several powers of that factor up to the highest. For instance, in splitting the fraction $\frac{3x^2 - 2x + 1}{(x+1)^2(x-2)}$ we must allow for the possible existence of a denominator $x + 1$ as well as $(x + 1)^2$ and $x - 2$.

It may be thought, however, that, since the numerator of the fraction the denominator of which is $(x + 1)^2$ may contain terms in x, we must allow for this in writing $Ax + B$ for its numerator, so that

$$\frac{3x^2 - 2x + 1}{(x+1)^2(x-2)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x+1} + \frac{D}{x-2}$$

If, however, we try to find A, B, C and D in this case, we fail, because we get four unknowns; and we have only three relations connecting them, yet

$$\frac{3x^2 - 2x + 1}{(x+1)^2(x-2)} = \frac{x-1}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x-2}$$

But if we write

$$\frac{3x^2 - 2x + 1}{(x+1)^2(x-2)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-2}$$

we get $3x^2 - 2x + 1 = A(x - 2) + B(x + 1)(x - 2) + C(x + 1)^2$.

For $x = -1$; $6 = -3A$, or $A = -2$

For $x = 2$; $9 = 9C$, or $C = 1$

For $x = 0$; $1 = -2A - 2B + C$

Putting in the values of A and C:

$$1 = 4 - 2B + 1, \text{ from which } B = 2$$

Hence the partial fractions are:

$$\frac{2}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{x-2}$$

instead of $\dfrac{1}{x+1}+\dfrac{x-1}{(x+1)^2}+\dfrac{1}{x-2}$ stated above as being the fractions from which $\dfrac{3x^2-2x+1}{(x+1)^2(x-2)}$ was obtained. The mystery is cleared if we observe that $\dfrac{x-1}{(x+1)^2}$ can itself be split into the two fractions $\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}$, so that the three fractions given are really equivalent to

$$\frac{1}{x+1}+\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2}=\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2}$$

which are the partial fractions obtained.

We see that it is sufficient to allow for one numerical term in each numerator, and that we always get the ultimate partial fractions.

When there is a power of a factor of x^2 in the denominator, however, the corresponding numerators must be of the form $Ax+B$; for example,

$$\frac{3x-1}{(2x^2-1)^2(x+1)}=\frac{Ax+B}{(2x^2-1)^2}+\frac{Cx+D}{2x^2-1}+\frac{E}{x+1}$$

which gives

$$3x-1=(Ax+B)(x+1)+(Cx+D)(x+1)(2x^2-1)+E(2x^2-1)^2.$$

For $x=-1$, this gives $E=-4$. Replacing, transposing, collecting like terms, and dividing by $x+1$, we get

$$16x^3-16x^2+3=2Cx^3+2Dx^2+x(A-C)+(B-D).$$

Hence $2C=16$ and $C=8$; $2D=-16$ and $D=-8$; $A-C=0$ or $A-8=0$ and $A=8$; and finally, $B-D=3$ or $B=-5$. So that we obtain as the partial fractions:

$$\frac{8x-5}{(2x^2-1)^2}+\frac{8(x-1)}{2x^2-1}-\frac{4}{x+1}$$

It is useful to check the results obtained. The simplest way is to replace x by a single value, say $+1$, both in the given expression and in the partial fractions obtained.

Whenever the denominator contains but a power of a single factor, a very quick method is as follows:

Taking, for example, $\dfrac{4x+1}{(x+1)^3}$, let $x+1=z$; then $x=z-1$.

Replacing, we get

$$\frac{4(z-1)+1}{z^3}=\frac{4z-3}{z^3}=\frac{4}{z^2}-\frac{3}{z^3}$$

The partial fractions are, therefore,

$$\frac{4}{(x+1)^2}-\frac{3}{(x+1)^3}$$

Applying this to differentiation, let it be required to differentiate $y=\dfrac{5-4x}{6x^2+7x-3}$; we have

$$\frac{dy}{dx}=-\frac{(6x^2+7x-3)\times 4+(5-4x)(12x+7)}{(6x^2+7x-3)^2}$$

$$=\frac{24x^2-60x-23}{(6x^2+7x-3)^2}$$

If we split the given expression into

$$\frac{1}{3x-1}-\frac{2}{2x+3}$$

we get, however,

$$\frac{dy}{dx}=-\frac{3}{(3x-1)^2}+\frac{4}{(2x+3)^2}$$

which is really the same result as above split into partial fractions. But the splitting, if done after differentiating, is more complicated, as will easily be seen. When we shall deal with the *integration* of such expressions, we shall find the splitting into partial fractions a precious auxiliary.

EXERCISES XI

Split into partial fractions:

(1) $\dfrac{3x+5}{(x-3)(x+4)}$ (2) $\dfrac{3x-4}{(x-1)(x-2)}$

(3) $\dfrac{3x+5}{x^2+x-12}$ (4) $\dfrac{x+1}{x^2-7x+12}$

(5) $\dfrac{x-8}{(2x+3)(3x-2)}$ (6) $\dfrac{x^2-13x+26}{(x-2)(x-3)(x-4)}$

(7) $\dfrac{x^2-3x+1}{(x-1)(x+2)(x-3)}$ (8) $\dfrac{5x^2+7x+1}{(2x+1)(3x-2)(3x+1)}$

(9) $\dfrac{x^2}{x^3-1}$ (10) $\dfrac{x^4+1}{x^3+1}$

(11) $\dfrac{5x^2+6x+4}{(x+1)(x^2+x+1)}$ (12) $\dfrac{x}{(x-1)(x-2)^2}$

(13) $\dfrac{x}{(x^2-1)(x+1)}$ (14) $\dfrac{x+3}{(x+2)^2(x-1)}$

(15) $\dfrac{3x^2+2x+1}{(x+2)(x^2+x+1)^2}$ (16) $\dfrac{5x^2+8x-12}{(x+4)^3}$

(17) $\dfrac{7x^2+9x-1}{(3x-2)^4}$ (18) $\dfrac{x^2}{(x^3-8)(x-2)}$

Derivative of an Inverse Function

Consider the function $y = 3x$; it can be expressed in the form $x = \dfrac{y}{3}$; this latter form is called the *inverse function* to the one originally given.

If $y = 3x$, $\dfrac{dy}{dx} = 3$; if $x = \dfrac{y}{3}$, $\dfrac{dx}{dy} = \dfrac{1}{3}$, and we see that

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \quad \text{or} \quad \frac{dy}{dx} \times \frac{dx}{dy} = 1$$

Consider $y = 4x^2$, $\dfrac{dy}{dx} = 8x$; the inverse function is

$$x = \frac{y^{\frac{1}{2}}}{2}, \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{4\sqrt{y}} = \frac{1}{4 \times 2x} = \frac{1}{8x}$$

Here again $$\frac{dy}{dx} \times \frac{dx}{dy} = 1$$

It can be shown that for all functions which can be put into the inverse form, one can always write

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

It follows that, being given a function, if it be easier to differentiate the inverse function, this may be done, and the reciprocal of the derivative of the inverse function gives the derivative of the given function itself.

As an example, suppose that we wish to differentiate $y = \sqrt{\dfrac{3}{x} - 1}$. We have seen one way of doing this, by writing $u = \dfrac{3}{x} - 1$, and finding $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. This gives

$$\frac{dy}{dx} = -\frac{3}{2x^2\sqrt{\dfrac{3}{x} - 1}}$$

If we had forgotten how to proceed by this method, or wished to check our result by some other way of obtaining the derivative, or for any other reason we could not use the ordinary method, we could proceed as follows: The inverse function is $x = \dfrac{3}{1+y^2}$.

$$\frac{dx}{dy} = -\frac{3 \times 2y}{(1+y^2)^2} = -\frac{6y}{(1+y^2)^2}$$

hence

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = -\frac{(1+y^2)^2}{6y} = -\frac{\left(1+\dfrac{3}{x}-1\right)^2}{6 \times \sqrt{\dfrac{3}{x}-1}} = -\frac{3}{2x^2\sqrt{\dfrac{3}{x}-1}}$$

Let us take, as another example, $y = \dfrac{1}{\sqrt[3]{\theta+5}}$

The inverse function $\theta = \dfrac{1}{y^3} - 5$ or $\theta = y^{-3} - 5$, and

$$\frac{d\theta}{dy} = -3y^{-4} = -3\sqrt[3]{(\theta+5)^4}$$

It follows that $\dfrac{dy}{d\theta} = -\dfrac{1}{3\sqrt[3]{(\theta+5)^4}}$, as might have been found otherwise.

We shall find this dodge most useful later on; meanwhile you are advised to become familiar with it by verifying by its means the results obtained in Exercises I (Chapter IV), Nos. 5, 6, 7; Examples (Chapter IX), Nos. 1, 2, 4; and Exercises VI (Chapter IX), Nos. 1, 2, 3 and 4.

You will surely realize from this chapter and the preceding, that in many respects the calculus is an *art* rather than a *science:* an art only to be acquired, as all other arts are, by practice. Hence you should work many examples, and set yourself other examples, to see if you can work them out, until the various artifices become familiar by use.

Chapter XIV

ON TRUE COMPOUND INTEREST AND THE LAW OF ORGANIC GROWTH

Let there be a quantity growing in such a way that the increment of its growth, during a given time, shall always be proportional to its own magnitude. This resembles the process of reckoning interest on money at some fixed rate; for the bigger the capital, the bigger the amount of interest on it in a given time.

Now we must distinguish clearly between two cases, in our calculation, according as the calculation is made by what the arithmetic books call "simple interest", or by what they call "compound interest". For in the former case the capital remains fixed, while in the latter the interest is added to the capital, which therefore increases by successive additions.

(1) *At simple interest.* Consider a concrete case. Let the capital at start be \$100, and let the rate of interest be 10 percent per annum. Then the increment to the owner of the capital will be \$10 every year. Let him go on drawing his interest every year, and hoard it by putting it by in a stocking or locking it up in his safe. Then, if he goes on for 10 years, by the end of that time he will have received 10 increments of \$10 each, or \$100, making, with the original \$100, a total of \$200 in all. His property will have doubled itself in 10 years. If the rate of interest had been 5 percent, he would have had to hoard for 20 years to double his property. If it had been only 2 percent, he would have had to hoard for 50 years. It is easy to see that if the value of the yearly interest is $\frac{1}{n}$ of the capital, he must go on hoarding for n years in order to double his property.

Or, if y be the original capital, and the yearly interest is $\frac{y}{n}$, then, at the end of n years, his property will be

$$y + n\frac{y}{n} = 2y$$

(2) *At compound interest.* As before, let the owner begin with a capital of \$100, earning interest at the rate of 10 percent per annum; but, instead of hoarding the interest, let it be added to the capital each year, so that the capital grows year by year. Then, at the end of one year, the capital will have grown to \$110; and in the second year (still at 10%) this will earn \$11 interest. He will start the third year with \$121, and the interest on that will be \$12.10; so that he starts the fourth year with \$133.10, and so on. It is easy to work it out, and find that at the end of the ten years the total capital will have grown to more than \$259. In fact, we see that at the end of each year, each dollar will have earned $\frac{1}{10}$ of a dollar, and therefore, if this is always added on, each year multiplies the capital by $\frac{11}{10}$; and if continued for ten years (which will multiply by this factor ten times over) the original capital will be multiplied by 2.59374. Let us put this into symbols. Put y_0 for the original capital; $\frac{1}{n}$ for the fraction added on at each of the n operations; and y_n for the value of the capital at the end of the n^{th} operation. Then

$$y_n = y_0\left(1 + \frac{1}{n}\right)^n$$

But this mode of reckoning compound interest once a year is really not quite fair; for even during the first year the \$100 ought to have been growing. At the end of half a year it ought to have been at least \$105, and it certainly would have been fairer had the interest for the second half of the year been calculated on \$105. This would be equivalent to calling it 5% per half-year; with 20 operations, therefore, at each of which the capital is multiplied by $\frac{21}{20}$. If reckoned this way, by the end of ten years the capital would have grown to more than \$265; for

$$\left(1 + \tfrac{1}{20}\right)^{20} = 2.653$$

But, even so, the process is still not quite fair; for, by the end of the first month, there will be some interest earned; and a half-yearly reckoning assumes that the capital remains stationary for six months at a time. Suppose we divided the year into 10 parts, and reckon a one percent interest for each tenth of the year. We now have 100 operations lasting over the ten years; or

$$y_n = \$100\left(1 + \tfrac{1}{100}\right)^{100}$$

which works out to \$270.48.

Even this is not final. Let the ten years be divided into 1000 periods, each of $\frac{1}{100}$ of a year; the interest being $\frac{1}{10}$ percent for each such period; then

$$y_n = \$100\left(1 + \tfrac{1}{1000}\right)^{1000}$$

which works out to a little more than \$271.69.

Go even more minutely, and divide the ten years into 10,000 parts, each $\frac{1}{1000}$ of a year, with interest at $\frac{1}{100}$ of 1 percent. Then

$$y_n = \$100\left(1 + \tfrac{1}{10{,}000}\right)^{10{,}000}$$

which amounts to about \$271.81.

Finally, it will be seen that what we are trying to find is in reality the ultimate value of the expression $\left(1 + \dfrac{1}{n}\right)^n$, which, as we see, is greater than 2; and which, as we take n larger and larger, grows closer and closer to a particular limiting value. However big you make n, the value of this expression grows nearer and nearer to the limit

$$2.71828\ldots,$$

a number *never to be forgotten.*[1]

1. This number is transcendental—an irrational number that is not the root of any polynomial equation with integer coefficients. It was named e by Euler (1707-1783), a famous Swiss mathematician. He was the first to prove that e is the limit of $(1 + 1/x)^x$ as x approaches infinity.

Like all irrationals, the decimal expansion of e (2.71828 18284 59045 . . .) never repeats. (The curious repetition of 1828 is entirely co-

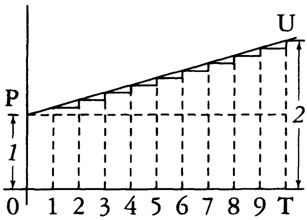


FIG. 36.

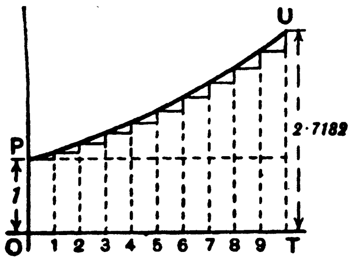


FIG. 37.

Let us take geometrical illustrations of these things. In Fig. 36, *OP* stands for the original value. *OT* is the whole time during which the value is growing. It is divided into 10 periods, in each of which there is an equal step up. Here $\frac{dy}{dx}$ is a constant; and if each step up is $\frac{1}{10}$ of the original *OP*, then, by 10 such steps, the height is doubled. If we had taken 20 steps, each of half the height shown, at the end the height would still be just doubled. Or n such steps, each of $\frac{1}{n}$ of the original height *OP*, would suffice to double the height. This is the case of simple interest. Here is 1 growing till it becomes 2.

In Fig. 37, we have the corresponding illustration of the geometrical progression. Each of the successive ordinates is to be $1 + \frac{1}{n}$, that is, $\frac{n+1}{n}$ times as high as its predecessor. The steps up are not equal, because each step up is now $\frac{1}{n}$ of the ordinate *at that part* of the curve. If we had literally 10 steps, with

incidental.) Its best fractional approximation with integers less than 1,000 is $878/323 = 2.71826 \dots$

The number e is as ubiquitous as pi, turning up everywhere, and especially in probability theory. If you randomly select real numbers between 0 and 1, and continue until their sum exceeds 1, the expected number of choices is e. See the chapter on e in my *Unexpected Hanging and Other Mathematical Diversions* (1969).—M.G.

$\left(1+\frac{1}{10}\right)$ for the multiplying factor, the final total would be $\left(1+\frac{1}{10}\right)^{10}$ or 2.594 times the original 1. But if only we take n sufficiently large (and the corresponding $\frac{1}{n}$ sufficiently small), then the final value $\left(1+\frac{1}{n}\right)^n$ to which unity will grow will be 2.71828. . . .

e. To this mysterious number 2.7182818. . . , the mathematicians have assigned the English letter *e*. All schoolboys know that the Greek letter π (called *pi*) stands for 3.141592. . . , but how many of them know that *e* stands for 2.71828. . . . Yet it is an even more important number than π!

What, then, is *e*?

Suppose we were to let 1 grow at simple interest till it became 2; then, if at the same nominal rate of interest, and for the same time, we were to let 1 grow at true compound interest, instead of simple, it would grow to the value *e*.

This process of growing proportionately, at every instant, to the magnitude at that instant, some people call an *exponential rate* of growing. Unit exponential rate of growth is that rate which in unit time will cause 1 to grow to 2.718281. . . . It might also be called the *organic rate* of growing: because it is characteristic of organic growth (in certain circumstances) that the increment of the organism in a given time is proportional to the magnitude of the organism itself.

If we take 100 percent as the unit of rate, and any fixed period as the unit of time, then the result of letting 1 grow *arithmetically* at unit rate, for unit time, will be 2, while the result of letting 1 grow *exponentially* at unit rate, for the same time, will be 2.71828. . . .

A Little More About e. We have seen that we require to know what value is reached by the expression $\left(1+\frac{1}{n}\right)^n$, when n becomes infinitely great. Arithmetically, here are tabulated a lot of values obtained by assuming $n = 2$; $n = 5$; $n = 10$; and so on, up to $n = 10{,}000$.

$$\left(1+\tfrac{1}{2}\right)^2 = 2.25$$
$$\left(1+\tfrac{1}{5}\right)^5 = 2.489$$
$$\left(1+\tfrac{1}{10}\right)^{10} = 2.594$$
$$\left(1+\tfrac{1}{20}\right)^{20} = 2.653$$
$$\left(1+\tfrac{1}{100}\right)^{100} = 2.705$$
$$\left(1+\tfrac{1}{1000}\right)^{1000} = 2.7169$$
$$\left(1+\tfrac{1}{10{,}000}\right)^{10{,}000} = 2.7181$$

It is, however, worth while to find another way of calculating this immensely important figure.

Accordingly, we will avail ourselves of the binomial theorem, and expand the expression $\left(1+\frac{1}{n}\right)^n$ in that well-known way.

The binomial theorem gives the rule that

$$(a+b)^n = a^n + n\,\frac{a^{n-1}b}{1!} + n(n-1)\frac{a^{n-2}b^2}{2!} + n(n-1)(n-2)\frac{a^{n-3}b^3}{3!} + \dots.$$

Putting $a = 1$ and $b = \frac{1}{n}$, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\,\frac{(n-1)(n-2)}{n^2} + \frac{1}{4!}\,\frac{(n-1)(n-2)(n-3)}{n^3} + \dots.$$

Now, if we suppose n to become infinitely great, say a billion, or a billion billions, then $n-1$, $n-2$, and $n-3$. etc., will all be sensibly equal to n; and then the series becomes

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots.$$

By taking this rapidly convergent series to as many terms as we please, we can work out the sum to any desired point of accuracy. Here is the working for ten terms:

	1.000000
dividing by 1	1.000000
dividing by 2	0.500000
dividing by 3	0.166667
dividing by 4	0.041667
dividing by 5	0.008333
dividing by 6	0.001389
dividing by 7	0.000198
dividing by 8	0.000025
dividing by 9	0.000003
Total	2.718282

e is incommensurable with 1, and resembles π in being an interminable nonrecurrent decimal.

The Exponential Series. We shall have need of yet another series.

Let us, again making use of the binomial theorem, expand the expression $\left(1+\frac{1}{n}\right)^{nx}$, which is the same as e^x when we make n indefinitely great.

$$e^x = 1^{nx} + nx\frac{1^{nx-1}\left(\frac{1}{n}\right)}{1!} + nx(nx-1)\frac{1^{nx-2}\left(\frac{1}{n}\right)^2}{2!}$$

$$+\, nx(nx-1)(nx-2)\frac{1^{nx-3}\left(\frac{1}{n}\right)^3}{3!} + \ldots .$$

$$= 1 + x + \frac{1}{2!} \cdot \frac{n^2x^2 - nx}{n^2} + \frac{1}{3!} \cdot \frac{n^3x^3 - 3n^2x^2 + 2nx}{n^3} + \ldots.$$

$$= 1 + x + \frac{x^2 - \frac{x}{n}}{2!} + \frac{x^3 - \frac{3x^2}{n} + \frac{2x}{n^2}}{3!} + \ldots.$$

But, when n is made infinitely great, this simplifies down to the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$$

This series is called *the exponential series.*

The great reason why e is regarded of importance is that e^x possesses a property, not possessed by any other function of x, that *when you differentiate it its value remains unchanged;* or, in other words, its derivative is the same as itself. This can be instantly seen by differentiating it with respect to x, thus:

$$\frac{d(e^x)}{dx} = 0 + 1 + \frac{2x}{1 \cdot 2} + \frac{3x^2}{1 \cdot 2 \cdot 3} + \frac{4x^3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{5x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots.$$

$$\text{or} \qquad = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots.$$

which is exactly the same as the original series.

Now we might have gone to work the other way, and said: Go to; let us find a function of x, such that its derivative is the same as itself. Or, is there any expression, involving only powers of x, which is unchanged by differentiation? Accordingly, let us *assume* as a general expression that

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots.$$

(in which the coefficients A, B, C, ... will have to be determined), and differentiate it.

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots.$$

Now, if this new expression is really to be the same as that from which it was derived, it is clear that A *must* $= B$; that $C = \frac{B}{2} = \frac{A}{1 \cdot 2}$; that $D = \frac{C}{3} = \frac{A}{1 \cdot 2 \cdot 3}$; that $E = \frac{D}{4} = \frac{A}{1 \cdot 2 \cdot 3 \cdot 4} \ldots.$

The law of change is therefore that

$$y = A\left(1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots.\right)$$

If, now, we take $A = 1$ for the sake of further simplicity, we have

$$y = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots.$$

Differentiating it any number of times will give always the same series over again.

If, now, we take the particular case of $A = 1$, and evaluate the series, we shall get simply

$$\begin{aligned} &\text{when } x = 1, \quad y = 2.718281 \ldots.; \quad &&\text{that is, } y = e; \\ &\text{when } x = 2, \quad y = (2.718281 \ldots.)^2; \quad &&\text{that is, } y = e^2; \\ &\text{when } x = 3, \quad y = (2.718281 \ldots.)^3; \quad &&\text{that is, } y = e^3; \end{aligned}$$

and therefore

$$\text{when } x = x, \quad y = (2.718281 \ldots.)^x; \quad \text{that is, } y = e^x,$$

thus finally demonstrating that

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots.$$

Natural or Napierian Logarithms.

Another reason why e is important is because it was made by Napier, the inventor of logarithms, the basis of his system. If y is the value of e^x, then x is the *logarithm,* to the base e, of y. Or, if

$$y = e^x$$

then $$x = \log_e y$$

The two curves plotted in Figs. 38 and 39 represent these equations.

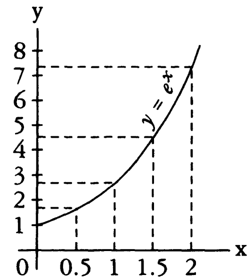


FIG. 38.

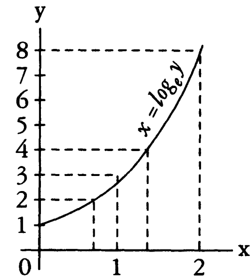


FIG. 39.

The points calculated are:

For Fig. 38

x	0	0.5	1	1.5	2
y	1	1.65	2.72	4.48	7.39

For Fig. 39

y	1	2	3	4	8
x	0	0.69	1.10	1.39	2.08

It will be seen that, though the calculations yield different points for plotting, yet the result is identical. The two equations really mean the same thing.

As many persons who use ordinary logarithms, which are calculated to base 10 instead of base e, are unfamiliar with the "natural" logarithms, it may be worth while to say a word about them.[2] The ordinary rule that adding logarithms gives the logarithm of the product still holds good; or

$$\ln a + \ln b = \ln ab$$

Also the rule of powers holds good;

$$n \times \ln a = \ln a^n$$

But as 10 is no longer the basis, one cannot multiply by 100 or 1000 by merely adding 2 or 3 to the index. A natural logarithm

2. It is customary today to write ln (pronounced el-en) rather than $\log_e$ for all natural logarithms. From here on I have replaced Thompson's $\log_e$ with ln—M.G.

is connected to the common logarithm of the same number by the relations:

$$\log_{10} x = \log_{10} e \times \ln x, \quad \text{and} \quad \ln x = \ln 10 \times \log_{10} x;$$

but $\log_{10} e = \log_{10} 2.718 = 0.4343$ and $\ln 10 = 2.3026$

$$\log_{10} x = 0.4343 \times \ln x$$

$$\ln x = 2.3026 \times \log_{10} x$$

A Useful Table of "Naperian Logarithms"

(Also called Natural Logarithms or Hyperbolic Logarithms)

Number	*Log$_e$*	*Number*	*Log$_e$*
1	0.0000	6	1.7918
1.1	0.0953	7	1.9459
1.2	0.1823	8	2.0794
1.5	0.4055	9	2.1972
1.7	0.5306	10	2.3026
2.0	0.6931	20	2.9957
2.2	0.7885	50	3.9120
2.5	0.9163	100	4.6052
2.7	0.9933	200	5.2983
2.8	1.0296	500	6.2146
3.0	1.0986	1,000	6.9078
3.5	1.2528	2,000	7.6009
4.0	1.3863	5,000	8.5172
4.5	1.5041	10,000	9.2103
5.0	1.6094	20,000	9.9035

Exponential and Logarithmic Equations.

Now let us try our hands at differentiating certain expressions that contain logarithms or exponentials.

Take the equation:

$$y = \ln x$$

First transform this into

$$e^y = x$$

whence, since the derivative of e^y with regard to y is the original function unchanged,

$$\frac{dx}{dy} = e^y$$

and, reverting from the inverse to the original function,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

Now this is a very curious result. It may be written

$$\frac{d(\ln x)}{dx} = x^{-1}$$

Note that x^{-1} is a result that we could never have got by the rule for differentiating powers. That rule is to multiply by the power, and reduce the power by 1. Thus, differentiating x^3 gave us $3x^2$; and differentiating x^2 gave $2x^1$. But differentiating x^0 gives us $0 \times x^{-1} = 0$, because x^0 is itself $= 1$, and is a constant. We shall have to come back to this curious fact that differentiating $\log_e x$ gives us $\frac{1}{x}$ when we reach the chapter on integrating.

Now, try to differentiate

$$y = \ln(x + a)$$

that is $\qquad e^y = x + a$

we have $\frac{d(x+a)}{dy} = e^y$, since the derivative of e^y remains e^y.

This gives $\qquad \frac{dx}{dy} = e^y = x + a;$

hence, reverting to the original function, we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x+a}$$

Next try $$y = \log_{10} x$$

First change to natural logarithms by multiplying by the modulus 0.4343. This gives us

$$y = 0.4343 \ln x$$

whence $$\frac{dy}{dx} = \frac{0.4343}{x}$$

The next thing is not quite so simple. Try this:

$$y = a^x$$

Taking the logarithm of both sides, we get

$$\ln y = x \ln a$$

or $$x = \frac{\ln y}{\ln a} = \frac{1}{\ln a} \times \ln y$$

Since $\frac{1}{\ln a}$ is a constant, we get

$$\frac{dx}{dy} = \frac{1}{\ln a} \times \frac{1}{y} = \frac{1}{a^x \times \ln a}$$

hence, reverting to the original function,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = a^x \times \ln a$$

We see that, since

$$\frac{dx}{dy} \times \frac{dy}{dx} = 1 \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{y} \times \frac{1}{\ln a}, \quad \frac{1}{y} \times \frac{dy}{dx} = \ln a$$

We shall find that whenever we have an expression such as $\ln y =$ a function of x, we always have $\frac{1}{y}\frac{dy}{dx} =$ the derivative of the function of x, so that we could have written at once, from $\ln y = x \ln a$

$$\frac{1}{y}\frac{dy}{dx} = \ln a \quad \text{and} \quad \frac{dy}{dx} = y \ln a = a^x \ln a$$

Let us now attempt further examples.

Examples.

(1) $y = e^{-ax}$. Let $z = -ax$; then $y = e^z$

$$\frac{dy}{dz} = e^z;\ \frac{dz}{dx} = -a;\ \text{hence}\ \frac{dy}{dx} = -ae^z = -ae^{-ax}$$

Or thus:

$$\ln y = -ax;\ \frac{1}{y}\frac{dy}{dx} = -a;\ \frac{dy}{dx} = -ay = -ae^{-ax}$$

(2) $y = e^{\frac{x^2}{3}}$. Let $z = \frac{x^2}{3}$; then $y = e^z$.

$$\frac{dy}{dz} = e^z;\ \frac{dz}{dx} = \frac{2x}{3};\ \frac{dy}{dx} = \frac{2x}{3}e^{\frac{x^2}{3}}$$

Or thus: $\quad \ln y = \frac{x^2}{3};\ \frac{1}{y}\frac{dy}{dx} = \frac{2x}{3};\ \frac{dy}{dx} = \frac{2x}{3}e^{\frac{x^2}{3}}$

(3) $y = e^{\frac{2x}{x+1}}$. $\ln y = \frac{2x}{x+1},\ \frac{1}{y}\frac{dy}{dx} = \frac{2(x+1) - 2x}{(x+1)^2}$

hence $\quad \frac{dy}{dx} = \frac{2y}{(x+1)^2} = \frac{2}{(x+1)^2}e^{\frac{2x}{x+1}}$

Check by writing $z = \frac{2x}{x+1}$

(4) $y=e^{\sqrt{x^2+a}}$. $\ln y=(x^2+a)^{\frac{1}{2}}$

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{(x^2+a)^{\frac{1}{2}}} \quad \text{and} \quad \frac{dy}{dx}=\frac{x\times e^{\sqrt{x^2+a}}}{(x^2+a)^{\frac{1}{2}}}$$

For if $u=(x^2+a)^{\frac{1}{2}}$ and $v=x^2+a$, $u=v^{\frac{1}{2}}$,

$$\frac{du}{dv}=\frac{1}{2v^{\frac{1}{2}}};\ \frac{dv}{dx}=2x;\ \frac{du}{dx}=\frac{x}{(x^2+a)^{\frac{1}{2}}}$$

Check by writing $z=\sqrt{x^2+a}$

(5) $y=\ln(a+x^3)$. Let $z=(a+x^3)$; then $y=\ln z$.

$$\frac{dy}{dz}=\frac{1}{z};\ \frac{dz}{dx}=3x^2;\ \text{hence}\ \frac{dy}{dx}=\frac{3x^2}{a+x^3}$$

(6) $y=\ln\left\{3x^2+\sqrt{a+x^2}\right\}$. Let $z=3x^2+\sqrt{a+x^2}$; then $y=\ln z$.

$$\frac{dy}{dz}=\frac{1}{z};\ \frac{dz}{dx}=6x+\frac{x}{\sqrt{x^2+a}}$$

$$\frac{dy}{dx}=\frac{6x+\dfrac{x}{\sqrt{x^2+a}}}{3x^2+\sqrt{a+x^2}}=\frac{x\left(1+6\sqrt{x^2+a}\right)}{\left(3x^2+\sqrt{x^2+a}\right)\sqrt{x^2+a}}$$

(7) $y=(x+3)^2\sqrt{x-2}$

$$\ln y=2\ln(x+3)+\tfrac{1}{2}\ln(x-2)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{(x+3)}+\frac{1}{2(x-2)}$$

$$\frac{dy}{dx}=(x+3)^2\sqrt{x-2}\left\{\frac{2}{x+3}+\frac{1}{2(x-2)}\right\}=\frac{5(x+3)(x-1)}{2\sqrt{x-2}}$$

(8) $y=(x^2+3)^3(x^3-2)^{\frac{2}{3}}$.

$$\ln y=3\ln(x^2+3)+\tfrac{2}{3}\ln(x^3-2)$$

$$\frac{1}{y}\frac{dy}{dx}=3\frac{2x}{x^2+3}+\frac{2}{3}\frac{3x^2}{x^3-2}=\frac{6x}{x^2+3}+\frac{2x^2}{x^3-2}$$

For if $u = \ln(x^2 + 3)$, $z = x^2 + 3$ and $u = \ln z$

$$\frac{du}{dz} = \frac{1}{z}; \ \frac{dz}{dx} = 2x; \ \frac{du}{dx} = \frac{2x}{z} = \frac{2x}{x^2 + 3}$$

Similarly, if $v = \ln(x^3 - 2)$, $\dfrac{dv}{dx} = \dfrac{3x^2}{x^3 - 2}$ and

$$\frac{dy}{dx} = (x^2 + 3)^3 (x^3 - 2)^{\frac{2}{3}} \left\{ \frac{6x}{x^2 + 3} + \frac{2x^2}{x^3 - 2} \right\}$$

(9) $y = \dfrac{\sqrt[2]{x^2 + a}}{\sqrt[3]{x^3 - a}}$

$$\ln y = \frac{1}{2} \ln(x^2 + a) - \frac{1}{3} \ln(x^3 - a)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \frac{2x}{x^2 + a} - \frac{1}{3} \frac{3x^2}{x^3 - a} = \frac{x}{x^2 + a} - \frac{x^2}{x^3 - a}$$

and $$\frac{dy}{dx} = \frac{\sqrt[2]{x^2 + a}}{\sqrt[3]{x^3 - a}} \left\{ \frac{x}{x^2 + a} - \frac{x^2}{x^3 - a} \right\}$$

(10) $y = \dfrac{1}{\ln x}$.

$$\frac{dy}{dx} = \frac{\ln x \times 0 - 1 \times \frac{1}{x}}{\ln^2 x} = -\frac{1}{x \ln^2 x}$$

(11) $y = \sqrt[3]{\ln x} = (\ln x)^{\frac{1}{3}}$. Let $z = \ln x$; $y = z^{\frac{1}{3}}$,

$$\frac{dy}{dz} = \frac{1}{3} z^{-\frac{2}{3}}; \ \frac{dz}{dx} = \frac{1}{x}; \ \frac{dy}{dx} = \frac{1}{3x\sqrt[3]{\ln^2 x}}$$

(12) $y = \left(\dfrac{1}{a^x} \right)^{ax}$

$$\ln y = -ax \ln a^x = -ax^2 \cdot \ln a$$

$$\frac{1}{y}\frac{dy}{dx} = -2ax \cdot \ln a$$

and
$$\frac{dy}{dx} = -2ax\left(\frac{1}{a^x}\right)^{ax} \cdot \ln a = -2xa^{1-ax^2} \cdot \ln a$$

Try now the following exercises.

EXERCISES XII

(1) Differentiate $y = b(e^{ax} - e^{-ax})$.

(2) Find the derivative with respect to t of the expression $u = at^2 + 2 \ln t$.

