(a) Prove that $y_0^2 + (y_0')^2$ is constant, and conclude that either $y_0(0) \neq 0$ or $y_0'(0) \neq 0$.

(b) Prove that there is a function s satisfying $s'' + s = 0$ and $s(0) = 0$ and $s'(0) = 1$. Hint: Try s of the form $ay_0 + by_0'$.

If we define $\sin = s$ and $\cos = s'$, then almost all facts about trigonometric functions become trivial. There is one point which requires work, however—producing the number π. This is most easily done using an exercise from the Appendix to Chapter 11:

(c) Use Problem 6 of the Appendix to Chapter 11 to prove that $\cos x$ cannot be positive for all $x > 0$. It follows that there is a smallest $x_0 > 0$ with $\cos x_0 = 0$, and we can define $\pi = 2x_0$.

(d) Prove that $\sin \pi/2 = 1$. (Since $\sin^2 + \cos^2 = 1$, we have $\sin \pi/2 = \pm 1$; the problem is to decide why $\sin \pi/2$ is positive.)

(e) Find $\cos \pi$, $\sin \pi$, $\cos 2\pi$, and $\sin 2\pi$. (Naturally you may use any addition formulas, since these can be derived once we know that $\sin' = \cos$ and $\cos' = -\sin$.)

(f) Prove that cos and sin are periodic with period 2π.

31. (a) After all the work involved in the definition of sin, it would be disconcerting to find that sin is actually a rational function. Prove that it isn't. (There is a simple property of sin which a rational function cannot possibly have.)

(b) Prove that sin isn't even defined implicitly by an algebraic equation; that is, there do not exist rational functions $f_0, \ldots, f_{n-1}$ such that

$$(\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \cdots + f_0(x) = 0 \quad \text{for all } x.$$

Hint: Prove that $f_0 = 0$, so that $\sin x$ can be factored out. The remaining factor is 0 except perhaps at multiples of π. But this implies that it is 0 for all x. (Why?) You are now set up for a proof by induction.

***32.** Suppose that ϕ_1 and ϕ_2 satisfy

$$\begin{aligned} \phi_1'' + g_1\phi_1 &= 0, \\ \phi_2'' + g_2\phi_2 &= 0, \end{aligned}$$

and that $g_2 > g_1$.

(a) Show that

$$\phi_1''\phi_2 - \phi_2''\phi_1 - (g_2 - g_1)\phi_1\phi_2 = 0.$$

(b) Show that if $\phi_1(x) > 0$ and $\phi_2(x) > 0$ for all x in (a, b), then

$$\int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] > 0,$$

and conclude that

$$[\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a)] - [\phi_1(b)\phi_2'(b) - \phi_1(a)\phi_2'(a)] > 0.$$

(c) Show that in this case we cannot have $\phi_1(a) = \phi_1(b) = 0$. Hint: Consider the sign of $\phi_1{}'(a)$ and $\phi_1{}'(b)$.

(d) Show that the equations $\phi_1(a) = \phi_1(b) = 0$ are also impossible if $\phi_1 > 0$, $\phi_2 < 0$ or $\phi_1 < 0$, $\phi_2 > 0$, or $\phi_1 < 0$, $\phi_2 < 0$ on (a, b). (You should be able to do this with almost no extra work.)

The net result of this problem may be stated as follows: if a and b are consecutive zeros of ϕ_1, then ϕ_2 must have a zero somewhere between a and b. This result, in a slightly more general form, is known as the Sturm Comparison Theorem. As a particular example, any solution of the differential equation

$$y'' + (x+1)y = 0$$

must have at least one zero in any interval $(n\pi, (n+1)\pi)$.

33. (a) Using the formula for $\sin x - \sin y$ derived in Problem 14, show that

$$\sin(k + \tfrac{1}{2})x - \sin(k - \tfrac{1}{2})x = 2 \sin \frac{x}{2} \cos kx.$$

(b) Conclude that

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x}{2 \sin \dfrac{x}{2}}.$$

Like two other results in this problem set, this equation is very important in the study of Fourier series, and we also make use of it in Problems 19-43 and 23-22.

(c) Similarly, derive the formula

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin\left(\dfrac{n+1}{2}x\right) \sin\left(\dfrac{n}{2}x\right)}{\sin \dfrac{x}{2}}.$$

(A more natural derivation of these formulas will be given in Problem 27-14.)

(d) Use parts (b) and (c) to find $\int_0^b \sin x \, dx$ and $\int_0^b \cos x \, dx$ directly from the definition of the integral.

*CHAPTER 16 π IS IRRATIONAL

This short chapter, diverging from the main stream of the book, is included to demonstrate that we are already in a position to do some sophisticated mathematics. This entire chapter is devoted to an elementary proof that π is irrational. Like many "elementary" proofs of deep theorems, the motivation for many steps in our proof cannot be supplied; nevertheless, it is still quite possible to follow the proof step-by-step.

Two observations must be made before the proof. The first concerns the function

$$f_n(x) = \frac{x^n(1-x)^n}{n!},$$

which clearly satisfies

$$0 < f_n(x) < \frac{1}{n!} \quad \text{for } 0 < x < 1.$$

An important property of the function f_n is revealed by considering the expression obtained by actually multiplying out $x^n(1-x)^n$. The lowest power of x appearing will be n and the highest power will be $2n$. Thus f_n can be written in the form

$$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i,$$

where the numbers c_i are integers. It is clear from this expression that

$$f_n{}^{(k)}(0) = 0 \quad \text{if } k < n \text{ or } k > 2n.$$

Moreover,

$$\begin{aligned}
f_n{}^{(n)}(x) &= \frac{1}{n!}[n!\,c_n + \text{ terms involving } x]\\
f_n{}^{(n+1)}(x) &= \frac{1}{n!}[(n+1)!\,c_{n+1} + \text{ terms involving } x]\\
&\;\;\vdots\\
f_n{}^{(2n)}(x) &= \frac{1}{n!}[(2n)!\,c_{2n}].
\end{aligned}$$

This means that

$$\begin{aligned} f_n^{(n)}(0) &= c_n, \\ f_n^{(n+1)}(0) &= (n+1)c_{n+1} \\ &\vdots \\ f_n^{(2n)}(0) &= (2n)(2n-1)\cdot\ldots\cdot(n+1)c_{2n}, \end{aligned}$$

where the numbers on the right are all integers. Thus

$$f_n^{(k)}(0) \text{ is an } \textit{integer} \text{ for all } k.$$

The relation

$$f_n(x) = f_n(1-x)$$

implies that

$$f_n^{(k)}(x) = (-1)^k f_n^{(k)}(1-x);$$

therefore,

$$f_n^{(k)}(1) \text{ is also an } \textit{integer} \text{ for all } k.$$

The proof that π is irrational requires one further observation: if a is any positive number, and $\varepsilon > 0$, then for sufficiently large n we will have

$$\frac{a^n}{n!} < \varepsilon.$$

To prove this, notice that if $n \geq 2a$, then

$$\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1} \cdot \frac{a^n}{n!} < \frac{1}{2} \cdot \frac{a^n}{n!}.$$

Now let n_0 be any natural number with $n_0 \geq 2a$. Then, whatever value

$$\frac{a^{n_0}}{(n_0)!}$$

may have, the succeeding values satisfy

$$\begin{aligned} \frac{a^{(n_0+1)}}{(n_0+1)!} &< \frac{1}{2} \cdot \frac{a^{n_0}}{(n_0)!} \\ \frac{a^{(n_0+2)}}{(n_0+2)!} &< \frac{1}{2} \cdot \frac{a^{(n_0+1)}}{(n_0+1)!} < \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{a^{n_0}}{(n_0)!} \\ &\vdots \\ \frac{a^{(n_0+k)}}{(n_0+k)!} &< \frac{1}{2^k} \cdot \frac{a^{n_0}}{(n_0)!}. \end{aligned}$$

If k is so large that $\dfrac{a^{n_0}}{(n_0)!\,\varepsilon} < 2^k$, then

$$\frac{a^{(n_0+k)}}{(n_0+k)!} < \varepsilon,$$

which is the desired result. Having made these observations, we are ready for the one theorem in this chapter.

THEOREM 1 The number π is irrational; in fact, π^2 is irrational. (Notice that the irrationality of π^2 implies the irrationality of π, for if π were rational, then π^2 certainly would be.)

PROOF Suppose π^2 were rational, so that

$$\pi^2 = \frac{a}{b}$$

for some positive integers a and b. Let

$$(1) \quad G(x) = b^n[\pi^{2n} f_n(x) - \pi^{2n-2} f_n{}''(x) + \pi^{2n-4} f_n{}^{(4)}(x) - \cdots + (-1)^n f_n{}^{(2n)}(x)].$$

Notice that each of the factors

$$b^n \pi^{2n-2k} = b^n(\pi^2)^{n-k} = b^n \left(\frac{a}{b}\right)^{n-k} = a^{n-k}b^k$$

is an integer. Since $f_n{}^{(k)}(0)$ and $f_n{}^{(k)}(1)$ are integers, this shows that

$$G(0) \text{ and } G(1) \text{ are integers.}$$

Differentiating G twice yields

$$(2) \qquad G''(x) = b^n[\pi^{2n} f_n{}''(x) - \pi^{2n-2} f_n{}^{(4)}(x) + \cdots + (-1)^n f_n{}^{(2n+2)}(x)].$$

The last term, $(-1)^n f_n{}^{(2n+2)}(x)$, is zero. Thus, adding (1) and (2) gives

$$(3) \qquad G''(x) + \pi^2 G(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x).$$

Now let

$$H(x) = G'(x)\sin \pi x - \pi G(x)\cos \pi x.$$

Then

$$\begin{aligned} H'(x) &= \pi G'(x)\cos \pi x + G''(x)\sin \pi x - \pi G'(x)\cos \pi x + \pi^2 G(x)\sin \pi x \\ &= [G''(x) + \pi^2 G(x)]\sin \pi x \\ &= \pi^2 a^n f_n(x)\sin \pi x, \text{ by (3).} \end{aligned}$$

By the Second Fundamental Theorem of Calculus,

$$\begin{aligned} \pi^2 \int_0^1 a^n f_n(x)\sin \pi x\, dx &= H(1) - H(0) \\ &= G'(1)\sin \pi - \pi G(1)\cos \pi - G'(0)\sin 0 + \pi G(0)\cos 0 \\ &= \pi[G(1) + G(0)]. \end{aligned}$$

Thus

$$\pi \int_0^1 a^n f_n(x)\sin \pi x\, dx \quad \text{is an } \textit{integer.}$$

On the other hand, $0 < f_n(x) < 1/n!$ for $0 < x < 1$, so

$$0 < \pi a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!} \quad \text{for } 0 < x < 1.$$

Consequently,

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!}.$$

This reasoning was completely independent of the value of n. Now if n is large enough, then

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!} < 1.$$

But this is absurd, because the integral is an integer, and there is no integer between 0 and 1. Thus our original assumption must have been incorrect: π^2 is irrational. ▌

This proof is admittedly mysterious; perhaps most mysterious of all is the way that π enters the proof—it almost looks as if we have proved π irrational without ever mentioning a definition of π. A close reexamination of the proof will show that precisely one property of π is essential—

$$\sin(\pi) = 0.$$

The proof really depends on the properties of the function sin, and proves the irrationality of the smallest positive number x with $\sin x = 0$. In fact, very few properties of sin are required, namely,

$$\begin{aligned} \sin' &= \cos, \\ \cos' &= -\sin, \\ \sin(0) &= 0, \\ \cos(0) &= 1. \end{aligned}$$

Even this list could be shortened; as far as the proof is concerned, cos might just as well be defined as $\sin'$. The properties of sin required in the proof may then be written

$$\begin{aligned} \sin'' + \sin &= 0, \\ \sin(0) &= 0, \\ \sin'(0) &= 1. \end{aligned}$$

Of course, this is not really very surprising at all, since, as we have seen in the previous chapter, these properties characterize the function sin completely.

PROBLEMS

1. (a) For the areas of triangles OAB and OAC in Figure 1, with $\angle AOB \le \pi/4$, show that we have

$$\text{area } OAC = \frac{1}{2}\sqrt{\frac{1-\sqrt{1-16(\text{area } OAB)^2}}{2}}.$$

Hint: Solve the equations $xy = 2(\text{area } OAB)$, $x^2 + y^2 = 1$, for y.

(b) Let P_m be the regular polygon of m sides inscribed in the unit circle. If A_m is the area of P_m show that

$$A_{2m} = \frac{m}{2}\sqrt{2 - 2\sqrt{1-(2A_m/m)^2}}.$$

This result allows one to obtain (more and more complicated) expressions for A_{2^n}, starting with $A_4 = 2$, and thus to compute π as accurately as desired (according to Problem 8-11). Although better methods will appear in Chapter 20, a slight variant of this approach yields a very interesting expression for π:

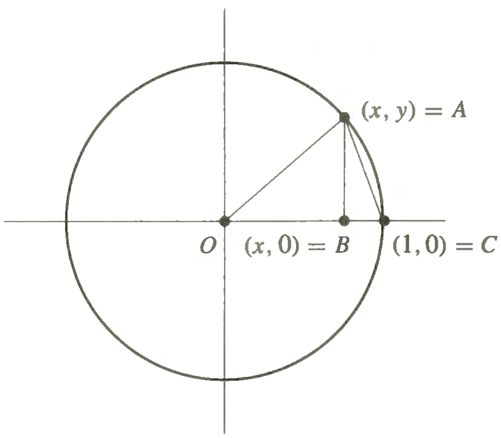


FIGURE 1

2. (a) Using the fact that

$$\frac{\text{area}(OAB)}{\text{area}(OAC)} = OB,$$

show that if α_m is the distance from O to one side of P_m, then

$$\frac{A_m}{A_{2m}} = \alpha_m.$$

(b) Show that

$$\frac{2}{A_{2^k}} = \alpha_4 \cdot \alpha_8 \cdot \ldots \cdot \alpha_{2^{k-1}}.$$

(c) Using the fact that

$$\alpha_m = \cos\frac{\pi}{m},$$

and the formula $\cos x/2 = \sqrt{\dfrac{1+\cos x}{2}}$ (Problem 15-15), prove that

$$\alpha_4 = \sqrt{\frac{1}{2}}$$

$$\alpha_8 = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}},$$

$$\alpha_{16} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}},$$

etc.

Together with part (b), this shows that $2/\pi$ can be written as an "infinite product"

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ldots ;$$

to be precise, this equation means that the product of the first n factors can be made as close to $2/\pi$ as desired, by choosing n sufficiently large. This product was discovered by François Viète in 1579, and is only one of many fascinating expressions for π, some of which are mentioned later.

*CHAPTER 17 PLANETARY MOTION

Nature and Nature's Laws lay hid in night
God said "Let Newton be," and all was light.

Alexander Pope

Unlike Chapter 16, a short chapter diverging from the main stream of the book, this long chapter diverges from the main stream of the book to demonstrate that we are already in a position to do some real physics.

In 1609 Kepler published his first two laws of planetary motion. The first law describes the shape of planetary orbits:

The planets move in ellipses, with the sun at one focus.

The second law involves the area swept out by the segment from the sun to the planet (the 'radius vector from the sun to the planet') in various time intervals (Figure 1):

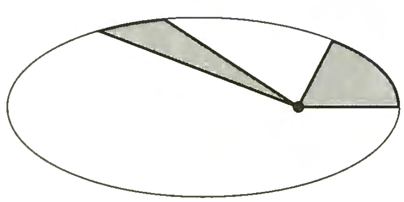


FIGURE 1

Equal areas are swept out by the radius vector in equal times. (Equivalently, the area swept out in time t is proportional to t.)

Kepler's third law, published in 1619, relates the motions of different planets. If a is the major axis of a planet's elliptical orbit and T is its period, the time it takes the planet to return to a given position, then:

The ratio a^3/T^2 is the same for all planets.

Newton's great accomplishment was to show (using his general law that the force on a body is its mass times its acceleration) that Kepler's laws follow from the assumption that the planets are attracted to the sun by a force (the gravitational force of the sun) always directed toward the sun, proportional to the mass of the planet, and satisfying an inverse square law; that is, by a force directed toward the sun whose magnitude varies inversely with the square of the distance from the sun to the planet and directly with the mass of the planet. Since force is mass times acceleration, this is equivalent simply to saying that the magnitude of the acceleration is a constant divided by the square of the distance from the sun.

Newton's analysis actually established three results that correlate with Kepler's individual laws. The first of Newton's results concerns Kepler's second law (which was actually discovered first, nicely preserving the symmetry of the situation):

Kepler's second law is true precisely for 'central forces', i.e., if and only the force between the sun and the planet always lies along the line between the sun and the planet.

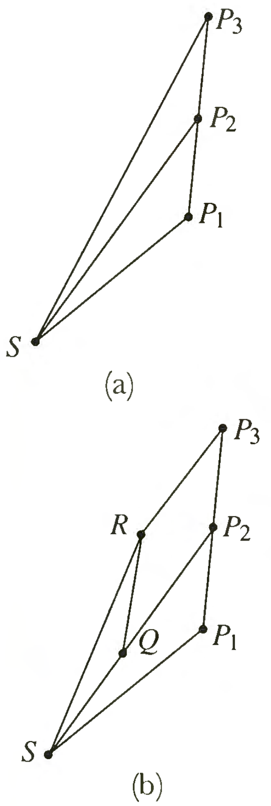


FIGURE 2

Although Newton is revered as the discoverer of calculus, and indeed invented calculus precisely in order to treat such problems, his derivation hardly seems to use calculus at all. Instead of considering a force that varies continuously as the planet moves, Newton first considers short equal time intervals and assumes that a momentary force is exerted at the ends of each of these intervals.

To be specific, let us imagine that during the first time interval the planet moves along the line P_1P_2, with uniform velocity (Figure 2a). If, during the next equal time interval, the planet continued to move along this line, it would end up at P_3, where the length of P_1P_2 equals the length of P_2P_3. This would imply that the triangle SP_1P_2 has the same area as the triangle SP_2P_3 (since they have equal bases, and the same height)—this just says that Kepler's law holds in the special case where the force is 0.

Now suppose (Figure 2b) that at the moment the planet arrives at P_2 it experiences a force exerted *along the line from S to P_2*, which by itself would cause the planet to move to the point Q. Combined with the motion that the planet already has, this causes the planet to move to R, the vertex opposite P_2 in the parallelogram whose sides are P_2P_3 and P_2Q.

Thus, the area swept out in the second time interval is actually the triangle SP_2R. But the area of triangle SP_2R is equal to the area of triangle SP_3P_2, since they have the same base SP_2, and the same heights (since RP_3 is parallel to SP_2). Hence, finally, the area of triangle SP_2R is the same as the area of the original triangle SP_1P_2! Conversely, if the triangle SRP_2 has the same area as SP_1P_2, and hence the same area as SP_3P_2, then RP_3 must be parallel to SP_2, and this implies that Q must lie along SP_2.

Of course, this isn't quite the sort of argument one would expect to find in a modern book, but in its own charming way it shows physically just *why* the result should be true.

To analyze planetary motion we will be using the material in the Appendix to Chapter 12, and the "determinant" det defined in Problem 4 of Appendix 1 to Chapter 4. We describe the motion of the planet by the parameterized curve

$$c(t) = r(t)(\cos\theta(t), \sin\theta(t)),$$

so that r always gives the length of the line from the sun to the planet, while θ gives the angle, which we will assume is increasing (the case where θ is decreasing then follows easily). It will be convenient to write this also as

$$(1) \qquad c(t) = r(t)\cdot \mathbf{e}(\theta(t)),$$

where

$$\mathbf{e}(t) = (\cos t, \sin t)$$

is just the parameterized curve that runs along the unit circle. Note that

$$\mathbf{e}'(t) = (-\sin t, \cos t)$$

is also a vector of unit length, but perpendicular to $\mathbf{e}(t)$, and that we also have

$$(2) \qquad \det\big(\mathbf{e}(t), \mathbf{e}'(t)\big) = 1.$$

Differentiating (1), using the formulas on page 247, we obtain

$$(3) \qquad c'(t) = r'(t)\cdot\mathbf{e}(\theta(t)) + r(t)\theta'(t)\cdot\mathbf{e}'(\theta(t)),$$

and combining with (1), together with the formulas in Problem 6 of Appendix 1 to Chapter 4, we get

$$\begin{aligned}\det\big(c(t), c'(t)\big) &= r(t)r'(t)\det\big(\mathbf{e}(\theta(t)), \mathbf{e}(\theta(t))\big) + r(t)^2\theta'(t)\det\big(\mathbf{e}(\theta(t)), \mathbf{e}'(\theta(t))\big)\\ &= r(t)^2\theta'(t)\det\big(\mathbf{e}(\theta(t)), \mathbf{e}'(\theta(t))\big),\end{aligned}$$

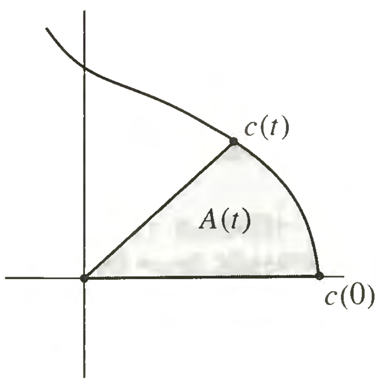


FIGURE 3

since $\det(v, v)$ is always 0. Using (2) we then get

$$(4) \qquad \det(c, c') = r^2\theta'.$$

As we will see, $r^2\theta'$ turns out to have another important interpretation.

Suppose that $A(t)$ is the area swept out from time 0 to t (Figure 3). We want to get a formula for $A'(t)$, and, in the spirit of Newton, we'll begin by making an educated guess. Figure 4 shows $A(t+h) - A(t)$, together with a straight line segment between $c(t)$ and $c(t+h)$. It is easy to write down a formula for the area of the triangle $\Delta(h)$ with vertices O, $c(t)$, and $c(t+h)$: according to Problems 4 and 5 of Appendix 1 to Chapter 4, the area is

$$\text{area}(\Delta(h)) = \tfrac{1}{2}\det\big(c(t), c(t+h) - c(t)\big).$$

Since the triangle $\Delta(h)$ has practically the same area as the region $A(t+h) - A(t)$, this shows (or practically shows) that

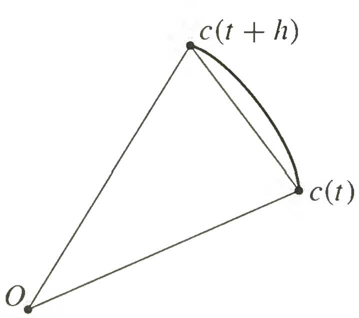


FIGURE 4

$$\begin{aligned}A'(t) &= \lim_{h\to 0}\frac{A(t+h) - A(t)}{h}\\ &= \lim_{h\to 0}\frac{\text{area}\,\Delta(h)}{h}\\ &= \tfrac{1}{2}\det\left(c(t), \lim_{h\to 0}\frac{c(t+h) - c(t)}{h}\right)\\ &= \tfrac{1}{2}\det\big(c(t), c'(t)\big).\end{aligned}$$

A rigorous derivation, establishing more in the process, can be made using Problem 13-24, which gives a formula for the area of a region determined by the graph of a function in polar coordinates. According to this Problem, we can write

$$(*) \qquad A(t) = \frac{1}{2}\int_{\theta(0)}^{\theta(t)} \rho(\phi)^2\,d\phi$$

if our parameterized curve $c(t) = r(t)\cdot\mathbf{e}(\theta(t))$ is the graph of the function ρ in polar coordinates (here we've used ϕ for the angular polar coordinate, to avoid confusion with the function θ used to describe the curve c).

Now the function ρ is just

$$\rho = r \circ \theta^{-1}$$

[for any particular angle ϕ, $\theta^{-1}(\phi)$ is the time at which the curve c has angular polar coordinate ϕ, so $r(\theta^{-1}(t))$ is the radius coordinate corresponding to ϕ]. Although the presence of the inverse function might look a bit forbidding, it's actually quite innocent: Applying the First Fundamental Theorem of Calculus and the Chain Rule to ($*$) we immediately get

$$\begin{aligned} A'(t) &= \tfrac{1}{2}\rho(\theta(t))^2 \cdot \theta'(t) \\ &= \tfrac{1}{2}r(t)^2\theta'(t), \qquad \text{since } \rho = r \circ \theta^{-1}. \end{aligned}$$

Briefly,

$$A' = \tfrac{1}{2}r^2\theta'.$$

Combining with (4), we thus have

$$\boxed{A' = \tfrac{1}{2}\det(c, c') = \tfrac{1}{2}r^2\theta'.} \tag{5}$$

Now we're ready to consider Kepler's second law. Notice that *Kepler's second law is equivalent to saying that A' is constant*, and thus it is equivalent to $A'' = 0$. But

$$\begin{aligned} A'' = \tfrac{1}{2}[\det(c, c')]' &= \tfrac{1}{2}\det(c', c') + \tfrac{1}{2}\det(c, c'') \qquad \text{(see page 248)} \\ &= \tfrac{1}{2}\det(c, c''). \end{aligned}$$

So

$$\boxed{\text{Kepler's second law is equivalent to } \det(c, c'') = 0.}$$

Putting this all together we have:

THEOREM 1 Kepler's second law is true if and only if the force is central, and in this case each planetary path $c(t) = r(t) \cdot \mathbf{e}(\theta(t))$ satisfies the equation

$$r^2\theta' = \det(c, c') = \text{constant}. \tag{K_2}$$

PROOF Saying that the force is central just means that it always points along $c(t)$. Since $c''(t)$ is in the direction of the force, that is equivalent to saying that $c''(t)$ always points along $c(t)$. And this is equivalent to saying that we always have

$$\det(c, c'') = 0.$$

We've just seen that this is equivalent to Kepler's second law.

Moreover, this equation implies that $[\det(c, c')]' = 0$, which by (5) gives (K_2). ▮

Newton next showed that if the gravitational force of the sun is a central force and also satisfies an inverse square law, then the path of any object in it will be a conic section having the sun at one focus. Planets, of course, correspond to the case where the conic section is an ellipse, and this is also true for comets that visit the sun periodically; parabolas and hyperbolas represent objects that come from outside the solar system, and eventually continue on their merry way back outside the system.

THEOREM 2 If the gravitational force of the sun is a central force that satisfies an inverse square law, then the path of any body in it will be a conic section having the sun at one focus (more precisely, either an ellipse, parabola, or one branch of an hyperbola).

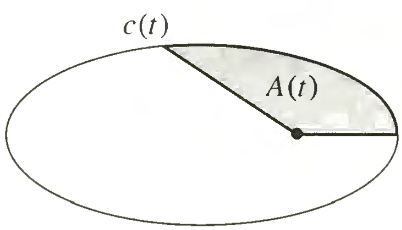


FIGURE 5

PROOF Notice that our conclusion specifies the shape of the path, not a particular parameterization. But this parameterization is essentially determined by Theorem 1: the hypothesis of a central force implies that the area $A(t)$ (Figure 5) is proportional to t, so determining $c(t)$ is essentially equivalent to determining A for arbitrary points on the ellipse. Unfortunately, the areas of such segments cannot be determined explicitly.* This means that we have to determine the *shape* of the path $c(t) = r(t) \cdot \mathbf{e}(\theta(t))$ without finding its parameterization! Since it is the function $r \circ \theta^{-1}$ which actually describes the shape of the path in polar coordinates, we shouldn't be surprised to find θ^{-1} entering into the proof.

By Theorem 1, the hypothesis of a central force implies that

$$(K_2) \qquad r^2\theta' = \det(c, c') = M$$

for some constant M. The hypothesis of an inverse square law can be written

$$(*) \qquad c''(t) = -\frac{H}{r(t)^2}\,\mathbf{e}(\theta(t))$$

for some constant H. Using (K_2), this can be written

$$\frac{c''(t)}{\theta'(t)} = -\frac{H}{M}\,\mathbf{e}(\theta(t)).$$

Notice that the left-hand side of this equation is

$$[c' \circ \theta^{-1}]'(\theta(t)).$$

So if we let

$$D = c' \circ \theta^{-1}$$

(this is the main trick—"we consider c' as a function of θ"), then the equation can be written as

$$D'(\theta(t)) = -\frac{H}{M}\,\mathbf{e}(\theta(t)) = -\frac{H}{M}\big(\cos\theta(t), \sin\theta(t)\big).$$

*More precisely, we can't write down a solution in terms of familiar "standard functions," like sin, arcsin, etc.

and we can write this simply as

$$D'(u) = -\frac{H}{M}(\cos u, \sin u) = \left(-\frac{H}{M}\cos u, -\frac{H}{M}\sin u\right)$$

[for all u of the form $\theta(t)$ for some t], completely eliminating θ.

The equation that we have just obtained is simply a pair of equations, for the components of D, each of which we can easily solve individually; we thus find that

$$D(u) = \left(\frac{H\cdot\sin u}{-M} + A, \frac{H\cdot\cos u}{M} + B\right)$$

for two constants A and B. Letting $u = \theta(t)$ again we thus have an explicit formula for c':

$$c' = \left(\frac{H\cdot\sin\theta}{-M} + A, \frac{H\cdot\cos\theta}{M} + B\right).$$

[Here $\sin\theta$ really stands for $\sin\circ\theta$, etc., abbreviations that we will use throughout.]

Although we can't get an explicit formula for c itself, if we substitute this equation, together with $c = r(\cos\theta, \sin\theta)$, into the equation

$$\det(c, c') = M \qquad \text{(equation } (K_2)),$$

we get

$$r\left[\frac{H}{M}\cos^2\theta + B\cos\theta + \frac{H}{M}\sin^2\theta - A\sin\theta\right] = M,$$

which simplifies to

$$r\left[\frac{H}{M^2} + \frac{B}{M}\cos\theta - \frac{A}{M}\sin\theta\right] = 1.$$

Problem 15-8 shows that this can be written in the form

$$r(t)\left[\frac{H}{M^2} + C\cos(\theta(t) + D)\right] = 1,$$

for some constants C and D. We can let $D = 0$, since this simply amounts to rotating our polar coordinate system (choosing which ray corresponds to $\theta = 0$), so we can write, finally,

$$r[1 + \varepsilon\cos\theta] = \frac{M^2}{H} = \Lambda.$$

But this is the formula for a conic section derived in Appendix 3 of Chapter 4 (together with Problems 5, 6, and 7 of that Appendix). ∎

In terms of the constant M in the equation

$$r^2\theta' = M$$

and the constant Λ in the equation of the orbit

$$r[1 + \varepsilon\cos\theta] = \Lambda$$

the last equation in our proof shows that we can rewrite (∗) as

$$(**) \qquad c''(t) = -\frac{M^2}{\Lambda} \cdot \frac{1}{r(t)^2}\,\mathbf{e}(\theta(t)).$$

Recall (page 87) that the major axis a of the ellipse is given by

$$\text{(a)} \qquad a = \frac{\Lambda}{1-\varepsilon^2},$$

while the minor axis b is given by

$$\text{(b)} \qquad b = \frac{\Lambda}{\sqrt{1-\varepsilon^2}}.$$

Consequently,

$$\text{(c)} \qquad \frac{b^2}{\Lambda} = a.$$

Remember that equation (5) gives

$$A'(t) = \tfrac{1}{2}r^2\theta' = \tfrac{1}{2}M,$$

and thus

$$A(t) = \tfrac{1}{2}Mt.$$

We can therefore interpret M in terms of the period T of the orbit. This period T is, by definition, the value of t for which we have $\theta(t) = 2\pi$, so that we obtain the complete ellipse. Hence

$$\text{area of the ellipse} = A(T) = \tfrac{1}{2}MT,$$

or

$$M = \frac{2(\text{area of the ellipse})}{T} = \frac{2\pi ab}{T} \qquad \text{by Problem 13-17.}$$

Hence the constant M^2/Λ in (∗∗) is

$$\begin{aligned} \frac{M^2}{\Lambda} &= \frac{4\pi^2a^2b^2}{T^2\Lambda} \\ &= \frac{4\pi^2a^3}{T^2}, \qquad \text{using (c).} \end{aligned}$$

This completes the final step of Newton's analysis:

THEOREM 3 Kepler's third law is true if and only if the accelerations $c''(t)$ of the various planets, moving on ellipses, satisfy

$$c''(t) = -G \cdot \frac{1}{r^2}\,\mathbf{e}(\theta(t))$$

for a constant G that does not depend on the planet.

It should be mentioned that the converse of Theorem 2 is also true. To prove this, we first want to establish one further consequence of Kepler's second law. Recall that for

$$\mathbf{e}(t) = (\cos t, \sin t)$$

we have

$$\mathbf{e}'(t) = (-\sin t, \cos t).$$

Consequently,

$$\mathbf{e}''(t) = (-\cos t, -\sin t) = -\mathbf{e}(t).$$

Now differentiating (3) gives

$$\begin{aligned} c''(t) = r''(t) \cdot \mathbf{e}(\theta(t)) + r'(t)\theta'(t) \cdot \mathbf{e}'(\theta(t)) \\ + r'(t)\theta'(t) \cdot \mathbf{e}'(\theta(t)) + r(t)\theta''(t) \cdot \mathbf{e}'(\theta(t)) + r(t)\theta'(t)\theta'(t) \cdot \mathbf{e}''(\theta(t)). \end{aligned}$$

Using $\mathbf{e}''(t) = -\mathbf{e}(t)$ we get

$$c''(t) = \left[r''(t) - r(t)\theta'(t)^2\right] \cdot \mathbf{e}(\theta(t)) + \left[2r'(t)\theta'(t) + r(t)\theta''(t)\right] \cdot \mathbf{e}'(\theta(t)).$$

Since Kepler's second law implies central forces, hence that $c''(t)$ is always a multiple of $c(t)$, and thus always a multiple of $\mathbf{e}(\theta(t))$, the coefficient of $\mathbf{e}'(\theta(t))$ must be 0 [as a matter of fact, we can see this directly by taking the derivative of formula (K_2)]. Thus Kepler's second law implies that

$$(6) \qquad c''(t) = \left[r''(t) - r(t)\theta'(t)^2\right] \cdot \mathbf{e}(\theta(t)).$$

THEOREM 4 If the path of a planet moving under a central gravitational force lies on a conic section with the sun as focus, then the force must satisfy an inverse square law.

PROOF As in Theorem 2, notice that the hypothesis on the shape of the path, together with the hypothesis of a central force, which is equivalent to Kepler's second law, essentially determines the parameterization. But we can't write down an explicit solution, so we have to obtain information about the acceleration without actually knowing what it is.

Once again, the hypothesis of a central force implies that

$$(K_2) \qquad r^2\theta' = M,$$

for some constant M, and the hypothesis that the path lies on a conic section with the sun as focus implies that it satisfies the equation

$$(A) \qquad r[1 + \varepsilon\cos\theta] = \Lambda,$$

for some ε and Λ. For our (not especially illuminating) proof, we will keep differentiating and substituting from these two equations.

First we differentiate (A) to obtain

$$r'[1 + \varepsilon\cos\theta] - \varepsilon r\theta'\sin\theta = 0.$$

Multiplying by r this becomes

$$rr'[1 + \varepsilon\cos\theta] - \varepsilon r^2\theta'\sin\theta = 0.$$

Using both (A) and (K_2), this becomes

$$\Lambda r' - \varepsilon M \sin\theta = 0.$$

Differentiating again, we get

$$\Lambda r'' - \varepsilon M \theta' \cos\theta = 0.$$

Using (K_2) we get

$$\Lambda r'' - \frac{\varepsilon M^2}{r^2}\cos\theta = 0,$$

and then using (A) we get

$$\Lambda r'' - \frac{M^2}{r^2}\left[\frac{\Lambda}{r} - 1\right] = 0.$$

Substituting from (K_2) yet again, we get

$$\Lambda[r'' - r(\theta')^2] + \frac{M^2}{r^2} = 0,$$

or

$$r'' - r(\theta')^2 = -\frac{M^2}{\Lambda r^2}.$$

Comparing with (6), we obtain

$$c''(t) = -\frac{M^2}{\Lambda r^2}\,\mathbf{e}(\theta(t)),$$

which is precisely what we wanted to show: the force is inversely proportional to the square of the distance from the sun to the planet. ▮

CHAPTER 18 THE LOGARITHM AND EXPONENTIAL FUNCTIONS

In Chapter 15 the integral provided a rigorous formulation for a preliminary definition of the functions sin and cos. In this chapter the integral plays a more essential role. For certain functions even a preliminary definition presents difficulties. For example, consider the function

$$f(x) = 10^x.$$

This function is assumed to be defined for all x and to have an inverse function, defined for positive x, which is the "logarithm to the base 10,"

$$f^{-1}(x) = \log_{10} x.$$

In algebra, 10^x is usually defined only for *rational* x, while the definition for irrational x is quietly ignored. A brief review of the definition for rational x will not only explain this omission, but also recall an important principle behind the definition of 10^x.

The symbol 10^n is first defined for natural numbers n. This notation turns out to be extremely convenient, especially for multiplying very large numbers, because

$$10^n \cdot 10^m = 10^{n+m}.$$

The extension of the definition of 10^x to rational x is motivated by the desire to preserve this equation; this requirement actually forces upon us the customary definition. Since we want the equation

$$10^0 \cdot 10^n = 10^{0+n} = 10^n$$

to be true, we must define $10^0 = 1$; since we want the equation

$$10^{-n} \cdot 10^n = 10^0 = 1$$

to be true, we must define $10^{-n} = 1/10^n$; since we want the equation

$$\underbrace{10^{1/n} \cdot \ldots \cdot 10^{1/n}}_{n \text{ times}} = 10^{\overbrace{}^{}\underbrace{1/n+\cdots+1/n}_{n \text{ times}}} = 10^1 = 10$$

to be true, we must define $10^{1/n} = \sqrt[n]{10}$; and since we want the equation

$$\underbrace{10^{1/n} \cdot \ldots \cdot 10^{1/n}}_{m \text{ times}} = 10^{\underbrace{1/n+\cdots+1/n}_{m \text{ times}}} = 10^{m/n}$$

to be true, we must define $10^{m/n} = (\sqrt[n]{10}\,)^m$.

Unfortunately, at this point the program comes to a dead halt. We have been guided by the principle that 10^x should be defined so as to ensure that $10^{x+y} = 10^x 10^y$; but this principle does not suggest any simple algebraic way of defining

10^x for irrational x. For this reason we will try some more sophisticated ways of finding a function f such that

$$(*) \qquad f(x+y) = f(x)\cdot f(y) \quad \text{for all } x \text{ and } y.$$

Of course, we are interested in a function which is not always zero, so we might add the condition $f(1) \neq 0$. If we add the more specific condition $f(1) = 10$, then $(*)$ will imply that $f(x) = 10^x$ for rational x, and 10^x could be *defined* as $f(x)$ for other x; in general $f(x)$ will equal $[f(1)]^x$ for rational x.

One way to find such a function is suggested if we try to solve an apparently more difficult problem: find a *differentiable* function f such that

$$f(x+y) = f(x)\cdot f(y) \quad \text{for all } x \text{ and } y,$$
$$f(1) = 10.$$

Assuming that such a function exists, we can try to find f'—knowing the derivative of f might provide a clue to the definition of f itself. Now

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)}{h} \\ &= f(x)\cdot \lim_{h\to 0} \frac{f(h)-1}{h}. \end{aligned}$$

The answer thus depends on

$$f'(0) = \lim_{h\to 0} \frac{f(h)-1}{h};$$

for the moment assume this limit exists, and denote it by α. Then

$$f'(x) = \alpha \cdot f(x) \quad \text{for all } x.$$

Even if α could be computed, this approach seems self-defeating. The derivative of f has been expressed in terms of f again.

If we examine the inverse function $f^{-1} = \log_{10}$, the whole situation appears in a new light:

$$\begin{aligned} \log_{10}{}'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{\alpha\cdot f(f^{-1}(x))} = \frac{1}{\alpha x}. \end{aligned}$$

The derivative of f^{-1} is about as simple as one could ask! And, what is even more interesting, of all the integrals $\int_a^b x^n\,dx$ examined previously, the integral $\int_a^b x^{-1}\,dx$ is the only one which we cannot evaluate. Since $\log_{10} 1 = 0$ we should have

$$\frac{1}{\alpha}\int_1^x \frac{1}{t}\,dt = \log_{10} x - \log_{10} 1 = \log_{10} x.$$

This suggests that we define $\log_{10} x$ as $(1/\alpha)\int_1^x t^{-1}\,dt$. The difficulty is that α is unknown. One way of evading this difficulty is to define

$$\log x = \int_1^x \frac{1}{t}\,dt,$$

and hope that this integral will be the logarithm to *some* base, which might be determined later. In any case, the function defined in this way is surely more reasonable, from a mathematical point of view, than $\log_{10}$. The usefulness of $\log_{10}$ depends on the important role of the number 10 in arabic notation (and thus ultimately on the fact that we have ten fingers), while the function log provides a notation for an extremely simple integral which cannot be evaluated in terms of any functions already known to us.

