, S_4 as illustrated in the figure 8-4. The surface integral can then be broken up and written as a summation of surface integrals. One can write

$$\iint\limits_S \vec{F}\cdot d\vec{S} = \iint\limits_{S_1} \vec{F}\cdot d\vec{S} + \iint\limits_{S_2} \vec{F}\cdot d\vec{S} + \iint\limits_{S_3} \vec{F}\cdot d\vec{S} + \iint\limits_{S_4} \vec{F}\cdot d\vec{S}.$$

Each surface integral can be evaluated as follows.

On S_1, $\quad z = 4, \;\; \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_3, \quad dS = dx\,dy$ and

$$\iint\limits_{S_1} \vec{F} \cdot d\vec{S} = \int_0^2 \int_0^{\sqrt{4-x^2}} 16\,dy\,dx = 16 \int_0^2 \sqrt{4-x^2}\,dx = 16\pi.$$

On S_2, $\quad y = 0, \;\; \hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_2, \quad dS = dx\,dz$ and

$$\iint\limits_{S_2} \vec{F} \cdot d\vec{S} = \iint -3y\,dx\,dz = 0.$$

On S_3, $\quad x = 0, \;\; \hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_1, \quad dS = dy\,dz$ and

$$\iint\limits_{S_3} \vec{F} \cdot d\vec{S} = \iint -2x\,dy\,dz = 0.$$

On S_4 the surface is defined by $\phi = x^2 + y^2 - z = 0$, and the normal is determined from

$$\hat{\mathbf{e}}_n = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{4(x^2+y^2)+1}} = \frac{2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{4z+1}},$$

and consequently the element of surface area can be represented by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \sqrt{4z+1}\,dx\,dy.$$

One can then write

$$\begin{aligned}
\iint\limits_{S_4} \vec{F} \cdot d\vec{S} = \iint\limits_{S_4} \vec{F} \cdot \hat{\mathbf{e}}_n\,dS &= \iint\limits_{S_4} (4x^2 - 6y^2 - 4z)\,dx\,dy \\
&= \iint\limits_{S_4} (4x^2 - 6y^2 - 4(x^2+y^2))\,dx\,dy \\
&= \int_0^2 \int_0^{\sqrt{4-x^2}} -10y^2\,dy\,dx = -\int_0^2 \frac{10}{3} y^3 \Big|_0^{\sqrt{4-x^2}} dx \\
&= -\int_0^2 \frac{10}{3} \sqrt{(4-x^2)^3}\,dx = -10\pi.
\end{aligned}$$

The total surface integral is the summation of the surface integrals over each section of the surface and produces the result 6π which agrees with our previous result.

Sometimes it is convenient to change the variables in a surface or volume integral. For example, the integral over the surface S_4 is not an integral which is easily evaluated. The geometry suggests a change to cylindrical coordinates. In cylindrical coordinates the following relations are satisfied:

$$\begin{aligned}
x = r\cos\theta, \quad y &= r\sin\theta, \quad z = x^2+y^2 = r^2\\
\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 &= r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + r^2\,\hat{\mathbf{e}}_3\\
E = 1+4r^2, \quad F &= 0, \quad G = r^2\\
\hat{\mathbf{e}}_n &= \frac{2r\cos\theta\,\hat{\mathbf{e}}_1 + 2r\sin\theta\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{1+4r^2}}\\
\vec{F}\cdot\hat{\mathbf{e}}_n &= \frac{4r^2\cos^2\theta - 6r^2\sin^2\theta - 4r^2}{\sqrt{1+4r^2}}, \qquad dS = r\sqrt{1+4r^2}\,dr d\theta.
\end{aligned}$$

The integral over the surface S_4 can then be expressed in the form

$$\iint_{S_4} \vec{F}\cdot d\vec{S} = \int_0^{\frac{\pi}{2}}\int_0^2 (4\cos^2\theta - 6\sin^2\theta - 4)r^3\,dr d\theta = -10\pi.$$

■

Physical Interpretation of Divergence

The divergence of a vector field is **a scalar field** which is interpreted as representing the flux per unit volume diverging from a small neighborhood of a point. In the limit as the volume of the neighborhood tends toward zero, the limit of the ratio of flux divided by volume is called **the instantaneous flux per unit volume at a point** or **the instantaneous flux density at a point.**

If $\vec{F}(x,y,z)$ defines a vector field which is continuous with continuous derivatives in a region R and if at some point P_0 of R, one finds that

div $\vec{F} > 0$, then a source is said to exist at point P_0.
div $\vec{F} < 0$, then a sink is said to exist at point P_0.
div $\vec{F} = 0$, then $\vec{F}$ is called solenoidal and no sources or sinks exist.

The Gauss divergence theorem states that if div $\vec{F} = 0$, then the flux $\varphi = \iint_S \vec{F}\cdot d\vec{S}$ over the closed surface vanishes. When the flux vanishes the vector field is called **solenoidal**, and in this case, the flux of the vector field $\vec{F}$ into a volume exactly equals the flux of the field $\vec{F}$ out of the volume. Consider the field lines discussed earlier and visualize a bundle of these field lines forming a tube. Cut the tube by two plane areas S_1 and S_2 normal to the field lines as in figure 8-5.

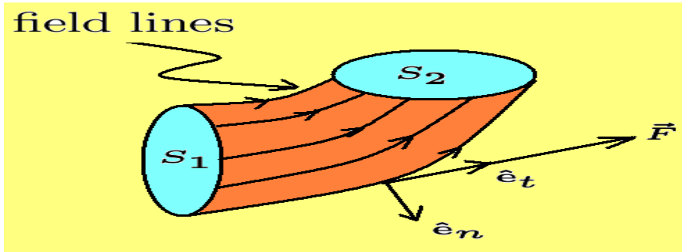


Figure 8-5. Tube of field lines.

The sides of the tube are composed of field lines, and at any point on a field line the direction of the tangents to the field lines are in the same direction as the vector field $\vec{F}$ at that point. The unit normal vector at a point on one of these field lines is perpendicular to the unit tangent vector and therefore perpendicular to the vector $\vec{F}$ so that the dot product $\vec{F} \cdot \hat{\mathbf{e}}_n = 0$ must be zero everywhere on the sides of the tube. The sides of the tube consist of field lines, and therefore there is no flux of the vector field across the sides of the tube and all the flux enters, through S_1, and leaves through S_2. In particular, if $\vec{F}$ is solenoidal and div $\vec{F} = 0$, then

$$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{Sides} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = 0 \quad \text{or} \quad \iint_{S_1} \vec{F} \cdot d\vec{S} = -\iint_{S_2} \vec{F} \cdot d\vec{S}$$

If the vector field is a velocity field then one can say that the flux or flow into S_1 must equal the flow leaving S_2.

Physically, the divergence assigns a number to each point of space where the vector field exists. The number assigned by the divergence is a scalar and represents the rate per unit volume at which the field issues (or enters) from (or toward) a point. In terms of figure 8-5, if more flux lines enter S_1 than leaves S_2, the divergence is negative and a sink is said to exist. If more flux lines leave S_2 than enter S_1, a source is said to exist.

Example 8-4. Consider the vector field

$$\vec{V} = \frac{kx}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_1 + \frac{ky}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_2 \qquad (x, y) \neq (0,0), \; k \text{ a constant}$$

A sketch of this vector field is illustrated in figure 8-6.

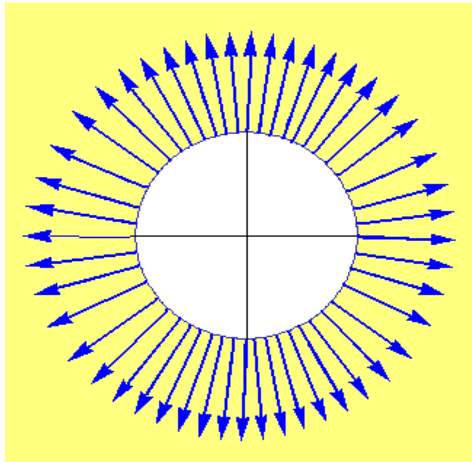


Figure 8-6. Vector field $\vec{V} = \frac{kx}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_1 + \frac{ky}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_2,\ k>0$ constant.

Observe that the magnitude of the vector field at any point $(x, y) \neq (0, 0)$ is given by $|\vec{V}| = k$. In polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, the vector field can be represented by

$$\vec{V} = k(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2).$$

Thus, on the circle $r =$ Constant, the vector field may be thought of as tiny needles of length $|k|$, which emanate outward (inward if k is negative) and are orthogonal to the circle $r =$ constant. The divergence of this vector field is

$$\text{div}\ \vec{V} = \nabla\cdot\vec{V} = \frac{ky^2}{(x^2+y^2)^{3/2}} + \frac{kx^2}{(x^2+y^2)^{3/2}} = \frac{k}{r},$$

where $r = \sqrt{x^2+y^2}$. The divergence of this field is positive if $k > 0$ and negative if $k < 0$. If the vector field $\vec{V}$ represents a velocity field and $k > 0$, the flow is said to emanate from a source at the origin. If $k < 0$, the flow is said to have a sink at the origin.

■

Green's Theorem in the Plane

Let C denote a simple closed curve enclosing a region R of the xy plane. If $M(x, y)$ and $N(x, y)$ are continuous function with continuous derivatives in the region R, then **Green's theorem in the plane** can be written as

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy, \tag{8.13}$$

where the line integral is taken in **a counterclockwise direction** around the simple closed curve C which encloses the region R.

To prove this theorem, let $y = y_2(x)$ and $y = y_1(x)$ be single-valued continuous functions which describe the upper and lower portions C_2 and C_1 of the simple closed curve C in the interval $x_1 \le x \le x_2$ as illustrated in figure 8-7(a).

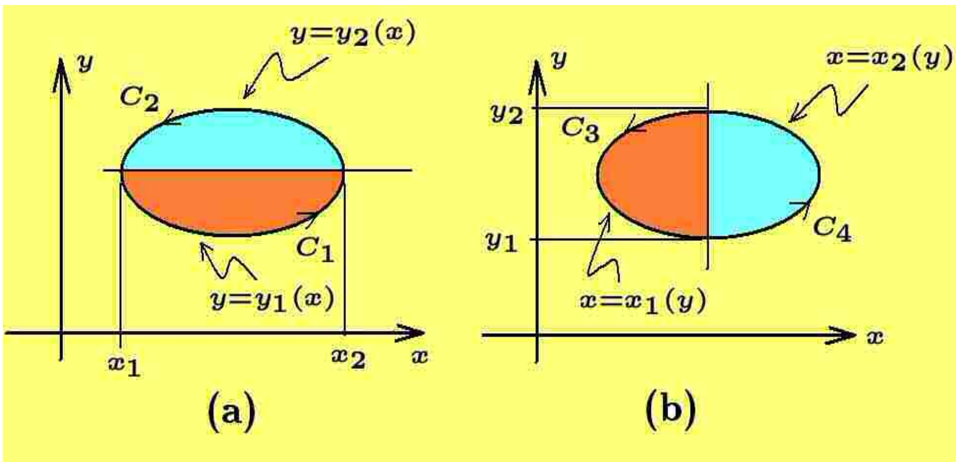


Figure 8-7. Simple closed curve for Green's theorem.

The right-hand side of equation (8.13) can be expressed

$$
\begin{aligned}
-\iint\limits_R \frac{\partial M}{\partial y}\,dx\,dy = & -\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} \frac{\partial M}{\partial y}\,dy\,dx = -\int_{x_1}^{x_2} M(x,y)\Big|_{y_1(x)}^{y_2(x)}\,dx \\
=& \int_{x_1}^{x_2} [M(x,y_1(x)) - M(x,y_2(x))]\,dx \\
=& \int_{x_1}^{x_2} M(x,y_1(x))\,dx + \int_{x_2}^{x_1} M(x,y_2(x))\,dx \\
=& \int_{C_1} M(x,y_1(x))\,dx + \int_{C_2} M(x,y_2(x))\,dx = \oint_C M(x,y)\,dx.
\end{aligned} \tag{8.14}
$$

Now let $x = x_1(y)$ and $x = x_2(y)$ be single-valued continuous functions which describe the left and right sections C_3 and C_4 of the curve C in the interval $y_1 \le y \le y_2$.

The remaining part of the right-hand side can then be expressed

$$\begin{aligned}\iint\limits_R \frac{\partial N}{\partial x}\,dx\,dy =& \int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} \frac{\partial N}{\partial x}\,dx\,dy = \int_{y_1}^{y_2} N(x,y)\Big|_{x_1(y)}^{x_2(y)}\,dy\\ =& \int_{y_1}^{y_2}\left[N(x_2(y),y)-N(x_1(y),y)\right]dy\\ =& \int_{y_1}^{y_2} N(x_2(y),y)\,dy + \int_{y_2}^{y_1} N(x_1(y),y)\,dy\\ =& \int_{C_3} N(x_1(y),y)\,dy + \int_{C_4} N(x_2(y),y)\,dy = \oint_C N(x,y)\,dy.\end{aligned} \tag{8.15}$$

Adding the results of equations (8.14) and (8.15) produces the desired result.

Example 8-5. Verify Green's theorem in the plane in the special case

$$M(x,y) = x^2+y^2 \qquad \text{and} \qquad N(x,y) = xy,$$

where C is the wedge shaped curve illustrated in figure 8-8.

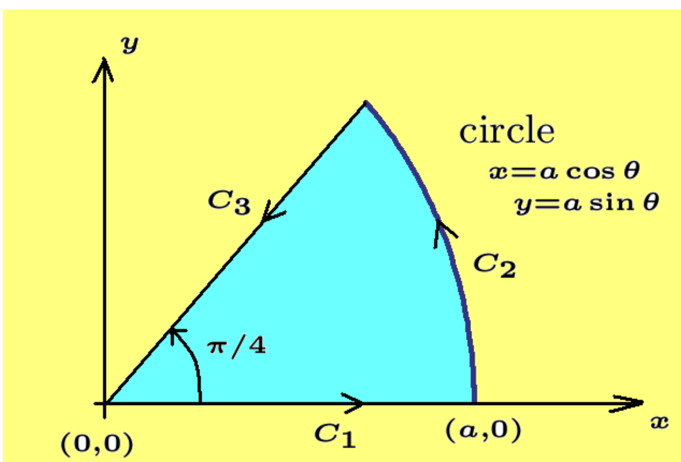


Figure 8-8. Wedge shaped path for Green's theorem example.

Solution The boundary curve can be broken up into three parts and the left-hand side of the Green's theorem can be expressed

$$\oint_C M\,dx + N\,dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy + \int_{C_3} M dx + N dy.$$

On the curve C_1, where $y=0,\ dy=0$, the first integral reduces to

$$\int_{C_1} M dx + N dy = \int_0^a x^2\,dx = \frac{a^3}{3}$$

On the curve C_2, where

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad 0 \le \theta \le \frac{\pi}{4}$$

the second integral reduces to

$$\begin{aligned}\int_{C_2} M dx + N dy =& \int_0^{\pi/4} -a^2\cdot a\sin\theta\,d\theta + a^2\sin\theta\cos\theta\cdot a\cos\theta\,d\theta\\ =& a^3\cos\theta\Big|_0^{\pi/4} - \frac{a^3}{3}\cos^3\theta\Big|_0^{\pi/4}\\ =& a^3(\frac{\sqrt{2}}{2}-1) - \frac{a^3}{3}\left(\frac{\sqrt{2}}{4}-1\right)\\ =& a^3\left(\frac{5\sqrt{2}}{12}-\frac{2}{3}\right).\end{aligned}$$

On the curve C_3, where $y=x,\ 0\le x\le \frac{\sqrt{2}}{2}a$, the third integral can be expressed as

$$\int_{C_3} M dx + N dy = \int_{\frac{\sqrt{2}}{2}a}^0 2x^2\,dx + x^2 dx = -\frac{\sqrt{2}}{4}a^3.$$

Adding the three integrals give us the line integral portion of Green's theorem which is

$$\oint_C M\,dx + N\,dy = \frac{1}{3}a^3 + a^3\left(\frac{5\sqrt{2}}{12}-\frac{2}{3}\right) - \frac{\sqrt{2}}{4}a^3 = \frac{a^3}{3}\left(\frac{\sqrt{2}}{2}-1\right).$$

The area integral representing the right-hand side of Green's theorem is now evaluated. One finds

$$\frac{\partial N}{\partial x} = y, \qquad \frac{\partial M}{\partial y} = 2y$$

and

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint -y\,dy\,dx$$

The geometry of the problem suggests a transformation to polar coordinates in order to evaluate the integral. Changing to polar coordinates the above integral becomes

$$\int_0^{\pi/4}\int_0^a (r\sin\theta)(r\,dr d\theta) = -\frac{r^3}{3}\Big|_0^a\int_0^{\pi/4}\sin\theta\,d\theta = \frac{a^3}{3}\left(\frac{\sqrt{2}}{2}-1\right).$$

■

Solution of Differential Equations by Line Integrals

The total differential of a function $\phi = \phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy. \tag{8.17}$$

When the right-hand side of this equation is set equal to zero, the resulting equation is called **an exact differential equation**, and $\phi = \phi(x, y) = Constant$ is called **a primitive or integral of this equation**. The set of curves $\phi(x, y) = C = constant$ represents a family of solution curves to the exact differential equation.

A differential equation of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{8.17}$$

is **an exact differential equation** if there exists a function $\phi = \phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y).$$

If such a function ϕ exists, then the mixed second partial derivatives must be equal and

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial M}{\partial y} = M_y = \frac{\partial^2 \phi}{\partial y\, \partial x} = \frac{\partial N}{\partial x} = N_x. \tag{8.19}$$

Hence **a necessary condition that the differential equation be exact** is that the **partial derivative of M with respect to y must equal the partial derivative of N with respect to x or $M_y = N_x$**. If the differential equation is exact, then Green's theorem tells us that the line integral of $M\, dx + N\, dy$ around a closed curve must equal zero, since

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = 0 \quad \text{because} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \tag{8.19}$$

For an arbitrary path of integration, such as the path illustrated in figure 8-9, the above integral can be expressed

$$\begin{aligned} \oint_C M\, dx + N\, dy = & \int_{x_0}^{x} M(x, y_1(x))\, dx + N(x, y_1(x))\, dy \\ & + \int_{x}^{x_0} M(x, y_2(x))\, dx + N(x, y_2(x))\, dy = 0 \end{aligned}$$

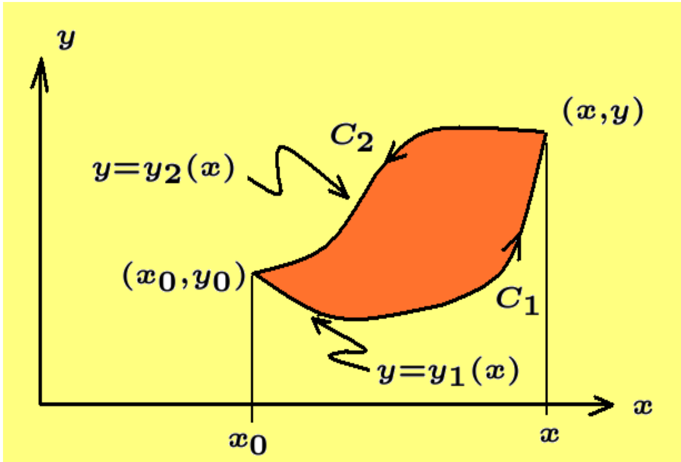


Figure 8-9. Arbitrary paths connecting points (x_0, y_0) and (x, y).

Consequently, one can write

$$\int_{x_0}^{x} M(x, y_1(x))\,dx + N(x, y_1(x))\,dy = \int_{x_0}^{x} M(x, y_2(x))\,dx + N(x, y_2(x))\,dy. \tag{8.20}$$

Equation (8.20) shows that the line integral of $M\,dx + N\,dy$ from (x_0, y_0) to (x, y) is **independent of the path joining these two points.**

It is now demonstrated that the line integral

$$\int_{(x_0,y_0)}^{(x,y)} M(x, y)\,dx + N(x, y)\,dy$$

is a function of x and y which is related to the solution of the exact differential equation $M\,dx + N\,dy = 0$. Observe that if $M\,dx + N\,dy$ is an exact differential, there exists a function $\phi = \phi(x, y)$ such that $\phi_x = M$ and $\phi_y = N$, and the above line integral reduces to

$$\int_{(x_0,y_0)}^{(x,y)} \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = \int_{(x_0,y_0)}^{(x,y)} d\phi = \phi\,\Big|_{(x_0,y_0)}^{(x,y)} = \phi(x, y) - \phi(x_0, y_0). \tag{8.21}$$

Thus the solution of the differential equation $M\,dx + N\,dy = 0$ can be represented as $\phi(x, y) = C = Constant$, where the function ϕ can be obtained from the integral

$$\phi(x, y) - \phi(x_0, y_0) = \int_{(x_0,y_0)}^{(x,y)} M(x, y)\,dx + N(x, y)\,dy. \tag{8.22}$$

Since the line integral is independent of the path of integration, it is possible to select **any convenient path of integration from** (x_0, y_0) **to** (x, y).

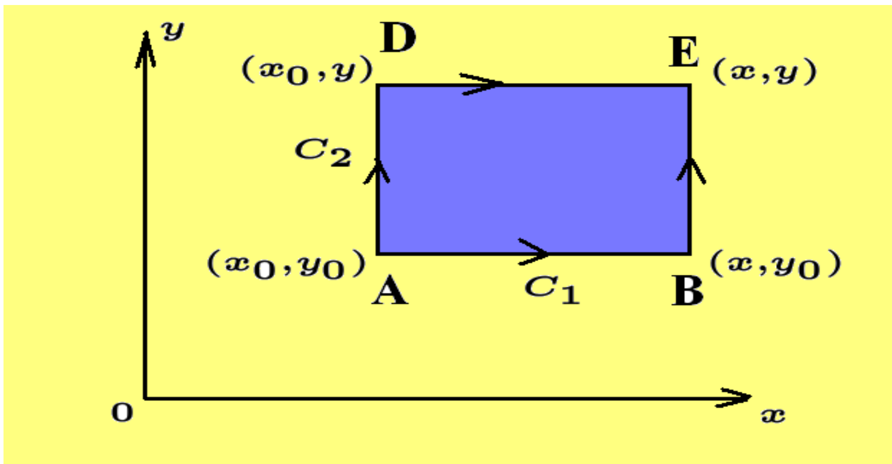


Figure 8-10. Path of integration for solution to exact differential equation.

Illustrated in figure 8-10 are two paths of integration consisting of straight-line segments. The point (x_0, y_0) may be chosen as any convenient point which guarantees that the functions M and N remain bounded and continuous along the line segments joining the point (x_0, y_0) to (x, y). If the path C_1 is selected, note that on the segment AB one finds $y = y_0$ is constant so $dy = 0$ and on the line segment BE one finds x is held constant so $dx = 0$. The line integral is then broken up into two parts and can be expressed in the form

$$\int_{(x_0,y_0)}^{(x,y)} d\phi = \phi(x,y) - \phi(x_0,y_0) = \int_{x_0}^{x} M(x,y_0)\,dx + \int_{y_0}^{y} N(x,y)\,dy, \tag{8.23}$$

where x is held constant in the second integral of equation (8.23). If the path C_2 is chosen as the path of integration note that on AD x is held constant so $dx = 0$ and on the segment DE y is held constant so that $dy = 0$. One should break up the line integral into two parts and express it in the form

$$\int_{(x_0,y_0)}^{(x,y)} d\phi = \phi(x,y) - \phi(x_0,y_0) = \int_{y_0}^{y} N(x_0,y)\,dy + \int_{x_0}^{x} M(x,y)\,dx, \tag{8.24}$$

where y is held constant in the second integral of equation (8.24).

Example 8-6. Find the solution of the exact differential equation

$$(2xy - y^2)\,dx + (x^2 - 2xy)\,dy = 0$$

Solution After verifying that $M_y = N_x$ one can state that the given differential equation is exact. Along the path C_1 illustrated in the figure 8-10 one finds

$$\begin{aligned}\phi(x,y) - \phi(x_0,y_0) &= \int_{x_0}^{x} (2xy_0 - y_0^2)\,dx + \int_{y_0}^{y} (x^2 - 2xy)\,dy \\ &= (x^2 y_0 - y_0^2 x)\Big|_{x_0}^{x} + (x^2 y - xy^2)\Big|_{y_0}^{y} \\ &= \left(x^2 y - xy^2\right) - (x_0^2 y_0 - x_0 y_0^2).\end{aligned}$$

Here $\phi(x,y) = x^2 y - xy^2 = Constant$ represents the solution family of the differential equation. It is left as an exercise to verify that this same result is obtained by performing the integration along the path C_2 illustrated in figure 8-10.

■

Area Inside a Simple Closed Curve.

A very interesting special case of Green's theorem concerns the area enclosed by a simple closed curve. Consider the simple closed curve such as the one illustrated in figure 8-11. Green's theorem in the plane allows one to find the area inside a simple closed curve if **one knows the values of x, y on the boundary of the curve.**

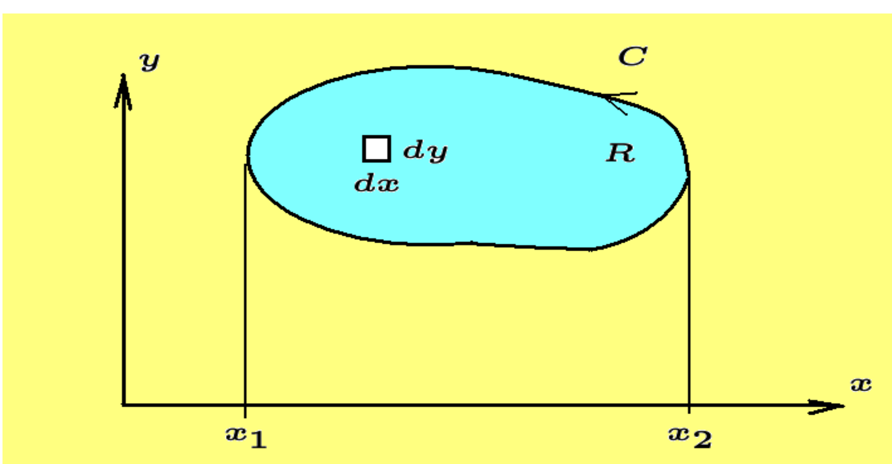


Figure 8-11. Area enclosed by a simple closed curve.

In Green's theorem the functions M and N are arbitrary. Therefore, in the special case $M = -y$ and $N = 0$ one obtains

$$\oint_C -y\,dx = \iint_R dx\,dy = A = \text{Area enclosed by C.} \tag{8.25}$$

Similarly, in the special case $M = 0$ and $N = x$, the Green's theorem becomes

$$\oint_C x\,dy = \iint_R dx\,dy = A = \text{Area enclosed by C.} \tag{8.26}$$

Adding the results from equations (8.25) and (8.26) produces

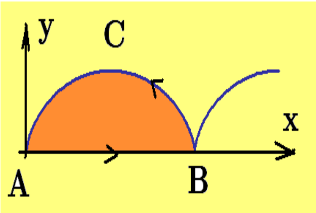
$$\begin{aligned} 2A &= \oint_C x\,dy - y\,dx \\ \text{or} \qquad A &= \frac{1}{2}\oint_C x\,dy - y\,dx = \iint_R dx\,dy = \text{Area enclosed by C.} \end{aligned} \tag{8.27}$$

Therefore the area enclosed by a simple closed curve C can be expressed as a line integral around the boundary of the region R enclosed by C. That is, by knowing the values of x and y on the boundary, one can calculate the area enclosed by the boundary. This is the concept of a device called **a planimeter**, which is a mechanical instrument used for measuring the area of a plane figure by moving a pointer around the surrounding boundary curve.

Example 8-7.

Find the area under the cycloid defined by $x = r(\phi - \sin\phi)$, $y = r(1 - \cos\phi)$ for $0 \le \phi \le 2\pi$ and $r > 0$ is a constant. Find the area illustrated by using a line integral around the boundary of the area moving from A to B to C to A.

Solution



Let BCA and the line AB denote the bounding curves of the area under the cycloid between $\phi = 0$ and $\phi = 2\pi$. The area is given by the relation

$$A = \frac{1}{2}\int_{C_1} (x\,dy - y\,dx) + \frac{1}{2}\int_{C_2} (x\,dy - y\,dx),$$

where C_1 is the straight-line from A to B and C_2 is the curve

B to C to A. On the straight-line C_1 where $y=0$ and $dy=0$ the first line integral has the value of zero. On the cycloid one finds

$$x = r(\phi - \sin\phi) \qquad y = r(1-\cos\phi)$$
$$dx = r(1-\cos\phi)\,d\phi \qquad dy = r\sin\phi\,d\phi$$

and ϕ varies from 2π to zero. By substituting the known values of x and y on the boundary of the cycloid, the line integral along C_2 becomes

$$2A = \int_{2\pi}^{0} r(\phi-\sin\phi)\,r\sin\phi\,d\phi - r(1-\cos\phi)\,r(1-\cos\phi)\,d\phi$$

or

$$\begin{aligned}
2A &= r^2\int_0^{2\pi}\left[1-2\cos\phi+\cos^2\phi+\sin^2\phi-\phi\sin\phi\right]\,d\phi\\
2A &= r^2\left[2\phi-2\sin\phi+\phi\cos\phi-\sin\phi\right]_0^{2\pi}\\
2A &= 6\pi r^2
\end{aligned}$$

Hence, the area under the curve is given by $A = 3\pi r^2$.

■

Change of Variable in Green's Theorem

Often it is convenient to change variables in an integration in order to make the integrals more tractable. If x,y are variables which are related to another set of variables u,v by a set of transformation equations

$$x = x(u,v) \qquad y = y(u,v) \tag{8.28}$$

and if these equations are continuous and have partial derivatives, then one can calculate

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv. \tag{8.29}$$

It is therefore possible to express the area integral (8.27) in the form

$$\begin{aligned}
\iint_R dx\,dy &= \frac{1}{2}\oint_C x\,dy - y\,dx\\
\iint_R dx\,dy &= \frac{1}{2}\oint_C x(u,v)\left[\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right] - y(u,v)\left[\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right]\\
&= \frac{1}{2}\oint_C \left[x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right]\,du + \left[x\frac{\partial y}{\partial v} - y\frac{\partial x}{\partial v}\right]\,dv.
\end{aligned} \tag{8.30}$$

where R is a region of the x,y-plane where the area is to be calculated.

Let $M(u,v) = \frac{1}{2}\left(x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right)$ and $N(u,v) = \frac{1}{2}\left(x\frac{\partial y}{\partial v} - y\frac{\partial x}{\partial v}\right)$ and apply Green's theorem to the integral (8.30). Using the results

$$\frac{\partial M}{\partial v} = \frac{1}{2}\left[x\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - y\frac{\partial^2 x}{\partial u\,\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right] \quad \text{and} \quad \frac{\partial N}{\partial u} = \frac{1}{2}\left[x\frac{\partial^2 y}{\partial v\,\partial u} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - y\frac{\partial^2 x}{\partial v\,\partial u} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right]$$

one finds

$$\left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = J\left(\frac{x,y}{u,v}\right), \tag{8.31}$$

where the determinant J is called **the Jacobian determinant** of the transformation from (x,y) to (u,v). The area integral can then be expressed in the form

$$A = \iint_R dx\,dy = \iint_{u,v} J\left(\frac{x,y}{u,v}\right)\,du\,dv, \tag{8.32}$$

where the limits of integration are over that range of the variables u, v which define the region R.

Example 8-8. In changing from rectangular coordinates (x,y) to polar coordinates (r,θ) the transformation equations are

$$x = x(r,\theta) = r\cos\theta \qquad y = y(r,\theta) = r\sin\theta,$$

and the Jacobian of this transformation is

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

and the area can be expressed as

$$\iint_{R_{xy}} dx\,dy = \iint_{R_{r\theta}} r\,dr\,d\theta \tag{8.33}$$

which is the familiar area integral from polar coordinates.

■

In general, an integral of the form $\iint_{R_{xy}} f(x,y)\,dx\,dy$ under a change of variables $x = x(u,v),\ y = y(u,v)$ becomes $\iint_{R_{uv}} f(x(u,v), y(u,v))\,J\left(\frac{x,y}{u,v}\right)\,du\,dv$, where the integrand is expressed in terms of u and v and the element of area $dx\,dy$ is replaced by the new element of area $J\left(\frac{x,y}{u,v}\right)\,du\,dv$.