(3) If $y = n^t$, find $\dfrac{d(\ln y)}{dt}$

(4) Show that if $y = \dfrac{1}{b} \cdot \dfrac{a^{bx}}{\ln a}$; $\dfrac{dy}{dx} = a^{bx}$

(5) If $w = pv^n$, find $\dfrac{dw}{dv}$

Differentiate

(6) $y = \ln x^n$ (7) $y = 3e^{-\frac{x}{x-1}}$ (8) $y = (3x^2 + 1)e^{-5x}$

(9) $y = \ln(x^a + a)$ (10) $y = (3x^2 - 1)(\sqrt{x} + 1)$

(11) $y = \dfrac{\ln(x+3)}{x+3}$ (12) $y = a^x \times x^a$

(13) It was shown by Lord Kelvin that the speed of signalling through a submarine cable depends on the value of the ratio of the external diameter of the core to the diameter of the enclosed copper wire. If this ratio is called y, then the number of signals s that can be sent per minute can be expressed by the formula

$$s = ay^2 \ln\frac{1}{y}$$

where a is a constant depending on the length and the quality of the materials. Show that if these are given, s will be a maximum if $1/e^{\frac{1}{2}}$.

(14) Find the maximum or minimum of

$$y = x^3 - \ln x$$

(15) Differentiate $y = \ln(axe^x)$

(16) Differentiate $y = (\ln ax)^3$

The Logarithmic Curve

Let us return to the curve which has its successive ordinates in geometrical progression, such as that represented by the equation $y = bp^x$.

We can see, by putting $x = 0$, that b is the initial height of y. Then when

$$x = 1,\ y = bp; \quad x = 2,\ y = bp^2; \quad x = 3,\ y = bp^3, \text{ etc.}$$

Also, we see that p is the numerical value of the ratio between the height of any ordinate and that of the next preceding it. In Fig. 40, we have taken p as $\frac{6}{5}$; each ordinate being $\frac{6}{5}$ as high as the preceding one.

If two successive ordinates are related together thus in a constant ratio, their logarithms will have a constant difference; so that, if we should plot out a new curve, Fig. 41, with values of $\ln y$ as ordinates, it would be a straight line sloping up by equal steps. In fact, it follows from the equation, that

$$\ln y = \ln b + x \cdot \ln p, \text{ whence } \ln y - \ln b = x \cdot \ln p.$$

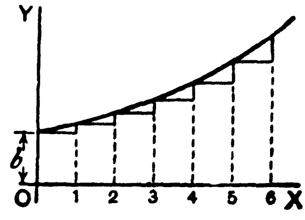


FIG. 40.

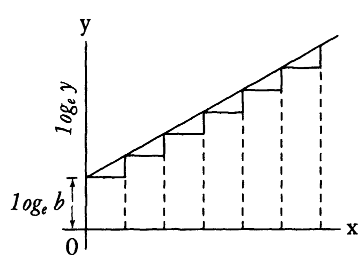


FIG. 41.

Now, since $\ln p$ is a mere number, and may be written as $\ln p = a$, it follows that

$$\ln \frac{y}{b} = ax$$

and the equation takes the new form

$$y = be^{ax}$$

The Die-away Curve

If we were to take p as a proper fraction (less than unity), the curve would obviously tend to sink downwards, as in Fig. 42, where each successive ordinate is $\frac{3}{4}$ of the height of the preceding one.

The equation is still

$$y = bp^x$$

but since p is less than one, $\ln p$ will be a negative quantity, and may be written $-a$; so that $p = e^{-a}$, and now our equation for the curve takes the form

$$y = be^{-ax}$$

The importance of this expression is that, in the case where the independent variable is *time,* the equation represents the course of a great many physical processes in which something is *gradually dying away.* Thus, the cooling of a hot body is represented (in Newton's celebrated "law of cooling") by the equation

$$\theta_t = \theta_0 e^{-at}$$

where θ_0 is the original excess of temperature of a hot body over that of its surroundings, θ_t the excess of temperature at the end of time t, and a is a constant—namely, the constant of decrement, depending on the amount of surface exposed by the body, and on its coefficients of conductivity and emissivity, etc.

A similar formula,

$$Q_t = Q_0 e^{-at}$$

is used to express the charge of an electrified body, originally having a charge Q_0, which is leaking away with a con

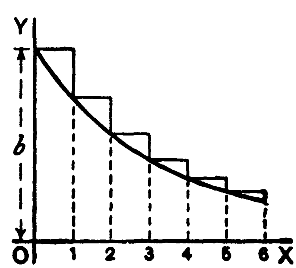


FIG. 42.

stant of decrement a; which constant depends in this case on the capacity of the body and on the resistance of the leakage-path.

Oscillations given to a flexible spring die out after a time, and the dying-out of the amplitude of the motion may be expressed in a similar way.

In fact e^{-at} serves as a *die-away factor* for all those phenomena in which the rate of decrease is proportional to the magnitude of that which is decreasing; or where, in our usual symbols, $\frac{dy}{dt}$ is proportional at every moment to the value that y has at that moment. For we have only to inspect the curve, Fig. 42, to see that at every part of it, the slope $\frac{dy}{dx}$ is proportional to the height y; the curve becoming flatter as y grows smaller. In symbols, thus

$$y = be^{-ax}$$

or
$$\ln y = \ln b - ax \ln e = \ln b - ax,$$

and, differentiating,
$$\frac{1}{y}\frac{dy}{dx} = -a$$

hence,
$$\frac{dy}{dx} = be^{-ax} \times (-a) = -ay$$

or, in words, the slope of the curve is downward, and proportional to y and to the constant a.

We should have got the same result if we had taken the equation in the form

$$y = bp^x$$

for then
$$\frac{dy}{dx} = bp^x \times \ln p$$

But
$$\ln p = -a$$

giving us
$$\frac{dy}{dx} = y \times (-a) = -ay$$

as before.

The Time-Constant. In the expression for the "die-away factor" e^{-at}, the quantity a is the reciprocal of another quantity known as "*the time-constant*", which we may denote by the symbol T. Then the die-away factor will be written $e^{-\frac{t}{T}}$; and it will be seen, by making $t = T$, that the meaning of T $\left(\text{or of } \frac{1}{a}\right)$ is that this is the length of time which it takes for the original quantity (called θ_0 or Q_0 in the preceding instances) to die away to $\frac{1}{e}$*th* part—that is to 0.3679—of its original value.

The values of e^x and e^{-x} are continually required in different branches of physics, and as they are given in very few sets of mathematical tables, some of the values are tabulated here for convenience.

x	e^x	e^{-x}	$1 - e^{-x}$
0.00	1.0000	1.0000	0.0000
0.10	1.1052	0.9048	0.0952
0.20	1.2214	0.8187	0.1813
0.50	1.6487	0.6065	0.3935
0.75	2.1170	0.4724	0.5276
0.90	2.4596	0.4066	0.5934
1.00	2.7183	0.3679	0.6321
1.10	3.0042	0.3329	0.6671
1.20	3.3201	0.3012	0.6988
1.25	3.4903	0.2865	0.7135
1.50	4.4817	0.2231	0.7769
1.75	5.755	0.1738	0.8262
2.00	7.389	0.1353	0.8647
2.50	12.182	0.0821	0.9179
3.00	20.086	0.0498	0.9502
3.50	33.115	0.0302	0.9698
4.00	54.598	0.0183	0.9817
4.50	90.017	0.0111	0.9889
5.00	148.41	0.0067	0.9933
5.50	244.69	0.0041	0.9959
6.00	403.43	0.00248	0.99752
7.50	1808.04	0.00055	0.99945
10.00	22026.5	0.000045	0.999955

As an example of the use of this table, suppose there is a hot body cooling, and that at the beginning of the experiment (*i.e.* when $t = 0$) it is 72° hotter than the surrounding objects, and if the time-constant of its cooling is 20 minutes (that is, if it takes 20 minutes for its excess of temperature to fall to $\frac{1}{e}$ part of 72°), then we can calculate to what it will have fallen in any given time t. For instance, let t be 60 minutes. Then $\frac{t}{T} = 60 \div 20 = 3$, and we shall have to find the value of e^{-3}, and then multiply the original 72° by this. The table shows that e^{-3} is 0.0498. So that at the end of 60 minutes the excess of temperature will have fallen to $72° \times 0.0498 = 3.586°$.

Further Examples.

(1) The strength of an electric current in a conductor at a time t secs. after the application of the electromotive force producing it is given by the expression $C = \frac{E}{R}\left\{1 - e^{-\frac{Rt}{L}}\right\}$.

The time constant is $\frac{L}{R}$.

If $E = 10$, $R = 1$, $L = 0.01$; then when t is very large the term $1 - e^{-\frac{Rt}{L}}$ becomes 1, and $C = \frac{E}{R} = 10$; also

$$\frac{L}{R} = T = 0.01$$

Its value at any time may be written:

$$C = 10 - 10e^{-\frac{t}{0.01}}$$

the time-constant being 0.01. This means that it takes 0.01 sec. for the variable term to fall to $\frac{1}{e} = 0.3679$ of its initial value $10e^{-\frac{0}{0.01}} = 10$.

To find the value of the current when $t = 0.001$ sec., say, $\frac{t}{T} = 0.1$, $e^{-0.1} = 0.9048$ (from table).

It follows that, after 0.001 sec., the variable term is

$$0.9048 \times 10 = 9.048$$

and the actual current is $10 - 9.048 = 0.952$.

Similarly, at the end of 0.1 sec.,

$$\frac{t}{T} = 10;\ e^{-10} = 0.000045$$

the variable term is $10 \times 0.000045 = 0.00045$, the current being 9.9995.

(2) The intensity I of a beam of light which has passed through a thickness l cm. of some transparent medium is $I = I_0 e^{-Kl}$, where I_0 is the initial intensity of the beam and K is a "constant of absorption".

This constant is usually found by experiments. If it be found, for instance, that a beam of light has its intensity diminished by 18% in passing through 10 cm. of a certain transparent medium, this means that $82 = 100 \times e^{-K \times 10}$ or $e^{-10K} = 0.82$, and from the table one sees that $10K = 0.20$ very nearly; hence $K = 0.02$.

To find the thickness that will reduce the intensity to half its value, one must find the value of l which satisfies the equality $50 = 100 \times e^{-0.02l}$, or $0.5 = e^{-0.02l}$. It is found by putting this equation in its natural logarithmic form, namely,

$$l = \frac{\ln 0.5}{-0.02} = 34.7 \text{ cm. nearly.}$$

(3) The quantity Q of a radio-active substance which has not yet undergone transformation is known to be related to the initial quantity Q_0 of the substance by the relation $Q = Q_0 e^{-\lambda t}$, where λ is a constant and t the time in seconds elapsed since the transformation began.

For "Radium A", if time is expressed in seconds, experiment shows that $\lambda = 3.85 \times 10^{-3}$. Find the time required for transforming half the substance. (This time is called the "half life" of the substance.)

We have $$0.5 = e^{-0.00385t}$$

$$\log_{10} 0.5 = -0.00385t \times \log_{10} e$$

and $$t = 3 \text{ minutes very nearly.}$$

EXERCISES XIII

(1) Draw the curve $y = be^{-\frac{t}{T}}$; where $b = 12$, $T = 8$, and t is given various values from 0 to 20.

(2) If a hot body cools so that in 24 minutes its excess of temperature has fallen to half the initial amount, deduce the time-constant, and find how long it will be in cooling down to 1 per-cent of the original excess.

(3) Plot the curve $y = 100(1 - e^{-2t})$.

(4) The following equations give very similar curves:

$$\text{(i) } y = \frac{ax}{x+b}; \quad \text{(ii) } y = a\left(1 - e^{-\frac{x}{b}}\right)$$

$$\text{(iii) } y = \frac{a}{90^\circ} \arctan\left(\frac{x}{b}\right)$$

Draw all three curves, taking $a = 100$ millimetres; $b = 30$ millimetres.

(5) Find the derivative of y with respect to x, if

$$(a)\ y = x^x; \quad (b)\ y = (e^x)^x; \quad (c)\ y = e^{x^x}$$

(6) For "Thorium A", the value of λ is 5; find the "half life", that is, the time taken by the transformation of a quantity Q of "Thorium A" equal to half the initial quantity Q_0 in the expression

$$Q = Q_0 e^{-\lambda t};$$

t being in seconds.

(7) A condenser of capacity $K = 4 \times 10^{-6}$, charged to a potential $V_0 = 20$, is discharging through a resistance of 10,000 ohms. Find the potential V after (a) 0.1 second; (b) 0.01 second; assuming that the fall of potential follows the rule $V = V_0 e^{-\frac{t}{KR}}$.

(8) The charge Q of an electrified insulated metal sphere is reduced from 20 to 16 units in 10 minutes. Find the coefficient μ of leakage, if $Q = Q_0 \times e^{-\mu t}$; Q_0 being the initial charge and t being in seconds. Hence find the time taken by half the charge to leak away.

(9) The damping on a telephone line can be ascertained from the relation $i = i_0 e^{-\beta l}$, where i is the strength, after t seconds, of a telephonic current of initial strength i_0; l is the length of the line in kilometres, and β is a constant. For the Franco-English submarine cable laid in 1910, $\beta = 0.0114$. Find the damping at the end of the cable (40 kilometres), and the length along which i is still 8% of the original current (limiting value of very good audition).

(10) The pressure p of the atmosphere at an altitude h kilometres is approximately $p = p_0\, e^{-kh}$ for some constant k; p_0 being the pressure at sea level (760 millimetres).

The pressures at 10, 20, and 50 kilometres being 199.2, 42.4, 0.32 millimetres respectively, find k in each case. Using the average of these three values of k, find the percentage error in the computed value of the pressure at the three heights.

(11) Find the minimum or maximum of $y = x^x$.

(12) Find the minimum or maximum of $y = x^{\frac{1}{x}}$.

(13) Find the minimum or maximum of $y = xa^{\frac{1}{x}}$, if $a > 1$.

Chapter XV

HOW TO DEAL WITH SINES AND COSINES

Greek letters being usual to denote angles, we will take as the usual letter for any variable angle the letter θ ("theta").

Let us consider the function

$$y = \sin\,\theta$$

What we have to investigate is the value of $\dfrac{d(\sin\,\theta)}{d\theta}$; or, in other words, if the angle θ varies, we have to find the relation between the increment of the sine and the increment of the angle, both increments being infinitely small in themselves. Examine Fig. 43, wherein, if the radius of the circle is unity, the height of y is the sine, and θ is the angle. Now, if θ is supposed to increase by the addition to it of the small angle $d\theta$—an element of angle—the height of y, the sine, will be increased by a small element dy. The new height $y + dy$ will be the sine of the new angle $\theta + d\theta$, or, stating it as an equation,

$$y + dy = \sin\,(\theta + d\theta)$$

and subtracting from this the first equation gives

$$dy = \sin\,(\theta + d\theta) - \sin\,\theta$$

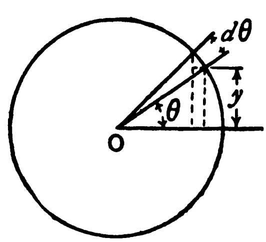


FIG. 43.

The quantity on the right-hand side is the difference between two sines, and books on trigonometry tell us how to work this out. For they tell us that if M and N are two different angles,

$$\sin M - \sin N = 2\cos\frac{M+N}{2}\cdot\sin\frac{M-N}{2}$$

If, then, we put $M=\theta+d\theta$ for one angle, and $N=\theta$ for the other, we may write

$$dy = 2\cos\frac{\theta+d\theta+\theta}{2}\cdot\sin\frac{\theta+d\theta-\theta}{2}$$

or, $$dy = 2\cos(\theta+\tfrac{1}{2}d\theta)\cdot\sin\tfrac{1}{2}d\theta$$

But if we regard $d\theta$ as infinitely small, then in the limit we may neglect $\frac{1}{2}d\theta$ by comparison with θ, and may also take $\sin\frac{1}{2}d\theta$ as being the same as $\frac{1}{2}d\theta$. The equation then becomes:

$$dy = 2\cos\theta\cdot\tfrac{1}{2}d\theta$$

$$dy = \cos\theta\cdot d\theta$$

and, finally, $$\frac{dy}{d\theta}=\cos\theta$$

The accompanying curves, Figs. 44 and 45, show, plotted to scale, the values of $y=\sin\theta$, and $\dfrac{dy}{d\theta}=\cos\theta$, for the corresponding values of θ.

Take next the cosine.

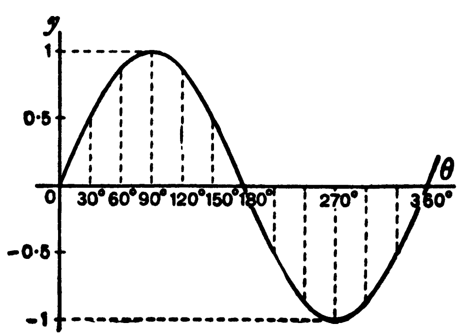


FIG. 44.

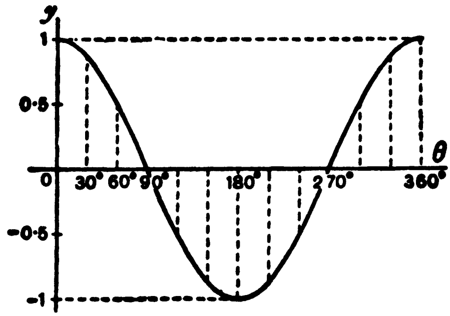


FIG. 45.

Let $\qquad y = \cos\theta$

Now $\quad \cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$

Therefore

$$dy = d\left(\sin\left(\frac{\pi}{2} - \theta\right)\right) = \cos\left(\frac{\pi}{2} - \theta\right) \times d(-\theta)$$

$$= \cos\left(\frac{\pi}{2} - \theta\right) \times (-d\theta)$$

$$\frac{dy}{d\theta} = -\cos\left(\frac{\pi}{2} - \theta\right)$$

And it follows that

$$\frac{dy}{d\theta} = -\sin\theta$$

Lastly, take the tangent.

Let
$$y = \tan\theta$$
$$= \frac{\sin\theta}{\cos\theta}$$

Applying the rule given in Chapter VI for differentiating a quotient of two functions, we get

$$\frac{dy}{d\theta} = \frac{\cos\theta \dfrac{d(\sin\theta)}{d\theta} - \sin\theta \dfrac{d(\cos\theta)}{d\theta}}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

or $$\frac{dy}{d\theta} = \sec^2\theta$$

Collecting these results, we have:

y	$\frac{dy}{d\theta}$
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$

Sometimes, in mechanical and physical questions, as, for example, in simple harmonic motion and in wave-motions, we have to deal with angles that increase in proportion to the time. Thus, if T be the time of one complete *period*, or movement round the circle, then, since the angle all round the circle is 2π radians, or $360°$, the amount of angle moved through in time t will be

$$\theta = 2\pi \frac{t}{T}, \text{ in radians}$$

or $$\theta = 360\frac{t}{T}, \text{ in degrees}$$

If the *frequency*, or number of periods per second, be denoted by n, then $n = \frac{1}{T}$, and we may then write:

$$\theta = 2\pi nt$$

Then we shall have $\qquad y = \sin 2\pi nt$

If, now, we wish to know how the sine varies with respect to time, we must differentiate with respect, not to θ, but to t. For this we must resort to the artifice explained in Chapter IX, and put

$$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$$

Now $\frac{d\theta}{dt}$ will obviously be $2\pi n$; so that

$$\frac{dy}{dt} = \cos\,\theta \times 2\pi n$$

$$= 2\pi n \cdot \cos\, 2\pi nt$$

Similarly, it follows that

$$\frac{d(\cos 2\pi nt)}{dt} = -2\pi n \cdot \sin\, 2\pi nt$$

Second Derivative of Sine or Cosine

We have seen that when $\sin\theta$ is differentiated with respect to θ it becomes $\cos\theta$; and that when $\cos\theta$ is differentiated with respect to θ it becomes $-\sin\theta$; or, in symbols,

$$\frac{d^2(\sin\theta)}{d\theta^2} = -\sin\theta$$

So we have this curious result that we have found a function such that if we differentiate it twice over, we get the same thing from which we started, but with the sign changed from + to −.

The same thing is true for the cosine; for differentiating $\cos\theta$ gives us $-\sin\theta$, and differentiating $-\sin\theta$ gives us $-\cos\theta$; or thus:

$$\frac{d^2(\cos\theta)}{d\theta^2} = -\cos\theta.$$

Sines and cosines furnish a basis for the only functions of which the second derivative is equal and of opposite sign to the original function.

Examples.

With what we have so far learned we can now differentiate expressions of a more complex nature.

(1) $y = \arcsin x$.

If y is the angle whose sine is x, then $x = \sin y$.

$$\frac{dx}{dy} = \cos y$$

Passing now from the inverse function to the original one, we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}$$

Now $$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

a rather unexpected result. Because by definition, $-\frac{\pi}{2} \leq \arcsin y \leq \frac{\pi}{2}$, we know that $\cos y$ is positive, so we use the positive square root here.

(2) $y = \cos^3 \theta$.

This is the same thing as $y = (\cos \theta)^3$

Let $v = \cos \theta$; then $y = v^3$; $\frac{dy}{dv} = 3v^2$

$$\frac{dv}{d\theta} = -\sin \theta$$

$$\frac{dy}{d\theta} = \frac{dy}{dv} \times \frac{dv}{d\theta} = -3 \cos^2\theta \sin \theta$$

(3) $y = \sin(x + a)$.
Let $v = x + a$; then $y = \sin v$.

$$\frac{dv}{dx} = 1; \quad \frac{dy}{dv} = \cos v \quad \text{and} \quad \frac{dy}{dx} = \cos(x + a)$$

(4) $y = log_e \sin\theta$
Let $v = \sin\theta$; $y = log_e v$.

$$\frac{dv}{d\theta} = \cos\theta; \quad \frac{dy}{dv} = \frac{1}{v}; \quad \frac{dy}{d\theta} = \frac{1}{\sin\theta} \times \cos\theta = \cot\theta.$$

(5) $y = \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$$\frac{dy}{d\theta} = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$
$$= -(1 + \cot^2\theta) = -\csc^2\theta$$

(6) $y = \tan 3\theta$

Let $v = 3\theta$; $y = \tan v$; $\dfrac{dv}{d\theta} = 3$; $\dfrac{dy}{dv} = \sec^2 v$

and
$$\frac{dy}{d\theta} = 3\sec^2 3\theta$$

(7) $y = \sqrt{1 + 3\tan^2\theta} = (1 + 3\tan^2\theta)^{\frac{1}{2}}$
Let $v = 3\tan^2\theta$, then $y = (1 + v)^{\frac{1}{2}}$;

$$\frac{dv}{d\theta} = 6\tan\theta\sec^2\theta; \quad \frac{dy}{dv} = \frac{1}{2\sqrt{1 + v}}$$

and
$$\frac{dy}{d\theta} = \frac{6\tan\theta\sec^2\theta}{2\sqrt{1 + v}} = \frac{6\tan\theta\sec^2\theta}{2\sqrt{1 + 3\tan^2\theta}}$$

for, if $u = \tan\theta$, $v = 3u^2$

$$\frac{du}{d\theta} = \sec^2\theta; \quad \frac{dv}{du} = 6u$$

hence $$\frac{dv}{d\theta} = 6 \tan \theta \sec^2\theta$$

hence $$\frac{dy}{d\theta} = \frac{6 \tan \theta \sec^2\theta}{2\sqrt{1 + 3 \tan^2\theta}}$$

(8) $y = \sin x \cos x$.

$$\frac{dy}{dx} = \sin x(-\sin x) + \cos x \times \cos x$$

$$= \cos^2 x - \sin^2 x$$

Closely associated with *sines, cosines* and *tangents* are three other very useful functions. They are the hyperbolic sine, cosine and tangent, and are written *sinh, cosh, tanh.* These functions are defined as follows:

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),\ \cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Between $\sinh x$ and $\cosh x$, there is an important relation, for

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}(e^x + e^{-x})^2 - \tfrac{1}{4}(e^x - e^{-x})^2$$
$$= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) = 1.$$

Now $$\frac{d}{dx}(\sinh x) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

$$\frac{d}{dx}(\cosh x) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

$$\frac{d}{dx}(\tanh x) = \frac{\cosh x \dfrac{d}{dx}(\sinh x) - \sinh x \dfrac{d}{dx}(\cosh x)}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

by the relation just proved.

EXERCISES XIV

(1) Differentiate the following:

$$\text{(i) } y = A \sin\left(\theta - \frac{\pi}{2}\right)$$

$$\text{(ii) } y = \sin^2\theta;\text{ and } y = \sin 2\theta$$

$$\text{(iii) } y = \sin^3\theta;\text{ and } y = \sin 3\theta$$

(2) Find the value of θ ($0 \leq \theta \leq 2\pi$) for which $\sin\theta \times \cos\theta$ is a maximum.

(3) Differentiate $y = \dfrac{1}{2\pi}\cos 2\pi nt$.

(4) If $y = \sin a^x$, find $\dfrac{dy}{dx}$. (5) Differentiate $y = \ln\cos x$.

(6) Differentiate $y = 18.2\sin(x + 26^\circ)$.

(7) Plot the curve $y = 100\sin(\theta - 15^\circ)$; and show that the slope of the curve at $\theta = 75^\circ$ is half the maximum slope.

(8) If $y = \sin\theta \cdot \sin 2\theta$, find $\dfrac{dy}{d\theta}$.

(9) If $y = a \cdot \tan^m(\theta^n)$, find the derivative of y with respect to θ.

(10) Differentiate $y = e^x \sin^2 x$.

(11) Differentiate the three equations of Exercises XIII, No. 4, and compare their derivatives, as to whether they are equal, or nearly equal, for very small values of x, or for very large values of x, or for values of x in the neighbourhood of $x = 30$.

(12) Differentiate the following:

(i) $y = \sec x$. (ii) $y = \arccos x$.

(iii) $y = \arctan x$. (iv) $y = \operatorname{arcsec} x$.

(v) $y = \tan x \times \sqrt{3\sec x}$.

(13) Differentiate $y = \sin(2\theta + 3)^{2.3}$.

(14) Differentiate $y = \theta^3 + 3\sin(\theta + 3) - 3^{\sin\theta} - 3^\theta$.

(15) Find the maximum or minimum of $y = \theta\cos\theta$, for $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$.

Chapter XVI

PARTIAL DIFFERENTIATION

We sometimes come across quantities that are functions of more than one independent variable. Thus, we may find a case where y depends on two other variable quantities, one of which we will call u and the other v. In symbols

$$y = f(u, v)$$

Take the simplest concrete case.

Let $$y = u \times v$$

What are we to do? If we were to treat v as a constant, and differentiate with respect to u, we should get

$$dy_v = v\,du$$

or if we treat u as a constant, and differentiate with respect to v, we should have:

$$dy_u = u\,dv$$

The little letters here put as subscripts are to show which quantity has been taken as constant in the operation.

Another way of indicating that the differentiation has been performed only *partially*, that is, has been performed only with-respect to *one* of the independent variables, is to write the derivatives with a symbol based on Greek small deltas, instead of little *d*.[1] In this way

$$\frac{\partial y}{\partial u} = v$$

$$\frac{\partial y}{\partial v} = u$$

If we put in these values for v and u respectively, we shall have

$$\left.\begin{aligned} dy_v &= \frac{\partial y}{\partial u}du, \\ dy_u &= \frac{\partial y}{\partial v}dv, \end{aligned}\right\} \text{ which are } \textit{partial derivatives.}$$

But, if you think of it, you will observe that the total variation of y depends on *both* these things at the same time. That is to say, if both are varying, the real dy ought to be written

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

and this is called a *total differential.* In some books it is written $dy = \left(\dfrac{dy}{du}\right)du + \left(\dfrac{dy}{dv}\right)dv.$

1. Robert Ainsley, in his amusing booklet *Bluff Your Way in Mathematics* (1988) defines partial derivatives as "derivatives biased toward x, y, or z instead of treating all three equally—the sign for this is six written backwards . . . ."

The adjective "partial" indicates that the derivative is partial toward one independent variable, the other or others being treated as constants. A derivative is called a "mixed partial derivative" if it is a partial derivative of order 2 or higher that involves more than one of the independent variables. Higher partial derivatives are partial derivatives of partial derivatives.—M.G.

Example (1). Find the partial derivatives of the expression $w = 2ax^2 + 3bxy + 4cy^3$. The answers are:

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= 4ax + 3by \\ \frac{\partial w}{\partial y} &= 3bx + 12cy^2 \end{aligned}\right\}$$

The first is obtained by supposing y constant, the second is obtained by supposing x constant; then the total differential is

$$dw = (4ax + 3by)dx + (3bx + 12cy^2)dy$$

Example (2). Let $z = x^y$. Then, treating first y and then x as constant, we get in the usual way

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= yx^{y-1} \\ \frac{\partial z}{\partial y} &= x^y \times \log_e x \end{aligned}\right\}$$

so that $dz = yx^{y-1}dx + x^y \log_e x dy$.

Example (3). A cone having height h and radius of base r, has volume $V = \frac{1}{3}\pi r^2 h$. If its height remains constant, while r changes, the ratio of change of volume, with respect to radius, is different from ratio of change of volume with respect to height which would occur if the height were varied and the radius kept constant, for

$$\left.\begin{aligned} \frac{\partial V}{\partial r} &= \frac{2\pi}{3} rh \\ \frac{\partial V}{\partial h} &= \frac{\pi}{3} r^2 \end{aligned}\right\}$$

The variation when both the radius and the height change is given by $dV = \dfrac{2\pi}{3} rh\, dr + \dfrac{\pi}{3} r^2 dh$.

Example (4). In the following example F and f denote two arbitrary functions of any form whatsoever. For example, they may be sine-functions, or exponentials, or mere algebraic functions of the two independent variables, t and x. This being understood, let us take the expression

$$y = F(x + at) + f(x - at)$$

or,

$$y = F(w) + f(v)$$

where

$$w = x + at, \quad \text{and} \quad v = x - at$$

Then

$$\frac{\partial y}{\partial x} = \frac{\partial F(w)}{\partial w} \cdot \frac{\partial w}{\partial x} + \frac{\partial f(v)}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= F'(w) \cdot 1 + f'(v) \cdot 1$$

(where the figure 1 is simply the coefficient of x in w and v);

and

$$\frac{\partial^2 y}{\partial x^2} = F''(w) + f''(v)$$

Also

$$\frac{\partial y}{\partial t} = \frac{\partial F(w)}{\partial w} \cdot \frac{\partial w}{\partial t} + \frac{\partial f(v)}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$= F'(w) \cdot a - f'(v)a$$

and

$$\frac{\partial^2 y}{\partial t^2} = F''(w)a^2 + f''(v)a^2$$

whence

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This differential equation is of immense importance in mathematical physics.

Maxima and Minima of Functions of two Independent Variables

Example (5). Let us take up again Exercises IX, No. 4.

Let x and y be the lengths of two of the portions of the string. The third is $30 - (x + y)$, and the area of the triangle is $A = \sqrt{s(s - x)(s - y)(s - 30 + x + y)}$, where s is the half perimeter, so that $s = 15$, and $A = \sqrt{15P}$, where

$$P = (15 - x)(15 - y)(x + y - 15)$$
$$= xy^2 + x^2y - 15x^2 - 15y^2 - 45xy + 450x + 450y - 3375$$

Clearly A is a maximum when P is maximum.

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

For a maximum (clearly it will not be a minimum in this case), one must have simultaneously

$$\frac{\partial P}{\partial x} = 0 \quad \text{and} \quad \frac{\partial P}{\partial y} = 0$$

that is,
$$\left.\begin{aligned} 2xy - 30x + y^2 - 45y + 450 = 0, \\ 2xy - 30y + x^2 - 45x + 450 = 0. \end{aligned}\right\}$$

Subtracting the second equation from the first and factoring gives

$$(y - x)(x + y - 15) = 0$$

so either $x = y$ or $x + y - 15 = 0$. In the latter case $P = 0$, which is not a maximum, hence $x = y$.

If we now introduce this condition in the value of P, we find

$$P = (15 - x)^2(2x - 15) = 2x^3 - 75x^2 + 900x - 3375$$

For maximum or minimum, $\frac{dP}{dx} = 6x^2 - 150x + 900 = 0$, which gives $x = 15$ or $x = 10$.

Clearly $x = 15$ gives zero area; $x = 10$ gives the maximum, for $\frac{d^2P}{dx^2} = 12x - 150$, which is $+30$ for $x = 15$ and -30 for $x = 10$.

Example (6). Find the dimensions of an ordinary railway coal truck with rectangular ends, so that, for a given volume V, the area of sides and floor together is as small as possible.

The truck is a rectangular box open at the top. Let x be the length and y be the width; then the depth is $\frac{V}{xy}$. The surface area is $S = xy + \frac{2V}{x} + \frac{2V}{y}$.

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy = \left(y - \frac{2V}{x^2}\right)dx + \left(x - \frac{2V}{y^2}\right)dy$$

For minimum (clearly it won't be a maximum here),

$$y - \frac{2V}{x^2} = 0 \quad x - \frac{2V}{y^2} = 0$$

Multiplying the first equation by x, the second by y, and subtracting gives $x = y$. So $x^3 = 2V$ and $x = y = \sqrt[3]{2V}$.

EXERCISES XV

(1) Differentiate the expression $\frac{x^3}{3} - 2x^3y - 2y^2x + \frac{y}{3}$ with respect to x alone, and with respect to y alone.

(2) Find the partial derivatives with respect to x, y, and z, of the expression

$$x^2yz + xy^2z + xyz^2 + x^2y^2z^2$$

(3) Let $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

Find the value of $\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}$. Also find the value of $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}$.

(4) Find the total derivative of $y = u^v$.

(5) Find the total derivative of $y = u^3 \sin v$; of $y = (\sin x)^u$; and of $y = \frac{\ln u}{v}$.

(6) Verify that the sum of three quantities x, y, z, whose product is a constant k, is minimum when these three quantities are equal.

(7) Does the function $u = x + 2xy + y$ have a maximum or minimum?

(8) A post office regulation once stated that no parcel is to be of such a size that its length plus its girth exceeds 6 feet. What is the greatest volume that can be sent by post (*a*) in the case of a package of rectangular cross-section; (*b*) in the case of a package of circular cross-section?

(9) Divide π into 3 parts such that the product of their sines may be a maximum or minimum.

(10) Find the maximum or minimum of $u = \dfrac{e^{x+y}}{xy}$.

(11) Find maximum and minimum of

$$u = y + 2x - 2 \ln y - \ln x$$

(12) A bucket of given capacity has the shape of a horizontal isosceles triangular prism with the apex underneath, and the opposite face open. Find its dimensions in order that the least amount of iron sheet may be used in its construction.

Chapter XVII

INTEGRATION

The great secret has already been revealed that this mysterious symbol $\int$, which is after all only a long S, merely means "the sum of", or "the sum of all such quantities as". It therefore resembles that other symbol Σ (the Greek *Sigma*), which is also a sign of summation. There is this difference, however, in the practice of mathematical men as to the use of these signs, that while Σ is generally used to indicate the sum of a number of finite quantities, the integral sign $\int$ is generally used to indicate the summing up of a vast number of small quantities of infinitely minute magnitude, mere elements in fact, that go to make up the total required. Thus $\int dy = y$, and $\int dx = x$.

Any one can understand how the whole of anything can be conceived of as made up of a lot of little bits; and the smaller the bits the more of them there will be. Thus, a line one inch long may be conceived as made up of 10 pieces, each $\frac{1}{10}$ of an inch long; or of 100 parts, each part being $\frac{1}{100}$ of an inch long; or of 1,000,000 parts, each of which is $\frac{1}{1,000,000}$ of an inch long; or, pushing the thought to the limits of conceivability, it may be regarded as made up of an infinite number of elements each of which is infinitesimally small.