DEFINITION

If $x > 0$, then

$$\mathbf{log}\ x = \int_1^x \frac{1}{t}\,dt.$$

The graph of log is shown in Figure 1. Notice that if $x > 1$, then $\log x > 0$, and if $0 < x < 1$, then $\log x < 0$, since, by our conventions,

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0.$$

For $x \le 0$, a number $\log x$ cannot be defined in this way, because $f(t) = 1/t$ is not bounded on $[x, 1]$.

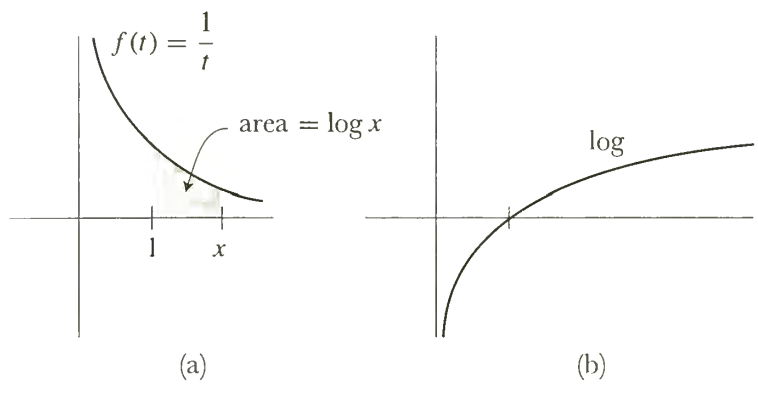


FIGURE 1

The justification for the notation "log" comes from the following theorem.

THEOREM 1 If $x, y > 0$, then

$$\log(xy) = \log x + \log y.$$

PROOF Notice first that $\log'(x) = 1/x$, by the Fundamental Theorem of Calculus. Now choose a number $y > 0$ and let

$$f(x) = \log(xy).$$

Then

$$f'(x) = \log'(xy) \cdot y = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

Thus $f' = \log'$. This means that there is a number c such that

$$f(x) = \log x + c \quad \text{for all } x > 0,$$

that is,

$$\log(xy) = \log x + c \quad \text{for all } x > 0.$$

The number c can be evaluated by noting that when $x = 1$ we obtain

$$\begin{aligned} \log(1 \cdot y) &= \log 1 + c \\ &= c. \end{aligned}$$

Thus

$$\log(xy) = \log x + \log y \quad \text{for all } x.$$

Since this is true for all $y > 0$, the theorem is proved. ▮

COROLLARY 1 If n is a natural number and $x > 0$, then

$$\log(x^n) = n \log x.$$

PROOF Let to you (use induction). ▮

COROLLARY 2 If $x, y > 0$, then

$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

PROOF This follows from the equations

$$\log x = \log\left(\frac{x}{y} \cdot y\right) = \log\left(\frac{x}{y}\right) + \log y. ▮$$

Theorem 1 provides some important information about the graph of log. The function log is clearly increasing, but since $\log'(x) = 1/x$, the derivative becomes very small as x becomes large, and log consequently grows more and more slowly. It is not immediately clear whether log is bounded or unbounded on **R**. Observe, however, that for a natural number n,

$$\log(2^n) = n \log 2 \quad (\text{and } \log 2 > 0);$$

it follows that log is, in fact, not bounded above. Similarly,

$$\log\left(\frac{1}{2^n}\right) = \log 1 - \log 2^n = -n \log 2;$$

therefore log is not bounded below on $(0, 1)$. Since log is continuous, it actually takes on all values. Therefore $\mathbf{R}$ is the domain of the function $\log^{-1}$. This important function has a special name, whose appropriateness will soon become clear.

DEFINITION The "exponential function," **exp**, is defined as $\log^{-1}$.

The graph of exp is shown in Figure 2. Since $\log x$ is defined only for $x > 0$, we always have $\exp(x) > 0$. The derivative of the function exp is easy to determine.

THEOREM 2 For all numbers x,

$$\exp'(x) = \exp(x).$$

PROOF

$$\begin{aligned}\exp'(x) = (\log^{-1})'(x) &= \frac{1}{\log'(\log^{-1}(x))} \\ &= \frac{1}{\dfrac{1}{\log^{-1}(x)}} \\ &= \log^{-1}(x) = \exp(x). \blacksquare\end{aligned}$$

A second important property of exp is an easy consequence of Theorem 1.

THEOREM 3 If x and y are any two numbers, then

$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

PROOF Let $x' = \exp(x)$ and $y' = \exp(y)$, so that

$$\begin{aligned} x &= \log x', \\ y &= \log y'. \end{aligned}$$

Then

$$x + y = \log x' + \log y' = \log(x'y').$$

This means that

$$\exp(x+y) = x'y' = \exp(x) \cdot \exp(y). \blacksquare$$

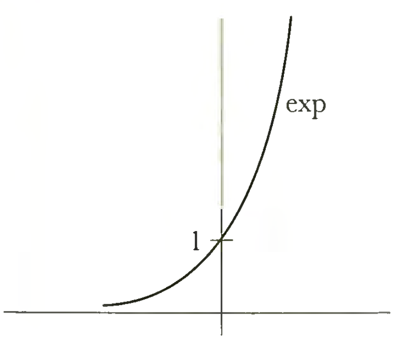


FIGURE 2

This theorem, and the discussion at the beginning of this chapter, suggest that $\exp(1)$ is particularly important. There is, in fact, a special symbol for this number.

DEFINITION $e = \exp(1)$.

This definition is equivalent to the equation

$$1 = \log e = \int_1^e \frac{1}{t}\,dt.$$

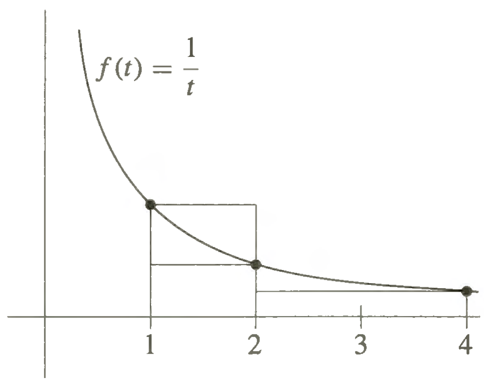


FIGURE 3

As illustrated in Figure 3,

$$\int_1^2 \frac{1}{t}\,dt < 1, \quad \text{since } 1\cdot(2-1) \text{ is an upper sum for } f(t) = 1/t \text{ on } [1,2],$$

and

$$\int_1^4 \frac{1}{t}\,dt > 1, \quad \text{since } \tfrac{1}{2}\cdot(2-1) + \tfrac{1}{4}\cdot(4-2) = 1 \text{ is a lower sum for } f(t) = 1/t \text{ on } [1,4].$$

Thus

$$\int_1^2 \frac{1}{t}\,dt < \int_1^e \frac{1}{t}\,dt < \int_1^4 \frac{1}{t}\,dt,$$

which shows that

$$2 < e < 4.$$

In Chapter 20 we will find much better approximations for e, and also prove that e is irrational (the proof is much easier than the proof that π is irrational!).

As we remarked at the beginning of the chapter, the equation

$$\exp(x + y) = \exp(x)\cdot\exp(y)$$

implies that

$$\begin{aligned}\exp(x) &= [\exp(1)]^x \\ &= e^x, \quad \text{for all } \textit{rational } x.\end{aligned}$$

Since exp is defined for all x and $\exp(x) = e^x$ for rational x, it is consistent with our earlier use of the exponential notation to *define* e^x as $\exp(x)$ for all x.

DEFINITION

For any number x,

$$e^x = \exp(x).$$

The terminology "exponential function" should now be clear. We have succeeded in defining e^x for an arbitrary (even irrational) exponent x. We have not yet defined a^x, if $a \neq e$, but there is a reasonable principle to guide us in the attempt. If x is *rational*, then

$$a^x = (e^{\log a})^x = e^{x\log a}.$$

But the last expression is defined for *all* x, so we can use it to define a^x.

DEFINITION

If $a > 0$, then, for any real number x,

$$a^x = e^{x \log a}.$$

(If $a = e$ this definition clearly agrees with the previous one.)

The requirement $a > 0$ is necessary, in order that $\log a$ be defined. This is not unduly restrictive since, for example, we would not even expect

$$(-1)^{1/2} \stackrel{?}{=} \sqrt{-1}$$

to be defined. (Of course, for certain rational x, the symbol a^x will make sense, according to the old definition; for example,

$$(-1)^{1/3} = \sqrt[3]{-1} = -1.)$$

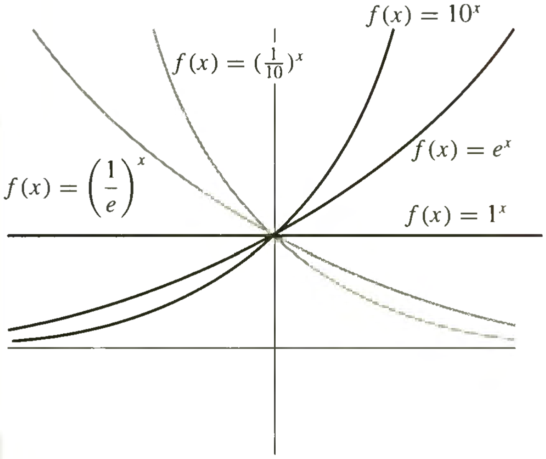


FIGURE 4

Our definition of a^x was designed to ensure that

$$(e^x)^y = e^{xy} \quad \text{for all } x \text{ and } y.$$

As we would hope, this equation turns out to be true when e is replaced by any number $a > 0$. The proof is a moderately involved unraveling of terminology. At the same time we will prove the other important properties of a^x.

THEOREM 4 If $a > 0$, then

(1) $(a^b)^c = a^{bc}$ for all b, c.

(Notice that a^b will automatically be positive, so $(a^b)^c$ will be defined);

(2) $a^1 = a$ and $a^{x+y} = a^x \cdot a^y$ for all x, y.

(Notice that (2) implies that this definition of a^x agrees with the old one for all rational x.)

PROOF

$$(1) \quad (a^b)^c = e^{c \log a^b} = e^{c \log(e^{b \log a})} = e^{c(b \log a)} = e^{cb \log a} = a^{bc}.$$

(Each of the steps in this string of equalities depends upon our last definition, or the fact that $\exp = \log^{-1}$.)

$$(2) \quad \begin{aligned} a^1 &= e^{1 \log a} = e^{\log a} = a, \\ a^{x+y} &= e^{(x+y)\log a} = e^{x \log a + y \log a} = e^{x \log a} \cdot e^{y \log a} = a^x \cdot a^y. \ \blacksquare \end{aligned}$$

Figure 4 shows the graphs of $f(x) = a^x$ for several different a. The behavior of the function depends on whether $a < 1$, $a = 1$, or $a > 1$. If $a = 1$, then

$f(x) = 1^x = 1$. Suppose $a > 1$. In this case $\log a > 0$. Thus,

$$\begin{array}{ll} \text{if} & x < y, \\ \text{then} & x \log a < y \log a, \\ \text{so} & e^{x \log a} < e^{y \log a}, \\ \text{i.e.,} & a^x < a^y. \end{array}$$

Thus the function $f(x) = a^x$ is increasing. On the other hand, if $0 < a < 1$, so that $\log a < 0$, the same sort of reasoning shows that the function $f(x) = a^x$ is decreasing. In either case, if $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is one-one. Since exp takes on every positive value it is also easy to see that a^x takes on every positive value. Thus the inverse function is defined for all positive numbers, and takes on all values. If $f(x) = a^x$, then f^{-1} is the function usually denoted by $\log_a$ (Figure 5).

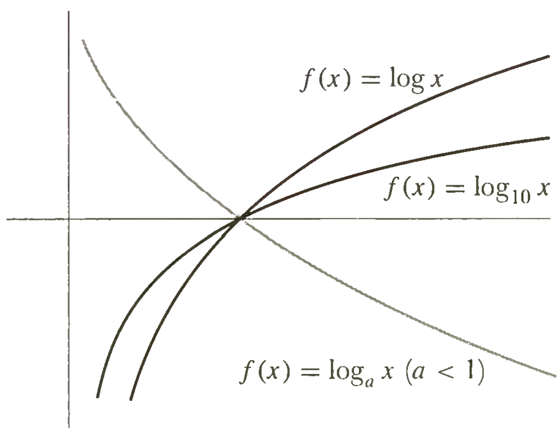


FIGURE 5

Just as a^x can be expressed in terms of exp, so $\log_a$ can be expressed in terms of log. Indeed,

$$\begin{array}{ll} \text{if} & y = \log_a x, \\ \text{then} & x = a^y = e^{y \log a}, \\ \text{so} & \log x = y \log a, \\ \text{or} & y = \dfrac{\log x}{\log a}. \end{array}$$

In other words,

$$\log_a x = \frac{\log x}{\log a}.$$

The derivatives of $f(x) = a^x$ and $g(x) = \log_a x$ are both easy to find:

$$f(x) = e^{x \log a}, \quad \text{so } f'(x) = \log a \cdot e^{x \log a} = \log a \cdot a^x,$$

$$g(x) = \frac{\log x}{\log a}, \quad \text{so } g'(x) = \frac{1}{x \log a}.$$

A more complicated function like

$$f(x) = g(x)^{h(x)}$$

is also easy to differentiate, if you remember that, *by definition*,

$$f(x) = e^{h(x) \log g(x)};$$

it follows from the Chain Rule that

$$\begin{aligned} f'(x) &= e^{h(x) \log g(x)} \cdot \left[h'(x) \log g(x) + h(x) \frac{g'(x)}{g(x)} \right] \\ &= g(x)^{h(x)} \cdot \left[h'(x) \log g(x) + h(x) \frac{g'(x)}{g(x)} \right]. \end{aligned}$$

There is no point in remembering this formula—simply apply the principle behind it in any specific case that arises; it does help, however, to remember that the first factor in the derivative will be $g(x)^{h(x)}$.

There is one special case of the above formula which *is* worth remembering. The function $f(x) = x^a$ was previously defined only for rational a. We can now define and find the derivative of the function $f(x) = x^a$ for any number a; the result is just what we would expect:

$$f(x) = x^a = e^{a\log x}$$

so

$$f'(x) = \frac{a}{x} \cdot e^{a\log x} = \frac{a}{x} \cdot x^a = ax^{a-1}.$$

Algebraic manipulations with the exponential functions will become second nature after a little practice—just remember that all the rules which ought to work actually do. The basic properties of exp are still those stated in Theorems 2 and 3:

$$\exp'(x) = \exp(x),$$
$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

In fact, each of these properties comes close to characterizing the function exp. Naturally, exp is not the only function f satisfying $f' = f$, for if $f = ce^x$, then $f'(x) = ce^x = f(x)$; these functions are the only ones with this property, however.

THEOREM 5 If f is differentiable and

$$f'(x) = f(x) \quad \text{for all } x,$$

then there is a number c such that

$$f(x) = ce^x \quad \text{for all } x.$$

PROOF Let

$$g(x) = \frac{f(x)}{e^x}.$$

(This is permissible, since $e^x \neq 0$ for all x.) Then

$$g'(x) = \frac{e^x f'(x) - f(x)e^x}{(e^x)^2} = 0.$$

Therefore there is a number c such that

$$g(x) = \frac{f(x)}{e^x} = c \quad \text{for all } x. \blacksquare$$

The second basic property of exp requires a more involved discussion. The function exp is clearly not the only function f which satisfies

$$f(x+y) = f(x) \cdot f(y).$$

In fact, $f(x) = 0$ or any function of the form $f(x) = a^x$ also satisfies this equation. But the true story is much more complex than this—there are infinitely many other functions which satisfy this property, but it is impossible, without appealing to more advanced mathematics, to prove that there is even one function other than those already mentioned! It is for this reason that the definition of 10^x is so difficult: there are infinitely many functions f which satisfy

$$\begin{aligned} f(x+y) &= f(x)\cdot f(y), \\ f(1) &= 10, \end{aligned}$$

but which are *not* the function $f(x) = 10^x$! One thing is true however—any *continuous* function f satisfying

$$f(x+y) = f(x)\cdot f(y)$$

must be of the form $f(x) = a^x$ or $f(x) = 0$. (Problem 38 indicates the way to prove this, and also has a few words to say about discontinuous functions with this property.)

In addition to the two basic properties stated in Theorems 2 and 3, the function exp has one further property which is very important—exp "grows faster than any polynomial." In other words,

THEOREM 6 For any natural number n,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

PROOF The proof consists of several steps.

Step 1. $e^x > x$ for all x, and consequently $\lim_{x\to\infty} e^x = \infty$ (this may be considered to be the case $n = 0$).

To prove this statement (which is clear for $x \le 0$) it suffices to show that

$$x > \log x \quad \text{for all } x > 0.$$

If $x < 1$ this is clearly true, since $\log x < 0$. If $x > 1$, then (Figure 6) $x - 1$ is an upper sum for $f(t) = 1/t$ on $[1, x]$, so $\log x < x - 1 < x$.

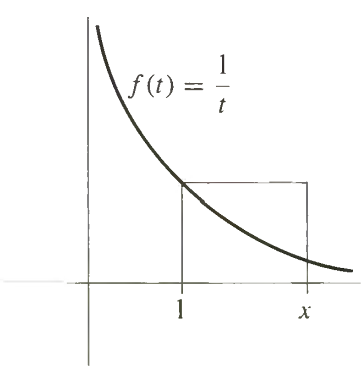


FIGURE 6

Step 2. $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.

To prove this, note that

$$\frac{e^x}{x} = \frac{e^{x/2}\cdot e^{x/2}}{\frac{x}{2}\cdot 2} = \frac{1}{2}\left(\frac{e^{x/2}}{\frac{x}{2}}\right)\cdot e^{x/2}.$$

By Step 1, the expression in parentheses is greater than 1, and $\lim_{x\to\infty} e^{x/2} = \infty$; this shows that $\lim_{x\to\infty} e^x/x = \infty$.

Step 3. $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

Note that

$$\frac{e^x}{x^n} = \frac{(e^{x/n})^n}{\left(\frac{x}{n}\right)^n \cdot n^n} = \frac{1}{n^n}\cdot\left(\frac{e^{x/n}}{\frac{x}{n}}\right)^n.$$

The expression in parentheses becomes arbitrarily large, by Step 2, so the nth power certainly becomes arbitrarily large. ▮

It is now possible to examine carefully the following very interesting function: $f(x) = e^{-1/x^2}$, $x \neq 0$. We have

$$f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}.$$

Therefore,

$$\begin{aligned} f'(x) &< 0 \quad \text{for } x < 0, \\ f'(x) &> 0 \quad \text{for } x > 0, \end{aligned}$$

so f is decreasing for negative x and increasing for positive x. Moreover, if $|x|$ is large, then x^2 is large, so $-1/x^2$ is close to 0, so e^{-1/x^2} is close to 1 (Figure 7).

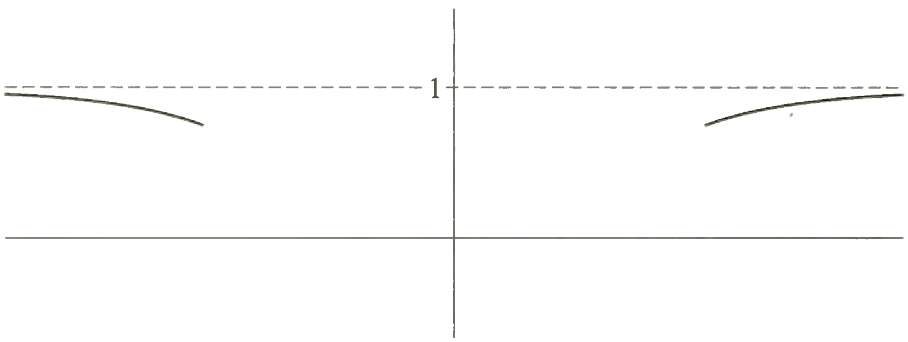


FIGURE 7

The behavior of f near 0 is more interesting. If x is small, then $1/x^2$ is large, so e^{1/x^2} is large, so $e^{-1/x^2} = 1/(e^{1/x^2})$ is small. This argument, suitably stated with ε's and δ's, shows that

$$\lim_{x \to 0} e^{-1/x^2} = 0.$$

Therefore, if we define

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

then the function f is continuous (Figure 8). In fact, f is actually differentiable

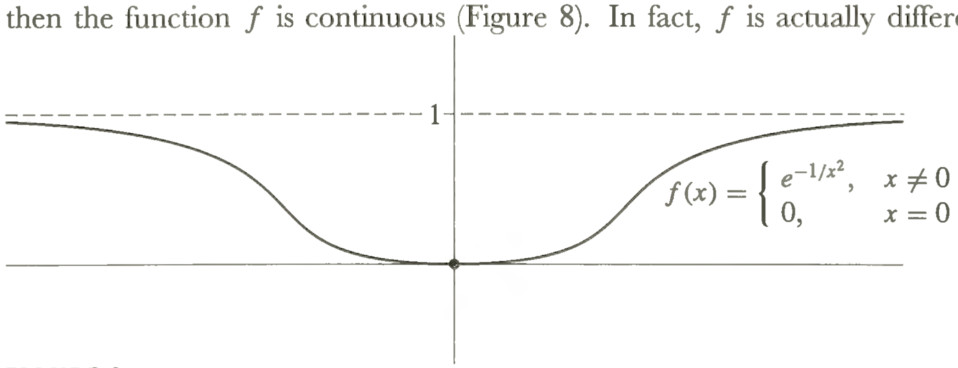


FIGURE 8

at 0: Indeed

$$f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1/h}{e^{(1/h)^2}},$$

and

$$\lim_{h\to 0^+} \frac{1/h}{e^{(1/h)^2}} = \lim_{x\to\infty} \frac{x}{e^{(x^2)}}, \qquad \text{while} \qquad \lim_{h\to 0^-} \frac{1/h}{e^{(1/h)^2}} = -\lim_{x\to\infty} \frac{x}{e^{(x^2)}}.$$

We already know that

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty;$$

it is all the more true that

$$\lim_{x\to\infty} \frac{e^{(x^2)}}{x} = \infty,$$

and this means that

$$\lim_{x\to\infty} \frac{x}{e^{(x^2)}} = 0.$$

Thus

$$f'(x) = \begin{cases} e^{-1/x^2} \cdot \dfrac{2}{x^3}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We can now compute that

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2} \cdot \dfrac{2}{h^3}}{h} \\ &= \lim_{h\to 0} \frac{2 \cdot e^{-1/h^2}}{h^4} = \lim_{h\to 0} \frac{2 \cdot \dfrac{1}{h^4}}{e^{1/h^2}} = \lim_{x\to\infty} \frac{2x^4}{e^{(x^2)}}; \end{aligned}$$

an argument similar to the one above shows that $f''(0) = 0$. Thus

$$f''(x) = \begin{cases} e^{-1/x^2} \cdot \dfrac{-6}{x^4} + e^{-1/x^2} \cdot \dfrac{4}{x^6}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This argument can be continued. In fact, using induction it can be shown (Problem 40) that $f^{(k)}(0) = 0$ for *every* k. The function f is *extremely* flat at 0, and approaches 0 so quickly that it can mask many irregularities of other functions. For example (Figure 9), suppose that

$$f(x) = \begin{cases} e^{-1/x^2} \cdot \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It can be shown (Problem 41) that for this function it is also true that $f^{(k)}(0) = 0$ for all k. This example shows, perhaps more strikingly than any other, just how bad a function can be, and still be infinitely differentiable. In Part IV we will investigate even more restrictive conditions on a function, which will finally rule out behavior of this sort.

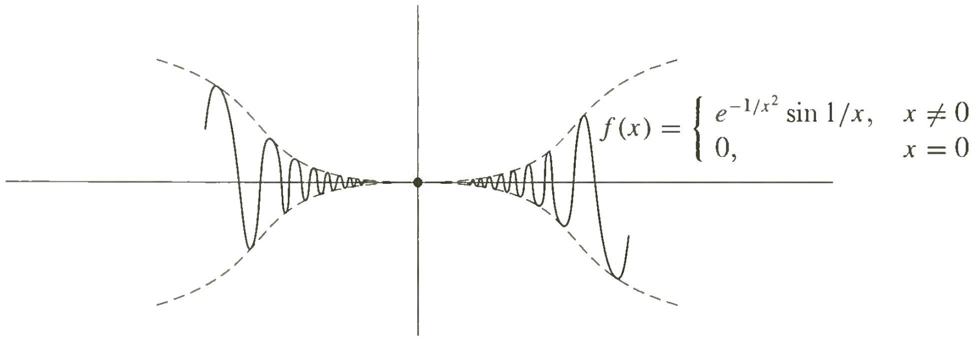


FIGURE 9

PROBLEMS

1. Differentiate each of the following functions (remember that a^{b^c} always denotes $a^{(b^c)}$).

(i) $f(x) = e^{e^{e^{e^x}}}$.

(ii) $f(x) = \log(1 + \log(1 + \log(1 + e^{1+e^{1+x}})))$.

(iii) $f(x) = (\sin x)^{\sin(\sin x)}$.

(iv) $f(x) = e^{\left(\int_0^x e^{-t^2}\, dt\right)}$.

(v) $f(x) = (\sin x)^{(\sin x)^{\sin x}}$.

(vi) $f(x) = \log_{(e^x)} \sin x$.

(vii) $f(x) = \left[\arcsin\left(\dfrac{x}{\sin x}\right)\right]^{\log(\sin e^x)}$.

(viii) $f(x) = (\log(3 + e^4))e^{4x} + (\arcsin x)^{\log 3}$.

(ix) $f(x) = (\log x)^{\log x}$.

(x) $f(x) = x^x$.

(xi) $f(x) = \sin\left(x^{\sin(x^{\sin x})}\right)$.

2. (a) Check that the derivative of $\log \circ f$ is f'/f.

This expression is called the *logarithmic derivative* of f. It is often easier to compute than f', since products and powers in the expression for f become sums and products in the expression for $\log \circ f$. The derivative f' can then be recovered simply by multiplying by f: this process is called *logarithmic differentiation.*

(b) Use logarithmic differentiation to find $f'(x)$ for each of the following.

(i) $f(x) = (1 + x)(1 + e^{x^2})$.

(ii) $f(x) = \dfrac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}$.

(iii) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$.

(iv) $f(x) = \dfrac{e^x - e^{-x}}{e^{2x}(1+x^3)}$.

3. Find

$$\int_a^b \frac{f'(t)}{f(t)}\,dt$$

(for $f > 0$ on $[a,b]$).

4. Graph each of the following functions.

(a) $f(x) = e^{x+1}$.
(b) $f(x) = e^{\sin x}$.
(c) $f(x) = e^x + e^{-x}$.
(d) $f(x) = e^x - e^{-x}$.
(Compare the graph with the graphs of exp and $1/\exp$.)

(e) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = 1 - \dfrac{2}{e^{2x}+1}$.

5. Find the following limits by l'Hôpital's Rule.

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2 - x^3/6}{x^3}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2}{x^2}$.

(v) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2}{x^3}$.

(vi) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2 - x^3/3}{x^3}$.

6. Find the following limits by l'Hôpital's Rule.

(i) $\displaystyle\lim_{x\to 0} (1-x)^{1/x}$.

(ii) $\displaystyle\lim_{x\to \frac{\pi}{4}} (\tan x)^{\tan 2x}$.

(iii) $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$.

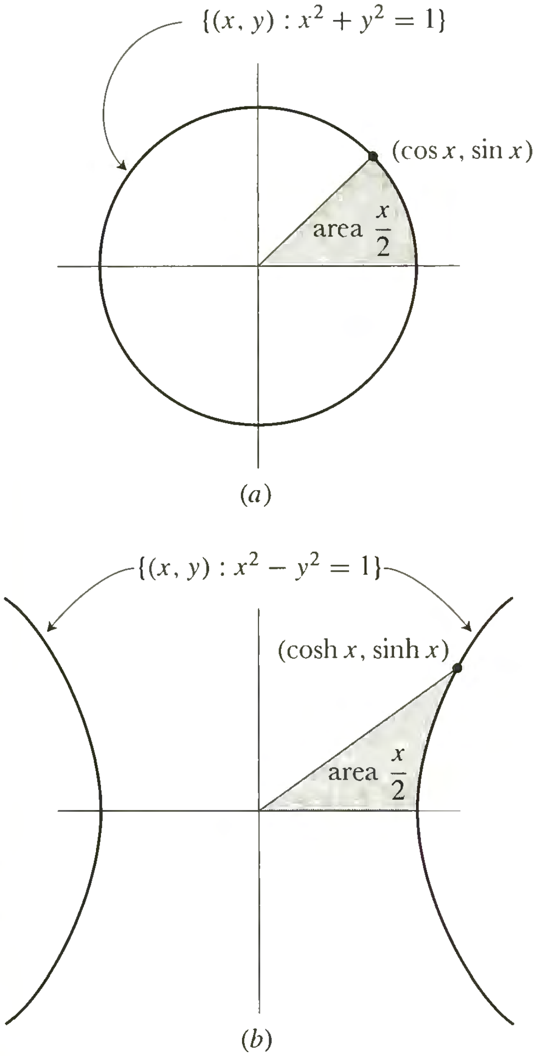


FIGURE 10

7. The functions

$$\sinh x = \frac{e^x - e^{-x}}{2},$$
$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{e^{2x} + 1},$$

are called the **hyperbolic sine**, **hyperbolic cosine**, and **hyperbolic tangent**, respectively (but usually read 'sinch,' 'cosh,' and 'tanch'). There are many analogies between these functions and their ordinary trigonometric counterparts. One analogy is illustrated in Figure 10; a proof that the region shown in Figure 10(b) really has area $x/2$ is best deferred until the next chapter, when we will develop methods of computing integrals. Other analogies are discussed in the following three problems, but the deepest analogies must wait until Chapter 27. If you have not already done Problem 4, graph the functions sinh, cosh, and tanh.

8. Prove that

(a) $\cosh^2 - \sinh^2 = 1$.
(b) $\tanh^2 + 1/\cosh^2 = 1$.
(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
(d) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
(e) $\sinh' = \cosh$.
(f) $\cosh' = \sinh$.
(g) $\tanh' = \dfrac{1}{\cosh^2}$.

9. The functions sinh and tanh are one-one; their inverses $\sinh^{-1}$ and $\tanh^{-1}$, are defined on $\mathbf{R}$ and $(-1, 1)$, respectively. These inverse functions are sometimes denoted by arg sinh and arg tanh (the "argument" of the hyperbolic sine and tangent). If cosh is restricted to $[0, \infty)$ it has an inverse, denoted by arg cosh, or simply $\cosh^{-1}$, which is defined on $[1, \infty)$. Prove, using the information in Problem 8, that

(a) $\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$.
(b) $\cosh(\sinh^{-1} x) = \sqrt{1 + x^2}$.
(c) $(\sinh^{-1})'(x) = \dfrac{1}{\sqrt{1 + x^2}}$.
(d) $(\cosh^{-1})'(x) = \dfrac{1}{\sqrt{x^2 - 1}}$ for $x > 1$.
(e) $(\tanh^{-1})'(x) = \dfrac{1}{1 - x^2}$ for $|x| < 1$.

10. (a) Find an explicit formula for $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ (by solving the equation $y = \sinh^{-1} x$ for x in terms of y, etc.).
(b) Find

$$\int_a^b \frac{1}{\sqrt{1+x^2}}\,dx,$$

$$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx \quad \text{for } a, b > 1 \text{ or } a, b < -1,$$

$$\int_a^b \frac{1}{1-x^2}\,dx \quad \text{for } |a|, |b| < 1.$$

Compare your answer for the third integral with that obtained by writing

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right].$$

11. Show that

$$F(x) = \int_2^x \frac{1}{\log t}\,dt$$

is not bounded on $[2, \infty)$.

12. Let f be a nondecreasing function on $[1, \infty)$, and define

$$F(x) = \int_1^x \frac{f(t)}{t}\,dt, \qquad x \geq 1.$$

Prove that f is bounded on $[1, \infty)$ if and only if $F/\log$ is bounded on $[1, \infty)$.

13. Find

(a) $\lim_{x\to\infty} a^x$ for $0 < a < 1$. (Remember the definition!)

(b) $\lim_{x\to\infty} \dfrac{x}{(\log x)^n}$.

(c) $\lim_{x\to\infty} \dfrac{(\log x)^n}{x}$.

(d) $\lim_{x\to 0^+} x(\log x)^n$. Hint: $x(\log x)^n = \dfrac{(-1)^n\left(\log \dfrac{1}{x}\right)^n}{\dfrac{1}{x}}$.

(e) $\lim_{x\to 0^+} x^x$.

14. Graph $f(x) = x^x$ for $x > 0$. (Use Problem 13(e).)

15. (a) Find the minimum value of $f(x) = e^x/x^n$ for $x > 0$, and conclude that $f(x) > e^n/n^n$ for $x > n$.
(b) Using the expression $f'(x) = e^x(x-n)/x^{n+1}$, prove that $f'(x) > e^{n+1}/(n+1)^{n+1}$ for $x > n+1$, and thus obtain another proof that $\lim_{x\to\infty} f(x) = \infty$.

16. Graph $f(x) = e^x/x^n$.

17. (a) Find $\lim_{y \to 0} \log(1 + y)/y$. (You can use l'Hôpital's Rule, but that would be silly.)
(b) Find $\lim_{x \to \infty} x \log(1 + 1/x)$.
(c) Prove that $e = \lim_{x \to \infty} (1 + 1/x)^x$.
(d) Prove that $e^a = \lim_{x \to \infty} (1 + a/x)^x$. (It is possible to derive this from part (c) with just a little algebraic fiddling.)
*(e) Prove that $\log b = \lim_{x \to \infty} x(b^{1/x} - 1)$.

18. Graph $f(x) = (1 + 1/x)^x$ for $x > 0$. (Use Problem 17(c).)

19. If a bank gives a percent interest per annum, then an initial investment I yields $I(1 + a/100)$ after 1 year. If the bank compounds the interest (counts the accrued interest as part of the capital for computing interest the next year), then the initial investment grows to $I(1 + a/100)^n$ after n years. Now suppose that interest is given twice a year. The final amount after n years is, alas, not $I(1 + a/100)^{2n}$, but merely $I(1 + a/200)^{2n}$—although interest is awarded twice as often, the interest must be halved in each calculation, since the interest is $a/2$ per half year. This amount is larger than $I(1 + a/100)^n$, but not that much larger. Suppose that the bank now compounds the interest continuously, i.e., the bank considers what the investment would yield when compounding k times a year, and then takes the least upper bound of all these numbers. How much will an initial investment of 1 dollar yield after 1 year?

20. (a) Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = 1/x$ for $x \neq 0$.
(b) If $f(x) \neq 0$ for all x, prove that $(\log|f|)' = f'/f$.

21. Suppose that on some interval the function f satisfies $f' = cf$ for some number c.

(a) Assuming that f is never 0, use Problem 20(b) to prove that $|f(x)| = le^{cx}$ for some number l (> 0). It follows that $f(x) = ke^{cx}$ for some k.
(b) Show that this result holds without the added assumption that f is never 0. Hint: Show that f can't be 0 at the endpoint of an open interval on which it is nowhere 0.
(c) Give a simpler proof that $f(x) = ke^{cx}$ for some k by considering the function $g(x) = f(x)/e^{cx}$.
(d) Suppose that $f' = fg'$ for some g. Show that $f(x) = ke^{g(x)}$ for some k.

***22.** A radioactive substance diminishes at a rate proportional to the amount present (since all atoms have equal probability of disintegrating, the total disintegration is proportional to the number of atoms remaining). If $A(t)$ is the amount at time t, this means that $A'(t) = cA(t)$ for some c (which represents the probability that an atom will disintegrate).

(a) Find $A(t)$ in terms of the amount $A_0 = A(0)$ present at time 0.

(b) Show that there is a number τ (the "half-life" of the radioactive element) with the property that $A(t+\tau) = A(t)/2$.

23. *Newton's law of cooling* states that an object cools at a rate proportional to the difference of its temperature and the temperature of the surrounding medium. Find the temperature $T(t)$ of the object at time t, in terms of its temperature T_0 at time 0, assuming that the temperature of the surrounding medium is kept at a constant, M. Hint: To solve the differential equation expressing Newton's law, remember that $T' = (T - M)'$.

24. Prove that if $f(x) = \int_0^x f(t)\,dt$, then $f = 0$.

25. Find all continuous functions f satisfying

(i) $\displaystyle\int_0^x f = e^x.$

(ii) $\displaystyle\int_0^{x^2} f = 1 - e^{2x^2}.$

26. Find all functions f satisfying $f'(t) = f(t) + \int_0^1 f(t)\,dt$.

27. Find all continuous functions f which satisfy the equation

$$(f(x))^2 = \int_0^x f(t)\frac{t}{1+t^2}\,dt.$$

28. (a) Let f and g be continuous functions on $[a,b]$ with g nonnegative. Suppose that for some C we have

$$f(x) \le C + \int_a^x fg, \qquad a \le x \le b.$$

Prove *Gronwall's inequality*:

$$f(x) \le Ce^{\int_a^x g}.$$

Hint: Consider the derivative of the function $h(x) = \left(C + \int_a^x fg\right)e^{-\int_a^x g}$.

(b) Let f and g be nonnegative functions with g continuous and f differentiable. Suppose that $f'(x) = g(x)f(x)$ and $f(0) = 0$. Prove that $f = 0$. (Compare Problem 21.)

29. (a) Prove that

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \le e^x \quad \text{for } x \ge 0.$$

Hint: Use induction on n, and compare derivatives.

(b) Give a new proof that $\lim_{x\to\infty} e^x/x^n = \infty$.

30. Give yet another proof of this fact, using the appropriate form of l'Hôpital's Rule. (See Problem 11-56.)

31. (a) Evaluate $\lim_{x\to\infty} e^{-x^2}\int_0^x e^{t^2}\,dt$. (You should be able to make an educated guess before doing any calculations.)

(b) Evaluate the following limits.

(i) $$\lim_{x\to\infty} e^{-x^2}\int_x^{x+(1/x)} e^{t^2}\,dt.$$

(ii) $$\lim_{x\to\infty} e^{-x^2}\int_x^{x+(\log x)/x} e^{t^2}\,dt.$$

(iii) $$\lim_{x\to\infty} e^{-x^2}\int_x^{x+(\log x)/2x} e^{t^2}\,dt.$$

32. This problem outlines the classical approach to logarithms and exponentials. To begin with, we will simply assume that the function $f(x) = a^x$, defined in an elementary way for rational x, can somehow be extended to a continuous one-one function, obeying the same algebraic rules, on the whole line. (See Problem 22-29 for a direct proof of this.) The inverse of f will then be denoted by $\log_a$.

(a) Show, directly from the definition, that

$$\begin{aligned}\log_a{}'(x) &= \lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{1/h}\\ &= \frac{1}{x}\cdot\log_a\left(\lim_{k\to 0}(1+k)^{1/k}\right).\end{aligned}$$

Thus, the whole problem has been reduced to the determination of $\lim_{h\to 0}(1+h)^{1/h}$. If we can show that this has a limit e, then $\log_e{}'(x) = \frac{1}{x}\cdot\log_e e = \frac{1}{x}$, and consequently $\exp = \log_e^{-1}$ has derivative $\exp'(x) = \exp(x)$.

(b) Let $a_n = \left(1+\frac{1}{n}\right)^n$ for natural numbers n. Using the binomial theorem, show that

$$a_n = 2 + \sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right).$$

Conclude that $a_n < a_{n+1}$.

(c) Using the fact that $1/k! \le 1/2^{k-1}$ for $k \ge 2$, show that all $a_n < 3$. Thus, the set of numbers $\{a_1, a_2, a_3, \ldots\}$ is bounded, and therefore has a least upper bound e. Show that for any $\varepsilon > 0$ we have $e - a_n < \varepsilon$ for large enough n.

(d) If $n \le x \le n+1$, then

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}.$$

Conclude that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. Also show that $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$, and conclude that $\lim_{h\to 0}(1+h)^{1/h} = e$.

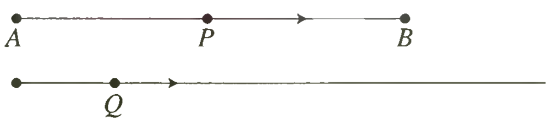


FIGURE 11

*33. A point P is moving along a line segment AB of length 10^7 while another point Q moves along an infinite ray (Figure 11). The velocity of P is always equal to the distance from P to B (in other words, if $P(t)$ is the position of P at time t, then $P'(t) = 10^7 - P(t)$), while Q moves with constant velocity $Q'(t) = 10^7$. The distance traveled by Q after time t is defined to be the *Napierian logarithm* of the distance from P to B at time t. Thus

$$10^7 t = \text{Nap log}[10^7 - P(t)].$$

This was the definition of logarithms given by Napier (1550–1617) in his publication of 1614, *Mirifici logarithmonum canonis description* (A Description of the Wonderful Law of Logarithms); work which was done *before* the use of exponents was invented! The number 10^7 was chosen because Napier's tables (intended for astronomical and navigational calculations), listed the logarithms of sines of angles, for which the best possible available tables extended to seven decimal places, and Napier wanted to avoid fractions. Prove that

$$\text{Nap log}\, x = 10^7 \log\frac{10^7}{x}.$$

Hint: Use the same trick as in Problem 23 to solve the equation for P.