The Curl of a Vector Field

Let $\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,x)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ denote a continuous vector field possessing continuous derivatives, and let P_0 denote a point in this vector field having coordinates (x_0,y_0,z_0). Insert into this field an arbitrary surface S which contains the point P_0 and construct a unit normal $\hat{\mathbf{e}}_n$ to the surface at point P_0. On the surface construct a simple closed curve C which encircles the point P_0. The work done in moving around the closed curve is called **the circulation at point** P_0. The circulation is a scalar quantity and is expressed as

$$\oint_C \vec{F}\cdot d\vec{r} = \text{Circulation of } \vec{F} \text{ around } C \text{ on the surface } S,$$

where the integration is taken counterclockwise. If the circulation is divided by the area ΔS enclosed by the simple closed curve C, then the limit of the ratio $\dfrac{\text{Circulation}}{\text{Area}}$ as the area ΔS tends toward zero, is called **the component of the** curl **of** $\vec{F}$ **in the direction** $\hat{\mathbf{e}}_n$ and is written as

$$(\,\text{curl}\,\vec{F}\,)\cdot\hat{\mathbf{e}}_n = \lim_{\Delta S\to 0}\frac{\oint_C \vec{F}\cdot d\vec{r}}{\Delta S}. \tag{8.34}$$

To evaluate one component of the curl of a vector field $\vec{F}$ at the point $P_0(x_0,y_0,z_0)$, construct the plane $z=z_0$ which passes through P_0 and is parallel to the xy plane. This plane has the unit normal $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_3$ at all points on the plane. In this plane, consider the circulation at P_0 due to a circle of radius ϵ centered at P_0. The equation of this circle in parametric form is

$$x = x_0+\epsilon\cos\theta,\qquad y = y_0+\epsilon\sin\theta,\qquad z = z_0$$

and in vector form $\vec{r} = (x_0+\epsilon\cos\theta)\,\hat{\mathbf{e}}_1 + (y_0+\epsilon\sin\theta)\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$. The circulation can be expressed as

$$I = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(x_0+\epsilon\cos\theta, y_0+\epsilon\sin\theta, z_0)\,[-\epsilon\sin\theta\,\hat{\mathbf{e}}_1 + \epsilon\cos\theta\,\hat{\mathbf{e}}_2]\;d\theta.$$

By expanding $\vec{F} = \vec{F}(x_0+\epsilon\cos\theta, y_0+\epsilon\sin\theta, z_0)$ in a Taylor series about $\epsilon=0$, one finds

$$\vec{F}(x_0+\epsilon\cos\theta, y_0+\epsilon\sin\theta, z_0) = \vec{F}(x_0,y_0,z_0) + \epsilon\frac{d\vec{F}}{d\epsilon} + \frac{\epsilon^2}{2!}\frac{d^2\vec{F}}{d\epsilon^2} + \cdots,$$

where all the derivatives are evaluated at $\epsilon=0$. The circulation can be written as

$$I = \oint_C \vec{F}\cdot d\vec{r} = \epsilon\mu_0 + \epsilon^2\mu_1 + \epsilon^3\mu_2 + \cdots,$$

where

$$\mu_0 = \int_0^{2\pi} \vec{F}_0 \cdot d\vec{\xi}, \qquad F_0 = F(x_0, y_0, z_0)$$
$$\mu_1 = \int_0^{2\pi} \frac{d\vec{F}}{d\epsilon} \cdot d\vec{\xi}$$
$$\mu_2 = \int_0^{2\pi} \frac{1}{2!} \frac{d^2\vec{F}}{d\epsilon^2} \cdot d\vec{\xi}$$
$$\cdots$$

where all the derivatives are evaluated at $\epsilon = 0$ and $d\vec{\xi} = (-\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2)\,d\theta$. The vector $\vec{F}_0$ is a constant and the integral μ_0 is easily shown to be zero. The vector $\frac{d\vec{F}}{d\epsilon}$ evaluated at $\epsilon = 0$, when expanded is given by

$$\begin{aligned}\frac{d\vec{F}}{d\epsilon} = \frac{\partial \vec{F}}{\partial x}\cos\theta + \frac{\partial \vec{F}}{\partial y}\sin\theta = &\left(\frac{\partial F_1}{\partial x}\cos\theta + \frac{\partial F_1}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_1 \\ &+ \left(\frac{\partial F_2}{\partial x}\cos\theta + \frac{\partial F_2}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_2 \\ &+ \left(\frac{\partial F_3}{\partial x}\cos\theta + \frac{\partial F_3}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_3,\end{aligned}$$

where the partial derivatives are all evaluated at $\epsilon = 0$. It is readily verified that the integral μ_1 reduces to

$$\mu_1 = \pi\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

The area of the circle surrounding P_0 is $\pi\epsilon^2$, and consequently the ratio of the circulation divided by the area in the limit as ϵ tends toward zero produces

$$(\text{ curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}. \tag{8.35}$$

Similarly, by considering other planes through the point P_0 which are parallel to the xz and yz planes, arguments similar to those above produce the relations

$$(\text{ curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \quad \text{and} \quad (\text{ curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}. \tag{8.36}$$

Adding these components gives the mathematical expression for curl $\vec{F}$. One finds the curl $\vec{F}$ can be written as

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{e}}_3. \tag{8.37}$$

The curl $\vec{F}$ can be expressed by using the operator ∇ in the determinant form

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{\mathbf{e}}_1[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}] - \hat{\mathbf{e}}_2[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}] + \hat{\mathbf{e}}_3[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}] \quad (8.38)$$

Example 8-9. Find the curl of the vector field

$$\vec{F} = x^2 y\, \hat{\mathbf{e}}_1 + (x^2 + y^2 z)\, \hat{\mathbf{e}}_2 + 4xyz\, \hat{\mathbf{e}}_3$$

Solution From the relation (8.38) one finds

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & x^2 + y^2 z & 4xyz \end{vmatrix}$$

$$\text{curl } \vec{F} = (4xz - y^2)\, \hat{\mathbf{e}}_1 - 4yz\, \hat{\mathbf{e}}_2 + (2x - x^2)\, \hat{\mathbf{e}}_3.$$

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Physical Interpretation of Curl

The curl of a vector field is itself a vector field. If curl $\vec{F} = \vec{0}$ at all points of a region R, where $\vec{F}$ is defined, then the vector field $\vec{F}$ is called an **irrotational vector field**, otherwise the vector field is called **rotational.**

The circulation $\oint_C \vec{F} \cdot d\vec{r}$ about a point P_0 can be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \oint_C \vec{F} \cdot \hat{\mathbf{e}}_t\, ds$$

where C is a simple closed curve about the point P_0 enclosing an area. The quantity $\vec{F} \cdot \hat{\mathbf{e}}_t$, evaluated at a point on the curve C, represents the projection of the vector $\vec{F}(x, y, z)$, onto the unit tangent vector to the curve C. If the summation of these tangential components around the simple closed curve is positive or negative, then this indicates that there is a moment about the point P_0 which causes a rotation. The circulation is **a measure of the forces tending to produce a rotation about a given point** P_0. The curl is the limit of the circulation divided by the area surrounding P_0 as the area tends toward zero. The curl can also be thought of as **a measure of the circulation density of the field** or as **a measure of the angular velocity produced by the vector field.**

Consider the two-dimensional velocity field $\vec{V} = V_0\, \hat{\mathbf{e}}_1, \quad 0 \le y \le h$, where V_0 is constant, which is illustrated in figure 8-12(a). The velocity field $\vec{V} = V_0\, \hat{\mathbf{e}}_1$ is uniform, and to each point (x, y) there corresponds a constant velocity vector in the $\hat{\mathbf{e}}_1$ direction. The curl of this velocity field is zero since the derivative of a constant is zero. The given velocity field is an example of an irrotational vector field.

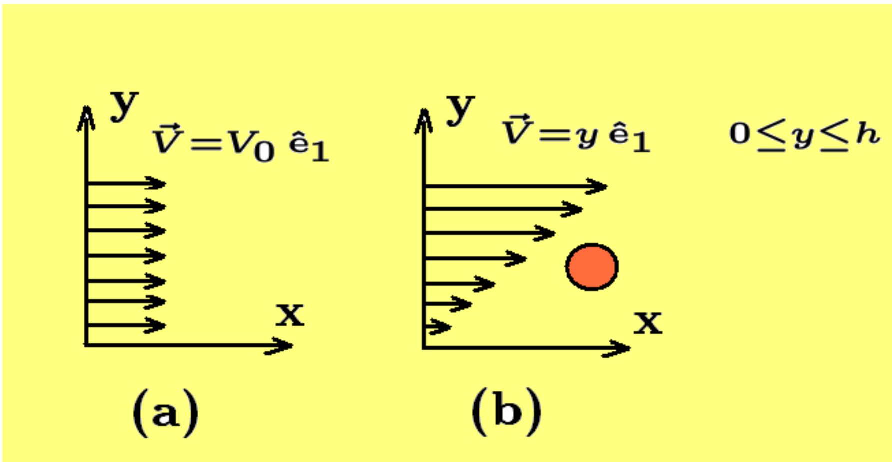


Figure 8-12. Comparison of two dimensional velocity fields.

In comparison, consider the two-dimensional velocity field $\vec{V} = y\,\hat{\mathbf{e}}_1, \quad 0 \le y \le h$, which is illustrated in figure 8-12(b). Here the velocity field may be thought of as representing the flow of fluid in a river. The curl of this velocity field is

$$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\,\hat{\mathbf{e}}_3.$$

In this example, the velocity field is rotational. Consider a spherical ball dropped into this velocity field. The curl $\vec{V}$ tells us that the ball rotates in a clockwise direction about an axis normal to the xy plane. Observe the difference in velocities of the water particles acting upon the upper and lower surfaces of the sphere which cause the clockwise rotation.

Using the right-hand rule, let the fingers of the right hand move in the direction of the rotation. The thumb then points in the $-\,\hat{\mathbf{e}}_3$ direction.

The curl tells us the direction of rotation, but it does not tell us the angular velocity associated with a point as the following example illustrates. Consider a basin of water in which the water is rotating with a constant angular velocity $\vec{\omega} = \omega_0\,\hat{\mathbf{e}}_3$. The velocity of a particle of fluid at a position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ is given by

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & \omega_0 \\ x & y & 0 \end{vmatrix} = -\omega_0 y\,\hat{\mathbf{e}}_1 + \omega_0 x\,\hat{\mathbf{e}}_2.$$

The curl of this velocity field is

$$\operatorname{curl} \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega_0 y & -\omega_0 x & 0 \end{vmatrix} = 2\omega_0 \, \hat{\mathbf{e}}_3.$$

The curl tells us the direction of the angular velocity but not its magnitude.

Stokes Theorem

Let $\vec{F} = \vec{F}(x, y, z)$ denote a vector field having continuous derivatives in a region of space. Let S denote an open two-sided surface in the region of the vector field. For any simple closed curve C lying on the surface S, the following integral relation holds

$$\iint_S (\operatorname{curl} \vec{F}\,) \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{\mathbf{e}}_n \, dS = \oint_C \vec{F} \cdot d\vec{r}, \tag{8.39}$$

where the surface integrations are understood to be over the portion of the surface enclosed by the simple closed curve C lying on S and the line integral around C is in the positive sense with respect to the normal vector to the surface bounded by the simple closed curve C. The above integral relation is known as **Stokes theorem.**[1] In scalar form, the line and surface integrals in Stokes theorem can be expressed as

$$\begin{aligned} \iint_S (\operatorname{curl} \vec{F}\,) \cdot d\vec{S} &= \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n \, dS \\ &= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\mathbf{e}}_3 \right] \cdot \hat{\mathbf{e}}_n \, dS \end{aligned} \tag{8.40}$$

and $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_1 \, dx + F_2 \, dy + F_3 \, dz,$$

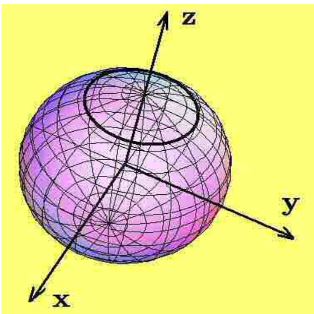
where $\hat{\mathbf{e}}_n$ is a unit normal to the surface S inside the closed curve C. In this case the path of integration C is counterclockwise with respect to this normal. By the right-hand rule if you place the thumb of your right hand in the direction of the normal, then your fingers indicate the direction of integration in the counterclockwise or positive sense.

[1] George Gabriel Stokes (1819-1903) An Irish mathematician who studied hydrodynamics.

Example 8-10. Illustrate Stokes theorem using the vector field

$$\vec{F} = yz\,\hat{\mathbf{e}}_1 + xz^2\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3,$$

where the surface S is a portion of a sphere of radius r inside a circle on the sphere.



The surface of the sphere can be described by the parametric equations.

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

for r constant, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The position vector to a point on the sphere being represented

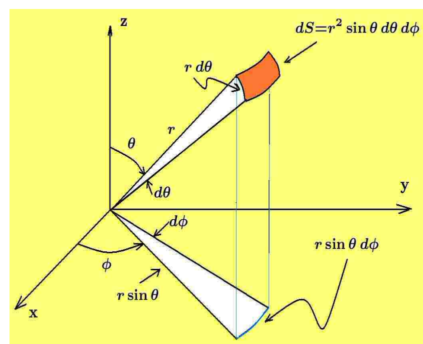
$$\vec{r} = \vec{r}(\theta,\phi) = r\sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + r\sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + r\cos\theta\,\hat{\mathbf{e}}_3 \qquad (r \text{ is constant})$$

From the previous chapter we found an element of surface area on the sphere can be represented

$$dS = \left|\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}\right| d\theta\,d\phi = \sqrt{EG-F^2}\,d\theta\,d\phi = r^2\sin\theta\,d\theta\,d\phi \qquad (r \text{ is constant}) \tag{8.41}$$

The physical interpretation of dS being that it is the area of the parallelogram having the sides $\frac{\partial\vec{r}}{\partial\theta}\,d\theta$ and $\frac{\partial\vec{r}}{\partial\phi}\,d\phi$ with diagonal vector

$$d\vec{r} = \frac{\partial\vec{r}}{\partial\theta}\,d\theta + \frac{\partial\vec{r}}{\partial\phi}\,d\phi$$



If one holds $\theta = \theta_0$ constant, one obtains a circle C on the sphere described by

$$\vec{r} = \vec{r}(\phi) = r\sin\theta_0\cos\phi\,\hat{\mathbf{e}}_1 + r\sin\theta_0\sin\phi\,\hat{\mathbf{e}}_2 + r\cos\theta_0\,\hat{\mathbf{e}}_3, \qquad 0 \le \phi \le 2\pi \tag{8.42}$$

A unit outward normal to the sphere and inside the circle C is given by

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3, \qquad \begin{array}{c} 0 \le \phi \le 2\pi \\ 0 \le \theta \le \theta_0 \end{array} \tag{8.43}$$

The vector $\operatorname{curl}\vec{F}$ is calculated from the determinant

$$\begin{aligned}\operatorname{curl}\vec{F} =& \nabla\times\vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz^2 & xy \end{vmatrix} \\ =& \,\hat{\mathbf{e}}_1(x-2xz) + \hat{\mathbf{e}}_2(0) + \hat{\mathbf{e}}_3(z^2-z)\end{aligned}$$

The left-hand side of Stokes theorem can be expressed

$$\begin{aligned}
\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint\limits_S \operatorname{curl}\vec{F}\cdot\hat{\mathbf{e}}_n\,dS = &\iint\limits_S \left[x(1-2z)\sin\theta\cos\phi + (z^2-z)\cos\theta\right]\, r^2\sin\theta\,d\theta\,d\phi\\
=&\int_0^{2\pi}\int_0^{\theta_0}\left[r\sin^2\theta\cos^2\phi(1-2r\cos\theta)+(r^2\cos^3\theta - r\cos^2\theta)\right]\, r^2\sin\theta\,d\theta\,d\phi\\
=&\int_0^{2\pi}\frac{r^3}{12}\Big[4(-1+\cos^3\theta_0)-3r(-1+\cos^4\theta_0)\\
&\qquad + 16(2+\cos\theta_0)\cos^2\phi\sin^4\left(\frac{\theta_0}{2}\right) - 6r\cos^2\phi\sin^4\theta_0\Big]\,d\phi\\
\iint\limits_S \operatorname{curl}\vec{F}\cdot d\vec{S} =&\pi r^3\cos\theta_0(-1+r\cos\theta_0)\sin^2\theta_0
\end{aligned}$$

The right-hand side of Stokes theorem can be expressed

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C yz\,dx + xz^2\,dy + xy\,dz$$

Substitute the values of x, y, z on the curve C using

$$\begin{aligned}
x =& r\sin\theta_0\cos\phi, \qquad & y =& r\sin\theta_0\sin\phi, \qquad & z =& r\cos\theta_0\\
dx =& -r\sin\theta_0\sin\phi\,d\phi, \qquad & dy =& r\sin\theta_0\cos\phi, \qquad & dz =& 0
\end{aligned}$$

The right-hand side of Stokes theorem then becomes

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} =& \oint_C \left[yz(-r\sin\theta_0\sin\phi) + xz^2(r\sin\theta_0\cos\phi) + xy(0)\right]\,d\phi\\
=&\int_0^{2\pi}\left[r\sin\theta_0\sin\phi(r\cos\theta_0)(-r\sin\theta_0\sin\phi) + (r\sin\theta_0\cos\phi)(r^2\cos^2\theta_0)(r\sin\theta_0\cos\phi)\right]\,d\phi\\
=&\pi r^3\cos\theta_0(-1+r\cos\theta_0)\sin^2\theta_0
\end{aligned}$$

■

Proof of Stokes Theorem

To prove Stokes theorem one could verify each of the following integral relations

$$\begin{aligned}
\iint\limits_S \left(\frac{\partial F_1}{\partial z}\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_n - \frac{\partial F_1}{\partial y}\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_n\right)dS =& \oint_C F_1\,dx\\
\iint\limits_S \left(\frac{\partial F_2}{\partial x}\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_n - \frac{\partial F_2}{\partial z}\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_n\right)dS =& \oint_C F_2\,dy\\
\iint\limits_S \left(\frac{\partial F_3}{\partial y}\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_n - \frac{\partial F_3}{\partial x}\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_n\right)dS =& \oint_C F_3\,dz.
\end{aligned}\tag{8.44}$$

Then an addition of these integrals would produce the Stokes theorem as given by equation (8.40). However, the arguments used in proving the above integrals are repetitious, and so only the first integral is verified.

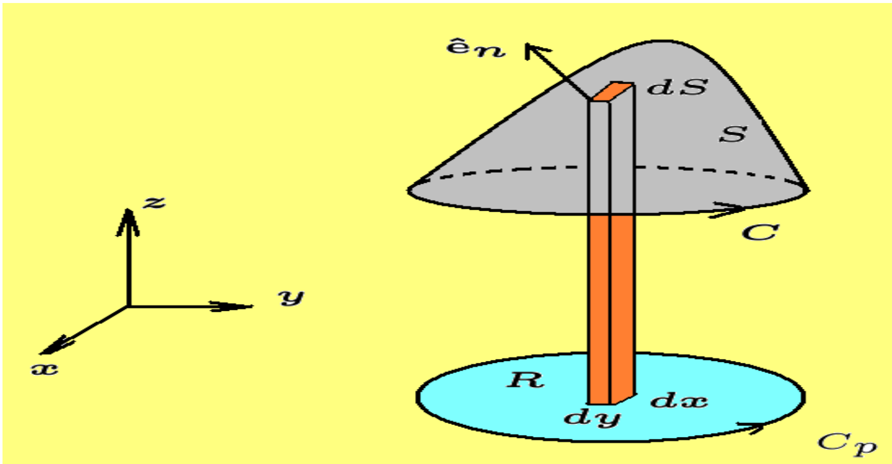


Figure 8-13. Surface S bounded by a simple closed curve C on the surface.

Let $z = z(x, y)$ define the surface S and consider the projections of the surface S and the curve C onto the plane $z = 0$ as illustrated in figure 8-13. Call these projections R and C_p. The unit normal to the surface has been shown to be of the form

$$\hat{\mathbf{e}}_n = \frac{-\dfrac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \dfrac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}} \tag{8.45}$$

Consequently, one finds

$$\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \frac{-\frac{\partial z}{\partial y}}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}}, \qquad \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}} \qquad (8.46))$$

The element of surface area can be expressed as

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

and consequently the integral on the left side of equation (8.44) can be simplified to the form

$$\iint\limits_S \left(\frac{\partial F_1}{\partial z}\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n - \frac{\partial F_1}{\partial y}\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\right) dS = \iint\limits_S \left(-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_1}{\partial y}\right) dx\, dy \tag{8.47}$$

Now on the surface S defined by $z = z(x, y)$ one finds $F_1 = F_1(x, y, z) = F_1(x, y, z(x, y))$ is a function of x and y so that a differentiation of the composite function F_1 with respect to y produces

$$\frac{\partial F_1(x, y, z(x, y))}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} \tag{8.48}$$

which is the integrand in the integral (8.47) with the sign changed. Therefore, one can write

$$\iint_S \left(-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_1}{\partial y}\right) dx\, dy = -\iint_S \frac{\partial F_1(x, y, z(x, y))}{\partial y}\, dx\, dy \tag{8.49}$$

Now by using Greens theorem with $M(x, y) = F_1(x, y, z(x, y))$ and $N(x, y) = 0$, the integral (8.49) can be expressed as

$$-\iint_S \frac{\partial F_1(x, y, z(x, y))}{\partial y}\, dx\, dy = \int_{C_p} F_1(x, y, z(x, y))\, dx = \int_C F_1(x, y, z)\, dx \tag{8.50}$$

which verifies the first integral of the equations (8.44). The remaining integrals in equations (8.44) may be verified in a similar manner.

Example 8-11. Verify Stokes theorem for the vector field

$$\vec{F} = 3x^2 y\, \hat{\mathbf{e}}_1 + x^2 y\, \hat{\mathbf{e}}_2 + z\, \hat{\mathbf{e}}_3,$$

where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution The given vector field has the curl vector

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 y & x^2 y & z \end{vmatrix} = (2xy - 3x^2)\, \hat{\mathbf{e}}_3.$$

The unit normal to the sphere at a general point (x, y, z) on the sphere is given by

$$\hat{\mathbf{e}}_n = x\, \hat{\mathbf{e}}_1 + y\, \hat{\mathbf{e}}_2 + z\, \hat{\mathbf{e}}_3$$

and the element of surface area dS when projected upon the xy plane is

$$dS = \frac{dx\, dy}{\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3} = \frac{dx\, dy}{z}.$$

The surface integral portion of Stokes theorem can therefore by expressed as

$$\begin{aligned}\iint\limits_S \text{curl } \vec{F}\, d\vec{S} =& \iint\limits_S (2xy - 3x^2)\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\, dS = \iint\limits_S (2xy - 3x^2)\, dx\, dy\\
=& \int_{-1}^{1} \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} x(2y - 3x)\, dy\, dx = \int_{-1}^{1} x(y^2 - 3xy)\Big|_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dx\\
=& \int_{-1}^{1} \left\{x\left[(1-x^2) - 3x\sqrt{1-x^2}\right] - x\left[(1-x^2) + 3x\sqrt{1-x^2}\right]\right\} dx\\
=& \int_{-1}^{1} -6x^2\sqrt{1-x^2}\, dx = -\frac{3\pi}{4}.\end{aligned}$$

For the line integral portion of Stokes theorem one should observe the boundary of the surface S is the circle

$$x = \cos\theta \qquad y = \sin\theta \qquad z = 0 \qquad 0 \le \theta \le 2\pi.$$

Consequently, there results

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} =& \oint_C 3x^2y\, dx + x^2y\, dy + z\, dz\\
=& \int_0^{2\pi} -3\cos^2\theta \sin^2\theta\, d\theta + \cos^3\theta \sin\theta\, d\theta = -\frac{3\pi}{4}.\end{aligned}$$

Think of the unit circle $x^2 + y^2 = 1$ with a rubber sheet over it. The hemisphere in this example is assumed to be formed by stretching this rubber sheet. All two-sided surfaces that result by deforming the rubber sheet in a continuous manner are surfaces for which Stokes theorem is applicable.

■

Related Integral Theorems

Let ϕ denote a scalar field and $\vec{F}$ a vector field. These fields are assumed to be continuous with continuous derivatives. For the volumes, surfaces, and simple closed curves of Stokes theorem and the divergence theorem, there exist the additional integral relationships

$$\iiint\limits_V \text{curl } \vec{F}\, dV = \iiint\limits_V \nabla \times \vec{F}\, dV = \iint\limits_S \hat{\mathbf{e}}_n \times \vec{F} dS \tag{8.51}$$

$$\iiint\limits_V \text{grad } \phi\, dV = \iiint\limits_V \nabla\, \phi\, dV = \iint\limits_S \phi\, \hat{\mathbf{e}}_n\, dS \tag{8.52}$$

$$\iint\limits_S \hat{\mathbf{e}}_n \times \text{grad } \phi\, dS = \iint\limits_S \hat{\mathbf{e}}_n \times \text{grad } \phi\, dS = -\iint\limits_S \text{grad } \phi\, \times d\vec{S} = \oint_C \phi\, d\vec{r}. \tag{8.53}$$

The integral relation (8.51) follows from the divergence theorem. In the divergence theorem, substitute $\vec{F} = \vec{H} \times \vec{C}$, where $\vec{C}$ is an arbitrary constant vector. By using the vector relations

$$\begin{aligned}\nabla \cdot (\vec{H} \times \vec{C}) =&\vec{C} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{C})\\ \text{div } \vec{F} = \nabla \cdot \vec{F} = \nabla \cdot (\vec{H} \times \vec{C}) &= \vec{C} \cdot (\nabla \times \vec{H}) \qquad \text{and} \\ \vec{F} \cdot \hat{\mathbf{e}}_n = (\vec{H} \times \vec{C}) \cdot \hat{\mathbf{e}}_n = \vec{H} \cdot (\vec{C} \times \hat{\mathbf{e}}_n) &= \vec{C} \cdot (\hat{\mathbf{e}}_n \times \vec{H}) \qquad \text{(triple scalar product)},\end{aligned} \tag{8.54}$$

the divergence theorem can be written as

$$\iiint_V \text{div } (\vec{H} \times \vec{C})\, dV = \iiint_V \vec{C} \cdot (\nabla \times \vec{H})\, dV = \iint_S (\vec{H} \times \vec{C}) \cdot \hat{\mathbf{e}}_n\, dS = \iint_S \vec{C} \cdot (\hat{\mathbf{e}}_n \times \vec{H})\, dS \tag{8.55}$$

Since $\vec{C}$ is a constant vector one may write

$$\vec{C} \cdot \iiint_V \nabla \times \vec{H}\, dV = \vec{C} \cdot \iint_S \hat{\mathbf{e}}_n \times \vec{H}\, dS. \tag{8.56}$$

For arbitrary $\vec{C}$ this relation implies

$$\iiint_V \nabla \times \vec{H}\, dV = \iint_S \hat{\mathbf{e}}_n \times \vec{H}\, dS. \tag{8.57}$$

In this integral replace $\vec{H}$ by $\vec{F}$ ($\vec{H}$ is arbitrary) to obtain the relation (8.51).

The integral (8.52) also is a special case of the divergence theorem. If in the divergence theorem one makes the substitution $\vec{F} = \phi\,\vec{C}$, where ϕ is a scalar function of position and $\vec{C}$ is an arbitrary constant vector, there results

$$\iint_V \text{div } \vec{F}\, dV = \iiint_V \nabla(\phi\,\vec{C})\, dV = \iiint_V \vec{C} \cdot \nabla\phi\, dV = \iint_S \vec{C}\phi\, d\vec{S}. \tag{8.58}$$

where the vector identity $\nabla(\phi\,\vec{C}) = (\nabla\phi)\cdot\vec{C} + \phi(\nabla\times\vec{C})$ has been employed. The relation given by equation (8.58), for an arbitrary constant vector $\vec{C}$, produces the integral relation (8.52).

The integral (8.53) is a special case of Stokes theorem. If in Stokes theorem one substitutes $\vec{F} = \phi\,\vec{C}$, where $\vec{C}$ is a constant vector, there results

$$\begin{aligned}\iint_S (\,\text{curl } \vec{F}\,) \cdot d\vec{S} = \iint_S \nabla \times (\phi\,\vec{C}) \cdot \hat{\mathbf{e}}_n\, dS &= \iint_S (\nabla\,\phi \times \vec{C}) \cdot \hat{\mathbf{e}}_n\, dS \\ = \iint_S (\,\hat{\mathbf{e}}_n \times \nabla\,\phi) \cdot \vec{C}\, dS &= \oint_C \vec{C}\,\phi\, d\vec{r}.\end{aligned} \tag{8.59}$$

For arbitrary $\vec{C}$, this integral implies the relation (8.53). That is, one can factor out the constant vector $\vec{C}$ as long as this vector is different from zero. Under these conditions the integral relation (8.53) must hold.

Region of Integration

Green's, Gauss' and Stokes theorems are valid only if certain conditions are satisfied. In these theorems it has been assumed that the integrands are continuous inside the region and on the boundary where the integrations occur. Also assumed is that all necessary derivatives of these integrands exist and are continuous over the regions or boundaries of the integration. In the study of the various vector and scalar fields arising in engineering and physics, there are times when discontinuities occur at points inside the regions or on the boundaries of the integration. Under these circumstances the above theorems are still valid but one must modify the theorems slightly. Modification is done by using superposition of the integrals over each side of a discontinuity and under these circumstances there usually results some kind of a jump condition involving the value of the field on either side of the discontinuity.

If a region of space has the property that every simple closed curve within the region can be deformed or shrunk in a continuous manner to a single point within the region, without intersecting a boundary of the region, then the region is said to be **simply connected.** If in order to shrink or reduce a simply closed curve to a point the curve must leave the region under consideration, then the region is said to be **a multiply connected region.** An example of a multiply connected region is the surface of a torus. Here a circle which encloses the hole of this doughnut-shaped region cannot be shrunk to a single point without leaving the surface, and so the region is called a multiply-connected region.

If a region is multiply connected it usually can be **modified by introducing imaginary cuts or lines within the region and requiring that these lines cannot be crossed.** By introducing appropriate cuts, one can usually modify a multiply connected region into a simply connected region. **The theorems of Gauss, Green, and Stokes are applicable to simply connected regions or multiply connected regions which can be reduced to simply connected regions by introducing suitable cuts.**

Example 8-12. Consider the evaluation of a line integral around a curve in a multiply connected region. Let the multiply connected region be bounded by curves like $C_0, C_1, \ldots, C_n$ as illustrated in figure 8-14(a).

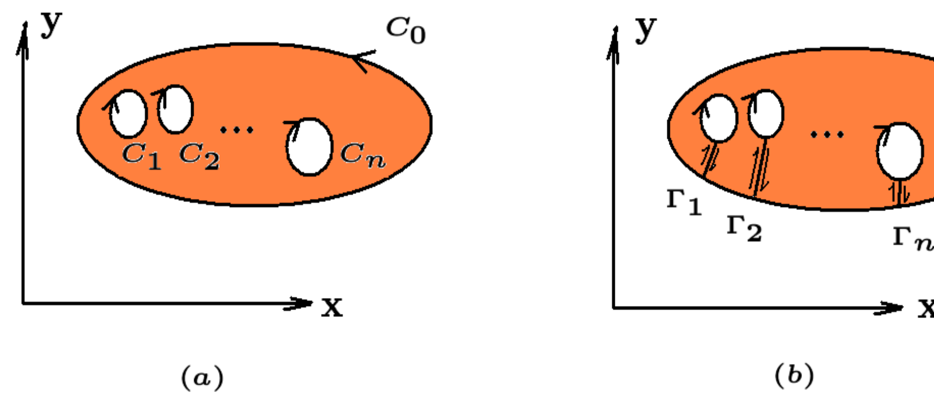


Figure 8-14. Multiply-connected region.

Such a region can be converted to a simply connected region by introducing cuts Γ_i, $i = 1, \ldots, n$. Observe that one can integrate along C_0 until one comes to a cut, say for example the cut Γ_1 in figure 8-14. Since it is not possible to cross a cut, one must integrate along Γ_1 to the curve C_1, then move about C_1 clockwise and then integrate along Γ_1 back to the curve C_0. Continue this process for each of the cuts one encounter as one moves around C_0. Note that the line integrals along the cuts add to zero in pairs (i.e. from C_0 to C_i and from C_i to C_0 for each $i = 1, 2, \ldots n$), then one is left with only the line integrals around the curves $C_0, C_1, \ldots, C_n$ in the sense illustrated in figure 8-14(b).

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Green's First and Second Identities

Two special cases of the divergence theorem, known as **Green's first and second identities**, are generated as follows.