Yes, you will say, but what is the use of thinking of anything that way? Why not think of it straight off, as a whole? The simple reason is that there are a vast number of cases in which one cannot calculate the bigness of the thing as a whole without reckoning up the sum of a lot of small parts. The process of *"integrating"* is to enable us to calculate totals that otherwise we should be unable to estimate directly.

Let us first take one or two simple cases to familiarize ourselves with this notion of summing up a lot of separate parts.

Consider the series:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \dots.$$

Here each member of the series is formed by taking half the value of the preceding one. What is the value of the total if we could go on to an infinite number of terms? Every schoolboy knows that the answer is 2. Think of it, if you like, as a line. Begin with one inch; add a half inch; add a quarter; add an eighth; and so on. If at any point of the operation we stop, there will still be a piece wanting to make up the whole 2 inches; and the piece wanting will always be the same size as the last piece added. Thus, if after having put together 1, $\frac{1}{2}$, and $\frac{1}{4}$, we stop, there will be $\frac{1}{4}$ wanting. If we go on till we have added $\frac{1}{64}$, there will still be $\frac{1}{64}$ wanting. The remainder needed will always be equal to the last term added. By an infinite number of operations only should we reach the actual 2 inches. Practically we should reach it when we got to pieces so small that they could not be drawn—that would be after about 10 terms, for the eleventh term is $\frac{1}{1024}$. If we want to go so far that no measuring machine could detect it, we should merely have to go to about 20 terms. A microscope would not show even the 18th term! So the infinite number of operations is no such dreadful thing after all. The *integral* is simply the whole lot. But, as we shall see, there are cases in which the integral calculus enables us to get at the *exact* total that there would be as the result of an infinite number of operations. In such cases the integral calculus gives us a *rapid* and easy way of getting at a result

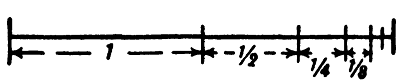


FIG. 46.

that would otherwise require an interminable lot of elaborate working out. So we had best lose no time in learning *how to integrate.*

Slopes of Curves, and the Curves Themselves

Let us make a little preliminary enquiry about the slopes of curves. For we have seen that differentiating a curve means finding an expression for its slope (or for its slopes at different points). Can we perform the reverse process of reconstructing the whole curve if the slope (or slopes) are prescribed for us?

Go back to Chapter 10, case (2). Here we have the simplest of curves, a sloping line with the equation

$$y = ax + b$$

We know that here b represents the initial height of y when $x = 0$, and that a, which is the same as $\frac{dy}{dx}$, is the "slope" of the line. The line has a constant slope. All along it the elementary

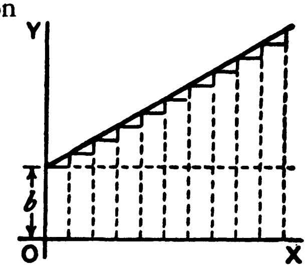
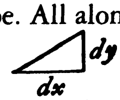
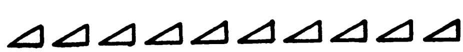


FIG. 47.

triangles  have the same proportion between height and base. Suppose we were to take the dx's and dy's of finite magnitude, so that 10 dx's made up one inch, then there would be ten little triangles like



Now, suppose that we were ordered to reconstruct the "curve", starting merely from the information that $\frac{dy}{dx} = a$. What could we do? Still taking the little d's as of finite size, we could draw 10 of them, all with the same slope, and then put them together, end to end, like this:

And, as the slope is the same for all, they would join to make, as in Fig. 48, a sloping line sloping with the correct slope

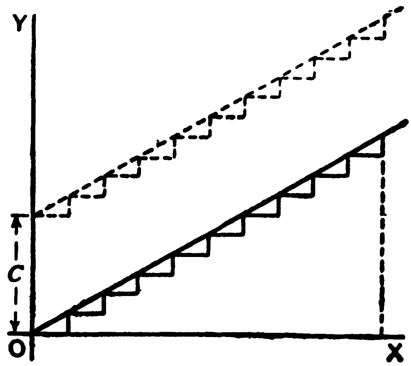


FIG. 48.

$\frac{dy}{dx} = a$. And whether we take the dy's and dx's as finite or infinitely small, as they are all alike, clearly $\frac{y}{x} = a$, if we reckon y as the total of all the dy's, and x as the total of all the dx's. But whereabouts are we to put this sloping line? Are we to start at the origin O, or higher up? As the only information we have is as to the slope, we are without any instructions as to the particular height above O; in fact the initial height is undetermined. The slope will be the same, whatever the initial height. Let us therefore make a shot at what may be wanted, and start the sloping line at a height C above O. That is, we have the equation

$$y = ax + C$$

It becomes evident now that in this case the added constant means the particular value that y has when $x = 0$.

Now let us take a harder case, that of a line, the slope of which is not constant, but turns up more and more. Let us assume that the upward slope gets greater and greater in proportion as x grows. In symbols this is:

$$\frac{dy}{dx} = ax$$

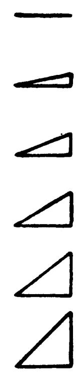
Or, to give a concrete case, take $a = \frac{1}{5}$, so that

$$\frac{dy}{dx} = \tfrac{1}{5}x$$

Then we had best begin by calculating a few of the values of the slope at different values of x, and also draw little diagrams of them.

When

$$x=0, \quad \frac{dy}{dx}=0,$$
$$x=1, \quad \frac{dy}{dx}=0{\cdot}2,$$
$$x=2, \quad \frac{dy}{dx}=0{\cdot}4,$$
$$x=3, \quad \frac{dy}{dx}=0{\cdot}6,$$
$$x=4, \quad \frac{dy}{dx}=0{\cdot}8,$$
$$x=5, \quad \frac{dy}{dx}=1{\cdot}0.$$



Now try to put the pieces together, setting each so that the middle of its base is the proper distance to the right, and so that they fit together at the corners; thus (Fig. 49). The result is, of course, not a smooth curve: but it is an approximation to one. If we had taken bits half as long, and twice as numerous, like Fig. 50, we should have a better approximation.[1]

1. Approximating a continuous curve by drawing smaller and smaller right triangles under the curve is today called the "trapezoidal rule" because the little triangles, joined to the narrow rectangles beneath them, form trapezoids as shown in Figure 47.

A closer approximation, though much more difficult to apply, is to draw tiny parabolas below (or above) a curve's segments. The sum is then approximated by the "parabolic rule" or what is also known as "Simpson's rule" after British mathematician Thomas Simpson (1710-1761). Thompson does not go into this, but you can read about Simpson's rule in modern calculus textbooks.—M.G.

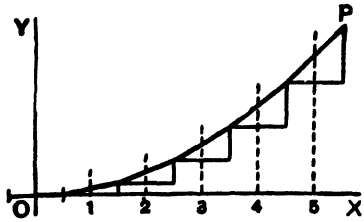


FIG. 49.

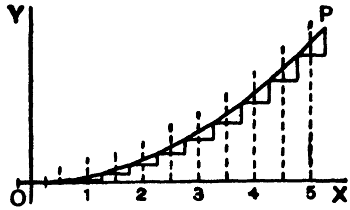


FIG. 50.

But for a perfect curve we ought to take each dx and its corresponding dy infinitesimally small, and infinitely numerous.

Then, how much ought the value of any y to be? Clearly, at any point P of the curve, the value of y will be the sum of all the little dy's from 0 up to that level, that is to say, $\int dy = y$. And as each dy is equal to $\frac{1}{5}x \cdot dx$, it follows that the whole y will be equal to the sum of all such bits as $\frac{1}{5}x \cdot dx$, or, as we should write it, $\int \frac{1}{5}x \cdot dx$.

Now if x had been constant, $\int \frac{1}{5}x \cdot dx$ would have been the same as $\frac{1}{5}x \int dx$, or $\frac{1}{5}x^2$. But x began by being 0, and increases to the particular value of x at the point P, so that its average value from 0 to that point is $\frac{1}{2}x$. Hence $\int \frac{1}{5}x dx = \frac{1}{10}x^2$; or $y = \frac{1}{10}x^2$.

But, as in the previous case, this requires the addition of an undetermined constant C, because we have not been told at what height above the origin the curve will begin, when $x = 0$. So we write, as the equation of the curve drawn in Fig. 51,

$$y = \tfrac{1}{10}x^2 + C$$

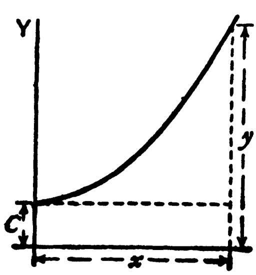


FIG. 51.

EXERCISES XVI

(1) Find the ultimate sum of $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\ldots.$

(2) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\ldots$, is convergent, and find its sum to 8 terms.

(3) If $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots$, find ln 1.3.

(4) Following a reasoning similar to that explained in this chapter, find y,

$$\text{if } \frac{dy}{dx}=\tfrac{1}{4}x$$

(5) If $\dfrac{dy}{dx}=2x+3$, find y

Chapter XVIII

INTEGRATING AS THE REVERSE OF DIFFERENTIATING

Differentiating is the process by which when y is given us as a function of x, we can find $\frac{dy}{dx}$.

Like every other mathematical operation, the process of differentiation may be reversed.[1] Thus, if differentiating $y = x^4$ gives us $\frac{dy}{dx} = 4x^3$, then, if one begins with $\frac{dy}{dx} = 4x^3$, one would say that reversing the process would yield $y = x^4$. But here comes in a curious point. We should get $\frac{dy}{dx} = 4x^3$ if we had begun with *any* of the following: x^4, or $x^4 + a$, or $x^4 + c$, or x^4 with *any* added constant. So it is clear that in working backwards from $\frac{dy}{dx}$ to y,

1. Familiar examples of inverse operations in arithmetic are subtraction as the reverse of addition, division as the reverse of multiplication, and root extraction as the reverse of raising to higher powers.

In arithmetic you can test a subtraction $A - B = C$ by adding B and C to see if you get A. In similar fashion one can test an integration or a differentiation by reversing the process to see if you return to the original expression.

In its geometrical model, differentiating a function gives a formula that determines the slope of a function's curve at any given point. Integration provides a method with which, given the formula for the slope you can determine the curve and its function. This in turn provides a quick way of calculating the area between intervals on a curve and the graph's x axis.

one must make provision for the possibility of there being an added constant, the value of which will be undetermined until ascertained in some other way. So, if differentiating x^n yields nx^{n-1}, going backwards from $\frac{dy}{dx} = nx^{n-1}$ will give us $y = x^n + C$; where C stands for the yet undetermined possible constant.

Clearly, in dealing with powers of x, the rule for working backwards will be: Increase the power by 1, then divide by that increased power, and add the undetermined constant.

So, in the case where

$$\frac{dy}{dx} = x^n$$

working backwards, we get

$$y = \frac{1}{n+1} x^{n+1} + C$$

If differentiating the equation $y = ax^n$ gives us

$$\frac{dy}{dx} = anx^{n-1}$$

it is a matter of common sense that beginning with

$$\frac{dy}{dx} = anx^{n-1}$$

and reversing the process, will give us

$$y = ax^n$$

So, when we are dealing with a multiplying constant, we must simply put the constant as a multiplier of the result of the integration.

Thus, if $\frac{dy}{dx} = 4x^2$, the reverse process gives us $y = \frac{4}{3}x^3$.

But this is incomplete. For we must remember that if we had started with

$$y = ax^n + C$$

where C is any constant quantity whatever, we should equally have found

$$\frac{dy}{dx} = anx^{n-1}$$

So, therefore, when we reverse the process we must always remember to add on this undetermined constant, even if we do not yet know what its value will be.[2]

This process, the reverse of differentiating, is called *integrating*; for it consists in finding the value of the whole quantity y when you are given only an expression for dy or for $\frac{dy}{dx}$. Hitherto we have as much as possible kept dy and dx together as a derivative: henceforth we shall more often have to separate them.

If we begin with a simple case,

$$\frac{dy}{dx} = x^2$$

We may write this, if we like, as

$$dy = x^2 dx$$

2. A joke about integration recently made the rounds of math students and teachers. Two students at a technical college, Bill and Joe, are having lunch at a campus hangout. Bill complains that math teaching has become so poor in the United States that most college students know next to nothing about calculus.

Joe disagrees. While Bill is in the men's room, Joe calls over a pretty blond waitress. He gives her five dollars to play a joke on Bill. When she brings the dessert he will ask her a question. He doesn't tell her the question, but he instructs her to answer "one third x cubed." The waitress smiles, pockets the fiver, and agrees.

When Bill returns to the booth, Joe proposes the following twenty-dollar bet. He will ask their waitress about an integral. If she responds correctly he wins the bet. Joe knows he can't lose. The two friends shake hands on the deal.

When the waitress comes to the table, Joe asks, "What's the integral of x squared?"

"One third x cubed," she replies. Then as she walks away she says over her shoulder, "plus a constant."

Now this is a "differential equation" which informs us that an element of y is equal to the corresponding element of x multiplied by x^2. Now, what we want is the integral; therefore, write down with the proper symbol the instructions to integrate both sides, thus:

$$\int dy = \int x^2 dx$$

[Note as to reading integrals: the above would be read thus:

"*Integral of dee-wy* equals *integral of eks-squared dee-eks.*"]

We haven't yet integrated: we have only written down instructions to integrate—if we can. Let us try. Plenty of other fools can do it—why not we also? The left-hand side is simplicity itself. The sum of all the bits of y is the same thing as y itself. So we may at once put:

$$y = \int x^2 dx$$

But when we come to the right-hand side of the equation we must remember that what we have got to sum up together is not all the dx's, but all such terms as $x^2 dx$; and this will *not* be the same as $x^2 \int dx$, because x^2 is not a constant. For some of the dx's will be multiplied by big values of x^2, and some will be multiplied by small values of x^2, according to what x happens to be. So we must bethink ourselves as to what we know about this process of integration being the reverse of differentiation. Now, our rule for this reversed process when dealing with x^n is "increase the power by one, and divide by the same number as this increased power". That is to say, $x^2 dx$ will be changed* to $\frac{1}{3}x^3$. Put this into

*You may ask: what has become of the little dx at the end? Well, remember that it was really part of the derivative, and when changed over to the right-hand side, as in the $x^2 dx$, serves as a reminder that x is the independent variable with respect to which the operation is to be effected; and, as the result of the product being totalled up, the power of x has increased by *one*. You will soon become familiar with all this.

the equation; but don't forget to add the "constant of integration" C at the end. So we get:

$$y = \tfrac{1}{3}x^3 + C$$

You have actually performed the integration. How easy!

Let us try another simple case

Let $$\frac{dy}{dx} = ax^{12}$$

where a is any constant multiplier. Well, we found when differentiating (see Chapter V) that any constant factor in the value of y reappeared unchanged in the value of $\dfrac{dy}{dx}$. In the reversed process of integrating, it will therefore also reappear in the value of y. So we may go to work as before, thus:

$$dy = ax^{12} \cdot dx$$

$$\int dy = \int ax^{12} \cdot dx$$

$$\int dy = a\int x^{12} dx$$

$$y = a \times \tfrac{1}{13}x^{13} + C$$

So that is done. How easy!

We begin to realize now that integrating is a process *of finding our way back,* as compared with differentiating. If ever, during differentiating, we have found any particular expression—in this example ax^{12}—we can find our way back to the y from which it was derived. The contrast between the two processes may be illustrated by the following illustration due to a well-known teacher. If a stranger to Manhattan were set down in Times Square, and told to find his way to Grand Central Station, he might find the task hopeless. But if he had previously been personally conducted from Grand Central Station to Times Square, it would be comparatively easy for him to find his way back to Grand Central Station.

Integration of the Sum or Difference of Two Functions

Let $$\frac{dy}{dx} = x^2 + x^3$$

then $$dy = x^2dx + x^3dx$$

There is no reason why we should not integrate each term separately; for, as may be seen in Chapter VI, we found that when we differentiated the sum of two separate functions, the derivative was simply the sum of the two separate differentiations. So, when we work backwards, integrating, the integration will be simply the sum of the two separate integrations.

Our instructions will then be:

$$\int dy = \int (x^2 + x^3)dx$$

$$= \int x^2dx + \int x^3dx$$

$$y = \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + C$$

If either of the terms had been a negative quantity, the corresponding term in the integral would have also been negative. So that differences are as readily dealt with as sums.

How to Deal with Constant Terms

Suppose there is in the expression to be integrated a constant term—such as this:

$$\frac{dy}{dx} = x^n + b$$

This is laughably easy. For you have only to remember that when you differentiated the expression $y = ax$, the result was $\frac{dy}{dx} = a$. Hence, when you work the other way and integrate, the constant reappears multiplied by x. So we get

$$dy = x^n dx + b \cdot dx$$

$$\int dy = \int x^n dx + \int b\, dx$$

$$y = \frac{1}{n+1} x^{n+1} + bx + C$$

Here are a lot of examples on which to try your newly acquired powers.

Examples.

(1) Given $\frac{dy}{dx} = 24x^{11}$. Find y. *Ans.* $y = 2x^{12} + C$

(2) Find $\int (a+b)(x+1)dx$. It is $(a+b)\int (x+1)dx$

or $\quad (a+b)\left[\int x\, dx + \int dx\right]$ or $(a+b)\left(\frac{x^2}{2} + x\right) + C$

(3) Given $\frac{du}{dt} = gt^{\frac{1}{2}}$ Find u. *Ans.* $u = \frac{2}{3}gt^{\frac{3}{2}} + C$

(4) $\frac{dy}{dx} = x^3 - x^2 + x$ Find y.

$dy = (x^3 - x^2 + x)\, dx$

or $\quad dy = x^3 dx - x^2 dx + x\, dx$; $y = \int x^b dx - \int x^2 dx + \int x\, dx$

and $\quad y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

(5) Integrate $9.75x^{2.25}dx$. *Ans.* $y = 3x^{3.25} + C$

All these are easy enough. Let us try another case.

Let $$\frac{dy}{dx} = ax^{-1}$$

Proceeding as before, we will write

$$dy = ax^{-1} \cdot dx, \qquad \int dy = a \int x^{-1} dx$$

Well, but what is the integral of $x^{-1}dx$?

If you look back amongst the results of differentiating x^2 and x^3 and x^n, etc., you will find we never got x^{-1} from any one of them as the value of $\frac{dy}{dx}$. We got $3x^2$ from x^3; we got $2x$ from x^2; we got 1 from x^1 (that is, from x itself); but we did not get x^{-1} from x^0, for a very good reason. Its derivative (got by slavishly following the usual rule) is $0 \times x^{-1}$, and that multiplication by zero gives it zero value! Therefore when we now come to try to integrate $x^{-1}dx$, we see that it does not come in anywhere in the powers of x that are given by the rule:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

It is an exceptional case.

Well; but try again. Look through all the various derivatives obtained from various functions of x, and try to find amongst them x^{-1}. A sufficient search will show that we actually did get $\frac{dy}{dx} = x^{-1}$ as the result of differentiating the function $y = \ln x$.

Then, of course, since we know that differentiating $\ln x$ gives us x^{-1}, we know that, by reversing the process, integrating $dy = x^{-1}dx$ will give us $y = \ln x$. But we must not forget the constant factor a that was given, nor must we omit to add the undetermined constant of integration. This then gives us as the solution to the present problem,

$$y = a \ln x + C$$

But this is valid only for $x > 0$. For $x < 0$, you should verify that the solution is

$$y = a \ln(-x) + C$$

These two cases are usually combined by writing

$$y = a \ln |x| + C$$

where $|x|$ is the absolute value of x, namely x if $x \geq 0$ and $-x$ if $x < 0$.

N.B.—Here note this very remarkable fact, that we could not have integrated in the above case if we had not happened to know the corresponding differentiation. If no one had found out that differentiating $\ln x$ gave x^{-1}, we should have been utterly stuck by the problem how to integrate $x^{-1}dx$. Indeed it should be frankly admitted that this is one of the curious features of the integral calculus:—that you can't integrate anything before the reverse process of differentiating something else has yielded that expression which you want to integrate.

Another Simple Case.

Find $\int (x+1)(x+2)dx$.

On looking at the function to be integrated, you remark that it is the product of two different functions of x. You could, you think, integrate $(x+1)dx$ by itself, or $(x+2)dx$ by itself. Of course you could. But what to do with a product? None of the differentiations you have learned have yielded you for the derivative a product like this. Failing such, the simplest thing is to multiply up the two functions, and then integrate. This gives us

$$\int (x^2 + 3x + 2)dx$$

And this is the same as

$$\int x^2 dx + \int 3x\,dx + \int 2\,dx$$

And performing the integrations, we get

$$\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x + C$$

Some Other Integrals

Now that we know that integration is the reverse of differentiation, we may at once look up the derivatives we already know, and see from what functions they were derived. This gives us the following integrals ready made:

$$x^{-1} \qquad \int x^{-1}dx \qquad = \ln\,|x| + C$$

$$\frac{1}{x+a} \qquad \int \frac{1}{x+a}dx \qquad = \ln\,|x+a| + C$$

$$e^x \qquad \int e^x dx \qquad = e^x + \mathrm{C}$$

$$e^{-x} \qquad \int e^{-x}dx \qquad = -e^{-x} + C$$

for if $y = \dfrac{-1}{e^x}$, $\dfrac{dy}{dx} = -\dfrac{e^x \times 0 - 1 \times e^x}{e^{2x}} = e^{-x}$

$$\sin x; \qquad \int \sin x\, dx \qquad = -\cos x + C$$

$$\cos x; \qquad \int \cos x\, dx \qquad = \sin x + C$$

Also we may deduce the following:

$$\ln x \qquad \int \ln x\, dx \qquad = x(\ln x - 1) + C$$

for if $y = x \ln x - x$, $\dfrac{dy}{dx} = \dfrac{x}{x} + \ln x - 1 = \ln x$

$$\log_{10} x \int \log_{10} \times\, dx = 0.4343x(\ln x - 1) + C$$

$$a^x \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\cos ax \qquad \int \cos ax\, dx = \frac{1}{a} \sin ax + C$$

for if $y = \sin ax$, $\dfrac{dy}{dx} = a \cos ax$; hence to get $\cos ax$ one must differentiate $y = \dfrac{1}{a} \sin ax$

$$\sin ax \qquad \int \sin ax\, dx = -\frac{1}{a} \cos ax + C$$

Try also $\cos^2 \theta$; a little dodge will simplify matters:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$$

hence $$\cos^2 \theta = \tfrac{1}{2}(\cos 2\theta + 1)$$

and $$\int \cos^2 \theta\, d\theta = \tfrac{1}{2} \int (\cos 2\theta + 1) d\theta$$

$$= \tfrac{1}{2} \int \cos 2\theta\, d\theta + \tfrac{1}{2} \int d\theta$$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$$

See also the Table of Standard Forms at the end of the last chapter. You should make such a table for yourself, putting in it only the general functions which you have successfully differentiated and integrated. See to it that it grows steadily!

EXERCISES XVII

(1) Find $\int y\,dx$ when $y^2 = 4ax$.

(2) Find $\int \frac{3}{x^4}\,dx$.

(3) Find $\int \frac{1}{a}\,x^3\,dx$.

(4) Find $\int (x^2 + a)dx$.

(5) Integrate $5x^{-\frac{7}{2}}$.

(6) Find $\int (4x^3 + 3x^2 + 2x + 1)dx$.

(7) If $\frac{dy}{dx} = \frac{ax}{2} + \frac{bx^2}{3} + \frac{cx^3}{4}$; find y.

(8) Find $\int \left(\frac{x^2 + a}{x + a}\right) dx$.

(9) Find $\int (x + 3)^3 dx$.

(10) Find $\int (x + 2)(x - a)dx$.

(11) Find $\int (\sqrt{x} + \sqrt[3]{x})3a^2 dx$.

(12) Find $\int (\sin\theta - \tfrac{1}{2})\frac{d\theta}{3}$.

(13) Find $\int \cos^2 a\theta\,d\theta$.

(14) Find $\int \sin^2\theta\,d\theta$.

(15) Find $\int \sin^2 a\theta\,d\theta$.

(16) Find $\int e^{3x}\,dx$.

(17) Find $\int \frac{dx}{1 + x}$.

(18) Find $\int \frac{dx}{1 - x}$.

Chapter XIX

ON FINDING AREAS BY INTEGRATING

One use of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let us try to get at the subject bit by bit.

Let AB be a curve, the equation to which is known. That is, y in this curve is some known function of x. (See Fig. 52.) Think of a piece of the curve from the point P to the point Q.

Let a perpendicular PM be dropped from P, and another QN from the point Q. Then call $OM = x_1$ and $ON = x_2$, and the ordinates $PM = y_1$ and $QN = y_2$. We have thus marked out the area $PQNM$ that lies beneath the piece PQ. The problem is, *how can we calculate the value of this area?*

The secret of solving this problem is to conceive the area as being divided up into a lot of narrow strips, each of them being of the width dx.[1] The smaller we take dx, the more of them there will be between x_1 and x_2. Now, the whole area is clearly equal to the sum of the areas of all such strips. Our business will then

1. Thompson explains the integral as the sum of a finite number of thin strips under the curve as their widths approach the limit of zero. When these strips are all below a curve their sum is called today a "lower Riemann sum" after German mathematician George Friedrich Bernhard Riemann (1826–1866). The same sum can be obtained by letting the strips extend above the curve as shown in Figure 42, in which case the sum is called an "upper Riemann sum." If the strips are drawn so that their tops cross the curve, their sum is called a "Riemann sum." Regardless of how the strips are drawn, they will have the same "Riemann integral" at the limit when they became infinite in number and their widths become infinitely small.—M.G.

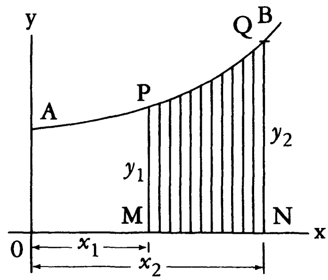


FIG. 52.

be to discover an expression for the area of any one narrow strip, and to integrate it so as to add together all the strips. Now think of any one of the strips. It will be like this: being bounded between two vertical sides, with a flat bottom dx, and with a slightly curved sloping top. Suppose we take its *average* height as being y; then, as its width is dx, its area will be ydx. And seeing that we may take the width as narrow as we please, if we only take it narrow enough its average height will be the same as the height at the middle of it. Now let us call the unknown value of the whole area S, meaning surface. The area of one strip will be simply a bit of the whole area, and may therefore be called dS. So we may write

$$\text{area of 1 strip} = dS = y\ dx$$

If then we add up all the strips, we get

$$\text{total area } S = \int dS = \int y\ dx$$

So then our finding S depends on whether we can integrate $y\ dx$ for the particular case, when we know what the value of y is as a function of x.

For instance, if you were told that for the particular curve in question $y = b + ax^2$, no doubt you could put that value into the expression and say: then I must find $\int (b + ax^2)dx$.

That is all very well; but a little thought will show you that something more must be done. Because the area we are trying to find is not the area under the whole length of the curve, but only the area limited on the left by PM, and on the right by QN, it follows that we must do something to define our area between those bounds.

This introduces us to a new notion, namely, that of *integrating between limits.*[2] We suppose x to vary, and for the present purpose we do not require any value of x below x_1 (that is OM), nor any value of x above x_2 (that is ON). When an integral is to be thus defined between two limits, we call the lower of the two values *the inferior limit,* and the upper value *the superior limit.* Any integral so limited we designate as a *definite integral,* by way of distinguishing it from a *general integral* to which no limits are assigned.[3]

2. The word "limit" is confusing here because it is not a limit in the sense of the sum of an infinite series. The word "bound" is much clearer. What Thompson calls inferior and superior limits of a closed interval along a continuous curve are lower and upper bounds, although today many textbooks call them "lower and upper limits of integration," or "left and right endpoints of integration."—M.G.

3. Thompson's term "general integral" is no longer used. In the past it was also called a "primitive integral" and later an "indefinite integral." Today it is usually called an "antiderivative." The reason is obvious. It is the inverse of a derivative. Writers differ on how to symbolize it. Thompson simply puts it inside brackets. One common symbol for it today is $F(x)$, using a capital F instead of a lower case f. In all that follows I have substituted "antiderivative" for Thompson's "general integral."

As Thompson makes clear, a derivative does not have a unique antiderivative because an antiderivative can have any one of an infinite number of added constants. These constants correspond to different heights a curve can have above the x axis. For example, the derivative of x^2 is $2x$. But $2x$ is also the derivative of $x^2 + 1$; $x^2 + 666$; $x^2 - \text{pi}$, and so on. It can be x^2 plus or minus any real number. Because there is an infinity of real numbers, if a derivative has one antiderivative it will have an infinite number of them. They differ only in what are called their "constants of integration." The antiderivative is "indefinite" because it is not unique.

In *Calculus Made Easy,* and in all calculus textbooks, when "integral" is used without an adjective, it means the definite integral. It is the fundamental concept of integration.—M.G.

In the symbols which give instructions to integrate, the limits are marked by putting them at the top and bottom respectively of the sign of integration. Thus the instruction

$$\int_{x=x_1}^{x=x_2} y \cdot dx$$

will be read: find the integral of $y \cdot dx$ between the inferior limit x_1 and the superior limit x_2.

Sometimes the thing is written more simply

$$\int_{x_1}^{x_2} y \cdot dx$$

Well, but *how* do you find an integral between limits when you have got these instructions?

Look again at Fig. 52. Suppose we could find the area under the larger piece of curve from A to Q, that is from $x = 0$ to $x = x_2$, naming the area $AQNO$. Then, suppose we could find the area under the smaller piece from A to P, that is from $x = 0$ to $x = x_1$, namely, the area $APMO$. If then we were to subtract the smaller area from the larger, we should have left as a remainder the area $PQNM$, which is what we want. Here we have the clue as to what to do; the definite integral between the two limits is *the difference* between the antiderivative worked out for the superior limit and the antiderivative worked out for the lower limit.

Let us then go ahead. First, find the antiderivative thus:

$$\int y\, dx$$

and, as $y = b + ax^2$ is the equation to the curve (Fig. 52),

$$\int (b + ax^2)dx$$

is the antiderivative which we must find.

Doing the integration in question, we get

$$bx + \frac{a}{3}x^3 + C$$

and this will be the whole area from 0 up to any value of x that we may assign. When x is 0, this area is 0, so $C = 0$.

Therefore, the larger area up to the superior limit x_2 will be

$$bx_2 + \frac{a}{3}x_2{}^3$$

and the smaller area up to the inferior limit x_1 will be

$$bx_1 + \frac{a}{3}x_1{}^3$$

Now, subtract the smaller from the larger, and we get for the area S the value,

$$\text{area } S = b(x_2 - x_1) + \frac{a}{3}(x_2{}^3 - x_1{}^3)$$

This is the answer we wanted. Let us give some numerical values. Suppose $b = 10$, $a = 0.06$, and $x_2 = 8$ and $x_1 = 6$. Then the area S is equal to

$$\begin{aligned} &10(8 - 6) + \frac{0.06}{3}(8^3 - 6^3) \\ &= 20 + 0.02(512 - 216) \\ &= 20 + 0.02 \times 296 \\ &= 20 + 5.92 \\ &= 25.92 \end{aligned}$$

Let us here put down a symbolic way of stating what we have ascertained about limits:

$$\int_{x=x_1}^{x=x_2} y\, dx = y_2 - y_1$$

where y_2 is the integrated value of $y\ dx$ corresponding to x_2, and y_1 that corresponding to x_1.

All integration between limits requires the difference between two values to be thus found. Also note that, in making the subtraction the added constant C has disappeared.[4]

4. Because the technique Thompson describes is at the heart of integral calculus, let me try to make it clearer.

To transform an antiderivative into a definite integral, bounds on the continuous curve must be specified. Each bound has a value for the curve's antiderivative. The definite integral is the difference between those two values. Simply subtract the value of the antiderivative at the left bound, where x is smaller, from the value of the antiderivative at the right bound, where x is larger. The result is the definite integral.

The definite integral is not a function. It is a number that is the limit sum of all the thin rectangles under the curve, between the curve's upper and lower bounds, as their widths approach zero and their number becomes infinite. The situation is analogous to cutting a piece of string. Suppose it is a foot long and you wish to obtain a 9-inch portion from the 3-inch mark to the 12-inch end. What do you do? You snip off the first three inches.

The fact that the definite integral is the difference between two values of the antiderivative is known as the "fundamental theorem of calculus." The theorem can be expressed in other ways, but this way is the simplest and most useful. It is an amazing theorem—one that unites differentiating with integrating. It works like sorcery, almost too good to be true!

Jerry P. King, in his *Art of Mathematics* (1992) likens the theorem to the cornerstone of an arch that holds together the two sides of calculus. Because there is one unified calculus, many mathematicians have recommended dropping the terms "differential calculus" and "integral calculus," replacing them with "calculus of derivatives" and "calculus of integrals."

It is impossible, King writes, to overestimate the importance of this arch. "Above the great arch and supported by it rests all of mathematical analysis and the significant parts of physics and the other sciences which calculus sustains and explains. Mathematics and science *stand* on calculus. . . ."

Newton was the first to construct the arch. Eric Temple Bell, in his chapter on Newton in *Men of Mathematics* (1937), calls the arch "surely one of the most astonishing things a mathematician ever discovered."

The mean value theorem (see the Postscript of Chapter X) defines a point p on a continuous function curve between bounds a and b. The point's y value is the mean value of the function. If you draw a horizontal line through this point, and drop vertical lines through a and b to the x axis, you form what is called the function's "mean value rectangle." (See Figure 53a). The area of this rectangle, shown shaded, is exactly equal to the area under the curve's interval from a to b.—M.G.

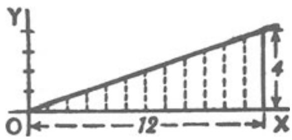


FIG. 53.

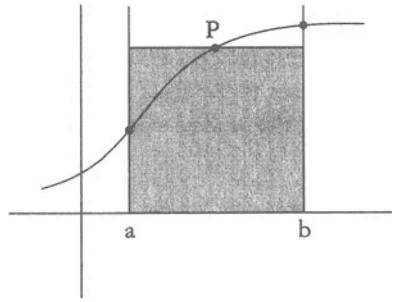


FIG. 53a. The mean-value rectangle.

Examples.

(1) To familiarize ourselves with the process, let us take a case of which we know the answer beforehand. Let us find the area of the triangle (Fig. 53), which has base $x = 12$ and height $y = 4$. We know beforehand, from obvious mensuration, that the answer will come to 24.

Now, here we have as the "curve" a sloping line for which the equation is

$$y = \frac{x}{3}$$

The area in question will be

$$\int_{x=0}^{x=12} y \cdot dx = \int_{x=0}^{x=12} \frac{x}{3} \cdot dx$$

Integrating $\frac{x}{3}\,dx$, and putting down the value of the antiderivative in square brackets with the limits marked above and below, we get

$$\text{area} = \left[\frac{1}{3} \cdot \frac{1}{2} x^2 + C\right]_{x=0}^{x=12}$$

$$= \left[\frac{x^2}{6} + C\right]_{x=0}^{x=12}$$

$$\text{area} = \left[\frac{12^2}{6} + C\right] - \left[\frac{0^2}{6} + C\right]$$
$$= \frac{144}{6} = 24$$

Note that, in dealing with definite integrals, the constant C always disappears by subtraction.