*34. (a) Sketch the graph of $f(x) = (\log x)/x$ (paying particular attention to the behavior near 0 and ∞).

(b) Which is larger, e^π or π^e?

(c) Prove that if $0 < x \le 1$, or $x = e$, then the only number y satisfying $x^y = y^x$ is $y = x$; but if $x > 1$, $x \ne e$, then there is precisely one number $y \ne x$ satisfying $x^y = y^x$; moreover, if $x < e$, then $y > e$, and if $x > e$, then $y < e$. (Interpret these statements in terms of the graph in part (a)!)

(d) Prove that if x and y are natural numbers and $x^y = y^x$, then $x = y$ or $x = 2$, $y = 4$, or $x = 4$, $y = 2$.

(e) Show that the set of all pairs (x, y) with $x^y = y^x$ consists of a curve and a straight line which intersect; find the intersection and draw a rough sketch.

**(f) For $1 < x < e$ let $g(x)$ be the unique number $> e$ with $x^{g(x)} = g(x)^x$. Prove that g is differentiable. (It is a good idea to consider separate functions,

$$f_1(x) = \frac{\log x}{x}, \quad 0 < x < e$$
$$f_2(x) = \frac{\log x}{x}, \quad e < x$$

and write g in terms of f_1 and f_2. You should be able to show that

$$g'(x) = \frac{[g(x)]^2}{1 - \log g(x)} \cdot \frac{1 - \log x}{x^2}$$

if you do this part properly.)

***35.** This problem uses the material from the Appendix to Chapter 11.

(a) Prove that exp is convex and log is concave.

(b) Prove that if $\sum_{i=1}^{n} p_i = 1$ and all $p_i > 0$, then for all $z_i > 0$ we have

$$z_1{}^{p_1} \cdot \ldots \cdot z_n{}^{p_n} < p_1 z_1 + \cdots + p_n z_n.$$

(Use Problem 8 from the Appendix to Chapter 11.)

(c) Deduce another proof that $G_n \le A_n$ (Problem 2-22).

36. (a) Let f be a positive function on $[a, b]$, and let P_n be the partition of $[a, b]$ into n equal intervals. Use Problem 2-22 to show that

$$\frac{1}{b-a} L(\log f, P_n) \le \log\left(\frac{1}{b-a} L(f, P_n)\right).$$

(b) Use the Appendix to Chapter 13 to conclude that for all integrable $f > 0$ we have

$$\frac{1}{b-a}\int_a^b \log f \le \log\left(\frac{1}{b-a}\int_a^b f\right).$$

A more direct approach is illustrated in the next part:

(c) In Problem 35, Problem 2-22 was deduced as a special case of the inequality

$$g\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i g(x_i)$$

for $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$ and g convex. For g concave we have the reverse inequality

$$\sum_{i=1}^{n} p_i g(x_i) \le g\left(\sum_{i=1}^{n} p_i x_i\right).$$

Apply this with $g = \log$ to prove the result of part (b) directly for any integrable f.

(d) State a general theorem of which part (b) is just a special case.

37. Suppose f satisfies $f' = f$ and $f(x + y) = f(x)f(y)$ for all x and y. Prove that $f = \exp$ or $f = 0$.

***38.** Prove that if f is continuous and $f(x+y) = f(x)f(y)$ for all x and y, then either $f = 0$ or $f(x) = [f(1)]^x$ for all x. Hint: Show that $f(x) = [f(1)]^x$ for rational x, and then use Problem 8-6. This problem is closely related to Problem 8-7, and the information mentioned at the end of Problem 8-7 can be used to show that there are discontinuous functions f satisfying $f(x+y) = f(x)f(y)$.

***39.** Prove that if f is a continuous function defined on the positive real numbers, and $f(xy) = f(x) + f(y)$ for all positive x and y, then $f = 0$ or $f(x) = f(e)\log x$ for all $x > 0$. Hint: Consider $g(x) = f(e^x)$.

***40.** Prove that if $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$, then $f^{(k)}(0) = 0$ for all k (you will encounter the same sort of difficulties as in Problem 10-21). Hint: Consider functions $g(x) = e^{-1/x^2}P(1/x)$ for a polynomial function P.

***41.** Prove that if $f(x) = e^{-1/x^2}\sin 1/x$ for $x \neq 0$, and $f(0) = 0$, then $f^{(k)}(0) = 0$ for all k.

42. (a) Prove that if α is a root of the equation

$$(*) \qquad a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,$$

then the function $y(x) = e^{\alpha x}$ satisfies the differential equation

$$(**) \qquad a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

*(b) Prove that if α is a double root of $(*)$, then $y(x) = xe^{\alpha x}$ also satisfies $(**)$. Hint: Remember that if α is a double root of a polynomial equation $f(x) = 0$, then $f'(\alpha) = 0$.

*(c) Prove that if α is a root of $(*)$ of order r, then $y(x) = x^ke^{\alpha x}$ is a solution for $0 \le k \le r - 1$.

If $(*)$ has n real numbers as roots (counting multiplicities), part (c) gives n solutions $y_1, \ldots, y_n$ of $(**)$.

(d) Prove that in this case the function $c_1y_1 + \cdots + c_ny_n$ also satisfies $(**)$.

It is a theorem that in this case these are the only solutions of $(**)$. Problem 21 and the next two problems prove special cases of this theorem, and the general case is considered in Problem 20-26. In Chapter 27 we will see what to do when $(*)$ does not have n real numbers as roots.

***43.** Suppose that f satisfies $f'' - f = 0$ and $f(0) = f'(0) = 0$. Prove that $f = 0$ as follows.

(a) Show that $f^2 - (f')^2 = 0$.

(b) Suppose that $f(x) \neq 0$ for all x in some interval (a, b). Show that either $f(x) = ce^x$ or else $f(x) = ce^{-x}$ for all x in (a, b), for some constant c.

**(c) If $f(x_0) \neq 0$ for $x_0 > 0$, say, then there would be a number a such that $0 \le a < x_0$ and $f(a) = 0$, while $f(x) \neq 0$ for $a < x < x_0$. Why? Use this fact and part (b) to deduce a contradiction.

***44.** (a) Show that if f satisfies $f'' - f = 0$, then $f(x) = ae^x + be^{-x}$ for some a and b. (First figure out what a and b should be in terms of $f(0)$ and $f'(0)$, and then use Problem 43.)

(b) Show also that $f = a \sinh + b \cosh$ for some (other) a and b.

45. Find all functions f satisfying

(a) $f^{(n)} = f^{(n-1)}$.

(b) $f^{(n)} = f^{(n-2)}$.

***46.** This problem, a companion to Problem 15-30, outlines a treatment of the exponential function starting from the assumption that the differential equation $f' = f$ has a nonzero solution.

(a) Suppose there is a function $f \neq 0$ with $f' = f$. Prove that $f(x) \neq 0$ for each x by considering the function $g(x) = f(x_0 + x)f(x_0 - x)$, where $f(x_0) \neq 0$.

(b) Show that there is a function f satisfying $f' = f$ and $f(0) = 1$.

(c) For this f show that $f(x + y) = f(x) \cdot f(y)$ by considering the function $g(x) = f(x + y)/f(x)$.

(d) Prove that f is one-one and that $(f^{-1})'(x) = 1/x$.

47. Let f and g be continuous functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We say that f *grows faster than* g ($f \gg g$) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

and we say that f and g *grow at the same rate* ($f \sim g$) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \text{ exists and is } \neq 0, \infty.$$

For example, for any polynomial function P with $\lim_{x\to\infty} P(x) = \infty$ (i.e., P is non-constant and has positive leading coefficient) we have $\exp \gg P$ and $P \gg \log^n$ for any positive integer n.

(a) Given f and g, with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, is it necessarily true that one of the three conditions $f \gg g$ or $g \gg f$ or $f \sim g$ holds?

(b) If $f \gg g$, then $f + g \sim f$.

(c) If

$$\frac{\log f}{\log g} \geq c > 1$$

for sufficiently large x, then $f \gg g$.

(d) If $f \gg g$ and $F(x) = \int_0^x f$, $G(x) = \int_0^x g$, does it necessarily follow that $F \gg G$?

(e) Arrange each of the following sets of functions in increasing order of growth (for convenience, we indicate each function simply by giving its value at x):

(i) x^3, e^x, $x^3 + \log(x^3)$, $\log 4x$, $(\log x)^x$, x^x, $x + e^{-5x}$, $x^3 \log x$.
(ii) $x \log^2 x$, e^{5x}, $\log(x^x)$, e^{x^2}, x^x, $x^{\log x}$, $(\log x)^x$.
(iii) e^x, x^e, x^x, e^{x^2}, 2^x, $e^{x/2}$, $(\log x)^{2x}$.

48. Suppose that $g_1, g_2, g_3, \ldots$ are continuous functions. Show that there is a continuous function f which grows faster than each g_i.

49. Prove that $\log_{10} 2$ is irrational.

CHAPTER 19 INTEGRATION IN ELEMENTARY TERMS

Every computation of a derivative yields, according to the Second Fundamental Theorem of Calculus, a formula about integrals. For example,

$$\text{if } F(x) = x(\log x) - x \quad \text{then } F'(x) = \log x;$$

consequently,

$$\int_a^b \log x\,dx = F(b) - F(a) = b(\log b) - b - [a(\log a) - a], \quad 0 < a, b.$$

Formulas of this sort are simplified considerably if we adopt the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

We may then write

$$\int_a^b \log x\,dx = x(\log x) - x\Big|_a^b.$$

This evaluation of $\int_a^b \log x\,dx$ depended on the lucky guess that log is the derivative of the function $F(x) = x(\log x) - x$. In general, a function F satisfying $F' = f$ is called a **primitive** of f. Of course, **a continuous function f always has a primitive**, namely,

$$F(x) = \int_a^x f,$$

but in this chapter we will try to find a primitive which can be written in terms of familiar functions like sin, log, etc. A function which can be written in this way is called an elementary function. To be precise,* an **elementary function** is one which can be obtained by addition, multiplication, division, and composition from the rational functions, the trigonometric functions and their inverses, and the functions log and exp.

It should be stated at the very outset that elementary primitives usually cannot be found. For example, there is no *elementary* function F such that

$$F'(x) = e^{-x^2} \quad \text{for all } x$$

(this is not merely a report on the present state of mathematical ignorance; it is a (difficult) theorem that no such function exists). And, what is even worse, you

*The definition which we will give is precise, but not really accurate, or at least not quite standard. Usually the elementary functions are defined to include "algebraic" functions, that is, functions g satisfying an equation

$$(g(x))^n + f_{n-1}(x)(g(x))^{n-1} + \cdots + f_0(x) = 0,$$

where the f_i are rational functions. But for our purposes these functions can be ignored.

will have no way of knowing whether or not an elementary primitive *can* be found (you will just have to hope that the problems for this chapter contain no misprints). Because the search for elementary primitives is so uncertain, finding one is often peculiarly satisfying. If we observe that the function

$$F(x) = x \arctan x - \frac{\log(1+x^2)}{2}$$

satisfies

$$F'(x) = \arctan x$$

(just how we would ever be led to such an observation is quite another matter), so that

$$\int_a^b \arctan x \, dx = x \arctan x - \left. \frac{\log(1+x^2)}{2} \right|_a^b,$$

then we may feel that we have "really" evaluated $\int_a^b \arctan x \, dx$.

This chapter consists of little more than methods for finding elementary primitives of given elementary functions (a process known simply as "integration"), together with some notation, abbreviations, and conventions designed to facilitate this procedure. This preoccupation with elementary functions can be justified by three considerations:

(1) Integration is a standard topic in calculus, and everyone should know about it.
(2) Every once in a while you might actually need to evaluate an integral, under conditions which do not allow you to consult any of the standard integral tables (for example, you might take a (physics) course in which you are expected to be able to integrate).
(3) The most useful "methods" of integration are actually very important theorems (that apply to all functions, not just elementary ones).

Naturally, the last reason is the crucial one. Even if you intend to forget how to integrate (and you probably will forget some details the first time through), you must never forget the basic methods.

These basic methods are theorems which allow us to express primitives of one function in terms of primitives of other functions. To begin integrating we will therefore need a list of primitives for *some* functions; such a list can be obtained simply by differentiating various well-known functions. The list given below makes use of a standard symbol which requires some explanation. The symbol

$$\int f \quad \text{or} \quad \int f(x)\,dx$$

means "a primitive of f" or, more precisely, "the collection of all primitives of f." The symbol $\int f$ will often by used in stating theorems, while $\int f(x)\,dx$ is most useful in formulas like the following:

$$\int x^3\,dx = \frac{x^4}{4}.$$

This "equation" means that the function $F(x) = x^4/4$ satisfies $F'(x) = x^3$. It cannot be interpreted literally because the right side is a number, not a function, but in this one context we will allow such discrepancies; our aim is to make the integration process as mechanical as possible, and we will resort to any possible device. Another feature of the equation deserves mention. Most people write

$$\int x^3\,dx = \frac{x^4}{4} + C$$

to emphasize that the primitives of $f(x) = x^3$ are precisely the functions of the form $F(x) = x^4/4 + C$ for some number C. Although it is possible (Problem 14) to obtain contradictions if this point is disregarded, in practice such difficulties do not arise, and concern for this constant is merely an annoyance.

There is one important convention accompanying this notation: the letter appearing on the right side of the equation should match with the letter appearing after the "d" on the left side—thus

$$\int u^3\,du = \frac{u^4}{4},$$
$$\int tx\,dx = \frac{tx^2}{2},$$
$$\int tx\,dt = \frac{xt^2}{2}.$$

A function in $\int f(x)\,dx$, i.e., a primitive of f, is often called an "indefinite integral" of f, while $\int_a^b f(x)\,dx$ is called, by way of contrast, a "definite integral." This suggestive notation works out quite well in practice, but it is important not to be led astray. At the risk of boring you, the following fact is emphasized once again: the integral $\int_a^b f(x)\,dx$ is *not* defined as "$F(b) - F(a)$, where F is an indefinite integral of f" (if you do not find this statement repetitious, it is time to reread Chapter 13).

We can verify the formulas in the following short table of indefinite integrals simply by differentiating the functions indicated on the right side.

$$\int a\,dx = ax$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \log x$$ $\left(\int \frac{1}{x}\,dx\right.$ is often written $\int \frac{dx}{x}$ for convenience; similar abbreviations are used in the last two examples of this table.)

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \frac{dx}{1+x^2} = \arctan x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

Two general formulas of the same nature are consequences of theorems about differentiation:

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx,$$

$$\int c \cdot f(x) \, dx = c \cdot \int f(x) \, dx.$$

These equations should be interpreted as meaning that a primitive of $f + g$ can be obtained by adding a primitive of f to a primitive of g, while a primitive of $c \cdot f$ can be obtained by multiplying a primitive of f by c.

Notice the consequences of these formulas for definite integrals: If f and g are continuous, then

$$\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx,$$

$$\int_a^b c \cdot f(x) \, dx = c \cdot \int_a^b f(x) \, dx.$$

These follow from the previous formulas, since each definite integral may be written as the difference of the values at a and b of a corresponding primitive. Continuity is required in order to know that these primitives exist. (Of course, the formulas are also true when f and g are merely integrable, but recall how much more difficult the proofs are in this case.)

The product formula for the derivative yields a more interesting theorem, which will be written in several different ways.

THEOREM 1 (INTEGRATION BY PARTS) If f' and g' are continuous, then

$$\int fg' = fg - \int f'g,$$

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx,$$

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) \, dx.$$

(Notice that in the second equation $f(x)g(x)$ denotes the *function* $f \cdot g$.)

PROOF The formula

$$(fg)' = f'g + fg'$$

can be written

$$fg' = (fg)' - f'g.$$

Thus

$$\int fg' = \int (fg)' - \int f'g,$$

and fg can be chosen as one of the functions denoted by $\int (fg)'$. This proves the first formula.

The second formula is merely a restatement of the first, and the third formula follows immediately from either of the first two. ▮

As the following examples illustrate, integration by parts is useful when the function to be integrated can be considered as a product of a function f, whose derivative is simpler than f, and another function which is obviously of the form g'.

$$\int \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g'}}{e^x}\,dx = \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g}}{e^x} - \int \underset{\substack{\downarrow\\ f'}}{1}\cdot \underset{\substack{\downarrow\\ g}}{e^x}\,dx$$

$$= xe^x - e^x$$

$$\int \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g'}}{\sin x}\,dx = \underset{\substack{\downarrow\\ f}}{x}\cdot\underset{\substack{\downarrow\\ g}}{(-\cos x)} - \int \underset{\substack{\downarrow\\ f'}}{1}\cdot\underset{\substack{\downarrow\\ g}}{(-\cos x)}\,dx$$

$$= -x\cos x + \sin x$$

There are two special tricks which often work with integration by parts. The first is to consider the function g' to be the factor 1, which can always be written in.

$$\int \log x\,dx = \int \underset{\substack{\downarrow\\ g'}}{1}\cdot\underset{\substack{\downarrow\\ f}}{\log x}\,dx = \underset{\substack{\downarrow\\ g}}{x}\,\underset{\substack{\downarrow\\ f}}{\log x} - \int \underset{\substack{\downarrow\\ g}}{x}\cdot\underset{\substack{\downarrow\\ f'}}{(1/x)}\,dx$$

$$= x(\log x) - x.$$

The second trick is to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$. A simple example is the calculation

$$\int \underset{\substack{\downarrow\\ g'}}{(1/x)}\cdot\underset{\substack{\downarrow\\ f}}{\log x}\,dx = \underset{\substack{\downarrow\\ g}}{\log x}\cdot\underset{\substack{\downarrow\\ f}}{\log x} - \int \underset{\substack{\downarrow\\ f'}}{(1/x)}\cdot\underset{\substack{\downarrow\\ g}}{\log x}\,dx,$$

which implies that

$$2\int \frac{1}{x}\log x\,dx = (\log x)^2$$

or

$$\int \frac{1}{x} \log x \, dx = \frac{(\log x)^2}{2}.$$

A more complicated calculation is often required:

$$\begin{aligned}
\int \underset{f}{\underset{\downarrow}{e^x}} \underset{g'}{\underset{\downarrow}{\sin x}} \, dx &= \underset{f}{\underset{\downarrow}{e^x}} \cdot \underset{g}{\underset{\downarrow}{(-\cos x)}} - \int \underset{f'}{\underset{\downarrow}{e^x}} \cdot \underset{g}{\underset{\downarrow}{(-\cos x)}} \, dx \\
&= -e^x \cos x + \int \underset{u}{\underset{\downarrow}{e^x}} \underset{v'}{\underset{\downarrow}{\cos x}} \, dx \\
&= -e^x \cos x + [\underset{u}{\underset{\downarrow}{e^x}} \cdot \underset{v}{\underset{\downarrow}{(\sin x)}} - \int \underset{u'}{\underset{\downarrow}{e^x}} \underset{v}{\underset{\downarrow}{(\sin x)}} \, dx];
\end{aligned}$$

therefore,

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

or

$$\int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2}.$$

Since integration by parts depends upon recognizing that a function is of the form g', the more functions you can already integrate, the greater your chances for success. It is frequently reasonable to do a preliminary integration before tackling the main problem. For example, we can use parts to integrate

$$\int (\log x)^2 \, dx = \int \underset{f}{\underset{\downarrow}{(\log x)}} \underset{g'}{\underset{\downarrow}{(\log x)}} \, dx$$

if we recall that $\int \log x \, dx = x(\log x) - x$ (this formula was itself derived by integration by parts); we have

$$\begin{aligned}
\int \underset{f}{\underset{\downarrow}{(\log x)}} \underset{g'}{\underset{\downarrow}{(\log x)}} \, dx &= \underset{f}{\underset{\downarrow}{(\log x)}} \underset{g}{\underset{\downarrow}{[x(\log x) - x]}} - \int \underset{f'}{\underset{\downarrow}{(1/x)}} \underset{g}{\underset{\downarrow}{[x(\log x) - x]}} \, dx \\
&= (\log x)[x(\log x) - x] - \int [\log x - 1] \, dx \\
&= (\log x)[x(\log x) - x] - \int \log x \, dx + \int 1 \, dx \\
&= (\log x)[x(\log x) - x] - [x(\log x) - x] + x \\
&= x(\log x)^2 - 2x(\log x) + 2x.
\end{aligned}$$

The most important method of integration is a consequence of the Chain Rule. The use of this method requires considerably more ingenuity than integrating by parts, and even the explanation of the method is more difficult. We will therefore develop this method in stages, stating the theorem for definite integrals first, and saving the treatment of indefinite integrals for later.

THEOREM 2 (THE SUBSTITUTION FORMULA) If f and g' are continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x)) \cdot g'(x)\,dx.$$

PROOF If F is a primitive of f, then the left side is $F(g(b)) - F(g(a))$. On the other hand,

$$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g) \cdot g',$$

so $F \circ g$ is a primitive of $(f \circ g) \cdot g'$ and the right side is

$$(F \circ g)(b) - (F \circ g)(a) = F(g(b)) - F(g(a)). \blacksquare$$

The simplest uses of the substitution formula depend upon recognizing that a given function is of the form $(f \circ g) \cdot g'$. For example, the integration of

$$\int_a^b \sin^5 x \cos x\,dx \quad \left(= \int_a^b (\sin x)^5 \cos x\,dx \right)$$

is facilitated by the appearance of the factor $\cos x$, which will be the factor $g'(x)$ for $g(x) = \sin x$; the remaining expression, $(\sin x)^5$, can be written as $(g(x))^5 = f(g(x))$, for $f(u) = u^5$. Thus

$$\int_a^b \sin^5 x \cos x\,dx \qquad \begin{bmatrix} g(x) = \sin x \\ f(u) = u^5 \end{bmatrix}$$

$$= \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$= \int_{\sin a}^{\sin b} u^5\,du = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}.$$

The integration of $\int_a^b \tan x\,dx$ can be treated similarly if we write

$$\int_a^b \tan x\,dx = -\int_a^b \frac{-\sin x}{\cos x}\,dx.$$

In this case the factor $-\sin x$ is $g'(x)$, where $g(x) = \cos x$; the remaining factor $1/\cos x$ can then be written $f(\cos x)$ for $f(u) = 1/u$. Hence

$$\int_a^b \tan x\,dx \qquad \begin{bmatrix} g(x) = \cos x \\ f(u) = \dfrac{1}{u} \end{bmatrix}$$

$$= -\int_a^b f(g(x))g'(x)\,dx = -\int_{g(a)}^{g(b)} f(u)\,du$$

$$= -\int_{\cos a}^{\cos b} \frac{1}{u}\,du = \log(\cos a) - \log(\cos b).$$

Finally, to find

$$\int_a^b \frac{1}{x\log x}\,dx,$$

notice that $1/x = g'(x)$ where $g(x) = \log x$, and that $1/\log x = f(g(x))$ for $f(u) = 1/u$. Thus

$$\int_a^b \frac{1}{x\log x}\,dx \qquad \begin{bmatrix} g(x) = \log x \\ f(u) = \dfrac{1}{u} \end{bmatrix}$$

$$= \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$= \int_{\log a}^{\log b} \frac{1}{u}\,du = \log(\log b) - \log(\log a).$$

Fortunately, these uses of the substitution formula can be shortened considerably. The intermediate steps, which involve writing

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

can easily be eliminated by noticing the following: To go from the left side to the right side,

$$\text{substitute} \begin{cases} u \text{ for } g(x) \\ du \text{ for } g'(x)\,dx \end{cases}$$

(and change the limits of integration);

the substitutions can be performed directly on the original function (accounting for the name of this theorem). For example,

$$\int_a^b \sin^5 x\cos x\,dx \left[\text{substitute} \begin{matrix} u \text{ for } \sin x \\ du \text{ for } \cos x\,dx \end{matrix}\right] = \int_{\sin a}^{\sin b} u^5\,du,$$

and similarly

$$\int_a^b \frac{-\sin x}{\cos x}\,dx \left[\text{substitute} \begin{matrix} u \text{ for } \cos x \\ du \text{ for } -\sin x\,dx \end{matrix}\right] = \int_{\cos a}^{\cos b} \frac{1}{u}\,du.$$

Usually we abbreviate this method even more, and say simply:

$$\text{“Let } \begin{aligned} u &= g(x) \\ du &= g'(x)\,dx.\text{”} \end{aligned}$$

Thus

$$\int_a^b \frac{1}{x\log x}\,dx \begin{bmatrix} \text{let } u = \log x \\ du = \dfrac{1}{x}\,dx \end{bmatrix} = \int_{\log a}^{\log b} \frac{1}{u}\,du.$$

In this chapter we are usually interested in primitives rather than definite integrals, but if we can find $\int_a^b f(x)\,dx$ for all a and b, then we can certainly find

$\int f(x)\,dx$. For example, since

$$\int_a^b \sin^5 x \cos x\,dx = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6},$$

it follows that

$$\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6}.$$

Similarly,

$$\int \tan x\,dx = -\log\cos x,$$

$$\int \frac{1}{x\log x}\,dx = \log(\log x).$$

It is quite uneconomical to obtain primitives from the substitution formula by first finding definite integrals. Instead, the two steps can be combined, to yield the following procedure:

(1) Let

$$u = g(x),$$
$$du = g'(x)\,dx;$$

(after this manipulation only the letter u should appear, *not* the letter x).

(2) Find a primitive (as an expression involving u).

(3) Substitute $g(x)$ back for u.

Thus, to find

$$\int \sin^5 x \cos x\,dx,$$

(1) let

$$u = \sin x,$$
$$du = \cos x\,dx$$

so that we obtain

$$\int u^5\,du;$$

(2) evaluate

$$\int u^5\,du = \frac{u^6}{6};$$

(3) remember to substitute $\sin x$ back for u, so that

$$\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6}.$$

Similarly, if

$$\begin{aligned} u &= \log x, \\ du &= \frac{1}{x}\,dx, \end{aligned}$$

then

$$\int \frac{1}{x\log x}\,dx \quad \text{becomes} \quad \int \frac{1}{u}\,du = \log u,$$

so that

$$\int \frac{1}{x\log x}\,dx = \log(\log x).$$

To evaluate

$$\int \frac{x}{1+x^2}\,dx,$$

let

$$\begin{aligned} u &= 1+x^2, \\ du &= 2x\,dx; \end{aligned}$$

the factor 2 which has just popped up causes no problem—the integral becomes

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log u,$$

so

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2).$$

(This result may be combined with integration by parts to yield

$$\begin{aligned} \int 1\cdot \arctan x\,dx &= x\arctan x - \int \frac{x}{1+x^2}\,dx \\ &= x\arctan x - \tfrac{1}{2}\log(1+x^2), \end{aligned}$$

a formula that has already been mentioned.)

These applications of the substitution formula* illustrate the most straightforward and least interesting types—once the suitable factor $g'(x)$ is recognized, the whole problem may even become simple enough to do mentally. The following three problems require only the information provided by the short table of indefinite integrals at the beginning of the chapter and, of course, the right substitution

*The substitution formula is often written in the form

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx, \qquad u = g(x).$$

This formula cannot be taken literally (after all, $\int f(u)\,du$ should mean a primitive of f and the symbol $\int f(g(x))g'(x)\,dx$ should mean a primitive of $(f \circ g)\cdot g'$; these are certainly not equal). However, it may be regarded as a symbolic summary of the procedure which we have developed. If we use Leibniz's notation, and a little fudging, the formula reads particularly well:

$$\int f(u)\,du = \int f(u)\frac{du}{dx}\,dx.$$

(the third problem has been disguised a little by some algebraic chicanery).

$$\int \sec^2 x \tan^5 x\,dx,$$

$$\int (\cos x)e^{\sin x}\,dx,$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx.$$

If you have not succeeded in finding the right substitutions, you should be able to guess them from the answers, which are $(\tan^6 x)/6$, $e^{\sin x}$, and $\arcsin e^x$. At first you may find these problems too hard to do in your head, but at least when g is of the very simple form $g(x) = ax + b$ you should not have to waste time writing out the substitution. The following integrations should all be clear. (The only worrisome detail is the proper positioning of the constant—should the answer to the second be $e^{3x}/3$ or $3e^{3x}$? I always take care of these problems as follows. Clearly $\int e^{3x}\,dx = e^{3x} \cdot$ (something). Now if I differentiate $F(x) = e^{3x}$, I get $F'(x) = 3e^{3x}$, so the "something" must be $\frac{1}{3}$, to cancel the 3.)

$$\int \frac{dx}{x+3} = \log(x+3),$$

$$\int e^{3x}\,dx = \frac{e^{3x}}{3},$$

$$\int \cos 4x\,dx = \frac{\sin 4x}{4},$$

$$\int \sin(2x+1)\,dx = \frac{-\cos(2x+1)}{2},$$

$$\int \frac{dx}{1+4x^2} = \frac{\arctan 2x}{2}.$$

More interesting uses of the substitution formula occur when the factor $g'(x)$ does *not* appear. There are two main types of substitutions where this happens. Consider first

$$\int \frac{1+e^x}{1-e^x}\,dx.$$

The prominent appearance of the expression e^x suggests the simplifying substitution

$$u = e^x,$$
$$du = e^x\,dx.$$

Although the expression $e^x\,dx$ does not appear, it can always be put in:

$$\int \frac{1+e^x}{1-e^x}\,dx = \int \frac{1+e^x}{1-e^x} \cdot \frac{1}{e^x} \cdot e^x\,dx.$$

We therefore obtain

$$\int \frac{1+u}{1-u} \cdot \frac{1}{u}\,du,$$

which can be evaluated by the algebraic trick

$$\int \frac{1+u}{1-u} \cdot \frac{1}{u}\,du = \int \frac{2}{1-u} + \frac{1}{u}\,du = -2\log(1-u) + \log u,$$

so that

$$\int \frac{1+e^x}{1-e^x}\,dx = -2\log(1-e^x) + \log e^x = -2\log(1-e^x) + x.$$

There is an alternative and preferable way of handling this problem, which does not require multiplying and dividing by e^x. If we write

$$\begin{aligned} u = e^x, \qquad x &= \log u, \\ dx &= \frac{1}{u}\,du, \end{aligned}$$

then

$$\int \frac{1+e^x}{1-e^x}\,dx \quad \text{immediately becomes} \quad \int \frac{1+u}{1-u} \cdot \frac{1}{u}\,du.$$

Most substitution problems are much easier if one resorts to this trick of expressing x in terms of u, and dx in terms of du, instead of vice versa. It is not hard to see why this trick always works (as long as the function expressing u in terms of x is one-one for all x under consideration): If we apply the substitution

$$\begin{aligned} u = g(x), \qquad x &= g^{-1}(u) \\ dx &= (g^{-1})'(u)\,du \end{aligned}$$

to the integral

$$\int f(g(x))\,dx,$$

we obtain

$$(1) \qquad \int f(u)(g^{-1})'(u)\,du.$$

On the other hand, if we apply the straightforward substitution

$$\begin{aligned} u &= g(x) \\ du &= g'(x)\,dx \end{aligned}$$

to the same integral,

$$\int f(g(x))\,dx = \int f(g(x)) \cdot \frac{1}{g'(x)} \cdot g'(x)\,dx,$$

we obtain

$$(2) \qquad \int f(u) \cdot \frac{1}{g'(g^{-1}(u))}\,du.$$

The integrals (1) and (2) are identical, since $(g^{-1})'(u) = 1/g'(g^{-1}(u))$.

As another concrete example, consider

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx.$$

In this case we will go the whole hog and replace the entire expression $\sqrt{e^x+1}$ by one letter. Thus we choose the substitution

$$\begin{aligned} u &= \sqrt{e^x+1}, \\ u^2 &= e^x+1, \\ u^2-1 = e^x, \quad x &= \log(u^2-1), \\ dx &= \frac{2u}{u^2-1}\,du. \end{aligned}$$

The integral then becomes

$$\int \frac{(u^2-1)^2}{u}\cdot\frac{2u}{u^2-1}\,du = 2\int u^2-1\,du = \frac{2u^3}{3}-2u.$$

Thus

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \frac{2}{3}(e^x+1)^{3/2}-2(e^x+1)^{1/2}.$$

Another example, which illustrates the second main type of substitution that can occur, is the integral

$$\int \sqrt{1-x^2}\,dx.$$

In this case, instead of replacing a complicated expression by a simpler one, we will replace x by $\sin u$, because $\sqrt{1-\sin^2 u} = \cos u$. This really means that we are using the substitution $u = \arcsin x$, but it is the expression for x in terms of u which helps us find the expression to be substituted for dx. Thus,

$$\begin{aligned} \text{let} \quad x &= \sin u, \qquad [u = \arcsin x] \\ dx &= \cos u\,du; \end{aligned}$$

then the integral becomes

$$\int \sqrt{1-\sin^2 u}\,\cos u\,du = \int \cos^2 u\,du.$$

The evaluation of this integral depends on the equation

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

(see the discussion of trigonometric functions below) so that

$$\int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2}+\frac{\sin 2u}{4},$$

and

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \frac{\arcsin x}{2}+\frac{\sin(2\arcsin x)}{4} \\ &= \frac{\arcsin x}{2}+\frac{1}{2}\sin(\arcsin x)\cdot\cos(\arcsin x) \\ &= \frac{\arcsin x}{2}+\frac{1}{2}x\sqrt{1-x^2}. \end{aligned}$$

Substitution and integration by parts are the only fundamental methods which you have to learn; with their aid primitives can be found for a large number of functions. Nevertheless, as some of our examples reveal, success often depends upon some additional tricks. The most important are listed below. Using these you should be able to integrate all the functions in Problems 1 to 10 (a few other interesting tricks are explained in some of the remaining problems).

1. TRIGONOMETRIC FUNCTIONS

Since

$$\sin^2 x + \cos^2 x = 1$$

and

$$\cos 2x = \cos^2 x - \sin^2 x,$$

we obtain

$$\begin{aligned} \cos 2x &= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1, \\ \cos 2x &= (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x, \end{aligned}$$

or

$$\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2}, \\ \cos^2 x &= \frac{1 + \cos 2x}{2}. \end{aligned}$$

These formulas may be used to integrate

$$\int \sin^n x\,dx,$$
$$\int \cos^n x\,dx,$$

if n is even. Substituting

$$\frac{(1 - \cos 2x)}{2} \quad \text{or} \quad \frac{(1 + \cos 2x)}{2}$$

for $\sin^2 x$ or $\cos^2 x$ yields a sum of terms involving lower powers of cos. For example,

$$\int \sin^4 x\,dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \int \frac{1}{4}\,dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx$$

and

$$\int \cos^2 2x\,dx = \int \frac{1 + \cos 4x}{2}\,dx.$$

If n is odd, $n = 2k + 1$, then

$$\int \sin^n x\,dx = \int \sin x(1 - \cos^2 x)^k\,dx;$$

the latter expression, multiplied out, involves terms of the form $\sin x \cos^l x$, all of which can be integrated easily. The integral for $\cos^n x$ is treated similarly. An integral

$$\int \sin^n x \cos^m x \, dx$$

is handled the same way if n or m is odd. If n and m are both even, use the formulas for $\sin^2 x$ and $\cos^2 x$.

A final important trigonometric integral is

$$\int \frac{1}{\cos x} \, dx = \int \sec x \, dx = \log(\sec x + \tan x).$$

Although there are several ways of "deriving" this result, by means of the methods already at our disposal (Problem 13), it is simplest to check this formula by differentiating the right side, and to memorize it.

2. REDUCTION FORMULAS

Integration by parts yields (Problem 21)

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$$

$$\int \frac{1}{(x^2+1)^n} \, dx = \frac{1}{2n-2} \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(x^2+1)^{n-1}} \, dx$$

and many similar formulas. The first two, used repeatedly, give a different method for evaluating primitives of $\sin^n$ or $\cos^n$. The third is very important for integrating a large general class of functions, which will complete our discussion.

3. RATIONAL FUNCTIONS

Consider a rational function p/q where

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \\ q(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0. \end{aligned}$$

We might as well assume that $a_n = b_m = 1$. Moreover, we can assume that $n < m$, for otherwise we may express p/q as a polynomial function plus a rational function which *is* of this form by dividing (the calculation

$$\frac{u^2}{u-1} = u + 1 + \frac{1}{u-1}$$

is a simple example). The integration of an arbitrary rational function depends on two facts; the first follows from the "Fundamental Theorem of Algebra" (see Chapter 26, Theorem 2 and Problem 26-3), but the second will not be proved in this book.

THEOREM Every polynomial function

$$q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

can be written as a product

$$q(x) = (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdot \ldots \cdot (x^2+\beta_l x+\gamma_l)^{s_l}$$

(where $r_1 + \cdots + r_k + 2(s_1 + \cdots + s_l) = m$).

(In this expression, identical factors have been collected together, so that all $x - \alpha_i$ and $x^2 + \beta_i x + \gamma_i$ may be assumed distinct. Moreover, we assume that each quadratic factor cannot be factored further. This means that

$$\beta_i{}^2 - 4\gamma_i < 0,$$

since otherwise we can factor

$$x^2 + \beta_i x + \gamma_i = \left[x - \left(\frac{-\beta_i + \sqrt{\beta_i{}^2 - 4\gamma_i}}{2}\right)\right] \cdot \left[x - \left(\frac{-\beta_i - \sqrt{\beta_i{}^2 - 4\gamma_i}}{2}\right)\right]$$

into linear factors.)

THEOREM If $n < m$ and

$$\begin{aligned} p(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_0, \\ q(x) &= x^m + b_{m-1}x^{m-1} + \cdots + b_0 \\ &= (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdot \ldots \cdot (x^2+\beta_l x+\gamma_l)^{s_l}, \end{aligned}$$

then $p(x)/q(x)$ can be written in the form

$$\begin{aligned} \frac{p(x)}{q(x)} = &\left[\frac{a_{1,1}}{(x-\alpha_1)} + \cdots + \frac{a_{1,r_1}}{(x-\alpha_1)^{r_1}}\right] + \cdots \\ &+ \left[\frac{a_{k,1}}{(x-\alpha_k)} + \cdots + \frac{a_{k,r_k}}{(x-\alpha_k)^{r_k}}\right] \\ &+ \left[\frac{b_{1,1}x + c_{1,1}}{(x^2+\beta_1 x+\gamma_1)} + \cdots + \frac{b_{1,s_1}x + c_{1,s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right] + \cdots \\ &+ \left[\frac{b_{l,1}x + c_{l,1}}{(x^2+\beta_l x+\gamma_l)} + \cdots + \frac{b_{l,s_l}x + c_{l,s_l}}{(x^2+\beta_l x+\gamma_l)^{s_l}}\right]. \end{aligned}$$

This expression, known as the "partial fraction decomposition" of $p(x)/q(x)$, is so complicated that it is simpler to examine the following example, which illustrates such an expression and shows how to find it. According to the theorem, it is possible to write

$$\begin{aligned} &\frac{2x^7 + 8x^6 + 13x^5 + 20x^4 + 15x^3 + 16x^2 + 7x + 10}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2} \\ &\quad = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+2x+2} + \frac{ex+f}{x^2+x+1} + \frac{gx+h}{(x^2+x+1)^2}. \end{aligned}$$

To find the numbers a, b, c, d, e, f, g, and h, write the right side as a polynomial over the common denominator $(x^2+x+1)^2(x^2+2x+3)(x-1)^2$; the numerator becomes

$$\begin{aligned} a(x-1)(x^2+2x+2)(x^2+x+1)^2 + b(x^2+2x+2)(x^2+x+1)^2 \\ + (cx+d)(x-1)^2(x^2+x+1)^2 + (ex+f)(x-1)^2(x^2+2x+2)(x^2+x+1) \\ + (gx+h)(x-1)^2(x^2+2x+2). \end{aligned}$$

Actually multiplying this out (!) we obtain a polynomial of degree 8, whose coefficients are combinations of $a, \dots, h$. Equating these coefficients with the coefficients of $2x^7+8x^6+13x^5+20x^4+15x^3+16x^2+7x+10$ (the coefficient of x^8 is 0) we obtain 8 equations in the eight unknowns $a, \dots, h$. After heroic calculations these can be solved to give

$$\begin{aligned} a &= 1, & b &= 2, & c &= 1, & d &= 3, \\ e &= 0, & f &= 0, & g &= 0, & h &= 1. \end{aligned}$$

Thus

$$\int \frac{2x^7+5x^6+13x^5+20x^4+17x^3+16x^2+7x+7}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2}\,dx$$
$$= \int \frac{1}{(x-1)}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x^2+x+1)^2}\,dx + \int \frac{x+3}{x^2+2x+2}\,dx.$$

(In simpler cases the requisite calculations may actually be feasible. I obtained this particular example by *starting* with the partial fraction decomposition and converting it into one fraction.)