In the divergence theorem, make the substitution $\vec{F} = \psi\nabla\phi$ to obtain

$$\iiint_V \nabla \vec{F}\, dV = \iiint_V \nabla\left(\psi\nabla\phi\right)\, dV = \iint_S \psi\nabla\phi\cdot d\vec{S} = \iint_S \psi\frac{\partial\phi}{\partial n}\, dS \tag{8.60}$$

where $\dfrac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{\mathbf{e}}_n$ is known as a normal derivative at the boundary. Using the relation

$$\nabla(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi$$

one can express equation (8.60) in the form

$$\iint\limits_V \left(\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi\right)\,dV = \iint\limits_S \psi\nabla\phi\cdot d\vec{S} = \iint\limits_S \psi\frac{\partial\phi}{\partial n}\,dS \tag{8.61}$$

This result is known as **Green's first identity.**

In Green's first identity interchange ψ and ϕ to obtain

$$\iint\limits_V \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)\,dV = \iint\limits_S \phi\nabla\psi\cdot d\vec{S} = \iint\limits_S \phi\frac{\partial\psi}{\partial n}\,dS \tag{8.62}$$

Subtracting equation (8.61) from equation (8.62) produces **Green's second identity**

$$\iiint\limits_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,dV = \iint\limits_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot d\vec{S} \tag{8.63}$$

or

$$\iiint\limits_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,dV = \iint\limits_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\,dS$$

Green's first and second identities have many uses in studying scalar and vector fields arising in science and engineering.

Additional Operators

The del operator in Cartesian coordinates

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3 \tag{8.64}$$

has been used to express the gradient of a scalar field and the divergence and curl of a vector field. There are other operators involving the operator ∇. In the following list of operators let $\vec{A}$ denote a vector function of position which is both continuous and differentiable.

1. The operator $\vec{A}\cdot\nabla$ is defined as

$$\begin{aligned}\vec{A}\cdot\nabla &= \left(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3\right)\cdot\left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\\ &= A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\end{aligned} \tag{8.65}$$

Note that $\vec{A}\cdot\nabla$ is an operator which can operate on vector or scalar quantities.

2. The operator $\vec{A}\times\nabla$ is defined as

$$\begin{aligned}\vec{A}\times\nabla &= \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ A_1 & A_2 & A_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\\ &= \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{\mathbf{e}}_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{\mathbf{e}}_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{\mathbf{e}}_3\end{aligned} \tag{8.66}$$

This operator is a vector operator.

3. The Laplacian operator $\nabla^2 = \nabla \cdot \nabla$ in rectangular Cartesian coordinates is given by

$$\nabla^2 = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot\left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2} \tag{8.67}$$

This operator can operate on vector or scalar quantities.

One must be careful in the use of operators because **in general, they are not commutative.** They operate only on the quantities to their immediate right.

Example 8-13. For the vector and scalar fields defined by

$$\begin{aligned} &\vec{B} = xyz\,\hat{\mathbf{e}}_1 + (x+y)\,\hat{\mathbf{e}}_2 + (z-x)\,\hat{\mathbf{e}}_3 = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3 \\ &\vec{A} = x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3 = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3 \\ \text{and}\quad &\phi = x^2y^2 + z^2yx \end{aligned}$$

evaluate each of the following.

$$\begin{array}{llll} (a) & (\vec{A}\cdot\nabla)\vec{B} & (b) & (\vec{A}\times\nabla)\cdot\vec{B} \\ (c) & \nabla^2\vec{A} & (d) & (\vec{A}\times\nabla)\phi \end{array}$$

Solution

(a)

$$\begin{aligned} (\vec{A}\cdot\nabla)\vec{B} =&\ x^2\frac{\partial\vec{B}}{\partial x} + xy\frac{\partial\vec{B}}{\partial y} + y^2\frac{\partial\vec{B}}{\partial z} \\ =&\ x^2\left(\frac{\partial B_1}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial x}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial x}\,\hat{\mathbf{e}}_3\right) \\ &+ xy\left(\frac{\partial B_1}{\partial y}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial y}\,\hat{\mathbf{e}}_3\right) \\ &+ y^2\left(\frac{\partial B_1}{\partial z}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial z}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial z}\,\hat{\mathbf{e}}_3\right) \\ =&\ \left[x^2(yz) + xy(xz) + y^2(xy)\right]\,\hat{\mathbf{e}}_1 + \left[x^2(1) + xy(1)\right]\,\hat{\mathbf{e}}_2 + \left[x^2(-1) + y^2(1)\right]\,\hat{\mathbf{e}}_3 \end{aligned}$$

(b)

$$\begin{aligned} (\vec{A}\times\nabla)\cdot\vec{B} =& \left[\left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\,\hat{\mathbf{e}}_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\,\hat{\mathbf{e}}_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\,\hat{\mathbf{e}}_3\right]\cdot\vec{B} \\ =&\ \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)B_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)B_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)B_3 \\ =&\ \left[xy\frac{\partial(xyz)}{\partial z} - y^2\frac{\partial(xyz)}{\partial y}\right] + \left[y^2\frac{\partial(x+y)}{\partial x} - x^2\frac{\partial(x+y)}{\partial z}\right] + \left[x^2\frac{\partial(z-x)}{\partial y} - xy\frac{\partial(z-x)}{\partial x}\right] \\ =&\ x^2y^2 - y^2xz + y^2 + xy \end{aligned}$$

(c)

$$\nabla^2 \vec{A} = \frac{\partial^2}{\partial x^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3) + \frac{\partial^2}{\partial y^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3) + \frac{\partial^2}{\partial z^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3)$$
$$= 2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_3$$

(d)

$$(\vec{A}\times\nabla)\phi = (xy\frac{\partial\phi}{\partial z} - y^2\frac{\partial\phi}{\partial y})\,\hat{\mathbf{e}}_1 + (y^2\frac{\partial\phi}{\partial x} - x^2\frac{\partial\phi}{\partial z})\,\hat{\mathbf{e}}_2 + (x^2\frac{\partial\phi}{\partial y} - xy\frac{\partial\phi}{\partial x})\,\hat{\mathbf{e}}_3,$$

where $\frac{\partial\phi}{\partial x} = 2xy^2 + yz^2, \qquad \frac{\partial\phi}{\partial y} = 2yx^2 + xz^2, \qquad \frac{\partial\phi}{\partial z} = 2xyz$

and
$$\begin{aligned}(\vec{A}\times\nabla)\phi = &(2x^2y^2z - 2x^2y^3 - xy^2z^2)\,\hat{\mathbf{e}}_1 \\ &+ (2xy^4 + y^3z^2 - 2x^3yz)\,\hat{\mathbf{e}}_2 \\ &+ (2x^4y + x^3z^2 - 2x^2y^3 - xy^2z^2)\,\hat{\mathbf{e}}_3\end{aligned}$$

■

Relations Involving the Del Operator

In summary, the following table illustrates a variety of relations involving the del operator. In these tables the functions f, g are assumed to be differentiable scalar functions of position and $\vec{A},\ \vec{B}$ are vector functions of position, which are continuous and differentiable.

The ∇ operator and differentiation

1. $\nabla(f+g) = \nabla f + \nabla g$ or $\operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g$
2. $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$ or $\operatorname{div}(\vec{A}+\vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$
3. $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$ or $\operatorname{curl}(\vec{A}+\vec{B}) = \operatorname{curl}\vec{A} + \operatorname{curl}\vec{B}$
4. $\nabla(f\vec{A}) = (\nabla f)\cdot\vec{A} + f(\nabla\cdot\vec{A})$
5. $\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f(\nabla\times\vec{A})$
6. $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}(\nabla\times\vec{A}) - \vec{A}(\nabla\times\vec{B})$
7. $(\vec{A}\times\nabla)f = \vec{A}\times\nabla f$
8. For $f = f(u)$ and $u = u(x,y,z)$, then $\nabla f = \frac{df}{du}\nabla u$
9. For $f = f(u_1, u_2, \ldots, u_n)$ and $u_i = u_i(x,y,z)$ for $i = 1, 2, \ldots, n$, then
$$\nabla f = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \cdots + \frac{\partial f}{\partial u_n}\nabla u_n$$
10. $\nabla\times(\vec{A}\times\vec{B}) = \nabla(\nabla\cdot\vec{A}) - \vec{B}(\nabla\cdot\vec{A})$
11. $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\nabla)\vec{A} + (\vec{A}\nabla)\vec{B}$
12. $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$
13. $\nabla\cdot(\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
14. $\nabla\times\nabla f) = \vec{0}$ The curl of a gradient is the zero vector.
15. $\nabla\cdot(\nabla\times\vec{A}) = 0$ The divergence of a curl is zero.

The ∇ operator and integration

16.	$\displaystyle\iiint_V \nabla f\, dV = \iint_S f\, \hat{\mathbf{e}}_n\, dS$	Special case of divergence theorem
17.	$\displaystyle\iiint_V \nabla \times \vec{A}\, dV = \iint_S \hat{\mathbf{e}}_n \times \vec{A}\, dS$	Special case of divergence theorem
18.	$\displaystyle\oint_C d\vec{r} \times \vec{A} = \iint_S (\hat{\mathbf{e}}_n \times \nabla) \times \vec{A}\, dS$	Special case of Stokes theorem
19.	$\displaystyle\oint_C f\, d\vec{r} = \iint_S d\vec{S} \times \nabla f$	Special case of Stokes theorem

Vector Operators in curvilinear coordinates

In this section the concept of curvilinear coordinates is introduced and the representation of scalars and vectors in these new coordinates are studied.

If associated with each point (x, y, z) of a rectangular coordinate system there is a set of variables (u, v, w) such that x, y, z can be expressed in terms of u, v, w by a set of functional relationships or transformations equations, then (u, v, w) are called the curvilinear coordinates[2] (x, y, z). Such transformation equations are expressible in the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w) \tag{8.68}$$

and the inverse transformation can be expressed as

$$u = u(x, y, z), \quad v = v(x, y, z) \quad w = w(x, y, z) \tag{8.69}$$

It is assumed that the transformation equations (8.68) and (8.69) are single valued and continuous functions with continuous derivatives. It is also assumed that the transformation equations (8.68) are such that the inverse transformation (8.69) exists, because this condition assures us that the correspondence between the variables (x, y, z) and (u, v, w) is a one-to-one correspondence.

The position vector

$$\vec{r} = x\, \hat{\mathbf{e}}_1 + y\, \hat{\mathbf{e}}_2 + z\, \hat{\mathbf{e}}_3 \tag{8.70}$$

[2] Note how coordinates are defined and the order of their representation because there are no standard representation of angles or directions. Depending upon how variables are defined and represented, sometime left-hand coordinates are confused with right-handed coordinates.

of a general point (x,y,z) can be expressed in terms of the curvilinear coordinates (u,v,w) by utilizing the transformation equations (8.68). The position vector $\vec{r}$, when expressed in terms of the curvilinear coordinates, becomes

$$\vec{r} = \vec{r}(u,v,w) = x(u,v,w)\,\hat{\mathbf{e}}_1 + y(u,v,w)\,\hat{\mathbf{e}}_2 + z(u,v,w)\,\hat{\mathbf{e}}_3 \tag{8.71}$$

and an element of arc length squared is $ds^2 = d\vec{r}\cdot d\vec{r}$. In the curvilinear coordinates one finds $\vec{r}=\vec{r}(u,v,w)$ as a function of the curvilinear coordinates and consequently

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u}\,du + \frac{\partial\vec{r}}{\partial v}\,dv + \frac{\partial\vec{r}}{\partial w}\,dw. \tag{8.72}$$

From the differential element $d\vec{r}$ one finds the element of arc length squared given by

$$\begin{aligned} d\vec{r}\cdot d\vec{r} = ds^2 = \; & \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial u}\,du\,du + \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial v}\,du\,dv + \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial w}\,du\,dw \\ & + \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial u}\,dvdu + \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial v}\,dvdv + \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial w}\,dv\,dw \\ & + \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial u}\,dwdu + \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial v}\,dwdv + \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial w}\,dwdw. \end{aligned} \tag{8.73}$$

The quantities

$$\begin{aligned} g_{11} &= \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial u} \qquad & g_{12} &= \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial v} \qquad & g_{13} &= \frac{\partial\vec{r}}{\partial u}\cdot\frac{\partial\vec{r}}{\partial w} \\ g_{21} &= \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial u} & g_{22} &= \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial v} & g_{23} &= \frac{\partial\vec{r}}{\partial v}\cdot\frac{\partial\vec{r}}{\partial w} \\ g_{31} &= \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial u} & g_{32} &= \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial v} & g_{33} &= \frac{\partial\vec{r}}{\partial w}\cdot\frac{\partial\vec{r}}{\partial w} \end{aligned} \tag{8.74}$$

are called the metric components of the curvilinear coordinate system. The metric components may be thought of as the elements of a symmetric matrix, since $g_{ij}=g_{ji}$, $i,j = 1,2,3$. These metrices play an important role in the subject area of tensor calculus.

The vectors $\frac{\partial\vec{r}}{\partial u}$, $\frac{\partial\vec{r}}{\partial v}$, $\frac{\partial\vec{r}}{\partial w}$, used to calculate the metric components g_{ij} have the following physical interpretation. The vector $\vec{r}=\vec{r}(u,c_2,c_3)$, where u is a variable and $v=c_2,\ w=c_3$ are constants, traces out a curve in space called a coordinate curve. Families of these curves create a coordinate system. Coordinate curves can also be viewed as being generated by the intersection of the coordinate surfaces $v(x,y,z)=c_2$ and $w(x,y,z)=c_3$. The tangent vector to the coordinate curve is calculated with the partial derivative $\frac{\partial \vec{r}}{\partial u}$. Similarly, the curves $\vec{r} = \vec{r}(c_1, v, c_3)$ and $\vec{r} = \vec{r}(c_1, c_2, w)$ are coordinate curves and have the respective tangent vectors $\frac{\partial \vec{r}}{\partial v}$ and $\frac{\partial \vec{r}}{\partial w}$. One can calculate the magnitude of these tangent vectors by defining the scalar magnitudes as

$$h_1 = h_u = |\frac{\partial \vec{r}}{\partial u}|, \qquad h_2 = h_v = |\frac{\partial \vec{r}}{\partial v}|, \qquad h_3 = h_w = |\frac{\partial \vec{r}}{\partial w}|. \tag{8.75}$$

The unit tangent vectors to the coordinate curves are given by the relations

$$\hat{\mathbf{e}}_u = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u}, \qquad \hat{\mathbf{e}}_v = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial v}, \qquad \hat{\mathbf{e}}_w = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial w}. \tag{8.76}$$

The coordinate surfaces and coordinate curves may be formed from the equations (8.68) and are illustrated in figure 8-15

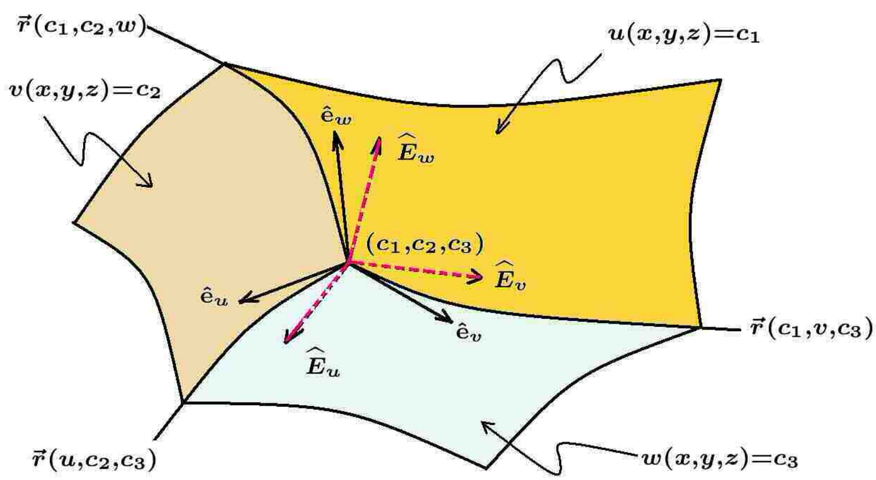


Figure 8-15. Coordinate curves and surfaces.

Consider the point $u = c_1,\ v = c_2,\ w = c_3$ in the curvilinear coordinate system. This point can be viewed as being created from the intersection of the three surfaces

$$u = u(x, y, z) = c_1$$
$$v = v(x, y, z) = c_2$$
$$w = w(x, y, z) = c_3$$

obtained from the inverse transformation equations (8.69).

For example, the figure 8-15 illustrates the surfaces $u = c_1$ and $v = c_2$ intersecting in the curve $\vec{r} = \vec{r}(c_1, c_2, w)$. The point where this curve intersects the surface $w = c_3$, is (c_1, c_2, c_3).

The vector grad $u(x, y, z)$ is a vector normal to the surface $u = c_1$. A unit normal to the $u = c_1$ surface has the form

$$\vec{E}_u = \frac{\text{grad } u}{|\text{grad } u|}.$$

Similarly, the vectors

$$\vec{E}_v = \frac{\text{grad } v}{|\text{grad } v|}, \qquad \text{and} \qquad \vec{E}_w = \frac{\text{grad } w}{|\text{grad } w|}$$

are unit normal vectors to the surfaces $v = c_2$ and $w = c_3$.

The unit tangent vectors $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$, $\hat{\mathbf{e}}_w$ and the unit normal vectors $\vec{E}_u$, $\vec{E}_v$, $\vec{E}_w$ are identical if and only if $g_{ij} = 0$ for $i \neq j$; for this case, the curvilinear coordinate system is called an orthogonal coordinate system.

Example 8-14. Consider the identity transformation between (x, y, z) and (u, v, w). We have $u = x$, $v = y$, and $w = z$. The position vector is

$$\vec{r}(x, y, z) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3,$$

and in this rectangular coordinate system, the element of arc length squared is given by $ds^2 = dx^2 + dy^2 + dz^2$. In this space the metric components are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the coordinate system is orthogonal.

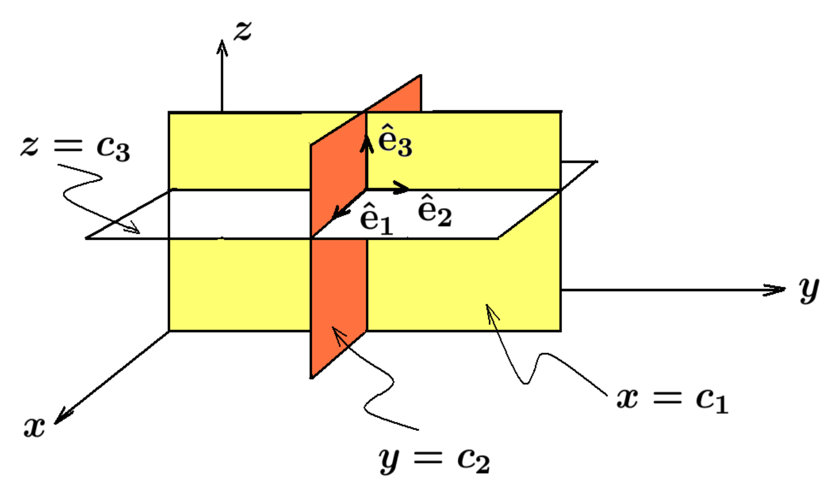


Figure 8-16. Cartesian coordinate system.

In rectangular coordinates consider the family of surfaces

$$x = c_1, \qquad y = c_2, \qquad z = c_3,$$

where c_1, c_2, c_3 take on the integer values $1, 2, 3, \ldots$. These surfaces intersect in lines which are the coordinate curves. The vectors

$$\text{grad } x = \hat{\mathbf{e}}_1, \qquad \text{grad } y = \hat{\mathbf{e}}_2, \quad \text{and} \quad \text{grad } z = \hat{\mathbf{e}}_3$$

are the unit vectors which are normal to the coordinate surfaces. The vectors

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1, \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2, \qquad \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$$

can also be viewed as being tangent to the coordinate curves. The situation is illustrated in figure 8-16.

■

Example 8-15. In cylindrical coordinates (r, θ, z), the transformation equations (8.68) become

$$\begin{aligned} x &= x(r, \theta, z) = r\cos\theta \\ y &= y(r, \theta, z) = r\sin\theta \\ z &= z(r, \theta, z) = z \end{aligned}$$

and the inverse transformation (8.69) can be written

$$\begin{aligned} r &= r(x, y, z) = \sqrt{x^2 + y^2} \\ \theta &= \theta(x, y, z) = \arctan\frac{y}{x} \\ z &= z(x, y, z) = z. \end{aligned}$$

where the substitutions $u = r,\ v = \theta, w = z$ have been made. The position vector (8.70) is then

$$\vec{r} = \vec{r}(r, \theta, z) = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3.$$

The curve

$$\vec{r} = \vec{r}(c_1, \theta, c_3) = c_1\cos\theta\,\hat{\mathbf{e}}_1 + c_1\sin\theta\,\hat{\mathbf{e}}_2 + c_3\,\hat{\mathbf{e}}_3,$$

where c_1 and c_3 are constants, represents the circle $x^2 + y^2 = c_1^2$ in the plane $z = c_3$ and is illustrated in figure 8-17. The curve

$$\vec{r} = \vec{r}(c_1, c_2, z) = c_1\cos c_2\,\hat{\mathbf{e}}_1 + c_1\sin c_2\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

represents a straight line parallel to the z-axis which is normal to the xy plane at the point $r = c_1,\ \theta = c_2$. The curve

$$\vec{r} = \vec{r}(r, c_2, c_3) = r\cos c_2\,\hat{\mathbf{e}}_1 + r\sin c_2\,\hat{\mathbf{e}}_2 + c_3\,\hat{\mathbf{e}}_3$$

represents a straight line in the plane $z = c_3$, which extends in the direction $\theta = c_2$.

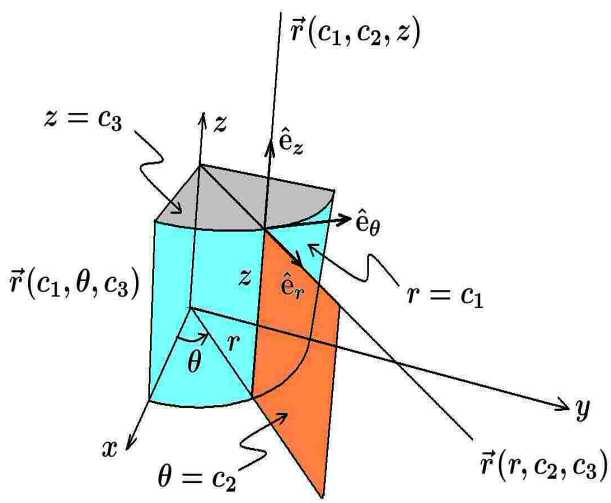


Figure 8-17. Cylindrical coordinates.

The tangent vectors to the coordinate curves are given by

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial r} &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \\
\frac{\partial \vec{r}}{\partial \theta} &= -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2 \\
\frac{\partial \vec{r}}{\partial z} &= \ \hat{\mathbf{e}}_3
\end{aligned}$$

and are illustrated in figure 8-17. The element of arc length squared is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

and the metric components of the space are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that this is an orthogonal system where $g_{ij} = 0$ for $i \neq j$. The surface $r = c_1$ is a cylinder, whereas the surface $\theta = c_2$ is a plane perpendicular to the xy plane and passing through the z-axis. The surface $z = c_3$ is a plane parallel to the xy plane. The cylindrical coordinate system is an orthogonal system.

■

Example 8-16. The spherical coordinates (ρ, θ, ϕ) are related to the rectangular coordinates through the transformation equations

$$
\begin{aligned}
x &= x(\rho, \theta, \phi) = \rho \sin\theta \cos\phi \\
y &= y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi \\
z &= z(\rho, \theta, \phi) = \rho \cos\theta
\end{aligned}
$$

which can be obtained from the geometry of figure 8-18.

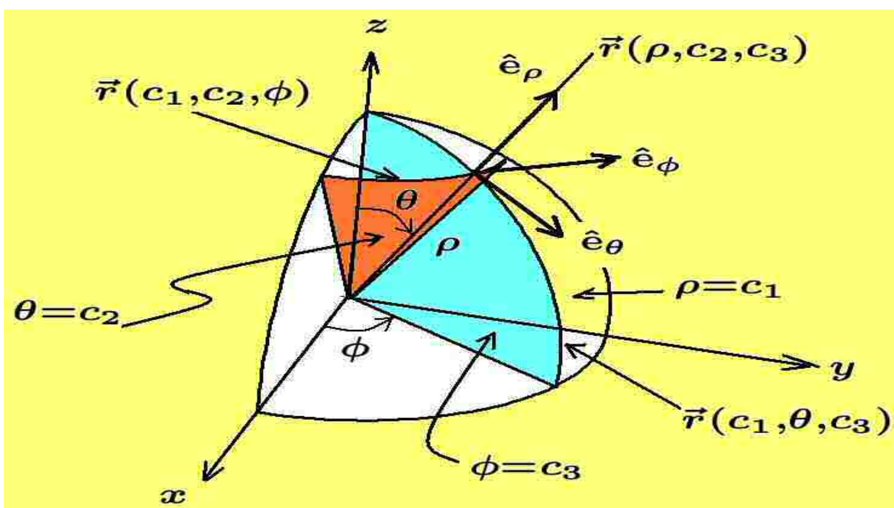


Figure 8-18. Spherical coordinate system.

The position vector (8.70) becomes

$$\vec{r} = \vec{r}(\rho, \theta, \phi) = \rho \sin\theta \cos\phi \, \hat{\mathbf{e}}_1 + \rho \sin\theta \sin\phi \, \hat{\mathbf{e}}_2 + \rho \cos\theta \, \hat{\mathbf{e}}_3,$$

and from this position vector one can generate the curves

$$\vec{r} = \vec{r}(c_1, c_2, \phi), \quad \vec{r} = \vec{r}(c_1, \theta, c_3), \quad \vec{r} = \vec{r}(\rho, c_2, c_3),$$

where c_1, c_2, c_3 are constants. These curves are, respectively, circles of radius $c_1 \sin c_2$, meridian lines on the surface of the sphere, and a line normal to the sphere. These curves are illustrated in figure 8-18. The surfaces $r = c_1$, $\theta = c_2$, and $\phi = c_3$ are, respectively, spheres, circular cones, and planes passing through the z-axis.

The unit tangent vectors to the coordinate curves and scale factors are given by

$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3, & h_1 &= h_\rho = 1 \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3, & h_2 &= h_\theta = \rho \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2, & h_3 &= h_\phi = \rho\sin\theta.
\end{aligned}$$

The element of arc length squared is

$$ds^2 = d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2,$$

and the metric components of this space are given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2\sin^2\theta \end{pmatrix}.$$

Note that the spherical coordinate system is an orthogonal system.

■

Example 8-17.

An example of a curvilinear coordinate system which is not orthogonal is the oblique cylindrical coordinate system (r, θ, η) illustrated in figure 8-19

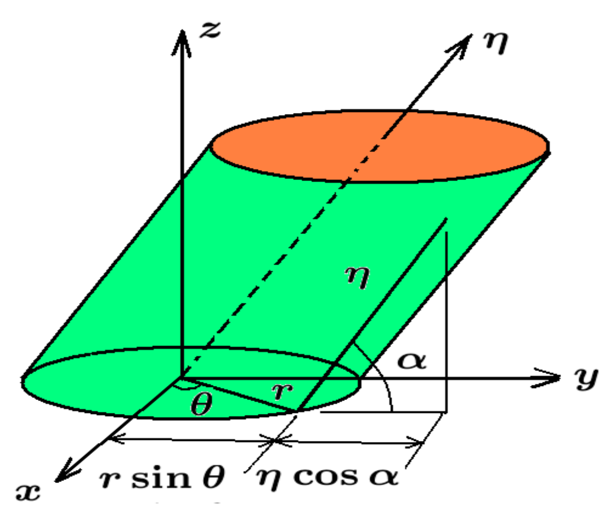


Figure 8-19. Oblique cylindrical coordinate system.

The transformation equations (8.68) are obtained from the geometry in figure 8-19. These equations are

$$x = r\cos\theta$$
$$y = r\sin\theta + \eta\cos\alpha$$
$$z = \eta\sin\alpha,$$

which for $\alpha = 90^\circ$ reduces to the transformation equations for cylindrical coordinates.

The unit tangent vectors are

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\eta = \cos\alpha\,\hat{\mathbf{e}}_2 + \sin\alpha\,\hat{\mathbf{e}}_3,$$

and the metric components of this space are

$$g_{ij} = \begin{pmatrix} 1 & 0 & \sin\theta\cos\alpha \\ 0 & r^2 & r\cos\theta\cos\alpha \\ \sin\theta\cos\alpha & r\cos\theta\cos\alpha & 1 \end{pmatrix}.$$

■

Orthogonal Curvilinear Coordinates

The following is a list of some orthogonal curvilinear coordinates which have applications in many different scientific investigations.

Cylindrical coordinates (r, θ, z) :

$$x = r\cos\theta \qquad 0 \le \theta \le 2\pi$$
$$y = r\sin\theta \qquad r \ge 0$$
$$z = z \qquad -\infty < z < \infty$$
$$ds^2 = h_r^2\,dr^2 + h_\theta^2\,d\theta^2 + h_z^2\,dz^2 \tag{8.77}$$
$$h_r = 1, \quad h_\theta = r, \quad h_z = 1$$
$$g_{ij} = \begin{pmatrix} h_r^2 & 0 & 0 \\ 0 & h_\theta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix}$$

Spherical coordinates (ρ, θ, ϕ) :

$$
\begin{aligned}
x &= \rho \sin\theta \cos\phi, \qquad \rho \ge 0 \\
y &= \rho \sin\theta \sin\phi, \qquad 0 \le \phi \le 2\pi \\
z &= \rho \cos\theta, \qquad 0 \le \theta \le \pi \\
ds^2 &= h_\rho^2\, d\rho^2 + h_\theta^2\, d\theta^2 + h_\phi^2\, d\phi^2 \\
h_\rho &= 1, \quad h_\theta = \rho, \quad h_\phi = \rho\sin\theta \\
g_{ij} &= \begin{pmatrix} h_\rho^2 & 0 & 0 \\ 0 & h_\theta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned}
\tag{8.78}
$$

Parabolic cylindrical coordinates (ξ, η, z) :

$$
\begin{aligned}
x &= \xi\eta, \qquad -\infty < \xi < \infty \\
y &= \frac{1}{2}\left(\xi^2 - \eta^2\right), \qquad -\infty < z < \infty \\
z &= z, \qquad \eta \ge 0 \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_z^2\, dz^2 \\
h_\eta &= h_\xi = \sqrt{\eta^2 + \xi^2}, \qquad h_z = 1 \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix}
\end{aligned}
\tag{8.79}
$$

Parabolic coordinates (ξ, η, ϕ) :

$$
\begin{aligned}
x &= \xi\eta\cos\phi, \qquad \xi \ge 0, \quad \eta \ge 0 \\
y &= \xi\eta\sin\phi, \qquad 0 < \phi < 2\pi \\
z &= \frac{1}{2}(\xi^2 - \eta^2) \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_\phi^2\, d\phi^2 \\
h_\xi &= h_\eta = \sqrt{\eta^2 + \xi^2}, \qquad h_\phi = \xi\eta \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned}
\tag{8.80}
$$

Elliptic cylindrical coordinates (ξ, η, z) :

$$
\begin{aligned}
x =& \ \cosh\xi\cos\eta, \qquad \xi \geq 0 \\
y =& \ \sinh\xi\sin\eta, \qquad 0 \leq \eta \leq 2\pi \\
z =& \ z, \qquad -\infty < z < \infty \\
ds^2 =& \ h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_z^2\, dz^2 \\
h_\xi =& \ h_\eta = \sqrt{\sinh^2\xi + \sin^2\eta}, \qquad h_z = 1 \\
g_{ij} =& \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix}
\end{aligned}
\tag{8.81}
$$

Elliptic coordinates (ξ, η, ϕ) :

$$
\begin{aligned}
x =& \sqrt{(1-\eta^2)(\xi^2-1)}\cos\phi, \qquad -1 \leq \eta \leq 1 \\
y =& \sqrt{(1-\eta^2)(\xi^2-1)}\sin\phi, \qquad 1 \leq \xi < \infty \\
z =& \ \xi\eta, \qquad 0 \leq \phi \leq 2\pi \\
ds^2 =& \ h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_\phi^2\, d\phi^2 \\
h_\xi =& \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \quad h_\eta = \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \quad h_\phi = \sqrt{(\xi^2-1)(1-\eta^2)} \\
g_{ij} =& \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned}
\tag{8.82}
$$

Transformation of Vectors

A vector field defined by

$$
\vec{A} = \vec{A}(x,y,z) = A_1(x,y,z)\,\hat{\mathbf{e}}_1 + A_2(x,y,z)\,\hat{\mathbf{e}}_2 + A_3(x,y,z)\,\hat{\mathbf{e}}_3
$$

represents a magnitude and direction associated to each point (x,y,z) in some region R or three dimensional cartesian coordinates. This vector field is to remain invariant under a coordinate transformation. However, the form used to represent the vector field will change. For example, under a transformation to cylindrical coordinates, where

$$
x = r\cos\theta \qquad y = r\sin\theta \qquad z = z, \tag{8.83}
$$

the above vector can be represented in terms of the unit orthogonal vectors $\hat{\mathbf{e}}_r,\ \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{e}}_z$ in the form

$$
\vec{A} = \vec{A}(r,\theta,z) = A_r(r,\theta,z)\,\hat{\mathbf{e}}_r + A_\theta(r,\theta,z)\,\hat{\mathbf{e}}_\theta + A_z(r,\theta,z)\,\hat{\mathbf{e}}_z. \tag{8.84}
$$

Here the quantities A_1, A_2, A_3 represent the components of the vector field $\vec{A}$ in rectangular coordinates, and A_r, A_θ, A_z represent the components of the same vector field $\vec{A}$ when referenced with respect to cylindrical coordinates. The unit vectors $\hat{\mathbf{e}}_r,\ \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{e}}_z$ are orthogonal unit vectors and hence

$$\begin{aligned}
\vec{A}\cdot\hat{\mathbf{e}}_r &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_r + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_r + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_r = A_r\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_r \text{ direction}\\
\vec{A}\cdot\hat{\mathbf{e}}_\theta &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_\theta + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_\theta + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_\theta = A_\theta\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_\theta \text{ direction}\\
\vec{A}\cdot\hat{\mathbf{e}}_z &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_z + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_z + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_z = A_z\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_z \text{ direction.}
\end{aligned}$$

These equations can be expressed in the matrix form as follows:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_z & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_z & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_z \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{8.85}$$

For example, it is known that the unit vectors in cylindrical coordinates are

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2\\
\hat{\mathbf{e}}_\theta &= -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2\\
\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_3,
\end{aligned}$$

and consequently the matrix (8.85) can be expressed as

$$\begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{8.86}$$

Equation (8.86) illustrates how to represent the vector field components A_r, A_θ, A_z of cylindrical coordinates, in terms of the components A_1, A_2, A_3 of rectangular coordinates. In using the above transformation equation, be sure to convert all x, y, z coordinates to r, θ, z cylindrical coordinates using the transformation equations (8.83). Note also that the coefficient matrix in equation (8.83) is an orthonormal matrix.