Let us satisfy ourselves about this rather surprising dodge of calculation, by testing it on a simple example, Get some squared paper, preferably some that is ruled in little squares of one-eighth or one-tenth inch each way. On this squared paper plot out the graph of this equation,

$$y = \frac{x}{3}$$

The values to be plotted will be:

x	0	3	6	9	12
y	0	1	2	3	4

The plot is given in Fig. 54.

Now reckon out the area beneath the curve *by counting the little squares* below the line, from $x = 0$ as far as $x = 12$ on the right. There are 18 whole squares and four triangles, each of which has an area equal to $1\frac{1}{2}$ squares; or, in total, 24 squares. Hence 24 is

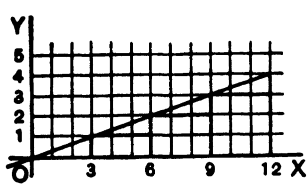


FIG. 54.

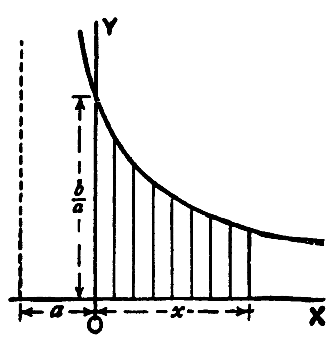


FIG. 55.

the numerical value of the integral of $\frac{x}{3}\,dx$ between the lower limit of $x=0$ and the higher limit of $x=12$.

As a further exercise, show that the value of the same integral between the limits of $x=3$ and $x=15$ is 36.

(2) Find the area, between limits $x=x_1$ and $x=0$, of the curve $y=\dfrac{b}{x+a}$

$$\text{Area}=\int_{x=0}^{x=x_1} y\cdot dx=\int_{x=0}^{x=x_1}\frac{b}{x+a}\,dx$$

$$=b\Big[\ln(x+a)+C\Big]_0^{x_1}$$

$$=b[\ln(x_1+a)+C-\ln(0+a)-C]$$

$$=b\ln\frac{x_1+a}{a}$$

Let it be noted that this process of subtracting one part from a larger to find the difference is really a common practice. How do you find the area of a plane ring (Fig. 56), the outer radius of which is r_2 and the inner radius is r_1? You know from mensuration that the area of the outer circle is πr_2^2; then you find the area of the inner circle πr_1^2; then you subtract the latter from the former, and find area of ring $=\pi(r_2^2-r_1^2)$; which may be written

$$\pi(r_2+r_1)(r_2-r_1)$$

$=$ mean circumference of ring $\times$ width of ring.

(3) Here's another case—that of *the die-away curve*. Find the area between $x=0$ and $x=a$ of the curve (Fig. 57) whose equation is

$$y=be^{-x}$$

$$\text{Area}=b\int_{x=0}^{x=a} e^{-x}\cdot dx$$

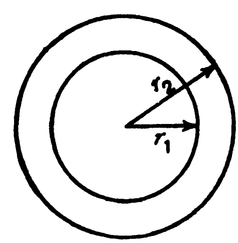


FIG. 56.

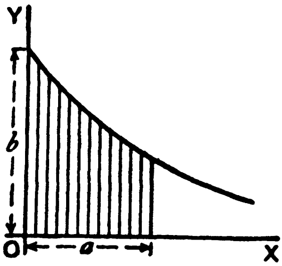


FIG. 57.

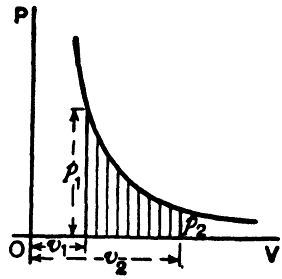


FIG. 58.

The integration gives

$$= b\left[-e^{-x}\right]_0^a$$

$$= b[-e^{-a} - (-e^{-0})]$$

$$= b(1 - e^{-a})$$

(4) Another example is afforded by the adiabatic curve of a perfect gas, the equation to which is $pv^n = c$, where p stands for pressure, v for volume, and n has the value 1.42 (Fig. 58).

Find the area under the curve (which is proportional to the work done in suddenly compressing the gas) from volume v_2 to volume v_1.

Here we have

$$\text{area} = \int_{v=v_1}^{v=v_2} cv^{-n} \cdot dv$$

$$= c\left[\frac{1}{1-n}\, v^{1-n}\right]_{v_1}^{v_2}$$

$$= c\,\frac{1}{1-n}(v_2^{\;1-n} - v_1^{\;1-n})$$

$$= \frac{-c}{0.42}\left(\frac{1}{v_2^{\;0.42}} - \frac{1}{v_1^{\;0.42}}\right)$$

An Exercise.

Prove the ordinary mensuration formula, that the area A of a circle whose radius is R, is equal to πR^2.

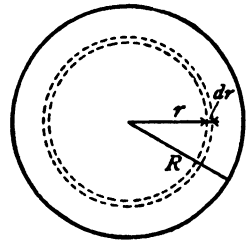


FIG. 59.

Consider an elementary zone or annulus of the surface (Fig. 59), of breadth dr, situated at a distance r from the centre. We may consider the entire surface as consisting of such narrow zones, and the whole area A will simply be the integral of all such elementary zones from centre to margin, that is, integrated from $r=0$ to $r=R$.

We have therefore to find an expression for the elementary area dA of the narrow zone. Think of it as a strip of breadth dr, and of a length that is the periphery of the circle of radius r, that is, a length of $2\pi r$. Then we have, as the area of the narrow zone,

$$dA = 2\pi r\, dr$$

Hence the area of the whole circle will be:

$$A = \int dA = \int_{r=0}^{r=R} 2\pi r \cdot dr = 2\pi \int_{r=0}^{r=R} r \cdot dr$$

Now, the antiderivative of $r \cdot dr$ is $\frac{1}{2}r^2$. Therefore,

$$A = 2\pi\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R}$$

or $\quad A = 2\pi\left[\tfrac{1}{2}R^2 - \tfrac{1}{2}(0)^2\right]$

whence $\quad A = \pi R^2$

Another Exercise.

Let us find the mean value of the positive part of the curve $y = x - x^2$, which is shown in Fig. 60. To find the mean ordinate, we shall have to find the area of the piece OMN, and then divide it by the length of the base ON. But before we can find the area we must ascertain the length of the base, so as

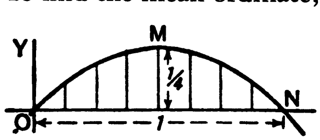


FIG. 60.

to know up to what limit we are to integrate. At N the ordinate y has zero value; therefore, we must look at the equation and see what value of x will make $y = 0$. Now, clearly, if x is 0, y will also be 0, the curve passing through the origin O; but also, if $x = 1$, $y = 0$: so that $x = 1$ gives us the position of the point N.

Then the area wanted is

$$= \int_{x=0}^{x=1} (x - x^2)dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \left[\tfrac{1}{2} - \tfrac{1}{3}\right] - [0 - 0] = \tfrac{1}{6}$$

But the base length is 1.

Therefore, the average ordinate of the curve $= \frac{1}{6}$.

[*N.B.*—It will be a pretty and simple exercise in maxima and minima to find by differentiation what is the height of the maximum ordinate. It *must* be greater than the average.]

The mean ordinate of any curve, over a range from $x = 0$ to $x = x_1$, is given by the expression,

$$\text{mean } y = \frac{1}{x_1} \int_{x=0}^{x=x_1} y \cdot dx$$

If the mean ordinate be required over a distance not beginning at the origin but beginning at a point distant x_1 from the origin and ending at a point distant x_2 from the origin, the value will be

$$\text{mean } y = \frac{1}{x_2 - x_1} \int_{x=x_1}^{x=x_2} y \, dx$$

Areas in Polar Coordinates

When the equation of the boundary of an area is given as a function of the distance r of a point of it from a fixed point O (see Fig. 61) called the *pole*, and of the angle which r makes with the positive horizontal direction OX, the process just explained can be applied just as easily, with a small modification. Instead of a strip of area, we consider a

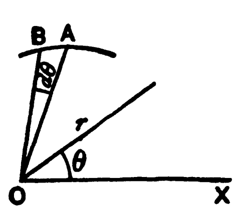


FIG. 61.

small triangle OAB, the angle at O being $d\theta$, and we find the sum of all the little triangles making up the required area.

The area of such a small triangle is approximately $\frac{r d\theta}{2} \times r$; hence the portion of the area included between the curve and two positions of r corresponding to the angles θ_1 and θ_2 is given by

$$\tfrac{1}{2}\int_{\theta=\theta_1}^{\theta=\theta_2} r^2\, d\theta$$

Examples.

(1) Find the area of the sector of 1 radian in a circumference of radius a inch.

The polar equation of the circumference is evidently $r = a$. The area is

$$\tfrac{1}{2}\int_{\theta=0}^{\theta=1} a^2\, d\theta = \frac{a^2}{2}\int_{\theta=0}^{\theta=1} d\theta = \frac{a^2}{2}$$

(2) Find the area of the first quadrant of the curve (known as a "cardioid"), the polar equation of which is

$$r = a(1 + \cos\theta)$$

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} a^2(1 + \cos\theta)^2 d\theta \\ &= \frac{a^2}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1 + 2\cos\theta + \cos^2\theta) d\theta \\ &= \frac{a^2}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} \\ &= \frac{a^2(3\pi + 8)}{8} \end{aligned}$$

Volumes by Integration

What we have done with the area of a little strip of a surface, we can, of course, just as easily do with the volume of a little strip of a solid. We can add up all the little strips that make up the total solid, and find its volume, just as we have added up all the small little bits that made up an area to find the final area of the figure operated upon.

Examples.

(1) Find the volume of a sphere of radius r.

A thin spherical shell has for volume $4\pi x^2\,dx$ (see Fig. 59). Summing up all the concentric shells which make up the sphere, we have

$$\text{volume sphere} = \int_{x=0}^{x=r} 4\pi x^2\,dx = 4\pi\left[\frac{x^3}{3}\right]_0^r = \tfrac{4}{3}\pi r^3$$

We can also proceed as follows: a slice of the sphere, of thickness dx, has for volume $\pi y^2\,dx$ (see Fig. 62). Also x and y are related by the expression

$$y^2 = r^2 - x^2$$

Hence $$\text{volume sphere} = 2\int_{x=0}^{x=r} \pi(r^2 - x^2)dx$$

$$= 2\pi\left[\int_{x=0}^{x=r} r^2\,dx - \int_{x=0}^{x=r} x^2\,dx\right]$$

$$= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = \frac{4\pi}{3}r^3$$

(2) Find the volume of the solid generated by the revolution of the curve $y^2 = 6x$ about the axis of x, between $x = 0$ and $x = 4$.

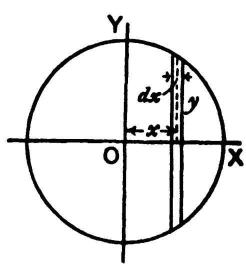


FIG. 62.

The volume of a slice of the solid is $\pi y^2 dx$.

Hence $\quad$ volume $= \displaystyle\int_{x=0}^{x=4} \pi y^2 dx = 6\pi \int_{x=0}^{x=4} x\, dx$

$$= 6\pi \left[\frac{x^2}{2} \right]_0^4 = 48\pi = 150.8.$$

On Quadratic Means

In certain branches of physics, particularly in the study of alternating electric currents, it is necessary to be able to calculate the *quadratic mean* of a variable quantity. By "quadratic mean" is denoted the square root of the mean of the squares of all the values between the limits considered. Other names for the quadratic mean of any quantity are its "virtual" value, or its "R.M.S." (meaning root-mean-square) value. The French term is *valeur efficace.* If y is the function under consideration, and the quadratic mean is to be taken between the limits of $x = 0$ and $x = k$; then the quadratic mean is expressed as

$$\sqrt[2]{\frac{1}{k} \int_0^k y^2\, dx}$$

Examples.

(1) To find the quadratic mean of the function $y = ax$ (Fig. 63).

Here the intergral is $\displaystyle\int_0^k a^2x^2 dx$ which is $\frac{1}{3}a^2k^3$. Dividing by k and taking the square root, we have

$$\text{quadratic mean} = \frac{1}{\sqrt{3}} ak$$

Here the arithmetical mean is $\frac{1}{2}ak$; and the ratio of quadratic to arithmetical mean (this ratio is called the *form-factor*) is $2/\sqrt{3} = 2\sqrt{3}/3 = 1.1547.\ .\ .\ .$

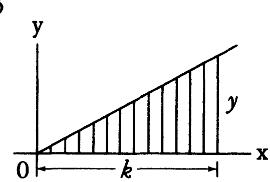


FIG. 63.

(2) To find the quadratic mean of the function $y = x^a$.

$$\text{The integral is } \int_{x=0}^{x=k} x^{2a}dx, \text{ that is } \frac{k^{2a+1}}{2a+1}$$

Hence

$$\text{quadratic mean} = \sqrt{\frac{k^{2a}}{2a+1}}$$

(3) To find the quadratic mean of the function $y = a^{\frac{x}{2}}$

$$\text{The integral is } \int_{x=0}^{x=k} \left(a^{\frac{x}{2}}\right)^2 dx, \text{ that is } \int_{x=0}^{x=k} a^x dx$$

or

$$\left[\frac{a^x}{\ln a}\right]_{x=0}^{x=k}, \text{ which is } \frac{a^k - 1}{\ln a}$$

$$\text{Hence the quadratic mean is } \sqrt{\frac{a^k - 1}{k \ln a}}$$

EXERCISES XVIII

(1) Find the area of the curve $y = x^2 + x + 5$ between $x = 0$ and $x = 6$, and the mean ordinate between these limits.

(2) Find the area of the parabola $y = 2a\sqrt{x}$ between $x = 0$ and $x = a$. Show that it is two-thirds of the rectangle of the limiting ordinate and of its abscissa.

(3) Find the area of the portion of a sine curve between $x = 0$ and $x = \pi$, and the mean ordinate.

(4) Find the area of the portion of the curve $y = \sin^2 x$ from 0° to 180°, and find the mean ordinate.

(5) Find the area included between the two branches of the curve $y = x^2 \pm x^{\frac{5}{2}}$ from $x = 0$ to $x = 1$, also the area of the positive portion of the lower branch of the curve (Fig. 30).

(6) Find the volume of a cone of radius of base r, and of height h.

(7) Find the area of the curve $y = x^3 - \ln x$ between $x = 0$ and $x = 1$.

(8) Find the volume generated by the curve $y = \sqrt{1 + x^2}$, as it revolves about the axis of x, between $x = 0$ and $x = 4$.

(9) Find the volume generated by a sine curve between $x = 0$ and $x = \pi$, revolving about the axis of x.

(10) Find the area of the portion of the curve $xy = a$ included between $x = 1$ and $x = a$, where $a > 1$. Find the mean ordinate between these limits.

(11) Show that the quadratic mean of the function $y = \sin x$, between the limits of 0 and π radians, is $\dfrac{\sqrt{2}}{2}$. Find also the arithmetical mean of the same function between the same limits; and show that the form-factor is =1.11.

(12) Find the arithmetical and quadratic means of the function $x^2 + 3x + 2$, from $x = 0$ to $x = 3$.

(13) Find the quadratic mean and the arithmetical mean of the function $y = A_1 \sin x + A_3 \sin 3x$ between $x = 0$ and $x = 2\pi$.

(14) A certain curve has the equation $y = 3.42e^{0.21x}$. Find the area included between the curve and the axis of x, from the ordinate at $x = 2$ to the ordinate at $x = 8$. Find also the height of the mean ordinate of the curve between these points.

(15) The curve whose polar equation is $r = a(1 - \cos \theta)$ is known as the cardioid. Show that the area enclosed by the axis and the curve between $\theta = 0$ and $\theta = 2\pi$ radians is equal to 1.5 times that of the circle whose radius is a.

(16) Find the volume generated by the curve

$$y = \pm\frac{x}{6}\sqrt{x(10 - x)}$$

rotating about the axis of x.

Chapter XX

DODGES, PITFALLS, AND TRIUMPHS

Dodges. A great part of the labor of integrating things consists in licking them into some shape that can be integrated. The books—and by this is meant the serious books—on the integral calculus are full of plans and methods and dodges and artifices for this kind of work. The following are a few of them.

Integration by Parts. This name is given to a dodge, the formula for which is

$$\int u\,dx = ux - \int x\,du + C$$

It is useful in some cases that you can't tackle directly, for it shows that if in any case $\int x\,du$ can be found, then $\int u\,dx$ can also be found. The formula can be deduced as follows.

$$d(ux) = u\,dx + x\,du$$

which may be written

$$u\,dx = d(ux) - x\,du$$

which by direct integration gives the above expression.

Examples.

(1) Find $\int w \cdot \sin w\,dw$

Write $u = w$, and dx for $\sin w \cdot dw$. We shall then have $du = dw$, while $x = \int \sin w \cdot dw = -\cos w$.

Putting these into the formula, we get

$$\int w \cdot \sin w \, dw = w(-\cos w) - \int -\cos w \, dw$$

$$= -w \cos w + \sin w + C.$$

(2) Find $\int xe^x dx$.

Write $\qquad u = x \quad dv = e^x dx$

then $\qquad du = dx \quad v = e^x$

and $\qquad \int xe^x dx = xe^x - \int e^x dx$ (by the formula)

$$= xe^x - e^x + C = e^x(x-1) + C$$

(3) Try $\int \cos^2\theta \, d\theta$.

$$u = \cos\theta \qquad dx = \cos\theta \, d\theta$$

Hence $\qquad du = -\sin\theta \, d\theta \qquad x = \sin\theta$

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int \sin^2\theta \, d\theta$$

$$= \frac{2\cos\theta \sin\theta}{2} + \int (1 - \cos^2\theta) d\theta$$

$$= \frac{\sin 2\theta}{2} + \int d\theta - \int \cos^2\theta \, d\theta$$

Hence $\qquad 2\int \cos^2\theta \, d\theta = \dfrac{\sin 2\theta}{2} + \theta + 2C$

and
$$\int \cos^2\theta\, d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C$$

(4) Find $\int x^2 \sin x\, dx$.

Write $u = x^2 \qquad dv = \sin x\, dx$

then $du = 2x\, dx \qquad v = -\cos x$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$$

Now find $\int x \cos x\, dx$, integrating by parts (as in Example 1 above):

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

Hence
$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C'$$
$$= (2 - x^2) \cos x + 2x \sin x + C'$$

(5) Find $\int \sqrt{1 - x^2}dx$.

Write $u = \sqrt{1 - x^2}, \quad dx = dv$

then $du = -\dfrac{x\, dx}{\sqrt{1 - x^2}}$ (see Chap. IX)

and $x = v$; so that

$$\int \sqrt{1 - x^2}dx = x\sqrt{1 - x^2} + \int \frac{x^2 dx}{\sqrt{1 - x^2}}$$

Here we may use a little dodge, for we can write

$$\int\sqrt{1-x^2}dx = \int\frac{(1-x^2)dx}{\sqrt{1-x^2}} = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{x^2dx}{\sqrt{1-x^2}}$$

Adding these two last equations, we get rid of $\int\frac{x^2dx}{\sqrt{1-x^2}}$, and we have

$$2\int\sqrt{1-x^2}dx = x\sqrt{1-x^2} + \int\frac{dx}{\sqrt{1-x^2}}$$

Do you remember meeting $\frac{dx}{\sqrt{1-x^2}}$? It is got by differentiating $y = \text{arc sin } x$; hence its integral is arc sin x, and so

$$\int\sqrt{1-x^2}dx = \frac{x\sqrt{1-x^2}}{2} + \tfrac{1}{2}\text{ arc sin } x + C$$

You can try now some exercises by yourself; you will find some at the end of this chapter.

Substitution. This is the same dodge as explained in Chap. IX. Let us illustrate its application to integration by a few examples.

(1) $\int\sqrt{3+x}\,dx$

Let $u = 3 + x, \quad du = dx$

replace: $\int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(3+x)^{\frac{3}{2}} + C$

(2) $\int\frac{dx}{e^x + e^{-x}}$

Let $$u = e^x,\ \frac{du}{dx} = e^x, \text{ and } dx = \frac{du}{e^x}$$

so that $$\int \frac{dx}{e^x + e^{-x}} = \int \frac{du}{e^x(e^x + e^{-x})} = \int \frac{du}{u\left(u + \frac{1}{u}\right)} = \int \frac{du}{u^2 + 1}$$

$\frac{du}{1 + u^2}$ is the result of differentiating arc tan u.

Hence the integral is arc tan $e^x + C$.

(3) $$\int \frac{dx}{x^2 + 2x + 3} = \int \frac{dx}{x^2 + 2x + 1 + 2} = \int \frac{dx}{(x+1)^2 + \left(\sqrt{2}\right)^2}.$$

Let $$u = x + 1, \quad du = dx;$$

then the integral becomes $\int \frac{du}{u^2 + \left(\sqrt{2}\right)^2}$; but $\frac{du}{u^2 + a^2}$ is the result of differentiating $\frac{1}{a}$ arc tan $\frac{u}{a}$.

Hence one has finally $\frac{1}{\sqrt{2}}$ arc tan $\frac{x+1}{\sqrt{2}} + C$ for the value of the given integral.

Rationalization, and *Factorization of Denominator* are dodges applicable in special cases, but they do not admit of any short or general explanation. Much practice is needed to become familiar with these preparatory processes.

The following example shows how the process of splitting into partial fractions, which we learned in Chap. XIII, can be made use of in integration.

Take $\int \frac{dx}{x^2 + 2x - 3}$; if we split $\frac{1}{x^2 + 2x - 3}$ into partial fractions, this becomes:

$$\frac{1}{4}\left[\int \frac{dx}{x-1} - \int \frac{dx}{x+3}\right] = \frac{1}{4}\,[\ln(x-1) - \ln(x+3)] + C$$
$$= \frac{1}{4} \ln \frac{x-1}{x+3} + C$$

Notice that the same integral can be expressed sometimes in more than one way (which are equivalent to one another).

Pitfalls. A beginner is liable to overlook certain points that a practised hand would avoid; such as the use of factors that are equivalent to either zero or infinity, and the occurence of indeterminate quantities such as $\frac{0}{0}$. There is no golden rule that will meet every possible case. Nothing but practice and intelligent care will avail. An example of a pitfall which had to be circumvented arose in Chap. XVIII, when we came to the problem of integrating $x^{-1}\,dx$.

Triumphs. By triumphs must be understood the successes with which the calculus has been applied to the solution of problems otherwise intractable. Often in the consideration of physical relations one is able to build up an expression for the law governing the interaction of the parts or of the forces that govern them, such expression being naturally in the form of a *differential equation*, that is an equation containing derivatives with or without other algebraic quantities. And when such a differential equation has been found, one can get no further until it has been integrated. Generally it is much easier to state the appropriate differential equation than to solve it: the real trouble begins then only when one wants to integrate, unless indeed the equation is seen to possess some standard form of which the integral is known, and then the triumph is easy. The equation which results from integrating a differential equation is called* its "solution"; and it is quite astonishing how in many cases the solution looks as if it had no relation to the differential equation of which it is the integrated form. The solution often seems as different from the original expression as a butterfly does from a

*This means that the actual result of solving it is called its "solution". But many mathematicians would say, with Professor A.R. Forsyth, "every differential equation *is considered as solved* when the value of the dependent variable is expressed as a function of the independent variable by means either of known functions, or of integrals, whether the integrations in the latter can or cannot be expressed in terms of functions already known."

caterpillar that it was. Who would have supposed that such an innocent thing as

$$\frac{dy}{dx} = \frac{1}{a^2 - x^2}$$

could blossom out into

$$y = \frac{1}{2a} \ln \frac{a + x}{a - x} + C?$$

yet the latter is the *solution* of the former.

As a last example, let us work out the above together.

By partial fractions,

$$\frac{1}{a^2 - x^2} = \frac{1}{2a(a + x)} + \frac{1}{2a\,(a - x)}$$

$$dy = \frac{dx}{2a\,(a + x)} + \frac{dx}{2a\,(a - x)}$$

$$y = \frac{1}{2a}\left(\int \frac{dx}{a + x} + \int \frac{dx}{a - x}\right)$$

$$= \frac{1}{2a}\,[\ln\,(a + x) - \ln\,(a - x)] + C$$

$$= \frac{1}{2a} \ln \frac{a + x}{a - x} + C$$

Not a very difficult metamorphosis!

There are whole treatises, such as George Boole's *Differential Equations*, devoted to the subject of finding the "solutions" for different original forms.

EXERCISES XIX

(1) Find $\int \sqrt{a^2 - x^2}\,dx$.

(2) Find $\int x \ln x\, dx$.

(3) Find $\int x^a \ln x\, dx$.

(4) Find $\int e^x \cos e^x\, dx$.

(5) Find $\int \frac{1}{x} \cos(\ln x)\, dx$.

(6) Find $\int x^2 e^x\, dx$.

(7) Find $\int \frac{(\ln x)^a}{x}\, dx$.

(8) Find $\int \frac{dx}{x \ln x}$.

(9) Find $\int \frac{5x+1}{x^2+x-2}\, dx$.

(10) Find $\int \frac{(x^2-3)dx}{x^3-7x+6}$.

(11) Find $\int \frac{b\, dx}{x^2-a^2}$.

(12) Find $\int \frac{4x\, dx}{x^4-1}$.

(13) Find $\int \frac{dx}{1-x^4}$.

(14) Find $\int \frac{x\, dx}{\sqrt{a^2-b^2x^2}}$.

(15) Use the substitution $\frac{1}{x} = \frac{b}{a} \cosh u$ to show that

$$\int \frac{dx}{x\sqrt{a^2-b^2x^2}} = \frac{1}{a} \ln \frac{a-\sqrt{a^2-b^2x^2}}{x} + C$$

Chapter XXI

FINDING SOLUTIONS

In this chapter we go to work finding solutions to some important differential equations, using for this purpose the processes shown in the preceding chapters.

The beginner, who now knows how easy most of those processes are in themselves, will here begin to realize that integration is *an art*. As in all arts, so in this, facility can be acquired only by diligent and regular practice. Those who would attain that facility must work out examples, and more examples, and yet more examples, such as are found abundantly in all the regular treatises on the calculus. Our purpose here must be to afford the briefest introduction to serious work.

Example 1. Find the solution of the differential equation

$$ay + b\frac{dy}{dx} = 0$$

Transposing, we have

$$b\frac{dy}{dx} = -ay$$

Now the mere inspection of this relation tells us that we have got to do with a case in which $\frac{dy}{dx}$ is proportional to y. If we think of the curve which will represent y as a function of x, it will be such that its slope at any point will be proportional to the ordinate at that point, and will be a negative slope if y is positive. So obviously the curve will be a die-away curve, and the solution will contain e^{-x} as a factor. But, without presuming on this bit of sagacity, let us go to work.

As both y and dy occur in the equation and on opposite sides, we can do nothing until we get both y and dy to one side, and dx to the other. To do this, we must split our usually inseparable companions dy and dx from one another.

$$\frac{dy}{y} = -\frac{a}{b}\,dx$$

Having done the deed, we now can see that both sides have got into a shape that is integrable, because we recognize $\frac{dy}{y}$, or $\frac{1}{y}\,dy$, as a differential that we have met with when differentiating logarithms. So we may at once write down the instructions to integrate,

$$\int \frac{dy}{y} = \int -\frac{a}{b}\,dx$$

and doing the two integrations, we have:

$$\ln y = -\frac{a}{b}x + \ln C$$

where $\ln C$ is the yet undetermined constant* of integration. Then, delogarizing, we get:

$$y = Ce^{-\frac{a}{b}x}$$

which is *the solution* required. Now, this solution looks quite unlike the original differential equation from which it was constructed: yet to an expert mathematician they both convey the same information as to the way in which y depends on x.

*We may write down any form of constant as the "constant of integration", and the form $\ln C$ is adopted here by preference, because the other terms in this line of equation are, or are treated as logarithms; and it saves complications afterward if the added constant be *of the same kind.*

Now, as to the C, its meaning depends on the initial value of y. For if we put $x = 0$ in order to see what value y then has, we find that this makes $y = Ce^{-0}$; and as $e^{-0} = 1$, we see that C is nothing else than the particular value* of y at starting. This we may call y_0 and so write the solution as

$$y = y_0 e^{-\frac{a}{b}x}$$

Example 2.
Let us take as an example to solve

$$ay + b\frac{dy}{dx} = g$$

where g is a constant. Again, inspecting the equation will suggest, (1) that somehow or other e^x will come into the solution, and (2) that if at any part of the curve y becomes either a maximum or a minimum, so that $\frac{dy}{dx} = 0$, then y will have the value $=$ $\frac{g}{a}$. But let us go to work as before, separating the differentials and trying to transform the thing into some integrable shape.

$$b\frac{dy}{dx} = g - ay$$

$$\frac{dy}{dx} = \frac{a}{b}\left(\frac{g}{a} - y\right)$$

$$\frac{dy}{y - \frac{g}{a}} = -\frac{a}{b}dx$$

Now we have done our best to get nothing but y and dy on one side, and nothing but dx on the other. But is the result on the left side integrable?

*Compare what was said about the "constant of integration", with reference to Fig. 48, and Fig. 51.

It is of the same form as the result in Chapter XIV, so, writing the instructions to integrate, we have:

$$\int \frac{dy}{y - \frac{g}{a}} = - \int \frac{a}{b}\,dx$$

and, doing the integration, and adding the appropriate constant,

$$\ln\left(y - \frac{g}{a}\right) = -\frac{a}{b}\,x + \ln C$$

whence $$y - \frac{g}{a} = Ce^{-\frac{a}{b}x}$$

and finally, $$y = \frac{g}{a} + Ce^{-\frac{a}{b}x}$$

which is *the solution.*

If the condition is laid down that $y = 0$ when $x = 0$ we can find C; for then the exponential becomes $=1$; and we have

$$0 = \frac{g}{a} + C$$

or $$C = -\frac{g}{a}$$

Putting in this value, the solution becomes

$$y = \frac{g}{a}(1 - e^{-\frac{a}{b}x})$$

But further, if x grows infinitely, y will grow to a maximum; for when $x = \infty$, the exponential $= 0$, giving $y_{\text{max.}} = \frac{g}{a}$. Substituting this, we get finally

$$y = y_{\text{max.}}(1 - e^{-\frac{a}{b}x})$$

This result is also of importance in physical science.

Before proceeding to the next example, it is necessary to discuss two integrals which are of great importance in physics and engineering. These seem to be very elusive as, when either of them is tackled, it turns partly into the other. Yet this very fact helps us to determine their values. Let us denote these integrals by S and C, where

$$S = \int e^{pt} \sin kt\, dt, \quad \text{and} \quad C = \int e^{pt} \cos kt\, dt,$$

where p and k are constants.

To tackle these formidable-looking integrals, we resort to the device of integrating by parts, the general formula of which is

$$\int u\, dv = uv - \int v\, du$$

For this purpose, write $u = e^{pt}$ and $dv = \sin\, kt\, dt$ in S; then $du = pe^{pt}\, dt$, and $v = \int \sin\, kt\, dt = -\frac{1}{k} \cos\, kt$, omitting temporarily the constant.

Inserting these values, the integral S becomes

$$\begin{aligned} S = \int e^{pt} \sin\, kt\, dt &= -\frac{1}{k} e^{pt} \cos\, kt - \int -\frac{1}{k} \cos\, kt\, pe^{pt}\, dt \\ &= -\frac{1}{k} e^{pt} \cos\, kt + \frac{p}{k} \int e^{pt} \cos\, kt\, dt \\ &= -\frac{1}{k} e^{pt} \cos\, kt + \frac{p}{k}\, \mathrm{C} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)} \end{aligned}$$

Thus the dodge of integrating by parts turns S partly into C. But let us look at C. Writing $u = e^{pt}$, as before, $dv = \cos\, kt\, dt$, then $v = \frac{1}{k} \sin\, kt$; hence, the rule for integrating by parts gives

$$\begin{aligned} C = \int e^{pt} \cos\, kt\, dt &= \frac{1}{k} e^{pt} \sin\, kt - \frac{p}{k} \int e^{pt} \sin\, kt\, dt \\ &= \frac{1}{k} e^{pt} \sin\, kt - \frac{p}{k}\, \mathrm{S}. \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)} \end{aligned}$$

The facts that S turns partly into C, and C partly into S might lead you to think that the integrals are intractable, but from the relations (i) and (ii), which may be regarded as two equations in S and C, the integrals themselves may be readily deduced.

Thus, substitute in (i) the value of C from (ii), then

$$S = -\frac{1}{k} e^{pt} \cos kt + \frac{p}{k}\left(\frac{1}{k} e^{pt} \sin kt - \frac{p}{k} S\right)$$

or $$S\left(\frac{p^2}{k^2} + 1\right) = \frac{1}{k^2} e^{pt} (p \sin kt - k \cos kt)$$

from which $$S = \frac{e^{pt}}{p^2 + k^2} (p \sin kt - k \cos kt)$$

The integral C may be obtained in like manner by inserting in (ii) the equivalent of S given by (i); the final result is

$$C = \frac{e^{pt}}{p^2 + k^2} (p \cos kt + k \sin kt)$$

We have, therefore, the following very important integrals to add to our list, namely:

$$\int e^{pt} \sin kt \, dt = \frac{e^{pt}}{p^2 + k^2} (p \sin kt - k \cos kt) + E$$

$$\int e^{pt} \cos kt \, dt = \frac{e^{pt}}{p^2 + k^2} (p \cos kt + k \sin kt) + F$$

where E and F are the constants of integration.

Example 3.

Let $$ay + b\frac{dy}{dt} = g \sin 2\pi nt$$

First divide through by b.

$$\frac{dy}{dt} + \frac{a}{b} y = \frac{g}{b} \sin 2\pi nt$$

Now, as it stands, the left side is not integrable. But it can be made so by the artifice—and this is where skill and practice suggest a plan—of multiplying all the terms by $e^{\frac{a}{b}t}$, giving us:

$$\frac{dy}{dt}e^{\frac{a}{b}t} + \frac{a}{b}ye^{\frac{a}{b}t} = \frac{g}{b}e^{\frac{a}{b}t}\sin 2\pi nt$$

For if $u = ye^{\frac{a}{b}t}$, $\dfrac{du}{dt} = \dfrac{dy}{dt}e^{\frac{a}{b}t} + \dfrac{a}{b}ye^{\frac{a}{b}t}$

The equation thus becomes

$$\frac{du}{dt} = \frac{g}{b}e^{\frac{a}{b}t}\sin 2\pi nt$$

Hence, integrating gives

$$u \text{ or } ye^{\frac{a}{b}t} = \frac{g}{b}\int e^{\frac{a}{b}t}\sin 2\pi nt\,dt + K$$

But the right-hand integral is of the same form as S which has just been evaluated; hence putting $p = \dfrac{a}{b}$ and $k = 2\pi n$;

$$ye^{\frac{a}{b}t} = \frac{ge^{\frac{a}{b}t}}{a^2 + 4\pi^2n^2b^2}(a\sin 2\pi nt - 2\pi nb\cos 2\pi nt) + K$$

or
$$y = g\left\{\frac{a\sin 2\pi nt - 2\pi nb\cos 2\pi nt}{a^2 + 4\pi^2n^2b^2}\right\} + Ke^{-\frac{a}{b}t}$$

To simplify still further, let us imagine an angle ϕ such that $\tan\phi = 2\pi nb/a$.