We are already in a position to find each of the integrals appearing in the above expression; the calculations will illustrate all the difficulties which arise in integrating rational functions.

The first two integrals are simple:

$$\int \frac{1}{x-1}\,dx = \log(x-1),$$
$$\int \frac{2}{(x-1)^2}\,dx = \frac{-2}{x-1}.$$

The third integration depends on "completing the square":

$$\begin{aligned} x^2+x+1 &= (x+\tfrac{1}{2})^2 + \tfrac{3}{4} \\ &= \frac{3}{4}\left[\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1\right]. \end{aligned}$$

(If we had obtained $-\frac{3}{4}$ instead of $\frac{3}{4}$ we could not take the square root, but in this case our original quadratic factor could have been factored into linear factors.) We can now write

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \frac{16}{9}\int \frac{1}{\left[\left(\dfrac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1\right]^2}\,dx.$$

The substitution

$$u = \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}},$$

$$du = \frac{1}{\sqrt{\frac{3}{4}}}\,dx,$$

changes this integral to

$$\frac{16}{9}\int \frac{\sqrt{\frac{3}{4}}}{(u^2+1)^2}\,du,$$

which can be computed using the third reduction formula given above.

Finally, to evaluate

$$\int \frac{x+3}{(x^2+2x+2)}\,dx$$

we write

$$\int \frac{x+3}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{2}{(x+1)^2+1}\,dx.$$

The first integral on the right side has been purposely constructed so that we can evaluate it by using the substitution

$$u = x^2+2x+2,$$

$$du = (2x+2)\,dx$$

The second integral on the right, which is just the difference of the other two, is simply $2\arctan(x+1)$. If the original integral were

$$\int \frac{x+3}{(x^2+2x+2)^n}\,dx = \frac{1}{2}\int \frac{2x+2}{(x^2+2x+2)^n}\,dx + \int \frac{2}{[(x+1)^2+1]^n}\,dx,$$

the first integral on the right would still be evaluated by the same substitution. The second integral would be evaluated by means of a reduction formula.

This example has probably convinced you that integration of rational functions is a theoretical curiosity only, especially since it is necessary to find the factorization of $q(x)$ before you can even begin. This is only partly true. We have already seen that simple rational functions sometimes arise, as in the integration

$$\int \frac{1+e^x}{1-e^x}\,dx;$$

another important example is the integral

$$\int \frac{1}{x^2-1}\,dx = \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx = \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1).$$

Moreover, if a problem has been reduced to the integration of a rational function, it is then certain that an elementary primitive exists, even when the difficulty or impossibility of finding the factors of the denominator may preclude writing this primitive explicitly.

PROBLEMS

1. This problem contains some integrals which require little more than algebraic manipulation, and consequently test your ability to discover algebraic tricks, rather than your understanding of the integration processes. Nevertheless, any one of these tricks might be an important preliminary step in an honest integration problem. Moreover, you want to have some feel for which integrals are easy, so that you can see when the end of an integration process is in sight. The answer section, if you resort to it, will only reveal what algebra you should have used.

(i) $\displaystyle\int \frac{\sqrt[5]{x^3} + \sqrt[6]{x}}{\sqrt{x}}\,dx.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1} + \sqrt{x+1}}.$

(iii) $\displaystyle\int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}}\,dx.$

(iv) $\displaystyle\int \frac{a^x}{b^x}\,dx.$

(v) $\displaystyle\int \tan^2 x\,dx.$ (Trigonometric integrals are always very touchy, because there are so many trigonometric identities that an easy problem can easily look hard.)

(vi) $\displaystyle\int \frac{dx}{a^2 + x^2}.$

(vii) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}.$

(viii) $\displaystyle\int \frac{dx}{1 + \sin x}.$

(ix) $\displaystyle\int \frac{8x^2 + 6x + 4}{x+1}\,dx.$

(x) $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\,dx.$

2. The following integrations involve simple substitutions, most of which you should be able to do in your head.

(i) $\displaystyle\int e^x \sin e^x\,dx.$

(ii) $\displaystyle\int xe^{-x^2}\,dx.$

(iii) $\displaystyle\int \frac{\log x}{x}\,dx.$ (In the text this was done by parts.)

(iv) $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+1}.$

(v) $\displaystyle\int e^{e^x}e^x\,dx.$

(vi) $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^4}}.$

(vii) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

(viii) $\displaystyle\int x\sqrt{1-x^2}\,dx.$

(ix) $\displaystyle\int \log(\cos x)\tan x\,dx.$

(x) $\displaystyle\int \frac{\log(\log x)}{x\log x}\,dx.$

3. Integration by parts.

(i) $\displaystyle\int x^2e^x\,dx.$

(ii) $\displaystyle\int x^3e^{x^2}\,dx.$

(iii) $\displaystyle\int e^{ax}\sin bx\,dx.$

(iv) $\displaystyle\int x^2\sin x\,dx.$

(v) $\displaystyle\int (\log x)^3\,dx.$

(vi) $\displaystyle\int \frac{\log(\log x)}{x}\,dx.$

(vii) $\displaystyle\int \sec^3 x\,dx.$ (This is a tricky and important integral that often comes up. If you do not succeed in evaluating it, be sure to consult the answers.)

(viii) $\displaystyle\int \cos(\log x)\,dx.$

(ix) $\displaystyle\int \sqrt{x}\log x\,dx.$

(x) $\displaystyle\int x(\log x)^2\,dx.$

4. The following integrations can all be done with substitutions of the form $x = \sin u$, $x = \cos u$, etc. To do some of these you will need to remember that

$$\int \sec x \, dx = \log(\sec x + \tan x)$$

as well as the following formula, which can also be checked by differentiation:

$$\int \csc x \, dx = -\log(\csc x + \cot x).$$

In addition, at this point the derivatives of all the trigonometric functions should be kept handy.

(i) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. (You already know this integral, but use the substitution $x = \sin u$ anyway, just to see how it works out.)

(ii) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$. (Since $\tan^2 u + 1 = \sec^2 u$, you want to use the substitution $x = \tan u$.)

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$.

(iv) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. (The answer will be a certain inverse function that was given short shrift in the text.)

(v) $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$.

(vi) $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$.

(vii) $\displaystyle\int x^3\sqrt{1-x^2}\, dx$.

(viii) $\displaystyle\int \sqrt{1-x^2}\, dx$.

You will need to remember the methods for integrating powers of sin and cos.

(ix) $\displaystyle\int \sqrt{1+x^2}\, dx$.

(x) $\displaystyle\int \sqrt{x^2-1}\, dx$.

5. The following integrations involve substitutions of various types. There is no substitute for cleverness, but there is a general rule to follow: substitute for an expression which appears frequently or prominently; if two different troublesome expressions appear, try to express them both in terms of some new expression. And don't forget that it usually helps to express x directly in terms of u, to find out the proper expression to substitute for dx.

(i) $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$.

(ii) $\displaystyle\int \frac{dx}{1+e^x}$.

(iii) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

(iv) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$. (The substitution $u = e^x$ leads to an integral requiring yet another substitution; this is all right, but both substitutions can be done at once.)

(v) $\displaystyle\int \frac{dx}{2+\tan x}$.

(vi) $\displaystyle\int \frac{dx}{\sqrt{\sqrt{x}+1}}$. (Another place where one substitution can be made to do the work of two.)

(vii) $\displaystyle\int \frac{4^x+1}{2^x+1}\,dx$.

(viii) $\displaystyle\int e^{\sqrt{x}}\,dx$.

(ix) $\displaystyle\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx$. (In this case two successive substitutions work out best; there are two obvious candidates for the first substitution, and either will work.)

*(x) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}\,dx$.

6. The previous problem provided gratis a haphazard selection of rational functions to be integrated. Here is a more systematic selection.

(i) $\displaystyle\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx$.

(ii) $\displaystyle\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx$.

(iii) $\displaystyle\int \frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3}\,dx$.

(iv) $\displaystyle\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx$.

(v) $\displaystyle\int \frac{x+4}{x^2+1}\,dx$.

(vi) $\displaystyle\int \frac{x^3+x+2}{x^4+2x^2+1}\,dx$.

(vii) $\displaystyle\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx$.

(viii) $\displaystyle\int \frac{dx}{x^4+1}$.

(ix) $\displaystyle\int \frac{2x}{(x^2+x+1)^2}\,dx$.

(x) $\displaystyle\int \frac{3x}{(x^2+x+1)^3}\,dx$.

***7.** Find $\displaystyle\int \frac{dx}{\sqrt{x^n - x^2}}$, which looks a little different from any of the previous problems. Hint: It helps to write $(x^n - x^2)^{1/2} = x(x^{n-2} - 1)^{1/2}$. Extra Hint 1: Use a substitution of the form $u^2 = \dots$ to obtain an answer involving arctan. Extra Hint 2: Use a substitution of the form $y = x^\alpha$ to obtain an answer involving arcsin.

***8.** Potpourri. (No holds barred.) The following integrations involve all the methods of the previous problems

(i) $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx.$

(ii) $\displaystyle\int \frac{x \arctan x}{(1+x^2)^2}\,dx.$

(iii) $\displaystyle\int \log\sqrt{1+x^2}\,dx.$

(iv) $\displaystyle\int x\log\sqrt{1+x^2}\,dx.$

(v) $\displaystyle\int \frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\,dx.$

(vi) $\displaystyle\int \arcsin\sqrt{x}\,dx.$

(vii) $\displaystyle\int \frac{x}{1+\sin x}\,dx.$

(viii) $\displaystyle\int e^{\sin x}\cdot\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx.$

(ix) $\displaystyle\int \sqrt{\tan x}\,dx.$

(x) $\displaystyle\int \frac{dx}{x^6+1}$. (To factor x^6+1, first factor y^3+1, using Problem 1-1.)

The following two problems provide still more practice at integration, if you need it (and can bear it). Problem 9 involves algebraic and trigonometric manipulations and integration by parts, while Problem 10 involves substitutions. (Of course, in many cases the resulting integrals will require still further manipulations.)

9. Find the following integrals.

(i) $\displaystyle\int \log(a^2+x^2)\,dx.$

(ii) $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx.$

(iii) $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}\,dx.$

(iv) $\displaystyle\int x\arctan x\,dx.$

(v) $\displaystyle\int \sin^3 x\, dx.$

(vi) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\, dx.$

(vii) $\displaystyle\int x^2 \arctan x\, dx.$

(viii) $\displaystyle\int \frac{x\, dx}{\sqrt{x^2 - 2x + 2}}.$

(ix) $\displaystyle\int \sec^3 x \tan x\, dx.$

(x) $\displaystyle\int x \tan^2 x\, dx.$

10. Find the following integrals.

(i) $\displaystyle\int \frac{dx}{(a^2 + x^2)^2}.$

(ii) $\displaystyle\int \sqrt{1 - \sin x}\, dx.$

(iii) $\displaystyle\int \arctan \sqrt{x}\, dx.$

(iv) $\displaystyle\int \sin \sqrt{x + 1}\, dx.$

(v) $\displaystyle\int \frac{\sqrt{x^3 - 2}}{x}\, dx.$

(vi) $\displaystyle\int \log\left(x + \sqrt{x^2 - 1}\right) dx.$

(vii) $\displaystyle\int \log(x + \sqrt{x})\, dx.$

There are obvious substitutions to try, but integration by parts is much easier. Comparing the answers obtained is, perhaps, instructive.

(viii) $\displaystyle\int \frac{dx}{x - x^{3/5}}.$

(ix) $\displaystyle\int (\arcsin x)^2\, dx.$

(x) $\displaystyle\int x^5 \arctan(x^2)\, dx.$

11. If you have done Problem 18-10, the integrals (ii) and (iii) in Problem 4 will look very familiar. In general, the substitution $x = \cosh u$ often works for integrals involving $\sqrt{x^2 - 1}$, while $x = \sinh u$ is the thing to try for integrals involving $\sqrt{x^2 + 1}$. Try these substitutions on the other integrals in Problem 4. (The method is not really recommended; it is easier to stick with trigonometric substitutions.)

***12.** The world's sneakiest substitution is undoubtedly

$$t = \tan \frac{x}{2}, \qquad x = 2 \arctan t, \qquad dx = \frac{2}{1 + t^2}\, dt.$$

As we found in Problem 15-17, this substitution leads to the expressions

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}.$$

This substitution thus transforms any integral which involves only sin and cos, combined by addition, multiplication, and division, into the integral of a rational function. Find

(i) $\displaystyle\int \frac{dx}{1+\sin x}$. (Compare your answer with Problem 1(viii).)

(ii) $\displaystyle\int \frac{dx}{1-\sin^2 x}$. (In this case it is better to let $t = \tan x$. Why?)

(iii) $\displaystyle\int \frac{dx}{a\sin x + b\cos x}$. (There is also another way to do this, using Problem 15-8.)

(iv) $\displaystyle\int \sin^2 x\, dx$. (An exercise to convince you that this substitution should be used only as a last resort.)

(v) $\displaystyle\int \frac{dx}{3+5\sin x}$. (A last resort.)

***13.** Derive the formula for $\int \sec x\, dx$ in the following two ways:

(a) By writing

$$\begin{aligned}\frac{1}{\cos x} &= \frac{\cos x}{\cos^2 x} \\ &= \frac{\cos x}{1-\sin^2 x} \\ &= \frac{1}{2}\left[\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right],\end{aligned}$$

an expression obviously inspired by partial fraction decompositions. Be sure to note that $\int \cos x/(1-\sin x)\, dx = -\log(1-\sin x)$; the minus sign is very important. And remember that $\frac{1}{2}\log\alpha = \log\sqrt{\alpha}$. From there on, keep doing algebra, and trust to luck.

(b) By using the substitution $t = \tan x/2$. One again, quite a bit of manipulation is required to put the answer in the desired form; the expression $\tan x/2$ can be attacked by using Problem 15-9, or both answers can be expressed in terms of t. There is another expression for $\int \sec x\, dx$, which is less cumbersome than $\log(\sec x + \tan x)$; using Problem 15-9, we obtain

$$\int \sec x\, dx = \log\left(\frac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}\right) = \log\left(\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right).$$

This last expression was actually the one first discovered, and was due, not to any mathematician's cleverness, but to a curious historical accident: In 1599 Wright computed nautical tables that amounted to definite integrals of sec. When the first tables for the logarithms of tangents were produced, the correspondence between the two tables was immediately noticed (but remained unexplained until the invention of calculus).

14. The derivation of $\int e^x \sin x\,dx$ given in the text seems to prove that the only primitive of $f(x) = e^x \sin x$ is $F(x) = e^x(\sin x - \cos x)/2$, whereas $F(x) = e^x(\sin x - \cos x)/2 + C$ is also a primitive for any number C. Where does C come from? (What is the meaning of the equation

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx?)$$

15. Suppose that f'' is continuous and that

$$\int_0^\pi [f(x) + f''(x)] \sin x\,dx = 2.$$

Given that $f(\pi) = 1$, compute $f(0)$.

16. (a) Find $\int \arcsin x\,dx$, using the same trick that worked for log and arctan.

*(b) Generalize this trick: Find $\int f^{-1}(x)\,dx$ in terms of $\int f(x)\,dx$. Compare with Problems 12-21 and 14-14.

17. (a) Find $\int \sin^4 x\,dx$ in two different ways: first using the reduction formula, and then using the formula for $\sin^2 x$.

(b) Combine your answers to obtain an impressive trigonometric identity.

18. Express $\int \log(\log x)\,dx$ in terms of $\int (\log x)^{-1}\,dx$. (Neither is expressible in terms of elementary functions.)

19. Express $\int x^2 e^{-x^2}\,dx$ in terms of $\int e^{-x^2}\,dx$.

20. Prove that the function $f(x) = e^x/(e^{5x} + e^x + 1)$ has an elementary primitive. (Do not try to find it!)

21. Prove the reduction formulas in the text. For the third one write

$$\int \frac{dx}{(x^2+1)^n} = \int \frac{dx}{(x^2+1)^{n-1}} - \int \frac{x^2\,dx}{(x^2+1)^n}$$

and work on the last integral. (Another possibility is to use the substitution $x = \tan u$.)

22. Find a reduction formula for

(a) $\displaystyle\int x^n e^x\,dx$

(b) $\displaystyle\int (\log x)^n\,dx.$

***23.** Prove that

$$\int_1^{\cosh x} \sqrt{t^2-1}\,dt = \frac{\cosh x \sinh x}{2} - \frac{x}{2}.$$

(See Problem 18-7 for the significance of this computation.)

24. Prove that

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

(A geometric interpretation makes this clear, but it is also a good exercise in the handling of limits of integration during a substitution.)

25. Prove that the area of a circle of radius r is πr^2. (Naturally you must remember that π is defined as the area of the unit circle.)

26. Let ϕ be a nonnegative integrable function such that $\phi(x) = 0$ for $|x| \geq 1$ and such that $\int_{-1}^{1} \phi = 1$. For $h > 0$, let

$$\phi_h(x) = \frac{1}{h}\phi(x/h).$$

(a) Show that $\phi_h(x) = 0$ for $|x| \geq h$ and that $\int_{-h}^{h} \phi_h = 1$.

(b) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \phi_h f = \lim_{h\to 0^+} \int_{-h}^{h} \phi_h f = f(0).$$

(c) Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2}\,dx = \pi.$$

The final part of this problem might appear, at first sight, to be an exact analogue of part (b), but it actually requires more careful argument.

(d) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx = \pi f(0).$$

Hint: If h is small, then $h/(h^2+x^2)$ will be small on most of $[-1, 1]$.

The next two problems use the formula

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta,$$

derived in Problem 13-24, for the area of a region bounded by the graph of f in polar coordinates.

27. For each of the following functions, find the area bounded by the graphs in polar coordinates. (Be careful about the proper range for θ, or you will get nonsensical results!)

(i) $f(\theta) = a\sin\theta$.
(ii) $f(\theta) = 2 + \cos\theta$.
(iii) $f(\theta)^2 = 2a^2\cos 2\theta$.
(iv) $f(\theta) = a\cos 2\theta$.

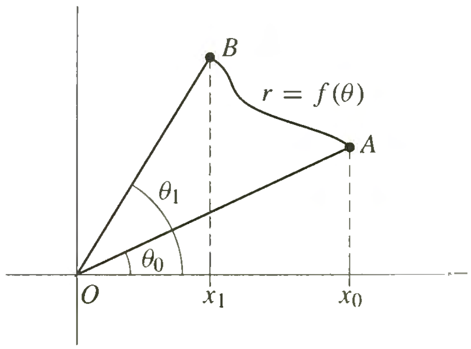


FIGURE 1

28. Figure 1 shows the graph of f in polar coordinates; the region OAB thus has area $\displaystyle\frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta$. Now suppose that this graph also happens to be the ordinary graph of some function g. Then the region OAB also has area

$$\text{area } \Delta O x_1 B + \int_{x_1}^{x_0} g - \text{ area } \Delta O x_0 A.$$

Prove analytically that these two numbers are indeed the same. Hint: The function g is determined by the equations

$$x = f(\theta)\cos\theta, \qquad g(x) = f(\theta)\sin\theta.$$

The next four problems use the formulas, derived in Problems 3 and 4 of the Appendix to Chapter 13, for the length of a curve represented parametrically (and, in particular, as the graph of a function in polar coordinates).

29. Let c be a curve represented parametrically by u and v on $[a,b]$, and let h be an increasing function with $h(\bar{a}) = a$ and $h(\bar{b}) = b$. Then on $[\bar{a},\bar{b}]$ the functions $\bar{u} = u \circ h$, $\bar{v} = v \circ h$ give a parametric representation of another curve $\bar{c}$; intuitively, $\bar{c}$ is just the same curve c traversed at a different rate.

(a) Show, directly from the definition of length, that the length of c on $[a,b]$ equals the length of $\bar{c}$ on $[\bar{a},\bar{b}]$.
(b) Assuming differentiability of any functions required, show that the lengths are equal by using the integral formula for length, and the appropriate substitution.

30. Find the length of the following curves, all described as the graphs of functions, except for (iii), which is represented parametrically.

(i) $f(x) = \dfrac{1}{3}(x^2+2)^{3/2}, \qquad 0 \le x \le 1.$
(ii) $f(x) = x^3 + \dfrac{1}{12x}, \qquad 1 \le x \le 2.$
(iii) $x = a^3\cos^3 t, \qquad y = a^3\sin^3 t, \qquad 0 \le t \le 2\pi.$
(iv) $f(x) = \log(\cos x), \qquad 0 \le x \le \pi/6.$
(v) $f(x) = \log x, \qquad 1 \le x \le e.$
(vi) $f(x) = \arcsin e^x, \qquad -\log 2 \le x \le 0.$

31. For the following functions, find the length of the graph in polar coordinates.

(i) $f(\theta) = a\cos\theta.$
(ii) $f(\theta) = a(1-\cos\theta).$
(iii) $f(\theta) = a\sin^2(\theta/2).$
(iv) $f(\theta) = \theta \qquad 0 \le \theta \le 2\pi.$
(v) $f(\theta) = 3\sec\theta \qquad 0 \le \theta \le \pi/3.$

32. In Problem 8 of the Appendix to Chapter 12 we described the cycloid, which has the parametric representation

$$x = u(t) = a(t - \sin t), \qquad y = v(t) = a(1 - \cos t).$$

(a) Find the length of one arch of the cycloid. [Answer: $8a$.]
(b) Recall that the cycloid is the graph of $v \circ u^{-1}$. Find the area under one arch of the cycloid by using the appropriate substitution in $\int f$ and evaluating the resultant integral. [Answer: $3\pi a^2$.]

33. Use induction and integration by parts to generalize Problem 14-10:

$$\int_0^x \frac{f(u)(x-u)^n}{n!}\,du = \int_0^x \left(\int_0^{u_n} \left(\ldots\left(\int_0^{u_1} f(t)\,dt\right) du_1\right)\ldots\right) du_n.$$

34. If f' is continuous on $[a, b]$, use integration by parts to prove the Riemann-Lebesgue Lemma for f:

$$\lim_{\lambda\to\infty} \int_a^b f(t)\sin(\lambda t)\,dt = 0.$$

This result is just a special case of Problem 15-26, but it can be used to prove the general case (in much the same way that the Riemann-Lebesgue Lemma was derived in Problem 15-26 from the special case in which f is a step function).

35. The Mean Value Theorem for Integrals was introduced in Problem 13-23. The "Second Mean Value Theorem for Integrals" states the following. Suppose that f is integrable on $[a, b]$ and that ϕ is either nondecreasing or nonincreasing on $[a, b]$. Then there is a number ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx + \phi(b)\int_\xi^b f(x)\,dx.$$

In this problem, we will assume that f is continuous and that ϕ is differentiable, with a continuous derivative ϕ'.

(a) Prove that if the result is true for nonincreasing ϕ, then it is also true for nondecreasing ϕ.
(b) Prove that if the result is true for nonincreasing ϕ satisfying $\phi(b) = 0$, then it is true for all nonincreasing ϕ.

Thus, we can assume that ϕ is nonincreasing and $\phi(b) = 0$. In this case, we have to prove that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx.$$

(c) Prove this by using integration by parts.
(d) Show that the hypothesis that ϕ is either nondecreasing or nonincreasing is needed.

From this special case of the Second Mean Value Theorem for Integrals, the general case could be derived by some approximation arguments, just as in the case of the Riemann-Lebesgue Lemma. But there is a more instructive way, outlined in the next problem.

36. (a) Given $a_1, \dots, a_n$ and $b_1, \dots, b_n$, let $s_k = a_1 + \dots + a_k$. Show that

$$(*) \qquad a_1b_1 + \dots + a_nb_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \dots + s_{n-1}(b_{n-1} - b_n) + s_nb_n$$

This disarmingly simple formula is sometimes called "Abel's formula for summation by parts." It may be regarded as an analogue for sums of the integration by parts formula

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx,$$

especially if we use Riemann sums (Chapter 13, Appendix). In fact, for a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, the left side is approximately

$$(1) \qquad \sum_{k=1}^{n} f'(t_k)g(t_{k-1})(t_k - t_{k-1}),$$

while the right side is approximately

$$f(b)g(b) - f(a)g(a) - \sum_{k=1}^{n} f(t_k)g'(t_k)(t_k - t_{k-1})$$

which is approximately

$$\begin{aligned} f(b)g(b) - f(a)g(a) &- \sum_{k=1}^{n} f(t_k)\frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^{n} f(t_k)[g(t_{k-1}) - g(t_k)] \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^{n}[f(t_k) - f(a)]\cdot[g(t_{k-1}) - g(t_k)] \\ &\qquad + f(a)\sum_{k=1}^{n} g(t_{k-1}) - g(t_k). \end{aligned}$$

Since the right-most sum is just $g(a) - g(b)$, this works out to be

$$(2) \qquad [f(b) - f(a)]g(b) + \sum_{k=1}^{n}[f(t_k) - f(a)]\cdot[g(t_{k-1}) - g(t_k)].$$

If we choose

$$a_k = f'(t_k)(t_k - t_{k-1}), \qquad b_k = g(t_{k-1})$$

then

$$(1) \qquad \text{is} \qquad \sum_{k=1}^{n} a_kb_k,$$

which is the left side of $(*)$, while

$$s_k = \sum_{i=1}^{k} f'(t_i)(t_i - t_{i-1}) \quad \text{is approximately} \quad \sum_{i=1}^{k} f(t_i) - f(t_{i-1}) = f(t_k) - f(a),$$

so

$$(2) \qquad \text{is approximately} \qquad s_nb_n + \sum_{k=1}^{n} s_k(b_k - b_{k-1}),$$

which is the right side of $(*)$.

This discussion is not meant to suggest that Abel's formula can actually be derived from the formula for integration by parts, or *vice versa*. But, as we shall see, Abel's formula can often be used as a substitute for integration by parts in situations where the functions in question aren't differentiable.

(b) Suppose that $\{b_n\}$ is nonincreasing, with $b_n \geq 0$ for each n, and that

$$m \leq a_1 + \cdots + a_n \leq M$$

for all n. Prove Abel's Lemma:

$$b_1 m \leq a_1 b_1 + \cdots + a_n b_n \leq b_1 M.$$

(And, moreover,

$$b_k m \leq a_k b_k + \cdots + a_n b_n \leq b_k M,$$

a formula which only looks more general, but really isn't.)

(c) Let f be integrable on $[a,b]$ and let ϕ be nonincreasing on $[a,b]$ with $\phi(b) = 0$. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a,b]$. Show that the sum

$$\sum_{i=1}^{n} f(t_{i-1})\phi(t_{i-1})(t_i - t_{i-1})$$

lies between the smallest and the largest of the sums

$$\phi(a)\sum_{i=1}^{k} f(t_{i-1})(t_i - t_{i-1}).$$

Conclude that

$$\int_a^b f(x)\phi(x)\,dx$$

lies between the minimum and the maximum of

$$\phi(a)\int_a^x f(t)\,dt,$$

and that it therefore equals $\phi(a)\displaystyle\int_a^{\xi} f(t)\,dt$ for some ξ in $[a,b]$.

37. (a) Show that the following improper integrals both converge.

(i) $\displaystyle\int_0^1 \sin\left(x + \frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_0^1 \sin^2\left(x + \frac{1}{x}\right) dx.$

(b) Decide which of the following improper integrals converge.

(i) $\displaystyle\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_1^{\infty} \sin^2\left(\frac{1}{x}\right) dx.$

38. (a) Compute the (improper) integral $\int_0^1 \log x \, dx$.

(b) Show that the improper integral $\int_0^\pi \log(\sin x)\, dx$ converges.

(c) Use the substitution $x = 2u$ to show that

$$\int_0^\pi \log(\sin x)\, dx = 2\int_0^{\pi/2} \log(\sin x)\, dx + 2\int_0^{\pi/2} \log(\cos x)\, dx + \pi \log 2.$$

(d) Compute $\int_0^{\pi/2} \log(\cos x)\, dx$.

(e) Using the relation $\cos x = \sin(\pi/2 - x)$, compute $\int_0^\pi \log(\sin x)\, dx$.

39. Prove the following version of integration by parts for improper integrals:

$$\int_a^\infty u'(x)v(x)\, dx = u(x)v(x)\Big|_a^\infty - \int_a^\infty u(x)v'(x)\, dx.$$

The first symbol on the right side means, of course,

$$\lim_{x\to\infty} u(x)v(x) - u(a)v(a).$$

***40.** One of the most important functions in analysis is the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\, dt.$$

(a) Prove that the improper integral $\Gamma(x)$ is defined if $x > 0$.
(b) Use integration by parts (more precisely, the improper integral version in the previous problem) to prove that

$$\Gamma(x+1) = x\Gamma(x).$$

(c) Show that $\Gamma(1) = 1$, and conclude that $\Gamma(n) = (n-1)!$ for all natural numbers n.

The gamma function thus provides a simple example of a continuous function which "interpolates" the values of $n!$ for natural numbers n. Of course there are infinitely many continuous functions f with $f(n) = (n-1)!$; there are even infinitely many continuous functions f with $f(x+1) = xf(x)$ for all $x > 0$. However, the gamma function has the important additional property that $\log \circ \Gamma$ is convex, a condition which expresses the extreme smoothness of this function. A beautiful theorem due to Harold Bohr and Johannes Mollerup states that Γ is the only function f with $\log \circ f$ convex, $f(1) = 1$ and $f(x+1) = xf(x)$. See reference [43] of the Suggested Reading.

***41.** (a) Use the reduction formula for $\int \sin^n x\, dx$ to show that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx.$$

(b) Now show that

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n},$$

and conclude that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\,\frac{\displaystyle\int_0^{\pi/2} \sin^{2n} x\,dx}{\displaystyle\int_0^{\pi/2} \sin^{2n+1} x\,dx}.$$

(c) Show that the quotient of the two integrals in this expression is between 1 and $1 + 1/2n$, starting with the inequalities

$$0 < \sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x \quad \text{for } 0 < x < \pi/2.$$

This result, which shows that the products

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$$

can be made as close to $\pi/2$ as desired, is usually written as an infinite product, known as Wallis' product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots.$$

(d) Show also that the products

$$\frac{1}{\sqrt{n}}\,\frac{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}$$

can be made as close to $\sqrt{\pi}$ as desired. (This fact is used in the next problem and in Problem 27-19.)

Wallis' procedure was quite different! He worked with the integral $\int_0^1 (1-x^2)^n\,dx$ (which appears in Problem 42), hoping to recover, from the values obtained for natural numbers n, a formula for

$$\frac{\pi}{4} = \int_0^1 (1-x^2)^{1/2}\,dx.$$

A complete account can be found in reference [49] of the Suggested Reading, but the following summary gives the basic idea. Wallis first obtained the formula

$$\begin{aligned}\int_0^1 (1-x^2)^n\,dx &= \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1}\\ &= \frac{(2\cdot 4\cdots 2n)^2}{2\cdot 3\cdot 4\cdots 2n(2n+1)} = \frac{2^n}{2n+1}\,\frac{(n!)^2}{(2n)!}.\end{aligned}$$

He then reasoned that $\pi/4$ should be

$$\int_0^1 (1-x^2)^{1/2}\,dx = \frac{2^1}{2}\,\frac{(\frac{1}{2}!)^2}{1!} = (\tfrac{1}{2}!)^2.$$

If we interpret $\frac{1}{2}!$ to mean $\Gamma(1+\frac{1}{2})$, this agrees with Problem 45, but Wallis did not know of the gamma function (which was invented by Euler, guided principally by Wallis' work). Since $(2n)!/(n!)^2$ is the binomial coefficient $\binom{2n}{n}$, Wallis hoped to find $\frac{1}{2}!$ by finding $\binom{p+q}{p}$ for $p=q=1/2$. Now

$$\binom{p+q}{p}=\frac{(p+q)(p+q-1)\cdots(p+1)}{q!}$$

and this makes sense even if p is not a natural number. Wallis therefore decided that

$$\binom{\frac{1}{2}+q}{\frac{1}{2}}=\frac{(\frac{1}{2}+q)\cdots(\frac{3}{2})}{q!}.$$

With this interpretation of $\binom{p+q}{p}$ for $p=1/2$, it is still true that

$$\binom{p+q+1}{p}=\frac{p+q+1}{q+1}\binom{p+q}{p}.$$

Denoting $\binom{\frac{1}{2}+q}{\frac{1}{2}}$ by $W(q)$ this equation can be written

$$W(q+1)=\frac{\frac{1}{2}+q+1}{q+1}W(q)=\frac{2q+3}{2q+2}W(q),$$

which leads to the table

q	1	2	3	
$W(q)$	$\frac{3}{2}$	$\frac{3}{2}\cdot\frac{5}{4}$	$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$	$\cdots$

But, since $W(\frac{1}{2})$ should be $4/\pi$, Wallis also constructs the table

q	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	
$W(q)$	$\frac{4}{\pi}$	$\frac{4}{\pi}\cdot\frac{4}{3}$	$\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}$	$\cdots$

Next Wallis notes that if a_1, a_2, a_3, a_4 are 4 successive values $W(q)$, $W(q+1)$, $W(q+2)$, $W(q+3)$, appearing in either of these tables, then

$$\frac{a_2}{a_1}>\frac{a_3}{a_2}>\frac{a_4}{a_3} \qquad \text{since } \frac{2q+3}{2q+2}>\frac{2q+5}{2q+4}\cdot\frac{2q+7}{2q+6}$$

(this says that $\log\circ(1/W)$ is convex, compare the remarks before Problem 41), which implies that

$$\sqrt{\frac{a_3}{a_1}}>\frac{a_3}{a_2}>\sqrt{\frac{a_4}{a_2}}.$$

Wallis then argues that this should still be true when a_1, a_2, a_3, a_4 are four successive values in a combined table where q is given *both* integer and half-integer values! Thus, taking as the four successive values $W(n+\frac{1}{2})$, $W(n)$, $W(n+\frac{3}{2})$, $W(n+1)$, he obtains

$$\sqrt{\frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+4}{2n+3}}{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+2}{2n+1}}}>\frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+2}{2n+1}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n}}>\sqrt{\frac{\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n+3}{2n+2}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n}}}$$

which yields simply

$$\sqrt{\frac{2n+4}{2n+3}}>\frac{4}{\pi}\cdot\left[\frac{2\cdot4\cdot4\cdot6\cdot6\cdots(2n)(2n)(2n+2)}{3\cdot3\cdot5\cdot5\cdots(2n+1)(2n+1)}\right]>\sqrt{\frac{2n+3}{2n+2}},$$

from which Wallis' product follows immediately.

****42.** It is an astonishing fact that improper integrals $\int_0^\infty f(x)\,dx$ can often be computed in cases where ordinary integrals $\int_a^b f(x)\,dx$ cannot. There is no elementary formula for $\int_a^b e^{-x^2}\,dx$, but we can find the value of $\int_0^\infty e^{-x^2}\,dx$ precisely! There are many ways of evaluating this integral, but most require some advanced techniques; the following method involves a fair amount of work, but no facts that you do not already know.

(a) Show that

$$\int_0^1 (1-x^2)^n\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\dots\cdot\frac{2n}{2n+1},$$

$$\int_0^\infty \frac{1}{(1+x^2)^n}\,dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\dots\cdot\frac{2n-3}{2n-2}.$$

(This can be done using reduction formulas, or by appropriate substitutions, combined with the previous problem.)

(b) Prove, using the derivative, that

$$1-x^2 \le e^{-x^2} \qquad \text{for } 0 \le x \le 1.$$

$$e^{-x^2} \le \frac{1}{1+x^2} \qquad \text{for } 0 \le x.$$

(c) Integrate the nth powers of these inequalities from 0 to 1 and from 0 to ∞, respectively. Then use the substitution $y = \sqrt{n}\,x$ to show that

$$\begin{aligned}\sqrt{n}\,\frac{2}{3}\cdot\frac{4}{5}\cdot\dots\cdot\frac{2n}{2n+1} &\le \int_0^{\sqrt{n}} e^{-y^2}\,dy \le \int_0^\infty e^{-y^2}\,dy \\ &\le \frac{\pi}{2}\sqrt{n}\,\frac{1}{2}\cdot\frac{3}{4}\cdot\dots\cdot\frac{2n-3}{2n-2}.\end{aligned}$$

(d) Now use Problem 41(d) to show that

$$\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}.$$

****43.** (a) Use integration by parts to show that

$$\int_a^b \frac{\sin x}{x}\,dx = \frac{\cos a}{a} - \frac{\cos b}{b} - \int_a^b \frac{\cos x}{x^2}\,dx,$$

and conclude that $\int_0^\infty (\sin x)/x\,dx$ exists. (Use the left side to investigate the limit as $a \to 0^+$ and the right side for the limit as $b \to \infty$.)

(b) Use Problem 15-33 to show that

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin\dfrac{t}{2}}\,dt = \pi$$

for any natural number n.

(c) Prove that

$$\lim_{\lambda\to\infty}\int_0^\pi \sin(\lambda+\tfrac{1}{2})t\left[\frac{2}{t}-\frac{1}{\sin\dfrac{t}{2}}\right]dt = 0.$$

Hint: The term in brackets is bounded by Problem 15-2(vi); the Riemann-Lebesgue Lemma then applies.

(d) Use the substitution $u = (\lambda+\frac{1}{2})t$ and part (b) to show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

44. Given the value of $\displaystyle\int_0^\infty (\sin x)/x\,dx$ from Problem 43, compute

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

by using integration by parts. (As in Problem 38, the formula for $\sin 2x$ will play an important role.)

***45.** (a) Use the substitution $u = t^x$ to show that

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-u^{1/x}}\,du.$$

(b) Find $\Gamma(\frac{1}{2})$.

***46.** (a) Suppose that $\dfrac{f(x)}{x}$ is integrable on every interval $[a,b]$ for $0 < a < b$, and that $\displaystyle\lim_{x\to 0} f(x) = A$ and $\displaystyle\lim_{x\to\infty} f(x) = B$. Prove that for all $\alpha, \beta > 0$ we have

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = (A-B)\log\frac{\beta}{\alpha}.$$

Hint: To estimate $\displaystyle\int_\varepsilon^N \frac{f(\alpha x)-f(\beta x)}{x}\,dx$ use two different substitutions.

(b) Now suppose instead that $\displaystyle\int_a^\infty \frac{f(x)}{x}\,dx$ converges for all $a > 0$ and that $\displaystyle\lim_{x\to 0} f(x) = A$. Prove that

$$\int_0^\infty \frac{f(\alpha x)-f(\beta x)}{x}\,dx = A\log\frac{\beta}{\alpha}.$$

(c) Compute the following integrals:

(i) $$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx.$$

(ii) $$\int_0^\infty \frac{\cos(\alpha x) - \cos(\beta x)}{x}\,dx.$$

In Chapter 13 we said, rather blithely, that integrals may be computed to any degree of accuracy desired by calculating lower and upper sums. But an applied mathematician, who really has to do the calculation, rather than just talking about doing it, may not be overjoyed at the prospect of computing lower sums to evaluate an integral to three decimal places, say (a degree of accuracy that might easily be needed in certain circumstances). The next three problems show how more refined methods can make the calculations much more efficient.

We ought to mention at the outset that computing upper and lower sums might not even be practical, since it might not be possible to compute the quantities m_i and M_i for each interval $[t_{i-1}, t_i]$. It is far more reasonable simply to pick points x_i in $[t_{i-1}, t_i]$ and consider $\sum_{i=1}^{n} f(x_i) \cdot (t_i - t_{i-1})$. This represents the sum of the areas of certain rectangles which partially overlap the graph of f—see Figure 1 in the Appendix to Chapter 13. But we will get a much better result if we instead choose the trapezoids shown in Figure 2.

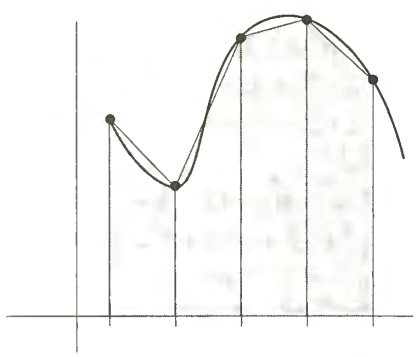


FIGURE 2

Suppose, in particular, that we divide $[a, b]$ into n equal intervals, by means of the points

$$t_i = a + i\left(\frac{b-a}{n}\right) = a + ih.$$

Then the trapezoid with base $[t_{i-1}, t_i]$ has area

$$\frac{f(t_{i-1}) + f(t_i)}{2} \cdot (t_i - t_{i-1})$$

and the sum of all these areas is simply

$$\Sigma_n = h\left[\frac{f(t_1)+f(a)}{2}+\frac{f(t_2)+f(t_1)}{2}+\cdots+\frac{f(b)+f(t_{n-1})}{2}\right]$$
$$= \frac{h}{2}\left[f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)\right], \qquad h=\frac{b-a}{n}.$$

This method of approximating an integral is called the *trapezoid rule.* Notice that to obtain Σ_{2n} from Σ_n it isn't necessary to recompute the old $f(t_i)$; their contribution to Σ_{2n} is just $\frac{1}{2}\Sigma_n$. So in practice it is best to compute Σ_2, Σ_4, Σ_8, ... to get approximations to $\int_a^b f$. In the next problem we will estimate $\int_a^b f - \Sigma_n$.