Example 8-18. Express the vector

$$\vec{A} = 2y\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + 2x\,\hat{\mathbf{e}}_3$$

in cylindrical coordinates.

Solution The rectangular components of $\vec{A}$ are $A_1 = 2y,\ A_2 = z,\ A_3 = 2x$, and from equation (8.86) the cylindrical components are

$$\begin{aligned} A_r &= 2y\cos\theta + z\sin\theta \\ A_\theta &= -2y\sin\theta + z\cos\theta \\ A_z &= 2x, \end{aligned}$$

where the variables x, y, z must be expressed in terms of the variables r, θ, z. From the transformation equations from rectangular to cylindrical coordinates one finds

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

so that

$$\begin{aligned} A_r &= 2r\sin\theta\cos\theta + z\sin\theta \\ A_\theta &= -2r\sin^2\theta + z\cos\theta \\ A_z &= 2r\cos\theta \end{aligned}$$

and the vector $\vec{A}$ in cylindrical coordinates can be represented as

$$\vec{A} = \vec{A}(r,\theta,z) = A_r\,\hat{\mathbf{e}}_r + A_\theta\,\hat{\mathbf{e}}_\theta + A_z\,\hat{\mathbf{e}}_z.$$

■

General Coordinate Transformations

In general, a vector in rectangular coordinates

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$$

can be expressed in terms of the orthogonal unit vectors $\hat{\mathbf{e}}_u,\ \hat{\mathbf{e}}_v,\ \hat{\mathbf{e}}_w$ associated with a set of orthogonal curvilinear coordinates defined by the transformation equations given in equation (8.68). Let the representation of this vector in the orthogonal curvilinear coordinates system be denoted by

$$\vec{A} = \vec{A}(u,v,w) = A_u\,\hat{\mathbf{e}}_u + A_v\,\hat{\mathbf{e}}_v + A_w\,\hat{\mathbf{e}}_w,$$

where A_u, A_v, A_w denote the components of $\vec{A}$ in the new coordinate system and are functions of these coordinates. The transformation equations from rectangular coordinates to curvilinear coordinates is represented by the matrix equation

$$\begin{pmatrix} A_u \\ A_v \\ A_w \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_u & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_u & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_u \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_v & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_v & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_w \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_w & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_w & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_w \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{8.87}$$

which is derived by taking projections of the vector $\vec{A}$ onto the u,v and w directions.

Let us find the representation of the gradient, divergence and curl in a general orthogonal curvilinear coordinate system. Recall that the gradient, divergence, and curl in rectangular coordinates are given by

$$\begin{aligned} \text{grad } \phi &= \frac{\partial \phi}{\partial x}\, \hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\, \hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\, \hat{\mathbf{e}}_3 \\ \text{div}\, \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ \text{curl } \vec{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \hat{\mathbf{e}}_3. \end{aligned}$$

Gradient in a General Orthogonal System of Coordinates

In an orthogonal curvilinear coordinate system, let the vector grad ϕ have the representation

$$\text{grad } \phi = A_u\, \hat{\mathbf{e}}_u + A_v\, \hat{\mathbf{e}}_v + A_w\, \hat{\mathbf{e}}_w.$$

By using the matrix equation (8.87), the component A_u in the curvilinear coordinates is

$$\text{grad } \phi \cdot \hat{\mathbf{e}}_u = A_u = \frac{\partial \phi}{\partial x} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_u + \frac{\partial \phi}{\partial y} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_u + \frac{\partial \phi}{\partial z} \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_u.$$

By employing equations (8.75) and (8.76), this result simplifies and

$$A_u = \frac{1}{h_1}\left[\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}\right] = \frac{1}{h_1}\frac{\partial \phi}{\partial u}. \tag{8.88}$$

In a similar manner, it can be shown that the other components have the form

$$\text{grad } \phi \cdot \hat{\mathbf{e}}_v = A_v = \frac{1}{h_2}\frac{\partial \phi}{\partial v} \qquad \text{and} \qquad \text{grad } \phi \cdot \hat{\mathbf{e}}_w = A_w = \frac{1}{h_3}\frac{\partial \phi}{\partial w}.$$

Thus the gradient can be represented in the curvilinear coordinate system as

$$\nabla \phi = \text{grad } \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u}\, \hat{\mathbf{e}}_u + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\, \hat{\mathbf{e}}_v + \frac{1}{h_3}\frac{\partial \phi}{\partial w}\, \hat{\mathbf{e}}_w. \tag{8.89}$$

Since

$$\begin{aligned}\frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1+\frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2+\frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3= &\left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}\right)\hat{\mathbf{e}}_1\\
&+\left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y}\right)\hat{\mathbf{e}}_2\\
&+\left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z}\right)\hat{\mathbf{e}}_3,\end{aligned}$$

equation (8.89) can be expressed in the form

$$\nabla \phi = \text{grad }\phi = \nabla u \frac{\partial \phi}{\partial u}+\nabla v\frac{\partial \phi}{\partial v}+\nabla w\frac{\partial \phi}{\partial w}. \tag{8.92}$$

Equation (8.92) suggests how the operator ∇ can be expressed in a general curvilinear coordinate system. In a general curvilinear coordinate system (u,v,w) one finds the operator ∇ has the form

$$\nabla = \nabla u\frac{\partial}{\partial u}+\nabla v\frac{\partial}{\partial v}+\nabla w\frac{\partial}{\partial w}. \tag{8.91}$$

Divergence in a General Orthogonal System of Coordinates

To find the divergence in an orthogonal curvilinear system, the following relations are employed:

$$\nabla u=\frac{1}{h_1}\,\hat{\mathbf{e}}_u,\qquad \nabla v=\frac{1}{h_2}\,\hat{\mathbf{e}}_v,\qquad \nabla w=\frac{1}{h_3}\,\hat{\mathbf{e}}_w \tag{8.92}$$

which are special cases of the result in equation (8.89). Equations (8.92) imply

$$\begin{aligned}\hat{\mathbf{e}}_u &= \hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w = h_2h_3(\nabla v)\times(\nabla w)\\
\hat{\mathbf{e}}_v &= \hat{\mathbf{e}}_w \times \hat{\mathbf{e}}_u = h_1h_3(\nabla w)\times(\nabla u)\\
\hat{\mathbf{e}}_w &= \hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = h_1h_2(\nabla u)\times(\nabla v)\end{aligned} \tag{8.93}$$

Example 8-19. Derive the divergence of a vector which is represented in the generalized orthogonal coordinates (u,v,w) in the form

$$\vec{F}=\vec{F}(u,v,w)=F_u\,\hat{\mathbf{e}}_u+F_v\,\hat{\mathbf{e}}_v+F_w\,\hat{\mathbf{e}}_w.$$

Solution: By using the properties of the del operator one finds

$$\nabla\cdot\vec{F}=\nabla(F_u\,\hat{\mathbf{e}}_u)+\nabla(F_v\,\hat{\mathbf{e}}_v)+\nabla(F_w\,\hat{\mathbf{e}}_w). \tag{8.94}$$

The first term in equation (8.94) can be expanded, and

$$\begin{aligned}\nabla(F_u\,\hat{\mathbf{e}}_u) &= \nabla(F_u)\cdot\,\hat{\mathbf{e}}_u + F_u\nabla(\,\hat{\mathbf{e}}_u)\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\nabla[h_2h_3(\nabla v)\times(\nabla w)] \qquad \text{(See eqs. (8.89) and (8.93))}\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\left\{\nabla(h_2h_3)\cdot[(\nabla v)\times(\nabla w)] + h_2h_3\nabla\cdot[(\nabla v)\times(\nabla w)]\right\},\end{aligned}$$

where properties of the del operator were used to obtain this result. With the result $\operatorname{div}(\text{grad } v \times \text{grad } w) = 0$, so that

$$\begin{aligned}\nabla(F_u\,\hat{\mathbf{e}}_u) &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\nabla(h_2h_3)\cdot\frac{\hat{\mathbf{e}}_u}{h_2h_3} \qquad \text{(See eq. (8.93)}\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + \frac{F_u}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3F_u)}{\partial u} \qquad \text{(See eq. (8.89)).}\end{aligned}$$

Similarly it can be verified that the remaining terms in equation (8.94) can be expressed as

$$\begin{aligned}\nabla(F_v\,\hat{\mathbf{e}}_v) &= \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2F_v)}{\partial v}\\ \text{and} \qquad \nabla(F_w\,\hat{\mathbf{e}}_w) &= \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2F_w)}{\partial w}.\end{aligned}$$

Hence, the divergence in generalized orthogonal curvilinear coordinates can be expressed as

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_u)}{\partial u} + \frac{\partial(h_1h_3F_v)}{\partial v} + \frac{\partial(h_1h_2F_w)}{\partial w}\right]. \tag{8.95}$$

■

Curl in a General Orthogonal System of Coordinates

Our problem is to derive an expression for the curl of a vector $\vec{F}$ which is represented in the generalized orthogonal coordinates (u, v, w) in the form

$$\vec{F} = \vec{F}(u, v, w) = F_u\,\hat{\mathbf{e}}_u + F_v\,\hat{\mathbf{e}}_v + F_w\,\hat{\mathbf{e}}_w$$

one can write

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \nabla\times(F_u\,\hat{\mathbf{e}}_u) + \nabla\times(F_v\,\hat{\mathbf{e}}_v) + \nabla\times(F_w\,\hat{\mathbf{e}}_w). \tag{8.96}$$

The first term in equation (8.96) can be expanded by using properties of the del operator and

$$\begin{aligned}\nabla\times(F_u\,\hat{\mathbf{e}}_u) &= \nabla\times(F_uh_1\nabla u) \quad \text{(See eq. (8.89))}\\ &= \nabla(F_uh_1)\times\nabla u + F_uh_1\nabla\times\nabla u.\end{aligned}$$

Since curl grad $u = 0$, the above simplifies to

$$
\begin{aligned}
\nabla \times (F_u\, \hat{\mathbf{e}}_u) &= \nabla (F_u h_1) \times \frac{\hat{\mathbf{e}}_u}{h_1} \\
&= \left[\frac{1}{h_1} \frac{\partial (F_u h_1)}{\partial u}\, \hat{\mathbf{e}}_u + \frac{1}{h_2} \frac{\partial (F_u h_1)}{\partial v}\, \hat{\mathbf{e}}_v + \frac{1}{h_3} \frac{\partial (F_u h_1)}{\partial w}\, \hat{\mathbf{e}}_w \right] \times \frac{\hat{\mathbf{e}}_u}{h_1} \\
&= \frac{1}{h_1 h_3} \frac{\partial (F_u h_1)}{\partial w}\, \hat{\mathbf{e}}_v - \frac{1}{h_1 h_2} \frac{\partial (F_u h_1)}{\partial v}\, \hat{\mathbf{e}}_w
\end{aligned}
$$

In a similar manner it may be verified that the remaining terms in equation (8.96) can be expressed as

$$
\begin{aligned}
\nabla \times (F_v\, \hat{\mathbf{e}}_v) &= \frac{1}{h_1 h_2} \frac{\partial (F_v h_2)}{\partial u}\, \hat{\mathbf{e}}_w - \frac{1}{h_2 h_3} \frac{\partial (F_v h_2)}{\partial w}\, \hat{\mathbf{e}}_u \\
\text{and} \qquad \nabla \times (F_w\, \hat{\mathbf{e}}_w) &= \frac{1}{h_2 h_3} \frac{\partial (F_w h_3)}{\partial v}\, \hat{\mathbf{e}}_u - \frac{1}{h_1 h_3} \frac{\partial (F_w h_3)}{\partial u}\, \hat{\mathbf{e}}_v .
\end{aligned}
$$

Hence, the curl of a vector in generalized curvilinear coordinates can be represented in the form

$$
\begin{aligned}
\nabla \times \vec{F} = &\frac{1}{h_2 h_3} \left[\frac{\partial (F_w h_3)}{\partial v} - \frac{\partial (F_v h_2)}{\partial w} \right] \hat{\mathbf{e}}_u \\
&+ \frac{1}{h_1 h_3} \left[\frac{\partial (F_u h_1)}{\partial w} - \frac{\partial (F_w h_3)}{\partial u} \right] \hat{\mathbf{e}}_v \\
&+ \frac{1}{h_1 h_2} \left[\frac{\partial (F_v h_2)}{\partial u} - \frac{\partial (F_u h_1)}{\partial v} \right] \hat{\mathbf{e}}_w .
\end{aligned}
\tag{8.97}
$$

Equation (8.97) can also be represented in the determinant form as

$$
\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\, \hat{\mathbf{e}}_u & h_2\, \hat{\mathbf{e}}_v & h_3\, \hat{\mathbf{e}}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ F_u h_1 & F_v h_2 & F_w h_3 \end{vmatrix} . \tag{8.98}
$$

The Laplacian in Generalized Orthogonal Coordinates

Using the definition $\nabla^2 \phi = \nabla \nabla \phi$ and the relation for the gradient given by equation (8.89) and show that

$$
\nabla \nabla \phi = \nabla \left[\frac{1}{h_1} \frac{\partial \phi}{\partial u}\ \hat{\mathbf{e}}_u + \frac{1}{h_2} \frac{\partial \phi}{\partial v}\ \hat{\mathbf{e}}_v + \frac{1}{h_3} \frac{\partial \phi}{\partial w}\ \hat{\mathbf{e}}_w \right] . \tag{8.99}
$$

The result of equation (8.95) simplifies equation (8.99) to the final form given as

$$
\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u} \right) = \frac{\partial}{\partial v} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial w} \right) \right] . \tag{8.100}
$$

The equation $\nabla^2 U = 0$ is known as Laplace's equation.

Example 8-20.

The Laplacian in rectangular coordinates (x, y, z) is given by

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \tag{8.101}$$

The Laplacian in cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned}\nabla^2 U =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\\ \nabla^2 U =& \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\end{aligned} \tag{8.102}$$

The Laplacian in spherical coordinates (ρ, θ, ϕ) is given by

$$\begin{aligned}\nabla^2 U =& \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}\\ \nabla^2 U =& \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial U}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}\end{aligned} \tag{8.103}$$

Special cases of the Laplace equation $\nabla^2 U = 0$ are easy to solve.

1. If $y = z = 0$ in rectangular coordinates, the Laplace equation, with Laplacian (8.101), reduces to $\dfrac{d^2U}{dx^2} = 0$. One integration produces $\dfrac{dU}{dx} = C_1$, where C_1 is a constant of integration. Another integration gives $U = C_1 x + C_2$, where C_2 is another constant of integration.
2. If $\theta = z = 0$ in cylindrical coordinates, the Laplace equation, with Laplacian (8.102), reduces to $\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dU}{dr}\right) = 0$. An integration of this equation gives $r\dfrac{dU}{dr} = C_1$, where C_1 is a constant of integration. Separate the variables in this equations and integrate again to show the solution of the special Laplace equation is given by $U = C_1 \ln r + C_2$, where C_2 is another constant of integration.
3. If $\theta = \phi = 0$ in spherical coordinates, the Laplace equation, with Laplacian (8.103), reduces to $\dfrac{1}{\rho^2}\left(\rho^2\dfrac{dU}{d\rho}\right) = 0$. An integration of this equation gives $\rho^2\dfrac{dU}{d\rho} = C_1$, where C_1 is a constant of integration. Separate the variables and perform another integration to show $U = \frac{-C_1}{\rho} + C_2$, where C_2 is another constant of integration.

■

Exercises

▶ **8-1.** Sketch some level curves $\phi = k$ for the given values of k and then find the gradient vector.

(i) $\phi = 4x - 2y, \quad k = -2, -1, 0, 1, 2$

(ii) $\phi = xy, \quad k = -2, -1, 0, 1, 2$

(iii) $\phi = x^2 + y^2, \quad k = 0, 1, 9, 25$

(iv) $\phi = 9x^2 + 4y^2, \quad k = 0, 36, 72$

▶ **8-2.** Find the gradient vector associated with the given functions and then evaluate the gradient at the points indicated.

(i) $\phi = 4x - 2y, \quad (4,9), (0,0), (-4,-9)$

(ii) $\phi = xy, \quad (0,1), (-1,0), (0,-1), (1,0), (1,1), (-1,1), (-1,-1), (1,-1)$

(iii) $\phi = x^2 + y^2, \quad (1,0), (3,4), (0,1), (-3,4), (-1,0), (-3,-4), (0,-1), (3,-4)$

(iv) $\phi = 9x^2 + 4y^2, \quad (2,0), (0,3), (-2,0), (0,-3)$

▶ **8-3.** Find a normal vector to the given surfaces at the point indicated and describe the surface.

(i) $4x + 3y + 6z = 13 \qquad P(1,1,1)$

(ii) $x^2 + y^2 + z^2 = 9 \qquad P(1,2,2)$

(iii) $z - x^2 - y^2 = 0 \qquad P(3,4,25)$

(iv) $z = xy \qquad P(2,3,6)$

▶ **8-4.** Discuss the critical points associated with the function $z = z(x,y) = xy$. Graph the level curves $z = k$, where $k = -2, -1, 0, 1, 2$ and describe the surface.

▶ **8-5.** Find the unit tangent vector at the point $P(3,2,6)$ on the curve of intersection of the surfaces

$$x^2 + y^2 + z^2 = 49, \qquad \text{and} \qquad x + y + z = 11.$$

▶ **8-6.** Let r denote the magnitude of the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$.

(i) Show that $\nabla(r^n) = nr^{n-2}\vec{r}$

(ii) Show that $\nabla(\ln r) = \frac{\vec{r}}{r^2}$

(iii) Show that $\nabla\left(f(r)\right) = f'(r)\frac{\vec{r}}{r}$, where f is differentiable.

(iv) Does the result in part (iii) check with the solutions given in parts (i) and (ii)?

▶ **8-7.** Find the minimum distance between the lines defined by the parametric equations

$$L_1: \quad x = \tau - 1, \quad y = -\tau + 16, \quad z = 2\tau - 2$$

$$L_2: \quad x = -t, \quad y = 2t, \quad z = 3t$$

▶ **8-8.** Find the minimum distance from the origin to the plane $x + y + z = 1$

▶ **8-9.** The special symbol $\frac{d\phi}{dn}$ is used to denote the normal derivative of a function ϕ on the boundary of a region R. The normal derivative is defined

$$\frac{d\phi}{dn} = \text{grad } \phi \cdot \hat{\mathbf{e}}_n = \nabla\phi \cdot \hat{\mathbf{e}}_n,$$

where $\hat{\mathbf{e}}_n$ is the unit exterior normal vector to the boundary of the region. Find the normal derivative of $\phi = x^3y + xy^2$ on the boundary of the regions given.

(i) The unit circle $x^2 + y^2 = 1$

(ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) The square with vertices $(0,0), (1,0), (1,1), (0,1)$

▶ **8-10.** Find the critical points associated with the given functions and test for relative maxima and minima.

(i) $z = (x-2)^2 + (y-3)^2$

(ii) $z = (x-2)^2 - (y-3)^2$

(iii) $z = -(x-2)^2 - (y-3)^2$

▶ **8-11.** Let $u(x, y, z)$ denote a scalar field which is continuous and differentiable. Let $x = x(t), y = y(t)$ and $z = z(t)$ denote the position vector of a particle moving through the scalar field. Show that on the path of the particle one finds

$$\frac{du}{dt} = (\text{grad } u) \cdot \frac{d\vec{r}}{dt}.$$

▶ **8-12.** Let $f(x, y, z, t)$ denote a scalar field which is changing with time as well as position. Let $x = x(t), y = y(t)$ and $z = z(t)$ denote the position vector of a particle moving through the scalar field. Show that on the path of the particle

$$\frac{df}{dt} = (\text{grad } f) \cdot \frac{d\vec{r}}{dt} + \frac{\partial f}{\partial t}.$$

In hydrodynamics, where $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ represents the velocity of the particle, the above derivative $\frac{df}{dt}$ is called **a material derivative** and is represented using the notation $\frac{Df}{Dt}$. Note that the material derivative represents the change of f as one follows the motion of the fluid.

▸ **8-13.** A force field $\vec{F}$ is said to be conservative if it is derivable from a scalar potential function V such that

$$\vec{F} = \pm \text{grad } V.$$

One uses either a plus sign or a minus sign depending upon the particular application being represented.

Consider the motion of a spring-mass system which oscillates in the x-direction. Assume the force acting on the mass m is derivable from the potential function $V = \frac{1}{2}kx^2$, where k is the spring constant. Use Newton's second law (vector form) and derive the equation of motion of the spring-mass system.

▸ **8-14.** (**Divergence of a vector quantity**)

Let

$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

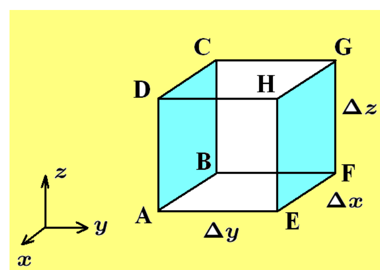
denote a vector field and consider a volume element $\Delta x \Delta y \Delta z$ located at the point (x,y,z) in this vector field.

(a) Use the first couple of terms of a Taylor series expansion to calculate the vector field at

(i) $\quad \vec{F}(x+\Delta x, y, z)$

(ii) $\quad \vec{F}(x, y+\Delta y, z)$

(iii) $\quad \vec{F}(x, y, z+\Delta z)$



(b) Use the results in part (a) and calculate the flux over the surface of the cubic volume element $\Delta V = \Delta x\,\Delta y\,\Delta z$ and then divided by the volume of this element in the limit as the volume tends toward zero.

▸ **8-15.** Determine whether the given vector fields are solenoidal or irrotational

(i) $\quad \vec{F} = (2xyz - z^2)\,\hat{\mathbf{e}}_1 + x^2 z\,\hat{\mathbf{e}}_2 + (x^2 y - 2xz)\,\hat{\mathbf{e}}_3$

(ii) $\quad \vec{F} = \hat{\mathbf{e}}_1 + (x^2 y - y^2 z)\,\hat{\mathbf{e}}_2 + (yz^2 - x^2 z)\,\hat{\mathbf{e}}_3$

(iii) $\quad \vec{F} = 2xy\,\hat{\mathbf{e}}_1 + (x^2 - 2yz)\,\hat{\mathbf{e}}_2 - y^2\,\hat{\mathbf{e}}_3$

(iv) $\quad \vec{F} = 2x(z-y)\,\hat{\mathbf{e}}_1 + (y^2 - yx^2)\,\hat{\mathbf{e}}_2 + (zx^2 - z^2)\,\hat{\mathbf{e}}_3$

▸ **8-16.** Show that div $(\text{curl } \vec{F}) = 0$

▸ **8-17.** Show that curl $(\text{grad } \phi) = \vec{0}$

▶ **8-18.** For $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and $r = |\vec{r}|$ show

(i) $\text{curl } r^n\vec{r} = \vec{0}$

(ii) $\text{div } \vec{r} = 3$

(iii) $\text{curl } \vec{r} = \vec{0}$

(iv) $\text{div } r^n\vec{r} = (n+3)r^n$

▶ **8-19.** Show that the vector field $\nabla\phi$ is both solenoidal and irrotational if ϕ is a scalar function of position which satisfies the Laplace equation $\nabla^2\phi = 0$.

▶ **8-20.** Show that the following functions are solutions of the Laplace equation in two dimensions.

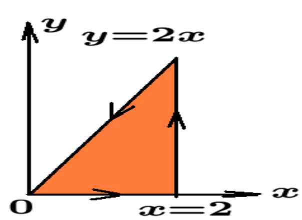
(i) $\phi = x^2 - y^2$

(ii) $\phi = 3x^2y - y^3$

(iii) $\phi = \ln(x^2 + y^2)$

▶ **8-21.** Verify the divergence theorem for $\vec{F} = xy\,\hat{\mathbf{e}}_1 + y^2\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ over the region bounded by the cylindrical surface $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 4$. Whenever possible, integrate by using cylindrical or polar coordinates. Find the sections of this surface which has a flux integral.

▶ **8-22.**



(i) Verify Green's theorem in the plane for

$$M(x,y) = x^2 + y^2 \quad \text{and} \quad N(x,y) = xy,$$

where C is the closed curve illustrated in the figure.

(ii) Use line integration and appropriate values for M and N in Green's theorem to determine the shaded area of the attached figure.

▶ **8-23.** Verify Stokes theorem for $\vec{F} = y\,\hat{\mathbf{e}}_3$ over that portion of the unit sphere in the first octant. Hint: Use spherical coordinates.

▶ **8-24.** Verify the given differential equations are exact and then use line integrals to find solutions.

(i) $(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$

(ii) $(3x^2y + 2xy^2)\,dx + (x^3 + 2yx^2 + 2)\,dy = 0$

▶ **8-25.** Use line integrals to find the area enclosed by the given curves.

(i) The ellipse, $x = a\cos t \quad y = b\sin t, \quad 0 \le t \le 2\pi.$

(ii) The circle, $x = \cos t \quad y = \sin t, \quad 0 < t < 2\pi.$

(iii) The unit square whose boundaries are $x = 0,\, x = 1,\, y = 0,\, y = 1.$

▶ **8-26.** Verify the divergence theorem in the case $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. Hint: Use spherical coordinates.

▶ **8-27.** Calculate the flux of the vector field $\vec{F} = z\,\hat{\mathbf{e}}_3$ entering and leaving the volume enclosed by the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Does the Gauss divergence theorem hold for this volume and surface?

▶ **8-28.** Calculate the flux of the vector field $\vec{F} = y\,\hat{\mathbf{e}}_2$ entering and leaving the volume enclosed by the two cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, bounded by the planes $z = 0$ and $z = 2$. Does the Gauss divergence theorem hold for this volume and surface?

▶ **8-29.**

Let S denote the surface of a rectangular parallelepiped with unit surface normals $\pm\,\hat{\mathbf{e}}_1, \pm\,\hat{\mathbf{e}}_2, \pm\,\hat{\mathbf{e}}_3$ and write the surface integral

$$I = \iint_S \vec{F}\cdot d\vec{S} = \int_{S_1} \vec{F}\cdot d\vec{S} + \int_{S_2} \vec{F}\cdot d\vec{S} + \int_{S_3} \vec{F}\cdot d\vec{S} + \int_{S_4} \vec{F}\cdot d\vec{S} + \int_{S_5} \vec{F}\cdot d\vec{S} + \int_{S_6} \vec{F}\cdot d\vec{S}$$

as a summation of the flux over the six faces of the parallelepiped. Calculate the above flux integral for $\vec{F} = y\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + x\,\hat{\mathbf{e}}_3$

(iii) Consider a unit cube with one vertex at the origin. Calculate the flux entering or leaving each face of the cube. Sum these fluxes and comment on your result.

▶ **8-30.** Let $\vec{F} = M(x,y)\,\hat{\mathbf{e}}_1 + N(x,y)\,\hat{\mathbf{e}}_2$ and use $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ to represent the position of the curve C and show Green's theorem in the plane can be represented in either of the forms

$$(a)\ \oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot\hat{\mathbf{e}}_3\,dxdy \quad \text{or} \quad (b)\ \oint_C (\vec{F}\times\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_n\,ds = \iint_R \nabla\cdot(\vec{F}\times\hat{\mathbf{e}}_3),dxdy$$

where $\hat{\mathbf{e}}_n$ is a unit outward normal to the boundary curve C.
Hint: Use triple scalar product.

► **8-31.** Let $\vec{r}$ denote a position vector to a general point on a closed surface S, which encloses a volume V. Evaluate the surface integral

$$\iint_S \vec{r} \cdot d\vec{S} = \iint_S \vec{r} \cdot \hat{\mathbf{e}}_n \, dS$$

► **8-32.** **The Gauss Theorem** Let $\vec{r}$ denote the position vector from the origin to a general point on a closed surface S. Show that

$$\iint_S \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r^3} \, dS = \begin{cases} 0, & \text{if the origin is outside the closed surface } S \\ 4\pi, & \text{if the origin is inside the closed surface } S \end{cases}$$

Hint: Use the divergence theorem and when the origin is inside S, construct a small sphere of radius ϵ about the origin.

► **8-33.** For $\vec{F} = x^2 z \, \hat{\mathbf{e}}_1 + xyz \, \hat{\mathbf{e}}_2 + yz \, \hat{\mathbf{e}}_3$ and $\phi = xyz^2$, calculate $\nabla(\phi \vec{F})$

► **8-34.** For $\vec{A}, \vec{B}$ vector fields and f a scalar field, verify each of the following:

(i) $\text{curl } (f\vec{A}) = (\text{grad } f) \times \vec{A} + f \, \text{curl } \vec{A}$

(ii) $\text{curl } (\vec{A} \times \vec{B}) = \vec{A}(\text{div } \vec{B}) - \vec{B}(\text{div } \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

(iii) $\text{div } (f\vec{A}) = (\text{grad } f) \cdot \vec{A} + f \, \text{div } \vec{A}$

(iv) $\text{grad } (\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl } \vec{A} - \vec{A} \cdot \text{curl } \vec{B}$

(v) $\text{grad } (fg) = f \, \text{grad } g + g \, \text{grad } f$

(vi) $\text{grad } (\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times \text{curl } \vec{B} + \vec{B} \times \text{curl } \vec{A}$

► **8-35.** Evaluate the line integral

$$A = \frac{1}{2} \oint_C x \, dy - y \, dx$$

around the triangle having the vertices $(0,0)$, $(b,0)$ and (c,h) where b, c, h are positive constants. Evaluate this integral using Green's theorem in the plane.

► **8-36.** Evaluate the integral

$$I = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S},$$

where $\vec{F} = (y - 2x) \, \hat{\mathbf{e}}_1 + (3x + 2y) \, \hat{\mathbf{e}}_2$ and S is the surface of the cone $x = u \cos v, \; y = u \sin v, \; z = u$ for $0 \leq u \leq 9$ and $0 \leq v \leq 2\pi$.
Hint: If you use Stoke's theorem be sure to note direction of integration.