Then $\sin\phi = \dfrac{2\pi nb}{\sqrt{a^2 + 4\pi^2n^2b^2}}$, and $\cos\phi = \dfrac{a}{\sqrt{a^2 + 4\pi^2n^2b^2}}$. Substituting these, we get:

$$y = g\frac{\cos\phi\sin 2\pi nt - \sin\phi\cos 2\pi nt}{\sqrt{a^2 + 4\pi^2n^2b^2}}$$

or
$$y = g\frac{\sin(2\pi nt - \phi)}{\sqrt{a^2 + 4\pi^2 n^2 b^2}}$$

which is *the solution* desired, omitting the constant which dies out.

This is indeed none other than the equation of an alternating electric current, where g represents the amplitude of the electromotive force, n the frequency, a the resistance, b the coefficient of induction of the circuit, and ϕ is the delay of a phase angle.

Example 4.

Suppose that
$$M\,dx + N\,dy = 0$$

We could integrate this expression directly, if M were a function of x only, and N a function of y only; but, if both M and N are functions that depend on both x and y, how are we to integrate it? Is it itself an exact differential? That is: have M and N each been formed by partial differentiations from some common function U, or not? If they have, then

$$\frac{\partial U}{\partial x} = M, \text{ and } \frac{\partial U}{\partial y} = N$$

And if such a common function exists, then

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

is an exact differential.

Now the test of the matter is this. If the expression is an exact differential, it must be true that

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

for then
$$\frac{\delta(\delta U)}{\delta x\,\delta y} = \frac{\delta(\delta U)}{\delta y\,\delta x}$$

which is necessarily true.

Take as an illustration the equation

$$(1 + 3xy)dx + x^2dy = 0$$

Is this an exact differential or not? Apply the test.

$$\frac{\delta(1+3xy)}{\delta y}=3x \qquad \frac{\delta(x^2)}{\delta x}=2x$$

which do not agree. Therefore, it is not an exact differential, and the two functions $1+3xy$ and x^2 have not come from a common original function.

It is possible in such cases to discover, however, *an integrating factor*, that is to say, a factor such that if both are multiplied by this factor, the expression will become an exact differential. There is no one rule for discovering such an integrating factor; but experience will usually suggest one. In the present instance $2x$ will act as such. Multiplying by $2x$, we get

$$(2x+6x^2y)dx+2x^3dy=0$$

Now apply the test to this.

$$\frac{\delta(2x+6x^2y)}{\delta y}=6x^2 \qquad \frac{\delta(2x^3)}{\delta x}=6x^2$$

which agrees. Hence this is an exact derivative, and may be integrated. Now, if $w=2x^3y$,

$$dw=6x^2y\ dx+2x^3dy$$

Hence $$\int 6x^2y\ dx+\int 2x^3\ dy=w=2x^3y$$

so that we get $$U=x^2+2x^3y+C$$

Example 5. Let $\dfrac{d^2y}{dt^2}+n^2\,y=0$

In this case we have a differential equation of the second degree, in which y appears in the form of a second derivative, as well as in person. Transposing, we have

$$\frac{d^2y}{dt^2}=-n^2y$$

It appears from this that we have to do with a function such that its second derivative is proportional to itself, but with reversed sign. In Chapter XV we found that there was such a function—namely, the *sine* (or the *cosine* also) which possessed this property. So, without further ado, we may guess that the solution will be of the form

$$y = A \sin (pt + q)$$

However, let us go to work.

Multiply both sides of the original equation by $2\dfrac{dy}{dt}$ and integrate, giving us $2\dfrac{d^2y}{dt^2}\dfrac{dy}{dt} + 2n^2y\dfrac{dy}{dt} = 0$, and, as

$$2\frac{d^2y}{dt^2}\frac{dy}{dt} = \frac{d\left(\dfrac{dy}{dt}\right)^2}{dt}, \quad \left(\frac{dy}{dt}\right)^2 + n^2(y^2 - C^2) = 0$$

C being a constant. Then, taking the square roots,

$$\frac{dy}{dt} = n\sqrt{C^2 - y^2} \quad \text{and} \quad \frac{dy}{\sqrt{C^2 - y^2}} = n \cdot dt$$

But it can be shown that

$$\frac{1}{\sqrt{C^2 - y^2}} = \frac{d\left(\text{arc sin}\dfrac{y}{C}\right)}{dy}$$

whence, passing from angles to sines,

$$\text{arc sin}\,\frac{y}{C} = nt + C_1 \text{ and } y = C \sin (nt + C_1)$$

where C_1 is a constant angle that comes in by integration.

Or, preferably, this may be written

$$y = A \sin nt + B \cos nt, \text{ which is the solution.}$$

Example 6.
$$\frac{d^2y}{dx^2} - n^2y = 0$$

Here we have obviously to deal with a function y which is such that its second derivative is proportional to itself. The only function we know that has this property is the exponential function, and we may be certain therefore that the solution of the equation will be of that form.

Proceeding as before, by multiplying through by $2\frac{dy}{dx}$, and integrating, we get $2\frac{d^2y}{dx^2}\frac{dy}{dx} - 2n^2y\,\frac{dy}{dx} = 0$, and, as

$$2\frac{d^2y}{dx^2}\frac{dy}{dx} = \frac{d\left(\frac{dy}{dx}\right)^2}{dx}, \quad \left(\frac{dy}{dx}\right)^2 - n^2(y^2 + c^2) = 0$$

$$\frac{dy}{dx} - n\sqrt{y^2 + c^2} = 0$$

where c is a constant, and $\frac{dy}{\sqrt{y^2 + c^2}} = n\,dx.$

To integrate this equation it is simpler to use hyperbolic functions.

Let $\qquad y = c \sinh u$, then $dy = c \cosh u\, du$, and

$$y^2 + c^2 = c^2\,(\sinh^2 u + 1) = c^2 \cosh^2 u.$$

$$\int \frac{dy}{\sqrt{y^2 + c^2}} = \int \frac{c \cosh u\, du}{c \cosh u} = \int du \;=\; u$$

Hence, the integral of the equation

$$n\int dx \;=\; \int \frac{dy}{\sqrt{y^2 + c^2}}$$

is
$$nx + K = u$$

where K is the constant of integration, and $c \sinh u = y$.

$$\sinh(nx+K) = \sinh u = \frac{y}{c}$$

or
$$\begin{aligned} y &= c\sinh(nx+K) \\ &= \tfrac{1}{2}c(e^{nx+K} - e^{-nx-K}) \\ &= Ae^{nx} + Be^{-nx} \end{aligned}$$

where $A = \frac{1}{2}ce^{K}$ and $B = -\frac{1}{2}ce^{-K}$.

This solution which at first sight does not look as if it had anything to do with the original equation, shows that y consists of two terms, one of which grows exponentially as x increases, while the other term dies away as x increases.

Example 7.

Let
$$b\frac{d^2y}{dt^2} + a\frac{dy}{dt} + gy = 0.$$

Examination of this expression will show that, if $b = 0$, it has the form of Example 1, the solution of which was a negative exponential. On the other hand, if $a = 0$, its form becomes the same as that of Example 6, the solution of which is the sum of a positive and a negative exponential. It is therefore not very surprising to find that the solution of the present example is

$$y = (e^{-mt})(Ae^{nt} + Be^{-nt})$$

where
$$m = \frac{a}{2b} \quad \text{and} \quad n = \frac{\sqrt{a^2 - 4bg}}{2b}$$

The steps by which this solution is reached are not given here; they may be found in advanced treatises.

Example 8.

$$\frac{\delta^2 y}{\delta t^2} = a^2\,\frac{\delta^2 y}{\delta x^2}$$

It was seen earlier that this equation was derived from the original

$$y = F(x + at) + f(x - at)$$

where F and f were any arbitrary functions of t.

Another way of dealing with it is to transform it by a change of variables into

$$\frac{\delta^2 y}{\delta u \cdot \delta v} = 0$$

where $u = x + at$, and $v = x - at$, leading to the same general solution. If we consider a case in which F vanishes, then we have simply

$$y = f(x - at)$$

and this merely states that, at the time $t = 0$, y is a particular function of x, and may be looked upon as denoting that the curve of the relation of y to x has a particular shape. Then any change in the value of t is equivalent simply to an alteration in the origin from which x is reckoned. That is to say, it indicates that, the form of the function being conserved, it is propagated along the x direction with a uniform velocity a; so that whatever the value of the ordinate y at any particular time t_0 at any particular point x_0, the same value of y will appear at the subsequent time t_1 at a point further along, the abscissa of which is $x_0 + a(t_1 - t_0)$. In this case the simplified equation represents the propagation of a wave (of any form) at a uniform speed along the x direction.

If the differential equation had been written

$$m\frac{d^2y}{dt^2} = k\frac{d^2y}{dx^2}$$

the solution would have been the same, but the velocity of propagation would have had the value

$$a = \sqrt{\frac{k}{m}}$$

EXERCISES XX

Try to solve the following equations.

(1) $\dfrac{dT}{d\theta} = \mu T$, given that μ is constant, and when $\theta = 0$, $T = T_0$.

(2) $\dfrac{d^2s}{dt^2} = a$, where a is constant. When $t = 0$, $s = 0$ and $\dfrac{ds}{dt} = u$.

(3) $\dfrac{di}{dt} + 2i = \sin 3t$, it being known that $i = 0$ when $t = 0$.

(*Hint.* Multiply out by e^{2t}.)

Chapter XXII

A LITTLE MORE ABOUT CURVATURE OF CURVES

In Chapter XII we have learned how we can find out which way a curve is curved, that is, whether it curves upwards or downwards towards the right. This gave us no indication whatever as to ***how much*** the curve is curved, or, in other words, what is its *curvature.*

By *curvature* of a curve, we mean the amount of bending or deflection taking place along a certain length of the curve, say along a portion of the curve the length of which is one unit of length (the same unit which is used to measure the radius, whether it be one inch, one foot, or any other unit). For instance, consider two circular paths of center O or O' and of equal lengths AB, $A'B'$ (see Fig. 64). When passing from A to B along the arc AB of the first one, one changes one's direction from AP to BQ, since at A

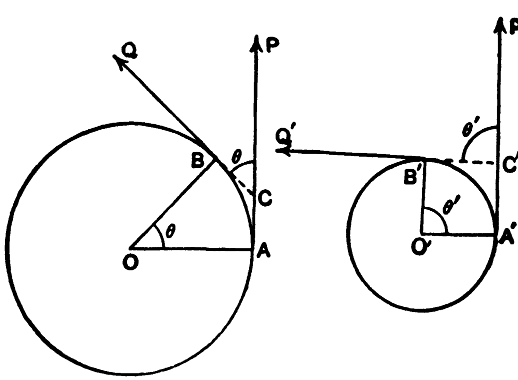


FIG. 64.

one faces in the direction AP and at B one faces in the direction BQ. In other words, in walking from A to B one unconsciously turns round through the angle PCQ, which is equal to the angle AOB. Similarly, in passing from A' to B', along the arc $A'B'$, of equal length to AB, on the second path, one turns round through the angle $P'C'Q'$, which is equal to the angle $A'O'B'$, obviously, *greater* than the corresponding angle AOB. The second path bends therefore more than the first for an equal length.

This fact is expressed by saying that the *curvature* of the second path is greater than that of the first one. The larger the circle, the lesser the bending, that is, the lesser the curvature. If the radius of the first circle is 2, 3, 4, . . . etc. times greater than the radius of the second, then the angle of bending or deflection along an arc of unit length will be 2, 3, 4, . . . etc. times less on the first circle than on the second, that is, it will be $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ etc. of the bending or deflection along the arc of same length on the second circle. In other words, the *curvature* of the first circle will be $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ etc. of that of the second circle. We see that, as the radius becomes 2, 3, 4, . . . etc. times greater, the curvature becomes 2, 3, 4, . . . etc. times smaller, and this is expressed by saying that *the curvature of a circle is inversely proportional to the radius of the circle,* or

$$\text{curvature} = k \times \frac{1}{\text{radius}}$$

where k is a constant. It is agreed to take $k = 1$, so that

$$\text{curvature} = \frac{1}{\text{radius}}$$

always.

If the radius becomes infinitely great, the curvature becomes $\dfrac{1}{\text{infinity}} = \text{zero}$, since when the denominator of a fraction is infinitely large, the value of the fraction is infinitely small. For this reason mathematicians sometimes consider a straight line as an arc of circle of infinite radius, or zero curvature.

In the case of a circle, which is perfectly symmetrical and uniform, so that the curvature is the same at every point of its circumference, the above method of expressing the curvature is perfectly definite. In the case of any other curve, however, the curvature is not the same at different points, and it may differ considerably even for two points fairly close to one another. It would not then be accurate to take the amount of bending or deflection between two points as a measure of the curvature of the arc between these points, unless this arc is very small, in fact, unless it is infinitely small.

If then we consider a very small arc such as AB (see Fig. 65), and if we draw such a circle that an arc AB of this circle coincides with the arc AB of the curve more closely than would be the case with any other circle, then the curvature of this circle may be taken as the curvature of the arc AB of the curve. The smaller the arc AB, the easier it will be to find a circle an arc of which most nearly coincides with the arc AB of the curve. When A and B are very near one another, so that AB is so small so that the length ds of the arc AB is practically negligible, then the coincidence of the two arcs, of circle and of curve, may be considered as being practically perfect, and the curvature of the curve at the point A (or B), being then the same as the curvature of the circle, will be expressed by the reciprocal of the radius of this circle, that is, by $\frac{1}{OA}$, according to our way of measuring curvature, explained above.

Now, at first, you may think that, if AB is very small, then the circle must be very small also. A little thinking will, however,

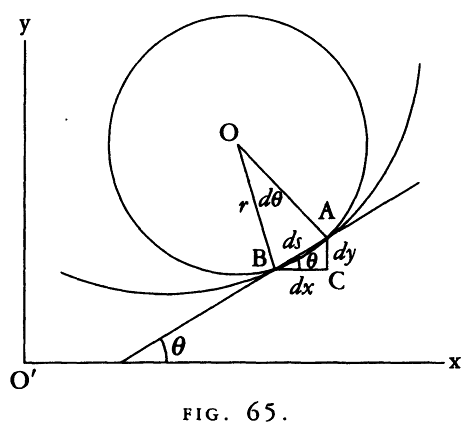


FIG. 65.

cause you to perceive that it is by no means necessarily so, and that the circle may have any size, according to the amount of bending of the curve along this very small arc AB. In fact, if the curve is almost flat at that point, the circle will be extremely large. This circle is called the *circle of curvature,* or the *osculating circle* at the point considered. Its radius is the *radius of curvature* of the curve at that particular point.

If the arc AB is represented by ds and the angle AOB by $d\theta$, then, if r is the radius of curvature,

$$ds = r\, d\theta \quad \text{or} \quad \frac{d\theta}{ds} = \frac{1}{r}$$

The secant AB makes with the axis OX the angle θ, and it will be seen from the small triangle ABC that $\dfrac{dy}{dx} = \tan\theta$. When AB is infinitely small, so that B practically coincides with A, the line AB becomes a tangent to the curve at the point A (or B).

Now, $\tan\theta$ depends on the position of the point A (or B, which is supposed to nearly coincide with it), that is, it depends on x, or, in other words, $\tan\theta$ is "a function" of x.

Differentiating with regard to x to get the slope, we get

$$\frac{d\left(\dfrac{dy}{dx}\right)}{dx} = \frac{d(\tan\theta)}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = \sec^2\theta\,\frac{d\theta}{dx} = \frac{1}{\cos^2\theta}\,\frac{d\theta}{dx}$$

hence $$\frac{d\theta}{dx} = \cos^2\theta\,\frac{d^2y}{dx^2}$$

But $\dfrac{dx}{ds} = \cos\theta$, and for $\dfrac{d\theta}{ds}$ one may write $\dfrac{d\theta}{dx} \times \dfrac{dx}{ds}$; therefore

$$\frac{1}{r} = \frac{d\theta}{ds} = \frac{d\theta}{dx} \times \frac{dx}{ds} = \cos^3\theta\,\frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dx^2}}{\sec^3\theta}$$

but $\sec\theta = \sqrt{1 + \tan^2\theta}$; hence

$$\frac{1}{r}=\frac{\dfrac{d^2y}{dx^2}}{\left(\sqrt{1+\tan^2\theta}\right)^3}=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

and finally,
$$r=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

The numerator, being a square root, may have the sign + or the sign −. One must select for it the same sign as the denominator, so as to have r positive always, as a negative radius would have no meaning.[1]

It has been shown (Chapter XII) that if $\dfrac{d^2y}{dx^2}$ is positive, the curve is concave upword, while if $\dfrac{d^2y}{dx^2}$ is negative, the curve is concave downward. If $\dfrac{d^2y}{dx^2}=0$, the radius of curvature is infinitely great, that is, the corresponding portion of the curve is a bit of straight line. This necessarily happens whenever a curve gradually changes from being convex to concave to the axis of x or vice versa. The point where this occurs is called a *point of inflection.*

The center of the circle of curvature is called the *center of curvature.* If its coordinates are x_1, y_1, then the equation of the circle is

$$(x-x_1)^2+(y-y_1)^2=r^2$$

hence
$$2(x-x_1)dx+2(y-y_1)dy=0$$

and
$$x-x_1+(y-y_1)\frac{dy}{dx}=0 \quad\text{.......................(1)}$$

1. Thompson is not always clear as to when a square root is to be taken as positive. In modern textbooks $\sqrt{x}$ (or in this case $x^{\frac{3}{2}}$) means the positive value. If we want to allow either the positive or the negative value we write $\pm\sqrt{x}$. The formula for the radius of curvature is usually written with $\left|\dfrac{d^2y}{dx^2}\right|$ in the denominator; since the numerator is positive, this makes r positive.—M.G.

Why did we differentiate? To get rid of the constant r. This leaves but two unknown constants x_1 and y_1; differentiate again; you shall get rid of one of them. This last differentiation is not quite as easy as it seems: let us do it together; we have:

$$\frac{d(x)}{dx} + \frac{d\left[(y - y_1)\dfrac{dy}{dx}\right]}{dx} = 0$$

the numerator of the second term is a product; hence differentiating it gives

$$(y - y_1)\frac{d\left(\dfrac{dy}{dx}\right)}{dx} + \frac{dy}{dx}\,\frac{d(y - y_1)}{dx} = (y - y_1)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

so that the result of differentiating (1) is

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - y_1)\frac{d^2y}{dx^2} = 0$$

from this we at once get

$$y_1 = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}$$

Replacing in (1), we get

$$(x - x_1) + \left\{ y - y - \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} \right\} \frac{dy}{dx} = 0$$

and finally,

$$x_1 = x - \frac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}}$$

x_1 and y_1 give the position of the center of curvature. The use of these formulae will be best seen by carefully going through a few worked-out examples.

Example 1. Find the radius of curvature and the coordinates of the center of curvature of the curve $y = 2x^2 - x + 3$ at the point $x = 0$.

We have $$\frac{dy}{dx} = 4x - 1 \quad \frac{d^2y}{dx^2} = 4$$

$$r = \frac{\pm\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\{1 + (4x - 1)^2\}^{\frac{3}{2}}}{4}$$

when $x = 0$; this becomes

$$\frac{\{1 + (-1)^2\}^{\frac{3}{2}}}{4} = \frac{\sqrt{8}}{4} = 0.707$$

If x_1, y_1 are the coordinates of the center of curvature, then

$$x_1 = x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}} = x - \frac{(4x - 1)\{1 + (4x - 1)^2\}}{4}$$

$$= 0 - \frac{(-1)\{1 + (-1)^2\}}{4} = \frac{1}{2}$$

when $x = 0$, $y = 3$, so that

$$y_1 = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1 + (4x - 1)^2}{4} = 3 + \frac{1 + (-1)^2}{4} = 3\tfrac{1}{2}$$

Plot the curve and draw the circle; it is both interesting and instructive. The values can be checked easily, as since when $x = 0$, $y = 3$, here

$$x_1^2 + (y_1 - 3)^2 = r^2 \quad \text{or} \quad 0.5^2 + 0.5^2 = 0.5 = 0.707^2$$

Example 2. Find the radius of curvature and the position of the center of curvature of the curve $y^2 = mx$ at the point for which $y = 0$.

Here $$y = m^{\frac{1}{2}}x^{\frac{1}{2}},\ \frac{dy}{dx} = \frac{1}{2}m^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{m^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2} \times \frac{m^{\frac{1}{2}}}{2}\, x^{-\frac{3}{2}} = -\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}$$

hence $$\frac{\pm\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\pm\left\{1 + \frac{m}{4x}\right\}^{\frac{3}{2}}}{-\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}} = \frac{(4x+m)^{\frac{3}{2}}}{2m^{\frac{1}{2}}}$$

taking the $-$ sign at the numerator, so as to have r positive.

Since, when $y = 0$, $x = 0$, we get $r = \frac{m^{\frac{3}{2}}}{2m^{\frac{1}{2}}} = \frac{m}{2}$

Also, if x_1, y_1 are the coordinates of the center,

$$x_1 = x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{d^2x}} = x - \frac{\frac{m^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\left\{1 + \frac{m}{4x}\right\}}{-\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}}$$

$$= x + \frac{4x+m}{2} = 3x + \frac{m}{2}$$

when $x = 0$, then $x_1 = \frac{m}{2}$

Also $$y_1 = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = m^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{1 + \frac{m}{4x}}{\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}} = -\frac{4x^{\frac{3}{2}}}{m^{\frac{1}{2}}}$$

when $x = 0$, $y_1 = 0$.

Example 3. Show that the circle is a curve of constant curvature.

If x_1, y_1 are the coordinates of the center, and R is the radius, the equation of the circle in rectangular coordinates is

$$(x - x_1)^2 + (y - y_1)^2 = R^2$$

Let $x - x_1 = R \cos \theta$, then

$$(y - y_1)^2 = R^2 - R^2 \cos^2\theta = R^2(1 - \cos^2\theta) = R^2 \sin^2\theta$$

$$y - y_1 = R \sin \theta$$

R, θ are thus the polar coordinates of any point on the circle referred to its center as pole.

Since $x - x_1 = R \cos \theta$, and $y - y_1 = R \sin \theta$,

$$\frac{dx}{d\theta} = -R \sin \theta, \quad \frac{dy}{d\theta} = R \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = -\cot \theta$$

Further, $$\frac{d^2y}{dx^2} = -(-\csc^2\theta)\,\frac{d\theta}{dx} = \csc^2\theta \cdot \left(-\frac{\csc \theta}{R}\right)$$

$$= -\frac{\csc^3 \theta}{R}. \quad \text{(See Ex. 5, \textit{Chapter XV.})}$$

Hence $$r = \frac{\pm(1 + \cot^2 \theta)^{\frac{3}{2}}}{-\dfrac{\csc^3\theta}{R}} = \frac{R \csc^3\theta}{\csc^3\theta} = R$$

Thus the radius of curvature is constant and equal to the radius of the circle.

Example 4. Find the radius of curvature of the curve $x = 2 \cos^3 t$, $y = 2 \sin^3 t$ at any point (x, y).

Here $dx = -6 \cos^2 t \sin t\, dt$ (see Ex. 2, Chapter XV)

and $dy = 6 \sin^2 t \cos t\, dt$.

$$\frac{dy}{dx} = -\frac{6 \sin^2 t \cos t\, dt}{6 \sin t \cos^2 t\, dt} = -\frac{\sin t}{\cos t} = -\tan t$$

Hence $$\frac{d^2y}{dx^2} = \frac{d}{dt}(-\tan t)\,\frac{dt}{dx} = \frac{-\sec^2 t}{-6\cos^2 t \sin t} = \frac{\sec^4 t}{6 \sin t}$$

$$r = \frac{\pm(1+\tan^2 t)^{\frac{3}{2}} \times 6 \sin t}{\sec^4 t} = \frac{6\sec^3 t \sin t}{\sec^4 t}$$

$$= 6 \sin t \cos t = 3 \sin 2t, \text{ for } 2 \sin t \cos t = \sin 2t$$

Example 5. Find the radius and the center of curvature of the curve $y = x^3 - 2x^2 + x - 1$ at points where $x = 0$, $x = 0.5$ and $x = 1$. Find also the position of the point of inflection of the curve.

Here $$\frac{dy}{dx} = 3x^2 - 4x + 1,\ \frac{d^2y}{dx^2} = 6x - 4$$

$$r = \frac{\{1+(3x^2-4x+1)^2\}^{\frac{3}{2}}}{6x-4}$$

$$x_1 = x - \frac{(3x^2-4x+1)\{1+(3x^2-4x+1)^2\}}{6x-4}$$

$$y_1 = y + \frac{1+(3x^2-4x+1)^2}{6x-4}$$

When $x = 0$, $y = -1$,

$$r = \frac{\sqrt{8}}{4} = 0.707,\ x_1 = 0 + \tfrac{1}{2} = 0.5,\ y_1 = -1 - \tfrac{1}{2} = -1.5.$$

When $x = 0.5$, $y = -0.875$:

$$r = \frac{-\{1+(-0.25)^2\}^{\frac{3}{2}}}{-1} = 1.09$$

$$x_1 = 0.5 - \frac{-0.25 \times 1.0625}{-1} = 0.23$$

$$y_1 = -0.875 + \frac{1.0625}{-1} = -1.94$$

When $x = 1$, $y = -1$.

$$r = \frac{(1+0)^{\frac{3}{2}}}{2} = 0.5$$

$$x_1 = 1 - \frac{0 \times (1+0)}{2} = 1$$

$$y_1 = -1 + \frac{1+0^2}{2} = -0.5$$

At the point of inflection $\frac{d^2y}{dx^2} = 0$, $6x - 4 = 0$, and $x = \frac{2}{3}$; hence $y = -0.926$.

Example 6. Find the radius and center of curvature of the curve $y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$, at the point for which $x = 0$. (This curve is called the *catenary,* as a hanging chain affects the same slope exactly.) The equation of the curve may be written

$$y = \frac{a}{2}e^{\frac{x}{a}} + \frac{a}{2}e^{-\frac{x}{a}}$$

then,

$$\frac{dy}{dx} = \frac{a}{2} \times \frac{1}{a}e^{\frac{x}{a}} - \frac{a}{2} \times \frac{1}{a}e^{-\frac{x}{a}} = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$$

Similarly

$$\frac{d^2y}{dx^2} = \frac{1}{2a}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\} = \frac{1}{2a} \times \frac{2y}{a} = \frac{y}{a^2}$$

$$r = \frac{\left\{1 + \frac{1}{4}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)^2\right\}^{\frac{3}{2}}}{\frac{y}{a^2}} = \frac{a^2}{8y}\sqrt{\left(2 + e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}\right)^3}$$

since $e^{\frac{x}{a}-\frac{x}{a}} = e^0 = 1$, or

$$r = \frac{a^2}{8y}\sqrt{\left(2e^{\frac{x}{a}-\frac{x}{a}} + e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}\right)^3} = \frac{a^2}{8y}\sqrt{\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^6} = \frac{y^2}{a}$$

when $$x = 0,\ y = \frac{a}{2}(e^0 + e^0) = a$$

hence $$r = \frac{a^2}{a} = a$$

The radius of curvature at the vertex is equal to the constant a.

Also when $$x = 0,\ x_1 = 0 - \frac{0(1+0)}{\frac{1}{a}} = 0$$

and $$y_1 = y + \frac{1+0}{\frac{1}{a}} = a + a = 2a$$

As defined earlier,

$$\frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = \cosh\frac{x}{a}$$

thus the equation of the catenary may be written in the form

$$y = a\cosh\frac{x}{a}$$

It will therefore be a useful exercise for you to verify the above results from this form of the equation.

You are now sufficiently familiar with this type of problem to work out the following exercises by yourself. You are advised to check your answers by careful plotting of the curve and construction of the circle of curvature, as explained in Example 4.

EXERCISES XXI

(1) Find the radius of curvature and the position of the center of curvature of the curve $y = e^x$ at the point for which $x = 0$.

(2) Find the radius and the center of curvature of the curve $y = x\left(\frac{x}{2} - 1\right)$ at the point for which $x = 2$.

(3) Find the point or points of curvature unity in the curve $y = x^2$.

(4) Find the radius and the center of curvature of the curve $xy = m$ at the point for which $x = \sqrt{m}$.

(5) Find the radius and the center of curvature of the curve $y^2 = 4ax$ at the point for which $x = 0$.

(6) Find the radius and the center of curvature of the curve $y = x^3$ at the points for which $x = \pm 0.9$ and also $x = 0$.

(7) Find the radius of curvature and the coordinates of the center of curvature of the curve

$$y = x^2 - x + 2$$

at the two points for which $x = 0$ and $x = 1$, respectively. Find also the maximum or minimum value of y. Verify graphically all your results.

(8) Find the radius of curvature and the coordinates of the center of curvature of the curve

$$y = x^3 - x - 1$$

at the points for which $x = -2$, $x = 0$, and $x = 1$.

(9) Find the coordinates of the point or points of inflection of the curve $y = x^3 + x^2 + 1$.

(10) Find the radius of curvature and the coordinates of the center of curvature of the curve

$$y = (4x - x^2 - 3)^{\frac{1}{2}}$$

at the points for which $x = 1.2$, $x = 2$ and $x = 2.5$. What is this curve?

(11) Find the radius and the center of curvature of the curve $y = x^3 - 3x^2 + 2x + 1$ at the points for which $x = 0$, $x = +1.5$. Find also the position of the point of inflection.

(12) Find the radius and center of curvature of the curve $y = \sin\theta$ at the points for which $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$. Find the position of the points of inflection.

(13) Draw a circle of radius 3, the center of which has for its

coordinates $x = 1$, $y = 0$. Deduce the equation of such a circle from first principles. Find by calculation the radius of curvature and the coordinates of the center of curvature for several suitable points, as accurately as possible, and verify that you get the known values.

(14) Find the radius and center of curvature of the curve $y = \cos\theta$ at the points for which $\theta = 0$, $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{2}$.

(15) Find the radius of curvature and the center of curvature of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the points for which $x = 0$ and at the points for which $y = 0$.

(16) When a curve is defined by equations in the form

$$x = F(\theta), \quad y = f(\theta)$$

the radius r of curvature is given by

$$r = \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}} \Bigg/ \left(\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}\right)$$

Apply the formula to find r for the curve

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

Chapter XXIII

HOW TO FIND THE LENGTH OF AN ARC ON A CURVE

Since an arc on any curve is made up of a lot of little bits of straight lines joined end to end, if we could add all these little bits, we would get the length of the arc. But we have seen that to add a lot of little bits together is precisely what is called integration, so that it is likely that, since we know how to integrate, we can find also the length of an arc on any curve, provided that the equation of the curve is such that it lends itself to integration.

If *MN* is an arc on any curve, the length s of which is required (see Fig. 66*a*), if we call "a little bit" of the arc ds, then we see at once that

$$(ds)^2 = (dx)^2 + (dy)^2$$

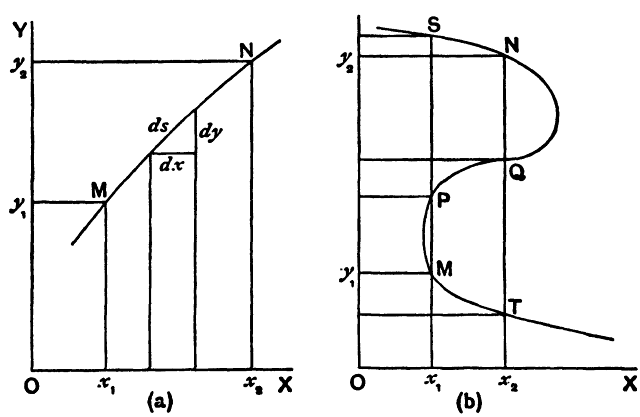


FIG. 66.

or either

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now the arc MN is made up of the sum of all the little bits ds between M and N, that is, between x_1 and x_2, or between y_1 and y_2, so that we get either

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

That is all!

The second integral is useful when there are several points of the curve corresponding to the given values of x (as in Fig. 66*b*). In this case the integral between x_1 and x_2 leaves a doubt as to the exact portion of the curve, the length of which is required. It may be ST, instead of MN, or SQ; by integrating between y_1 and y_2 the uncertainty is removed, and in this case one should use the second integral.

If instead of x and y coordinates—or Cartesian coordinates, as they are named from the French mathematician Descartes, who invented them—we have r and θ coordinates (or polar coordinates); then, if MN be a small arc of length ds on any curve, the length s of which is required (see Fig. 67), O being the pole, then the distance ON will generally differ from OM by a small amount dr. If the small angle MON is called $d\theta$, then, the polar coordinates of the point M being θ and r, those of N are $(\theta + d\theta)$ and $(r + dr)$. Let MP be perpendicular to ON, and let $OR = OM$; then

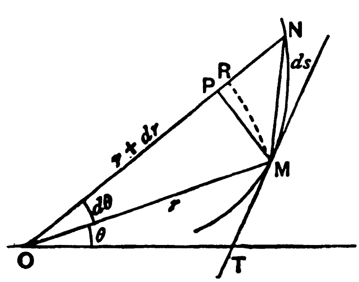


FIG. 67.

$RN = dr$, and this is very nearly the same as PN, as long as $d\theta$ is a very small angle. Also $RM = r\,d\theta$, and RM is very nearly equal to PM, and the arc MN is very nearly equal to the chord MN. In fact we can write $PN = dr$, $PM = r\,d\theta$, and arc MN = chord MN without appreciable error, so that we have:

$$(ds)^2 = (\text{chord } MN)^2 = \overline{PN}^2 + \overline{PM}^2 = dr^2 + r^2 d\theta^2$$

Dividing by $d\theta^2$ we get $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2$; hence

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad \text{and} \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

hence, since the length s is made up of the sum of all the little bits ds, between values of $\theta = \theta_1$ and $\theta = \theta_2$, we have

$$s = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

We can proceed at once to work out a few examples.

Example 1. The equation of a circle, the center of which is at the origin—or intersection of the axis of x with the axis of y—is $x^2 + y^2 = r^2$; find the length of an arc of one quadrant.

$$y^2 = r^2 - x^2 \text{ and } 2y\,dy = -2x\,dx, \text{ so that } \frac{dy}{dx} = -\frac{x}{y}$$

hence

$$s = \int \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dx = \int \sqrt{\left(1 + \frac{x^2}{y^2}\right)} dx$$

and since $y^2 = r^2 - x^2$,

$$s = \int \sqrt{\left(1 + \frac{x^2}{r^2 - x^2}\right)} dx = \int \frac{r\,dx}{\sqrt{r^2 - x^2}}$$

The length we want—one quadrant—extends from a point for which $x = 0$ to another point for which $x = r$. We express this by writing

$$s = \int_{x=0}^{x=r} \frac{r\,dx}{\sqrt{r^2 - x^2}}$$

or, more simply, by writing

$$s = \int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}}$$

the 0 and r to the right of the sign of integration merely meaning that the integration is only to be performed on a portion of the curve, namely, that between $x = 0$, $x = r$, as we have seen.