47. (a) Suppose that f'' is continuous. Let P_i be the linear function which agrees with f at t_{i-1} and t_i. Using Problem 11-46, show that if n_i and N_i are the minimum and maximum of f'' on $[t_{i-1}, t_i]$ and

$$I = \int_{t_{i-1}}^{t_i} (x-t_{i-1})(x-t_i)\,dx$$

then

$$\frac{n_i I}{2} \geq \int_{t_{i-1}}^{t_i} (f-P_i) \geq \frac{N_i I}{2}.$$

(b) Evaluate I to get

$$-\frac{n_i h^3}{12} \geq \int_{t_{i-1}}^{t_i} (f-P_i) \geq -\frac{N_i h^3}{12}.$$

(c) Conclude that there is some c in (a,b) with

$$\int_a^b f = \Sigma_n - \frac{(b-a)^3}{12n^2} f''(c).$$

Notice that the "error term" $(b-a)^3 f''(c)/12n^2$ varies as $1/n^2$ (while the error obtained using ordinary sums varies as $1/n$).

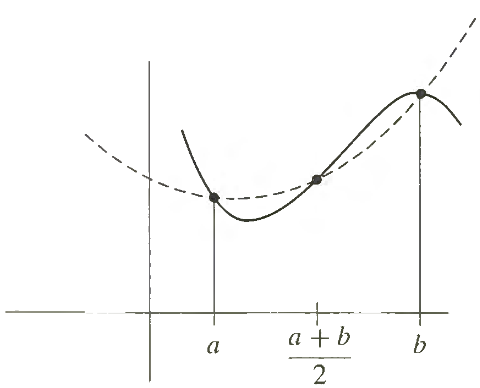


FIGURE 3

We can obtain still more accurate results if we approximate f by quadratic functions rather than by linear functions. We first consider what happens when the interval $[a,b]$ is divided into two equal intervals (Figure 3).

48. (a) Suppose first that $a=0$ and $b=2$. Let P be the polynomial function of degree ≤ 2 which agrees with f at 0, 1, and 2 (Problem 3-6). Show that

$$\int_0^2 P = \frac{1}{3}[f(0)+4f(1)+f(2)].$$

(b) Conclude that in the general case

$$\int_a^b P = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

(c) Naturally $\int_a^b P = \int_a^b f$ when f is a quadratic polynomial. But, remarkably enough, this same relation holds when f is a cubic polynomial! Prove this, using Problem 11-46; note that f''' is a constant.

The previous problem shows that we do not have to do any new calculations to compute $\int_a^b Q$ when Q is a *cubic* polynomial which agrees with f at a, b, and $\frac{a+b}{2}$: we still have

$$\int_a^b Q = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

But there is much more lee-way in choosing Q, which we can use to our advantage:

49. (a) Show that there is a cubic polynomial function Q satisfying

$$Q(a) = f(a), \qquad Q(b) = f(b), \qquad Q\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$$

$$Q'\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right).$$

Hint: Clearly $Q(x) = P(x) + A(x-a)(x-b)\left(x - \frac{a+b}{2}\right)$ for some A.

(b) Prove that if $f^{(4)}$ is defined on $[a,b]$, then for every x in $[a,b]$ we have

$$f(x) - Q(x) = (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)\frac{f^{(4)}(\xi)}{4!}$$

for some ξ in (a,b). Hint: Imitate the proof of Problem 11-46.

(c) Conclude that if $f^{(4)}$ is continuous, then

$$\int_a^b f = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(c)$$

for some c in (a,b).

(d) Now divide $[a,b]$ into $2n$ intervals by means of the points

$$t_i = a + ih, \qquad h = \frac{b-a}{2n}.$$

Prove *Simpson's rule*:

$$\int_a^b f = \frac{b-a}{6n}\left(f(a) + 4\sum_{i=1}^{n} f(t_{2i-1}) + 2\sum_{i=1}^{n-1} f(t_{2i}) + f(b)\right)$$

$$- \frac{(b-a)^5}{2880n^4} f^{(4)}(\bar{c})$$

for some $\bar{c}$ in (a,b).

APPENDIX. THE COSMOPOLITAN INTEGRAL

We originally introduced integrals in order to find the area under the graph of a function, but the integral is considerably more versatile than that. For example, Problem 13-24 used the integral to express the area of a region of quite another sort. Moreover, Problem 13-25 showed that the integral can also be used to express the lengths of curves—though, as we've seen in Appendix to Chapter 13, a lot of work may be necessary to consider the general case! This result was probably a little more surprising, since the integral seems, at first blush, to be a very two-dimensional creature. Actually, the integral makes its appearance in quite a few geometric formulas, which we will present in this Appendix. To derive these formulas we will assume some results from elementary geometry (and allow a little fudging).

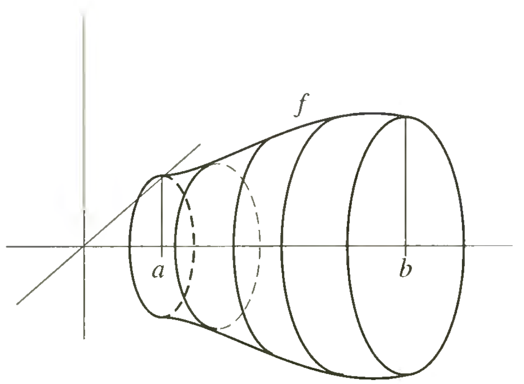


FIGURE 1

Instead of going down to one-dimensional objects, we'll begin by tackling some three-dimensional ones. There are some very special solids whose volumes can be expressed by integrals. The simplest such solid V is a "solid of revolution," obtained by revolving the region under the graph of $f \geq 0$ on $[a, b]$ around the horizontal axis, when we regard the plane as situated in space (Figure 1). If $P = \{t_0, \dots, t_n\}$ is any partition of $[a, b]$, and m_i and M_i have their usual meanings, then

$$\pi m_i{}^2(t_i - t_{i-1})$$

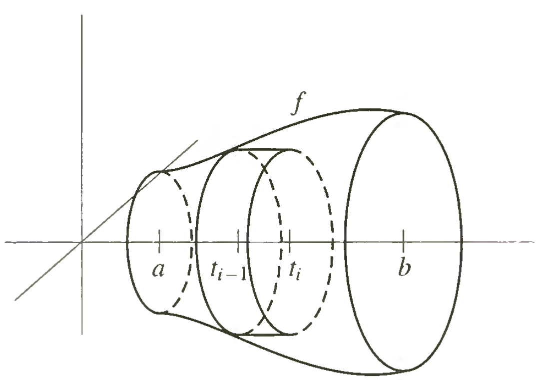


FIGURE 2

is the volume of a disc that lies inside the solid V (Figure 2). Similarly, $\pi M_i{}^2(t_i - t_{i-1})$ is the volume of a disc that contains the part of V between t_{i-1} and t_i. Consequently,

$$\pi \sum_{i=1}^{n} m_i{}^2(t_i - t_{i-1}) \leq \text{ volume } V \leq \pi \sum_{i=1}^{n} M_i{}^2(t_i - t_{i-1}).$$

But the sums on the ends of this inequality are just the lower and upper sums for f^2 on $[a, b]$:

$$\pi \cdot L(f^2, P) \leq \text{volume } V \leq \pi \cdot U(f^2, P).$$

Consequently, the volume of V must be given by

$$\text{volume } V = \pi \int_a^b f(x)^2\, dx.$$

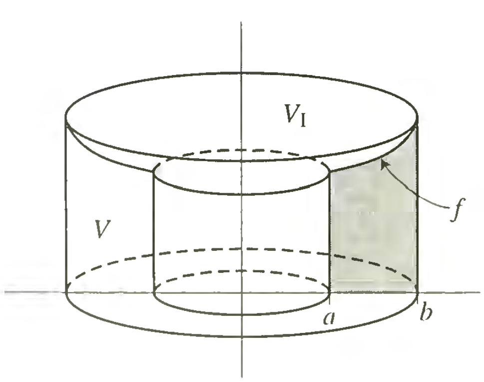


FIGURE 3

This method of finding volumes is affectionately referred to as the "disc method."

Figure 3 shows a more complicated solid V obtained by revolving the region under the graph of f around the *vertical* axis (V is the solid left over when we start with the big cylinder of radius b and take away both the small cylinder of radius a and the solid V_1 sitting right on top of it). In this case we assume $a \geq 0$ as well as $f \geq 0$. Figures 4 and 5 indicate some other possible shapes for V.

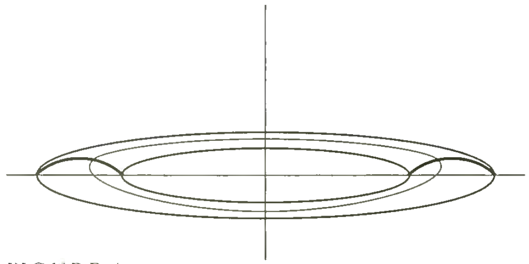


FIGURE 4

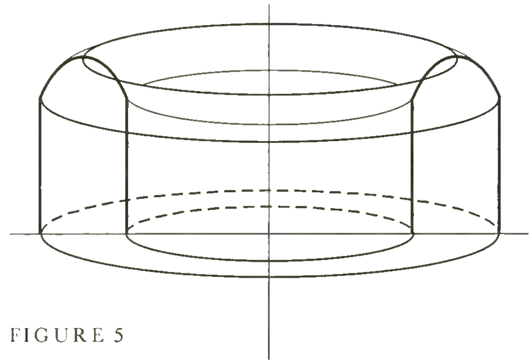


FIGURE 5

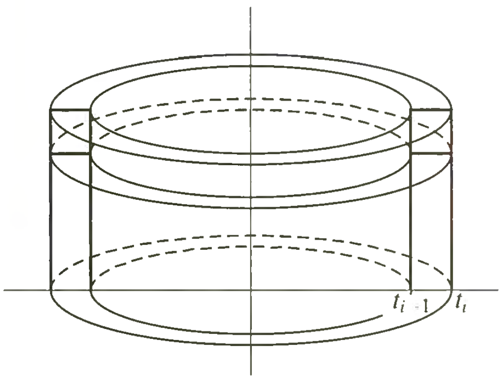


FIGURE 6

For a partition $P = \{t_0, \ldots, t_n\}$ we consider the "shells" obtained by rotating the rectangle with base $[t_{i-1}, t_i]$ and height m_i or M_i (Figure 6). Adding the volumes of these shells we obtain

$$\pi \sum_{i=1}^{n} m_i(t_i{}^2 - t_{i-1}{}^2) \leq \text{volume } V \leq \pi \sum_{i=1}^{n} M_i(t_i{}^2 - t_{i-1}{}^2),$$

which we can write as

$$\pi \sum_{i=1}^{n} m_i(t_i + t_{i-1})(t_i - t_{i-1}) \leq \text{volume } V \leq \pi \sum_{i=1}^{n} M_i(t_i + t_{i-1})(t_i - t_{i-1}).$$

Now these sums are not lower or upper sums of anything. But Problem 1 of the Appendix to Chapter 13 shows that each sum

$$\sum_{i=1}^{n} m_i t_i(t_i - t_{i-1}) \qquad \text{and} \qquad \sum_{i=1}^{n} m_i t_{i-1}(t_i - t_{i-1})$$

can be made as close as desired to $\int_a^b xf(x)\,dx$ by choosing the lengths $t_i - t_{i-1}$ small enough. The same is true of the sums on the right, so we find that

$$\text{volume } V = 2\pi \int_a^b xf(x)\,dx;$$

this is the so-called "shell method" of finding volumes.

The surface area of certain curved regions can also be expressed in terms of integrals. Before we tackle complicated regions, a little review of elementary geometric formulas may be appreciated here.

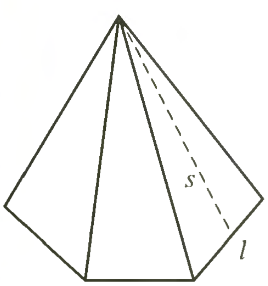


FIGURE 7

Figure 7 shows a right pyramid made up of triangles with bases of length l and altitude s. The total surface area of the sides of the pyramid is thus

$$\frac{1}{2}ps,$$

where p is the perimeter of the base. By choosing the base to be a regular polygon

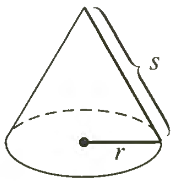


FIGURE 8

with a large number of sides we see that the area of a right circular cone (Figure 8) must be

$$\frac{1}{2}(2\pi r)s = \pi r s,$$

where s is the "slant height." Finally, consider the frustum of a cone with slant height s and radii r_1 and r_2 shown in Figure 9(a). Completing this to a cone, as in Figure 9(b), we have

$$\frac{s_1}{r_1} = \frac{s_1 + s}{r_2},$$

so

$$s_1 = \frac{r_1 s}{r_2 - r_1}, \qquad s_1 + s = \frac{r_2 s}{r_2 - r_1}.$$

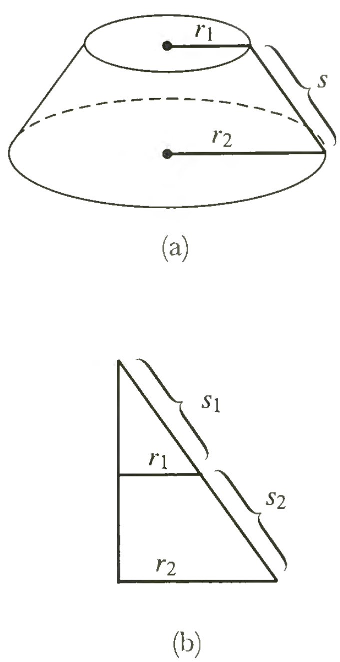


FIGURE 9

Consequently, the surface area is

$$\pi r_2(s_1 + s) - \pi r_1 s_1 = \pi s \frac{r_2{}^2 - r_1{}^2}{r_2 - r_1} = \pi s(r_1 + r_2).$$

Now consider the surface formed by revolving the graph of f around the horizontal axis. For a partition $P = \{t_0, \dots, t_n\}$ we can inscribe a series of frustums of cones, as in Figure 10. The total surface area of these frustums is

$$\pi \sum_{i=1}^{n} [f(t_{i-1}) + f(t_i)] \sqrt{(t_i - t_{i-1})^2 + [f(t_i) - f(t_{i-1})]^2}$$

$$= \pi \sum_{i=1}^{n} [f(t_{i-1}) + f(t_i)] \sqrt{1 + \left(\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right)^2} (t_i - t_{i-1}).$$

By the Mean Value Theorem, this is

$$\pi \sum_{i=1}^{n} [f(t_{i-1}) + f(t_i)] \sqrt{1 + f'(x_i)^2}\,(t_i - t_{i-1})$$

for some x_i in (t_{i-1}, t_i). Appealing to Problem 1 of the Appendix to Chapter 13, we conclude that the surface area is

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

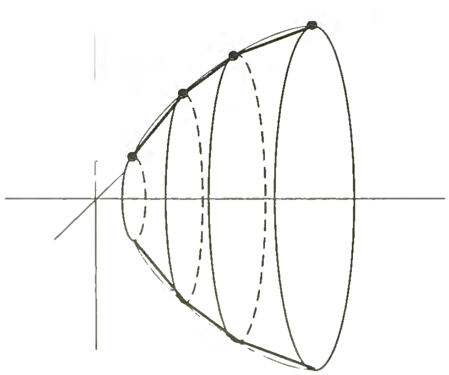


FIGURE 10

PROBLEMS

1. (a) Find the volume of the solid obtained by revolving the region bounded by the graphs of $f(x) = x$ and $f(x) = x^2$ around the horizontal axis.
(b) Find the volume of the solid obtained by revolving this same region around the vertical axis.

2. Find the volume of a sphere of radius r.

3. When the ellipse consisting of all points (x, y) with $x^2/a^2 + y^2/b^2 = 1$ is rotated around the horizontal axis we obtain an "ellipsoid of revolution" (Figure 11). Find the volume of the enclosed solid.

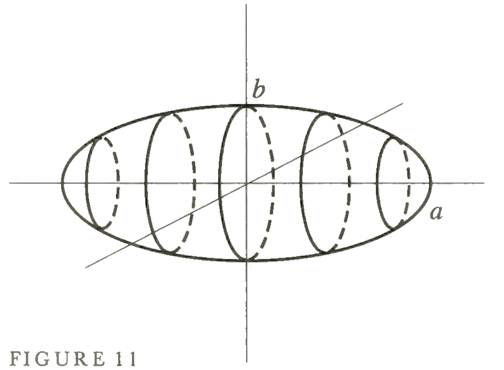


FIGURE 11

4. Find the volume of the "torus" (Figure 12), obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ $(a > b)$ around the vertical axis.

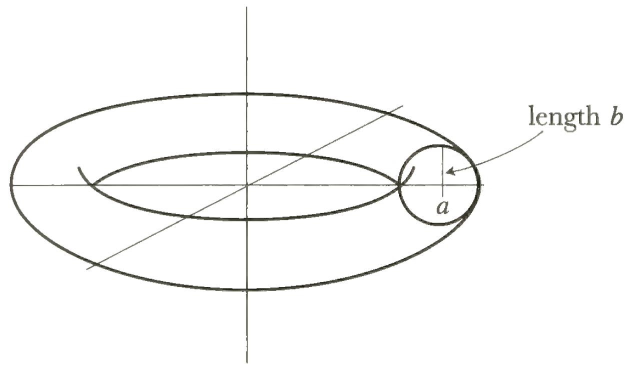


FIGURE 12

5. A cylindrical hole of radius a is bored through the center of a sphere of radius $2a$ (Figure 13). Find the volume of the remaining solid.

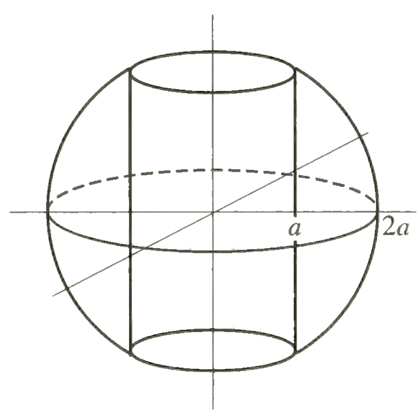


FIGURE 13

6. (a) For the solid shown in Figure 14, find the volume by the shell method.

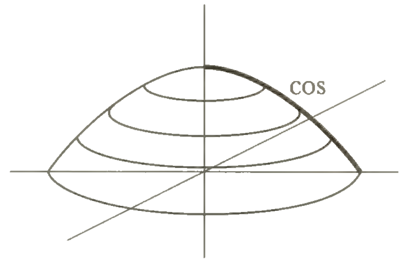


FIGURE 14

(b) This volume can also be evaluated by the disc method. Write down the integral which must be evaluated in this case; notice that it is more complicated. The next problem takes up a question which this might suggest.

7. Figure 15 shows a cylinder of height b and radius $f(b)$, divided into three solids, one of which, V_1, is a cylinder of height a and radius $f(a)$. If f is one-one, then a comparison of the disk method and the shell method of computing volumes leads us to believe that

$$\begin{aligned}\pi b f(b)^2 - \pi a f(a)^2 - \pi \int_a^b f(x)^2\,dx &= \text{volume } V_2 \\ &= 2\pi \int_{f(a)}^{f(b)} y f^{-1}(y)\,dy.\end{aligned}$$

Prove this analytically, using the formula for $\displaystyle\int f^{-1}$ from Problem 19-16, or more simply by going through the steps by which this formula was derived.

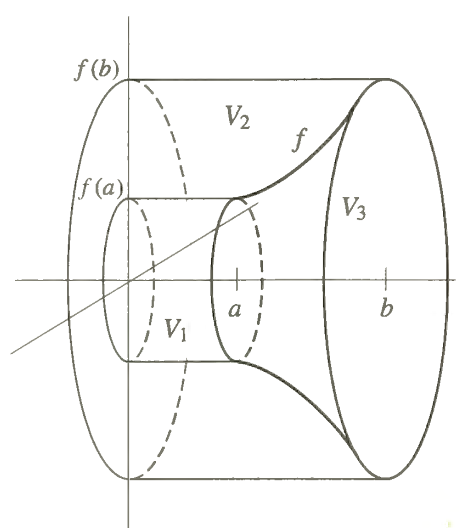


FIGURE 15

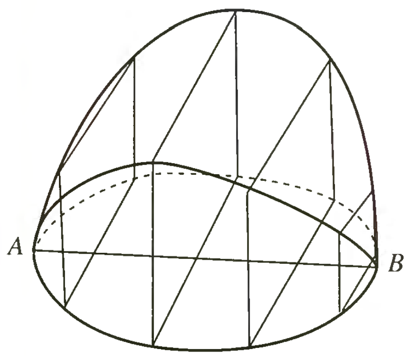


FIGURE 16

8. (a) Figure 16 shows a solid with a circular base of radius a. Each plane perpendicular to the diameter AB intersects the solid in a square. Using arguments similar to those already used in this Appendix, express the volume of the solid as an integral, and evaluate it.
(b) Same problem if each plane intersects the solid in an equilateral triangle.

9. Find the volume of a pyramid (Figure 17) in terms of its height h and the area A of its base.

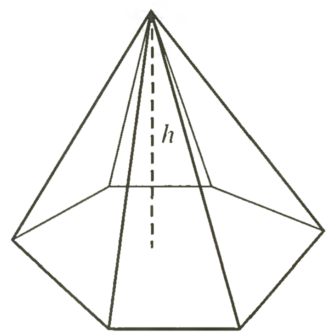


FIGURE 17

10. Find the volume of the solid which is the intersection of the two cylinders in Figure 18. Hint: Find the intersection of this solid with each horizontal plane.

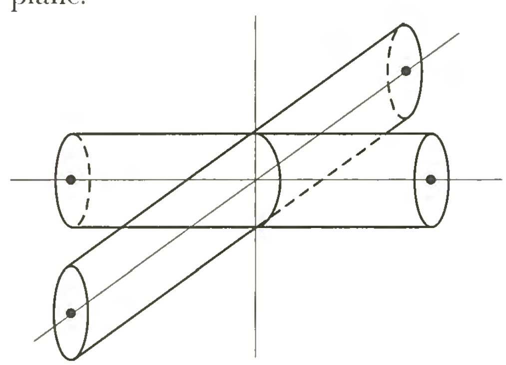


FIGURE 18

11. (a) Prove that the surface area of a sphere of radius r is $4\pi r^2$.
(b) Prove, more generally, that the area of the portion of the sphere shown in Figure 19 is $2\pi rh$. (Notice that this depends only on h, not on the position of the planes.)

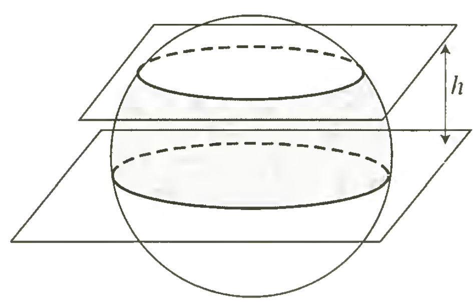


FIGURE 19

(c) A circular mud puddle can just be covered by a parallel collection of boards of length at least the radius of the circle, as in Figure 20 (a). Prove that it cannot be covered by the same boards if they are arranged in any non-parallel configuration, as in (b).

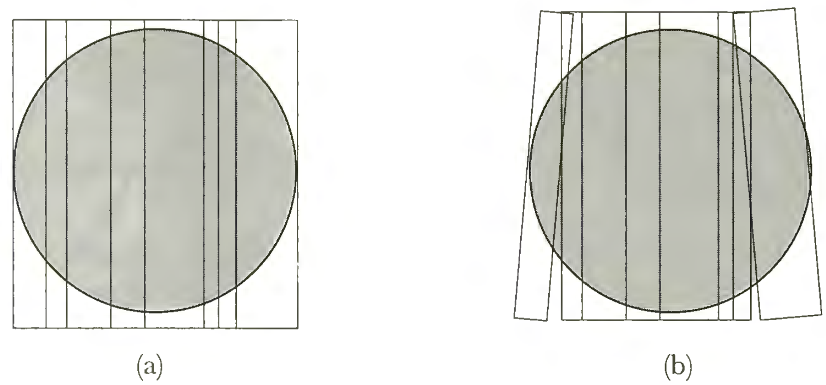


FIGURE 20

12. (a) Find the surface area of the ellipsoid of revolution in Problem 19-3.
    (b) Find the surface area of the torus in Problem 19-4.

13. The graph of $f(x) = 1/x$, $x \geq 1$ is revolved around the horizontal axis (Figure 21).

    (a) Find the volume of the enclosed "infinite trumpet."
    (b) Show that the surface area is infinite.
    (c) Suppose that we fill up the trumpet with the finite amount of paint found in part (a). It would seem that we have thereby coated the infinite inside surface area with only a finite amount of paint. How is this possible?

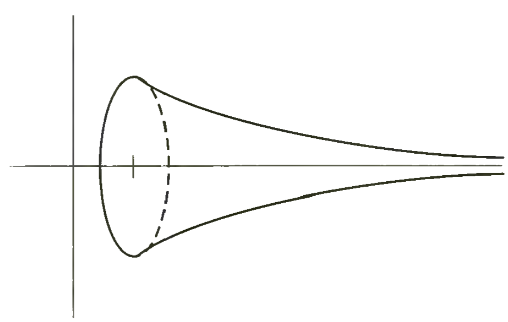


FIGURE 21

PART 4

INFINITE SEQUENCES AND INFINITE SERIES

One of the most remarkable series of algebraic analysis is the following:

$$1 + \frac{m}{1} x + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \cdots + \frac{m(m-1) \cdots [m-(n-1)]}{1 \cdot 2 \cdot \cdots \cdots \cdot n} x^n + \cdots$$

When m is a positive whole number
the sum of the series,
which is then finite, can be expressed,
as is known, by $(1 + x)^m$.
When m is not an integer,
the series goes on to infinity, and it will
converge or diverge according
as the quantities
m and x have this or that value.
In this case, one writes the same equality

$$(1 + x)^m = 1 + \frac{m}{1} x + \frac{m(m-1)}{1 \cdot 2} x^2 + \cdots \textit{etc.}$$

. . . *It is assumed that*
the numerical equality will always occur
whenever the series is convergent, but
this has never yet been proved.

NIELS HENRIK ABEL

CHAPTER 20 APPROXIMATION BY POLYNOMIAL FUNCTIONS

There is one sense in which the "elementary functions" are not elementary at all. If p is a polynomial function,

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

then $p(x)$ can be computed easily for any number x. This is not at all true for functions like sin, log, or exp. At present, to find $\log x = \int_1^x 1/t\,dt$ approximately, we must compute some upper or lower sums, and make certain that the error involved in accepting such a sum for $\log x$ is not too great. Computing $e^x = \log^{-1}(x)$ would be even more difficult: we would have to compute $\log a$ for many values of a until we found a number a such that $\log a$ is approximately x—then a would be approximately e^x.

In this chapter we will obtain important theoretical results which reduce the computation of $f(x)$, for many functions f, to the evaluation of polynomial functions. The method depends on finding polynomial functions which are close approximations to f. In order to guess a polynomial which is appropriate, it is useful to first examine polynomial functions themselves more thoroughly.

Suppose that

$$p(x) = a_0 + a_1x + \cdots + a_nx^n.$$

It is interesting, and for our purposes very important, to note that the coefficients a_i can be expressed in terms of the value of p and its various derivatives at 0. To begin with, note that

$$p(0) = a_0.$$

Differentiating the original expression for $p(x)$ yields

$$p'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

Therefore,

$$p'(0) = p^{(1)}(0) = a_1.$$

Differentiating again we obtain

$$p''(x) = 2a_2 + 3 \cdot 2 \cdot a_3x + \cdots + n(n-1) \cdot a_nx^{n-2}.$$

Therefore,

$$p''(0) = p^{(2)}(0) = 2a_2.$$

In general, we will have

$$p^{(k)}(0) = k!\,a_k \quad \text{or} \quad a_k = \frac{p^{(k)}(0)}{k!}.$$

If we agree to define $0! = 1$, and recall the notation $p^{(0)} = p$, then this formula holds for $k = 0$ also.

If we had begun with a function p that was written as a "polynomial in $(x-a)$,"

$$p(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n,$$

then a similar argument would show that

$$a_k = \frac{p^{(k)}(a)}{k!}.$$

Suppose now that f is a function (not necessarily a polynomial) such that

$$f^{(1)}(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \qquad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n.$$

The polynomial $P_{n,a}$ is called the **Taylor polynomial of degree n for f at a.** (Strictly speaking, we should use an even more complicated expression, like $P_{n,a,f}$, to indicate the dependence on f; at times this more precise notation will be useful.) The Taylor polynomial has been defined so that

$$P_{n,a}{}^{(k)}(a) = f^{(k)}(a) \quad \text{for } 0 \le k \le n;$$

in fact, it is clearly the only polynomial *of degree* $\le n$ with this property.

Although the coefficients of $P_{n,a,f}$ seem to depend upon f in a fairly complicated way, the most important elementary functions have extremely simple Taylor polynomials. Consider first the function sin. We have

$$\begin{aligned}
\sin(0) &= 0, \\
\sin'(0) &= \cos 0 = 1, \\
\sin''(0) &= -\sin 0 = 0, \\
\sin'''(0) &= -\cos 0 = -1, \\
\sin^{(4)}(0) &= \sin 0 = 0.
\end{aligned}$$

From this point on, the derivatives repeat in a cycle of 4. The numbers

$$a_k = \frac{\sin^{(k)}(0)}{k!}$$

are

$$0,\ 1,\ 0,\ -\frac{1}{3!},\ 0,\ \frac{1}{5!},\ 0,\ -\frac{1}{7!},\ 0,\ \frac{1}{9!},\ \ldots.$$

Therefore the Taylor polynomial $P_{2n+1,0}$ of degree $2n+1$ for sin at 0 is

$$P_{2n+1,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(Of course, $P_{2n+1,0} = P_{2n+2,0}$).

The Taylor polynomial $P_{2n,0}$ of degree $2n$ for cos at 0 is (the computations are left to you)

$$P_{2n,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

The Taylor polynomial for exp is especially easy to compute. Since $\exp^{(k)}(0) = \exp(0) = 1$ for all k, the Taylor polynomial of degree n at 0 is

$$P_{n,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}.$$

The Taylor polynomial for log must be computed at some point $a \neq 0$, since log is not even defined at 0. The standard choice is $a = 1$. Then

$$\begin{aligned}
\log'(x) &= \frac{1}{x}, & \log'(1) &= 1;\\
\log''(x) &= -\frac{1}{x^2}, & \log''(1) &= -1;\\
\log'''(x) &= \frac{2}{x^3}, & \log'''(1) &= 2;
\end{aligned}$$

in general

$$\log^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}, \qquad \log^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

Therefore the Taylor polynomial of degree n for log at 1 is

$$P_{n,1}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}.$$

It is often more convenient to consider the function $f(x) = \log(1+x)$. In this case we can choose $a = 0$. We have

$$f^{(k)}(x) = \log^{(k)}(1+x),$$

so

$$f^{(k)}(0) = \log^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

Therefore the Taylor polynomial of degree n for f at 0 is

$$P_{n,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n}.$$

There is one other elementary function whose Taylor polynomial is important—arctan. The computations of the derivatives begin

$$\begin{aligned}
\arctan'(x) &= \frac{1}{1+x^2} & \arctan'(0) &= 1;\\
\arctan''(x) &= \frac{-2x}{(1+x^2)^2}, & \arctan''(0) &= 0;\\
\arctan'''(x) &= \frac{(1+x^2)^2 \cdot (-2) + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4}, & \arctan'''(0) &= -2.
\end{aligned}$$

It is clear that this brute force computation will never do. However, the Taylor polynomials of arctan will be easy to find after we have examined the properties of Taylor polynomials more closely—although the Taylor polynomial $P_{n,a,f}$ was simply defined so as to have the same first n derivatives at a as f, the connection between f and $P_{n,a,f}$ will actually turn out to be much deeper.

One line of evidence for a closer connection between f and the Taylor polynomials for f may be uncovered by examining the Taylor polynomial of degree 1, which is

$$P_{1,a}(x) = f(a) + f'(a)(x-a).$$

Notice that

$$\frac{f(x) - P_{1,a}(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Now, by the definition of $f'(a)$ we have

$$\lim_{x\to a} \frac{f(x) - P_{1,a}(x)}{x-a} = 0.$$

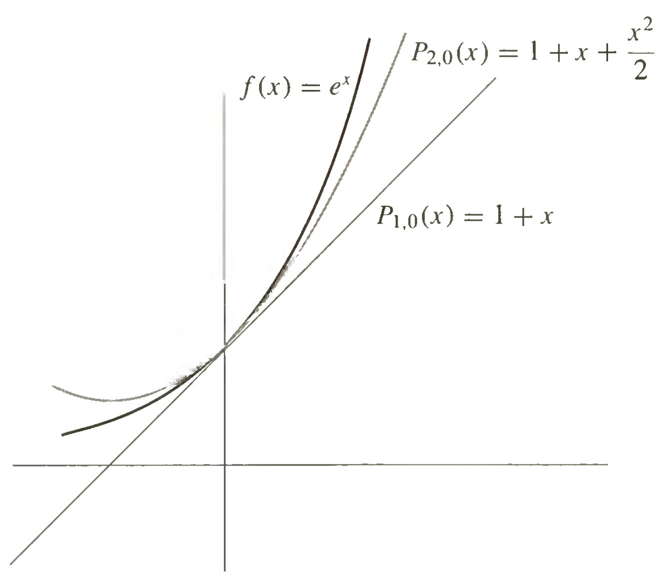


FIGURE 1

In other words, as x approaches a the difference $f(x) - P_{1,a}(x)$ not only becomes small, but actually becomes small even compared to $x - a$. Figure 1 illustrates the graph of $f(x) = e^x$ and of

$$P_{1,0}(x) = f(0) + f'(0)x = 1 + x,$$

which is the Taylor polynomial of degree 1 for f at 0. The diagram also shows the graph of

$$P_{2,0}(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2},$$

which is the Taylor polynomial of degree 2 for f at 0. As x approaches 0, the difference $f(x) - P_{2,0}(x)$ seems to be getting small even faster than the difference

$f(x) - P_{1,0}(x)$. As it stands, this assertion is not very precise, but we are now prepared to give it a definite meaning. We have just noted that in general

$$\lim_{x \to a} \frac{f(x) - P_{1,a}(x)}{x - a} = 0.$$

For $f(x) = e^x$ and $a = 0$ this means that

$$\lim_{x \to 0} \frac{f(x) - P_{1,0}(x)}{x} = \lim_{x \to 0} \frac{e^x - 1 - x}{x} = 0.$$

On the other hand, an easy double application of l'Hôpital's Rule shows that

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2} \neq 0.$$

Thus, although $f(x) - P_{1,0}(x)$ becomes small compared to x, as x approaches 0, it does *not* become small compared to x^2. For $P_{2,0}(x)$ the situation is quite different; the extra term $x^2/2$ provides just the right compensation:

$$\begin{aligned}\lim_{x \to 0} \frac{e^x - 1 - x - \dfrac{x^2}{2}}{x^2} &= \lim_{x \to 0} \frac{e^x - 1 - x}{2x} \\ &= \lim_{x \to 0} \frac{e^x - 1}{2} = 0.\end{aligned}$$

This result holds in general—if $f'(a)$ and $f''(a)$ exist, then

$$\lim_{x \to a} \frac{f(x) - P_{2,a}(x)}{(x - a)^2} = 0;$$

in fact, the analogous assertion for $P_{n,a}$ is also true.

THEOREM 1 Suppose that f is a function for which

$$f'(a), \dots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \leq k \leq n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$

PROOF Writing out $P_{n,a}(x)$ explicitly, we obtain

$$\frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \frac{f(x) - \displaystyle\sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} - \frac{f^{(n)}(a)}{n!}.$$

It will help to introduce the new functions

$$Q(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i \quad \text{and} \quad g(x) = (x-a)^n;$$

now we must prove that

$$\lim_{x \to a} \frac{f(x) - Q(x)}{g(x)} = \frac{f^{(n)}(a)}{n!}.$$

Notice that

$$\begin{aligned} Q^{(k)}(a) &= f^{(k)}(a), \quad k \le n-1, \\ g^{(k)}(x) &= n!(x-a)^{n-k}/(n-k)!. \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{x \to a} [f(x) - Q(x)] = f(a) - Q(a) = 0, \\ &\lim_{x \to a} [f'(x) - Q'(x)] = f'(a) - Q'(a) = 0, \\ &\quad \vdots \\ &\lim_{x \to a} [f^{(n-2)}(x) - Q^{(n-2)}(x)] = f^{(n-2)}(a) - Q^{(n-2)}(a) = 0. \end{aligned}$$

and

$$\lim_{x \to a} g(x) = \lim_{x \to a} g'(x) = \cdots = \lim_{x \to a} g^{(n-2)}(x) = 0.$$

We may therefore apply l'Hôpital's Rule $n-1$ times to obtain

$$\lim_{x \to a} \frac{f(x) - Q(x)}{(x-a)^n} = \lim_{x \to a} \frac{f^{(n-1)}(x) - Q^{(n-1)}(x)}{n!\,(x-a)}.$$

Since Q is a polynomial of degree $n-1$, its $(n-1)$st derivative is a constant; in fact, $Q^{(n-1)}(x) = f^{(n-1)}(a)$. Thus

$$\lim_{x \to a} \frac{f(x) - Q(x)}{(x-a)^n} = \lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{n!\,(x-a)}$$

and this last limit is $f^{(n)}(a)/n!$ by definition of $f^{(n)}(a)$. ∎

One simple consequence of Theorem 1 allows us to perfect the test for local maxima and minima which was developed in Chapter 11. If a is a critical point of f, then, according to Theorem 11-5, the function f has a local minimum at a if $f''(a) > 0$, and a local maximum at a if $f''(a) < 0$. If $f''(a) = 0$ no conclusion was possible, but it is conceivable that the sign of $f'''(a)$ might give further information; and if $f'''(a) = 0$, then the sign of $f^{(4)}(a) = 0$ might be significant. Even more generally, we can ask what happens when

$$(*) \qquad \begin{aligned} &f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0, \\ &f^{(n)}(a) \neq 0. \end{aligned}$$

The situation in this case can be guessed by examining the functions

$$\begin{aligned} f(x) &= (x-a)^n, \\ g(x) &= -(x-a)^n, \end{aligned}$$

which satisfy $(*)$. Notice (Figure 2) that if n is odd, then a is neither a local maximum nor a local minimum point for f or g. On the other hand, if n is even, then f, with a positive nth derivative, has a local minimum at a, while g, with a negative nth derivative, has a local maximum at a. Of all functions satisfying $(*)$, these are about the simplest available; nevertheless they indicate the general situation exactly. In fact, the whole point of the next proof is that any function satisfying $(*)$ looks very much like one of these functions, in a sense that is made precise by Theorem 1.

THEOREM 2 Suppose that

$$\begin{aligned} &f'(a) = \cdots = f^{(n-1)}(a) = 0, \\ &f^{(n)}(a) \neq 0. \end{aligned}$$

(1) If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.
(2) If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.
(3) If n is odd, then f has neither a local maximum nor a local minimum at a.