▶ **8-37.** Let $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and evaluate the surface integral

$$I = \iint\limits_S \vec{F} \cdot d\vec{S},$$

where S is the surface enclosing the volume bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $2x + 3y + 4z = 12$ Hint: The volume of a tetrahedron having sides $a,\ b$ and c is given by $V = \frac{1}{6}abc$.

▶ **8-38.** Use Stokes theorem to evaluate the integral

$$I = \oint_C \vec{F} \cdot d\vec{r}, \quad \text{where} \quad \vec{F} = y\,\hat{\mathbf{e}}_1 + 2z\,\hat{\mathbf{e}}_2 + (4y + 2x)\,\hat{\mathbf{e}}_3$$

and C is the simple closed curve consisting of the line segments

$$\overline{P_1P_2} + \overline{P_2P_3} + \overline{P_3P_1}$$

connecting the points $P_1(0,0,0),\ P_2(1,1,0)$, and $P_3(0,0,2\sqrt{2})$.

▶ **8-39.** Let $\vec{v} = \vec{v}(x,y,z,t)$ denote the velocity of a fluid having density$\rho = \rho(x,y,z,t)$. Construct an imaginary volume of fluid V enclosed by a surface S lying within the fluid.

(a) Show the mass of the fluid inside V is given by $M = \iiint \rho(x,y,z,t)\,dV$

(b) Show the time rate of change of mass is $\dfrac{\partial M}{\partial t} = \iint \dfrac{\partial \rho}{\partial t}\,dV$

(c) Show the mass of fluid leaving V per unit of time is given by $\dfrac{\partial M}{\partial t} = -\iint_S \rho\vec{v}\cdot\vec{n}\,dS$

(d) Use the divergence theorem to show $\iiint \dfrac{\partial \rho}{\partial t}\,dV = -\iint \rho\vec{v}\cdot\vec{n}\,dS = -\iiint_V \nabla(\rho\vec{v})\,dV$

(e) Since V is an arbitrary volume show that $\nabla\vec{J} + \dfrac{\partial \rho}{\partial t} = 0$, where $\vec{J} = \rho\vec{v}$. This equation is known as the continuity equation of fluid dynamics.

▶ **8-40.** In parabolic cylindrical coordinates (ξ,η,z), find

(a) The unit vectors $\hat{\mathbf{e}}_\xi,\ \hat{\mathbf{e}}_\eta,\ \hat{\mathbf{e}}_z$

(b) The metric components g_{ij}

▶ **8-41.** In the paraboloidal coordinates (ξ,η,ϕ), find

(a) The unit vectors $\hat{\mathbf{e}}_\xi, \hat{\mathbf{e}}_\eta, \hat{\mathbf{e}}_\phi$

(b) The metric components g_{ij}

▶ **8-42.** In cylindrical coordinates (r, θ, z), show that

$$\text{curl}\ \vec{F} = \nabla \times \vec{F} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

▶ **8-43.** In cylindrical coordinates (r, θ, z), show that

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(rA_z)\right]$$

▶ **8-44.** In cylindrical coordinates (r, θ, z), show that

$$\text{grad}\ u = \nabla u = \frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{\partial u}{\partial z}\,\hat{\mathbf{e}}_z$$

▶ **8-45.** In cylindrical coordinates (r, θ, z), show that

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

▶ **8-46.** In spherical coordinates (r, θ, ϕ), show that

$$\text{curl}\ \vec{F} = \nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

▶ **8-47.** In spherical coordinates (r, θ, ϕ), show that

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta F_r\right) + \frac{\partial}{\partial \theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right]$$

▶ **8-48.** In spherical coordinates (r, θ, ϕ), show that

$$\text{grad}\ u = \frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{r\sin^2\theta}\frac{\partial u}{\partial \phi}\,\hat{\mathbf{e}}_\phi$$

▶ **8-49.** In spherical coordinates (r, θ, ϕ), show that

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

▶ **8-50.** Show that

(a) In cylindrical coordinates (r, θ, z), the element of volume is $dV = r\,dr\,d\theta dz$.

(b) In spherical coordinates (r, θ, ϕ), the element of volume is $dV = r^2\ \sin\theta\,dr\,d\phi\,d\theta$.

(c) In a general orthogonal curvilinear coordinate system (u, v, w), the element of volume can be expressed as $dV = h_u h_v h_w\,du\,dv\,dw$.

► **8-51.** Show in a general orthogonal coordinate system $\operatorname{div}(\operatorname{grad} v \times \operatorname{grad} w) = 0$

► **8-52.** For $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ independent orthogonal unit vectors (base vectors), one can express any vector $\vec{A}$ as

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3,$$

where A_1, A_2, A_3 are the coordinates of $\vec{A}$ relative to the base vectors chosen.

(a) Show that these components are the projection of $\vec{A}$ onto the base vectors and

$$\vec{A} = (\vec{A}\cdot\hat{\mathbf{e}}_1)\,\hat{\mathbf{e}}_1 + (\vec{A}\cdot\hat{\mathbf{e}}_2)\,\hat{\mathbf{e}}_2 + (\vec{A}\cdot\hat{\mathbf{e}}_3)\,\hat{\mathbf{e}}_3.$$

(b) By selecting any three independent orthogonal vectors,$\vec{E}_1, \vec{E}_2, \vec{E}_3$, not necessarily of unit length, show that one can write

$$\vec{A} = \left(\frac{\vec{A}\cdot\vec{E}_1}{\vec{E}_1\cdot\vec{E}_1}\right)\vec{E}_1 + \left(\frac{\vec{A}\cdot\vec{E}_2}{\vec{E}_2\cdot\vec{E}_2}\right)\vec{E}_2 + \left(\frac{\vec{A}\cdot\vec{E}_3}{\vec{E}_3\cdot\vec{E}_3}\right)\vec{E}_3.$$

Consequently,

$$\frac{\vec{A}\cdot\vec{E}_i}{\vec{E}_i\cdot\vec{E}_i}, \qquad i = 1, 2, \text{or}\, 3$$

are the components of $\vec{A}$ relative to the chosen base vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

► **8-53.** Two bases $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ are said to be reciprocal if they satisfy the condition

$$\vec{E}_i\cdot\vec{E}^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(i.e., A vector from one basis is orthogonal to two of the vectors from the other basis). Show that if $\vec{E}_1, \vec{E}_2, \vec{E}_3$ is a given set of base vectors, then

$$\vec{E}^1 = \frac{1}{V}\vec{E}_2\times\vec{E}_3, \qquad \vec{E}^2 = \frac{1}{V}\vec{E}_3\times\vec{E}_1, \qquad \vec{E}^3 = \frac{1}{V}\vec{E}_1\times\vec{E}_2$$

is a reciprocal basis, where $V = \vec{E}_1\cdot(\vec{E}_2\times\vec{E}_3)$ is a triple scalar product and represents the volume of the parallelepiped having the basis vectors for its sides. Show also that $\vec{E}^1\cdot(\vec{E}^2\times\vec{E}^3) = \dfrac{1}{V}$

▶ **8-54.** Let $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ be a system of reciprocal basis. (See previous problem).

(a) If $\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3$ find the components A^1, A^2, A^3 of $\vec{A}$ relative to the base vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

(b) If $\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3$ find the components A_1, A_2, A_3 relative to the basis $\vec{E}^1, \vec{E}^2, \vec{E}^3$. The numbers A^i are called the contravariant components of $\vec{A}$ and the numbers A_i are called the covariant components of $\vec{A}$.

(c) Using the notation

$$\vec{E}_i \cdot \vec{E}_j = g_{ij} = g_{ji}, \qquad \text{and} \qquad \vec{E}^i \cdot \vec{E}^j = g^{ij} = g^{ji},$$

where $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ is a reciprocal system of basis, show that

$$A_i = \sum_{k=1}^{3} g_{ik}A^k \qquad \text{and} \qquad A^i = \sum_{k=1}^{3} g_{ik}A_k,$$

where i is called the free index and k is a summation index. Here g^{ij} are called the conjugate metric components of the space and satisfy $\sum_{j=1}^{3} g_{ij}g^{jk} = \delta_i^k$ is the **Kronecker delta**.

(d) Show that

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \sum_{j=1}^{3} g_{ij}g^{jk} = \delta_i^k$$

▶ **8-55.** Show that in an orthogonal curvilinear coordinate system (u, v, w), the vectors

$$(\vec{E}_1, \vec{E}_2, \vec{E}_3) = \left(\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}\right)$$
$$\text{and} \qquad (\vec{E}^1, \vec{E}^2, \vec{E}^3) = (\text{grad } u,\ \text{grad } v,\ \text{grad } w)$$

are a reciprocal system of basis.

Chapter 9

Applications of Vectors

The use of vectors in mathematics, physics, engineering and the sciences is extensive. The applications presented within these pages have been selected mainly from the study areas of physics and engineering.

Approximation of Vector Field

The Kriging[1] method is a numerical method to **approximate a quantity using a statistical weighting of known data values**. The Kriging method can be used to approximate many different kinds of quantities. The following illustrates an application for the **approximation of a vector field using interpolation**. The weighted average associated with a set of data values $\{Q_1, Q_2, Q_3, \ldots, Q_n\}$ is defined

$$Q = \frac{w_1 Q_1 + w_2 Q_2 + w_3 Q_3 + \cdots + w_n Q_n}{w_1 + w_2 + w_3 + \cdots w_n} \tag{9.1}$$

where $w_1, \ldots, w_n$ are the assigned weighting factors. Note that if all the weights equal unity, then equation (9.1) reduces down to a regular average of the given data values.

The following discussion illustrates how the Kriging method can be used to **approximate a vector field** in the neighborhood of **known points and known vectors associated with these points**. Note that the discussion presented can be generalized and made applicable to any quantity $Q = Q(x, y, z)$ which is a function of position that one wants to approximate.

Given a finite number of **known vectors**

$$\vec{F}_1 = \vec{F}(x_1, y_1, z_1), \vec{F}_2 = \vec{F}(x_2, y_2, z_2), \; \cdots, \vec{F}_n = \vec{F}(x_n, y_n, z_n)$$

which are associated with the **known points** $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$. It is assumed that these known vectors are associated with a vector field $\vec{F} = \vec{F}(x, y, z)$, but we don't know the form for $\vec{F}$. In order to approximate the representation of the vector field $\vec{F} = \vec{F}(x, y, z)$ in some neighborhood of the known points (x_i, y_i, z_i), $i = 1, \ldots, n$ and known vectors at these points one can proceed as follows. In order to use the known data values to estimate the value of $\vec{F} = \vec{F}(x, y, z)$ at a general point (x, y, z) one can define the distances

[1] Danie Gerhardus Krige(1919-) A South African geologist and mining engineer.

$$
\begin{aligned}
d_1 =& \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}\\
d_2 =& \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}\\
&\vdots\\
d_n =& \sqrt{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2}
\end{aligned}
\tag{9.2}
$$

of a general point (x, y, z) from each of the known data points. It is then possible to use these distances to construct the weights

$$
w_1 = d_2 d_3 d_4 \cdots d_n, \quad w_2 = d_1 d_3 d_4 \ldots d_n, \quad w_3 = d_1 d_2 d_4 \ldots d_n, \quad \ldots \quad w_n = d_1 d_2 \cdots d_{n-1} \tag{9.3}
$$

Note that to form the weight w_i, for some fixed value of i in the range $1 \le i \le n$, one can form a product of all the distances $\prod_{j=1}^{n} d_j = d_1 d_2 d_3 \cdots d_{i-1} d_i d_{i+1} \cdots d_n$ and then remove the term d_i from this product to form the weight $w_i = d_1 d_2 d_3 \cdots d_{i-1} d_{i+1} \cdots d_n$. A shorthand notation to represent the above weights is given by the product formula

$$
w_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} d_j \quad \text{where} \quad d_j = \sqrt{(x-x_j)^2+(y-y_j)^2+(z-z_j)^2} \tag{9.4}
$$

for $j = 1, 2, 3, \ldots, n$. The vector $\vec{F}$ at the interpolation position (x, y, z) is then approximated by the weighted average

$$
\vec{F} = \vec{F}(x, y, z) = \frac{w_1 \vec{F}_1 + w_2 \vec{F}_2 + \cdots + w_n \vec{F}_n}{w_1 + w_2 + \cdots + w_n} \tag{9.5}
$$

Observe that if $(x, y, z) = (x_i, y_i, z_i)$ for some fixed value of i in the range $1 \le i \le n$, then $d_i = 0$ and equation (9.5) reduces to the identity $\vec{F}(x_i, y_i, z_i) = \vec{F}_i$. The Kriging method examines the distances between the coordinates of the known quantities and the selected interpolation point (x, y, z). It then forms weights where points closest to the interpolation point have the highest weight. This can be seen by writing the coefficients of the vectors in equation (9.5) in the form

$$
Coefficient_i = \frac{\frac{1}{d_i}}{\sum_{j=1}^{n} \frac{1}{d_j}} \tag{9.6}
$$

so that the smaller the $d_i \neq 0$, the higher the weighting coefficient. If $d_i = 0$, then all the coefficients with index different from i are zero so that an identity with the known value at i occurs. This approximation method is **an interpolation method** associated with the given data values and is known as **a weighted prediction method.**

A modification of the above method is obtained as follows. The Kriging weights can be generalized by requiring the coefficients in equation (9.6) to have the form

$$Coefficient_i = \frac{\frac{1}{d_i^\beta}}{\sum_{j=1}^{n} \frac{1}{d_j^\beta}} \qquad \beta > 0 \text{ a constant} \tag{9.7}$$

and then adjusting the parameter β to achieve some kind of desired result.

Spherical Trigonometry

The figure 9-1 illustrates three points A, B, C on the surface of **a unit sphere** with a great circle passing through any two of the selected points. This forms a spherical triangle. Let α, β, γ denote the angles[2] at the points A, B, C and let a, b, c denote the length of the sides opposite these angles. On a circle of radius r the arc length s swept out by an angle θ is given by $s = r\theta$. The sphere being a unit sphere dictates that the arc lengths $a = \angle B0C$, $b = \angle A0C$, $c = \angle A0B$. **One of the basic problems in spherical trigonometry is to find a relation between the angles** α, β, γ **and the sides of arc lengths** a, b, c **of a spherical triangle.** The following illustrates how vectors can be employed to find such relationships.

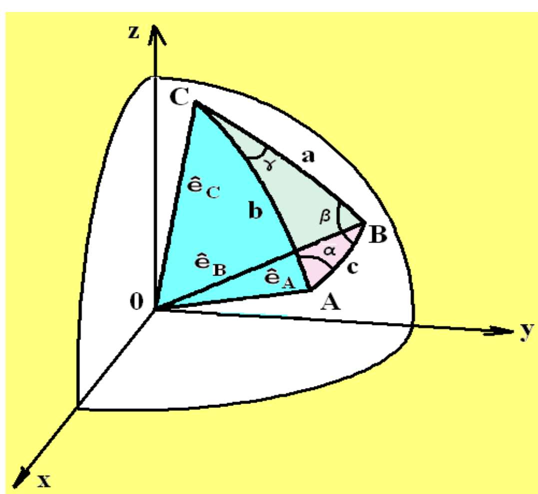


Figure 9-1. Spherical triangle with unit vectors $\hat{\mathbf{e}}_A$, $\hat{\mathbf{e}}_B$, $\hat{\mathbf{e}}_C$

[2] The angles are the same as the angles between the tangent lines to the great circles.

Define the **unit vectors** $\hat{\mathbf{e}}_A$, $\hat{\mathbf{e}}_B$, $\hat{\mathbf{e}}_C$ from **the center of the unit sphere** to the points A, B, C on the surface of the sphere and observe that by using the definition of a cross product and dot product one obtains

$$\begin{aligned}|\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C| &= \sin b, \qquad & |\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B| &= \sin c, \qquad & |\,\hat{\mathbf{e}}_C \times \hat{\mathbf{e}}_B| &= \sin a \\ \hat{\mathbf{e}}_A \cdot \hat{\mathbf{e}}_C &= \cos b, & \hat{\mathbf{e}}_A \cdot \hat{\mathbf{e}}_B &= \cos c, & \hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_B &= \cos a\end{aligned} \tag{9.8}$$

Note that since **the sphere is a unit sphere** the angles a, b, c are given respectively by the arcs $\widehat{BC}$, $\widehat{AC}$ and $\widehat{AB}$ or arcs opposite the vertices A, B, C.

The angle between two intersecting planes is called a **dihedral angle**. The dihedral angle can be calculated from the unit normal vectors to the intersecting planes. In figure 9-1 , let

$$\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C = \sin a\ \hat{\mathbf{e}}_{0BC}, \qquad \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C = \sin b\ \hat{\mathbf{e}}_{0AC}, \qquad \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B = \sin c\ \hat{\mathbf{e}}_{0AB} \tag{9.9}$$

define the unit vectors $\hat{\mathbf{e}}_{0BC}$, $\hat{\mathbf{e}}_{0AC}$, $\hat{\mathbf{e}}_{0AB}$ which are perpendicular to the planes defining the dihedral angles α, β, γ. The cross product relations given by the equation (9.8) together with the unit normal vectors can be used to calculate the cosines associated with the angle α, β, γ. One finds that

$$\hat{\mathbf{e}}_{0BC} \cdot \hat{\mathbf{e}}_{0AB} = \cos\beta, \qquad \hat{\mathbf{e}}_{0BC} \cdot \hat{\mathbf{e}}_{0AC} = \cos\gamma, \qquad \hat{\mathbf{e}}_{0AC} \cdot \hat{\mathbf{e}}_{0AB} = \cos\alpha \tag{9.10}$$

and with the aid of equations (9.9) one can write

$$\cos\gamma = \frac{|(\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C) \cdot (\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B)|}{|\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C|\,|\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C|} \tag{9.11}$$

with similar expressions for the representation of $\cos\alpha$ and $\cos\beta$. The relation (9.11) can be simplified using the dot product relation (6.32) which is repeated here

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{9.12}$$

The numerator in equation (9.11) can then be expressed

$$\begin{aligned}(\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C) \cdot (\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C) =& (\,\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_A)(\,\hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_C) - (\,\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_C)(\,\hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_A) \\ =& \cos c - \cos a \cos b\end{aligned} \tag{9.13}$$

The results from equations (9.8) and (9.13) show that the equation (9.11) can be expressed in the form

$$\cos c = \cos a\ \cos b + \sin a\ \sin b\ \cos\gamma \tag{9.14}$$

Using similar arguments associated with the representation of $\cos\alpha$ and $\cos\beta$, one can show

$$\begin{aligned} \cos b =& \cos c\, \cos a + \sin c\, \sin a\, \cos\beta \\ \cos a =& \cos b\, \cos c + \sin b\, \sin c\, \cos\alpha \end{aligned} \tag{9.15}$$

The equations (9.14) and (9.15)) are known as the **law of cosines for the spherical triangle** ABC.

Replace the dot product in equation (9.11) by a cross product and show

$$\sin\gamma = \frac{|(\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C)|}{|\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C|\,|\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C|} \tag{9.16}$$

The cross product relation (6.30), repeated here as

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{C}\left[\vec{D} \cdot (\vec{A} \times \vec{B})\right] - \vec{D}\left[\vec{C} \cdot (\vec{A} \times \vec{B})\right] \tag{9.17}$$

can be used to simplify the numerator of equation (9.16). One can use properties of the scalar triple product to write

$$\begin{aligned} (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C) =& \hat{\mathbf{e}}_A\,[\,\hat{\mathbf{e}}_C \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] - \,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] \\ =& \hat{\mathbf{e}}_A\,[\,\hat{\mathbf{e}}_B \cdot (\,\hat{\mathbf{e}}_C \times\, \hat{\mathbf{e}}_C)] - \,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] \\ =& -\,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] \end{aligned} \tag{9.18}$$

so that

$$|(\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C)| = \,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)$$

The triple scalar product relation shows that

$$\begin{aligned} \sin\gamma\, \sin a\, \sin b =& |\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)| \\ \sin\alpha\, \sin b\, \sin c =& |\,\hat{\mathbf{e}}_B \cdot (\,\hat{\mathbf{e}}_C \times\, \hat{\mathbf{e}}_A)| \\ \sin\beta\, \sin c\, \sin a =& |\,\hat{\mathbf{e}}_C \cdot (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_B)| \end{aligned} \tag{9.19}$$

and the scalar triple product relation implies that

$$\sin\alpha\, \sin b\, \sin c = \sin\beta\, \sin c\, \sin a = \sin\gamma\, \sin a\, \sin b \tag{9.20}$$

Divide each term in equation (9.20) by $\sin a\, \sin b\, \sin c$ to show

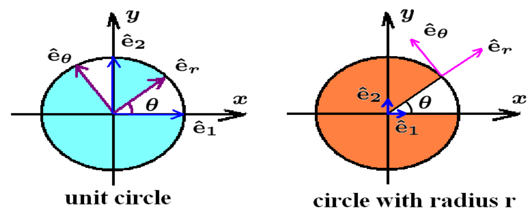
$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} \tag{9.21}$$

which is known as **the law of sines from spherical trigonometry**.

Velocity and Acceleration in Polar Coordinates

In polar coordinates (r, θ) one can employ the orthogonal unit vectors

$$\begin{aligned} \hat{\mathbf{e}}_r &= \cos\theta\, \hat{\mathbf{e}}_1 + \sin\theta\, \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_\theta &= -\sin\theta\, \hat{\mathbf{e}}_1 + \cos\theta\, \hat{\mathbf{e}}_2 \end{aligned} \tag{9.22}$$



to represent the position of a moving particle.

If in two-dimensional polar coordinates the position vector of a moving particle is represented in the form

$$\vec{r} = r\, \hat{\mathbf{e}}_r$$

where r, θ and consequently, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ are changing with respect to time t, then the velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r \tag{9.23}$$

Differentiate the vectors in equation (9.22) with respect to time t and show

$$\begin{aligned} \frac{d\,\hat{\mathbf{e}}_r}{dt} &= -\sin\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 + \cos\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = \frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta \\ \frac{d\,\hat{\mathbf{e}}_\theta}{dt} &= -\cos\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 - \sin\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_r \end{aligned} \tag{9.24}$$

The first equation in (9.24) simplifies the equation (9.23) to the form

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta \tag{9.25}$$

where $\dot{} = \frac{d}{dt}$. Here $v_r = \dot{r}$ is called **the radial component of the velocity** and the term $v_\theta = r\dot{\theta}$ is called **the transverse component of the velocity** or **tangential component of the velocity.** The **speed of the particle** is given by

$$v = |\vec{v}| = \sqrt{(r\dot{\theta})^2 + \dot{r}^2}$$

which represents the magnitude of the velocity.

The acceleration of the particle is obtained by differentiating the velocity. Differentiate the equation (9.25) and show

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \dot{r}\frac{d\,\hat{\mathbf{e}}_r}{dt} + \ddot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + \frac{d}{dt}(r\dot{\theta})\,\hat{\mathbf{e}}_\theta \\ =& \dot{r}(\dot{\theta}\,\hat{\mathbf{e}}_\theta) + \ddot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}(-\dot{\theta}\,\hat{\mathbf{e}}_r) + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta \\ \vec{a} =& (\ddot{r} - r(\dot{\theta})^2)\,\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta \end{aligned}$$

Here the **radial component of acceleration** is $(\ddot{r}-r(\dot{\theta})^2)$ and the **transverse component of acceleration** or **tangential component** is $(r\ddot{\theta}+2\dot{r}\dot{\theta})$. The **magnitude of the acceleration** is given by

$$a=|\vec{a}|=\sqrt{(\ddot{r}-r(\dot{\theta})^2)^2+(r\ddot{\theta}+2\dot{r}\dot{\theta})^2}$$

Velocity and Acceleration in Cylindrical Coordinates

In rectangular (x,y,z) coordinates the position vector, velocity vector and acceleration vector of a moving particle are given by

$$\begin{aligned}\vec{v}=\vec{r}=&x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3\\ \vec{v}=\frac{d\vec{r}}{dt}=&\frac{dx}{dt}\,\hat{\mathbf{e}}_1+\frac{dy}{dt}\,\hat{\mathbf{e}}_2+\frac{dz}{dt}\,\hat{\mathbf{e}}_3\\ \vec{a}=\frac{d\vec{r}}{dt}=\frac{d^2\vec{r}}{dt^2}=&\frac{d^2x}{dt^2}\,\hat{\mathbf{e}}_1+\frac{d^2y}{dt^2}\,\hat{\mathbf{e}}_2+\frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_3\end{aligned}$$

Upon changing to a cylindrical coordinates (r,θ,z) using the transformation equations

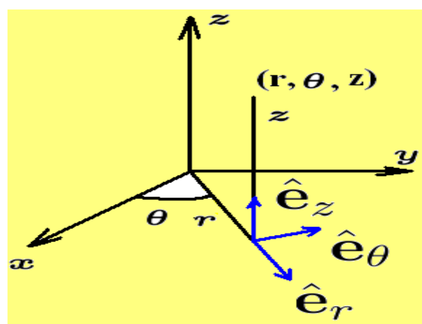
$$x=r\cos\theta,\qquad y=r\sin\theta,\qquad z=z$$

one can represent the position vector of the particle as

$$\vec{r}=r\cos\theta\,\hat{\mathbf{e}}_1+r\sin\theta\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3 \tag{9.26}$$

Using the orthogonal unit vectors

$$\begin{aligned}\hat{\mathbf{e}}_r=&\frac{\partial\vec{r}}{\partial r}=\cos\theta\,\hat{\mathbf{e}}_1+\sin\theta\,\hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_\theta=&\frac{1}{r}\frac{\partial\vec{r}}{\partial\theta}=-\sin\theta\,\hat{\mathbf{e}}_1+\cos\theta\,\hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_z=&\frac{\partial\vec{r}}{\partial z}=\,\hat{\mathbf{e}}_3\end{aligned} \tag{9.27}$$



obtained from equations (7.107), the position vector of a moving particle can be expressed in cylindrical coordinates as

$$\vec{r}=r\,\hat{\mathbf{e}}_r+z\,\hat{\mathbf{e}}_z \tag{9.28}$$

To obtain the velocity vector in cylindrical coordinates one must differentiate equation (9.28) with respect to time t to obtain

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{dr}{dt}\,\hat{\mathbf{e}}_r+r\frac{d\,\hat{\mathbf{e}}_r}{dt}+\frac{dz}{dt}\,\hat{\mathbf{e}}_z \tag{9.29}$$

since $\hat{\mathbf{e}}_r$ changes with time, but $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_3$ remains constant. From equation (9.27) one can calculate the derivatives

$$\begin{aligned} \frac{d\,\hat{\mathbf{e}}_r}{dt} =& -\sin\theta\,\frac{d\theta}{dt}\;\hat{\mathbf{e}}_1 + \cos\theta\,\frac{d\theta}{dt}\;\hat{\mathbf{e}}_2 = \frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta \\ \frac{d\,\hat{\mathbf{e}}_\theta}{dt} =& -\cos\theta\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 - \sin\theta\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta \\ \frac{d\,\hat{\mathbf{e}}_z}{dt} =& 0 \end{aligned} \tag{9.30}$$

as these derivatives will be useful in simplifying any derivatives with respect to time of vectors in cylindrical coordinates. The equations (9.30) allows one to obtain the result

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta + \frac{dz}{dt}\;\hat{\mathbf{e}}_z \tag{9.31}$$

which can also be represented in the form

$$\vec{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta + \dot{z}\,\hat{\mathbf{e}}_z$$

where the dot notation is used to represent time differentiation. Here $v_r = \dot{r}$ is **the radial component of the velocity**, $v_\theta = r\dot{\theta}$ is **the azimuthal component of velocity** and $v_z = \dot{z}$ is **the vertical component of the velocity**.

The acceleration in cylindrical coordinates is obtained by differentiating the velocity. Differentiate the equation (9.31) with respect to time t and show

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \frac{d}{dt}\left[\frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta + \frac{dz}{dt}\;\hat{\mathbf{e}}_z\right] \\ =& \frac{dr}{dt}\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{d^2r}{dt^2}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + r\frac{d^2\theta}{dt^2}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z \\ =& \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2r}{dt^2}\,\hat{\mathbf{e}}_r - r\left(\frac{d\theta}{dt}\right)^2\;\hat{\mathbf{e}}_r + r\frac{d^2\theta}{dt^2}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z \\ =& \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\;\hat{\mathbf{e}}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\;\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z \\ \vec{a} =& (\ddot{r} - r(\dot{\theta})^2)\,\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta + \ddot{z}\,\hat{\mathbf{e}}_z \end{aligned} \tag{9.32}$$

where $\dot{} = \frac{d}{dt}$ and $\ddot{} = \frac{d^2}{dt^2}$ are shorthand notations for the first and second derivatives with respect to time t. In calculating the derivatives in equation (9.32) make note that the results from equation (9.30) have been employed.

Velocity and Acceleration in Spherical Coordinates

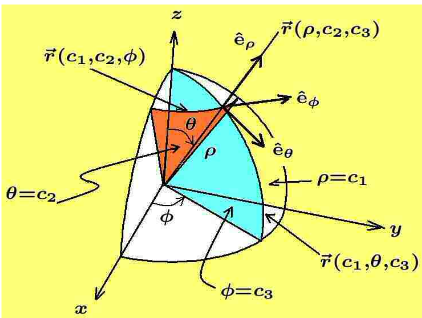
Upon changing to spherical[3] coordinates (ρ, θ, ϕ) the transformation equations are

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\theta$$

and consequently the position vector describing the position of a moving particle is given by

$$\vec{r} = \rho \sin\theta \cos\phi\, \hat{\mathbf{e}}_1 + \rho \sin\theta \sin\phi\, \hat{\mathbf{e}}_2 + \rho \cos\theta\, \hat{\mathbf{e}}_3 \tag{9.33}$$

Using the unit orthogonal vector $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$ in spherical coordinates obtained from the equations (7.102) and having the representation



$$\begin{aligned}
\hat{\mathbf{e}}_\rho =& \frac{\partial \vec{r}}{\partial \rho} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\\
\hat{\mathbf{e}}_\theta =& \frac{1}{\rho}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3\\
\hat{\mathbf{e}}_\phi =& \frac{1}{\rho\sin\theta} = -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2
\end{aligned} \tag{9.34}$$

the position vector $\vec{r}$ can be expressed in spherical coordinates by the equation

$$\vec{r} = \rho\, \hat{\mathbf{e}}_\rho \tag{9.35}$$

In order to obtain the first and second derivatives of equation (9.35) with respect to time t it is necessary that one first differentiate the equations (9.34) with respect to time t. As an exercise show that the derivatives of the equations (9.34) can be represented

$$\begin{aligned}
\frac{d\,\hat{\mathbf{e}}_\rho}{dt} =& \frac{\partial\,\hat{\mathbf{e}}_\rho}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\rho}{\partial\phi}\frac{d\phi}{dt} = \frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi\\
\frac{d\,\hat{\mathbf{e}}_\theta}{dt} =& \frac{\partial\,\hat{\mathbf{e}}_\theta}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\theta}{\partial\phi}\frac{d\phi}{dt} = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\rho + \cos\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi\\
\frac{d\,\hat{\mathbf{e}}_\phi}{dt} =& \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial\phi}\frac{d\phi}{dt} = -\sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\rho - \cos\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\theta
\end{aligned} \tag{9.36}$$

One can then differentiate equation (9.35) and show the velocity in spherical coordinates has the form

[3] Note (ρ, θ, ϕ) gives a right-handed coordinate system, whereas the ordering (ρ, ϕ, θ) gives a left-handed coordinate system. Be aware that European textbooks, many times use left-handed coordinate systems.

$$
\begin{aligned}
\vec{v} = \frac{d\vec{r}}{dt} =& \rho \frac{d\,\hat{\mathbf{e}}_\rho}{dt} + \frac{d\rho}{dt}\,\hat{\mathbf{e}}_\rho \\
=& \rho\left(\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \sin\theta \frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi\right) + \frac{d\rho}{dt}\,\hat{\mathbf{e}}_\rho \\
=& \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\theta}\,\hat{\mathbf{e}}_\theta + \rho\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\phi
\end{aligned}
\tag{9.38}
$$

Here $v_\rho = \dot{\rho}$ is **the radial component of the velocity**, $v_\theta = \rho\dot{\theta}$ is **the polar component of velocity** and $v_\phi = \rho\dot{\phi}\sin\theta$ is **the azimuthal component of velocity**.