Here is a fresh integral for you! Can you manage it?

In Chapter XV we differentiated $y = \arcsin x$ and found $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$. If you have tried all sorts of variations of the given examples (as you ought to have done!), you perhaps tried to differentiate something like $y = a \arcsin \dfrac{x}{a}$, which gave

$$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 - x^2}} \quad \text{or} \quad dy = \frac{a\,dx}{\sqrt{a^2 - x^2}}$$

that is, just the same expression as the one we have to integrate here.

Hence $s = \displaystyle\int \frac{r\,dx}{\sqrt{r^2 - x^2}} = r \arcsin \frac{x}{r} + C$, C being a constant.

As the integration is only to be made between $x = 0$ and $x = r$, we write

$$s = \int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}} = \left[r \arcsin \frac{x}{r} + C \right]_0^r$$

proceeding then as explained in Example (1), Chapter XIX, we get

$$s = r \arcsin \frac{r}{r} + C - r \arcsin \frac{0}{r} - C, \quad \text{or} \quad s = r \times \frac{\pi}{2}$$

since arcsin 1 is 90° or $\frac{\pi}{2}$ and arcsin 0 is zero, and the constant C disappears, as has been shown.

The length of the quadrant is therefore $\frac{\pi r}{2}$, and the length of the circumference, being four times this, is $4 \times \frac{\pi r}{2} = 2\pi r$.

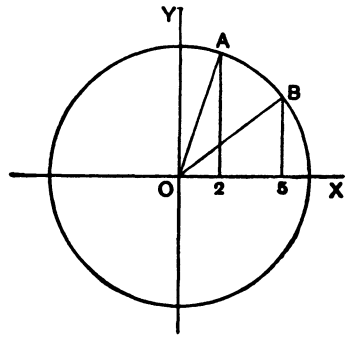


FIG. 68.

Example 2. Find the length of the arc AB between $x_1 = 2$ and $x_2 = 5$, in the circumference $x^2 + y^2 = 6^2$ (see Fig. 68).

Here, proceeding as in the previous example,

$$s = \left[r \arcsin \frac{x}{r} + C \right]_{x_1}^{x_2} = \left[6 \arcsin \frac{x}{6} + C \right]_2^5$$

$$= 6\left[\arcsin \frac{5}{6} - \arcsin \frac{2}{6} \right] = 6(0.9851 - 0.3398)$$

$$= 3.8716 \text{ (the arcs being expressed in radians).}$$

It is always well to check results obtained by a new and yet unfamiliar method. This is easy, for

$$\cos AOX = \tfrac{2}{6} = \tfrac{1}{3} \text{ and } \cos BOX = \tfrac{5}{6}$$

hence $AOB = AOX - BOX = \arccos \frac{1}{3} = \arccos \frac{5}{6} = 0.6453$ radians, and the length is $6 \times 0.6453 = 3.8716$.

Example 3. Find the length of an arc of the curve

$$y = \frac{a}{2}\left\{ e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right\}$$

between $x = 0$ and $x = a$. (This curve is the *catenary*.)

$$y = \frac{a}{2}e^{\frac{x}{a}} + \frac{a}{2}e^{-\frac{x}{a}}, \quad \frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$$

$$s = \int \sqrt{1 + \frac{1}{4}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}^2}\, dx$$

$$= \frac{1}{2}\int \sqrt{4 + e^{\frac{2x}{a}} + e^{-\frac{2x}{a}} - 2e^{\frac{x}{a}-\frac{x}{a}}}\, dx$$

Now

$$e^{\frac{x}{a}-\frac{x}{a}} = e^0 = 1, \text{ so that } s = \frac{1}{2}\int \sqrt{2 + e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}}\, dx$$

we can replace 2 by $2 \times e^0 = 2 \times e^{\frac{x}{a}-\frac{x}{a}}$; then

$$s = \frac{1}{2}\int \sqrt{e^{\frac{2x}{a}} + 2e^{\frac{x}{a}-\frac{x}{a}} + e^{-\frac{2x}{a}}}\, dx$$

$$= \frac{1}{2}\int \sqrt{\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2}\, dx = \frac{1}{2}\int \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx$$

$$= \frac{1}{2}\int e^{\frac{x}{a}}\, dx + \frac{1}{2}\int e^{-\frac{x}{a}}\, dx = \frac{a}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right]$$

Here $\quad s = \frac{a}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right]_0^a = \frac{a}{2}[e^1 - e^{-1} - 1 + 1]$

and $\quad s = \frac{a}{2}\left(e - \frac{1}{e}\right) = 1.1752a.$

Example 4. A curve is such that the length of the tangent at any point P (see Fig. 69) from P to the intersection T of the tangent with a fixed line AB is a constant length a. Find an expression for

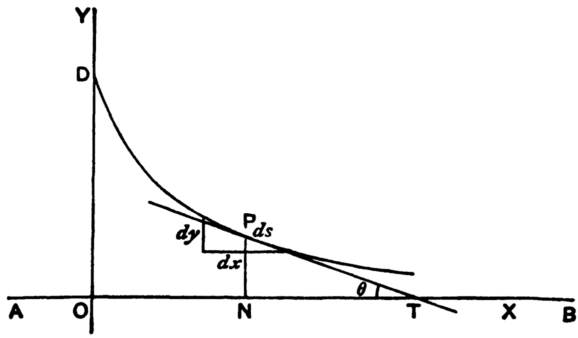


FIG. 69.

an arc of this curve—which is called the tractrix[1]—and find the length, when $a = 3$, between the ordinates $y = a$ and $y = 1$.

We shall take the fixed line for the axis of x. The point D, with $DO = a$, is a point on the curve, which must be tangent to OD at D. We take OD as the axis of y. $PT = a$, $PN = y$, $ON = x$.

If we consider a small portion ds of the curve, at P, then $\sin\,\theta = \frac{dy}{ds} = -\frac{y}{a}$ (minus because the curve slopes *downwards* to the right.

Hence $\frac{ds}{dy} = -\frac{a}{y}$, $ds = -a\frac{dy}{y}$ and $s = -a\int\frac{dy}{y}$, that is,

$$s = -a \ln y + C$$

When $x = 0$, $s = 0$, $y = a$, so that $0 = -a \ln a + C$, and $C = a \ln a$.

It follows that $s = a \ln a - a \ln y = a \ln \frac{a}{y}$.

1. The tractrix is the involute of the catenary. The involute of a curve is the curve traced by the end of a taut thread as it is unwound from a given curve.

The origin of the name tractrix is interesting. It can be demonstrated by the following experiment. Attach a length of string to a small object. Place the object at the center of a large rectangular table, then extend the string horizontally until it hangs over, say, the table's right edge. Grab the string where it touches the table edge, then move it along the edge. The track generated by the object as it is dragged along the table is a tractrix.—M.G.

When $a=3$, s between $y=a$ and $y=1$ is therefore

$$s=3\left[\ln\frac{3}{y}\right]_1^3=3(\ln 1-\ln 3)=3\times(0-1.0986)$$

$$=-3.296 \text{ or } 3.296,$$

as the sign $-$ refers merely to the direction in which the length was measured, from D to P, or from P to D.

Note that this result has been obtained without a knowledge of the equation of the curve. This is sometimes possible. In order to get the length of an arc between two points given by their abscissae, however, it is necessary to know the equation of the curve; this is easily obtained as follows:

$$\frac{dy}{dx}=-\tan\theta=-\frac{y}{\sqrt{a^2-y^2}}, \text{ since } PT=a$$

hence $dx=-\dfrac{\sqrt{a^2-y^2}\,dy}{y}$, and $x=-\displaystyle\int\frac{\sqrt{a^2-y^2}\,dy}{y}$

The integration will give us a relation between x and y, which is the equation of the curve.

To effect the integration, let $u^2=a^2-y^2$, then

$$2u\,du=-2y\,dy, \quad\text{or}\quad u\,du=-y\,dy$$

$$x=\int\frac{u^2\,du}{y^2}=\int\frac{u^2\,du}{a^2-u^2}=\int\frac{a^2-(a^2-u^2)}{a^2-u^2}\cdot du$$

$$=a^2\int\frac{du}{a^2-u^2}-\int du$$

$$=a^2\cdot\frac{1}{2a}\ln\frac{a+u}{a-u}-u+C$$

$$=\tfrac{1}{2}a\ln\frac{(a+u)(a+u)}{(a-u)(a+u)}-u+C$$

$$=a\ln\frac{a+u}{\sqrt{a^2-u^2}}-u+C$$

We have then, finally,

$$x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} + C$$

When $x = 0$, $y = a$, so that $0 = a \ln 1 - 0 + C$, and $C = 0$; the equation of the tractrix is therefore

$$x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$$

Example 5. Find the length of an arc of the logarithmic spiral $r = e^\theta$ between $\theta = 0$ and $\theta = 1$ radian.

Do you remember differentiating $y = e^x$? It is an easy one to remember, for it remains always the same whatever is done to it: $\dfrac{dy}{dx} = e^x$.

Here, since $\qquad r = e^\theta, \dfrac{dr}{d\theta} = e^\theta = r$

If we reverse the process and integrate $\int e^\theta \, d\theta$ we get back to $r + C$, the constant C being always introduced by such a process, as we have seen in Chap. XVII.

It follows that

$$s = \int \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta = \int \sqrt{(r^2 + r^2)} d\theta$$

$$= \sqrt{2} \int r \, d\theta = \sqrt{2} \int e^\theta \, d\theta = \sqrt{2} \, (e^\theta + C)$$

Integrating between the two given values $\theta = 0$ and $\theta = 1$, we get

$$s = \int_0^1 \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta = \left[\sqrt{2}(e^\theta + C)\right]_0^1$$

$$= \sqrt{2}e^1 - \sqrt{2}e^0 = \sqrt{2}(e - 1)$$

$$= 1.41 \times 1.718 = 2.43$$

Example 6. Find the length of an arc of the logarithmic spiral $r = e^{\theta}$ between $\theta = 0$ and $\theta = \theta_1$.

As we have just seen,

$$s = \sqrt{2}\int_0^{\theta_1} e^{\theta}\,d\theta = \sqrt{2}[e^{\theta_1} - e^0] = \sqrt{2}(e^{\theta_1} - 1)$$

Example 7. As a last example let us work fully a case leading to a typical integration which will be found useful for several of the exercises found at the end of this chapter. Let us find the expression for the length of an arc of the curve $y = \frac{a}{2}x^2 + 3$.

$$\frac{dy}{dx} = ax, \quad s = \int\sqrt{1 + a^2x^2}\,dx$$

To work out this integral, let $ax = \sinh z$, then $a\,dx = \cosh z\,dz$, and $1 + a^2x^2 = 1 + \sinh^2 z = \cosh^2 z$;

$$s = \frac{1}{a}\int \cosh^2 z\,dz = \frac{1}{4a}\int (e^{2z} + 2 + e^{-2z})dz$$

$$= \frac{1}{4a}\left[\tfrac{1}{2}e^{2z} + 2z - \tfrac{1}{2}e^{-2z}\right] = \frac{1}{8a}[(e^z)^2 - (e^{-z})^2 + 4z]$$

$$= \frac{1}{8a}(e^z - e^{-z})(e^z + e^{-z}) + \frac{z}{2a}$$

$$= \frac{1}{2a}(\sinh z \cosh z + z) = \frac{1}{2a}\left(ax\sqrt{1 + a^2x^2} + z\right)$$

To turn z back into terms of x, we have

$$ax = \sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

Multiply out by $2e^z$,

$$2axe^z = e^{2z} - 1$$

or $$(e^z)^2 - 2ax(e^z) - 1 = 0$$

This is a quadratic equation in e^z, and taking the positive root:

$$e^z = \tfrac{1}{2}\left(2ax + \sqrt{4a^2x^2 + 4}\right) = ax + \sqrt{1 + a^2x^2}$$

Taking natural logarithms:

$$z = \ln\left(ax + \sqrt{1 + a^2x^2}\right)$$

Hence, the integral becomes finally:

$$s = \int \sqrt{1 + a^2x^2}\,dx = \frac{x}{2}\sqrt{1 + a^2x^2} + \frac{1}{2a}\ln\left(ax + \sqrt{1 + a^2x^2}\right)$$

From several of the foregoing examples, some very important integrals and relations have been worked out. As these are of great use in solving many other problems, it will be an advantage to collect them here for future reference.

Inverse Hyperbolic Functions

If $x = \sinh z$, z is written inversely as $\sinh^{-1}x$;

and $$z = \sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Similarly, if $x = \cosh z$,

$$z = \cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Irrational Quadratic Integrals

(i) $$\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a\ln\frac{a + \sqrt{a^2 - x^2}}{x} + C$$

(ii) $$\int \sqrt{a^2 + x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2 + x^2} + \tfrac{1}{2}a^2\ln\left(x + \sqrt{a^2 + x^2}\right) + C$$

To these may be added:

(iii) $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right) + C$$

For, if $x = a \sinh u$, $dx = a \cosh u\, du$, and

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int du = u + C' = \sinh^{-1}\frac{x}{a} + C'$$

$$= \ln \frac{x+\sqrt{a^2+x^2}}{a} + C'$$

$$= \ln\left(x+\sqrt{a^2+x^2}\right) + C$$

You ought now to be able to attempt with success the following exercises. You will find it interesting as well as instructive to plot the curves and verify your results by measurement where possible.

EXERCISE XXII

(1) Find the length of the line $y = 3x + 2$ between the two points for which $x = 1$ and $x = 4$.

(2) Find the length of the line $y = ax + b$ between the two points for which $x = -1$ and $x = a^2$.

(3) Find the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between the two points for which $x = 0$ and $x = 1$.

(4) Find the length of the curve $y = x^2$ between the two points for which $x = 0$ and $x = 2$.

(5) Find the length of the curve $y = mx^2$ between the two points for which $x = 0$ and $x = \dfrac{1}{2m}$.

(6) Find the length of the curve $x = a \cos\theta$ and $y = a \sin\theta$ between $\theta = \theta_1$ and $\theta = \theta_2$.

(7) Find the length of the arc of the curve $r = a \sec\theta$ from $\theta = 0$ to an arbitrary point on the curve.

(8) Find the length of the arc of the curve $y^2 = 4ax$ between $x = 0$ and $x = a$.

(9) Find the length of the arc of the curve $y = x\left(\dfrac{x}{2} - 1\right)$ between $x = 0$ and $x = 4$.

(10) Find the length of the arc of the curve $y = e^x$ between $x = 0$ and $x = 1$.

(*Note.* This curve is in rectangular coordinates, and is not the same curve as the logarithmic spiral $r = e^\theta$ which is in polar coordinates. The two equations are similar, but the curves are quite different.)

(11) A curve is such that the coordinates of a point on it are $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, θ being a certain angle which varies between 0 and 2π. Find the length of the curve. (It is called a *cycloid.*)[2]

(12) Find the length of an arc of the curve $y = \ln \sec x$ between $x = 0$ and $x = \dfrac{\pi}{4}$ radians.

(13) Find the expression for the length of an arc of the curve $y^2 = \dfrac{x^3}{a}$.

(14) Find the length of the curve $y^2 = 8x^3$ between the two points for which $x = 1$ and $x = 2$.

(15) Find the length of the curve $y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$ between $x = 0$ and $x = a$.

(16) Find the length of the curve $r = a(1 - \cos\theta)$ between $\theta = 0$ and $\theta = \pi$.

You have now been personally conducted over the frontiers into the enchanted land. And in order that you may have a handy reference to the principal results, the author, in bidding you farewell, begs to present you with a passport in the shape of a convenient collection of standard forms. In the middle column are set down a number of the functions which most commonly occur. The results of differentiating them are set down on the left; the results of integrating them are set down on the right. May you find them useful!

2. See the paragraph on the cycloid in this book's appendix.—M.G.

Table of Standard Forms

$\dfrac{dy}{dx}$	y	$\int y\,dx$
	Algebraic	
1	x	$\frac{1}{2}x^2+C$
0	a	$ax+C$
1	$x\pm a$	$\frac{1}{2}x^2\pm ax+C$
a	ax	$\frac{1}{2}ax^2+C$
$2x$	x^2	$\frac{1}{3}x^3+C$
nx^{n-1}	x^n	$\dfrac{1}{n+1}x^{n+1}+C$
$-x^{-2}$	x^{-1}	$\ln x+C$
$\dfrac{du}{dx}\pm\dfrac{dv}{dx}\pm\dfrac{dw}{dx}$	$u\pm v\pm w$	$\int u\,dx\pm\int v\,dx\pm\int w\,dx$
$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$	uv	Put $v=\dfrac{dy}{dx}$ and integrate by parts
$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$	$\dfrac{u}{v}$	No general form known
$\dfrac{du}{dx}$	u	$\int u\,dx=ux-\int x\,du+C$
	Exponential and Logarithmic	
e^x	e^x	e^x+C
x^{-1}	$\ln x$	$x(\ln x-1)+C$
$0.4343x^{-1}$	$\log_{10}x$	$0.4343x(\ln x-1)+C$
$a^x\ln a$	a^x	$\dfrac{a^x}{\ln a}+C$
	Trigonometrical	
$\cos x$	$\sin x$	$-\cos x+C$
$-\sin x$	$\cos x$	$\sin x+C$
$\sec^2 x$	$\tan x$	$-\ln\cos x+C$

Circular (Inverse)

$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$x \arcsin x + \sqrt{1-x^2} + C$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	$x \arccos x - \sqrt{1-x^2} + C$
$\frac{1}{1+x^2}$	$\arctan x$	$x \arctan x - \frac{1}{2} \ln (1+x^2) + C$

Hyperbolic

$\cosh x$	$\sinh x$	$\cosh x + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$\operatorname{sech}^2 x$	$\tanh x$	$\ln \cosh x + C$

Miscellaneous

$-\frac{1}{(x+a)^2}$	$\frac{1}{x+a}$	$\ln \lvert x+a \rvert + C$
$-\frac{x}{(a^2+x^2)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left(x+\sqrt{a^2+x^2}\right)+C$
$\pm\frac{b}{(a \pm bx)^2}$	$\frac{1}{a \pm bx}$	$\pm\frac{1}{b} \ln \lvert a \pm bx \rvert + C$
$\frac{-3a^2x}{(a^2+x^2)^{\frac{5}{2}}}$	$\frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}$	$\frac{x}{\sqrt{a^2+x^2}} + C$
$a \cos ax$	$\sin ax$	$-\frac{1}{a} \cos ax + C$
$-a \sin ax$	$\cos ax$	$\frac{1}{a} \sin ax + C$
$a \sec^2 ax$	$\tan ax$	$-\frac{1}{a} \ln \lvert \cos ax \rvert + C$
$\sin 2x$	$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4} + C$
$-\sin 2x$	$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4} + C$

Miscellaneous

$n \cdot \sin^{n-1}x \cdot \cos x$	$\sin^n x$	$-\dfrac{\cos x}{n}\sin^{n-1}x + \dfrac{n-1}{n}\displaystyle\int \sin^{n-2}x\,dx + C$
$-\dfrac{\cos x}{\sin^2 x}$	$\dfrac{1}{\sin x}$	$\ln\left\lvert \tan\dfrac{x}{2}\right\rvert + C$
$-\dfrac{\sin 2x}{\sin^4 x}$	$\dfrac{1}{\sin^2 x}$	$-\cot x + C$
$\dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x}$	$\dfrac{1}{\sin x \cdot \cos x}$	$\ln \lvert\tan x\rvert + C$
$n \cdot \sin mx \cdot \cos nx + m \cdot \sin nx \cdot \cos mx$	$\sin mx \cdot \sin nx$	$\dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2\,(m+n)} + C$
$a \sin 2ax$	$\sin^2 ax$	$\dfrac{x}{2} - \dfrac{\sin 2ax}{4a} + C$
$-a \sin 2ax$	$\cos^2 ax$	$\dfrac{x}{2} + \dfrac{\sin 2ax}{4a} + C$

EPILOGUE AND APOLOGUE

It may be confidently assumed that when this book *Calculus Made Easy* falls into the hands of the professional mathematicians, they will (if not too lazy) rise up as one man, and damn it as being a thoroughly bad book. Of that there can be, from their point of view, no possible manner of doubt whatever. It commits several most grievous and deplorable errors.

First, it shows how ridiculously easy most of the operations of the calculus really are.

Secondly, it gives away so many trade secrets. By showing you that *what one fool can do, other fools can do also,* it lets you see that these mathematical swells, who pride themselves on having mastered such an awfully difficult subject as the calculus, have no such great reason to be puffed up. They like you to think how terribly difficult it is, and don't want that superstition to be rudely dissipated.

Thirdly, among the dreadful things they will say about "So Easy" is this: that there is an utter failure on the part of the author to demonstrate with rigid and satisfactory completeness the validity of sundry methods which he has presented in simple fashion, and has even *dared to use* in solving problems! But why should he not? You don't forbid the use of a watch to every person who does not know how to make one? You don't object to the musician playing on a violin that he has not himself constructed. You don't teach the rules of syntax to children until they have already become fluent in the *use* of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus.

One other thing will the professed mathematicians say about this thoroughly bad and vicious book: that the reason why it is *so easy* is because the author has left out all the things that are really difficult. And the ghastly fact about this accusation is that—*it is true!* That is, indeed, why the book has been written—written for the legion of innocents who have hitherto been deterred from acquiring the elements of the calculus by the stupid way in which its teaching is almost always presented. Any subject can be made repulsive by presenting it bristling with difficulties. The aim of this book is to enable beginners to learn its language, to acquire familiarity with its endearing simplicities, and to grasp its powerful methods of solving problems, without being compelled to toil through the intricate out-of-the-way (and mostly irrelevant) mathematical gymnastics so dear to the unpractical mathematician.

There are amongst young engineers a number on whose ears the adage that *what one fool can do, another can,* may fall with a familiar sound. They are earnestly requested not to give the author away, nor to tell the mathematicians what a fool he really is.

ANSWERS

Exercises I

(1) $\dfrac{dy}{dx} = 13x^{12}$ (2) $\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{5}{2}}$ (3) $\dfrac{dy}{dx} = 2ax^{2a-1}$

(4) $\dfrac{du}{dt} = 2.4t^{1.4}$ (5) $\dfrac{dz}{du} = \dfrac{1}{3}u^{-\frac{2}{3}}$ (6) $\dfrac{dy}{dx} = -\dfrac{5}{3}x^{-\frac{8}{3}}$

(7) $\dfrac{du}{dx} = -\dfrac{8}{5}x^{-\frac{13}{5}}$ (8) $\dfrac{dy}{dx} = 2ax^{a-1}$

(9) $\dfrac{dy}{dx} = \dfrac{3}{q}x^{\frac{3-q}{q}}$ (10) $\dfrac{dy}{dx} = -\dfrac{m}{n}x^{-\frac{m+n}{n}}$

Exercises II

(1) $\dfrac{dy}{dx} = 3ax^2$ (2) $\dfrac{dy}{dx} = 13 \times \frac{3}{2}x^{\frac{1}{2}}$ (3) $\dfrac{dy}{dx} = 6x^{-\frac{1}{2}}$

(4) $\dfrac{dy}{dx} = \dfrac{1}{2}c^{\frac{1}{2}}x^{-\frac{1}{2}}$ (5) $\dfrac{du}{dz} = \dfrac{an}{c}z^{n-1}$ (6) $\dfrac{dy}{dt} = 2.36t$

(7) $\dfrac{dl_t}{dt} = 0.000012 \times l_0$

(8) $\dfrac{dc}{dV} = abV^{b-1}$, 0.98, 3.00 and 7.46 candle power per volt respectively.

(9) $\dfrac{dn}{dD} = -\dfrac{1}{LD^2}\sqrt{\dfrac{gT}{\pi\sigma}},\ \dfrac{dn}{dL} = -\dfrac{1}{DL^2}\sqrt{\dfrac{gT}{\pi\sigma}}$

$\dfrac{dn}{d\sigma} = -\dfrac{1}{2DL}\sqrt{\dfrac{gT}{\pi\sigma^3}},\ \dfrac{dn}{dT} = \dfrac{1}{2DL}\sqrt{\dfrac{g}{\pi\sigma T}}$

(10) $\dfrac{\text{Rate of change of } P \text{ when } t \text{ varies}}{\text{Rate of change of } P \text{ when } D \text{ varies}} = -\dfrac{D}{t}$

(11) 2π, $2\pi r$, πl, $\frac{2}{3}\pi rh$, $8\pi r$, $4\pi r^2$.

Exercises III

(1) (*a*) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ (*b*) $2ax + b$ (*c*) $2x + 2a$

(*d*) $3x^2 + 6ax + 3a^2$

(2) $\frac{dw}{dt} = a - bt$ (3) $\frac{dy}{dx} = 2x$

(4) $14110x^4 - 65404x^3 - 2244x^2 + 8192x + 1379$

(5) $\frac{dx}{dy} = 2y + 8$ (6) $185.9022654x^2 + 154.36334$

(7) $\frac{-5}{(3x+2)^2}$ (8) $\frac{6x^4 + 6x^3 + 9x^2}{(1 + x + 2x^2)^2}$

(9) $\frac{ad - bc}{(cx + d)^2}$ (10) $\frac{anx^{-n-1} + bnx^{n-1} + 2nx^{-1}}{(x^{-n} + b)^2}$

(11) $b + 2ct$

(12) $R_0(a + 2bt), R_0\left(a + \frac{b}{2\sqrt{t}}\right), -\frac{R_0(a + 2bt)}{(1 + at + bt^2)^2}$ or $-\frac{R^2(a + 2bt)}{R_0{}^2}$

(13) 1.4340(0.000014t − 0.001024), −0.00117, −0.00107, −0.00097

(14) (*a*) $\frac{dE}{dl} = b + \frac{k}{i}$, (*b*) $\frac{dE}{di} = -\frac{c + kl}{i^2}$

Exercises IV

(1) $17 + 24x$; 24 (2) $\frac{x^2 + 2ax - a}{(x + a)^2}$; $\frac{2a(a + 1)}{(x + a)^3}$

(3) $1 + x + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3}$; $1 + x + \frac{x^2}{1 \times 2}$

(4) *(Exercises III):*

(1) (*a*) $\dfrac{d^2u}{dx^2} = \dfrac{d^3u}{dx^3} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots.$

(*b*) $2a$, 0 (*c*) 2, 0 (*d*) $6x + 6a$, 6

(2) $-b$, 0 (3) 2, 0

(4) $56440x^3 - 196212x^2 - 4488x + 8192.$
$169320x^2 - 392424x - 4488$

(5) 2, 0 (6) $371.80453x$, 371.80453

(7) $\dfrac{30}{(3x+2)^3}, \quad -\dfrac{270}{(3x+2)^4}$

(Examples):

(1) $\dfrac{6a}{b^2}x, \dfrac{6a}{b^2}$ (2) $\dfrac{3a\sqrt{b}}{2\sqrt{x}} - \dfrac{6b\sqrt[3]{a}}{x^3}, \dfrac{18b\sqrt[3]{a}}{x^4} - \dfrac{3a\sqrt{b}}{4\sqrt{x^3}}$

(3) $\dfrac{2}{\sqrt[3]{\theta^8}} - \dfrac{1.056}{\sqrt[5]{\theta^{11}}}, \quad \dfrac{2.3232}{\sqrt[5]{\theta^{16}}} - \dfrac{16}{3\sqrt[3]{\theta^{11}}}$

(4) $810t^4 - 648t^3 + 479.52t^2 - 139.968t + 26.64$
$3240t^3 - 1944t^2 + 959.04t - 139.968$

(5) $12x + 2$, 12 (6) $6x^2 - 9x$, $12x - 9$

(7) $\dfrac{3}{4}\left(\dfrac{1}{\sqrt{\theta}} + \dfrac{1}{\sqrt{\theta^5}}\right) + \dfrac{1}{4}\left(\dfrac{15}{\sqrt{\theta^7}} - \dfrac{1}{\sqrt{\theta^3}}\right)$

$\dfrac{3}{8}\left(\dfrac{1}{\sqrt{\theta^5}} - \dfrac{1}{\sqrt{\theta^3}}\right) - \dfrac{15}{8}\left(\dfrac{7}{\sqrt{\theta^9}} + \dfrac{1}{\sqrt{\theta^7}}\right)$

Exercises V

(2) 64; 147.2; and 0.32 feet per second

(3) $\dot{x} = a - gt$; $\ddot{x} = -g$ (4) 45.1 feet per second

(5) 12.4 feet per second per second. Yes.

(6) Angular velocity = 11.2 radians per second;
angular acceleration = 9.6 radians per second per second.

(7) $v = 20.4t^2 - 10.8, \quad a = 40.8t \quad 172.8$ in./sec., 122.4 in./sec.^{2}

(8) $v = \dfrac{1}{30\sqrt[3]{(t-125)^2}},\ a = -\dfrac{1}{45\sqrt[3]{(t-125)^5}}$

(9) $v = 0.8 - \dfrac{8t}{(4+t^2)^2},\ a = \dfrac{24t^2 - 32}{(4+t^2)^3}$, 0.7926 and 0.00211

(10) $n = 2,\ n = 11$

Exercises VI

(1) $\dfrac{x}{\sqrt{x^2+1}}$ (2) $\dfrac{x}{\sqrt{x^2+a^2}}$ (3) $-\dfrac{1}{2\sqrt{(a+x)^3}}$

(4) $\dfrac{ax}{\sqrt{(a-x^2)^3}}$ (5) $\dfrac{2a^2-x^2}{x^3\sqrt{x^2-a^2}}$

(6) $\dfrac{\frac{3}{2}x^2\left[\frac{8}{9}x(x^3+a)-(x^4+a)\right]}{(x^4+a)^{\frac{2}{3}}(x^3+a)^{\frac{3}{2}}}$ (7) $\dfrac{2a(x-a)}{(x+a)^3}$

(8) $\frac{5}{2}y^3$ (9) $\dfrac{1}{(1-\theta)\sqrt{1-\theta^2}}$

Exercises VII

(1) $\dfrac{dw}{dx} = -\dfrac{3x^2(3+3x^3)}{27\left(\frac{1}{2}x^3+\frac{1}{4}x^6\right)^3}$

(2) $\dfrac{dv}{dx} = -\dfrac{12x}{\sqrt{1+\sqrt{2}+3x^2}\left(\sqrt{3}+4\sqrt{1+\sqrt{2}+3x^2}\right)^2}$

(3) $\dfrac{du}{dx} = -\dfrac{x^2(\sqrt{3}+x^3)}{\sqrt{\left[1+\left(1+\dfrac{x^3}{\sqrt{3}}\right)^2\right]^3}}$

(5) $\dfrac{dx}{d\theta} = a(1 - \cos\,\theta) = 2a\,\sin^2 \tfrac{1}{2}\theta$

$\dfrac{dy}{d\theta} = a\,\sin\,\theta = 2a\,\sin\,\tfrac{1}{2}\theta\,\cos\,\tfrac{1}{2}\theta;\ \dfrac{dy}{dx} = \cot\,\tfrac{1}{2}\theta$

(6) $\dfrac{dx}{d\theta} = -3a\,\cos^2\,\theta\,\sin\,\theta,\ \dfrac{dy}{d\theta} = 3a\,\sin^2\,\theta\,\cos\,\theta;$

$\dfrac{dy}{dx} = -\tan\,\theta$

(7) $\dfrac{dy}{dx} = 2x\,\cot\,(x^2 - a^2)$

(8) Write $y = u - x$; find $\dfrac{dx}{du}$, $\dfrac{dy}{du}$, and then $\dfrac{dy}{dx}$

Exercises VIII

(2) 1.44

(4) $\dfrac{dy}{dx} = 3x^2 + 3$; and the numerical values are: 3, $3\tfrac{3}{4}$, 6, and 15.

(5) $\pm\sqrt{2}$

(6) $\dfrac{dy}{dx} = -\dfrac{4}{9}\dfrac{x}{y}$. Slope is zero where $x = 0$; and is $\pm\dfrac{1}{3\sqrt{2}}$ where $x = 1$.

(7) $m = 4$, $n = -3$

(8) Intersections at $x = 1$, $x = -3$. Angles $153°\ 26'$, $2°\ 28'$

(9) Intersection at $x = y = \frac{25}{7}$. Angle $16°\ 16'$

(10) $x = \frac{1}{3}$, $y = 2\frac{1}{3}$, $b = -\frac{5}{3}$

Exercises IX

(1) Min.: $x = 0$, $y = 0$; max.: $x = -2$, $y = -4$

(2) $x = a$

(4) $25\sqrt{3}$ square inches.

(5) $\dfrac{dy}{dx} = -\dfrac{10}{x^2} + \dfrac{10}{(8-x)^2}$; $x = 4$; $y = 5$

(6) Max. for $x = -1$; min. for $x = 1$.

(7) Join the middle points of the four sides.

(8) $r = \frac{2}{3}R$, $r = \dfrac{R}{2}$, no max.

(9) $r = R\sqrt{\dfrac{2}{3}}$, $r = \dfrac{R}{\sqrt{2}}$, $r = 0.8507R$

(10) At the rate of $8\sqrt{\pi}$ square feet per second.

(11) $r = \dfrac{R\sqrt{8}}{3}$

Exercises X

(1) Max.: $x = -2.19$, $y = 24.19$; min.: $x = 1.52$, $y = -1.38$

(2) $\dfrac{dy}{dx} = \dfrac{b}{a} - 2cx$; $\dfrac{d^2y}{dx^2} = -2c$; $x = \dfrac{b}{2ac}$ (a *maximum*)

(3) (*a*) One maximum and two minima.
(*b*) One maximum. ($x = 0$; other points unreal.)

(4) Min.: $x = 1.71$, $y = 6.13$ (5) Max.: $x = -.5$, $y = 4$

(6) Max.: $x = 1.414$, $y = 1.7678$. Min.: $x = -1.414$, $y = -1.7678$

(7) Max.: $x = -3.565$, $y = 2.12$. Min.: $x = +3.565$, $y = 7.88$

(8) $0.4N$, $0.6N$ (9) $x = \sqrt{\dfrac{a}{c}}$

(10) Speed 8.66 nautical miles per hour. Time taken 115.44 hours, total cost is $2,251.11.

(11) Max. and min. for $x = 7.5$, $y = \pm\ 5.413$.