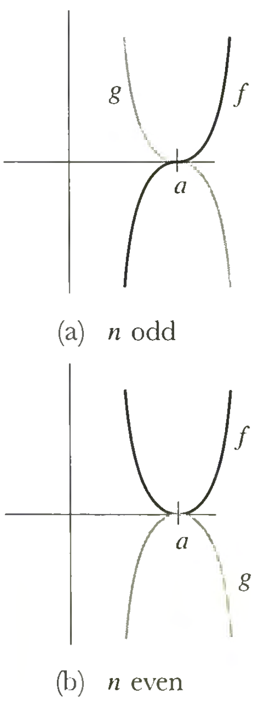


FIGURE 2

PROOF There is clearly no loss of generality in assuming that $f(a) = 0$, since neither the hypotheses nor the conclusion are affected if f is replaced by $f - f(a)$. Then, since the first $n-1$ derivatives of f at a are 0, the Taylor polynomial $P_{n,a}$ of f is

$$\begin{aligned} P_{n,a}(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Thus, Theorem 1 states that

$$0 = \lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \lim_{x \to a} \left[\frac{f(x)}{(x-a)^n} - \frac{f^{(n)}(a)}{n!} \right].$$

Consequently, if x is sufficiently close to a, then

$$\frac{f(x)}{(x-a)^n} \quad \text{has the same sign as} \quad \frac{f^{(n)}(a)}{n!}.$$

Suppose now that n is even. In this case $(x-a)^n > 0$ for all $x \neq a$. Since $f(x)/(x-a)^n$ has the same sign as $f^{(n)}(a)/n!$ for x sufficiently close to a, it follows

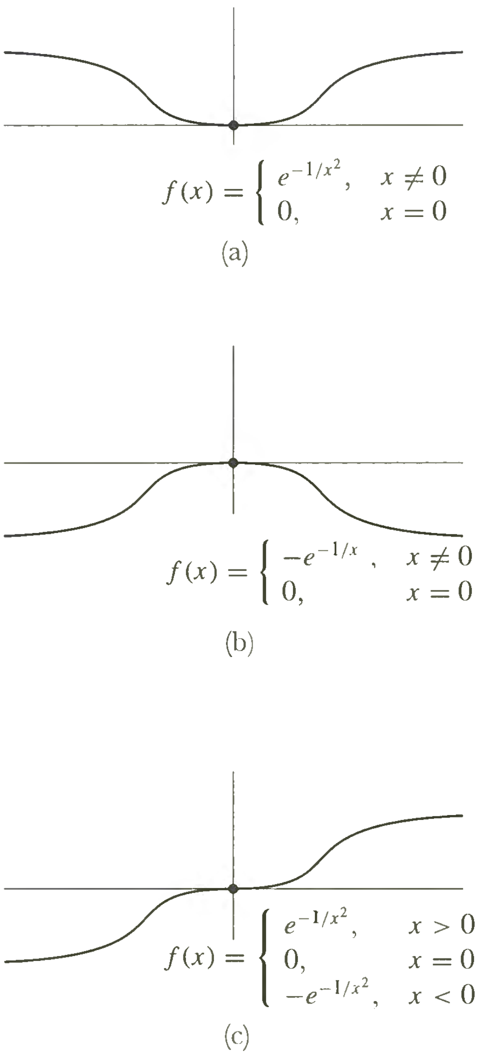


FIGURE 3

that $f(x)$ itself has the same sign as $f^n(a)/n!$ for x sufficiently close to a. If $f^{(n)}(a) > 0$, this means that

$$f(x) > 0 = f(a)$$

for x close to a. Consequently, f has a local minimum at a. A similar proof works for the case $f^{(n)}(a) < 0$.

Now suppose that n is odd. The same argument as before shows that if x is sufficiently close to a, then

$$\frac{f(x)}{(x-a)^n} \quad \text{always has the same sign.}$$

But $(x-a)^n > 0$ for $x > a$ and $(x-a)^n < 0$ for $x < a$. Therefore $f(x)$ has *different* signs for $x > a$ and $x < a$. This proves that f has neither a local maximum nor a local minimum at a. ▌

Although Theorem 2 will settle the question of local maxima and minima for just about any function which arises in practice, it does have some theoretical limitations, because $f^{(k)}(a)$ may be 0 for *all* k. This happens (Figure 3(a)) for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

which has a minimum at 0, and also for the negative of this function (Figure 3(b)), which has a maximum at 0. Moreover (Figure 3(c)), if

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x = 0 \\ -e^{-1/x^2}, & x < 0, \end{cases}$$

then $f^{(k)}(0) = 0$ for all k, but f has neither a local minimum nor a local maximum at 0.

The conclusion of Theorem 1 is often expressed in terms of an important concept of "order of equality." Two functions f and g are **equal up to order n at a** if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

In the language of this definition, Theorem 1 says that the Taylor polynomial $P_{n,a,f}$ equals f up to order n at a. The Taylor polynomial might very well have been designed to make this fact true, because there is at most one polynomial of degree $\leq n$ with this property. This assertion is a consequence of the following elementary theorem.

THEOREM 3 Let P and Q be two polynomials in $(x-a)$, of degree $\leq n$, and suppose that P and Q are equal up to order n at a. Then $P = Q$.

PROOF Let $R = P - Q$. Since R is a polynomial of degree $\leq n$, it is only necessary to prove that if

$$R(x) = b_0 + \cdots + b_n(x-a)^n$$

satisfies

$$\lim_{x \to a} \frac{R(x)}{(x-a)^n} = 0,$$

then $R = 0$. Now the hypothesis on R surely imply that

$$\lim_{x \to a} \frac{R(x)}{(x-a)^i} = 0 \quad \text{for } 0 \le i \le n.$$

For $i = 0$ this condition reads simply $\lim\limits_{x \to a} R(x) = 0$; on the other hand,

$$\begin{aligned} \lim_{x \to a} R(x) &= \lim_{x \to a} [b_0 + b_1(x-a) + \cdots + b_n(x-a)^n] \\ &= b_0. \end{aligned}$$

Thus $b_0 = 0$ and

$$R(x) = b_1(x-a) + \cdots + b_n(x-a)^n.$$

Therefore,

$$\frac{R(x)}{x-a} = b_1 + b_2(x-a) + \cdots + b_n(x-a)^{n-1}$$

and

$$\lim_{x \to a} \frac{R(x)}{x-a} = b_1.$$

Thus $b_1 = 0$ and

$$R(x) = b_2(x-a)^2 + \cdots + b_n(x-a)^n.$$

Continuing in this way, we find that

$$b_0 = \cdots = b_n = 0. \blacksquare$$

COROLLARY Let f be n-times differentiable at a, and suppose that P is a polynomial in $(x-a)$ of degree $\le n$, which equals f up to order n at a. Then $P = P_{n,a,f}$.

PROOF Since P and $P_{n,a,f}$ both equal f up to order n at a, it is easy to see that P equals $P_{n,a,f}$ up to order n at a. Consequently, $P = P_{n,a,f}$ by the Theorem. $\blacksquare$

At first sight this corollary appears to have unnecessarily complicated hypotheses; it might seem that the existence of the polynomial P would automatically imply that f is sufficiently differentiable for $P_{n,a,f}$ to exist. But in fact this is not so. For example (Figure 4), suppose that

$$f(x) = \begin{cases} x^{n+1}, & x \text{ irrational} \\ 0, & x \text{ rational.} \end{cases}$$

If $P(x) = 0$, then P is certainly a polynomial of degree $\le n$ which equals f up to order n at 0. On the other hand, $f'(a)$ does not exist for any $a \neq 0$, so $f''(0)$ is undefined.

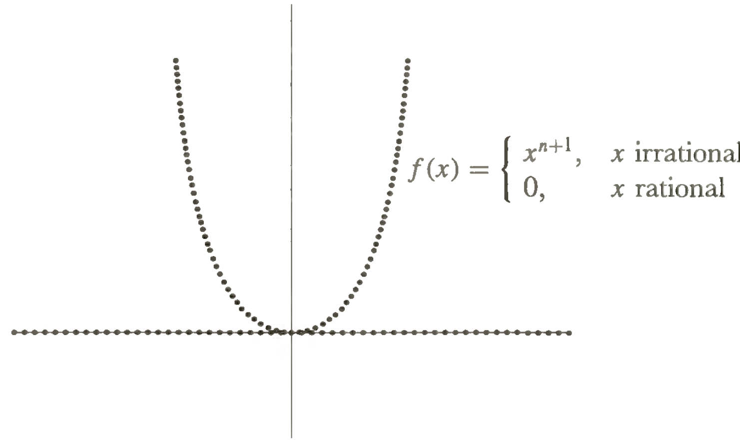


FIGURE 4

When f does have n derivatives at a, however, the corollary may provide a useful method for finding the Taylor polynomial of f. In particular, remember that our first attempt to find the Taylor polynomial for arctan ended in failure. The equation

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

suggests a promising method of finding a polynomial close to arctan—divide 1 by $1+t^2$, to obtain a polynomial plus a remainder:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}.$$

This formula, which can be checked easily by multiplying both sides by $1+t^2$, shows that

$$\begin{aligned} \arctan x &= \int_0^x 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n}\,dt + (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt. \end{aligned}$$

According to our corollary, the polynomial which appears here will be the Taylor polynomial of degree $2n+1$ for arctan at 0, provided that

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt}{x^{2n+1}} = 0.$$

Since

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3},$$

this is clearly true. Thus we have found that the Taylor polynomial of degree $2n+1$ for arctan at 0 is

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

By the way, now that we have discovered the Taylor polynomials of arctan, it is possible to work backwards and find $\arctan^{(k)}(0)$ for all k: Since

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1},$$

and since this polynomial is, by definition,

$$\arctan^{(0)}(0) + \arctan^{(1)}(0) + \frac{\arctan^{(2)}(0)}{2!}x^2 + \cdots + \frac{\arctan^{(2n+1)}(0)}{(2n+1)!}x^{2n+1},$$

we can find $\arctan^{(k)}(0)$ by simply equating the coefficients of x^k in these two polynomials:

$$\frac{\arctan^{(k)}(0)}{k!} = 0 \quad \text{if } k \text{ is even},$$

$$\frac{\arctan^{(2l+1)}(0)}{(2l+1)!} = \frac{(-1)^l}{2l+1} \quad \text{or} \quad \arctan^{(2l+1)}(0) = (-1)^l \cdot (2l)!\,.$$

A much more interesting fact emerges if we go back to the original equation

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt,$$

and remember the estimate

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \frac{|x|^{2n+3}}{2n+3}.$$

When $|x| \le 1$, this expression is at most $1/(2n+3)$, and we can make this as small as we like simply by choosing n large enough. In other words, for $|x| \le 1$ *we can use the Taylor polynomials for* arctan *to compute* $\arctan x$ *as accurately as we like.* The most important theorems about Taylor polynomials extend this isolated result to other functions, and the Taylor polynomials will soon play quite a new role. The theorems proved so far have always examined the behavior of the Taylor polynomial $P_{n,a}$ for *fixed* n, as x approaches a. Henceforth we will compare Taylor polynomials $P_{n,a}$ for *fixed* x, and different n. In anticipation of the coming theorem we introduce some new notation.

If f is a function for which $P_{n,a}(x)$ exists, we define the **remainder term** $R_{n,a}(x)$ by

$$\begin{aligned} f(x) &= P_{n,a}(x) + R_{n,a}(x) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x). \end{aligned}$$

We would like to have an expression for $R_{n,a}(x)$ whose size is easy to estimate. There is such an expression, involving an integral, just as in the case for arctan. One way to guess this expression is to begin with the case $n = 0$:

$$f(x) = f(a) + R_{0,a}(x).$$

The Fundamental Theorem of Calculus enables us to write

$$f(x) = f(a) + \int_a^x f'(t)\,dt,$$

so that

$$R_{0,a}(x) = \int_a^x f'(t)\,dt.$$

A similar expression for $R_{1,a}(x)$ can be derived from this formula using integration by parts in a rather tricky way: Let

$$u(t) = f'(t) \quad \text{and} \quad v(t) = t - x$$

(notice that x represents some fixed number in the expression for $v(t)$, so $v'(t) = 1$); then

$$\begin{aligned}\int_a^x f'(t)\,dt &= \int_a^x \underset{\substack{\downarrow\\ u(t)}}{f'(t)} \cdot \underset{\substack{\downarrow\\ v'(t)}}{1}\,dt\\ &= u(t)v(t)\Big|_a^x - \int_a^x \underset{\substack{\downarrow\\ u'(t)}}{f''(t)}\underset{\substack{\downarrow\\ v(t)}}{(t-x)}\,dt.\end{aligned}$$

Since $v(x) = 0$, we obtain

$$\begin{aligned}f(x) &= f(a) + \int_a^x f'(t)\,dt\\ &= f(a) - u(a)v(a) + \int_a^x f''(t)(x-t)\,dt\\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt.\end{aligned}$$

Thus

$$R_{1,a}(x) = \int_a^x f''(t)(x-t)\,dt.$$

It is hard to give any motivation for choosing $v(t) = t - x$, rather than $v(t) = t$. It just happens to be the choice which works out, the sort of thing one might discover after sufficiently many similar but futile manipulations. However, it is now easy to guess the formula for $R_{2,a}(x)$. If

$$u(t) = f''(t) \quad \text{and} \quad v(t) = \frac{-(x-t)^2}{2},$$

then $v'(t) = (x - t)$, so

$$\begin{aligned}\int_a^x f''(t)(x-t)\,dt &= u(t)v(t)\Big|_a^x - \int_a^x f'''(t)\cdot\frac{-(x-t)^2}{2}\,dt\\ &= \frac{f''(a)(x-a)^2}{2} + \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt.\end{aligned}$$

This shows that

$$R_{2,a}(x) = \int_a^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt.$$

You should now have little difficulty giving a rigorous proof, by induction, that if $f^{(n+1)}$ is continuous on $[a, x]$, then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$$

This formula is called the integral form of the remainder, and we can easily estimate it in terms of estimates for $f^{(n+1)}/n!$ on $[a, x]$. If m and M are the minimum and maximum of $f^{(n+1)}/n!$ on $[a, x]$, then $R_{n,a}(x)$ satisfies

$$m\int_a^x (x-t)^n\,dt \le R_{n,a}(x) \le M\int_a^x (x-t)\,dt,$$

so we can write

$$R_{n,a}(x) = \alpha \cdot \frac{(x-a)^{n+1}}{n+1}$$

for some number α between m and M. Since we've assumed that $f^{(n+1)}$ is continuous, this means that for some t in (a, x) we can also write

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}\,\frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}}{(n+1)!}(x-a)^{n+1},$$

which is called the Lagrange form of the remainder (these manipulations will look familiar to those who have done Problem 13-23).

The Lagrange form of the remainder is the one we will need in almost all cases, and we can even give a proof that doesn't require $f^{(n+1)}$ to be continuous (a refinement admittedly of little importance in most applications, where we often assume that f has derivatives of all orders). This is the form of the remainder that we will choose in our statement of the next theorem (Taylor's Theorem).

LEMMA Suppose that the function R is $(n+1)$-times differentiable on $[a, b]$, and

$$R^{(k)}(a) = 0 \quad \text{for } k = 0, 1, 2, \dots, n.$$

Then for any x in $(a, b]$ we have

$$\frac{R(x)}{(x-a)^{n+1}} = \frac{R^{(n+1)}(t)}{(n+1)!} \quad \text{for some } t \text{ in } (a, x).$$

PROOF For $n = 0$, this is just the Mean Value Theorem, and we will prove the theorem for all n by induction on n. To do this we use the Cauchy Mean Value Theorem to write

$$\frac{R(x)}{(x-a)^{n+2}} = \frac{R'(z)}{(n+2)(z-a)^{n+1}} = \frac{1}{n+2}\,\frac{R'(z)}{(z-a)^{n+1}} \quad \text{for some } z \text{ in } (a, x),$$

and then apply the induction hypothesis *to R' on the interval* $[a, z]$ to get

$$\begin{aligned}\frac{R(x)}{(x-a)^{n+2}} &= \frac{1}{n+2}\,\frac{(R')^{(n+1)}(t)}{(n+1)!} \quad \text{for some } t \text{ in } (a, z)\\ &= \frac{R^{(n+2)}(t)}{(n+2)!}. \ \blacksquare\end{aligned}$$

THEOREM 4 (TAYLOR'S THEOREM) Suppose that $f', \dots, f^{(n+1)}$ are defined on $[a, x]$, and that $R_{n,a}(x)$ is defined by

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Then

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } t \text{ in } (a, x)$$

(Lagrange form of the remainder).

PROOF The function $R_{n,a}$ satisfies the conditions of the Lemma by the very definition of the Taylor polynomial, so

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{{R_{n,a}}^{(n+1)}(t)}{(n+1)!}$$

for some t in (a, x). But

$${R_{n,a}}^{(n+1)} = f^{(n+1)},$$

since $R_{n,a} - f$ is a polynomial of degree n. ∎

Applying Taylor's Theorem to the functions sin, cos, and exp, with $a = 0$, we obtain the following formulas:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \frac{\sin^{(2n+2)}(t)}{(2n+2)!}x^{2n+2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \frac{\cos^{(2n+1)}(t)}{(2n+1)!}x^{2n+1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^t}{(n+1)!}x^{n+1}$$

(of course, we could actually go one power higher in the remainder terms for sin and cos).

Estimates for the first two are especially easy. Since

$$|\sin^{(2n+2)}(t)| \le 1 \quad \text{for all } t,$$

we have

$$\left| \frac{\sin^{(2n+2)}(t)}{(2n+2)!}x^{2n+2} \right| \le \frac{|x|^{2n+2}}{(2n+2)!}.$$

Similarly, we can show that

$$\left| \frac{\cos^{(2n+1)}(t)}{(2n+1)!}x^{2n+1} \right| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

These estimates are particularly interesting, because (as proved in Chapter 16) for any $\varepsilon > 0$ we can make

$$\frac{x^n}{n!} < \varepsilon$$

by choosing n large enough (how large n must be will depend on x). This enables us to compute $\sin x$ to any degree of accuracy desired simply by evaluating the proper Taylor polynomial $P_{n,0}(x)$. For example, suppose we wish to compute $\sin 2$ with an error of less than 10^{-4}. Since

$$\sin 2 = P_{2n+1,0}(2) + R, \quad \text{where } |R| \le \frac{2^{2n+2}}{(2n+2)!},$$

we can use $P_{2n+1,0}(2)$ as our answer, provided that

$$\frac{2^{2n+2}}{(2n+2)!} < 10^{-4}.$$

A number n with this property can be found by a straightforward search—it obviously helps to have a table of values for $n!$ and 2^n (see page 432). In this case it happens that $n = 5$ works, so that

$$\begin{aligned}\sin 2 &= P_{11,0}(2) + R\\ &= 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \frac{2^{11}}{11!} + R,\\ &\qquad \text{where } |R| < 10^{-4}.\end{aligned}$$

It is even easier to calculate $\sin 1$ approximately, since

$$\sin 1 = P_{2n+1,0}(1) + R, \quad \text{where } |R| < \frac{1}{(2n+2)!}.$$

To obtain an error less than ε we need only find an n such that

$$\frac{1}{(2n+2)!} < \varepsilon,$$

and this requires only a brief glance at a table of factorials. (Moreover, the individual terms of $P_{2n+1,0}(1)$ will be easier to handle.)

For very small x the estimates will be even easier. For example,

$$\sin\frac{1}{10} = P_{2n+1,0}\left(\frac{1}{10}\right) + R, \quad \text{where } |R| < \frac{1}{10^{2n+2}(2n+2)!}.$$

To obtain $|R| < 10^{-10}$ we can clearly take $n = 4$ (and we could even get away with $n = 3$). These methods can actually be used to compute tables of sin and cos; a high-speed computer can compute $P_{2n+1,0}(x)$ for many different x in almost no time at all. Nowadays, computers, and even cheap calculators, determine the values of such functions "on-the-fly", though by specialized methods that are even faster.

Estimating the remainder for e^x is only slightly harder. For simplicity assume that $x \geq 0$ (the estimates for $x \leq 0$ are obtained in Problem 15). On the interval $[0, x]$ the maximum value of e^t is e^x, since exp is increasing, so

$$R_{n,0} \leq \frac{e^x x^{n+1}}{(n+1)!}.$$

Since we already know that $e < 4$, we have

$$\frac{e^x x^{n+1}}{(n+1)!} < \frac{4^x x^{n+1}}{(n+1)!},$$

which can be made as small as desired by choosing n sufficiently large. How large n must be will depend on x (and the factor 4^x will make things more difficult). Once again, the estimates are easier for small x. If $0 \leq x \leq 1$, then

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R, \quad \text{where } 0 < R < \frac{4}{(n+1)!}.$$

In particular, if $n = 4$, then

$$0 < R < \frac{4}{5!} < \frac{1}{10},$$

so

$$\begin{aligned} e = e^1 &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + R, \quad \text{where } 0 < R < \frac{1}{10} \\ &= 2 + \frac{17}{24} \end{aligned}$$

which shows that

$$2 < e < 3.$$

(This then shows that

$$0 < R < \frac{3^x x^{n+1}}{(n+1)!},$$

allowing us to improve our estimate of R slightly.) By taking $n = 7$ you can compute that the first 3 decimals for e are

$$e = 2.718\ldots$$

(you should check that $n = 7$ does give this degree of accuracy, but it would be cruel to insist that you actually do the computations).

The function arctan is also important but, as you may recall, an expression for $\arctan^{(k)}(x)$ is hopelessly complicated, so that our expressions for the remainder are pretty useless. On the other hand, our derivation of the Taylor polynomial for arctan automatically provided a formula for the remainder:

$$\arctan x = x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt.$$

As we have already estimated,

$$\left| \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \, dt \right| \le \left| \int_0^x t^{2n+2} \, dt \right| = \frac{|x|^{2n+3}}{2n+3}.$$

For the moment we will consider only numbers x with $|x| \le 1$. In this case, the remainder term can clearly be made as small as desired by choosing n sufficiently large. In particular,

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + R, \quad \text{where } |R| < \frac{1}{2n+3}.$$

With this estimate it is easy to find an n which will make the remainder less than any preassigned number; on the other hand, n will usually have to be so large as to make computations hopelessly long. To obtain a remainder $< 10^{-4}$, for example, we must take $n > (10^4 - 3)/2$. This is really a shame, because $\arctan 1 = \pi/4$, so the Taylor polynomial for arctan should allow us to compute π. Fortunately, there are some clever tricks which enable us to surmount these difficulties. Since

$$|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3},$$

much smaller n's will work for only somewhat smaller x's. The trick for computing π is to express arctan 1 in terms of $\arctan x$ for smaller x; Problem 6 shows how this can be done in a convenient way.

From the calculations on page 413, we see that for $x \ge 0$ we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n}{n+1} t^{n+1}$$

where

$$\left| \frac{(-1)^n}{n+1} t^{n+1} \right| \le \frac{x^{n+1}}{n+1}$$

and there is a slightly more complicated estimate when $-1 < x < 0$ (Problem 16). For this function the remainder term can be made as small as desired by choosing n sufficiently large, provided that $-1 < x \le 1$.

The behavior of the remainder terms for arctan and $f(x) = \log(x+1)$ is quite another matter when $|x| > 1$. In this case, the estimates

$$|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3} \quad \text{for arctan},$$

$$|R_{n,0}(x)| < \frac{x^{n+1}}{n+1} \quad (x > 0) \text{ for } f,$$

are of no use, because when $|x| > 1$ the bounds x^m/m become large as m becomes large. This predicament is unavoidable, and is not just a deficiency of our estimates. It is easy to get estimates in the other direction which show that the remainders actually do remain large. To obtain such an estimate for arctan, note that if t is in $[0, x]$ (or in $[x, 0]$ if $x < 0$), then

$$1 + t^2 \le 1 + x^2 \le 2x^2, \quad \text{if } |x| \ge 1,$$

so

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt \right| \ge \frac{1}{2x^2} \left| \int_0^x t^{2n+2}\, dt \right| = \frac{|x|^{2n+1}}{4n+6}.$$

To get a similar estimate for $\log(1 + x)$, we can use the formula

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t};$$

to get

$$\log(1+x) = \int_0^x \frac{1}{1+t}\, dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\, dt.$$

If $x > 0$, then for t in $[0, x]$ we have

$$1 + t \le 1 + x \le 2x, \quad \text{if } x \ge 1,$$

so

$$\int_0^x \frac{t^n}{t+1}\, dt \ge \frac{1}{2x} \int_0^x t^n\, dt = \frac{x^n}{2n+2}.$$

These estimates show that if $|x| > 1$, then the remainder terms become large as n becomes large. In other words, for $|x| > 1$, the Taylor polynomials for arctan and f *are of no use whatsoever in computing* $\arctan x$ *and* $\log(x + 1)$. This is no tragedy, because the values of these functions can be found for any x once they are known for all x with $|x| < 1$.

This same situation occurs in a spectacular way for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

We have already seen that $f^{(k)}(0) = 0$ for every natural number k. This means that the Taylor polynomial $P_{n,0}$ for f is

$$\begin{aligned} P_{n,0}(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 0. \end{aligned}$$

In other words, the remainder term $R_{n,0}(x)$ always equals $f(x)$, and the Taylor polynomial is useless for computing $f(x)$, except for $x = 0$. Eventually we will be able to offer some explanation for the behavior of this function, which is such a disconcerting illustration of the limitations of Taylor's Theorem.

The word "compute" has been used so often in connection with our estimates for the remainder term, that the significance of Taylor's Theorem might be misconstrued. It is true that Taylor's Theorem can be used as a computational aid (despite its ignominious failure in the previous example), but it has even more important theoretical consequences. Most of these will be developed in succeeding chapters, but two proofs will illustrate some ways in which Taylor's Theorem may be used. The first illustration will be particularly impressive to those who have waded through the proof, in Chapter 16, that π is irrational.

THEOREM 5 e is irrational.

PROOF We know that, for any n,

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \quad \text{where } 0 < R_n < \frac{3}{(n+1)!}.$$

Suppose that e were rational, say $e = a/b$, where a and b are positive integers. Choose $n > b$ and also $n > 3$. Then

$$\frac{a}{b} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n,$$

so

$$\frac{n!a}{b} = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!} + n!R_n.$$

Every term in this equation other than $n!R_n$ is an integer (the left side is an integer because $n > b$). Consequently, $n!R_n$ must be an integer also. But

$$0 < R_n < \frac{3}{(n+1)!},$$

so

$$0 < n!R_n < \frac{3}{n+1} < \frac{3}{4} < 1,$$

which is impossible for an integer. ∎

The second illustration is merely a straightforward demonstration of a fact proved in Chapter 15: If

$$\begin{aligned} f'' + f &= 0, \\ f(0) &= 0, \\ f'(0) &= 0, \end{aligned}$$

then $f = 0$. To prove this, observe first that $f^{(k)}$ exists for every k; in fact

$$\begin{aligned} &f^{(3)} = (f'')' = -f', \\ &f^{(4)} = (f^{(3)})' = (-f')' = -f'' = f, \\ &f^{(5)} = (f^{(4)})' = f', \\ &\text{etc.} \end{aligned}$$

This shows, not only that all $f^{(k)}$ exist, but also that there are at most 4 different ones: f, f', $-f$, $-f'$. Since $f(0) = f'(0) = 0$, all $f^{(k)}(0)$ are 0. Now Taylor's Theorem states, for any n, that

$$f(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^n$$

for some t in $[0, x]$. Each function $f^{(n+1)}$ is continuous (since $f^{(n+2)}$ exists), so for any particular x there is a number M such that

$$|f^{(n+1)}(t)| \leq M \quad \text{for } 0 \leq t \leq x, \text{ and } \textit{all } n$$

(we can add the phrase "and *all* n" because there are only four different $f^{(k)}$). Thus

$$|f(x)| \leq \frac{M|x|^{n+1}}{(n+1)!}.$$

Since this is true for every n, and since $|x|^n/n!$ can be made as small as desired by choosing n sufficiently large, this shows that $|f(x)| \leq \varepsilon$ for any $\varepsilon > 0$; consequently, $f(x) = 0$.

The other uses to which Taylor's Theorem will be put in succeeding chapters are closely related to the computational considerations which have concerned us for much of this chapter. If the remainder term $R_{n,a}(x)$ can be made as small as desired by choosing n sufficiently large, then $f(x)$ can be computed to any degree of accuracy desired by using the polynomials $P_{n,a}(x)$. As we require greater and greater accuracy we must add on more and more terms. If we are willing to add up infinitely many terms (in theory at least!), then we ought to be able to ignore the remainder completely. There should be "infinite sums" like

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{if } |x| \leq 1, \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{if } -1 < x \leq 1.
\end{aligned}$$

We are almost completely prepared for this step. Only one obstacle remains—we have never even defined an infinite sum. Chapters 22 and 23 contain the necessary definitions.

PROBLEMS

1. Find the Taylor polynomials (of the indicated degree, and at the indicated point) for the following functions.

(i) $f(x) = e^{e^x}$; degree 3, at 0.

(ii) $f(x) = e^{\sin x}$ degree 3, at 0.

(iii) sin; degree $2n$, at $\dfrac{\pi}{2}$.

(iv) cos; degree $2n$, at π.

(v) exp; degree n, at 1.

(vi) log; degree n, at 2.

(vii) $f(x) = x^5 + x^3 + x$; degree 4, at 0.

(viii) $f(x) = x^5 + x^3 + x$; degree 4, at 1.

(ix) $f(x) = \dfrac{1}{1+x^2}$; degree $2n+1$, at 0.

(x) $f(x) = \dfrac{1}{1+x}$; degree n, at 0.

2. Write each of the following polynomials in x as a polynomial in $(x-3)$. (It is only necessary to compute the Taylor polynomial at 3, of the same degree as the original polynomial. Why?)

(i) $x^2 - 4x - 9$.

(ii) $x^4 - 12x^3 + 44x^2 + 2x + 1$.

(iii) x^5.

(iv) $ax^2 + bx + c$.

3. Write down a sum (using $\sum$ notation) which equals each of the following numbers to within the specified accuracy. To minimize needless computation, consult the tables for 2^n and $n!$ on the next page.

(i) $\sin 1$; error $< 10^{-17}$.

(ii) $\sin 2$; error $< 10^{-12}$.

(iii) $\sin \frac{1}{2}$; error $< 10^{-20}$.

(iv) e; error $< 10^{-4}$.

(v) e^2; error $< 10^{-5}$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720
7	128	5,040
8	256	40,320
9	512	362,880
10	1,024	3,628,800
11	2,048	39,916,800
12	4,096	479,001,600
13	8,192	6,227,020,800
14	16,384	87,178,291,200
15	32,768	1,307,674,368,000
16	65,536	20,922,789,888,000
17	131,072	355,687,428,096,000
18	262,144	6,402,373,705,728,000
19	524,888	121,645,100,408,832,000
20	1,048,576	2,432,902,008,176,640,000

***4.** This problem is similar to the previous one, except that the errors demanded are so small that the tables cannot be used. You will have to do a little thinking, and in some cases it may be necessary to consult the proof, in Chapter 16, that $x^n/n!$ can be made small by choosing n large—the proof actually provides a method for finding the appropriate n. In the previous problem it was possible to find rather short sums; in fact, it was possible to find the smallest n which makes the estimate of the remainder given by Taylor's Theorem less than the desired error. But in this problem, finding *any* specific sum is a moral victory (provided you can demonstrate that the sum works).

(i) $\sin 1$; error $< 10^{-(10^{10})}$.
(ii) e; error $< 10^{-1,000}$.
(iii) $\sin 10$; error $< 10^{-20}$.
(iv) e^{10}; error $< 10^{-30}$.
(v) $\arctan \frac{1}{10}$; error $< 10^{-(10^{10})}$.

5. (a) In Problem 11-41 you showed that the equation $x^2 = \cos x$ has precisely two solutions. Use the third degree Taylor polynomial of cos to show that the solutions are approximately $\pm\sqrt{2/3}$, and find bounds on the error. Then use the fifth degree Taylor polynomial to get a better approximation.
(b) Similarly, estimate the solutions of the equation $2x^2 = x \sin x + \cos^2 x$.

6. (a) Prove, using Problem 15-9, that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(b) Show that $\pi = 3.14159\ldots$. (Every budding mathematician should verify a few decimals of π, but the purpose of this exercise is not to set you off on an immense calculation. If the second expression in part (a) is used, the first 5 decimals for π can be computed with remarkably little work.)

7. Suppose that a_i and b_i are the coefficients in the Taylor polynomials at a of f and g, respectively. In other words, $a_i = f^{(i)}(a)/i!$ and $b_i = g^{(i)}(a)/i!$. Find the coefficients c_i of the Taylor polynomials at a of the following functions, in terms of the a_i's and b_i's.

(i) $f + g$.
(ii) fg.
(iii) f'.
(iv) $h(x) = \displaystyle\int_a^x f(t)\,dt$.
(v) $k(x) = \displaystyle\int_0^x f(t)\,dt$.

8. (a) Prove that the Taylor polynomial of $f(x) = \sin(x^2)$ of degree $4n+2$ at 0 is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Hint: If P is the Taylor polynomial of degree $2n+1$ for sin at 0, then $\sin x = P(x) + R(x)$, where $\lim\limits_{x\to 0} R(x)/x^{2n+1} = 0$. What does this imply about $\lim\limits_{x\to 0} R(x^2)/x^{4n+2}$?

(b) Find $f^{(k)}(0)$ for all k.

(c) In general, if $f(x) = g(x^m)$, find $f^{(k)}(0)$ in terms of the derivatives of g at 0.

The ideas in this problem can be extended significantly, in ways that are explored in the next three problems.

9. (a) Problem 7 (i) amounts to the equation

$$P_{n,a,f+g} = P_{n,a,f} + P_{n,a,g}.$$

Give a more direct proof by writing

$$f(x) = P_{n,a,f}(x) + R_{n,a,f}(x)$$
$$g(x) = P_{n,a,g}(x) + R_{n,a,g}(x),$$

and using the obvious fact about $R_{n,a,f} + R_{n,a,g}$.

(b) Similarly, Problem 7 (ii) could be used to show that

$$P_{n,a,fg} = [P_{n,a,f} \cdot P_{n,a,g}]_n,$$

where $[P]_n$ denotes the **truncation** of P to degree n, the sum of all terms of P of degree $\le n$ [with P written as a polynomial in $x - a$]. Again, give a more direct proof, using obvious facts about products involving terms of the form R_n.

(c) Prove that if p and q are polynomials in $x-a$ and $\lim_{x\to 0} R(x)/(x-a)^n = 0$, then

$$p(q(x) + R(x)) = p(q(x)) + \bar{R}(x)$$

where

$$\lim_{x\to 0} \bar{R}(x)/(x-a)^n = 0.$$

Also note that if p is a polynomial in $x - a$ having only terms of degree $> n$, and q is a polynomial in $x - a$ whose constant term is 0, then all terms of $p(q(x - a))$ are of degree $> n$.

(d) If $a = 0$ and $b = g(a) = 0$, then

$$P_{n,a,f\circ g} = [P_{n,b,f} \circ P_{n,a,g}]_n.$$

(Problem 8 is a special case.)

(e) The same result actually holds for all a and any value of $g(a)$. Hint: Consider $F(x) = f(x+g(a))$, $G(x) = g(x+a)$ and $H(x) = G(x)-g(a)$.

(f) If $g(a) = 0$, then

$$P_{n,a,\frac{1}{1-g}} = \left[1 + P_{n,a,g} + (P_{n,a,g})^2 + \cdots + (P_{n,a,g})^n\right]_n.$$

10. For the following applications of Problem 9, we assume $a = 0$ for simplicity, and just write $P_{n,f}$ instead of $P_{n,a,f}$.

(a) For $f(x) = e^x$ and $g(x) = \sin x$, find $P_{5,f+g}(x)$.

(b) For the same f and g, find $P_{5,fg}$.

(c) Find $P_{5,\tan}(x)$. Hint: First use Problem 9 (f) and the value of $P_{5,\cos}(x)$ to find $P_{5,1/\cos}(x)$. (Answer: $x + \frac{x^3}{3} + \frac{2x^5}{15}$)

(d) Find $P_{4,f}$ for $f(x) = e^{2x}\cos x$. (Answer: $1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4$)

(e) Find $P_{5,f}$ for $f(x) = \sin x/\cos 2x$. (Answer: $x + \frac{11}{6}x^3 + \frac{361}{120}x^5$)

(f) Find $P_{6,f}$ for $f(x) = x^3/[(1+x^2)e^x]$. (Answer: $x^3 - x^4 - \frac{1}{2}x^5 + \frac{5}{6}x^6$)

11. Calculations of this sort may be used to evaluate limits that we might otherwise try to find through laborious use of l'Hôpital's Rule. Find the following:

(a) $$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x - \sin x} = \lim_{x\to 0} \frac{N(x)}{D(x)}.$$

Hint: First find $P_{3,0,N}(x)$ and $P_{3,0,D}(x)$ for the numerator and denominator $N(x)$ and $D(x)$.

(b) $\displaystyle\lim_{x\to 0} \frac{\dfrac{e^x}{1+x} - 1 - \frac{1}{2}x^2}{x - \sin x}$.

Hint: For the term $e^x/(1+x)$, first write $1/(1+x) = 1 - x + x^2 - x^3 + \cdots$.

(c) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$.

(d) $\displaystyle\lim_{x\to 0} \frac{1-\cos(x^2)}{x^2 \sin^2 x}$.

(e) $\displaystyle\lim_{x\to 0} \frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}$.

(f) $\displaystyle\lim_{x\to 0} \frac{(\sin x)(\arctan x) - x^2}{1-\cos(x^2)}$.

12. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Starting with the Taylor polynomial of degree $2n+1$ for $\sin x$, together with the estimate for the remainder term derived on page 424, show that

$$f(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!}\right) + R_{2n,0,f}(x)$$

where

$$|R_{2n,0,f}(x)| \leq \frac{|x|^{2n+1}}{(2n+2)!},$$

and use this to conclude that

$$\int_0^1 f \approx \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx = \frac{1703}{1800} \approx .946$$

with an error of less than 10^{-3}.

13. Let

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

(a) Find the Taylor polynomial of degree n for f at 0, compute $f^{(k)}(0)$, and give an estimate for the remainder term $R_{n,0,f}$.

(b) Compute $\displaystyle\int_0^1 f$ with an error of less than 10^{-4}.

14. Estimate $\displaystyle\int_0^{0.1} \exp(x^2)\,dx$ with an error of less than 10^{-5}.

15. Prove that if $x \le 0$, then the remainder term $R_{n,0}$ for e^x satisfies

$$|R_{n,0}| \le \frac{|x|^{n+1}}{(n+1)!}.$$

16. Prove that if $-1 < x \le 0$, then the remainder term $R_{n,0}$ for $\log(1+x)$ satisfies

$$|R_{n,0}| \le \frac{|x|^{n+1}}{(1+x)(n+1)}.$$

***17.** (a) Show that if $|g'(x)| \le M|x-a|^n$ for $|x-a| < \delta$, then $|g(x) - g(a)| \le M|x-a|^{n+1}/(n+1)$ for $|x-a| < \delta$.

(b) Use part (a) to show that if $\lim_{x \to a} g'(x)/(x-a)^n = 0$, then

$$\lim_{x \to a} \frac{g(x) - g(a)}{(x-a)^{n+1}} = 0.$$

(c) Show that if $g(x) = f(x) - P_{n,a,f}(x)$, then $g'(x) = f'(x) - P_{n-1,a,f'}(x)$.

(d) Give an inductive proof of Theorem 1, without using l'Hôpital's Rule.

18. Deduce Theorem 1 as a corollary of Taylor's Theorem, with any form of the remainder. (The catch is that it will be necessary to assume one more derivative than in the hypotheses for Theorem 1.)

19. Lagrange's method for proving Taylor's Theorem used the following device. We consider a fixed number x and write

$$(*) \quad f(x) = f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + S(t)$$

for $S(t) = R_{n,t}(x)$. The notation is a tip-off that we are going to consider the right side as giving the value of some function for a given t, and then write down the fact the derivative of this function is 0, since it equals the constant function whose value is always $f(x)$. To make sure you understand the roles of x and t, check that if

$$g(t) = \frac{f^{(k)}(t)}{k!}(x-t)^k,$$

then

$$\begin{aligned} g'(t) &= \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1}(-1) + \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\ &= -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k. \end{aligned}$$

(a) Show that

$$0 = f'(t) + \left[-f'(t) + \frac{f''(t)}{1!}(x-t)\right]$$
$$+ \left[\frac{-f''(t)}{1!}(x-t) + \frac{f^{(3)}(t)}{2!}(x-t)^2\right]$$
$$+ \cdots$$
$$+ \left[\frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$
$$+ S'(t),$$

and notice that everything collapses to

$$S'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Noting that

$$S(x) = R_{n,x}(x) = 0,$$
$$S(a) = R_{n,a}(x),$$

apply the Cauchy Mean Value Theorem to the functions S and $h(t) = (x-t)^{n+1}$ on $[a,x]$ to obtain the Lagrange form of the remainder (Lagrange actually handled this part of the argument differently).

(b) Similarly, apply the regular Mean Value Theorem to S to obtain the strange hybrid formula

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a).$$

This is called the Cauchy form of the remainder.

20. Deduce the Cauchy and Lagrange forms of the remainder from the integral form on page 423, using Problem 13-23. There will be the same catch as in Problem 18.

I know of only one situation where the Cauchy form of the remainder is used. The next problem is preparation for that eventuality.

21. For every number α, and every natural number n, we define the "binomial coefficient"

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{n!},$$

and we define $\binom{\alpha}{0} = 1$, as usual. If α is not an integer, then $\binom{\alpha}{n}$ is never 0, and alternates in sign for $n > \alpha$. Show that the Taylor polynomial of degree n for $f(x) = (1+x)^\alpha$ at 0 is $P_{n,0}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k$, and that the Cauchy and Lagrange forms of the remainder are the following:

Cauchy form:

$$R_{n,0}(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{n!}x(x-t)^n(1+t)^{\alpha-n-1}$$

$$= \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{n!}x(1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n$$

$$= (n+1)\binom{\alpha}{n+1}x(1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n, \qquad t \text{ in } (0,x) \text{ or } (x,0).$$

Lagrange form:

$$R_{n,0}(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1+t)^{\alpha-n-1}$$

$$= \binom{\alpha}{n+1}x^{n+1}(1+t)^{\alpha-n-1}, \qquad t \text{ in } (0,x) \text{ or } (x,0).$$

Estimates for these remainder terms are rather difficult to handle, and are postponed to Problem 23-21.