Differentiating the velocity with respect to time gives the acceleration vector

$$
\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \frac{d}{dt}\left(\dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\theta}\,\hat{\mathbf{e}}_\theta + \rho\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\phi\right) \\
=& \dot{\rho}\frac{d\,\hat{\mathbf{e}}_\rho}{dt} + \ddot{\rho}\,\hat{\mathbf{e}}_\rho + (\rho\dot{\theta})\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + \frac{d}{dt}(\rho\dot{\theta})\,\hat{\mathbf{e}}_\theta + (\rho\dot{\phi}\sin\theta)\frac{d\,\hat{\mathbf{e}}_\phi}{dt} + \frac{d}{dt}(\rho\dot{\phi}\sin\theta)\,\hat{\mathbf{e}}_\phi
\end{aligned}
\tag{9.38}
$$

Substitute the derivatives from equation (9.36) into the equation (9.38) and simplify the results to show the acceleration vector in spherical coordinates is represented

$$
\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& (\ddot{\rho} - \rho(\dot{\theta})^2 - \rho(\dot{\phi})^2\sin^2\theta)\,\hat{\mathbf{e}}_\rho \\
&+ (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} - \rho(\dot{\phi})^2\sin\theta\cos\theta)\,\hat{\mathbf{e}}_\theta \\
&+ (\rho\ddot{\phi}\sin\theta + 2\dot{\rho}\dot{\phi}\sin\theta + 2\rho\dot{\theta}\dot{\phi}\cos\theta)\,\hat{\mathbf{e}}_\phi
\end{aligned}
\tag{9.39}
$$

where $\dot{} = \frac{d}{dt}$ and $\ddot{} = \frac{d^2}{dt^2}$ is the dot notation for the first and second time derivatives.

In spherical coordinates an element of volume is given by $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

Introduction to Potential Theory

In this section some properties of irrotational and/or solenoidal vector fields are derived. Recall that a vector field $\vec{F} = \vec{F}(x,y,z)$ which is continuous and differentiable in a region R is called irrotational if $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \vec{0}$ at all points of R and it is called solenoidal if $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = 0$ at all points of R.

Some properties of **irrotational vector fields** are now considered. If a vector field $\vec{F}$ is an irrotational vector field, then $\nabla\times\vec{F} = \vec{0}$ and under these conditions the vector field $\vec{F}$ is **derivable from a scalar field** $\phi = \phi(x,y,z)$ and can be calculated by the operation[4]

$$
\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3 = \nabla\phi = \text{grad }\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{e}}_3
$$

Note that it you have a choice to solve for three quantities

$$
F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)
$$

[4] Sometimes $\vec{F} = -\text{grad }\phi$. The selection of either a + or - sign in front of the gradient depends upon how the vector field is being used.

or to solve for one quantity $\phi = \phi(x,y,z)$, then it should be obvious that it would be easier to solve for the one quantity ϕ and then calculate the components F_1, F_2, F_3 by calculating the gradient $\operatorname{grad}\phi$. The function ϕ, which defines the scalar field from which $\vec{F}$ is derivable is called the **potential function** associated with the irrotational vector field $\vec{F}$.

In a **simply-connected**[5] **region** R, let $\vec{F}$ define an irrotational vector field which is continuous with derivatives which are also continuous. The following statements are then equivalent.

1. $\nabla \times \vec{F} = \operatorname{curl}\vec{F} = \vec{0}$ and the vector field $\vec{F}$ is irrotational.
2. $\vec{F} = \nabla\phi = \operatorname{grad}\phi$ and $\vec{F}$ is derivable from a scalar potential function $\phi = \phi(x,y,z)$ by taking the gradient of this function.
3. The dot product $\vec{F}\cdot d\vec{r} = d\phi$, where $d\phi$ is an exact differential.
4. The line integral $W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ is the work done in moving through the vector field $\vec{F}$ between two points P_1 and P_2, and this work done is independent of the curve selected for connecting the points P_1 and P_2.
5. The line integral $\oint_C \vec{F}\cdot d\vec{r} = 0$, which implies that the work done in moving around a simple closed path is zero.

If a vector field $\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ is derivable from a scalar function $\phi = \phi(x,y,z)$ such that $\vec{F} = \operatorname{grad}\phi = \nabla\phi$ (sometimes $\vec{F}$ is defined as the negative of the gradient due to a particular application that requires a negative sign), then $\vec{F}$ is called **a conservative vector field**, and ϕ is called **the potential function** from which the field is derivable. Set $\vec{F} = \operatorname{grad}\phi$, and equate the like components of these vectors and obtain the scalar equations

$$F_1(x,y,z) = \frac{\partial\phi}{\partial x}, \qquad F_2(x,y,z) = \frac{\partial\phi}{\partial y}, \qquad F_3(x,y,z) = \frac{\partial\phi}{\partial z}.$$

These equations imply that

$$\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = d\phi \tag{9.40}$$

is an exact differential. Consequently the statement 2 implies the statement 3.

If $\vec{F} = \operatorname{grad}\phi$, then the line integral $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ is independent of the path of integration joining the points P_1 and P_2. To show this, let $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$

[5] A region R where a closed curve can by continuously shrunk to a point, without the curve leaving the region, is called a simply-connected region.

denote two points in the simply connected region R of the vector field $\vec{F}$. The work done can be expressed by performing a line integral of the equation (9.40) to obtain

$$\begin{aligned} W =& \int_{P_1(x_1,y_1,z_1)}^{P_2(x_2,y_2,z_2)} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\vec{r} = \int_{P_1}^{P_2} d\phi = \phi\big|_{P_1}^{P_2} \\ \text{or} \qquad W =& \int_{P_1(x_1,y_1,z_1)}^{P_2(x_2,y_2,z_2)} \vec{F}\cdot d\vec{r} = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1) \end{aligned} \tag{9.41}$$

which implies that the work done depends only on the end points P_1 and P_2 and is thus independent of the path which joins these two points. Thus statement 2 above implies statement 4. Note that this result does not necessarily hold for multiply connected regions.

The line integral given by equation (9.41) being independent of the path of integration which joins P_1 and P_2 can be expressed as

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}, \tag{9.42}$$

where the integral on the left is along a path C_1 and the integral on the right is along a path C_2, where both paths go from P_1 to P_2 as illustrated in figure 9-2.

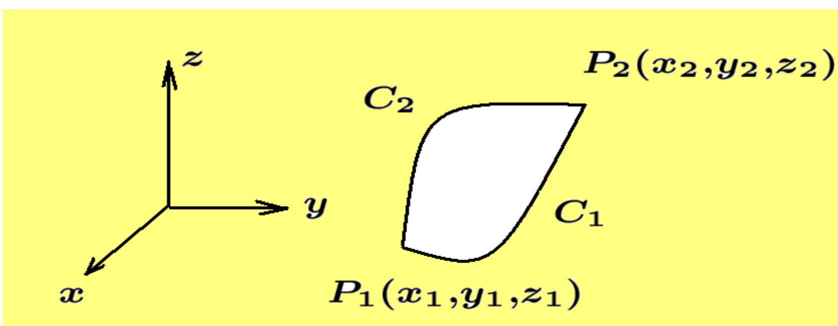


Figure 9-2. Paths of integration.

The integral (9.42) can be expressed in the form

$$\oint_C \vec{F}\cdot d\vec{r} = 0, \tag{9.43}$$

where the closed path C goes from P_1 to P_2 along the path C_1 and then from P_2 to P_1 along the path C_2. The curves C_1 and C_2 are arbitrary so that the work done in going around an arbitrary closed path is zero. Note that Stokes' theorem, with $\nabla \times \vec{F} = \vec{0}$, implies that the line integral around an arbitrary simple closed path is zero.

To show that statement 2 implies statement 1, let $\vec{F} = \text{grad}\ \phi = \nabla \phi$. In this case it is readily verified that

$$\text{curl}\ \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla \phi = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \vec{0}. \tag{9.44}$$

The relation (9.41) establishes that in a conservative vector field the line integral between any two points is independent of the path of integration. In this case one can write

$$\int_{P_0(x_0,y_0,z_0)}^{P(x,y,z)} \vec{F} \cdot d\vec{r} = \phi(x,y,z) - \phi(x_0,y_0,z_0), \tag{9.45}$$

and this line integral is independent of the path of integration which joins the two end points. The function ϕ can be evaluated from $\vec{F}$ by selecting the special path of integration which is the piecewise smooth curve constructed from straight line segments parallel to the coordinate axes. This special path of integration is illustrated in figure 9-3.

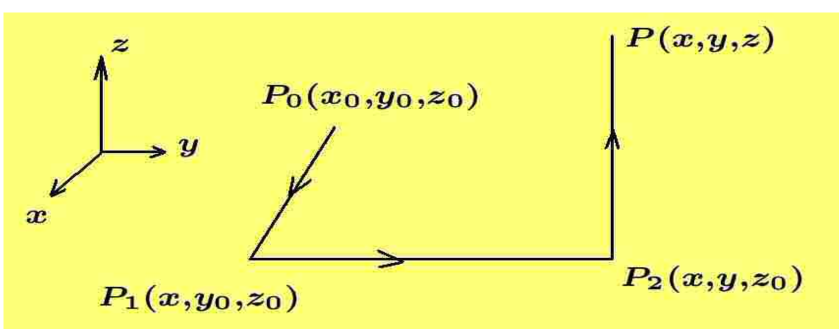


Figure 9-3. Straight line segments connecting end points of integration.

Along the sectionally continuous straight-line paths of integration illustrated in figure 9-3 the line integral (9.45) can be expressed in the component form as

$$\int_{P_0}^{P} \vec{F} \cdot d\vec{r} = \int_{P_0}^{P} F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz = \phi(x,y,z) - \phi(x_0,y_0,z_0).$$

The line integral (9.45) can be expressed as the sum of the line integrals along the straight line paths $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\overline{P_2P}$ illustrated in figure 9-3, where

Along $\overline{P_0P_1}$, one finds $\quad dy = dz = 0,\; y = y_0,\; z = z_0$

Along $\overline{P_1P_2}$, there exists the conditions $\quad dx = dz = 0,\; z = z_0,\; x$ held constant

Along $\overline{P_2P}$, use $\quad dx = dy = 0,\; x$ and y both held constant.

This produces the integral

$$
\begin{aligned}
\int_{P_0}^{P} \vec{F}\cdot d\vec{r} =& \int_{x_0}^{x} F_1(x,y_0,z_0)\,dx + \int_{y_0}^{y} F_2(x,y,z_0)\,dy + \int_{z_0}^{z} F_3(x,y,z)\,dz \\
=& \phi(x,y,z) - \phi(x_0,y_0,z_0).
\end{aligned}
\tag{9.46}
$$

If $\vec{F}$ is irrotational, then $\nabla\times\vec{F} = \vec{0}$ which implies that

$$
\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{9.47}
$$

This hypothesis leads to the result $\vec{F} = \text{grad}\;\phi = \nabla\phi$ or its equivalence

$$
\frac{\partial\phi}{\partial x} = F_1, \qquad \frac{\partial\phi}{\partial y} = F_2, \qquad \frac{\partial\phi}{\partial z} = F_3.
$$

To demonstrate this take the partial derivatives of both sides of the equation (9.46) and show

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} =& F_1(x,y_0,z_0) + \int_{y_0}^{y} \frac{\partial F_2(x,y,z_0)}{\partial x}\,dy + \int_{z_0}^{z} \frac{\partial F_3(x,y,z)}{\partial x}\,dz \\
\frac{\partial\phi}{\partial y} =& F_2(x,y,z_0) + \int_{z_0}^{z} \frac{\partial F_3(x,y,z)}{\partial y}\,dz \\
\frac{\partial\phi}{\partial z} =& F_3(x,y,z).
\end{aligned}
\tag{9.48}
$$

Use the results from equation (9.47), to simplify the first set of integrals and find

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} =& F_1(x,y_0,z_0) + \int_{y_0}^{y} \frac{\partial F_1(x,y,z_0)}{\partial y}\,dy + \int_{z_0}^{z} \frac{\partial F_1(x,y,z)}{\partial z}\,dz \\
=& F_1(x,y_0,z_0) + F_1(x,y,z_0)\Big|_{y_0}^{y} + F_1(x,y,z)\Big|_{z_0}^{z} \\
=& F_1(x,y_0,z_0) + F_1(x,y,z_0) - F_1(x,y_0,z_0) + F_1(x,y,z) - F_1(x,y,z_0) \\
=& F_1(x,y,z).
\end{aligned}
$$

Similarly, the relations of equations (9.47) can be used to simplify the second integral of equation (9.48) and one can show

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= F_2(x,y,z_0) + \int_{z_0}^{z} \frac{\partial F_2(x,y,z)}{\partial z}\,dz \\
&= F_2(x,y,z_0) + F_2(x,y,z)\Big|_{z_0}^{z} \\
&= F_2(x,y,z_0) + F_2(x,y,z) - F_2(x,y,z_0) \\
&= F_2(x,y,z).
\end{aligned}
$$

Thus, from the hypothesis that $\nabla \times \vec{F} = \vec{0}$, one finds that $\vec{F} = \text{grad}\,\phi = \nabla\phi$. Consequently, one can say that an irrotational vector field is derivable from a potential function ϕ.

Example 9-1. Show that

$$
\vec{F} = (y^2 + z)\,\hat{\mathbf{e}}_1 + (2xy + z^2)\,\hat{\mathbf{e}}_2 + (2yz + x)\,\hat{\mathbf{e}}_3
$$

is an irrotational vector field and find the corresponding potential function from which $\vec{F}$ is derivable.

Solution: It is readily verified that $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2+z) & (2xy+z^2) & (2yz+x) \end{vmatrix} = \vec{0}$ and hence $\vec{F}$ is irrotational. Two methods of finding the corresponding potential function are as follows.

Method 1 By line integral integration, where the path of integration consists of the straight-line segments illustrated in figure 9-3, one can show

$$
\begin{aligned}
\phi(x,y,z) - \phi(x_0,y_0,z_0) &= \int_{x_0}^{x} (y_0^2 + z_0)\,dx + \int_{y_0}^{y} (2xy + z_0^2)\,dy + \int_{z_0}^{z} (2yz + x)\,dz \\
&= (y_0^2 x + z_0 x)\Big|_{x_0}^{x} + (xy^2 + z_0^2 y)\Big|_{y_0}^{y} + (yz^2 + xz)\Big|_{z_0}^{z} \\
&= (xy^2 + yz^2 + xz) - (x_0 y_0^2 + y_0 z_0^2 + x_0 z_0)
\end{aligned}
$$

where in the second integral x is held constant and in the third integral both x and y are held constant. The resulting integral implies

$$
\phi(x,y,z) = xy^2 + yz^2 + xz.
$$

Method 2 The components of the relation $\vec{F} = \text{grad}\,\phi$ produce the scalar equations

$$
\frac{\partial \phi}{\partial x} = y^2 + z, \qquad \frac{\partial \phi}{\partial y} = 2xy + z^2, \qquad \frac{\partial \phi}{\partial z} = 2yz + x.
$$

Integrating the first equation with respect to x, the second equation with respect to y and the third equation with respect to z produces

$$
\phi = y^2 x + zx + f_1(y,z), \qquad \phi = y^2 x + z^2 y + f_2(x,z), \qquad \phi = xz + z^2 y + f_3(x,y)
$$

where f_1, f_2, f_3 are arbitrary functions which have been held constant during the partial differentiation process. Now add the first and second equations, add the first and third equation and add the second and third equations to obtain

$$\begin{aligned} 2\phi =& 2xy^2 + xz + yz^2 + f_1 + f_2 \\ 2\phi =& xy^2 + 2xz + yz^2 + f_1 + f_3 \\ 2\phi =& xy^2 + xz + 2yz^2 + f_2 + f_3 \end{aligned}$$

In order that these three equations be the same, require that

$$f_1 + f_2 = xz + yz^2, \qquad f_1 + f_3 = xy^2 + yz^2, \qquad f_2 + f_3 = xy^2 + xz \tag{9.49}$$

Now solve the equations (9.49) for f_1, f_2 and f_3 to show that

$$f_1 = z^2 y, \qquad f_2 = xz, \qquad f_3 = xy^2,$$

The potential function can then be expressed as

$$\phi = xy^2 + xz + yz^2.$$

Observe that any constant can be added to this potential function to obtain a more general result, since the derivative of a constant is zero.

■

Solenoidal Fields

A vector field which is **solenoidal** satisfies the property that **the divergence of the vector field is zero**. An alternate definition of a solenoidal vector field is obtained from Gauss' divergence theorem

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}. \tag{9.50}$$

If $\nabla \cdot \vec{F} = 0$, then $\iint_S \vec{F} \cdot d\vec{S} = 0$ which implies that the total flux, through the simple closed surface surrounding the volume V, is zero.

It has been shown that an irrotational vector field is derivable from a potential function. An analogous result holds for solenoidal vector fields. That is, if $\vec{F}$ is **a solenoidal vector field which is continuous and differentiable**, then there exists **a vector potential $\vec{V}$ such that $\vec{F} = \nabla \times \vec{V}$** $=$ curl $\vec{V}$. However, this vector potential **is not unique**, for if $\vec{V}$ is a vector satisfying $\vec{F} =$ curl $\vec{V}$, then the vector potential $\vec{V}^* = \vec{V} + \nabla\psi$, where

ψ is any scalar function, is also a vector satisfying $\vec{F} = \text{curl}\,\vec{V}^*$. This result is verified by using the distributive property of the curl since

$$\vec{F} = \text{curl}\,\vec{V}^* = \text{curl}\,(\vec{V} + \nabla\psi) = \text{curl}\,\vec{V} + \text{curl}\,(\nabla\psi) = \text{curl}\,\vec{V}. \tag{9.51}$$

together with the fact that $\text{curl}\,(\nabla\psi) = \text{curl grad}\,\psi = \vec{0}$.

Since the vector potential $\vec{V}$ is not uniquely determined, it is only necessary to exhibit **one vector potential** of $\vec{F}$. Toward this end make note of the fact that $\vec{F}$ can be expressed in the form

$$\begin{aligned}\vec{F} &= F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3 = \nabla\times\vec{V}\\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{\mathbf{e}}_3.\end{aligned} \tag{9.52}$$

Show that if the component $V_3 = 0$, then the components of $\vec{F}$ must satisfy the equations

$$F_1 = -\frac{\partial V_2}{\partial z}, \qquad F_2 = \frac{\partial V_1}{\partial z}, \qquad F_3 = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}. \tag{9.53}$$

An integration of the first two equations in (9.53) produces

$$\begin{aligned}V_1 &= \int_{z_0}^{z} F_2\,dz + f_1(x,y)\\ V_2 &= -\int_{z_0}^{z} F_1\,dz + f_2(x,y),\end{aligned} \tag{9.54}$$

where f_1, f_2 are arbitrary functions which are held constant during the partial differentiation processes used to calculate $\frac{\partial V_1}{\partial z}$ and $\frac{\partial V_2}{\partial z}$. The functions f_1 and f_2 must be selected in such a way that the last equation in (9.53) is also satisfied for all values of x, y, and z. Substitution of equations (9.54) into the last equation of (9.53) informs us that

$$\begin{aligned}\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} &= -\int_{z_0}^{z} \frac{\partial F_1}{\partial x}\,dz - \int_{z_0}^{z} \frac{\partial F_2}{\partial y}\,dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\\ &= -\int_{z_0}^{z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.\end{aligned} \tag{9.55}$$

Now by assumption, $\vec{F}$ is a solenoidal vector field and consequently

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0.$$

We therefore can write

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = -\frac{\partial F_3}{\partial z}$$

and thereby simplify the integral (9.55) to the form

$$\begin{aligned}\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} &= \int_{z_0}^{z} \frac{\partial F_3}{\partial z}\, dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \\ &= F_3(x,y,z) - F_3(x,y,z_0) + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.\end{aligned} \tag{9.56}$$

This equation tells us that if f_1, f_2 are selected to satisfy

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = F_3(x,y,z_0),$$

then the last equation of (9.53) is satisfied. One choice of f_1 and f_2 which satisfies the required condition is $f_1(x,y) = 0$ and

$$f_2(x,y) = \int_{x_0}^{x} F_3(x,y,z_0)\, dx.$$

For the special conditions assumed, the constructed vector potential $\vec{V}$ has the components

$$\begin{aligned}V_1 &= \int_{z_0}^{z} F_2(x,y,z)\, dz \\ V_2 &= -\int_{z_0}^{z} F_1(x,y,z)\, dz + \int_{x_0}^{x} F_3(x,y,z_0)\, dx \\ V_3 &= 0.\end{aligned} \tag{9.57}$$

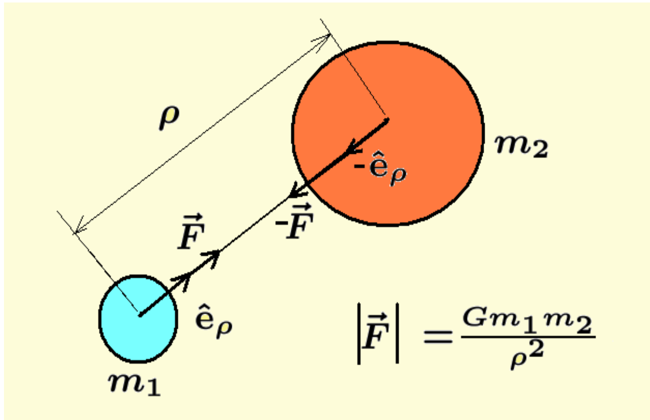
Other vector potential functions may be constructed by utilizing different assumptions on the components of $\vec{V}$ and performing similar integrations to those illustrated. Alternatively one could add the gradient of any arbitrary scalar function to the vector potential $\vec{V}$ and obtain other potential functions $\vec{V}^* = \vec{V} + \nabla\psi$.

Example 9-2.

By Newton's inverse square law, the force of attraction between two masses m_1 and m_2 is given by

$$\vec{F} = \frac{G m_1 m_2}{\rho^2}\, \hat{\mathbf{e}}_\rho \tag{9.58}$$

where $G = 6.6730\,(10)^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is called the gravitational constant, ρ is the distance between the center of mass of each body and $\hat{\mathbf{e}}_\rho$ is a unit vector pointing along the line connecting the center of mass of the two bodies. The force is an attractive force and so the direction of $\hat{\mathbf{e}}_\rho$ depends upon which center of mass is selected to sketch this force.



This force is derivable from the potential function

$$\phi = \frac{k}{\rho}, \qquad \text{where} \qquad \rho = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$$

is the distance between the two masses and $k = Gm_1m_2$ is a constant. In the above relation the coordinates of the center of mass of the bodies 1 and 2 are respectively $P_1(x_1, y_1, z_1)$ and $P_2(x, y, z)$. The quantity k is a constant, and $\hat{\mathbf{e}}_\rho$ is a unit vector with origin at P_1 and pointing toward P_2. The force of attraction of mass m_1 toward mass m_2 is calculated by the vector operation $\vec{F} = -\text{grad}\,\phi$. To calculate this force, first calculate the partial derivatives

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{-k}{\rho^2}\frac{(x-x_1)}{\rho} \\
\frac{\partial \phi}{\partial y} &= \frac{\partial \phi}{\partial \rho}\frac{\partial \rho}{\partial y} = \frac{-k}{\rho^2}\frac{(y-y_1)}{\rho} \\
\frac{\partial \phi}{\partial z} &= \frac{\partial \phi}{\partial \rho}\frac{\partial \rho}{\partial z} = \frac{-k}{\rho^2}\frac{(z-z_1)}{\rho}
\end{aligned}$$

and then the gradient is calculated and one obtains

$$\vec{F} = -\text{grad}\,\phi = \frac{k}{\rho^2}\left[\frac{(x-x_1)}{\rho}\,\hat{\mathbf{e}}_1 + \frac{(y-y_1)}{\rho}\,\hat{\mathbf{e}}_2 + \frac{(z-z_1)}{\rho}\,\hat{\mathbf{e}}_3\right] = \frac{k}{\rho^2}\,\hat{\mathbf{e}}_\rho$$

Here $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ is a position vector for the point P_2 and $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ is a position vector for the point P_1. The vector $\vec{r} - \vec{r}_1$ is a vector pointing from P_1 to P_2 and the vector $\dfrac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|} = \hat{\mathbf{e}}_\rho$ is a unit vector pointing from P_1 to P_2. Here **the vector field is called conservative since the force field is derivable from a potential function**. The potential function for Newton's law of gravitation is called **the gravitational potential**. By using the relation $\vec{F} = +\text{grad}\,\phi$ one obtains the force of attraction of mass m_2 toward mass m_1.

■

Example 9-3. Multiply both sides of Newton's second law $\vec{F} = m\vec{a} = m\dfrac{d^2\vec{r}}{dt^2}$ by $\dfrac{d\vec{r}}{dt}$ and then integrate from $P_0(x_0, y_0, z_0)$ to $P(x, y, z)$, to obtain

$$\begin{aligned}\int_{P_0}^{P} \vec{F}\cdot d\vec{r} = \int_{P_0}^{P} \vec{F}\cdot \frac{d\vec{r}}{dt}\,dt = &\int_{P_0}^{P} m\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}\,dt \\ =&\int_{P_0}^{P} \frac{m}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 dt = \int_{P_0}^{P}\frac{m}{2}\frac{d}{dt}\left(V^2\right)\,dt \\ =& \left.\frac{mV^2}{2}\right|_{P_0}^{P} = \left.\frac{mV^2}{2}\right|_{P} - \left.\frac{mV^2}{2}\right|_{P_0}\end{aligned} \tag{9.59}$$

which states that the work done in moving from P_0 to P_1 equals the change in kinetic energy. Now if $\vec{F}$ is derivable from a potential function ϕ such that $\vec{F} = -\nabla\phi$, then

$$\int_{P_0}^{P} \vec{F}\cdot d\vec{r} = \int_{P_0}^{P} -\nabla\phi\, d\vec{r} = \int_{P_0}^{P} -d\phi = -\phi\Big|_{P_0}^{P} = \phi(x_0, y_0, z_0) - \phi(x, y, z) \tag{9.60}$$

Equating the results from equations (9.59) and (9.60) and rearranging terms shows that

$$\phi(x_0, y_0, z_0) + \frac{m}{2}V^2(x_0, y_0, z_0) = \phi(x, y, z) + \frac{m}{2}V^2(x, y, z) \tag{9.61}$$

This equation states that **the sum of the kinetic energy and the potential energy has a constant value**. A result which is known as **the principal of conservation of energy**. As a result, any force fields which are derivable from potential functions are called **conservative force fields.**

■

Example 9-4. If $\vec{F}$ is a solenoidal vector field, then $\operatorname{div}\vec{F} = 0$ and one can write $\vec{F} = \operatorname{curl}\vec{V}$ for some vector potential $\vec{V}$. Consider an arbitrary region enclosed by a surface S and then select a simple closed curve C on this surface which divides the surface into two regions, call these regions S_1 and S_2. The flux through this volume is given by

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S}_1 + \iint_{S_2} \vec{F}\cdot d\vec{S}_2 = \iiint_V \operatorname{div}\vec{F}\, dV = 0$$

which implies that

$$\iint_{S_1} \vec{F}\cdot d\vec{S}_1 = -\iint_{S_2} \vec{F}\cdot d\vec{S}_2$$

By Stokes' theorem

$$\iint_{S_1} \vec{F}\cdot d\vec{S}_1 = \iint_{S_1} (\nabla \times \vec{V})\cdot d\vec{S}_1 = \oint_C \vec{V}\cdot d\vec{r}$$

and

$$\iint_{S_2} \vec{F}\cdot d\vec{S}_2 = \iint_{S_2} (\nabla \times \vec{V})\cdot d\vec{S}_2 = \oint_C -\vec{V}\cdot d\vec{r},$$

where the negative sign is due to the relative directions associated with the line integrals relative to the normals $\hat{\mathbf{e}}_{n_1}$ and $\hat{\mathbf{e}}_{n_2}$ to the respective surfaces S_1 and S_2. That is, Stokes theorem requires the line integral around the closed curve C be in the positive direction with respect to the normal on the surface. When the above integrals are added, the result is the net flux through an arbitrary closed surface is zero.

■

Two-dimensional Conservative Vector Fields

If corresponding to each point (x, y) in a region R of the plane $z = 0$, there corresponds a vector

$$\vec{F} = \vec{F}(x, y) = M(x, y)\,\hat{\mathbf{e}}_1 + N(x, y)\,\hat{\mathbf{e}}_2, \tag{9.62}$$

a vector field is said to exist in the region. Further, this field is said to be conservative if a scalar function of position $\phi(x, y)$ exists such that

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = M(x, y)\,\hat{\mathbf{e}}_1 + N(x, y)\,\hat{\mathbf{e}}_2 = \vec{F}. \tag{9.63}$$

The scalar function ϕ is called a potential function for the vector field $\vec{F}$. (Again, note that sometimes $\vec{F} = -\text{grad}\,\phi$ is more convenient to use.) The vector $\vec{F}$ is also referred to as an irrotational vector field and is derivable from the scalar potential function ϕ which satisfies

$$\frac{\partial \phi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N.$$

Differentiating these relations produces

$$\frac{\partial^2 \phi}{\partial x\,\partial y} = \frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial y\,\partial x} = \frac{\partial N}{\partial x} \tag{9.64}$$

so that a necessary condition that $\vec{F} = M\,\hat{\mathbf{e}}_1 + N\,\hat{\mathbf{e}}_2$ be a conservative field is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

An equivalent statement is that $\text{curl}\ \vec{F} = \vec{0}$.

Definition: (Equipotential curves) ***If $\vec{F} = \vec{F}(x,y)$ is a given conservative vector field with potential $\phi(x,y)$, then the family of curves $\phi(x,y) = c$ are called equipotential curves.***

By selecting a constant value c and graphing the equipotential curves

$$\phi = c, \quad \phi = c+1, \quad \phi = c+2, \ldots,$$

one can determine by the spacing of these curves **an estimate of the field intensity** in a given region.

The equipotential family of curves $\phi(x,y) = c$ satisfies the differential equation

$$\begin{aligned} d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0 \\ \text{or} \qquad M(x,y)\, dx + N(x,y)\, dy = 0. \end{aligned} \tag{9.65}$$

If this differential equation is exact, then it can be expressed in the form

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = M(x,y)\, dx + N(x,y)\, dy$$

and using the figure 8-10, its solution may be expressed as a line integral in either of the forms

$$\begin{aligned} \phi(x,y) = \int_{x_0}^{x} M(x,y_0)\, dx + \int_{y_0}^{y} N(x,y)\, dy = c \\ \text{or} \qquad \phi(x,y) = \int_{x_0}^{x} M(x,y)\, dx + \int_{y_0}^{y} N(x_0,y)\, dy = c \end{aligned} \tag{9.66}$$

depending upon the straight-line path of integration from P_0 to P. At any point (x,y), except singular points, where M and N are undefined, there is a tangent vector and a normal vector to the point (x,y) on the curve $\phi(x,y) = c$. The vector $\vec{F} = \vec{F}(x,y)$ lies in the direction of the normal to the curve since $\operatorname{grad}\phi = \vec{F}$ is a vector normal to $\phi(x,y) = c$ at the point (x,y)

Field Lines and Orthogonal Trajectories

Field lines are lines or curves such that **at each point on these curves the direction of the tangent vector to the curve is the same as the direction of the vector field at that point**. An **orthogonal trajectory** of a family of plane curves is a curve which intersects every member of the family at right angles. The set of all curves which intersect every member of $\phi(x,y)$ orthogonally are called **the orthogonal trajectories** of the family. Let $\psi(x,y)=c^*$ denote the family of orthogonal trajectories to the family of equipotential curves $\phi(x,y)=c$. The family of curves $\psi(x,y)=c^*$ describes the field lines associated with the vector field $\vec{F}$. That is, every orthogonal trajectory of the family of equipotential curves $\phi(x,y)=c$ has a tangent vector which lies along the same direction as the vector $\vec{F}$ (same direction as the normal to $\phi(x,y)=c$.) If $\vec{r}=\vec{r}(x,y)$ defines a field line, then $d\vec{r}$ points in the direction of vector field so that the slope of the field lines are in direct proportion to the components of $\vec{F}$. If $d\vec{r}=k\vec{F}$, where k is some constant, then one can write $dx\,\hat{\mathbf{e}}_1+dy\,\hat{\mathbf{e}}_2=k[M(x,y)\,\hat{\mathbf{e}}_1+N(x,y)\,\hat{\mathbf{e}}_2]$ or after equating like components

$$\frac{dx}{M}=\frac{dy}{N}=k \qquad \text{or} \qquad -N\,dx+M\,dy=0. \tag{9.67}$$

This gives the differential equation which defines the field lines. An equivalent statement is that $d\vec{r}\times\vec{F}=\vec{0}$, where $\vec{r}$ is the position vector to a point on the field line curve $\psi(x,y)=c$.