(12) Min.: $x = \frac{1}{2}$, $y = 0.25$; max.: $x = -\frac{1}{3}$, $y = 1.407$

Exercises XI

(1) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$ (2) $\dfrac{1}{x-1}+\dfrac{2}{x-2}$ (3) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$

(4) $\dfrac{5}{x-4}-\dfrac{4}{x-3}$ (5) $\dfrac{19}{13(2x+3)}-\dfrac{22}{13(3x-2)}$

(6) $\dfrac{2}{x-2}+\dfrac{4}{x-3}-\dfrac{5}{x-4}$

(7) $\dfrac{1}{6(x-1)}+\dfrac{11}{15(x+2)}+\dfrac{1}{10(x-3)}$

(8) $\dfrac{7}{9(3x+1)}+\dfrac{71}{63(3x-2)}-\dfrac{5}{7(2x+1)}$

(9) $\dfrac{1}{3(x-1)}+\dfrac{2x+1}{3(x^2+x+1)}$

(10) $x+\dfrac{2}{3(x+1)}+\dfrac{1-2x}{3(x^2-x+1)}$

(11) $\dfrac{3}{x+1}+\dfrac{2x+1}{x^2+x+1}$ (12) $\dfrac{1}{x-1}-\dfrac{1}{x-2}+\dfrac{2}{(x-2)^2}$

(13) $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}+\dfrac{1}{2(x+1)^2}$

(14) $\dfrac{4}{9(x-1)}-\dfrac{4}{9(x+2)}-\dfrac{1}{3(x+2)^2}$

(15) $\dfrac{1}{x+2}-\dfrac{x-1}{x^2+x+1}-\dfrac{1}{(x^2+x+1)^2}$

(16) $\dfrac{5}{x+4}-\dfrac{32}{(x+4)^2}+\dfrac{36}{(x+4)^3}$

(17) $\dfrac{7}{9(3x-2)^2}+\dfrac{55}{9(3x-2)^3}+\dfrac{73}{9(3x-2)^4}$

(18) $\dfrac{1}{6(x-2)}+\dfrac{1}{3(x-2)^2}-\dfrac{x}{6(x^2+2x+4)}$

Exercises XII

(1) $ab(e^{ax}+e^{-ax})$ (2) $2at+\frac{2}{t}$ (3) $\ln n$ (5) npv^{n-1}

(6) $\frac{n}{x}$ (7) $\frac{3e^{-\frac{x}{x-1}}}{(x-1)^2}$ (8) $6xe^{-5x}-5(3x^2+1)e^{-5x}$

(9) $\frac{ax^{a-1}}{x^a+a}$ (10) $\frac{15x^2+12x\sqrt{x}-1}{2\sqrt{x}}$ (11) $\frac{1-\ln(x+3)}{(x+3)^2}$

(12) $a^x(ax^{a-1}+x^a \ln a)$ (14) Min.: $y=0.7$ for $x=0.693$

(15) $\frac{1+x}{x}$ (16) $\frac{3}{x}(\ln ax)^2$

Exercises XIII

(2) $T=34.625$; 159.45 minutes

(5) (*a*) $x^x(1+\ln x)$; (*b*) $2x\,(e^x)^x$; (*c*) $e^{x^x}\times x^x\,(1+\ln x)$

(6) 0.14 second (7) (*a*) 1.642; (*b*) 15.58

(8) $\mu=0.00037$, 31.06 min

(9) i is 63.4% of i_0, 221.56 kilometers

(10) $k=0.1339$, 0.1445, 0.1555, mean $=0.1446$; percentage errors:—10.2%, 0.18% nil, +71.8%.

(11) Min. for $x=\frac{1}{e}$ (12) Max. for $x=e$

(13) Min. for $x=\ln a$

Exercises XIV

(1) (i) $\frac{dy}{d\theta}=A\cos\left(\theta-\frac{\pi}{2}\right)$

(ii) $\frac{dy}{d\theta}=2\sin\theta\cos\theta=\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$

(iii) $\dfrac{dy}{d\theta} = 3 \sin^2\theta \cos\theta$ and $\dfrac{dy}{d\theta} = 3 \cos 3\theta$

(2) $\theta = 45°$ or $\dfrac{\pi}{4}$ radians

(3) $\dfrac{dy}{dt} = -n \sin 2\pi nt$

(4) $a^x \, l_n \, a \cos a^x$

(5) $\dfrac{-\sin x}{\cos x} = -\tan x$

(6) $18.2 \cos (x + 26°)$

(7) The slope is $\dfrac{dy}{d\theta} = 100 \cos (\theta - 15°)$, which is a maximum when $(\theta - 15°) = 0$, or $\theta = 15°$; the value of the slope being then $= 100$. When $\theta = 75°$ the slope is $100 \cos (75° - 15°)$ $= 100 \cos 60° = 100 \times \frac{1}{2} = 50$

(8) $\cos\theta \sin 2\theta + 2 \cos 2\theta \sin\theta = 2 \sin\theta (\cos^2\theta + \cos 2\theta)$
$= 2 \sin\theta (3 \cos^2\theta - 1)$

(9) $amn\theta^{n-1} \tan^{m-1} (\theta^n) \sec^2 (\theta^n)$

(10) $e^x (\sin^2 x + \sin 2x)$

(11) (i) $\dfrac{dy}{dx} = \dfrac{ab}{(x+b)^2}$ (ii) $\dfrac{a}{b} e^{-\frac{x}{b}}$ (iii) $\dfrac{1}{90°} \times \dfrac{ab}{(b^2 + x^2)}$

(12) (i) $\dfrac{dy}{dx} = \sec x \tan x$ (ii) $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

(iii) $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ (iv) $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$

(v) $\dfrac{dy}{dx} = \dfrac{\sqrt{3 \sec x}\,(3 \sec^2 x - 1)}{2}$

(13) $\dfrac{dy}{d\theta} = 4.6 (2\theta + 3)^{1.3} \cos (2\theta + 3)^{2.3}$

(14) $\dfrac{dy}{d\theta} = 3\theta^2 + 3\cos(\theta + 3) - \ln 3(\cos\theta \times 3^{\sin\theta} + 3^{\theta})$

(15) $\theta = \cot\theta$; $\theta = \pm 0.86$; $y = \pm 0.56$; is max. for $+\theta$, min. for $-\theta$

Exercises XV

(1) $x^2 - 6x^2y - 2y^2$; $\frac{1}{3} - 2x^3 - 4xy$

(2) $2xyz + y^2z + z^2y + 2xy^2z^2$
$2xyz + x^2z + xz^2 + 2x^2yz^2$
$2xyz + x^2y + xy^2 + 2x^2y^2z$

(3) $\dfrac{1}{r}\{(x-a) + (y-b) + (z-c)\} = \dfrac{(x+y+z) - (a+b+c)}{r}$; $\dfrac{2}{r}$

(4) $dy = v\,u^{v-1}\,du + u^v \ln u\,dv$

(5) $dy = 3\sin v\,u^2\,du + u^3 \cos v\,dv$
$dy = u\,(\sin x)^{u-1} \cos x\,dx + (\sin x)^u \ln \sin x\,du$
$dy = \dfrac{1}{v}\,\dfrac{1}{u}\,du - \ln u\,\dfrac{1}{v^2}\,dv$

(7) There is no minimum or maximum.

(8) (*a*) Length 2 feet, width = depth = 1 foot, vol. = 2 cubic feet

(*b*) Radius $= \dfrac{2}{\pi}$ feet = 7.64 in., length = 2 feet,
vol. = 2.55

(9) All three parts equal; the product is maximum.

(10) Minimum $= e^2$ for $x = y = 1$

(11) Min. = 2.307 for $x = \frac{1}{2}$, $y = 2$

(12) Angle at apex = 90°; equal sides = length $= \sqrt[3]{2V}$

Exercises XVI

(1) $1\frac{1}{3}$. (2) 0.6345 (3) 0.2624

(4) $y = \frac{1}{8}x^2 + C$ (5) $y = x^2 + 3x + C$

Exercises XVII

(1) $\dfrac{4\sqrt{a}\,x^{\frac{3}{2}}}{3}+C$ (2) $-\dfrac{1}{x^3}+C$ (3) $\dfrac{x^4}{4a}+C$

(4) $\frac{1}{3}x^3+ax+C$ (5) $-2x^{-\frac{5}{2}}+C$

(6) $x^4+x^3+x^2+x+C$ (7) $\dfrac{ax^2}{4}+\dfrac{bx^3}{9}+\dfrac{cx^4}{16}+C$

(8) $\dfrac{x^2+a}{x+a}=x-a+\dfrac{a^2+a}{x+a}$ by division. Therefore the answer is $\frac{1}{2}x^2-ax+a(a+1)\ln(x+a)+C$

(9) $\dfrac{x^4}{4}+3x^3+\dfrac{27}{2}x^2+27x+C$

(10) $\dfrac{x^3}{3}+\dfrac{2-a}{2}x^2-2ax+C$

(11) $a^2\left(2x^{\frac{3}{2}}+\frac{9}{4}x^{\frac{4}{3}}\right)+C$ (12) $-\frac{1}{3}\cos\theta-\frac{1}{6}\theta+C$

(13) $\frac{1}{2}\theta+\dfrac{\sin 2a\theta}{4a}+C$ (14) $\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta+C$

(15) $\frac{1}{2}\theta-\dfrac{\sin 2a\theta}{4a}+C$ (16) $\frac{1}{3}e^{3x}+C$

(17) $\ln|1+x|+C$ (18) $-\ln|1-x|+C$

Exercises XVIII

(1) Area = 120; mean ordinate = 20. (2) Area $=\frac{4}{3}a^{\frac{5}{2}}$

(3) Area = 2; mean ordinate $= 2/\pi = 0.637$

(4) Area = 1.57; mean ordinate = 0.5 (5) 0.571, 0.0476

(6) Volume $=\frac{1}{3}\pi r^2 h$ (7) 1.25 (8) 79.6

(9) Volume = 4.935, from 0 to π (10) $a\ln a$, $\dfrac{a}{a-1}\ln a$

(12) A.M. = 9.5; Q.M. = 10.85

(13) Quadratic mean $= \dfrac{1}{\sqrt{2}}\sqrt{A_1^{\,2} + A_3^{\,2}}$;

arithmetical mean = 0.

The first involves the integral

$$\int (A_1^{\,2}\sin^2 x + 2A_1A_3 \sin x \sin 3x + A_3^{\,2}\sin^2 3x)dx$$

which may be evaluated by putting $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$ and $2 \sin x \sin 3x = \cos 2x - \cos 4x$.

(14) Area is 62.6 square units. Mean ordinate is 10.43.

(16) 436.3 (This solid is pear-shaped.)

Exercises XIX

(1) $\dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a} + C$

(2) $\dfrac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$

(3) $\dfrac{x^{a+1}}{a+1}\left(\ln x - \dfrac{1}{a+1}\right) + C$

(4) $\sin e^x + C$

(5) $\sin(\ln x) + C$

(6) $e^x(x^2 - 2x + 2) + C$

(7) $\dfrac{1}{a+1}(\ln x)^{a+1} + C$

(8) $\ln|\ln x| + C$

(9) $2\ln|x - 1| + 3\ln|x + 2| + C$

(10) $\frac{1}{2}\ln|x - 1| + \frac{1}{5}\ln|x - 2| + \frac{3}{10}\ln|x + 3| + C$

(11) $\dfrac{b}{2a}\ln\left|\dfrac{\mathrm{x} - \mathrm{a}}{\mathrm{x} + \mathrm{a}}\right| + \mathrm{C}$

(12) $\ln\left|\dfrac{\mathrm{x}^2 - 1}{\mathrm{x}^2 + 1}\right| + \mathrm{C}$

(13) $\frac{1}{4}\ln\left|\dfrac{1 + x}{1 - x}\right| + \frac{1}{2}\arctan x + C$

(14) $-\dfrac{\sqrt{a^2 - b^2x^2}}{b^2} + C$

Exercises XX

(1) $T = T_0 e^{\mu\theta}$ (2) $s = ut + \frac{1}{2}at^2$

(3) Multiplying out by e^{2t} gives $\dfrac{d}{dt}(ie^{2t}) = e^{2t} \sin 3t$, so that,

$$ie^{2t} = \int e^{2t} \sin 3t \, dt = \tfrac{1}{13}e^{2t}(2 \sin 3t - 3 \cos 3t) + E$$

Since $i = 0$ when $t = 0$, $E = \frac{3}{13}$; hence the solution becomes $i = \frac{1}{13}(2 \sin 3t - 3 \cos 3t + 3e^{-2t})$.

Exercises XXI

(1) $r = 2\sqrt{2}$, $x_1 = -2$, $y_1 = 3$ (2) $r = 2.83$, $x_1 = 0$, $y_1 = 2$

(3) $x = \pm 0.383$, $y = 0.147$ (4) $r = \sqrt{2|m|}$, $x_1 = y_1 = 2\sqrt{m}$

(5) $r = 2a$, $x_1 = 2a$, $y_1 = 0$

(6) When $x = 0$, $r = y_1 =$ infinity, $x_1 = 0$
When $x = +0.9$, $r = 3.36$, $x_1 = -2.21$, $y_1 = +2.01$
When $x = -0.9$, $r = 3.36$, $x_1 = +2.21$, $y_1 = -2.01$

(7) When $x = 0$, $r = 1.41$, $x_1 = 1$, $y_1 = 3$
When $x = 1$, $r = 1.41$, $x_1 = 0$, $y_1 = 3$
Minimum $= 1.75$

(8) For $x = -2$, $r = 112.3$, $x_1 = 109.8$, $y_1 = -17.2$
For $x = 0$, $r = x_1 = y_1 =$ infinity
For $x = 1$, $r = 1.86$, $x_1 = -0.67$, $y_1 = -0.17$

(9) $x = -0.33$, $y = +1.07$

(10) $r = 1$, $x = 2$, $y = 0$ for all points. A circle.

(11) When $x = 0$, $r = 1.86$, $x_1 = 1.67$, $y_1 = 0.17$
When $x = 1.5$, $r = 0.365$, $x_1 = 1.59$, $y_1 = 0.98$
$x = 1$, $y = 1$ for zero curvature.

(12) When $\theta = \dfrac{\pi}{2}$, $r = 1$, $x_1 = \dfrac{\pi}{2}$, $y_1 = 0$.

When $\theta = \dfrac{\pi}{4}$, $r = 2.598$, $x_1 = 2.285$, $y_1 = -1.414$

(14) When $\theta = 0$, $r = 1$, $x_1 = 0$, $y_1 = 0$

When $\theta = \dfrac{\pi}{4}$, $r = 2.598$, $x_1 = -0.715$, $y_1 = -1.414$

When $\theta = \dfrac{\pi}{2}$, $r = x_1 = y_1 =$ infinity

(15) $r = \dfrac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, when $x = 0$, $y = \pm b$, $r = \dfrac{a^2}{b}$,

$x_1 = 0$, $y_1 = \pm \dfrac{b^2 - a^2}{b}$; when $y = 0$, $x = \pm a$, $r = \dfrac{b^2}{a}$,

$x_1 = \pm \dfrac{a^2 - b^2}{a}$, $y_1 = 0$

(16) $r = 4a\ |\sin \tfrac{1}{2}\theta|$

Exercises XXII

(1) $s = 9.487$ (2) $s = (1 + a^2)^{\frac{3}{2}}$ (3) $s = 1.22$

(4) $s = \displaystyle\int_0^2 \sqrt{1 + 4x^2}\, dx$

$$= \left[\frac{x}{2}\sqrt{1 + 4x^2} + \tfrac{1}{4}\ln\left(2x + \sqrt{1 + 4x^2}\right)\right]_0^2 = 4.65$$

(5) $s = \dfrac{0.57}{m}$ (6) $s = a(\theta_2 - \theta_1)$ (7) $s = \sqrt{r^2 - a^2}$

(8) $s = \displaystyle\int_0^a \sqrt{1 + \frac{a}{x}}\, dx$ and $s = a\sqrt{2} + a\ln\left(1 + \sqrt{2}\right) = 2.30a$

(9) $s = \dfrac{x-1}{2}\sqrt{(x-1)^2+1} + \frac{1}{2}\ln\left\{(x-1)+\sqrt{(x-1)^2+1}\right\}$
and $s = 6.80$

(10) $s = \displaystyle\int_1^e \frac{\sqrt{1+y^2}}{y}\,dy$. Put $u^2 = 1+y^2$; this leads to

$$s = \sqrt{1+y^2} + \ln \frac{y}{1+\sqrt{1+y^2}} \text{ and } s = 2.00$$

(11) $s = 4a\displaystyle\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta$ and $s = 8a$

(12) $s = \displaystyle\int_0^{\frac{1}{4}\pi} \sec x\,dx$. Put $u = \sin x$; this leads to

$s = \ln\left(1+\sqrt{2}\right) = 0.8814$

(13) $s = \dfrac{8a}{27}\left\{\left(1+\dfrac{9x}{4a}\right)^{\frac{3}{2}} - 1\right\}$

(14) $s = \displaystyle\int_1^2 \sqrt{1+18x}\,dx$. Let $1+18x = z$, express s in terms of z and integrate between the values of z corresponding to $x = 1$ and $x = 2$. $s = 5.27$

(15) $s = \dfrac{3a}{2}$ (16) $4a$

All earnest students are exhorted to manufacture more examples for themselves at every stage, so as to test their powers. When integrating he can always test their answer by differentiating it, to see whether they gets back the expression from which they started.

APPENDIX

SOME RECREATIONAL PROBLEMS RELATED TO CALCULUS

THE OLD OAKEN CALCULUS PROBLEM

How dear to my heart are cylindrical wedges,
when fond recollection presents them once more,
and boxes from tin by upturning the edges,
and ships landing passengers where on the shore.
The ladder that slid in its slanting projection,
the beam in the corridor rounding the ell,
but rarest of all in that antique collection
the leaky old bucket that hung in the well—
the leaky old bucket, the squeaky old bucket,
the leaky old bucket that hung in the well.

Katherine O'Brien, in
The *American Mathematical Monthly*
(vol. 73, 1966, p. 881).

Problems solvable only by calculus are extremely rare in popular puzzle books. One exception is a puzzle that usually involves an animal A that is a certain distance from a man or another animal B. Assume the animal is a cat directly north of a dog. Both run at a constant rate, but B goes faster than A. If both animals run due north, the dog will of course eventually catch the cat. In this form it is equivalent to Zeno's famous paradox of Achilles and the tortoise. It is the simplest example of what are called "pursuit paths." Finding the distance traveled by pursuer and pursued, in this linear form, is an easy task involving only arithmetic.

Pursuit paths on the plane become more interesting. Assume that the cat travels in a straight line due east, and that the dog always runs directly toward the cat. Both go at a constant rate. If the dog is faster than the cat it can always catch the cat. Given its initial distance south of the cat, and the ratio of their speeds, how far does the cat go before it is caught? How far does the dog go, along its curved path, before it catches the cat?

Versions of this two-dimensional pursuit problem are not quite so easy to solve. Such puzzles were popular in the eighteenth century when they usually involved one ship pursuing another. Later versions took the form of running animals and persons. I will cite two examples from classic twentieth-century puzzle collections.

Henry Dudeney, in *Puzzles and Curious Problems* (1931, Problem 210) describes the situation this way. Pat is 100 yards south of a pig. The pig runs due west. Pat goes twice as fast as the pig. If he always runs directly toward the pig, how far does each go before the pig is caught?

Sam Loyd, in his *Cyclopedia of Puzzles* (1914, p. 217), also makes the pursued a pig, but now it is being chased by Tom the Piper's Son. (He stole a pig, remember, in the old Mother Goose rhyme). Tom is 250 yards south of the pig and the pig runs due east. Both go at uniform rates with Tom running $\frac{4}{3}$ as fast as the pig. Tom always runs directly toward the pig. Again, how far does each travel before the pig is caught?

Such problems can be solved the hard way by integrating. Dudeney answers his puzzle with no explanation of how to solve it. He adds: "The curve of Pat's line is one of those curves the length of which may be exactly measured. But we have not space to go into the method."

Like so many calculus problems, it turns out that pursuit problems of this type often can be handled by simple formulas, although calculus is needed to prove them. One such method is given by L.A. Graham in his *Ingenious Mathematical Problems and Methods* (1959, Problem 74). His version of the puzzle involves a dog chasing a cat. The dog, 60 yards south of the cat, runs $\frac{5}{4}$ as fast as the cat who runs due east. Graham first solves the problem by integrating, then provides the following surprising formula. The distance traveled by the cat equals the initial distance between dog and cat multiplied by the fraction that expresses the ratio of their speeds, divided by a number one less than the square of the ratio.

Using the parameters of Graham's problem, the distance traveled by the cat is:

$$\frac{60 \times 5}{4} \div \left(\frac{5^2}{4^2} - 1\right)$$

which works out to 133 and $\frac{1}{3}$ yards. Because the dog goes $\frac{5}{4}$ as fast as the cat, it travels $\frac{5}{4}$ times 133 and $\frac{1}{3}$, or 166 and $\frac{2}{3}$ yards.

Applying the formula to Dudeney's version shows that the pig runs 66 and $\frac{2}{3}$ yards, and Pat, who goes twice as fast, runs 133 and $\frac{1}{3}$ yards. In Loyd's version the pig goes 428 and $\frac{4}{7}$ yards. Tom travels $\frac{4}{3}$ times that distance, or 571 and $\frac{3}{7}$ yards.

Another well known pursuit path problem, covered in many books on recreational mathematics, involves *n* bugs at the corners of a regular *n*-sided polygon of unit side. The bugs simultaneously start to crawl directly toward their nearest neighbor on, say, the left. (It doesn't matter whether they go clockwise or counterclockwise). All bugs move at the same constant rate. It is intuitively obvious that they will travel spiral paths that come together at the polygon's center. When they all meet, how far has each bug traveled?

Although the problem can be posed with any regular polygon, it is usually based on a square. At any given moment the four bugs will be at the corners of a square that steadily diminishes and rotates until the four bugs meet at the center. Their paths are logarithmic spirals. Although the length of each path can be determined by calculus, the problem can be solved quickly without calculus if you have the following insight.

At all times the path of each bug is perpendicular to the bug it is pursuing. It follows that there is no component of a pursued bug's motion that carries it toward or away from its pursuer. Consequently, the pursuer will reach the pursued in the same time it would take if the pursued bug remained stationary and its pursuer moved directly toward it. Each spiral path will therefore have the same length as the side of the unit square.

A geometrical way of measuring the distance traveled by each bug starting at the corner of a regular polygon is shown in Figure 11 where it is applied to a regular hexagon. Draw line AO from a corner to the polygon's center, then extend OX at right angles to AO until it meets either side AB or its extension as shown. AX, which is r times the secant of angle theta, is then the length of the path traveled by each bug. In the case of the square, AX is the square's side. In the case of a triangle, OX meets AB at a spot two-thirds of the way from A, indicating that the bug travels two-thirds of the length of the triangle's side. For the hexagon, theta is 60°, and the bug travels twice the distance of a side.

Minimum path problems that require differentiating occasionally turn up in books on puzzles as well as in calculus textbooks. They usually take the form of a swimmer who is in a lake at distance x from a straight shoreline. He wants to get to a certain point p along the coast. Given a steady speed at which he swims, and his steady speed while walking on land, what spot along the coast should he swim to so as to minimize the total time it takes him to swim to shore and then walk to point p along the coast?

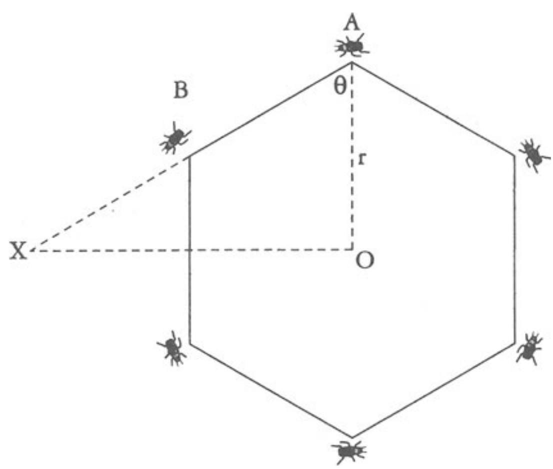


FIG. 11. How to calculate a bug's pursuit path on a regular hexagon.

In his *Cyclopedia of Puzzles* (page 165), Sam Loyd offers a version of such a problem. His text and its accompanying illustration are shown in Figure 12. Loyd's constants make the problem tedious to calculate, but the solution by equating a derivative to zero is straightforward.

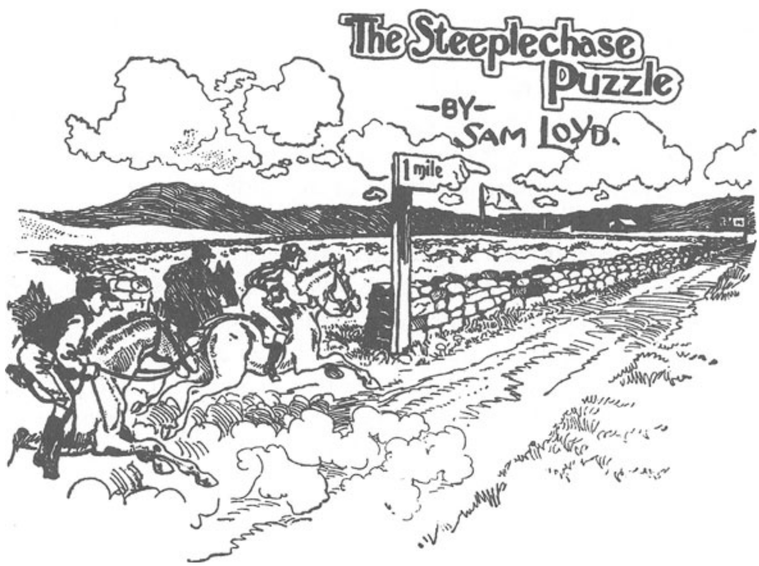


FIG. 12. Sam Loyd's steeplechase puzzle from his *Cyclopedia of Puzzles.* Here is a little cross-country steeplechase problem which developed during the recent meeting, which will interest turfites as well as puzzlists. It appears that toward the end of a well-contested course, when there was but a mile and three quarters yet to run, the leaders were so closely bunched together that victory turned upon the selection of the best or shortest road. The sketch shows the judges' stand to be at the opposite end of a rectangular field, bounded by a road of a mile long on one side by three-quarters of a mile on the other.

By the road, therefore, it would be a mile and three-quarters, which all of the horses could finish in three minutes. They are at liberty, however, to cut across lots at any point they wish, but over the rough ground they could not go so fast. So while they would lessen the distance, they would lose twenty-five per cent. in speed. By going directly across on the bias, or along the line of the hypotenuse as the mathematicians would term it, the distance would be a mile and a quarter exactly. What time can the winner make by selecting the most judicious route?

Figure 13 diagrams the relevant rectangle. Let x be the distance from the jump point to the rectangle's corner B, and $(1 - x)$ be the distance from where the horses start to where they should jump the wall. The path the horses take on rough ground is the hypotenuse of a right triangle with a length equal to the square root of $(x^2 + .75^2)$. The horses run on the smoother ground at a speed of $1.75/3 = .58333+$ miles per minute. Their speed on rougher ground is a fourth less, or .4375 miles per minute.

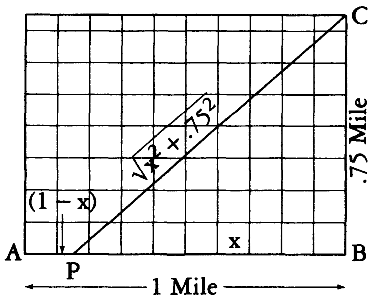


FIG. 13. Diagram of Sam Loyd's steeplechase puzzle.

To solve the problem we must first write an expression for the total time as a function of x. The time it takes the horses to go on rough ground from P to C is:

$$\frac{\sqrt{x^2 + .75^2}}{.4375}$$

The time it takes them to cover distance $1 - x$, on the smoother ground, to the point where they jump the wall is:

$$\frac{1 - x}{.5833}$$

The total time is the sum of these two expressions:

$$t = \frac{\sqrt{x^2 + .75^2}}{.4375} + \frac{1 - x}{.5833}$$

We next differentiate by the "sum rule" (see Chapter VI):

$$\frac{dt}{dx} = \frac{x}{.4375\sqrt{x^2 + .75^2}} - \frac{1}{.5833}$$

We equate the derivative to zero and solve for x. This gives x a value of .8504+ miles. Subtracting this from 1 gives the distance on smooth ground, from the start to the jump point, as about .15 miles. The times are easily obtained by substituting .8504 for x in the two time expressions. The total time is 2.85 minutes or about 2 minutes and 51 seconds.

Because the ratio of the two speeds is $\frac{2.25}{3}$ or $\frac{3}{4}$, we can greatly simplify the calculation by substituting 3 and 4 for the respective velocities:

$$\frac{\sqrt{x^2+.75^2}}{3}+\frac{1-x}{4}$$

The derivative of this function becomes:

$$\frac{x}{3\sqrt{x^2+.75^2}}-\frac{1}{4}$$

Equate to zero:

$$\frac{x}{3\sqrt{x^2+.75^2}}-\frac{1}{4}=0$$

$$\frac{x}{3\sqrt{x^2+.75^2}}=\frac{1}{4}$$

Square both sides:

$$\frac{x^2}{9(x^2+.75^2)}=\frac{1}{16}$$

$$16x^2 = 9(x^2 + .75^2) = 9x^2 + 5.0625$$

$$16x^2 - 9x^2 = 5.0625$$

$$7x^2 = 5.0625$$

$$x^2 = 5.0625/7 = .72321428 \ldots.$$

$$x = .85042 \ldots.$$

Occasionally what seems to be a very difficult path problem, involving the sum of an infinite converging series, can be solved in a flash if you have the right insight. A classic instance is the brainteaser about two locomotives that face each other on the same track, 100 miles apart. Each locomotive travels at a speed of 50 miles per hour. On the front of one engine is a fly that flies back and forth between the two engines on a zigzag path that ends when the locomotives collide. If the fly's speed is 80 miles per hour, how far does it travel before the trains crash?

It is not easy to sum the infinite series of zigzags, but this is not necessary. The trains collide in one hour, so if the fly's speed is 80 miles per hour, it will have gone 80 miles.

There is an anecdote about how the great mathematician John von Neumann solved this problem. He is said to have thought for a moment before giving the right answer. "Correct," said the proposer, "but most people think you have to sum an infinite series." Von Neumann looked surprised and said that was how he solved it!

A joke version of this problem gives the fly a normal speed of 40 miles per hour instead of 80. How far does it go? The answer is *not* 40 miles!

In Chapter 8 of my *Wheels, Life, and Other Mathematical Amusements* I published a peculiar path problem invented by A.K. Austin, a British mathematician. Here is how he phrased it:

> "A boy, a girl and a dog are at the same spot on a straight road. The boy and the girl walk forward—the boy at four miles per hour, the girl at three miles per hour. As they proceed, the dog trots back and forth between them at 10 miles per hour. Assume that each reversal of its direction is instantaneous. An hour later, where is the dog and which way is it facing?"
>
> Answer: "The dog can be at any point between the boy and the girl, facing either way. Proof: At the end of one hour, place the dog anywhere between the boy and the girl, facing in either direction. Time-reverse all motions and the three will return at the same instant to the starting point."

The problem generated considerable controversy centering around the question of whether the boy, girl, and dog could ever get started. Philosopher of science Wesley Salmon wrote a *Scientific American* article about it which you'll find reprinted in the book previously cited. The problem is a strange instance of an event that can be precisely defined in forward time, but becomes ambiguous when time is reversed.

Another example of such a paradox involves a spiral curve on a sphere. Imagine a point starting at the earth's equator and moving northeast at constant speed. It traces a spiral path called a loxodrome. The point will circle the north pole an infinite number of times before it finally, after a finite time, strangles the pole as the limit of its path. The point starts from a precise spot on the equator. But if the event is time reversed, the point can end at *any* spot on the equator. Are the time reversed versions of the loxodrome and Austin's dog self-contradictory because they require starting an infinite series at its limit? Can the two situations be resolved by applying nonstandard analysis—a form of calculus mentioned in my chapter on limits? For more details, see Professor Salmon's article.

Limits can lead to a variety of mind-twisting fallacies and paradoxes. Consider the doubling series $x = 1 + 2 + 4 + 8 + \ldots.$ Apply to it the trick I explained in my chapter on limits. Multiply each side by 2:

$$2x = 2 + 4 + 8 + 16 + \ldots.$$

The right side clearly is the original series minus 1, therefore $2x = x - 1$. We seem to have proved that the series has a final sum of -1.

A less obvious fallacy occurs when we consider the following two series, each with terms that get progressively smaller so it seems as if each converges:

$$x = \tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \ldots.$$

$$y = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \ldots.$$

The first series is the harmonic series with all even denominator fractions dropped. The second is the harmonic series with all odd denominator fractions omitted.

Multiply both sides of the second series by 2:

$$2y = \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots.$$

This is the harmonic series. Clearly it is the sum of x and y. If $x + y = 2y$, then $x = y$, and we seem to have shown that $x - y = 0$.

We now group the terms of the harmonic series so that each even denominator fraction is subtracted from an odd denominator one:

$$x - y = (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \ldots .$$

Each term inside parentheses is greater than zero. In other words $x - y$ is greater than zero, so apparently we have shown that 0 is greater than 0.

One encounters similar fallacies when terms in a nonconverging series are grouped in various ways.

An even wilder fallacy, which bewildered many mathematicians in the early days of calculus, involves the oscillating series:

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots .$$

If terms are grouped like this:

$x = (1 - 1) + (1 - 1) + (1 - 1) + \ldots$, then the series becomes $0 + 0 + 0 + \ldots$ and $x = 0$.

But if we group like so:

$$x = 1 - (1 - 1) - (1 - 1) - (1 - 1) - \ldots .$$

$$x = 1 - 0 - 0 - 0 - \ldots .$$

$$x = 1$$

Leonhard Euler, by the way, argued that the sum was $\frac{1}{2}$.

Mathematicians distinguish between infinite series that are *absolutely* convergent and those that are *conditionally* convergent. If grouping or rearranging terms has no effect on a series' limit, the series is absolutely convergent. This is always the case if in a series with a mix of plus and minus signs the plus terms taken separately converge, and so do the terms with minus signs taken separately. For example: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ is absolutely convergent, with a limit of $\frac{2}{3}$.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is conditionally convergent because its plus and minus terms, taken separately, do not converge. By rearranging its terms it can have any limit you please. If the terms are not rearranged, their partial sums hop back and forth to finally converge on .693147 . . . . , the natural logarithm of 2.

An ancient geometrical paradox involving a series of line segments that appear to reach a limit, but do not, is the fallacious

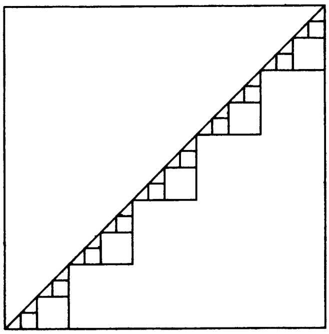


FIG. 14. "Proof" that a square's diagonal equals the sum of its two sides.

"proof" that a square's diagonal is equal to the sum of its two sides. Figure 14 shows a series of stairsteps along a square's diagonal. Because each new series of steps is smaller than the previous series, it looks as if the stairsteps eventually get so tiny that they become the diagonal. Because each stairstep has a length equal to the sum of two sides of the square, have we not shown that the diagonal is the sum of two sides?

Of course we haven't. It is true that the stairsteps, as they get smaller, converge on the diagonal as a limit, but this is a case where they never actually reach the limit. No matter how far we carry the procedure, the total length of the steps remains twice the square's side.