22. (a) Suppose that f is twice differentiable on $(0,\infty)$ and that $|f(x)| \le M_0$ for all $x > 0$, while $|f''(x)| \le M_2$ for all $x > 0$. Use an appropriate Taylor polynomial to prove that for any $x > 0$ we have

$$|f'(x)| \le \frac{2}{h}M_0 + \frac{h}{2}M_2 \qquad \text{for all } h > 0.$$

(b) Show that for all $x > 0$ we have

$$|f'(x)| \le 2\sqrt{M_0 M_2}.$$

Hint: Consider the smallest value of the expression appearing in (a).

(c) If f is twice differentiable on $(0,\infty)$, f'' is bounded, and $f(x)$ approaches 0 as $x \to \infty$, then also $f'(x)$ approaches 0 as $x \to \infty$.

(d) If $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f'(x) = 0$. (Compare Problem 11-34.)

23. (a) Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

The limit on the right is called the *Schwarz second derivative* of f at a. Hint: Use the Taylor polynomial $P_{2,a}(x)$ with $x = a+h$ and with $x = a-h$.

(b) Let $f(x) = x^2$ for $x \ge 0$, and $-x^2$ for $x \le 0$. Show that

$$\lim_{h\to 0} \frac{f(0+h) + f(0-h) - 2f(0)}{h^2}$$

exists, even though $f''(0)$ does not.

(c) Prove that if f has a local maximum at a, and the Schwarz second derivative of f at a exists, then it is ≤ 0.

(d) Prove that if $f'''(a)$ exists, then

$$\frac{f'''(a)}{3} = \lim_{h \to 0} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3}.$$

24. Use the Taylor polynomial $P_{1,a,f}$, together with the remainder, to prove a weak form of Theorem 2 of the Appendix to Chapter 11: If $f'' > 0$, then the graph of f always lies above the tangent line of f, except at the point of contact.

***25.** Problem 18-43 presented a rather complicated proof that $f = 0$ if $f'' - f = 0$ and $f(0) = f'(0) = 0$. Give another proof, using Taylor's Theorem. (This problem is really a preliminary skirmish before doing battle with the general case in Problem 26, and is meant to convince you that Taylor's Theorem is a good tool for tackling such problems, even though tricks work out more neatly for special cases.)

****26.** Consider a function f which satisfies the differential equation

$$f^{(n)} = \sum_{j=0}^{n-1} a_j f^{(j)},$$

for certain numbers $a_0, \dots, a_{n-1}$. Several special cases have already received detailed treatment, either in the text or in other problems; in particular, we have found all functions satisfying $f' = f$, or $f'' + f = 0$, or $f'' - f = 0$. The trick in Problem 18-42 enables us to find many solutions for such equations, but doesn't say whether these are the only solutions. This requires a *uniqueness* result, which will be supplied by this problem. At the end you will find some (necessarily sketchy) remarks about the general solution.

(a) Derive the following formula for $f^{(n+1)}$ (let us agree that "a_{-1}" will be 0):

$$f^{(n+1)} = \sum_{j=0}^{n-1} (a_{j-1} + a_{n-1}a_j) f^{(j)}.$$

(b) Deduce a formula for $f^{(n+2)}$.

The formula in part (b) is not going to be used; it was inserted only to convince you that a general formula for $f^{(n+k)}$ is out of the question. On the other hand, as part (c) shows, it is not very hard to obtain estimates on the size of $f^{(n+k)}(x)$.

(c) Let $N = \max(1, |a_0|, \dots, |a_{n-1}|)$. Then $|a_{j-1} + a_{n-1}a_j| \le 2N^2$; this means that

$$f^{(n+1)} = \sum_{j=0}^{n-1} b_j{}^1 f^{(j)}, \quad \text{where } |b_j{}^1| \le 2N^2.$$

Show that

$$f^{(n+2)} = \sum_{j=0}^{n-1} b_j{}^2 f^{(j)}, \quad \text{where } |b_j{}^2| \le 4N^3,$$

and, more generally,

$$f^{(n+k)} = \sum_{j=0}^{n-1} b_j{}^k f^{(j)}, \quad \text{where } |b_j{}^k| \le 2^k N^{k+1}.$$

(d) Conclude from part (c) that, for any particular number x, there is a number M such that

$$|f^{(n+k)}(x)| \le M \cdot 2^k N^{k+1} \quad \text{for all } k.$$

(e) Now suppose that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Show that

$$|f(x)| \le \frac{M \cdot 2^{k+1} N^{k+2} |x|^{n+k+1}}{(n+k+1)!} \le \frac{M \cdot |2Nx|^{n+k+1}}{(n+k+1)!},$$

and conclude that $f = 0$.

(f) Show that if f_1 and f_2 are both solutions of the differential equation

$$f^{(n)} = \sum_{j=0}^{n-1} a_j f^{(j)},$$

and $f_1{}^{(j)}(0) = f_2{}^{(j)}(0)$ for $0 \le j \le n-1$, then $f_1 = f_2$.

In other words, the solutions of this differential equation are determined by the "initial conditions" (the values $f^{(j)}(0)$ for $0 \le j \le n-1$). This means that we can find *all* solutions once we can find enough solutions to obtain any given set of initial conditions. If the equation

$$x^n - a_{n-1}x^{n-1} - \cdots - a_0 = 0$$

has n distinct roots $\alpha_1, \ldots, \alpha_n$, then any function of the form

$$f(x) = c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x}$$

is a solution, and

$$\begin{aligned} f(0) &= c_1 + \cdots + c_n, \\ f'(0) &= \alpha_1 c_1 + \cdots + \alpha_n c_n, \\ &\vdots \\ f^{(n-1)}(0) &= \alpha_1{}^{n-1} c_1 + \cdots + \alpha_n{}^{n-1} c_n. \end{aligned}$$

As a matter of fact, every solution is of this form, because we can obtain any set of numbers on the left side by choosing the c's properly, but we will not try to prove this last assertion. (It is a purely algebraic fact, which you can easily check for $n = 2$ or 3.) These remarks are also true if some of the roots are multiple roots, and even in the more general situation considered in Chapter 27.

**27. (a) Suppose that f is a continuous function on $[a, b]$ with $f(a) = f(b)$ and that for all x in (a, b) the Schwarz second derivative of f at x is 0 (Problem 23). Show that f is constant on $[a, b]$. Hint: Suppose that $f(x) > f(a)$ for some x in (a, b). Consider the function

$$g(x) = f(x) - \varepsilon(x-a)(b-x)$$

with $g(a) = g(b) = f(a)$. For sufficiently small $\varepsilon > 0$ we will have $g(x) > g(a)$, so g will have a maximum point y in (a, b). Now use Problem 23(c) (the Schwarz second derivative of $(x-a)(b-x)$ is simply its ordinary second derivative).

(b) If f is a continuous function on $[a, b]$ whose Schwarz second derivative is 0 at all points of (a, b), then f is linear.

*28. (a) Let $f(x) = x^4 \sin 1/x^2$ for $x \neq 0$, and $f(0) = 0$. Show that $f = 0$ up to order 2 at 0, even though $f''(0)$ does not exist.

This example is slightly more complex, but also slightly more impressive, than the example in the text, because both $f'(a)$ and $f''(a)$ exist for $a \neq 0$. Thus, for each number a there is another number $m(a)$ such that

$$(*) \quad f(x) = f(a) + f'(a)(x-a) + \frac{m(a)}{2}(x-a)^2 + R_a(x),$$

$$\text{where } \lim_{x \to a} \frac{R_a(x)}{(x-a)^2} = 0;$$

namely, $m(a) = f''(a)$ for $a \neq 0$, and $m(0) = 0$. Notice that the function m defined in this way is not continuous.

(b) Suppose that f is a differentiable function such that $(*)$ holds for all a, with $m(a) = 0$. Use Problem 27 to show that $f''(a) = m(a) = 0$ for all a.

(c) Now suppose that $(*)$ holds for all a, and that m is continuous. Prove that for all a the second derivative $f''(a)$ exists and equals $m(a)$.

*CHAPTER 21 e IS TRANSCENDENTAL

The irrationality of e was so easy to prove that in this optional chapter we will attempt a more difficult feat, and prove that the number e is not merely irrational, but actually much worse. Just how a number might be even worse than irrational is suggested by a slight rewording of definitions. A number x is irrational if it is not possible to write $x = a/b$ for any integers a and b, with $b \neq 0$. This is the same as saying that x does not satisfy any equation

$$bx - a = 0$$

for integers a and b, except for $a = 0$, $b = 0$. Viewed in this light, the irrationality of $\sqrt{2}$ does not seem to be such a terrible deficiency; rather, it appears that $\sqrt{2}$ just barely manages to be irrational—although $\sqrt{2}$ is not the solution of an equation

$$a_1x + a_0 = 0,$$

it *is* the solution of the equation

$$x^2 - 2 = 0,$$

of one higher degree. Problem 2-18 shows how to produce many irrational numbers x which satisfy higher-degree equations

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

where the a_i are integers not all 0. A number which satisfies an "algebraic" equation of this sort is called an **algebraic number**, and practically every number we have ever encountered is defined in terms of solutions of algebraic equations (π and e are the great exceptions in our limited mathematical experience). All roots, such as

$$\sqrt{2}, \quad \sqrt[10]{3}, \quad \sqrt[4]{7},$$

are clearly algebraic numbers, and even complicated combinations, like

$$\sqrt[3]{3 + \sqrt{5} + \sqrt[4]{1 + \sqrt{2}} + \sqrt[5]{6}}$$

are algebraic (although we will not try to prove this). Numbers which cannot be obtained by the process of solving algebraic equations are called **transcendental**; the main result of this chapter states that e is a number of this anomalous sort.

The proof that e is transcendental is well within our grasp, and was theoretically possible even before Chapter 20. Nevertheless, with the inclusion of this proof, we can justifiably classify ourselves as something more than novices in the study of higher mathematics; while many irrationality proofs depend only on elementary properties of numbers, the proof that a number is transcendental usually involves some really high-powered mathematics. Even the dates connected with the transcendence of e are impressively recent—the first proof that e is transcendental, due to Hermite, dates from 1873. The proof that we will give is a simplification, due to Hilbert.

Before tackling the proof itself, it is a good idea to map out the strategy, which depends on an idea used even in the proof that e is irrational. Two features of the expression

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n$$

were important for the proof that e is irrational: On the one hand, the number

$$1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$$

can be written as a fraction p/q with $q \le n!$ (so that $n!\,(p/q)$ is an integer); on the other hand, $0 < R_n < 3/(n+1)!$ (so $n!R_n$ is not an integer). These two facts show that e can be approximated particularly well by rational numbers. Of course, every number x can be approximated arbitrarily closely by rational numbers—if $\varepsilon > 0$ there is a rational number r with $|x - r| < \varepsilon$; the catch, however, is that it may be necessary to allow a very large denominator for r, as large as $1/\varepsilon$ perhaps. For e we are assured that this is not the case: there is a fraction p/q within $3/(n+1)!$ of e, whose denominator q is at most $n!$. If you look carefully at the proof that e is irrational, you will see that only this fact about e is ever used. The number e is by no means unique in this respect: generally speaking, the *better* a number can be approximated by rational numbers, the *worse* it is (some evidence for this assertion is presented in Problem 3). The proof that e is transcendental depends on a natural extension of this idea: not only e, but any finite number of powers $e, e^2, \ldots, e^n$, can be simultaneously approximated especially well by rational numbers. In our proof we will begin by assuming that e is algebraic, so that

$$(*) \qquad a_n e^n + \cdots + a_1 e + a_0 = 0, \qquad a_0 \neq 0$$

for some integers $a_0, \ldots, a_n$. In order to reach a contradiction we will than find certain integers $M, M_1, \ldots, M_n$ and certain "small" numbers $\epsilon_1, \ldots, \epsilon_n$ such that

$$\begin{aligned} e^1 &= \frac{M_1 + \epsilon_1}{M}, \\ e^2 &= \frac{M_2 + \epsilon_2}{M}, \\ &\ \ \vdots \\ e^n &= \frac{M_n + \epsilon_n}{M}. \end{aligned}$$

Just how small the ϵ's must be will appear when these expressions are substituted into the assumed equation $(*)$. After multiplying through by M we obtain

$$[a_0 M + a_1 M_1 + \cdots + a_n M_n] + [\epsilon_1 a_1 + \cdots + \epsilon_n a_n] = 0.$$

The first term in brackets is an integer, and we will choose the M's so that it will necessarily be a *nonzero* integer. We will also manage to find ϵ's so small that

$$|\epsilon_1 a_1 + \cdots + \epsilon_n a_n| < \tfrac{1}{2};$$

this will lead to the desired contradiction—the sum of a nonzero integer and a number of absolute value less than $\frac{1}{2}$ cannot be zero!

As a basic strategy this is all very reasonable and quite straightforward. The remarkable part of the proof will be the way that the M's and ϵ's are defined. In order to read the proof you will need to know about the gamma function! (This function was introduced in Problem 19-40.)

THEOREM 1 e is transcendental.

PROOF Suppose there were integers $a_0, \ldots, a_n$, with $a_0 \neq 0$, such that

$$(*) \quad a_n e^n + a_{n-1} e^{n-1} + \cdots + a_0 = 0.$$

Define numbers $M, M_1, \ldots, M_n$ and $\epsilon_1, \ldots, \epsilon_n$ as follows:

$$M = \int_0^\infty \frac{x^{p-1}[(x-1)\cdot\ldots\cdot(x-n)]^p e^{-x}}{(p-1)!}\,dx,$$

$$M_k = e^k \int_k^\infty \frac{x^{p-1}[(x-1)\cdot\ldots\cdot(x-n)]^p e^{-x}}{(p-1)!}\,dx,$$

$$\epsilon_k = e^k \int_0^k \frac{x^{p-1}[(x-1)\cdot\ldots\cdot(x-n)]^p e^{-x}}{(p-1)!}\,dx.$$

The unspecified number p represents a prime number* which we will choose later. Despite the forbidding aspect of these three expressions, with a little work they will appear much more reasonable. We concentrate on M first. If the expression in brackets,

$$[(x-1)\cdot\ldots\cdot(x-n)],$$

is actually multiplied out, we obtain a polynomial

$$x^n + \cdots \pm n!$$

*The term "prime number" was defined in Problem 2-17. An important fact about prime numbers will be used in the proof, although it is not proved in this book: If p is a prime number which does not divide the integer a, and which does not divide the integer b, then p also does not divide ab. The Suggested Reading mentions references for this theorem (which is crucial in proving that the factorization of an integer into primes is unique). We will also use the result of Problem 2-17(d), that there are infinitely many primes—the reader is asked to determine at precisely which points this information is required.

with integer coefficients. When raised to the pth power this becomes an even more complicated polynomial

$$x^{np} + \cdots \pm (n!)^p.$$

Thus M can be written in the form

$$M = \sum_{\alpha=0}^{np} \frac{1}{(p-1)!} C_\alpha \int_0^\infty x^{p-1+\alpha} e^{-x}\, dx,$$

where the C_α are certain integers, and $C_0 = \pm(n!)^p$. But

$$\int_0^\infty x^k e^{-x}\, dx = k!.$$

Thus

$$M = \sum_{\alpha=0}^{np} C_\alpha \frac{(p-1+\alpha)!}{(p-1)!}.$$

Now, for $\alpha = 0$ we obtain the term

$$\pm(n!)^p \frac{(p-1)!}{(p-1)!} = \pm(n!)^p.$$

We will now consider only primes $p > n$; then this term is an integer which is *not* divisible by p. On the other hand, if $\alpha > 0$, then

$$C_\alpha \frac{(p-1+\alpha)!}{(p-1)!} = C_\alpha (p+\alpha-1)(p+\alpha-2)\cdot \ldots \cdot p,$$

which *is* divisible by p. Therefore M itself is an integer which is *not* divisible by p.

Now consider M_k. We have

$$\begin{aligned} M_k &= e^k \int_k^\infty \frac{x^{p-1}[(x-1)\cdot \ldots \cdot (x-n)]^p e^{-x}}{(p-1)!}\, dx \\ &= \int_k^\infty \frac{x^{p-1}[(x-1)\cdot \ldots \cdot (x-n)]^p e^{-(x-k)}}{(p-1)!}\, dx. \end{aligned}$$

This can be transformed into an expression looking very much like M by the substitution

$$\begin{aligned} u &= x - k \\ du &= dx. \end{aligned}$$

The limits of integration are changed to 0 and ∞, and

$$M_k = \int_0^\infty \frac{(u+k)^{p-1}[(u+k-1)\cdot \ldots \cdot u \cdot \ldots \cdot (u+k-n)]^p e^{-u}}{(p-1)!}\, du.$$

There is one very significant difference between this expression and that for M. The term in brackets contains the factor u in the kth place. Thus the pth power contains the factor u^p. This means that the entire expression

$$(u+k)^{p-1}[(u+k-1)\cdot \ldots \cdot (u+k-n)]^p$$

is a polynomial with integer coefficients, *every term of which* has degree at least p. Thus

$$M_k = \sum_{\alpha=1}^{np} \frac{1}{(p-1)!} D_\alpha \int_0^\infty u^{p-1+\alpha} e^{-u}\, du = \sum_{\alpha=1}^{np} D_\alpha \frac{(p-1+\alpha)!}{(p-1)!},$$

where the D_α are certain integers. Notice that the summation begins with $\alpha = 1$; in this case *every* term in the sum is divisible by p. Thus each M_k is an integer which *is* divisible by p.

Now it is clear that

$$e^k = \frac{M_k + \epsilon_k}{M}, \qquad k = 1, \dots, n.$$

Substituting into (*) and multiplying by M we obtain

$$[a_0 M + a_1 M_1 + \cdots + a_n M_n] + [a_1\epsilon_1 + \cdots + a_n\epsilon_n] = 0.$$

In addition to requiring that $p > n$ let us also stipulate that $p > |a_0|$. This means that both M and a_0 are not divisible by p, so $a_0 M$ is also not divisible by p. Since each M_k is divisible by p, it follows that

$$a_0 M + a_1 M_1 + \cdots + a_n M_n$$

is *not* divisible by p. In particular it is a *nonzero* integer.

In order to obtain a contradiction to the assumed equation (*), and thereby prove that e is transcendental, it is only necessary to show that

$$|a_1\epsilon_1 + \cdots + a_n\epsilon_n|$$

can be made as small as desired, by choosing p large enough; it is clearly sufficient to show that each $|\epsilon_k|$ can be made as small as desired. This requires nothing more than some simple estimates; for the remainder of the argument remember that n is a certain fixed number (the degree of the assumed polynomial equation (*)). To begin with, if $1 \le k \le n$, then

$$\begin{aligned} |\epsilon_k| &\le e^k \int_0^k \frac{|x^{p-1}[(x-1)\cdot\ldots\cdot(x-n)]^p|\, e^{-x}}{(p-1)!}\, dx \\ &\le e^n \int_0^n \frac{n^{p-1}|(x-1)\cdot\ldots\cdot(x-n)|^p e^{-x}}{(p-1)!}\, dx. \end{aligned}$$

Now let A be the maximum of $|(x-1)\cdot\ldots\cdot(x-n)|$ for x in $[0, n]$. Then

$$\begin{aligned} |\epsilon_k| &\le \frac{e^n n^{p-1} A^p}{(p-1)!} \int_0^n e^{-x}\, dx \\ &\le \frac{e^n n^{p-1} A^p}{(p-1)!} \int_0^\infty e^{-x}\, dx \\ &= \frac{e^n n^{p-1} A^p}{(p-1)!} \\ &\le \frac{e^n n^p A^p}{(p-1)!} = \frac{e^n (nA)^p}{(p-1)!}. \end{aligned}$$

But n and A are fixed; thus $(nA)^p/(p-1)!$ can be made as small as desired by making p sufficiently large. ▮

This proof, like the proof that π is irrational, deserves some philosophic afterthoughts. At first sight, the argument seems quite "advanced"—after all, we use integrals, and integrals from 0 to ∞ at that. Actually, as many mathematicians have observed, integrals can be eliminated from the argument completely; the only integrals essential to the proof are of the form

$$\int_0^\infty x^k e^{-x}\, dx$$

for integral k, and these integrals can be replaced by $k!$ whenever they occur. Thus M, for example, could have been defined initially as

$$M = \sum_{\alpha=0}^{np} C_\alpha \frac{(p-1+\alpha)!}{(p-1)!},$$

where C_α are the coefficients of the polynomial

$$[(x-1)\cdot \ldots \cdot (x-n)]^p.$$

If this idea is developed consistently, one obtains a "completely elementary" proof that e is transcendental, depending only on the fact that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Unfortunately, this "elementary" proof is harder to understand than the original one—the whole structure of the proof must be hidden just to eliminate a few integral signs! This situation is by no means peculiar to this specific theorem—"elementary" arguments are frequently more difficult than "advanced" ones. Our proof that π is irrational is a case in point. You probably remember nothing about this proof except that it involves quite a few complicated functions. There is actually a more advanced, but much more conceptual proof, which shows that π is *transcendental*, a fact which is of great historical, as well as intrinsic, interest. One of the classical problems of Greek mathematics was to construct, with compass and straightedge alone, a square whose area is that of a circle of radius 1. This requires the construction of a line segment whose length is $\sqrt{\pi}$, which can be accomplished if a line segment of length π is constructible. The Greeks were totally unable to decide whether such a line segment could be constructed, and even the full resources of modern mathematics were unable to settle this question until 1882. In that year Lindemann proved that π is transcendental; since the length of any segment that can be constructed with straightedge and compass can be written in terms of $+$, $\cdot$, $-$, $\div$, and $\sqrt{}$, and is therefore algebraic, this proves that a line segment of length π cannot be constructed.

The proof that π is transcendental requires a sizable amount of mathematics which is too advanced to be reached in this book. Nevertheless, the proof is not much more difficult than the proof that e is transcendental. In fact, the proof for π is practically the same as the proof for e. This last statement should certainly surprise you. The proof that e is transcendental seems to depend so thoroughly on particular properties of e that it is almost inconceivable how any modifications could ever be used for π; after all, what does e have to do with π? Just wait and see!

PROBLEMS

1. (a) Prove that if $\alpha > 0$ is algebraic, then $\sqrt{\alpha}$ is algebraic.
(b) Prove that if α is algebraic and r is rational, then $\alpha + r$ and αr are algebraic.

Part (b) can actually be strengthened considerably: the sum, product, and quotient of algebraic numbers is algebraic. This fact is too difficult for us to prove here, but some special cases can be examined:

2. Prove that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2}(1 + \sqrt{3})$ are algebraic, by actually finding algebraic equations which they satisfy. (You will need equations of degree 4.)

***3.** (a) Let α be an algebraic number which is not rational. Suppose that α satisfies the polynomial equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

and that no polynomial function of lower degree has this property. Show that $f(p/q) \neq 0$ for any rational number p/q. Hint: Use Problem 3-7(b).
(b) Now show that $|f(p/q)| \geq 1/q^n$ for all rational numbers p/q with $q > 0$. Hint: Write $f(p/q)$ as a fraction over the common denominator q^n.
(c) Let $M = \sup\{|f'(x)| : |x - \alpha| < 1\}$. Use the Mean Value Theorem to prove that if p/q is a rational number with $|\alpha - p/q| < 1$, then $|\alpha - p/q| > 1/Mq^n$. (It follows that for $c = \min(1, 1/M)$ we have $|\alpha - p/q| > c/q^n$ for all rational p/q.)

***4.** Let

$$\alpha = 0.110001000000000000000001000\ldots,$$

where the 1's occur in the $n!$ place, for each n. Use Problem 3 to prove that α is transcendental. (For each n, show that α is not the root of an equation of degree n.)

Although Problem 4 mentions only one specific transcendental number, it should be clear that one can easily construct infinitely many other numbers α which do not satisfy $|\alpha - p/q| > c/q^n$ for any c and n. Such numbers were first considered by Liouville (1809–1882), and the inequality in Problem 3 is often called Liouville's inequality. None of the transcendental numbers constructed in this way happens to be particularly interesting, but for a long time Liouville's transcendental numbers were the only ones known. This situation was changed quite radically by the work of Cantor (1845–1918), who showed, without exhibiting a single transcendental number, that *most* numbers are transcendental. The next two problems provide an introduction to the ideas that allow us to make sense of such statements. The basic definition with which we must work is the following: A set A is called **countable** if its elements can be arranged in a sequence

$$a_1, a_2, a_3, a_4, \ldots.$$

The obvious example (in fact, more or less the Platonic ideal of) a countable set is $\mathbf{N}$, the set of natural numbers; clearly the set of even natural numbers is also countable:

$$2, 4, 6, 8, \ldots.$$

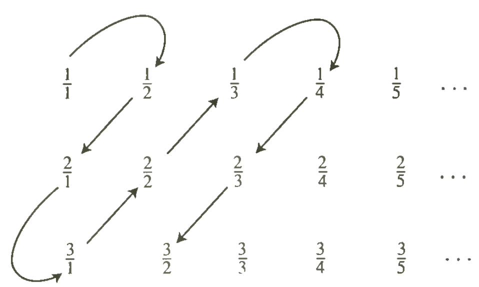
It is a little more surprising to learn that $\mathbf{Z}$, the set of all integers (positive, negative and 0) is also countable, but seeing is believing:

$$0, 1, -1, 2, -2, 3, -3, \ldots.$$

The next two problems, which outline the basic features of countable sets, are really a series of examples to show that (1) a lot more sets are countable than one might think and (2) nevertheless, some sets are not countable.

***5.** (a) Show that if A and B are countable, then so is $A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\}$. Hint: Use the same trick that worked for $\mathbf{Z}$.

(b) Show that the set of positive rational numbers is countable. (This is really quite startling, but the figure below indicates the path to enlightenment.)



(c) Show that the set of all pairs (m, n) of integers is countable. (This is practically the same as part (b).)

(d) If $A_1, A_2, A_3, \ldots$ are each countable, prove that

$$A_1 \cup A_2 \cup A_3 \cup \ldots$$

is also countable. (Again use the same trick as in part (b).)

(e) Prove that the set of all triples (l, m, n) of integers is countable. (A triple (l, m, n) can be described by a pair (l, m) and a number n.)

(f) Prove that the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ is countable. (If you have done part (e), you can do this, using induction.)

(g) Prove that the set of all roots of polynomial functions of degree n with integer coefficients is countable. (Part (f) shows that the set of all these polynomial functions can be arranged in a sequence, and each has at most n roots.)

(h) Now use parts (d) and (g) to prove that the set of all algebraic numbers is countable.

***6.** Since so many sets turn out to be countable, it is important to note that the set of all real numbers between 0 and 1 is *not* countable. In other words, there is no way of listing all these real numbers in a sequence

$$\begin{aligned} \alpha_1 &= 0.a_{11}a_{12}a_{13}a_{14}\dots \\ \alpha_2 &= 0.a_{21}a_{22}a_{23}a_{24}\dots \\ \alpha_3 &= 0.a_{31}a_{32}a_{33}a_{34}\dots \\ &\dots \end{aligned}$$

(decimal notation is being used on the right). To prove that this is so, suppose such a list were possible and consider the decimal

$$0.\bar{a}_{11}\bar{a}_{22}\bar{a}_{33}\bar{a}_{44}\dots,$$

where $\bar{a}_{nn} = 5$ if $a_{nn} \neq 5$ and $\bar{a}_{nn} = 6$ if $a_{nn} = 5$. Show that this number cannot possibly be in the list, thus obtaining a contradiction.

Problems 5 and 6 can be summed up as follows. The set of algebraic numbers is countable. If the set of transcendental numbers were also countable, then the set of all real numbers would be countable, by Problem 5(a), and consequently the set of real numbers between 0 and 1 would be countable. But this is false. Thus, the set of algebraic numbers is countable and the set of transcendental numbers is not ("there are more transcendental numbers than algebraic numbers"). The remaining two problems illustrate further how important it can be to distinguish between sets which are countable and sets which are not.

***7.** Let f be a nondecreasing function on $[0, 1]$. Recall (Problem 8-8) that $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist.

(a) For any $\varepsilon > 0$ prove that there are only finitely many numbers a in $[0, 1]$ with $\lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x) > \varepsilon$. Hint: There are, in fact, at most $[f(1) - f(0)]/\varepsilon$ of them.

(b) Prove that the set of points at which f is discontinuous is countable. Hint: If $\lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x) > 0$, then it is $> 1/n$ for some natural number n.

This problem shows that a nondecreasing function is automatically continuous at most points. For differentiability the situation is more difficult to analyze and also more interesting. A nondecreasing function can fail to be differentiable at a set of points which is not countable, but it is still true that nondecreasing functions are differentiable at most points (in a different sense of the word "most"). Reference [38] of the Suggested Reading gives a proof using the Rising Sun Lemma of Problem 8-20.

For those who have done Problem 9 of the Appendix to Chapter 11, it is possible to provide at least one application to differentiability of the ideas already developed in this problem set: If f is convex, then f is differentiable except at those points where its right-hand derivative $f_+{}'$ is discontinuous; but the function $f_+{}'$ is increasing, so a convex function is automatically differentiable except at a countable set of points.

***8.** (a) Problem 11-70 showed that if every point is a local maximum point for a *continuous* function f, then f is a constant function. Suppose now that the hypothesis of continuity is dropped. Prove that f takes on only a countable set of values. Hint: For each x choose *rational* numbers a_x and b_x such that $a_x < x < b_x$ and x is a maximum point for f on (a_x, b_x). Then every value $f(x)$ is the maximum value of f on some interval (a_x, b_x). How many such intervals are there?

(b) Deduce Problem 11-70(a) as a corollary.

(c) Prove the result of Problem 11-70(b) similarly.

CHAPTER 22 INFINITE SEQUENCES

The idea of an infinite sequence is so natural a concept that it is tempting to dispense with a definition altogether. One frequently writes simply "an infinite sequence

$$a_1, a_2, a_3, a_4, a_5, \ldots,"$$

the three dots indicating that the numbers a_i continue to the right "forever." A rigorous definition of an infinite sequence is not hard to formulate, however; the important point about an infinite sequence is that for each natural number, n, there is a real number a_n. This sort of correspondence is precisely what functions are meant to formalize.

DEFINITION

An **infinite sequence** of real numbers is a function whose domain is **N**.

From the point of view of this definition, a sequence should be designated by a single letter like a, and particular values by

$$a(1), a(2), a(3), \ldots,$$

but the subscript notation

$$a_1, a_2, a_3, \ldots$$

is almost always used instead, and the sequence itself is usually denoted by a symbol like $\{a_n\}$. Thus $\{n\}$, $\{(-1)^n\}$, and $\{1/n\}$ denote the sequences α, β, and γ defined by

$$\alpha_n = n,$$
$$\beta_n = (-1)^n,$$
$$\gamma_n = \frac{1}{n}.$$

A sequence, like any function, can be graphed (Figure 1) but the graph is usually rather unrevealing, since most of the function cannot be fit on the page.

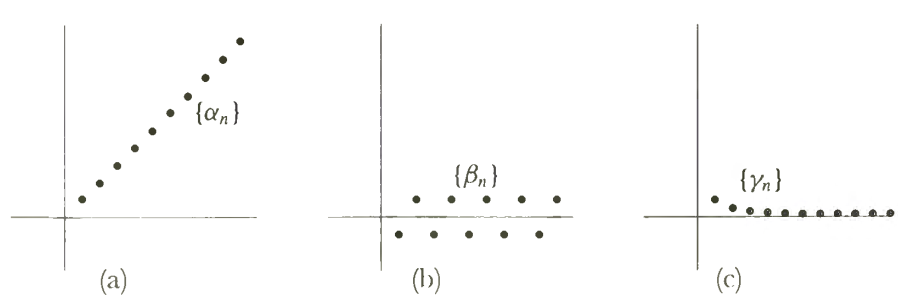


FIGURE 1

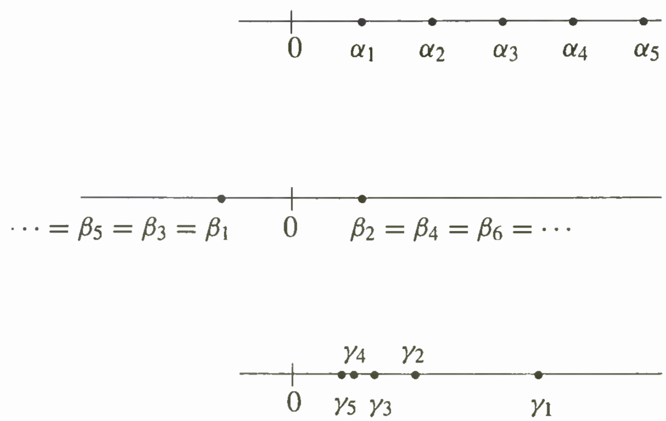


FIGURE 2

A more convenient representation of a sequence is obtained by simply labeling the points a_1, a_2, a_3, ... on a line (Figure 2). This sort of picture shows where the sequence "is going." The sequence $\{\alpha_n\}$ "goes out to infinity," the sequence $\{\beta_n\}$ "jumps back and forth between -1 and 1," and the sequence $\{\gamma_n\}$ "converges to 0." Of the three phrases in quotation marks, the last is the crucial concept associated with sequences, and will be defined precisely (the definition is illustrated in Figure 3).

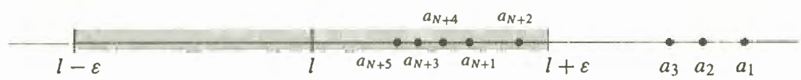


FIGURE 3

DEFINITION

A sequence $\{a_n\}$ **converges to l** (in symbols $\lim\limits_{n\to\infty} a_n = l$) if for every $\varepsilon > 0$ there is a natural number N such that, for all natural numbers n,

$$\text{if } n > N, \text{ then } |a_n - l| < \varepsilon.$$

In addition to the terminology introduced in this definition, we sometimes say that the sequence $\{a_n\}$ **approaches l** or has the **limit l**. A sequence $\{a_n\}$ is said to **converge** if it converges to l for some l, and to **diverge** if it does not converge.

To show that the sequence $\{\gamma_n\}$ converges to 0, it suffices to observe the following. If $\varepsilon > 0$, there is a natural number N such that $1/N < \varepsilon$. Then, if $n > N$ we have

$$\gamma_n = \frac{1}{n} < \frac{1}{N} < \varepsilon, \qquad \text{so } |\gamma_n - 0| < \varepsilon.$$

The limit

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$$

will probably seem reasonable after a little reflection (it just says that $\sqrt{n+1}$ is practically the same as $\sqrt{n}$ for large n), but a mathematical proof might not be so obvious. To estimate $\sqrt{n+1}-\sqrt{n}$ we can use an algebraic trick:

$$\begin{aligned}\sqrt{n+1}-\sqrt{n} &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}}.\end{aligned}$$

It is also possible to estimate $\sqrt{n+1}-\sqrt{n}$ by applying the Mean Value Theorem to the function $f(x)=\sqrt{x}$ on the interval $[n, n+1]$. We obtain

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{1} &= f'(x)\\ &= \frac{1}{2\sqrt{x}}, \qquad \text{for some } x \text{ in } (n, n+1)\\ &< \frac{1}{2\sqrt{n}}.\end{aligned}$$

Either of these estimates may be used to prove the above limit; the detailed proof is left to you, as a simple but valuable exercise.

The limit

$$\lim_{n\to\infty}\frac{3n^3+7n^2+1}{4n^3-8n+63}=\frac{3}{4}$$

should also seem reasonable, because the terms involving n^3 are the most important when n is large. If you remember the proof of Theorem 7-9 you will be able to guess the trick that translates this idea into a proof—dividing top and bottom by n^3 yields

$$\frac{3n^3+7n^2+1}{4n^3-8n+63}=\frac{3+\dfrac{7}{n}+\dfrac{1}{n^3}}{4-\dfrac{8}{n^2}+\dfrac{63}{n^3}}.$$

Using this expression, the proof of the above limit is not difficult, especially if one uses the following facts:

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist, then

$$\begin{aligned}\lim_{n\to\infty}(a_n+b_n) &= \lim_{n\to\infty} a_n+\lim_{n\to\infty} b_n,\\ \lim_{n\to\infty}(a_n\cdot b_n) &= \lim_{n\to\infty} a_n\cdot\lim_{n\to\infty} b_n;\end{aligned}$$

moreover, if $\lim\limits_{n\to\infty} b_n\neq 0$, then $b_n\neq 0$ for all n greater than some N, and

$$\lim_{n\to\infty} a_n/b_n=\lim_{n\to\infty} a_n/\lim_{n\to\infty} b_n.$$

(If we wanted to be utterly precise, the third statement would have to be even more complicated. As it stands, we are considering the limit of the sequence $\{c_n\} = \{a_n/b_n\}$, where the numbers c_n might not even be defined for certain $n < N$. This doesn't really matter—we could define c_n any way we liked for such n—because the limit of a sequence is not changed if we change the sequence at a finite number of points.)

Although these facts are very useful, we will not bother stating them as a theorem—you should have no difficulty proving these results for yourself, because the definition of $\lim_{n\to\infty} a_n = l$ is so similar to previous definitions of limits, especially $\lim_{x\to\infty} f(x) = l$.

The similarity between the definitions of $\lim_{n\to\infty} a_n = l$ and $\lim_{x\to\infty} f(x) = l$ is actually closer than mere analogy; it is possible to define the first in terms of the second. If f is the function whose graph (Figure 4) consists of line segments joining

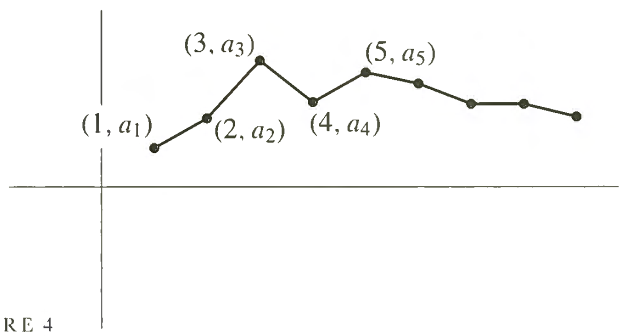


FIGURE 4

the points in the graph of the sequence $\{a_n\}$, so that

$$f(x) = (a_{n+1} - a_n)(x - n) + a_n \qquad n \le x \le n+1,$$

then

$$\lim_{n\to\infty} a_n = l \quad \text{if and only if} \quad \lim_{x\to\infty} f(x) = l.$$

Conversely, if f satisfies $\lim_{x\to\infty} f(x) = l$, and we set $a_n = f(n)$, then $\lim_{x\to\infty} a_n = l$.

This second observation is frequently very useful. For example, suppose that $0 < a < 1$. Then

$$\lim_{n\to\infty} a^n = 0.$$

To prove this we note that

$$\lim_{x\to\infty} a^x = \lim_{x\to\infty} e^{x \log a} = 0,$$

since $\log a < 0$, so that $x \log a$ is a negative and large in absolute value for large x. Notice that we actually have

$$\lim_{n\to\infty} a^n = 0 \quad \text{if } |a| < 1;$$

for if $a < 0$ we can write

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} (-1)^n |a|^n = 0.$$

The behavior of the logarithm function also shows that if $a > 1$, then a^n becomes arbitrarily large as n becomes large. This assertion is often written

$$\lim_{n\to\infty} a^n = \infty, \qquad a > 1,$$

and it is sometimes even said that $\{a^n\}$ approaches ∞. We also write equations like

$$\lim_{n\to\infty} -a^n = -\infty,$$

and say that $\{-a^n\}$ approaches $-\infty$. Notice, however, that if $a < -1$, then $\lim\limits_{n\to\infty} a^n$ does not exist, even in this extended sense.

Despite this connection with a familiar concept, it is more important to visualize convergence in terms of the picture of a sequence as points on a line (Figure 3). There is another connection between limits of functions and limits of sequences which is related to *this* picture. This connection is somewhat less obvious, but considerably more interesting, than the one previously mentioned—instead of defining limits of sequences in terms of limits of functions, it is possible to reverse the procedure.

THEOREM 1 Let f be a function defined in an open interval containing c, except perhaps at c itself, with

$$\lim_{x\to c} f(x) = l.$$

Suppose that $\{a_n\}$ is a sequence such that

$$\begin{array}{ll} (1) & \text{each } a_n \text{ is in the domain of } f, \\ (2) & \text{each } a_n \neq c, \\ (3) & \lim\limits_{n\to\infty} a_n = c. \end{array}$$

Then the sequence $\{f(a_n)\}$ satisfies

$$\lim_{n\to\infty} f(a_n) = l.$$

Conversely, if this is true for every sequence $\{a_n\}$ satisfying the above conditions, then $\lim\limits_{x\to c} f(x) = l$.