Example 9-5. Show that the vector field

$$\vec{F}=M(x,y)\,\hat{\mathbf{e}}_1+N(x,y)\,\hat{\mathbf{e}}_2=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2$$

is conservative and sketch **the equipotential curves** and **field lines** associated with this vector field.

Solution: The vector field is conservative, since $\text{curl}\,\vec{F}=\vec{0}$. If $\phi(x,y)=c$ is a family of equipotential curves, then $d\phi=\text{grad}\,\phi\cdot d\vec{r}=\vec{F}\cdot d\vec{r}=0$ produces the differential equation of the equipotential curves and one can write

$$d\phi=M\,dx+N\,dy=0 \qquad \text{or} \qquad d\phi=x\,dx+y\,dy=0.$$

By integrating this equation, there results the equipotential curves

$$\phi(x,y)=\frac{x^2}{2}+\frac{y^2}{2}=c,$$

which are circles centered at the origin.

If $\vec{r}$ is the position vector to a point on a field line, then $d\vec{r}$ is in the direction of the tangent to the field line and must have the same direction as the vector field $\vec{F}$ so that one can write $d\vec{r}=k\vec{F}$, where k is a proportionality constant. Equating like components one then finds the differential equation describing the field lines as

$$d\vec{r}=dx\,\hat{\mathbf{e}}_1+dy\,\hat{\mathbf{e}}_2=kF_1\,\hat{\mathbf{e}}_1+kF_2\,\hat{\mathbf{e}}_2 \quad \text{or} \quad \frac{dx}{F_1}=\frac{dy}{F_2}=k \quad \text{or} \quad \frac{dx}{x}=\frac{dy}{y}=k.$$

This differential equation is derived by requiring the direction of the vector field at an arbitrary point (x, y) have the same direction as the tangent to the field line curve which passes through the same point (x, y). An integration of the differential equation defining the field lines produces

$$\ln x = \ln y + \ln c$$

or the family curves defining the field lines is given by

$$\psi(x, y) = \frac{y}{x} = k,$$

where k is an arbitrary constant.

The equipotential curves and field lines associated with the given vector field $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ are illustrated in figure 9-4. In two dimensions, the vector fields are best visualized by sketches of the equipotential curves and field lines. In sketching the vector fields **be sure to distinguish the field lines from the equipotential curves by placing arrows at various points on the field lines**. These arrows indicate the **direction of the vector field** at various points.

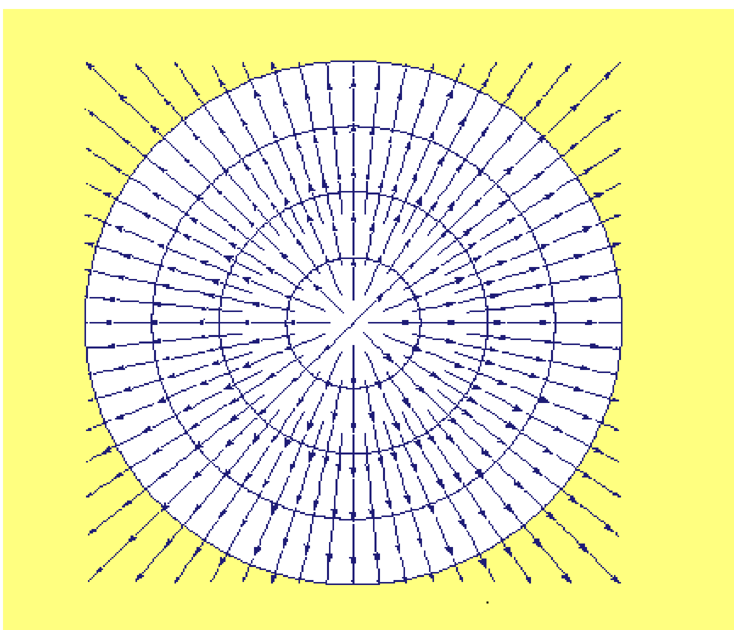


Figure 9-4. Equipotential curves and field lines for $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$

■

Vector Fields Irrotational and Solenoidal

If in addition to being conservative, the two-dimensional vector field given by $\vec{F} = M(x,y)\,\hat{\mathbf{e}}_1 + N(x,y)\,\hat{\mathbf{e}}_2$ is also solenoidal and

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0,$$

then

(i) The equipotential curves $\phi(x,y) = c$, c constant, are obtained from the exact differential equation

$$d\phi = \operatorname{grad}\phi \cdot d\vec{r} \quad \text{or} \quad d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0 \quad \text{or} \quad d\phi = M(x,y)\,dx + N(x,y)\,dy = 0$$

(ii) The family of field lines $\psi(x,y) = c^*$, c^* constant, are obtained from the differential equation

$$d\vec{r} = k\vec{F} \quad \text{or} \quad \frac{dx}{M} = \frac{dy}{N} = k \quad \text{or} \quad -N\,dx + M\,dy = 0$$

The solution of the differential equation defining the field lines is easily obtained since it also is an exact differential equation. The solution can be represented as a line integral in either of the forms

$$\begin{aligned} \psi(x,y) &= \int_{x_0}^{x} -N(x,y_0)\,dx + \int_{y_0}^{y} M(x,y)\,dy = c \\ \text{or} \quad \psi(x,y) &= \int_{x_0}^{x} -N(x,y)\,dx + \int_{y_0}^{y} M(x_0,y)\,dy = c, \end{aligned} \tag{9.68}$$

depending upon the choice of the path connecting the end points. These curves represent the field lines associated with the vector field $\vec{F}$, where

$$\frac{\partial \psi}{\partial x} = -N \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = M.$$

It follows that **if the vector field is both irrotational and solenoidal**, then the equipotential curves $\phi(x,y) = c$ and the field lines $\psi(x,y) = c^*$ are such that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{9.69}$$

These equations are called **the Cauchy–Riemann equations.** In vector form these equations may be expressed as

$$\text{grad } \phi = (\text{grad } \psi) \times \hat{\mathbf{e}}_3.$$

That is

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = \frac{\partial \psi}{\partial y}\,\hat{\mathbf{e}}_1 - \frac{\partial \psi}{\partial x}\,\hat{\mathbf{e}}_2. \tag{9.70}$$

Differentiate the Cauchy–Riemann equations (9.69) and show

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial x\,\partial y}. \tag{9.71}$$

Addition of these equations produces **the Laplace equation**

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{9.72}$$

Similarly, it can be shown that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y\,\partial x}$$

so that by addition

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{9.73}$$

Hence, both ϕ and ψ are solutions of Laplace's equation.

> **Definition: (Harmonic function)** ***Any function which is a solution of Laplace's equation $\nabla^2 \omega = 0$ and which has continuous second-order derivatives is called a harmonic function.***

Orthogonality of Equipotential Curves and Field Lines

To show that the equipotential curves $\phi = c_1$ and the field lines $\psi = c^*$ are orthogonal, consider the dot product of the vectors normal to these curves at a common point of intersection. These normal vectors are

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 \quad \text{and} \quad \text{grad } \psi = \frac{\partial \psi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \psi}{\partial y}\,\hat{\mathbf{e}}_2$$

and their dot product produces

$$\text{grad } \phi \cdot \text{grad } \psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}.$$

With the use of the Cauchy-Riemann equations it can be shown that this dot product is zero. Thus the vector $\text{grad}\,\psi$ is perpendicular to the vector $\text{grad}\,\phi$ and the equipotential curves and field lines are orthogonal.

In various branches of science and engineering, the quantities ϕ and ψ have many different physical interpretations. For example, in fluid dynamics, the velocity field is derivable from a velocity potential ϕ, and the field lines are called streamlines. In the study of heat flow, the heat flow vector is derivable from a potential ϕ which represents temperature and the equipotential curves $\phi =$ Constant are called isothermal curves (curves of constant temperature.) The field lines associated with this vector field are termed heat flow lines. In the study of electric and magnetic fields the potential functions from which these fields are derivable are termed, respectively, the electric and magnetic field potentials. The field lines associated with these potentials are called lines of electric and lines of magnetic force. Usually the harmonic functions ϕ and ψ are expressed as the real and imaginary parts of a function of a complex variable.

Laplace's Equation

For $\vec{F} = \vec{F}(x, y, z)$, a vector field which is both irrotational and solenoidal, then $\vec{F}$ satisfies

$$\text{curl } \vec{F} = \vec{0} \qquad \text{and} \qquad \text{div } \vec{F} = 0. \tag{9.74}$$

It has been shown that for these circumstances $\vec{F}$ is derivable from a scalar potential function Φ. In particular, $\vec{F} = \text{grad } \Phi = \nabla\, \Phi$. Hence, Φ must be a solution of Laplace's equation $\nabla^2\Phi = 0$. That is,

$$\text{div } \vec{F} = \text{div}\,(\text{grad}\,\Phi) = \nabla \cdot (\nabla\,\Phi) = \nabla^2\Phi = 0.$$

In expanded form the Laplace equation is expressed

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\,\Phi = 0$$

$$\text{or} \qquad \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

This partial differential equation has many physical applications associated with it and arises in many areas of science, physics and engineering. The Laplace equation can be expressed in different forms depending upon the coordinate system in which it is represented.

Three-dimensional Representations

In **a rectangular right-handed** (x, y, z) **system of coordinates**, the Laplace equation is expressed as

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad U = U(x, y, z) \tag{9.75}$$

In **a cylindrical** (r,θ,z) **coordinate system**, the Laplace equation takes the form

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad U = U(r,\theta,z) \tag{9.76}$$

and in **a spherical** (ρ,θ,ϕ) **coordinate system**, the Laplace equation is represented has the form

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{cot\,\theta}{\rho^2}\frac{\partial U}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} = 0, \qquad U = U(\rho,\theta,\phi) \tag{9.80}$$

Two-dimensional Representations

In a two-dimensional (x,y) coordinate system the Laplace equation is represented

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \qquad U = U(x,y) \tag{9.78}$$

In a polar (r,θ) coordinate system the Laplace equation becomes

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0, \qquad U = U(r,\theta) \tag{9.79}$$

In spherical coordinates, where there is symmetry with respect to the variable ϕ, the Laplacian is represented

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{cot\,\theta}{\rho^2}\frac{\partial U}{\partial \theta} = 0, \qquad U = U(\rho,\theta) \tag{9.80}$$

One-dimensional Representations

In one-dimension, the Laplace equation becomes

$$\begin{aligned} \frac{d^2 U}{dx^2} &= 0, && U = U(x) \quad \text{rectangular} \\ \frac{d^2 U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) &= 0, && U = U(r) \quad \text{polar} \\ \frac{d^2 U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dU}{d\rho}\right) &= 0, && U = U(\rho) \quad \text{spherical} \end{aligned} \tag{9.162}$$

Three-dimensional Conservative Vector Fields

Analogous to what has been done in studying two-dimensional vector fields, one can state that if a three-dimensional vector field

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

is derivable from a potential function $\phi(x,y,z)$ such that $\vec{F} = \text{grad}\ \phi$,, then the family of surfaces $\phi(x,y,z) = c$ are called equipotential surfaces. The differential equation satisfied by the equipotential surfaces is obtained by differentiating $\phi(x,y,z)=c$ to obtain the exact differential

$$\begin{aligned} d\phi = &\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z} = 0 \\ \text{or} \quad &F_1\,dx + F_2\,dy + F_3\,dz = \vec{F}\cdot d\vec{r} = 0 \end{aligned} \tag{9.82}$$

The solution of this differential equation may be obtained by line integration methods.

The field lines associated with the vector field $\vec{F}$ are those curves which are everywhere tangent to the field vectors. The direction of the field lines are in direct proportion to the components of $\vec{F}$ and thus the differential equation satisfied by the field lines is $d\vec{r} = k\vec{F}$, where k is a proportionality constant. Equating like components produces the equations

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)} = k \tag{9.83}$$

which is equivalent to the statement that $d\vec{r}\times\vec{F} = \vec{0}$ since $d\vec{r}$ has the same direction as $\vec{F}$. Another way of picturing this is to let $\vec{r}$ denote the position vector to a point (x,y,z) on a field line. The differential element $d\vec{r}$ will then be in the direction of the tangent to the field line which, by definition, is also in the same direction as $\vec{F}$ at the common point (x,y,z). Thus, $d\vec{r} = k\vec{F}$, where k is a proportionality constant. This equation can be written in the component form as

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3 = k\left[F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3\right].$$

Equating like components produces the differential relation (9.83). Geometrically, the field lines defined by equation (9.83) are orthogonal to the equipotential surfaces defined by equation (9.82). That is, $\text{grad}\,\phi$ is perpendicular to the tangent element $d\vec{r}$.

A solution of the differential system (9.83) consists of two independent relations or integrals of the form

$$\mu_1(x,y,z) = c_1 \qquad \text{and} \qquad \mu_2(x,y,z) = c_2,$$

which represents two families of surfaces having c_1 and c_2 as parameters. The field lines are the curves of intersection of these two family of surfaces, and these curves (field lines) are called a two-parameter family of curves, where the constants c_1 and

c_2 are the two parameters. Two methods for obtaining independent integrals of equations (9.83) are now presented.

Theory of Proportions

From the theory of proportions one can make use of the following result:

For constants $\alpha,\ \beta,\ \gamma$, not all zero, one can write

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} = \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{\alpha F_1 + \beta F_2 + \gamma F_3}.$$

In many instances one can choose appropriate values for the constants $\alpha,\ \beta,\ \gamma$ to construct equations which can be easily integrated to produce a family of surfaces representing a solution of the differential equations. Using the method of proportions, by trial and error, one tries to construct two independent family of solutions. Consider one surface from each family. These surfaces intersect and the curve of intersection defines a field line. The two family of surfaces intersect in a family of field lines.

Example 9-6. Find the field lines associated with the vector field

$$\vec{F} = \vec{F}(x,y,z) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

Solution The field lines are obtained from the differential equations

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \tag{9.84}$$

In this equation make note that an addition of the numerators and denominators of the first two fractions produces an exact differential and

$$\frac{dx+dy}{x+y} = \frac{dz}{z} = \frac{d(x+y)}{x+y}.$$

In this new equation the variables are separated and then an integration produces $\ln(x+y) = \ln z + \ln c_1$ where $\ln c_1$ is selected for the constant of integration in order to simplify the algebra. This result can be expressed as

$$\mu_1(x,y,z) = \frac{x+y}{z} = c_1$$

and represents one family of solution surfaces. Return to the equations (9.84) defining the field lines and observe that from the first two fractions one can write

$$\frac{dx}{y} = \frac{dy}{x}.$$

This is an equation where the variables can be separated and then an integration produces another independent family of surfaces

$$\mu_2(x,y,z) = \frac{x^2}{2} - \frac{y^2}{2} = c_2.$$

Hence the field lines are the intersection of the family of cylindrical surfaces, defined by hyperbola with rulings parallel to the z-axis, with the family of planes $x+y-c_1z=0$.

■

Method of Tangents.

Observe that if the field lines are defined as the intersection of two families of surfaces

$$\mu_1(x,y,z) = c_1 \qquad \text{and} \qquad \mu_2(x,y,z) = c_2,$$

then by differentiation one obtains

$$\frac{\partial \mu_1}{\partial x}\,dx + \frac{\partial \mu_1}{\partial y}\,dy + \frac{\partial \mu_1}{\partial z}\,dz = \text{grad}\,\mu_1 \cdot d\vec{r} = 0$$

In a similar fashion one can show

$$\frac{\partial \mu_2}{\partial x}\,dx + \frac{\partial \mu_2}{\partial y}\,dy + \frac{\partial \mu_2}{\partial z}\,dz = \text{grad}\,\mu_2 \cdot d\vec{r} = 0.$$

Note that at a point (x,y,z) on a curve of intersection of two surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$, the tangential direction $d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3$ is the same as the direction of the field line at that point. Therefore $d\vec{r}$ must be proportional to $\vec{F}$. At the common point (x,y,z) on both surfaces the gradient vectors $\text{grad}\,\mu_1$ and $\text{grad}\,\mu_2$ are perpendicular to the surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$ respectively. These vectors must therefore be perpendicular to the vector field $\vec{F}$ at this common point. Consequently, one can write $\text{grad}\,\mu_1 \cdot \vec{F} = 0$ or

$$\frac{\partial \mu_1}{\partial x}F_1 + \frac{\partial \mu_1}{\partial y}F_2 + \frac{\partial \mu_1}{\partial z}F_3 = 0$$

and similarly $\text{grad}\,\mu_2 \cdot \vec{F} = 0$ or

$$\frac{\partial \mu_2}{\partial x}F_1 + \frac{\partial \mu_2}{\partial y}F_2 + \frac{\partial \mu_2}{\partial z}F_3 = 0.$$

These equations are the basis for the method of tangents. One tries to find, by using a trial and error method, two vector functions $\vec{V} = \text{grad}\,\mu_1$ and $\vec{W} = \text{grad}\,\mu_2$ such that $\vec{V} \cdot \vec{F} = 0$ and $\vec{W} \cdot \vec{F} = 0$. Then the equations

$$\vec{V} \cdot d\vec{r} = \text{grad}\,\mu_1 \cdot d\vec{r} = 0 \quad \text{and} \quad \vec{W} \cdot d\vec{r} = \text{grad}\,\mu_2 \cdot d\vec{r} = 0$$

are exact differential equations which are easily integrated. From these integrations one finds two independent family of surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$.

Example 9-7. The field lines of the vector field $\vec{F} = \vec{F}(x,y,z) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ are determined from the differential system

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}.$$

By trial and error one can construct the functions

$$V_1 = \frac{1}{z} \qquad V_2 = \frac{1}{z} \qquad V_3 = -\frac{(x+y)}{z^2}$$

so that $\vec{V} \cdot \vec{F} = 0$. One can then construct the exact differential equation

$$\vec{V} \cdot d\vec{r} = \frac{1}{z}\,dx + \frac{1}{z}\,dy - \frac{(x+y)}{z^2}\,dz = \text{grad}\;\mu_1 \cdot d\vec{r} = d\mu_1 = 0$$

from which to determine

$$\mu_1 = \frac{x+y}{z} = c_1$$

Similarly, by using trial and error, one can show that the functions

$$W_1 = x \qquad W_2 = -y \qquad W_3 = 0$$

are such that $\vec{W} \cdot \vec{F} = 0$. This produces the exact differential equation

$$\vec{W} \cdot d\vec{r} = x\,dx - y\,dy = \text{grad}\;\mu_2 \cdot d\vec{r} = d\mu_2 = 0$$

which is easily integrated. One finds that

$$\mu_2 = \frac{x^2}{2} - \frac{y^2}{2} = c_2.$$

Note also that the trial and error method might produce all kinds of results. For example, let

$$P_1 = \frac{1}{2}z \qquad P_2 = -\frac{1}{2}z \qquad P_3 = \frac{1}{2}(x-y),$$

then one can show $\vec{P} \cdot \vec{F} = 0$. Consequently,

$$\vec{P} \cdot d\vec{r} = \frac{1}{2}z\,dx - \frac{1}{2}z\,dy + \frac{1}{2}(x-y)\,dz = \text{grad}\;\mu_3 \cdot d\vec{r} = d\mu_3 = 0 \tag{9.85}$$

is an exact differential which can be integrated. The equation (9.85) implies that

$$\frac{\partial \mu_3}{\partial x} = \frac{1}{2}z, \qquad \frac{\partial \mu_3}{\partial y} = -\frac{1}{2}z, \qquad \frac{\partial \mu_3}{\partial z} = \frac{1}{2}(x-y)$$

and an integration of each of these functions produces

$$\mu_3 = \frac{1}{2}xz + f(y,z)$$
$$\mu_3 = -\frac{1}{2}yz + g(x,z)$$
$$\mu_3 = \frac{1}{2}(x-y)z + h(x,y),$$

where $f(y,z), g(x,z), h(x,y)$ are treated as constants of integration during the integration of partial derivatives. One finds that by selecting

$$f = -\frac{1}{2}yz, \qquad g = \frac{1}{2}xz, \qquad h = 0$$

there results the family of surfaces

$$\mu_3 = \frac{1}{2}(x-y)z = c_3$$

At first glance it appears that $\mu_3 = c_3$ is a solution family different from $\mu_1 = c_1$ and $\mu_2 = c_2$. However, from $\mu_1 = c_1$ there results

$$z = \frac{x+y}{c_1}$$

which can be substituted into μ_3 to produce

$$\mu_3 = \frac{1}{2c_1}(x^2 - y^2).$$

Hence the solution $\mu_3 =$ Constant reduces to the solution $\mu_2 =$ Constant. When one of the surfaces $\mu_i = c_i$, $(i = 1 \text{ or } 2)$ has been obtained, this known solution may be used to determine the second surface. The known solution can be used to eliminate one of the variables in the differential system and thereby reduce it to a two-dimensional equation which theoretically can be solved. Three-dimensional field lines are in general more difficult to obtain and illustrate than their two-dimensional counterparts. ■

Solid Angles

A cone is described as a family of intersecting lines. A right circular cone is an example which is easily recognized, however, this is only one special kind of a cone.

A general cone is described by a line having one point fixed in space which is free to rotate. The figure 9-5 illustrates two cones which differ from a right circular cone.

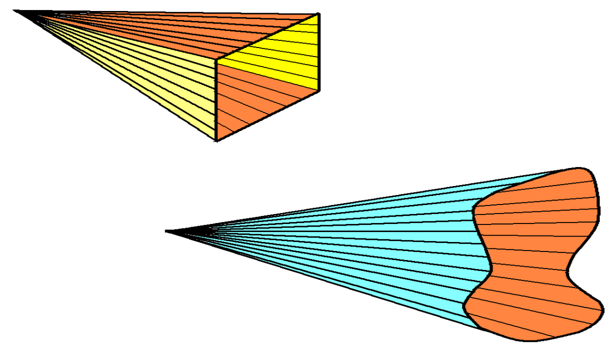
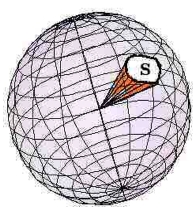


Figure 9-5. Cones generated by moving line about fixed point.



Consider a sphere of radius r and use the origin of the sphere to construct a cone which intersects the sphere and cuts out an area S on the surface as illustrated in the accompanying figure. The area S on the surface of the sphere of radius r will be proportional to r^2 since S is some fraction of the total surface area $4\pi r^2$. The ratio $\dfrac{S}{r^2}$ is therefore a dimensionless ratio and the quantity $\Omega = \dfrac{S}{r^2}$ is called **the solid angle subtended at the center of the sphere by the cone**. The **solid angle is a measure of how large an object appears to be when viewed from the origin of the sphere**. Solid angles are measured in units called steradians[6] (abbreviated sr) and by definition 1 steradian is the solid angle represented by the surface area of a sphere equal to the radius of the sphere squared. For example, if the area S in the accompanying figure equals r^2, then the solid angle subtended at the center of the sphere is said to be 1 steradian. The total solid angle about the center of the sphere being 4π steradians.

For a given oriented surface make the following constructions.

(i) A position vector $\vec{r}$ from the origin to the point on the oriented surface.

(ii) An element of surface area dS at the terminus of the vector $\vec{r}$.

(iii) A unit normal $\hat{\mathbf{e}}_n$ to the surface at the terminus of the vector $\vec{r}$.

(iv) A sphere of radius $r = |\vec{r}|$ centered at the origin 0.

[6] The solid angle is really dimensionless and sometimes the terminology of steradians is not used.

(v) A unit sphere about the origin.

(vi) Connect the points on the boundary of dS to the origin and form a cone which will intersect both the unit sphere and the sphere of radius r.

(vii) Use the dot product given by $\hat{\mathbf{e}}_n \cdot \vec{r} = r\cos\theta$ to find the angle θ between the unit normal to the surface and the position vector $\vec{r}$.

An example using the above constructions is illustrated in the figure 9-6. In the figure 9-6, the cone, constructed using the boundary of the element of surface area dS, intersects the sphere of radius r to produce an element of surface area $d\Omega$. The element of surface area $d\Omega$ can also be thought of as the projection of dS onto the sphere of radius r. This projection is given by $d\Omega = \cos\theta\, dS$ where θ is the angle between the normal to the surface and the position vector $\vec{r}$.

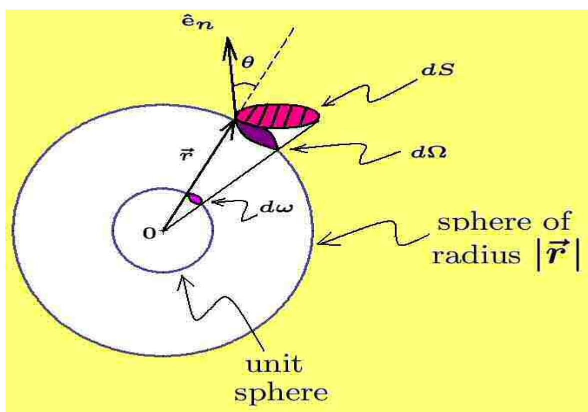


Figure 9-6. Solid angle as surface area on unit sphere.

The solid angle subtended at the origin does not depend upon the size of the sphere about the origin and so one can write

$$\frac{d\omega}{(1)^2} = \frac{d\Omega}{r^2} \qquad \Longrightarrow \qquad d\omega = \frac{d\Omega}{r^2}$$

Using the result $\hat{\mathbf{e}}_n \cdot \vec{r} = r\cos\theta$ or $\cos\theta = \dfrac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r}$ one obtains

$$d\Omega = \cos\theta\, dS = \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r}\, dS = \frac{\vec{r}\cdot d\vec{S}}{r}$$

where $d\vec{S} = \hat{\mathbf{e}}_n\, dS$ is a vector element of area.

Consider the special surface integral

$$\iint_S d\omega = \iint_S \frac{d\Omega}{r^2} = \iint_S \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r^3}\, dS = \iint_S \frac{\vec{r}\cdot d\vec{S}}{r^3}$$

where the surface S encloses a bounded, closed, simply connected region. Surface integrals of this type represent the total sum of the solid angles subtended by the element dS, summed over the surface S. For the solid angle summed about a point $0'$ outside the surface, the resulting sum of the solid angles is zero. This is because for each positive sum $+d\omega$ there is a corresponding negative sum $-d\omega$, and these add to zero in pairs. If the solid angle is summed about a point 0 inside the surface, the resulting sum is not zero. Here the sum of the areas $d\omega$ on the unit sphere, subtended by the elements dS, do not add together in pairs to produce zero but instead give the total surface area of the unit sphere which is 4π steradians. From these discussions one obtains the following

$$\iint_S d\omega = \iint_S \frac{\vec{r}\cdot d\vec{S}}{r^3} = \begin{cases} 0 & \text{if origin is outside closed surface} \\ 4\pi & \text{if origin is inside closed surface} \end{cases} \tag{9.86}$$

This result is **utilized in the study of inverse square law potentials and is known as Gauss' theorem.**

Example 9-8. Find the solid angle subtended by a right circular cone of radius r and height h.

Solution Let $\tan\theta_0 = \frac{r}{h}$ and construct a sphere of radius r which intersects the circular cone to form a spherical cap. On this spherical cap construct a ring-shaped element of area where the thickness of the ring is $ds = r\,d\theta$ and this element of thickness is rotated about the cone axis to form a ring as illustrated in the figure 9-7.

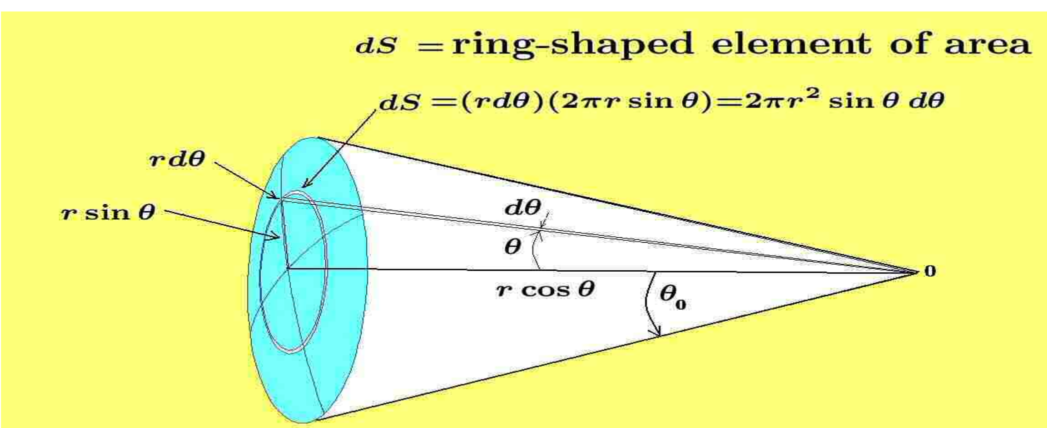


Figure 9-7. Area of spherical cap using ring-shaped element of area.

This produces an element of area

$$dS = (r d\theta)(2\pi r \sin\theta) = 2\pi r^2 \sin\theta\, d\theta \qquad \text{for} \qquad 0 \le \theta \le \theta_0$$

The total surface area of the spherical cap is obtained by a summation of the ring elements to produce the integral

$$S = \int_0^{\theta_0} 2\pi r^2 \sin\theta\, d\theta = 2\pi r^2 \left[-\cos\theta\right]_0^{\theta_0} = 2\pi r^2 (1 - \cos\theta_0)$$

The solid angle subtended by this right circular cone is therefore

$$\Omega = \frac{S}{r^2} = 2\pi(1 - \cos\theta_0)$$

■

Potential Theory

Potential theory is concerned with the solutions of Laplace's equation $\nabla^2 u = 0$, which satisfy **prescribed boundary conditions**. Two important problems of potential theory are **the Dirichlet problem** and **the Neumann problem**.

The Dirichlet problem deals with finding a solution U of Laplace's equation throughout a region R such that U takes on certain **pre assigned values on the boundary of the region** R.

The Neumann problem is concerned with obtaining a solution of Laplace's equation in a region R such that **on the boundary of** R**, the normal derivative**

$$\frac{\partial U}{\partial n} = \text{grad}\, U \cdot \hat{\mathbf{e}}_n$$

has prescribed values. Here $\hat{\mathbf{e}}_n$ is the unit outward normal to the boundary of the region R.

In obtaining a solution to a Dirichlet or Neumann problem in an infinite region there is the additional requirement that U satisfy certain conditions far from the origin.

Velocity Fields and Fluids

Let $\vec{V}$ denote the velocity field of a fluid in motion and let $\rho(x, y, z, t)$ denote the density of this fluid. Place within the fluid an arbitrary closed surface and consider an element of surface area dS on this surface. Let the mass of fluid flowing in a normal direction across this element of surface, in a time interval Δt, be denoted by

ΔM. It is assumed that the velocity is the same at all points over the tiny element of surface area. In a time interval Δt, the amount of fluid which crosses the element dS is given by $\Delta M = \rho \vec{V} \cdot \hat{\mathbf{e}}_n dS \Delta t$. The total mass of fluid flowing out of the volume $\mathcal{V}$ bounded by the surface S is given by

$$\Delta M = \Delta t \iint\limits_{S} \rho \vec{V} \cdot d\vec{S} = \Delta t \iiint\limits_{\mathcal{V}} \operatorname{div}(\rho \vec{V})\, dV.$$

Also the total mass of the fluid enclosed within the volume $\mathcal{V}$ bounded by S can be represented as the integral

$$M = \iiint\limits_{\mathcal{V}} \rho dV. \tag{9.87}$$

The rate of change of the mass with time is

$$\frac{\partial M}{\partial t} = \iiint\limits_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV. \tag{9.88}$$

Hence, in a time interval Δt, the amount of fluid in the volume $\mathcal{V}$ diminishes by the amount

$$\Delta M = -\Delta t \iiint\limits_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV. \tag{9.89}$$

The amount of fluid flowing out of the arbitrary volume is equated to the amount of fluid decreasing within the volume to obtain

$$\begin{aligned} \Delta t \iiint\limits_{\mathcal{V}} \operatorname{div}(\rho \vec{V})\, dV &= -\Delta t \iiint\limits_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV \\ \text{or} \qquad \iiint\limits_{\mathcal{V}} \left[\operatorname{div}(\rho \vec{V}) + \frac{\partial \rho}{\partial t} \right] dV &= 0. \end{aligned} \tag{9.90}$$

For an arbitrary volume $\mathcal{V}$ within the fluid, the relation (2.42) must hold and consequently

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) = 0. \tag{9.91}$$

This equation is called **the continuity equation of hydrodynamics** which can also be expressed in the form

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{V} + \rho \nabla \vec{V} = 0. \tag{9.92}$$

The first two terms on the left-hand side of this last equation represents the time rate of change of the density ρ, that is,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{V}. \tag{9.93}$$

If $\frac{d\rho}{dt} = 0$, then the fluid is an incompressible fluid, and the velocity field is solenoidal.