In a similar way we can "prove" that half the circumference of a circle equals the circle's diameter. The "proof" is based on the yin-yang symbol of the Orient, as shown in Figure 15. As the semicircles get smaller and smaller they appear to become the circle's diameter. However, as in the previous fallacy, they never actually reach this limit. The construction along the circle's diameter, and the construction along the square's diagonal, are fractals. Magnify any portion of either "curve" and it always looks the same.

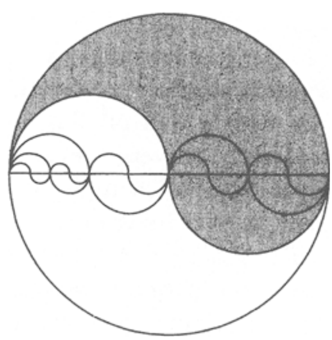


FIG. 15. A yin-yang "proof" that half a circle's circumference equals its diameter.

A beautiful example of a fractal of infinite length even though it surrounds a finite area is the snowflake curve. It is produced by attaching a small equilateral triangle to the central third of each side of an equilateral triangle, then continuing to add smaller and smaller triangles to the new sides. Imagine this process carried to infinity. When the snowflake reaches its limit, its perimeter becomes infinitely long. At the limit it has no tangent at any point, therefore it is a curve without a derivative. Figure 16 shows what the curve looks like after a small number of steps.

When I wrote about the snowflake in my *Scientific American* column I said it had a solid analog with very similar properties. Imagine a regular tetrahedron with a smaller regular tetrahedron attached to the central fourth of each face. Assume this procedure is repeated, with smaller and smaller tetrahedra, to infinity. The resulting surface, I wrote, is a fractal surface, crinkly like the snowflake, infinite in area even though it surrounds a finite volume of space.

My intuition was wrong. William Gosper, a computer scientist famed for his discovery of the glider gun in John Conway's fabled game of Life, was suspicious of my remarks. He wrote a computer program to trace the progress of the solid snowflake. He found that at the limit the polyhedron converged on the surface of a cube! Although the cube is crosscrossed with an infinity of lines, at the limit the lines have no thickness, so the cube's surface is perfectly smooth.

Another striking example of a solid with a finite volume but an infinite surface area is provided by the sequence of attached cubes shown in Figure 17. The top cube has a side of 1. The sides di-

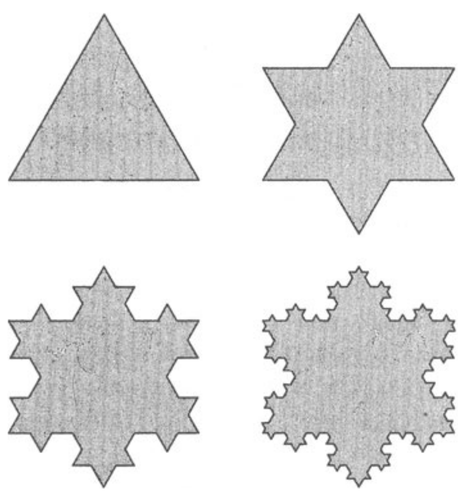


FIG. 16. How the snowflake curve grows.

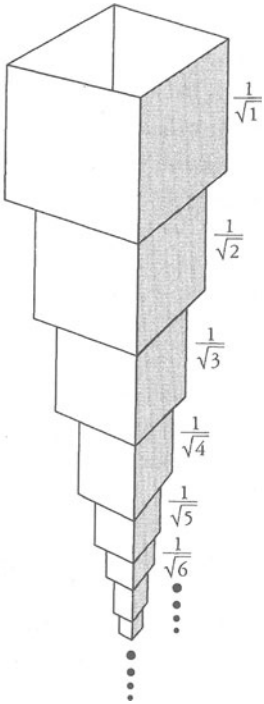


FIG. 17. A solid of finite volume but infinite surface area.

minish in the series: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots.$

The series diverges, which means that the polyhedron grows to an infinite length. The sum of the areas of the faces also diverges. Consider only the faces shown shaded. Their areas are in the series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \ldots.$

Do you recognize it? It is the harmonic series which we know diverges. It would require an infinite supply of paint just to paint one side of each cube! On the other hand, the volumes of the cubes fall into the series:

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots + \frac{1}{n\sqrt{n}} + \ldots.$$

This series converges. The total volume of the cubes is a finite number, but the sum of their surface areas is infinite!

EXTREMUM PROBLEMS

Examples of extrema (maximum and minimum values) abound in physics. Soap films form minimal surface areas. Refracted light minimizes the time it takes to go from A to B. In general relativity bodies moving freely in space-time follow geodesics (shortest paths), and planets and stars seek to minimize their surface area, though mountains and rotational bulges go the other way. Examples are endless. The fact that at extreme points in functions their derivatives equal zero is a striking tribute to the simplicity which nature seems to favor in its fundamental laws.

A maximum value of one variable in a function often minimizes another variable, and vice versa. A circle's circumference, for instance, is a minimum length that surrounds a given area.

Conversely, the circle is the closed curve of given length that maximizes the area it surrounds. A miniskirt is designed both to minimize a skirt's length and maximize the amount of exposed legs. A maxiskirt maximizes its length and minimizes the exposed legs.

In Chapter XI Thompson asks how a number n should be divided into two parts so that the product of the parts is maximum. He shows how easy it is to find the answer by equating a derivative to zero. The number must be divided in half, giving a product equal to $n^2/4$.

A father tosses a handful of coins on a table and says to his son: "Separate those coins into two parts, and your weekly allowance will be the product of the amount in each part. The son maximizes his allowance by dividing the coins so that the two amounts are as nearly equal as possible.

The problem of dividing n into two parts to get a maximum product is equivalent to the following geometrical problem. Given a rectangle's perimeter, what sides will maximize its area? The answer of course is a square. For example, suppose the perimeter is 14. The sum of the two adjacent sides is 7. To maximize the rectangle's area each side must be half of 7 or 3.5, forming a square of area 12.25.

Paul Halmos, in his *Problems for Mathematicians Young and Old* (1991), asks in Problem 51 for the shortest curve that bisects the area of an equilateral triangle. You might guess it to be a straight line parallel to a side, a line of length .707. . . . The correct answer is a circular arc of length .673. . . . You can prove this with calculus, but there is an easier way. Consider the regular hexagon shown in Figure 18. The figure of smallest perimeter that bisects the hexagon's area is the circle shown. Its arc within each of the six equilateral triangles must therefore bisect each triangle. The hexagon's area, assuming its side is 1, is $3\sqrt{3}/2$. The circle of half that area is $3\sqrt{3}/4 = 1.299. . . .$ It is not hard

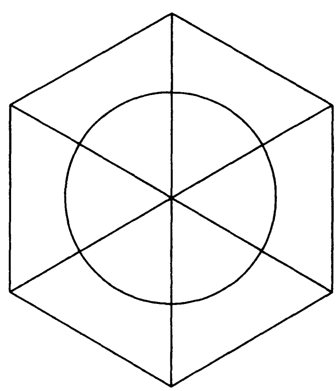


FIG. 18. The shortest curve bisecting an equilateral triangle.

now to find its radius to be .643 . . . , and the length of the arc that bisects each triangle to be

$$\frac{\pi}{3}\sqrt{\frac{3\sqrt{3}}{4\pi}} = .673. \ldots$$

The proof of course assumes the fact that a circle is the closed curve of given length that maximizes its interior area.

Thompson extends his proof that a number divided into two equal parts gives a maximum product to a proof that if a number is broken into n parts, the maximum product of the parts results when all n parts are equal. For example, if the number is 15, the 3-part partition with a maximum product is $5 \times 5 \times 5 = 125$.

This provides a neat proof that given the perimeter of a triangle, the largest area results if the triangle is equilateral. The proof rests on an elegant formula for determining the area of any triangle when given its three sides. It is called Hero's (or Heron's) formula after Hero (Heron) of Alexandria, an ancient Greek mathematician. (The formula has several algebraic proofs, but they are complicated.) Let a,b,c stand for a triangle's sides, and s for its semiperimeter (half the perimeter). Hero's famous formula is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

It is easy to see that the area is maximized when the three values inside the parentheses are equal. This is true only if $a = b = c$, making the triangle equilateral. If each side is 1, the triangle's area will be $\sqrt{3}/4$.

Suppose we are given the perimeter of a triangle and its base. What sides will maximize its area? Calculus will tell you. Here's another way. Imagine two pins separated by the distance of the triangle's base. A string with a length equal to the perimeter circles the pins as shown in Figure 19. A pencil point at c keeps the string taut. As you move the pencil point from side to side, clearly the highest altitude of the triangle formed by the string is obtained when the sides are equal, making the triangle isosceles. (Continuing to move the pencil around the pins will trace an ellipse.)

Assume that the sides of an isosceles triangle remain constant, but the angle they make between them is allowed to vary. This of

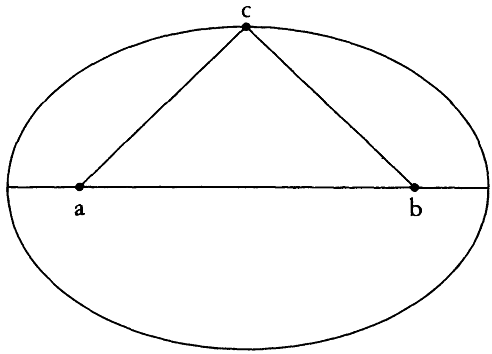


FIG. 19. A string proof of a maximum problem.

course also varies the length of the triangle's third side. What angle maximizes the triangle's area? Knowing that for a given perimeter the equilateral triangle has the largest area, one is tempted to guess that the angle will be 60 degrees. Surprisingly, this is not the case. The maximizing angle is 90 degrees, making the triangle an isosceles right triangle.

Calculus will prove this, but here is a simpler proof. Consider the isosceles right triangle shown on its side in Figure 20. The equal sides *a* and *b* are fixed. By moving the top vertex (c) left or right, keeping *b* the same length, we can vary the angle theta. Note that regardless of which direction the vertex moves, the altitude of the triangle is shortened. Because a triangle's area is half

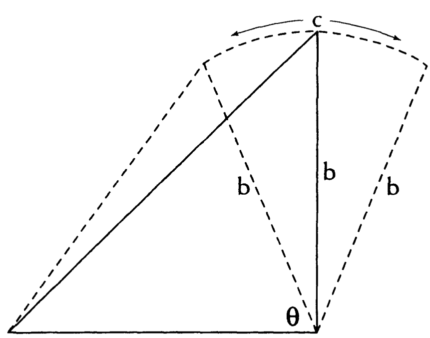


FIG. 20. A simple proof of an isosceles triangle problem.

the product of its base and altitude, the largest area results when the altitude is maximum. This occurs when the angle theta is neither acute nor obtuse, but a right angle.

Another simple proof that the angle is 90° is provided by imagining that the triangle's variable side is a mirror. As Figure 21 shows, the reflection produces a rhombus. As we have learned (and Thompson proves by calculus in Chapter XI), the rectangle of given perimeter which has the largest area is a square. Figure 21B shows that when the reflection forms a square, the area of the isosceles triangle is maximized by having a right angle between its two equal sides.

A farmer wishes to build a fence of three sides to enclose a rectangular plot of land with a fixed wall as its fourth side. What three lengths of the fence will maximize the plot's area? This problem is a favorite of calculus textbooks, but knowing that a square maximizes a rectangle's area, given the perimeter, solves the problem quickly. Imagine the wall to be a mirror. The plot of land plus its mirror image is a larger rectangle. The largest area of this larger rectangle results if it is a square. The plot of land is half that square. Its longer side will be twice the length of each of the other two sides.

A related problem can be solved using the correct assumption that a regular polygon of *n* sides encloses the largest area for its perimeter. If the farmer's fence has *n* sides, the mirror trick shows that the plot's area is maximized by making the sides half of a regular polygon of 2*n* sides. As the sides of a regular polygon increase, they approach the circle as their limit. (This was how

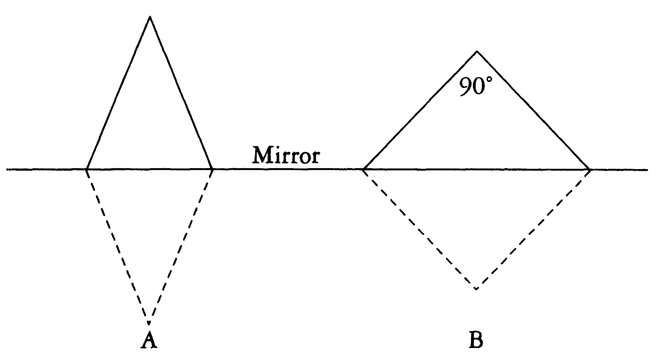


FIG. 21. A mirror reflection proof.

Archimedes found a good approximation for the value of π.) So, if the farmer wishes to surround his plot with a *curved* fence on the side of the wall, the mirror trick shows that he maximizes the plot's area by making the fence the arc of a semicircle.

Sometimes more than one mirror reflection will solve a problem easier than by using calculus. Suppose you have a screen made of two identical halves that are hinged together so you can vary the angle between them. You wish to place them at the corner of a room so as to maximize the enclosed area. Figure 22 shows how two mirrors solve the problem. A polygon of eight equal sides has the largest area when it is regular. The screens must be placed so they form a quadrant of the regular octagon which has angles of 135°.

A maximum problem given in most calculus textbooks (I'm surprised Thompson does not include it) concerns a square to be cut and folded to make a square box without a lid. This is done by cutting out little squares at the four corners of the large square, then folding up the rectangular sides. The question is: What sizes should the removed squares be to produce an open box of maximum volume?

For example, suppose the square is 12 inches on the side. Let x be the side of each corner square to be removed. The box's square base will have a side of $12-2x$, or $2(6-x)$. The area of the square base $= [2(6-x)]^2 = 4(6-x)^2$. The volume of the box is x times the area of the base, or $4x(6-x)^2$.

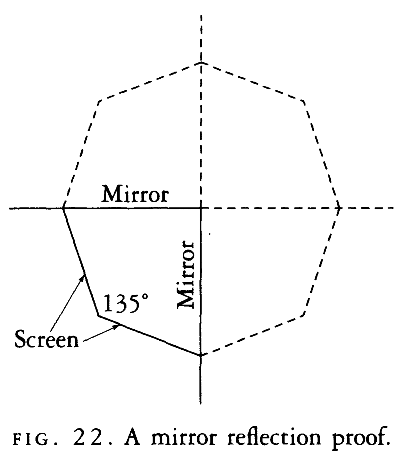


FIG. 22. A mirror reflection proof.

The derivative simplifies to $12(6 - x)(2 - x)$. When this is equated to zero, we find that x has a value of either 6 or 2. It can't be 6, because cutting out six-inch squares would leave nothing to fold up, therefore $x = 2$. The box of maximum volume has a square base of 8 inches on the side, a depth of 2 inches, and a volume of $2 \times 64 = 128$ cubic inches.

Note that the area of the base is 64 inches, and the total area of the sides is also 64 inches. This equality holds regardless of the size of the original square.

S. King, in "Maximizing a Polygonal Box" (*The Mathematical Gazette,* Vol. 81, March 1997, pp. 96-99) shows that this is true for any convex polygon whose corners are cut and the rectangular sides folded to make an open box. If the polygon is a triangle or a regular polygon the volume is maximized when the ratio of the combined area of the sides to the area of the discarded corners is 4 to 1.

The task of determining the largest square that will go inside a cube is involved in a famous problem known as Prince Rupert's problem after a nephew of England's King Charles. (The Prince lived from 1619 to 1682.) Rupert asked: Is it possible to cut a hole through a cube large enough for a slightly larger cube to be passed through the tunnel?

If you hold a cube so a corner points directly toward you you will see the regular hexagon shown in Figure 23, left. It is possible to inscribe within this hexagon a square that has a side a trifle larger than the cube's face. Figure 24, right, shows the square from a different perspective. Note that two of the

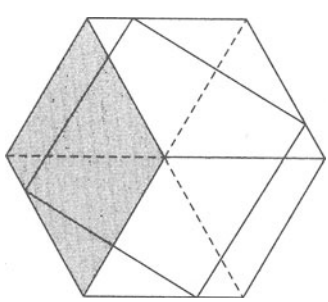


FIG. 23.

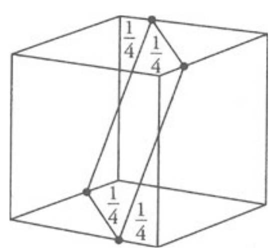


FIG. 24. Two views of Prince Rupert's hole through cube.

square's sides are visible on outside faces of the cube, whereas the other two sides are inside the cube and invisible if the cube is opaque.

The square is the largest possible square that fits inside a cube. Each corner, as indicated, is one-fourth the distance along an edge from a corner. If the cube's edge is 1, the side of the inscribed square is $3/\sqrt{8} = 1.060660. \ldots$ This is three-fourth's of the cube's face diagonal, a tiny bit longer than the cube's edge. This allows a tunnel to be cut through a solid cube—a tunnel with the inscribed square as its cross-section. Through the tunnel can be passed a cube whose edge is a trifle less than $1.060660. \ldots$ The square's area is exactly $9/8 = 1.125$.

I do not know how to prove by differentiating that this square is the largest that fits in a cube, and would enjoy hearing from any reader who can tell me. The problem generalizes to cubes of higher dimension. It is not easy to determine the largest cube that will fit within a hypercube of four dimensions. For higher dimensions the general problem remains unsolved. See Problem B4 in *Unsolved Problems in Geometry* (1991) by Hallard Croft, Kenneth Falconer, and Richard Guy.

The largest rectangle, by the way, that fits inside a cube is the cross-section that goes through the diagonals of two opposite faces. On the unit cube it is a rectangle of sides $1 \times \sqrt{2}$, with an area of $\sqrt{2}$.

CYLINDERS

Problems about right circular cylinders abound in calculus textbooks. Here is a delightful little-known puzzle that I found in *Problems for Puzzlebusters* (1992, pp 25-26), by David L. Book.

A right circular cylindrical can is open at the top. The can's diameter is 4 inches, its height is 6 inches. A sphere is dropped into the can. The task is to find the size of the ball that maximizes the amount of liquid that must be poured into the can to cover the ball exactly.

You might think that the largest possible ball, four inches in diameter, would do the trick. With more reflection you realize that a smaller ball might require more liquid because there would be more space around it to fill.

Let x be the ball's diameter. The ball, plus the liquid that just covers it, forms a right circular cylinder with a height equal to the ball's diameter. The height of this cylinder (ball plus liquid) is also x. The cylinder's volume is $4\pi x$. The ball's volume is $4\pi\left(\dfrac{x}{2}\right)^3/3 = \pi x^3/6$.

The amount of liquid required to cover the ball is the difference between the ball's volume and the cylinder's volume, so we can write:

$$v = 4\pi x - \frac{\pi x^3}{6}$$

$$= \frac{24\pi x}{6} - \frac{\pi x^3}{6}$$

$$= \frac{\pi}{6}(24x - x^3)$$

The above expression gives the amount of liquid needed to cover the ball as a function of x, the ball's diameter. To find the largest size of the ball, $\pi/6$ is a constant so it can be dropped. Only $24x - x^3$ need be differentiated and equated to zero. The derivative is $24 - 3x^2$. Equating to zero and solving for x gives the ball's diameter as $\sqrt{8} = 2.828+$.

In Chapter V Thompson considers the rate at which a right circular cylinder's volume varies as a function of its radius. Susan Jane Colley, writing on "Calculus in the Brewery" in *The College Mathematics Journal* (May 1994) points out how manufacturers of products that come in cylindrical cans make use of this function to deceive consumers. She noticed that beer cans which hold exactly the same amount often have different heights but seem to have the same radius. One expects the taller can to hold more beer when actually it does not.

Colley shows by differentiating the formula for the volume of a cylinder (the area of base times height) that if a cylindrical beer can has a very slight decrease in radius, a decrease imperceptible to the eye, it grows taller by ten times the decrease in radius while keeping its volume the same. The taller cans are optical illusions. They seem to hold more beer than the shorter cans, but actually do not. "Smart people, those marketing sales types," Cooley concludes.

Another delightful cylinder problem that deserves to be in calculus textbooks is solved by Aparna W. Higgins in "What is the Lowest Position of the Center of Mass of a Soda Can?", in *Primus* (Vol. 7, March 1997, pp. 35-42). A full can's center of mass is near the can's geometrical center, and the same is true of an empty can. As the drink is consumed, the center of mass steadily lowers. When the can is empty, it is back up to the geometric center. Obviously it cannot go to the bottom, then jump suddenly up to the center, so there must be a point at which the center of mass reaches its lowest point before it starts to rise.

The problem is to find that minimum. Three cylinders are involved: the can, the liquid, and the air above the liquid, assuming, of course, the can is vertical.

Dr. Higgins, a mathematician at the University of Dayton in Ohio, shows how to solve the problem neatly by differentiating and equating to zero. Surprise! The can's center of mass reaches its lowest point when it exactly reaches the level of the liquid!

In Chapter 16 of my *Wheels, Life, and Other Mathematical Amusements* I give this problem along with a reader's clever way of solving it without calculus. However, as Dr. Higgins points out, using differential calculus to find the answer is an excellent exercise for calculus students. I might add that unlike so many textbook problems it is directly related to a student's experience.

Problems of integration that routinely turn up in calculus textbooks can often be solved by simpler methods. A classic example involves two circular cylinders, each with a unit radius, which intersect at right angles as shown in Figure 25, top. What is the volume of the shaded portion that is common to both cylinders?

To answer this question you need know only that the area of a circle is πr^2, and the volume of a sphere is $(4\pi r^3)/3$.

Imagine a sphere of unit radius inside the volume common to the two cylinders and having as its center the point where the axes of the cylinders intersect. Suppose that the cylinders and sphere are sliced in half by a plane through the sphere's center and both axes of the cylinders (See Figure 26, bottom left). The cross section of the volume common to the cylinders will be a

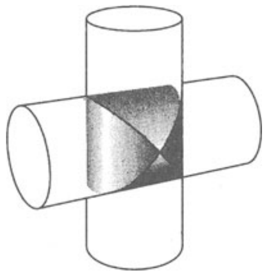


FIG. 25. Archimedes Problem of the crossed cylinders

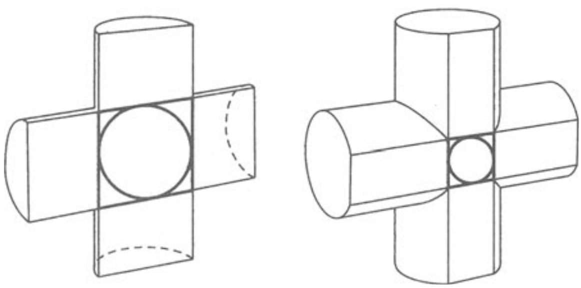


FIG. 26. Crossections of the cylinders.

square. The cross section of the sphere will be a circle inscribed in a square.

Now suppose that the cylinders and sphere are sliced by a plane parallel to the previous one but that shaves off only a small portion of each cylinder (Figure 26, bottom right). This will produce parallel tracks on each cylinder, which intersect as before to form a square cross section of the volume common to both cylinders. Also as before, the cross section of the sphere will be a circle inside the square. It is not hard to see (with a little imagination and pencil doodling) that any plane section through the cylinders, parallel to the cylinders' axes, will always have the same result: a square cross section of the volume common to the cylinders, enclosing a circular cross section of the sphere.

Think of all these plane sections as being packed together like the leaves of a book. Clearly, the volume of the sphere will be the sum of all circular cross sections, and the volume of the solid common to both cylinders will be the sum of all the square cross sections. We conclude, therefore that the ratio of the volume of the sphere to the volume of the solid common to the cylinders is the same as the ratio of the area of a circle to the area of a circumscribed square. A brief calculation shows that the latter ratio is $\pi/4$. This allows the following equation, in which x is the volume we seek:

$$\frac{4\pi r^3/3}{x} = \frac{\pi}{4}$$

The π's drop out, giving x a value of $16r^3/3$. The radius in this case is 1, so the volume common to both cylinders is 16/3. As Archimedes pointed out, it is exactly 2/3 the volume of a cube that encloses the sphere; that is, a cube with an edge equal to the diameter of each cylinder.

This solution is a famous application of what is known as "Cavalieri's principle" after Francesco Bonaventura Cavalieri (1598-1647), an Italian mathematical physicist and student of Galileo. In essence it says that solids, such as prisms, cones, cylinders, and pyramids which have the same height and corresponding cross sections of the same area, have the same volume. In proving his principle Cavalieri anticipated integral calculus by building up a volume by summing to a limit an infinite set of infinitesimal cross sections. The principle was known to Archimedes. In a lost book called *The Method* that was not found until 1906 (it is the book in which Archimedes answers the crossed cylinders problem), he attributes the principle to Democritus who used it for calculating the volume of a pyramid and a cone.

The problem can be generalized in various ways, notably to n mutually perpendicular cylinders of the same radius that intersect in spaces of three and higher dimensions. Even the case of three cylinders that intersect at right angles is beyond the power of Cavalieri's principle and requires the use of integral calculus. The volume common to three cylinders is $8r^3(2 - \sqrt{2})$.

The two-cylinder case has applications in architecture to what are called "barrel vaults" that form ceilings. When two such vaults intersect they create a cross vault—a portion of the surface of the solid shared by two crossed cylinders. There also are important applications in crystallography and engineering.

Another application of Cavalieri's principle is to an old brain teaser about a cork plug that will fit smoothly into three holes: a circle, square, and an almost equilateral triangle. (See Figure 27, left.) Put another way, what shape can, if turned properly, cast shadows that are circles, squares, and isosceles triangles?

The plug solving the problem is shown on the right. Assume its circular base has a radius of 1, its height 2, and the top edge, directly above a diameter of the base, is also 2. You can also think of the surface as generated by a straight line joining the sharp edge to the circumference of the base, and moving so that at all times it is parallel to a plane perpendicular to the sharp edge. Calculus will give the plug's volume, but Cavalieri's method again solves it more simply. All you need know is that the volume of a right circular cylinder is the area of its base times its altitude.

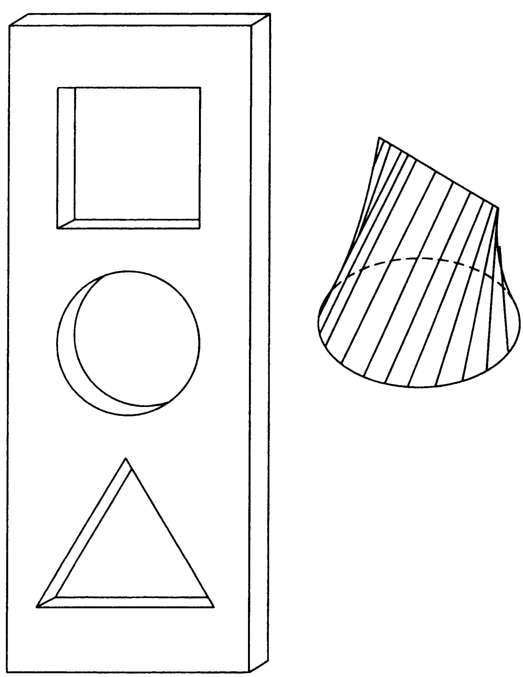


FIG. 27. The cork plug problem.

Here is how I gave the solution in my *Second Scientific American Book of Mathematical Puzzles and Diversions* (1961):

> Any vertical cross section of the cork plug at right angles to the top edge and perpendicular to the base will be a triangle. If the cork were a cylinder of the same height, corresponding cross sections would be rectangles. Each triangular cross section is obviously 1/2 the area of the corresponding rectangular cross section. Since all the triangular sections combine to make up the cylinder, the plug must be 1/2 the volume of the cylinder. The cylinder's volume is 2π, so our answer is simply π.
>
> Actually, the cork can have an infinite number of shapes and still fit the three holes. The shape described has the least volume of any convex solid that will fit the holes. The largest volume is obtained by the simple procedure of slicing the cylinder with two plane cuts as shown in Figure 28. This is the shape given in most puzzle books that include the plug problem. Its volume is equal to $2\pi - 8/3$.

Cavalieri's principle also applies to plane figures. If shapes between parallel lines A and B, as in Figure 29, have cross sections of the same area when cut by every line parallel to A and B, then they have the same area. As in the solid case, this generalizes to the theorem that if corresponding cross sections are always in the same ratio, their areas will have the same ratio.

CYCLOIDS AND HYPOCYCLOIDS

The curve generated by a point on a circle as it rolls along a straight line (see Figure 30) is called a cycloid. The length of the straight line segment from A to B is, of course, pi. Because pi is irrational, many mathematicians in days before calculus suspected that the length of the arc from cusp to cusp was also irrational. Today, almost all calculus textbooks show

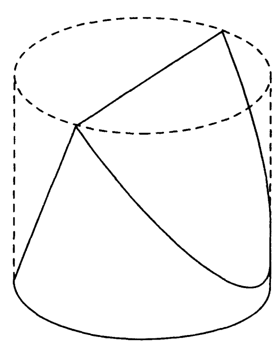


FIG. 28. Another way to slice the cork plug.

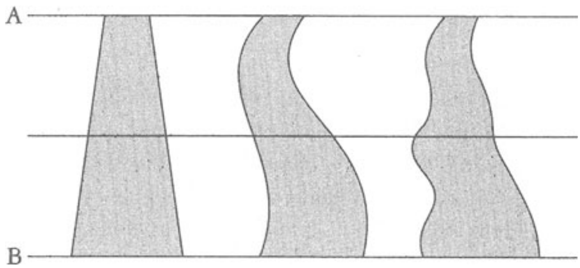


FIG. 29. Cavalieri's principle applied to plane figures.

how easy it is to determine that the arc's length is exactly four times the circle's diameter. It is also easy to determine by integration that the area below the arch is exactly three times the area of the circle. (For some of the cycloid's remarkable properties see Chapter 13 of my *Sixth Book of Mathematical Games,* 1971.)

Hypocycloids are closed curves generated by a point on a circle as it rolls inside a larger circle. They are discussed in many books on recreational mathematics as well as in calculus textbooks. Figure 31 shows two hypocycloids that have names. The deltoid is a three-cusped hypocycloid traced by a point on a rolling circle with a radius one-third or two-thirds that of the larger circle. The astroid of four cusps is produced by a point on a rolling circle one-fourth or three-fourths that of the containing circle.

A circle is not a graph of a function because a vertical line can cross it at two points, thereby giving two values to y for a given value of x. However, sectors of it can be integrated to obtain areas by using what are called "parametric" equations because they involve a third variable called their parameter. In the circle's case the parameter is the angle theta shown in Thompson's Figure 43, commonly designated by t or the Greek θ.

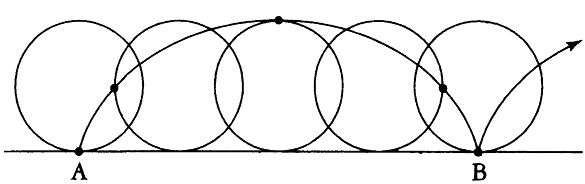
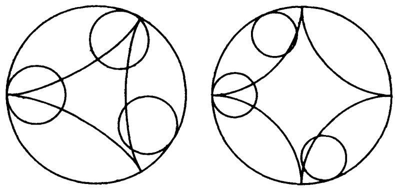


FIG. 30. How a cycloid is generated by a point on a rolling circle.



The Deltoid The Astroid

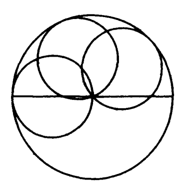
FIG. 31.

If r is the radius of a circle centered at the graph's origin, its parametric equations are the trigonometric functions $x = r \cos \theta$, and $y = r \sin \theta$. Each value of theta, from 0 to 360°, determines a point on the circle.

Like the circle, the deltoid and astroid are not graphs of functions. Their areas are best obtained from their parametric equations based on the angle theta. Thompson does not discuss parametric equations, but you'll find them in modern textbooks. Integrating such equations gives the deltoid's area 2/9 the area of the larger circle, or $2\pi a^2$ where a is a rolling circle's radius that is 1/3 that of the larger circle. If that radius is 1, the deltoid's area is 2π. The astroid's area is 3/8 of its outer circle, and six times the rolling circle's area. The rolling circle has a radius 1/4 of the larger circle, or 2/3 times the rolling circle's area if the rolling circle has 3/4 the radius of the larger one.

I mention all this because of two connections with recreational geometry. What is the area of the two-cusped hypocycloid traced by a point on a rolling circle with half the radius of the larger circle? (See Figure 32) The surprising answer is zero! The hypocycloid is a straight line that is the large circle's diameter!

The minimum-size convex figure in which a needle (a straight line segment) of length 1 can be rotated 180 degrees is the equilateral triangle with a unit altitude and area of $\sqrt{3}/3$. What is the minimum *nonconvex* figure in which a unit needle can be rotated?



"Two-cusped" Hypocycloid

FIG. 32.

For many decades it was believed to be the deltoid. To everyone's surprise it was proved that the area could be as small as desired! I give the history of what is called "Kakeya's needle problem" in Chapter 18 of my *Unexpected Hanging and Other Mathematical Diversions* (1969). Hypocycloids and their sisters the epicycloids (generated by points on a circle that rolls *outside* another circle) are discussed in the first chapter of my *Wheels, Life, and Other Mathematical Amusements* (1983).

I have, of course, covered only a tiny fraction of calculus related problems that can be found in the literature of recreational mathematics. As you have seen, many calculus problems can be solved more easily by not using calculus. Indeed, there is a tendency of some mathematicians to look down their noses on non-calculus solutions to calculus problems, as though they were undignified tricks. On the contrary, they are just as useful and just as elegant as calculus solutions. The two classic references on this are: "No Calculus, Please," an entertaining paper by J. H. Butchart and Leo Moser in *Scripta Mathematica* (Vol. 18, September-December 1952, pp. 221-226), and Ivan Niven's *Maxima and Minima Without Calculus* (1981).

"The position taken by this book," Niven writes (p. 242), "is that, while calculus offers a powerful technique for solving extremal problems, there are other methods of great power that should not be overlooked. . . . Many students . . . try to solve extremal questions . . . by seeking some function to differentiate, even though most of these problems are handled best by other methods. Moreover, students will often pursue the differentiation process through thick and thin, in spite of hopelessly complicated functions at hand."

Let me close this haphazard selection of problems by introducing a joke that physicist Richard Feynman once inflicted on fellow students at M.I.T. I quote from his autobiography *Surely You're Joking, Mr. Feynman* (1985):

> I often liked to play tricks on people when I was at MIT. One time, in mechanical drawing class, some joker picked up a French curve (a piece of plastic for drawing smooth curves—a curly, funny-looking thing) and said, "I wonder if the curves on this thing have some special formula?"
>
> I thought for a moment and said, "Sure they do. The curves are very special curves. Lemme show ya," and I picked up my French curve and began to turn it slowly.

"The French curve is made so that at the lowest point on each curve, no matter how you turn it, the tangent is horizontal."

All the guys in the class were holding their French curve up at different angles, holding their pencil up to it at the lowest point and laying it along, and discovering that, sure enough, the tangent is horizontal. They were all excited by this "discovery"—even though they had already gone through a certain amount of calculus and had already "learned" that the derivative (tangent) of the minimum (lowest point) of *any* curve is zero (horizontal). They didn't put two and two together. They didn't even know what they "knew."