If the fluid flow is also irrotational, then $\vec{V}$ is derivable from a potential function Φ called the velocity potential of the fluid flow. The potential function must be a solution of Laplace's equation. The field lines associated with the velocity field $\vec{V}$ produce a family of curves which are termed streamlines.

Heat Conduction

In the basic equations describing heat conduction in materials, the following assumptions and terminology are employed:

1. Let $T = T(x, y, z, t)$ denote the temperature (°C) at a point (x, y, z) within the material at time t.
2. Heat flow within the material is denoted by the vector $\vec{q}$ having units $[\vec{q}] = \frac{\text{J}}{\text{cm}^2\cdot\text{sec}}$
3. Heat flows from regions of higher temperature to regions of lower temperature and the direction of heat flow is in the direction of the greatest rate of change of the temperature. Expressing this as a mathematics statement, we write

$$\vec{q} = -k \,\text{grad}\, T, \tag{9.94}$$

   where k is a proportionality constant having units of $\frac{\text{J}}{\text{cm}\cdot\text{sec}\cdot{}^\circ\text{C}}$ and is called the thermal conductivity of the material. Since the gradient of temperature points in the direction of increasing temperature, the negative sign in the relation (9.94) indicates that heat is flowing in the direction of decreasing temperature.
4. The symbol c is used to denote the specific heat of the material which is a measure of the heat capacity per unit mass of material. The specific heat c is measured in units $\frac{\text{J}}{\text{g}\,{}^\circ\text{C}}$.
5. The symbol ρ is used to denote the density of the material $[\frac{\text{g}}{\text{cm}^3}]$.
6. The total amount of heat in an arbitrary volume $\mathcal{V}$ bounded by a closed surface S is given by

$$H = \iiint_{\mathcal{V}} c\rho T \, dV, \tag{9.95}$$

   where H is in joules.

If an imaginary closed surface S enclosing a volume $\mathcal{V}$ is placed within a body in which heat is flowing, then the heat flux across this surface is given by the integral

$$\iint_S \vec{q}\, d\vec{S} = \iint_S \vec{q} \cdot \hat{\mathbf{e}}_n \, dS \tag{9.96}$$

which by the divergence theorem can be expressed in the form

$$\iint_S \vec{q} \cdot d\vec{S} = \iiint_{\mathcal{V}} \text{div } \vec{q}\, dV. \tag{9.97}$$

Substituting the heat flow given by equation (2.46) into equation (2.49) produces the relation

$$\iint_S \vec{q} \cdot d\vec{S} = -\iiint_{\mathcal{V}} k \,\text{div }(\text{grad } T)\, dV, \tag{9.98}$$

which depicts the total amount of heat leaving the arbitrary volume $\mathcal{V}$ enclosed by S. From equation (2.47), one can calculate the rate of change of decreasing heat within the volume. Such a change is given by

$$-\frac{\partial H}{\partial t} = -\iint_{\mathcal{V}} c\rho \frac{\partial T}{\partial t}\, dV \tag{9.99}$$

and must equal the change given by the flux integral (2.50). Equating these quantities produces the relation

$$\iiint_{\mathcal{V}} \left[c\rho \frac{\partial T}{\partial t} - k \,\text{div }(\text{grad } T)\right] dV = 0 \tag{9.100}$$

which must hold for any arbitrary volume $\mathcal{V}$ within the material. Since the volume is arbitrary, it is required that the integrand be identically zero and write

$$c\rho \frac{\partial T}{\partial t} - k \,\text{div }(\text{grad } T) = 0 \implies \frac{c\rho}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \implies \frac{c\rho}{k}\frac{\partial T}{\partial T} = \nabla^2 T \tag{9.101}$$

This result is known as the heat equation.

For steady-state temperature distributions, write $\frac{\partial T}{\partial t} = 0$, and consequently equation (9.101) reduces to Laplace's equation.

In the study of heat flow the level curves $T(x,y,z) = c$ are called isothermal surfaces, and the field lines associated with the heat flow $\vec{q}$ within the material are called heat flow lines.

Two-Body Problem

Newton's law of gravitation states that two masses m and M, are attracted toward each other with a force of magnitude $\frac{GmM}{r^2}$, where G is a constant and r is the distant between the masses. Let M represent the mass of the Sun and m represent the mass of a planet and assume that the motion of one mass with respect to the other mass takes place in a plane. Construct a set of x, y axes with origin located at the center of mass of M. Further, let $\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$ denote a unit vector at the origin of our coordinate system and pointing in the direction of the mass m. One can then express the vector force of attraction of mass M on mass m by the equation

$$\vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r \tag{9.102}$$

To find the equation of motion of mass m with respect to mass M, use Newton's second law. Let $\vec{r} = r\,\hat{\mathbf{e}}_r$ denote the position vector of mass m with respect to our origin. The equation of motion of mass m is determined from Newton's second law and is

$$\vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{V}}{dt} \tag{9.103}$$

From this equation it can be shown that the motion of mass m can be described as a conic section. In order to accomplish this, let us review some facts about conic sections.

Recall that a conic section was defined as a locus of points $P(x, y)$ such that the distance of P from a fixed point (or points), called a focus, is proportional to the distance of P from a fixed line, called the directrix. The constant of proportionality is called the eccentricity and is denoted by the symbol ϵ. If $\epsilon = 1$, a parabola results; if $0 < \epsilon < 1$, an ellipse results; if $\epsilon > 1$, a hyperbola results; and if $\epsilon = 0$, the conic section is a circle.

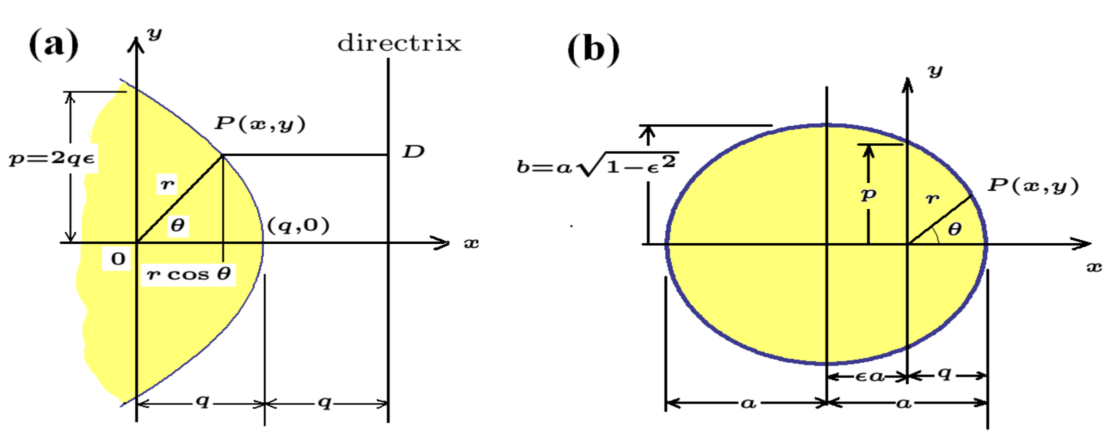


Figure 9-8. Parabolic and elliptic conic sections.

With reference to figure 9-8, a conic section is defined in terms of the ratio $\frac{\overline{OP}}{\overline{PD}} = \epsilon$, where $\overline{OP} = r$, and $\overline{PD} = 2q - r\cos\theta$. From this ratio, solve for the radius r and obtain the representation

$$r = \frac{p}{1 + \epsilon\cos\theta}, \tag{9.104}$$

where $p = 2q\epsilon$. Equation (9.104) is the equation of a conic section. The following terminology is applied to the variables and parameters in this equation:

1. The angle θ is called the true anomaly associated with the orbit.
2. The symbol a is introduced to denote the semi-major axis of an elliptical orbit. The symbol a can be shown to be related to r, p and ϵ.
3. The quantity p is called the semiparameter of the conic section and is illustrated in figure 9-8. Note that when θ has the value $\pi/2$, then $r = p$.

An important relation connecting the parameters p, a and ϵ is obtained from equation (9.104) by setting θ equal to zero. This gives

$$r = \frac{p}{1+\epsilon} = q = a(1-\epsilon) \qquad \text{which implies} \qquad p = a(1-\epsilon^2). \tag{9.105}$$

In order to demonstrate that the motion of mass m with respect to mass M is a conic section, show that the magnitude r of the position vector $\vec{r}$ satisfies an equation having the exact same form as equation (9.104).

Kepler's Laws

Johannes Kepler[7], an astronomer and mathematician, discovered three laws concerning the motion of the planets. **He discovered these laws from experimental data without the aid of calculus or vector analysis**. Newton, using calculus, verified these laws with the model for the inverse square law of attraction. These three laws are now derived.

To derive Kepler's three laws one can make use of the following vector identities:

$$\vec{r} \times \hat{\mathbf{e}}_r = r\,\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_r = \vec{0} \tag{9.106}$$

$$\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} \tag{9.107}$$

$$\hat{\mathbf{e}}_r \cdot \frac{d\,\hat{\mathbf{e}}_r}{dt} = 0 \tag{9.108}$$

$$\hat{\mathbf{e}}_r \times (\,\hat{\mathbf{e}}_r \times \frac{d\,\hat{\mathbf{e}}_r}{dt}) = -\frac{d\,\hat{\mathbf{e}}_r}{dt} \tag{9.109}$$

[7] Johannes Kepler (1571-1630), German astronomer and mathematician.

Note that the Newton law of gravitation implies that the derivative given by equation (9.107) is zero. That is, if

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = \vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r$$

$$\text{then} \quad \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = -\frac{GM}{r^2}\vec{r}\times\hat{\mathbf{e}}_r = \vec{0}. \tag{9.110}$$

An integration of this equation produces the result

$$\vec{r}\times\frac{d\vec{r}}{dt} = \vec{h} = \text{Constant} \tag{9.111}$$

Recall that the vector $\vec{H} = \vec{r}\times m\vec{v}$ is defined as **the angular momentum.** The quantity $\vec{h} = \frac{1}{m}\vec{H} = \vec{r}\times\frac{d\vec{r}}{dt}$ appearing in equation (9.111) is called **the angular momentum per unit mass.** Equation (9.111) tells us that the angular momentum is a constant for the two-body system under consideration. Since $\vec{h}$ is a constant vector, it can be verified that

$$\begin{aligned}\frac{d}{dt}\left(\vec{v}\times\vec{h}\right) &= \frac{d\vec{v}}{dt}\times\vec{h} = -\frac{GM}{r^2}\,\hat{\mathbf{e}}_r\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\\ &= -\frac{GM}{r^2}\,\hat{\mathbf{e}}_r\times\left[r\,\hat{\mathbf{e}}_r\times\left(r\frac{d\,\hat{\mathbf{e}}_r}{dt}+\frac{dr}{dt}\,\hat{\mathbf{e}}_r\right)\right]\\ &= -GM\,\hat{\mathbf{e}}_r\times\left(\hat{\mathbf{e}}_r\times\frac{d\,\hat{\mathbf{e}}_r}{dt}\right)\\ &= GM\frac{d\,\hat{\mathbf{e}}_r}{dt}.\end{aligned} \tag{9.112}$$

Note that the result (9.112) was obtained by making use of the equations (9.106) and (9.109). An integration of the result (9.112) gives us the relation

$$\vec{v}\times\vec{h} = GM\,\hat{\mathbf{e}}_r + \vec{C}, \tag{9.113}$$

where $\vec{C}$ is a constant vector of integration. Using the triple scalar product formula it is readily verified that

$$\vec{r}\cdot\left(\vec{v}\times\vec{h}\right) = \vec{h}\cdot\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = h^2 = \vec{r}\cdot\left(\vec{v}\times\vec{h}\right) = GM\vec{r}\cdot\,\hat{\mathbf{e}}_r + \vec{r}\cdot\vec{C}$$

or

$$h^2 = GMr + Cr\cos\theta, \tag{9.114}$$

where θ is the angle between the vectors $\vec{C}$ and $\vec{r}$. In the equation (9.114) one can solve for r and find

$$r = \frac{p}{1+\epsilon\cos\theta}, \tag{9.115}$$

where $p = h^2/GM$ and $\epsilon = C/GM$. This result is known as **Kepler's first law** and implies that all the planets of the solar system describe elliptical paths with the sun at one focus.

Kepler's second law states that the position vector $\vec{r}$ sweeps out equal areas in equal time intervals. Consider the area swept out by the position vector of a planet during a time interval Δt. This element of area, in polar coordinates, is written as

$$dA = \frac{1}{2} r^2 d\theta$$

and therefore the rate of change of this area with respect to time is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

It has been demonstrated that the angular momentum per unit mass $\vec{h} = \vec{r} \times \vec{v}$ is a constant. For $\vec{r} = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2$, the angular momentum has components which can be calculated from the determinant

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dot{\theta} + \dot{r}\cos\theta & r\cos\theta\dot{\theta} + \dot{r}\sin\theta & 0 \end{vmatrix}$$

By expanding the above determinant and simplifying one can verify that

$$\vec{h} = r^2 \frac{d\theta}{dt}\,\hat{\mathbf{e}}_3 = h\,\hat{\mathbf{e}}_3 = \text{Constant} \tag{9.116}$$

which in turn implies

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{is a constant.} \tag{9.117}$$

This result is known as Kepler's second law. Analysis of this second law informs us that the position vector sweeps out equal areas during equal time intervals.

The time it takes for mass m to complete one orbit about mass M is called the period of the motion. Denote this period by the Greek letter τ. Note that equation (9.117) tells us that when r^2 is small $\frac{d\theta}{dt}$ becomes large and, conversely, when $\frac{d\theta}{dt}$ is small r^2 becomes large. The resulting motion is for planets to move faster when they are closer to the Sun and slower when they are farther away. Express equation (9.117) in the form $dA = \frac{1}{2} h\, dt$ and integrate the result from $t = 0$ to $t = \tau$, to show

$$A = \frac{h}{2}\tau, \tag{9.118}$$

where A is the area of the ellipse and τ is the period of one orbit. The area of an ellipse is given by the formula $A = \pi ab$, where a is the semi-major axis and $b = a\sqrt{1-\epsilon^2}$ is the semi-minor axis. Equation (9.118) can therefore be expressed in the form

$$A = \pi a^2 \sqrt{1-\epsilon^2} = \frac{h}{2}\tau$$

from which the period of the orbit is

$$\tau = \frac{2\pi a^2}{h}\sqrt{1-\epsilon^2}.$$

With the substitutions

$$1-\epsilon^2 = \frac{p}{a} \qquad \text{and} \qquad p = \frac{h^2}{GM},$$

the period of the orbit can be expressed

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{GM}} \qquad \text{or} \qquad \tau^2 = \frac{4\pi^2 a^3}{GM}. \tag{9.119}$$

This result is known as **Kepler's third law** and depicts the fact that the square of the period of one revolution is proportional to the cube of the semi-major axis of the elliptical orbit.

Planets, comets, and asteroids have either elliptic, parabolic or hyperbolic orbits about the sun.

Vector Differential Equations

A homogeneous vector differential equation, such as

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{0} \tag{9.135}$$

where α and β are scalar constants is solved by first solving the homogeneous scalar differential equation

$$\frac{d^2 y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = 0 \tag{9.137}$$

The solution of the homogeneous differential equation is called the complementary solution and is expressed using the notation y_c. By assuming an exponential solution $y = e^{\lambda t}$ and substituting it into the homogeneous equation one obtains the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0 \tag{9.136}$$

There are three cases to consider.

Case 1 The roots of characteristic equation (9.136) are real and unique. If r_1, r_2 are these roots, then the scalar homogeneous differential equation (9.137) has the fundamental set of solutions $\{e^{r_1 t}, e^{r_2 t}\}$ and the general of equation (9.137)is $y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where c_1 and c_2 are arbitrary scalar constants.

Case 2 The roots of the characteristic equation (9.136) are real and equal, say $r_1 = r_2$. In this case the fundamental set of solutions is given by $\{e^{r_1 t}, te^{r_1 t}\}$ and the general solution of equation (9.137) is $y_c = c_1 e^{r_1 t} + c_2\, te^{r_1 t}$ where c_1 and c_2 are arbitrary scalar constants.

Case 3 The roots of the characteristic equation (9.136) are complex roots, say $r_1 = a + ib$ and $r_2 = a - ib$. In this case the fundamental set of solutions can be represented in the form $\{e^{(a+ib)t},\ e^{(a-ib)t}\}$ or one can make use of Euler's equation $e^{ibt} = \cos bt + i \sin bt$ and take appropriate linear combination of solutions to write the fundamental set of solutions in the form $\{e^{at}\cos bt,\ e^{at}\sin bt\}$. The general solution to the scalar homogeneous equation can then be expressed in either of the forms

$$\begin{aligned} y =& c_1 e^{(a+ib)t} + c_2 e^{(a-ib)t} \\ \text{or} \qquad y_c =& e^{at}\left(c_1 \cos bt + c_2 \sin bt\right) \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

If $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the homogeneous scalar differential equation (9.137), then

$$\vec{y}_c = \vec{c}_1\, y_1(t) + \vec{c}_2\, y_2(t) \tag{9.138}$$

where $\vec{c}_1$ and $\vec{c}_2$ are arbitrary vector constants, is the representation of the general solution to the vector differential equation (9.135). Substitute equation (9.138) into equation (9.135) and show there results the vector equation

$$\vec{c}_1\left(\frac{d^2 y_1}{dt^2} + \alpha\frac{dy_1}{dt} + \beta y_1\right) + \vec{c}_2\left(\frac{d^2 y_2}{dt^2} + \alpha\frac{dy_2}{dt} + \beta y_2\right) = \vec{0} \tag{9.139}$$

Observe that if $\vec{c}_1$ and $\vec{c}_2$ are arbitrary independent vectors, then in order for equation (9.139) to be satisfied, the scalar components of the arbitrary vectors $\vec{c}_1$ and $\vec{c}_2$ must equal zero.

The solution of the nonhomogeneous vector differential equation

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{F}(t) \tag{9.125}$$

where α and β are scalar constants is obtained by first solving the homogeneous equation

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{0}$$

given by equation (9.138) and then finding any particular solution $\vec{y}_p = \vec{y}_p(t)$ which satisfies

$$\frac{d^2\vec{y}_p}{dt^2} + \alpha\frac{d\vec{y}_p}{dt} + \beta\vec{y}_p = \vec{F}(t)$$

The general solution to the vector differential equation (9.125) is then given by

$$\vec{y} = \vec{y}_c + \vec{y}_p = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t) + \vec{y}_p(t) \tag{9.126}$$

Example 9-9. Solve the vector differential equation

$$\frac{d^2\vec{y}}{dt^2} + 2\alpha\frac{d\vec{y}}{dt} + \beta^2\vec{y} = \sin 3t\ \hat{\mathbf{e}}_3 \tag{9.133}$$

Solution First solve the homogeneous vector differential equation

$$\vec{y}'' + 2\alpha\vec{y}' + \beta^2\vec{y} = \vec{0} \tag{9.128}$$

If $\vec{y} = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t)$ is the general solution of equation (9.128), then $y_1(t)$ and $y_2(t)$ must be independent solutions of the scalar differential equation

$$\frac{d^2 y}{dt^2} + 2\alpha\frac{dy}{dt} + \beta^2 y = 0 \tag{9.129}$$

This is an equation with constant coefficients. The general procedure to solve a differential equation with constant coefficients is to assume an exponential solution $y = e^{\lambda t}$. Substituting the assumed exponential solution into the differential equation (9.129) produce the characteristic equation

$$\lambda^2 + 2\alpha\lambda + \beta^2 = 0 \tag{9.130}$$

for determining values of λ to be substituted into the assumed solution. Solving equation (9.130) for λ gives the characteristic roots

$$\lambda = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\beta^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \beta^2} \tag{9.131}$$

Case 1 If $\alpha^2 - \beta^2 = \omega^2 > 0$, then a fundamental set of solutions is given by $\{e^{-(\alpha-\omega)t}, e^{-(\alpha+\omega)t}\}$ and the general solution to equation (9.128) is

$$\vec{y} = \vec{c}_1 e^{-(\alpha-\omega)t} + \vec{c}_2 e^{-(\alpha+\omega)t}$$

Case 2 If $\alpha^2 - \beta^2 = 0$, the characteristic equation has the repeated roots $\lambda = -\alpha$. The first root gives the first member of the fundamental set as $e^{-\alpha t}$ and using the rule for repeated roots, the second member of the fundamental set of solutions is $te^{-\alpha t}$. The general solution to equation (9.128) can then be expressed in the form

$$\vec{y} = \vec{c}_1 e^{-\alpha t} + \vec{c}_2\, t e^{-\alpha t}$$

Case 3 If $\alpha^2 - \beta^2 = -\omega^2 < 0$, then the fundamental set of solutions is $\{e^{-\alpha t}\cos\omega t, e^{-\alpha t}\sin\omega t\}$ and the general solution to equation (9.128) is given by

$$\vec{y} = \vec{c}_1 e^{-\alpha t}\cos\omega t + \vec{c}_2 e^{-\alpha t}\sin\omega t$$

The solution to the homogeneous vector equation is then given by

$$\vec{y}_c = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t)$$

where $\{y_1(t), y_2(t)\}$ are the functions from one of the cases previously examined.

To find a particular solution which gives the right-hand side $\sin 3t\,\hat{\mathbf{e}}_3$ examine this function and its first couple of derivatives $3\cos 3t\,\hat{\mathbf{e}}_3,\ -9\sin 3t\,\hat{\mathbf{e}}_3$. The basic terms in the set containing the function and its derivatives are the terms $\sin 3t$ and $\cos 3t$ multiplied by some constant. One can then assume a particular solution has the form

$$\vec{y}_p = \vec{y}_p(t) = A\sin 3t\ \hat{\mathbf{e}}_3 + B\cos 3t\ \hat{\mathbf{e}}_3 \tag{9.132}$$

where A, B are undetermined coefficients. Substitute the equation (9.132) into the nonhomogeneous equation (9.133) and show there results after simplification

$$[6\alpha A + (\beta^2 - 9)B]\cos 3t + [(\beta^2 - 9)A - 6\alpha B]\sin 3t = \sin 3t \tag{9.133}$$

Compare like terms in equation (9.133) and show A and B must be selected to satisfy the simultaneous equations

$$\begin{aligned} 6\alpha A + (\beta^2 - 9)B =& 0 \\ (\beta^2 - 9)A - 6\alpha B =& 1 \end{aligned}$$

One finds

$$A = \frac{\beta^2 - 9}{(\beta^2 - 9)^2 + 36\alpha^2} \qquad\qquad B = \frac{-6\alpha}{(\beta^2 - 9)^2 + 36\alpha^2}$$

and so the particular solution is given by

$$\vec{y}_p = \vec{y}_p(t) = \left[\frac{(\beta^2 - 9)}{(\beta^2 - 9)^2 + 36\alpha^2}\sin 3t - \frac{6\alpha}{(\beta^2 - 9)^2 + 36\alpha^2}\cos 3t\right]\ \hat{\mathbf{e}}_3$$

The general solution can then be represented

$$\vec{y} = \vec{y}(t) = \vec{y}_c + \vec{y}_p$$

■

Maxwell's Equations

James Clerk Maxwell (1831-1874), a Scottish mathematician, studied properties of electric and magnetic fields and came up with a set of 20 partial differential equations in 20 unknowns which described mathematically how electric and magnetic fields interact. Much later, an English electrical engineer by the name of Oliver Heaviside (1850-1925), greatly simplified Maxwell's equations to four equations in two unknowns. A modern day version of the **Maxwell equations** in SI units[8] are

$$\begin{aligned}
\nabla \cdot \vec{E} =& \frac{\rho}{\epsilon_0} \\
\nabla \times \vec{E} =& -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} =& 0 \\
\nabla \times \vec{B} =& \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{9.134}$$

In the Maxwell equations (9.134) one finds the following quantities

$$\begin{aligned}
\vec{E} =& \vec{E}(x,y,z,t) \quad \text{Electric field intensity (N/coul)} \\
\vec{J} =& \vec{J}(x,y,z) \quad \text{Total current density (amp/m}^2\text{)} \\
\vec{B} =& \vec{B}(x,y,z,t) \quad \text{Magnetic field intensity (N/amp}\cdot\text{m)} \\
\rho =& \rho(x,y,z) \quad \text{charge density (coul/m}^3\text{)} \\
\mu_0 =& 4\pi \times 10^{-7} \left(\frac{\text{N}}{\text{amp}^2}\right) \quad \text{the permeability of free space} \\
\epsilon_0 =& 8.85 \times 10^{-12} \left(\frac{\text{coul}^2}{\text{N}\cdot\text{m}^2}\right) \quad \text{the permittivity of free space}
\end{aligned}$$

It is left as an exercise to show that the Maxwell equations are dimensionally homogeneous.

Note 1: Warning! The symbols $\vec{B}$ and $\vec{H}$ occur in the study of electromagnetism. The symbol $\vec{H}$ is used to denote magnetic fields within a material medium. It has no name, but some textbooks call it a magnetic induction– which is wrong. To make matters worse many textbooks interchange the roles of $\vec{B}$ and $\vec{H}$. My only suggestion is be aware of these conflicts and study any textbook carefully and **see how things are defined.**

[8] There are two popular sets of units used to represent the Maxwell equations. These two popular units are the **International System of Units** or Système international dúnit es, designated **SI** (mks) in all languages and the **Gaussian** (cgs) set of units. The main advantage of the Gaussian units is that they simplify many of the basic equations of electricity and magnetism more so than the SI units.

Note 2: The product $\mu_0\epsilon_0 = \frac{1}{c^2}$, where $c = 3 \times (10)^{10}$ cm/sec is the speed of light. It will be demonstrated later in this chapter that the vector fields describing $\vec{E}$ and $\vec{B}$ of Maxwell equations are solutions of the wave equation.

Electrostatics

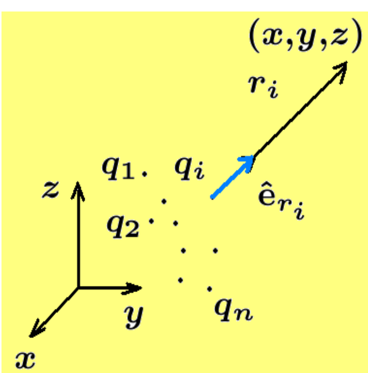
Coulomb's law[9] states that the force on a single test charge Q due to a single point charge q is given by

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,\hat{\mathbf{e}}_r \tag{9.135}$$

where $\epsilon_0 = 8.85 \times 10^{-12} \dfrac{\text{coul}^2}{\text{N}\cdot\text{m}^2}$ is the permittivity of free space, r is the distance between the charges and $\hat{\mathbf{e}}_r$ is a unit vector along the line connecting the charges. If q and Q have the same sign, the force is a repulsive force and if q and Q have opposite signs, then the force is attractive.

If there are many charges $q_1, q_2, \ldots, q_n$ at distances $r_1, r_2, \ldots, r_n$ from the test charge Q at the point (x, y, z), then one can use superposition to calculate the total force acting on the test charge. One finds

$$\vec{F} = \vec{F}(x,y,z) = \sum_{i=1}^{n} F_i = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\,\hat{\mathbf{e}}_{r_1} + \frac{q_2 Q}{r_2^2}\,\hat{\mathbf{e}}_{r_2} + \cdots + \frac{q_n Q}{r_n^2}\,\hat{\mathbf{e}}_{r_n}\right) = Q\vec{E} \tag{9.136}$$



where $\hat{\mathbf{e}}_{r_i}$, for $i = 1, \ldots, n$, are unit vectors pointing from charge q_i to the point (x, y, z) of the test charge Q. The quantity

$$\vec{E} = \vec{E}(x,y,z) = \frac{1}{Q}\vec{F}(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\,\hat{\mathbf{e}}_{r_i} \tag{9.137}$$

is called the electric field produced by the n-charges.

[9] Charles Augustin de Coulomb (1736-1806) A French engineer who studied electricity and magnetism.

Example 9-10.

Consider the special case of a single point charge q located at the origin. The electric field due to this point charge is

$$\vec{E} = \vec{E}(x,y,z) = \frac{1}{Q}\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{e}}_r \tag{9.138}$$

where $\hat{\mathbf{e}}_r$ is a unit vector in spherical coordinates and r is the distance of a test charge Q from the origin to the position (x,y,z). This vector field produces field lines and the strength of the vector field is proportional to the flux across some surface placed within the electric field. In the case where a sphere of radius r and centered at the origin is placed within the electric field, then the flux is calculated from the surface integral

$$\text{Flux} = \iint_S \vec{E}\cdot d\vec{S} = \iint_S \vec{E}\cdot \hat{\mathbf{e}}_n\, dS \tag{9.139}$$

where $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_r$ in spherical coordinates. Also the element of area dS in spherical coordinates is given by $d\vec{S} = r^2 \sin\theta\, d\theta d\phi\;\hat{\mathbf{e}}_r$ so that equation (9.139) can be expressed as

$$\text{Flux} = \iint_S \vec{E}\cdot d\vec{S} = \int_0^{2\pi}\left[\int_0^{\pi} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\, r^2\sin\theta\, d\theta\right] d\phi = \frac{q}{\epsilon_0} \tag{9.140}$$

This result states that the flux is a constant no matter what size sphere is placed about the point charge. If the sphere were made of rubber and could be deformed into some other simple closed surface, the number of field lines passing through the new surface would also be the same constant as above. This is because the dot product $\vec{E}\cdot\hat{\mathbf{e}}_n$ selects an element of area perpendicular to the field lines and the flux is proportional to the number of these lines. Note that if the point charge were outside the closed surface, then the flux would be zero, since field lines entering the surface at one point must exist at some other point and then the sum of the flux would be zero.

One can say that if there were n-charges $q_1, q_2, \ldots, q_n$ inside a simple closed surface and $\vec{E}_i$ was the electric field associated with the ith charge, then $\vec{E} = \sum_{i=1}^{n} \vec{E}_i$ would represent the total electric field and the flux across any simple closed surface due to this total electric field would be

$$\iint_S \vec{E}\cdot d\vec{S} = \sum_{i=1}^{n}\left(\iint_S \vec{E}_i\cdot d\vec{S}\right) = \sum_{i=1}^{n}\frac{q_i}{\epsilon_0} \tag{9.141}$$

If the discrete number of n-charges $q_1, \ldots, q_n$ were replaced by a continuous distribution of charges inside the surface, then the right-hand side of equation (9.141) would be replaced by $\iiint\limits_V \frac{\rho}{\epsilon_0}\, dV$ where dV is an element of volume $\left(\text{meter}^3\right)$ and ρ is a charge density $\left(\frac{\text{coulomb}}{\text{meter}^3}\right)$ and the equation (9.141) would then be written

$$\iint\limits_S \vec{E} \cdot d\vec{S} = \iiint\limits_V \frac{\rho}{\epsilon_0}\, dV \tag{9.142}$$

Using the divergence theorem of Gauss, the equation (9.142) can also be expressed as

$$\iint\limits_V \left(\nabla \cdot \vec{E} - \frac{\rho}{\epsilon_0}\right) dV = 0 \tag{9.143}$$

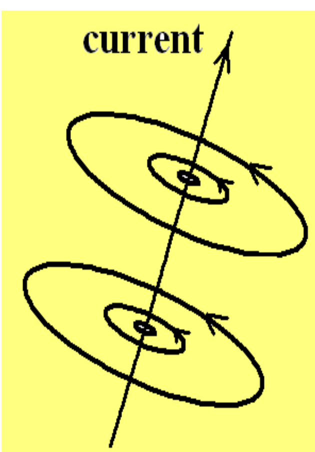
If the equation (9.143) is to hold for all arbitrary simple closed surfaces, then one must require that

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{9.144}$$

This is the first of Maxwell's equations (9.134) and is called the Gauss law of electrostatics.

■

Magnetostatics



A moving charge produces a current and a moving current produces a magnetic field. Consider a current moving along a wire considered as a line. The magnetic field created is described by circles around the wire. The strength of the magnetic field falls off as the perpendicular distance from the line increases. One can use the right-hand rule of letting the thumb point in the direction of the current flow, then the fingers of the right-hand point in the direction of the magnetic field lines.

The magnetic force on a charge Q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by

$$\vec{F}_m = Q\,(\vec{v} \times \vec{B}) \qquad (\text{coul})\left(\frac{\text{m}}{\text{s}}\right)\left(\frac{\text{N}}{\text{amp} \cdot \text{m}}\right) \tag{9.145}$$