PROOF Suppose first that $\lim\limits_{x\to c} f(x) = l$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x,

$$\text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - l| < \varepsilon.$$

If the sequence $\{a_n\}$ satisfies $\lim\limits_{n\to\infty} a_n = c$, then (Figure 3) there is a natural number N such that,

$$\text{if } n > N, \text{ then } |a_n - c| < \delta.$$

By our choice of δ, this means that

$$|f(a_n) - l| < \varepsilon,$$

showing that

$$\lim_{n\to\infty} f(a_n) = l.$$

Suppose, conversely, that $\lim\limits_{n\to\infty} f(a_n) = l$ for every sequence $\{a_n\}$ with $\lim\limits_{n\to\infty} a_n = c$. If $\lim\limits_{x\to c} f(x) = l$ were *not* true, there would be some $\varepsilon > 0$ such that for *every* $\delta > 0$ there is an x with

$$0 < |x - c| < \delta \quad \text{but} \quad |f(x) - l| > \varepsilon.$$

In particular, for each n there would be a number x_n such that

$$0 < |x_n - c| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - l| > \varepsilon.$$

Now the sequence $\{x_n\}$ clearly converges to c but, since $|f(x_n) - l| > \varepsilon$ for all n, the sequence $\{f(x_n)\}$ does not converge to l. This contradicts the hypothesis, so $\lim\limits_{x\to c} f(x) = l$ must be true. ▮

Theorem 1 provides many examples of convergent sequences. For example, the sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$a_n = \sin\left(13 + \frac{1}{n^2}\right)$$
$$b_n = \cos\left(\sin\left(1 + (-1)^n \cdot \frac{1}{n}\right)\right),$$

clearly converge to $\sin(13)$ and $\cos(\sin(1))$, respectively. It is important, however, to have some criteria guaranteeing convergence of sequences which are not obviously of this sort. There is one important criterion which is very easy to prove, but which is the basis for all other results. This criterion is stated in terms of concepts defined for functions, which therefore apply also to sequences: a sequence $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$ for all n, **nondecreasing** if $a_{n+1} \geq a_n$ for all n, and **bounded above** if there is a number M such that $a_n \leq M$ for all n; there are similar definitions for sequences which are decreasing, nonincreasing, and bounded below.

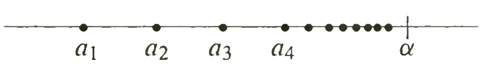


FIGURE 5

THEOREM 2 If $\{a_n\}$ is nondecreasing and bounded above, then $\{a_n\}$ converges (a similar statement is true if $\{a_n\}$ is nonincreasing and bounded below).

PROOF The set A consisting of all numbers a_n is, by assumption, bounded above, so A has a least upper bound α. We claim that $\lim\limits_{n\to\infty} a_n = \alpha$ (Figure 5). In fact, if $\varepsilon > 0$, there is some a_N satisfying $\alpha - a_N < \varepsilon$, since α is the least upper bound of A. Then if $n > N$ we have

$$a_n \geq a_N, \quad \text{so} \quad \alpha - a_n \leq \alpha - a_N < \varepsilon.$$

This proves that $\lim\limits_{n\to\infty} a_n = \alpha$. ▮

The hypothesis that $\{a_n\}$ is bounded above is clearly essential in Theorem 2: if $\{a_n\}$ is not bounded above, then (whether or not $\{a_n\}$ is nondecreasing) $\{a_n\}$ clearly diverges. Upon first consideration, it might appear that there should be little trouble deciding whether or not a given nondecreasing sequence $\{a_n\}$ is bounded above, and consequently whether or not $\{a_n\}$ converges. In the next chapter such sequences will arise very naturally and, as we shall see, deciding whether or not they converge is hardly a trivial matter. For the present, you might try to decide whether or not the following (obviously increasing) sequence is bounded above:

$$1,\ 1+\tfrac{1}{2},\ 1+\tfrac{1}{2}+\tfrac{1}{3},\ 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4},\ \ldots .$$

Although Theorem 2 treats only a very special class of sequences, it is more useful than might appear at first, because it is always possible to extract from an arbitrary sequence $\{a_n\}$ another sequence which is either nonincreasing or else nondecreasing. To be precise, let us define a **subsequence** of the sequence $\{a_n\}$ to be a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

where the n_j are natural numbers with

$$n_1 < n_2 < n_3 \cdots .$$

Then every sequence contains a subsequence which is either nondecreasing or nonincreasing. It is possible to become quite befuddled trying to prove this assertion, although the proof is very short if you think of the right idea; it is worth recording as a lemma.

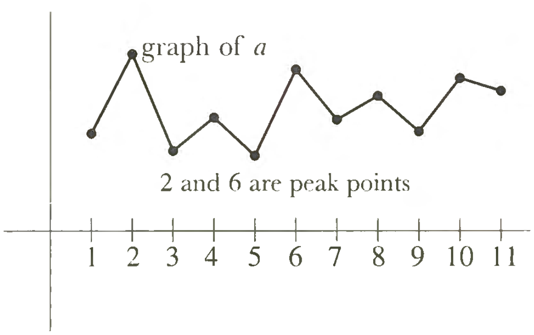


FIGURE 6

LEMMA Any sequence $\{a_n\}$ contains a subsequence which is either nondecreasing or nonincreasing.

PROOF Call a natural number n a "peak point" of the sequence $\{a_n\}$ if $a_m < a_n$ for all $m > n$ (Figure 6).

Case 1. The sequence has infinitely many peak points. In this case, if $n_1 < n_2 < n_3 < \cdots$ are the peak points, then $a_{n_1} > a_{n_2} > a_{n_3} > \cdots$, so $\{a_{n_k}\}$ is the desired (nonincreasing) subsequence.

Case 2. The sequence has only finitely many peak points. In this case, let n_1 be greater than all peak points. Since n_1 is not a peak point, there is some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Since n_2 is not a peak point (it is greater than n_1, and hence greater than all peak points) there is some $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continuing in this way we obtain the desired (nondecreasing) subsequence. ▮

If we assume that our original sequence $\{a_n\}$ is bounded, we can pick up an extra corollary along the way.

COROLLARY (THE BOLZANO-WEIERSTRASS THEOREM) Every bounded sequence has a convergent subsequence.

Without some additional assumptions this is as far as we can go: it is easy to construct sequences having many, evenly infinitely many, subsequences converging to different numbers (see Problem 3). There is a reasonable assumption to add, which yields a necessary and sufficient condition for convergence of any sequence. Although this condition will not be crucial for our work, it does simplify many proofs. Moreover, this condition plays a fundamental role in more advanced investigations, and for this reason alone it is worth stating now.

If a sequence converges, so that the individual terms are eventually all close to the same number, then the difference of any two such individual terms should be very small. To be precise, if $\lim_{n\to\infty} a_n = l$ for some l, then for any $\varepsilon > 0$ there is an N such that $|a_n - l| < \varepsilon/2$ for $n > N$; now if both $n > N$ and $m > N$, then

$$|a_n - a_m| \le |a_n - l| + |l - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This final inequality, $|a_n - a_m| < \varepsilon$, which eliminates mention of the limit l, can be used to formulate a condition (the Cauchy condition) which is clearly necessary for convergence of a sequence.

DEFINITION

A sequence $\{a_n\}$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there is a natural number N such that, for all m and n,

$$\text{if } m, n > N, \text{ then } |a_n - a_m| < \varepsilon.$$

(This condition is usually written $\lim_{m,n\to\infty} |a_m - a_n| = 0$.)

The beauty of the Cauchy condition is that it is also sufficient to ensure convergence of a sequence. After all our preliminary work, there is very little left to do in order to prove this.

THEOREM 3 A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.

PROOF We have already shown that $\{a_n\}$ is a Cauchy sequence if it converges. The proof of the converse assertion contains only one tricky feature: showing that every Cauchy sequence $\{a_n\}$ is bounded. If we take $\varepsilon = 1$ in the definition of a Cauchy sequence we find that there is some N such that

$$|a_m - a_n| < 1 \qquad \text{for } m, n > N.$$

In particular, this means that

$$|a_m - a_{N+1}| < 1 \qquad \text{for all } m > N.$$

Thus $\{a_m : m > N\}$ is bounded; since there are only finitely many other a_i's the whole sequence is bounded.

The corollary to the Lemma thus implies that some subsequence of $\{a_n\}$ converges.

Only one point remains, whose proof will be left to you: if a subsequence of a Cauchy sequence converges, then the Cauchy sequence itself converges. ▮

PROBLEMS

1. Verify each of the following limits.

(i) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1.$

(ii) $\lim\limits_{n\to\infty} \dfrac{n+3}{n^3+4} = 0.$

(iii) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0.$ Hint: You should at least be able to prove that $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0.$

(iv) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0.$

(v) $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1, \qquad a > 0.$

(vi) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

(vii) $\lim\limits_{n\to\infty} \sqrt[n]{n^2+n} = 1.$

(viii) $\lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n} = \max(a,b), \qquad a, b \geq 0.$

(ix) $\lim\limits_{n\to\infty} \dfrac{\alpha(n)}{n} = 0$, where $\alpha(n)$ is the number of primes which divide n. Hint: The fact that each prime is ≥ 2 gives a very simple estimate of how small $\alpha(n)$ must be.

*(x) $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} k^p}{n^{p+1}} = \dfrac{1}{p+1}.$

2. Find the following limits.

(i) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} - \dfrac{n+1}{n}.$

(ii) $\lim\limits_{n\to\infty} n - \sqrt{n+a}\sqrt{n+b}.$

(iii) $\lim\limits_{n\to\infty} \dfrac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}}.$

(iv) $\lim\limits_{n\to\infty} \dfrac{(-1)^n \sqrt{n} \sin(n^n)}{n+1}.$

(v) $\lim\limits_{n\to\infty} \dfrac{a^n - b^n}{a^n + b^n}.$

(vi) $\lim\limits_{n\to\infty} nc^n, \qquad |c| < 1.$

(vii) $\lim_{n\to\infty} \dfrac{2^{n^2}}{n!}$.

3. (a) What can be said about the sequence $\{a_n\}$ if it converges and each a_n is an integer?

(b) Find all convergent subsequences of the sequence $1, -1, 1, -1, 1, -1, \dots$. (There are infinitely many, although there are only two limits which such subsequences can have.)

(c) Find all convergent subsequences of the sequence $1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots$. (There are infinitely many limits which such subsequences can have.)

(d) Consider the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \dots.$$

For which numbers α is there a subsequence converging to α?

4. (a) Prove that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence.

(b) Prove that any subsequence of a convergent sequence converges.

5. (a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

(b) Prove that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \dots$$

converges.

(c) Find the limit. Hint: Notice that if $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2l}$, by Theorem 1.

6. Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}.$$

(a) Prove that the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

(b) Prove that they have the same limit.

7. In Problem 2-16 we saw that any rational approximation k/l to $\sqrt{2}$ can be replaced by a better approximation $(k + 2l)/(k + l)$. In particular, starting with $k = l = 1$, we obtain

$$1, \frac{3}{2}, \frac{7}{5}, \dots.$$

(a) Prove that this sequence is given recursively by

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{1 + a_n}.$$

(b) Prove that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the so-called *continued fraction expansion*

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

Hint: Consider separately the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$.

(c) Prove that for any natural numbers a and b,

$$\sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}.$$

8. Identify the function $f(x) = \lim_{n\to\infty} (\lim_{k\to\infty} (\cos n!\pi x)^{2k})$. (It has been mentioned many times in this book.)

9. Many impressive looking limits can be evaluated easily (especially by the person who makes them up), because they are really lower or upper sums in disguise. With this remark as hint, evaluate each of the following. (Warning: the list contains one red herring which can be evaluated by elementary considerations.)

(i) $$\lim_{n\to\infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^n}}{n}.$$

(ii) $$\lim_{n\to\infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^{2n}}}{n}.$$

(iii) $$\lim_{n\to\infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right).$$

(iv) $$\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right).$$

(v) $$\lim_{n\to\infty} \left(\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(n+n)^2}\right).$$

(vi) $$\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right).$$

10. Although limits like $\lim_{n\to\infty} \sqrt[n]{n}$ and $\lim_{n\to\infty} a^n$ can be evaluated using facts about the behavior of the logarithm and exponential functions, this approach is vaguely dissatisfying, because integral roots and powers can be defined without using the exponential function. Some of the standard "elementary" arguments for such limits are outlined here; the basic tools are inequalities derived from the binomial theorem, notably

$$(1+h)^n \ge 1 + nh, \qquad \text{for } h > 0;$$

and, for part (e),

$$(1+h)^n \ge 1 + nh + \frac{n(n-1)}{2}h^2 \ge \frac{n(n-1)}{2}h^2, \qquad \text{for } h > 0.$$

(a) Prove that $\lim_{n\to\infty} a^n = \infty$ if $a > 1$, by setting $a = 1 + h$, where $h > 0$.
(b) Prove that $\lim_{n\to\infty} a^n = 0$ if $0 < a < 1$.
(c) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $a > 1$, by setting $\sqrt[n]{a} = 1+h$ and estimating h.
(d) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $0 < a < 1$.
(e) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

11. (a) Prove that a convergent sequence is always bounded.
(b) Suppose that $\lim_{n\to\infty} a_n = 0$, and that some $a_n > 0$. Prove that the set of all numbers a_n actually has a maximum member.

12. (a) Prove that

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

(b) If

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n,$$

show that the sequence $\{a_n\}$ is decreasing, and that each $a_n \geq 0$. It follows that there is a number

$$\gamma = \lim_{n\to\infty} \left(1 + \cdots + \frac{1}{n} - \log n\right).$$

This number, known as Euler's number, has proved to be quite refractory; it is not even known whether γ is rational.

13. (a) Suppose that f is increasing on $[1, \infty)$. Show that

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + \cdots + f(n).$$

(b) Now choose $f = \log$ and show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n};$$

it follows that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

This result shows that $\sqrt[n]{n!}$ is approximately n/e, in the sense that the ratio of these two quantities is close to 1 for large n. But we cannot conclude that $n!$ is close to $(n/e)^n$ in this sense; in fact, this is false. An estimate for $n!$ is very desirable, even for concrete computations, because $n!$ cannot be calculated easily even with logarithm tables. The standard (and difficult) theorem which provides the right information will be found in Problem 27-19.

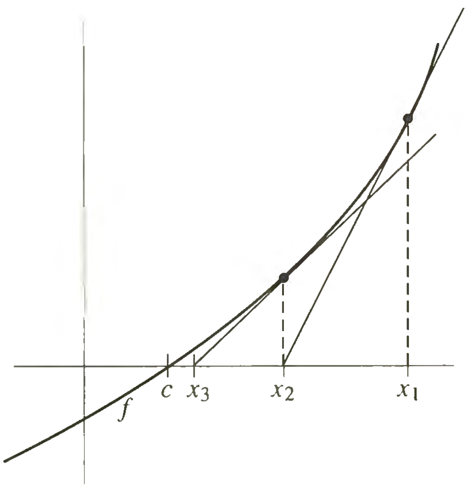


FIGURE 7

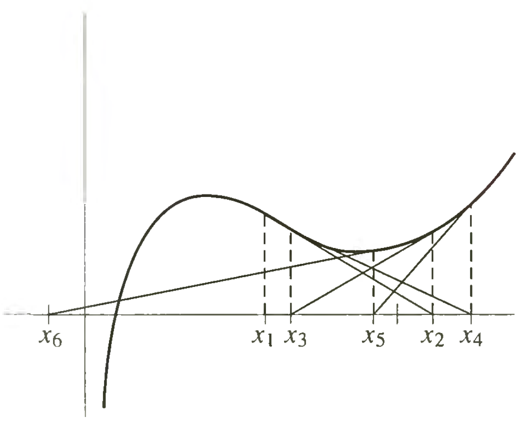


FIGURE 8

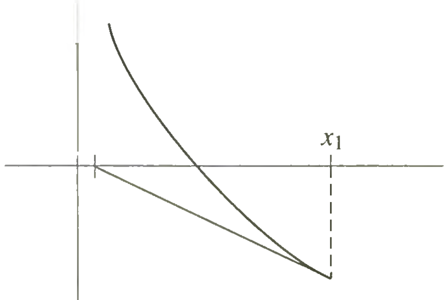


FIGURE 9

14. (a) Show that the tangent line to the graph of f at $(x_1, f(x_1))$ intersects the horizontal axis at $(x_2, 0)$, where

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

This intersection point may be regarded as a rough approximation to the point where the graph of f intersects the horizontal axis. If we now start at x_2 and repeat the process to get x_3, then use x_3 to get x_4, etc., we have a sequence $\{x_n\}$ defined inductively by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Figure 7 suggests that $\{x_n\}$ will converge to a number c with $f(c) = 0$; this is called *Newton's method* for finding a zero of f. In the remainder of this problem we will establish some conditions under which Newton's method works (Figures 8 and 9 show two cases where it doesn't). A few facts about convexity may be found useful; see Chapter 11, Appendix.

(b) Suppose that $f,' f'' > 0$, and that we choose x_1 with $f(x_1) > 0$. Show that $x_1 > x_2 > x_3 > \cdots > c$.

(c) Let $\delta_k = x_k - c$. Then

$$\delta_k = \frac{f(x_k)}{f'(\xi_k)}$$

for some ξ_k in (c, x_k). Show that

$$\delta_{k+1} = \frac{f(x_k)}{f'(\xi_k)} - \frac{f(x_k)}{f'(x_k)}.$$

Conclude that

$$\delta_{k+1} = \frac{f(x_k)}{f'(\xi_k)f'(x_k)} \cdot f''(\eta_k)(x_k - \xi_k)$$

for some η_k in (c, x_k), and then that

$$(*) \qquad \delta_{k+1} \le \frac{f''(\eta_k)}{f'(x_k)}\delta_k{}^2.$$

(d) Let $m = f'(c) = \inf f'$ on $[c, x_1]$ and let $M = \sup f''$ on $[c, x_1]$. Show that Newton's method works if $x_1 - c < m/M$.

(e) What is the formula for x_{n+1} when $f(x) = x^2 - A$?

If we take $A = 2$ and $x_1 = 1.4$ we get

$$\begin{aligned} x_1 &= 1.4 \\ x_2 &= 1.4142857 \\ x_3 &= 1.4142136 \\ x_4 &= 1.4142136, \end{aligned}$$

which is already correct to 7 decimals! Notice that the number of correct decimals at least doubled each time. This is essentially guaranteed by the inequality $(*)$ when $M/m < 1$.

15. Use Newton's method to estimate the zeros of the following functions.

(i) $f(x) = \tan x - \cos^2 x$ near 0.
(ii) $f(x) = \cos x - x^2$ near 0.
(iii) $f(x) = x^3 + x - 1$ on $[0, 1]$.
(iv) $f(x) = x^3 - 3x^2 + 1$ on $[0, 1]$.

***16.** Prove that if $\lim_{n\to\infty} a_n = l$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = l.$$

Hint: This problem is very similar to (in fact, it is actually a special case of) Problem 13-40.

17. (a) Prove that if $\lim_{n\to\infty} a_{n+1} - a_n = l$, then $\lim_{n\to\infty} a_n/n = l$. Hint: See the previous problem.
(b) Suppose that f is continuous and $\lim_{x\to\infty} f(x+1) - f(x) = l$. Prove that $\lim_{x\to\infty} f(x)/x = l$. Hint: Let a_n and b_n be the inf and sup of f on $[n, n+1]$.

***18.** Suppose that $a_n > 0$ for each n and that $\lim_{n\to\infty} a_{n+1}/a_n = l$. Prove that $\lim_{n\to\infty} \sqrt[n]{a_n} = l$. Hint: This requires the same sort of argument that works in Problem 16, except using multiplication instead of addition, together with the fact that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, for $a > 0$.

19. (a) Suppose that $\{a_n\}$ is a convergent sequence of points all in $[0, 1]$. Prove that $\lim_{n\to\infty} a_n$ is also in $[0, 1]$.
(b) Find a convergent sequence $\{a_n\}$ of points all in $(0, 1)$ such that $\lim_{n\to\infty} a_n$ is not in $(0, 1)$.

20. Suppose that f is continuous and that the sequence

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ \ldots$$

converges to l. Prove that l is a "fixed point" for f, i.e., $f(l) = l$. Hint: Two special cases have occurred already.

21. (a) Suppose that f is continuous on $[0, 1]$ and that $0 \le f(x) \le 1$ for all x in $[0, 1]$. Problem 7-11 shows that f has a fixed point (in the terminology of Problem 20). If f is *increasing*, a much stronger statement can be made: For any x in $[0, 1]$, the sequence

$$x,\ f(x),\ f(f(x)), \ldots$$

has a limit (which is necessarily a fixed point, by Problem 20). Prove this assertion, by examining the behavior of the sequence for $f(x) > x$ and $f(x) < x$, or by looking at Figure 10. A diagram of this sort is used in Littlewood's *Mathematician's Miscellany* to preach the value of drawing pictures: "For the professional the only proof needed is [this Figure]."

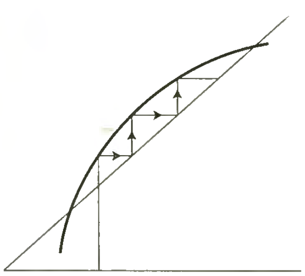


FIGURE 10

*(b) Suppose that f and g are two continuous functions on $[0, 1]$, with $0 \leq f(x) \leq 1$ and $0 \leq g(x) \leq 1$ for all x in $[0, 1]$, which satisfy $f \circ g = g \circ f$. Suppose, moreover, that f is increasing. Show that f and g have a common fixed point; in other words, there is a number l such that $f(l) = l = g(l)$. Hint: Begin by choosing a fixed point for g.

For a long time mathematicians amused themselves by asking whether the conclusion of part (b) holds without the assumption that f is increasing, but two independent announcements in the *Notices* of the American Mathematical Society, Volume 14, Number 2 give counterexamples, so it was probably a pretty silly problem all along.

The trick in Problem 20 is really much more valuable than Problem 20 might suggest, and some of the most important "fixed point theorems" depend upon looking at sequences of the form $x, f(x), f(f(x)), \ldots$. A special, but representative, case of one such theorem is treated in Problem 23 (for which the next problem is preparation).

22. (a) Use Problem 2-5 to show that if $c \neq 1$, then

$$c^m + c^{m+1} + \cdots + c^n = \frac{c^m - c^{n+1}}{1 - c}.$$

(b) Suppose that $|c| < 1$. Prove that

$$\lim_{m,n\to\infty} c^m + \cdots + c^n = 0.$$

(c) Suppose that $\{x_n\}$ is a sequence with $|x_n - x_{n+1}| \leq c^n$, where $0 < c < 1$. Prove that $\{x_n\}$ is a Cauchy sequence.

23. Suppose that f is a function on $\mathbf{R}$ such that

$$(*) \qquad |f(x) - f(y)| \leq c|x - y|, \qquad \text{for all } x \text{ and } y,$$

where $c < 1$. (Such a function is called a *contraction*.)

(a) Prove that f is continuous.
(b) Prove that f has at most one fixed point.
(c) By considering the sequence

$$x,\ f(x),\ f(f(x)),\ \ldots,$$

for any x, prove that f does have a fixed point. (This result, in a more general setting, is known as the "contraction lemma.")

24. (a) Prove that if f is differentiable and $|f'| < 1$, then f has at most one fixed point.
(b) Prove that if $|f'(x)| \leq c < 1$ for all x, then f has a fixed point.
(c) Give an example to show that the hypothesis $|f'(x)| \leq 1$ is not sufficient to ensure that f has a fixed point.

25. This problem is a sort of converse to the previous problem. Let b_n be a sequence defined by $b_1 = a$, $b_{n+1} = f(b_n)$. Prove that if $b = \lim_{n\to\infty} b_n$ exists and f' is continuous at b, then $|f'(b)| \leq 1$ (provided that we don't already have $b_n = b$ for some n). Hint: If $|f'(b)| > 1$, then $|f'(x)| > 1$ for all x in an interval around b, and b_n will be in this interval for large enough n. Now consider f on the interval $[b, b_n]$.

26. This problem investigates for which $a > 0$ the symbol

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$$

makes sense. In other words, if we define $b_1 = a$, $b_{n+1} = a^{b_n}$, when does $b = \lim_{n\to\infty} b_n$ exist? Note that if b exists, then $a^b = b$ by Problem 20.

(a) If b exists, then a can be written in the form $y^{1/y}$ for some y. Describe the graph of $g(y) = y^{1/y}$ and conclude that $0 < a \leq e^{1/e}$.
(b) Suppose that $1 \leq a \leq e^{1/e}$. Show that $\{b_n\}$ is increasing and also $b_n \leq e$. This proves that b exists (and also that $b \leq e$).

The analysis for $a < 1$ is more difficult.

(c) Using Problem 25, show that if b exists, then $e^{-1} \leq b \leq e$. Then show that $e^{-e} \leq a \leq e^{1/e}$.

From now on we will suppose that $e^{-e} \leq a < 1$.

(d) Show that the function

$$f(x) = \frac{a^x}{\log x}$$

is decreasing on the interval $(0, 1)$.
(e) Let b be the unique number such that $a^b = b$. Show that $a < b < 1$. Using part (e), show that if $0 < x < b$, then $x < a^{a^x} < b$. Conclude that $l = \lim_{n\to\infty} a_{2n+1}$ exists and that $a^{a^l} = l$.
(f) Using part (e) again, show that $l = b$.
(g) Finally, show that $\lim_{n\to\infty} a_{2n+2} = b$, so that $\lim_{n\to\infty} b_n = b$.

27. Let $\{x_n\}$ be a sequence which is bounded, and let

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

(a) Prove that the sequence $\{y_n\}$ converges. The limit $\lim_{n\to\infty} y_n$ is denoted by $\overline{\lim_{n\to\infty}}\, x_n$ or $\limsup_{n\to\infty} x_n$, and called the **limit superior**, or **upper limit**, of the sequence $\{x_n\}$.
(b) Find $\overline{\lim_{n\to\infty}}\, x_n$ for each of the following:

(i) $x_n = \dfrac{1}{n}$.

(ii) $x_n = (-1)^n \dfrac{1}{n}$.

(iii) $x_n = (-1)^n \left[1 + \dfrac{1}{n}\right]$.

(iv) $x_n = \sqrt[n]{n}$.

(c) Define $\underline{\lim}_{n\to\infty} x_n$ (or $\liminf_{n\to\infty} x_n$) and prove that

$$\underline{\lim}_{n\to\infty} x_n \le \overline{\lim}_{n\to\infty} x_n.$$

(d) Prove that $\lim_{n\to\infty} x_n$ exists if and only if $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$ and that in this case $\lim_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$.

(e) Recall the definition, in Problem 8-18, of $\overline{\lim} A$ for a bounded set A. Prove that if the numbers x_n are distinct, then $\overline{\lim}_{n\to\infty} x_n = \overline{\lim} A$, where $A = \{x_n : n \text{ in } \mathbf{N}\}$.

28. In the Appendix to Chapter 8 we defined uniform continuity of a function on an interval. If $f(x)$ is defined only for rational x, this concept still makes sense: we say that f is uniformly continuous on an interval if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, if x and y are rational numbers in the interval and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

(a) Let x be any (rational or irrational) point in the interval, and let $\{x_n\}$ be a sequence of *rational* points in the interval such that $\lim_{n\to\infty} x_n = x$. Show that the sequence $\{f(x_n)\}$ converges.

(b) Prove that the limit of the sequence $\{f(x_n)\}$ doesn't depend on the choice of the sequence $\{x_n\}$.

We will denote this limit by $\bar{f}(x)$, so that $\bar{f}$ is an extension of f to the whole interval.

(c) Prove that the extended function $\bar{f}$ is uniformly continuous on the interval.

29. Let $a > 0$, and for rational x let $f(x) = a^x$, as defined in the usual elementary algebraic way. This problem shows directly that f can be extended to a continuous function $\bar{f}$ on the whole line. Problem 28 provides the necessary machinery.

(a) For rational $x < y$, show that $a^x < a^y$ for $a > 1$ and $a^x > a^y$ for $a < 1$.

(b) Using Problem 10, show that for any $\varepsilon > 0$ we have $|a^x - 1| < \varepsilon$ for rational numbers x close enough to 0.

(c) Using the equation $a^x - a^y = a^y(a^{x-y} - 1)$, prove that on any closed interval f is uniformly continuous, in the sense of Problem 28.

(d) Show that the extended function $\bar{f}$ of Problem 28 is increasing for $a > 1$ and decreasing for $a < 1$ and satisfies $\bar{f}(x + y) = \bar{f}(x)\bar{f}(y)$.

***30.** The Bolzano-Weierstrass Theorem is usually stated, and also proved, quite differently than in the text—the classical statement uses the notion of limit points. A point x is a **limit point** of the set A if for every $\varepsilon > 0$ there is a point a in A with $|x - a| < \varepsilon$ but $x \neq a$.

(a) Find all limit points of the following sets.

(i) $\left\{\dfrac{1}{n} : n \text{ in } \mathbf{N}\right\}$.

(ii) $\left\{\dfrac{1}{n} + \dfrac{1}{m} : n \text{ and } m \text{ in } \mathbf{N}\right\}$.

(iii) $\left\{(-1)^n \left[1 + \dfrac{1}{n}\right] : n \text{ in } \mathbf{N}\right\}$.

(iv) $\mathbf{Z}$.

(v) $\mathbf{Q}$.

(b) Prove that x is a limit point of A if and only if for every $\varepsilon > 0$ there are infinitely many points a of A satisfying $|x - a| < \varepsilon$.

(c) Prove that $\overline{\lim}\, A$ is the largest limit point of A, and $\underline{\lim}\, A$ the smallest.

The usual form of the Bolzano-Weierstrass Theorem states that if A is an infinite set of numbers contained in a closed interval $[a, b]$, then some point of $[a, b]$ is a limit point of A. Prove this in two ways:

(d) Using the form already proved in the text. Hint: Since A is infinite, there are distinct numbers $x_1, x_2, x_3, \ldots$ in A.

(e) Using the Nested Intervals Theorem. Hint: If $[a, b]$ is divided into two intervals, at least one must contain infinitely many points of A.

31. (a) Use the Bolzano-Weierstrass Theorem to prove that if f is continuous on $[a, b]$, then f is bounded above on $[a, b]$. Hint: If f is not bounded above, then there are points x_n in $[a, b]$ with $f(x_n) > n$.

(b) Also use the Bolzano-Weierstrass Theorem to prove that if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ (see Chapter 8, Appendix).

****32.** (a) Let $\{a_n\}$ be the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \ldots.$$

Suppose that $0 \leq a < b \leq 1$. Let $N(n; a, b)$ be the number of integers $j \leq n$ such that a_j is in (a, b). (Thus $N(2; \frac{1}{3}, \frac{2}{3}) = 2$, and $N(4; \frac{1}{3}, \frac{2}{3}) = 3$.) Prove that

$$\lim_{n\to\infty} \frac{N(n; a, b)}{n} = b - a.$$

(b) A sequence $\{a_n\}$ of numbers in $[0, 1]$ is called **uniformly distributed** in $[0, 1]$ if

$$\lim_{n\to\infty} \frac{N(n; a, b)}{n} = b - a$$

for all a and b with $0 \le a < b \le 1$. Prove that if s is a step function defined on $[0, 1]$, and $\{a_n\}$ is uniformly distributed in $[0, 1]$, then

$$\int_0^1 s = \lim_{n\to\infty} \frac{s(a_1) + \cdots + s(a_n)}{n}.$$

(c) Prove that if $\{a_n\}$ is uniformly distributed in $[0, 1]$ and f is integrable on $[0, 1]$, then

$$\int_0^1 f = \lim_{n\to\infty} \frac{f(a_1) + \cdots + f(a_n)}{n}.$$

****33.** (a) Let f be a function defined on $[0, 1]$ such that $\lim_{y\to a} f(y)$ exists for all a in $[0, 1]$. For any $\varepsilon > 0$ prove that there are only finitely many points a in $[0, 1]$ with $|\lim_{y\to a} f(y) - f(a)| > \varepsilon$. Hint: Show that the set of such points cannot have a limit point x, by showing that $\lim_{y\to x} f(y)$ could not exist.

(b) Prove that, in the terminology of Problem 21-5, the set of points where f is discontinuous is countable. This finally answers the question of Problem 6-17: If f has only removable discontinuities, then f is continuous except at a countable set of points, and in particular, f cannot be discontinuous everywhere.

CHAPTER 23 INFINITE SERIES

Infinite sequences were introduced in the previous chapter with the specific intention of considering their "sums"

$$a_1 + a_2 + a_3 + \cdots$$

in this chapter. This is not an entirely straightforward matter, for the sum of infinitely many numbers is as yet completely undefined. What can be defined are the "partial sums"

$$s_n = a_1 + \cdots + a_n,$$

and the infinite sum must presumably be defined in terms of these partial sums. Fortunately, the mechanism for formulating this definition has already been developed in the previous chapter. If there is to be any hope of computing the infinite sum $a_1 + a_2 + a_3 + \cdots$, the partial sums s_n should represent closer and closer approximations as n is chosen larger and larger. This last assertion amounts to little more than a sloppy definition of limits: the "infinite sum" $a_1 + a_2 + a_3 + \cdots$ ought to be $\lim\limits_{n\to\infty} s_n$. This approach will necessarily leave the "sum" of many sequences undefined, since the sequence $\{s_n\}$ may easily fail to have a limit. For example, the sequence

$$1,\ -1,\ 1,\ -1,\ \ldots$$

with $a_n = (-1)^{n+1}$ yields the new sequence

$$\begin{aligned}
s_1 &= a_1 = 1,\\
s_2 &= a_1 + a_2 = 0,\\
s_3 &= a_1 + a_2 + a_3 = 1,\\
s_4 &= a_1 + a_2 + a_3 + a_4 = 0,\\
&\ldots,
\end{aligned}$$

for which $\lim\limits_{n\to\infty} s_n$ does not exist. Although there happen to be some clever extensions of the definition suggested here (see Problems 12 and 24-20) it seems unavoidable that some sequences will have no sum. For this reason, an acceptable definition of the sum of a sequence should contain, as an essential component, terminology which distinguishes sequences for which sums can be defined from less fortunate sequences.

DEFINITION

The sequence $\{a_n\}$ is **summable** if the sequence $\{s_n\}$ converges, where

$$s_n = a_1 + \cdots + a_n.$$

In this case, $\lim_{n\to\infty} s_n$ is denoted by

$$\sum_{n=1}^{\infty} a_n \qquad \text{(or, less formally, } a_1 + a_2 + a_3 + \cdots\text{)}$$

and is called the **sum** of the sequence $\{a_n\}$.

The terminology introduced in this definition is usually replaced by less precise expressions; indeed the title of this chapter is derived from such everyday language. An infinite sum $\sum_{n=1}^{\infty} a_n$ is usually called an *infinite series*, the word "series" emphasizing the connection with the infinite sequence $\{a_n\}$. The statement that $\{a_n\}$ is, or is not, summable is conventionally replaced by the statement that the series $\sum_{n=1}^{\infty} a_n$ does, or does not, converge. This terminology is somewhat peculiar, because at best the symbol $\sum_{n=1}^{\infty} a_n$ denotes a number (so it can't "converge"), and it doesn't denote anything at all unless $\{a_n\}$ is summable. Nevertheless, this informal language is convenient, standard, and unlikely to yield to attacks on logical grounds.

Certain elementary arithmetical operations on infinite series are direct consequences of the definition. It is a simple exercise to show that if $\{a_n\}$ and $\{b_n\}$ are summable, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n.$$

As yet these equations are not very interesting, since we have no examples of summable sequences (except for the trivial examples in which the terms are eventually all 0). Before we actually exhibit a summable sequence, some general conditions for summability will be recorded.

There is one necessary and sufficient condition for summability which can be stated immediately. The sequence $\{a_n\}$ is summable if and only if the sequence $\{s_n\}$ converges, which happens, according to Theorem 22-3, if and only if $\lim_{m,n\to\infty} s_m - s_n = 0$; this condition can be rephrased in terms of the original sequence as follows.

THE CAUCHY CRITERION The sequence $\{a_n\}$ is summable if and only if

$$\lim_{m,n\to\infty} a_{n+1} + \cdots + a_m = 0.$$

Although the Cauchy criterion is of theoretical importance, it is not very useful for deciding the summability of any particular sequence. However, one simple consequence of the Cauchy criterion provides a *necessary* condition for summability which is too important not to be mentioned explicitly.

THE VANISHING CONDITION If $\{a_n\}$ is summable, then

$$\lim_{n\to\infty} a_n = 0.$$

This condition follows from the Cauchy criterion by taking $m = n + 1$; it can also be proved directly as follows. If $\lim_{n\to\infty} s_n = l$, then

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= l - l = 0.\end{aligned}$$

Unfortunately, this condition is far from sufficient. For example, $\lim_{n\to\infty} 1/n = 0$, but the sequence $\{1/n\}$ is not summable; in fact, the following grouping of the numbers $1/n$ shows that the sequence $\{s_n\}$ is not bounded:

$$1 + \tfrac{1}{2} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\substack{\geq \frac{1}{2} \\ \text{(2 terms,} \\ \text{each } \geq \frac{1}{4})}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{\substack{\geq \frac{1}{2} \\ \text{(4 terms,} \\ \text{each } \geq \frac{1}{8})}} + \underbrace{\tfrac{1}{9} + \cdots + \tfrac{1}{16}}_{\substack{\geq \frac{1}{2} \\ \text{(8 terms,} \\ \text{each } \geq \frac{1}{16})}} + \cdots .$$

The method of proof used in this example, a clever trick which one might never see, reveals the need for some more standard methods for attacking these problems. These methods shall be developed soon (one of them will give an alternate proof that $\sum_{n=1}^{\infty} 1/n$ does not converge) but it will be necessary to first procure a few examples of convergent series.

The most important of all infinite series are the "geometric series"

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots .$$

Only the cases $|r| < 1$ are interesting, since the individual terms do not approach 0 if $|r| \geq 1$. These series can be managed because the partial sums

$$s_n = 1 + r + \cdots + r^n$$

can be evaluated in simple terms. The two equations

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ rs_n &= \phantom{1 + {}} r + r^2 + \cdots + r^n + r^{n+1} \end{aligned}$$

lead to

$$s_n(1-r) = 1 - r^{n+1}$$

or

$$s_n = \frac{1 - r^{n+1}}{1 - r}$$

(division by $1 - r$ is valid since we are not considering the case $r = 1$). Now $\lim_{n\to\infty} r^n = 0$, since $|r| < 1$. It follows that

$$\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1-r}, \qquad |r| < 1.$$

In particular,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 = \frac{1}{1 - \frac{1}{2}} - 1 = 1,$$

that is,

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots = 1,$$

an infinite sum which can always be remembered from the picture in Figure 1.

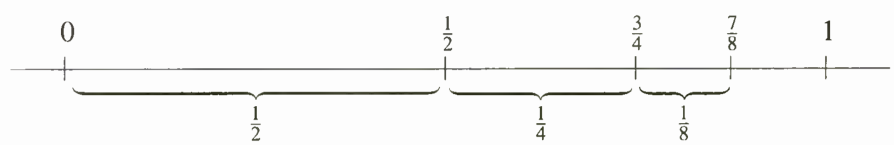


FIGURE 1

Special as they are, geometric series are standard examples from which important tests for summability will be derived.

For a while we shall consider only sequences $\{a_n\}$ with each $a_n \geq 0$; such sequences are called **nonnegative**. If $\{a_n\}$ is a nonnegative sequence, then the sequence $\{s_n\}$ is clearly nondecreasing. This remark, combined with Theorem 22-2, provides a simple-minded test for summability:

THE BOUNDEDNESS CRITERION

A nonnegative sequence $\{a_n\}$ is summable if and only if the set of partial sums s_n is bounded.

By itself, this criterion is not very helpful—deciding whether or not the set of all s_n is bounded is just what we are unable to do. On the other hand, if some convergent series are already available for comparison, this criterion can be used to obtain a result whose simplicity belies its importance (it is the basis for almost all other tests).

THEOREM 1 (THE COMPARISON TEST)

Suppose that

$$0 \leq a_n \leq b_n \qquad \text{for all } n.$$

Then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

PROOF If

$$s_n = a_1 + \cdots + a_n,$$
$$t_n = b_1 + \cdots + b_n,$$

then

$$0 \leq s_n \leq t_n \qquad \text{for all } n.$$

Now $\{t_n\}$ is bounded, since $\sum_{n=1}^{\infty} b_n$ converges. Therefore $\{s_n\}$ is bounded; consequently, by the boundedness criterion $\sum_{n=1}^{\infty} a_n$ converges. ▮

Quite frequently the comparison test can be used to analyze very complicated looking series in which most of the complication is irrelevant. For example,

$$\sum_{n=1}^{\infty} \frac{2 + \sin^3(n+1)}{2^n + n^2}$$

converges because

$$0 \leq \frac{2 + \sin^3(n+1)}{2^n + n^2} < \frac{3}{2^n},$$

and

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = 3 \sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a convergent (geometric) series.

Similarly, we would expect the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1 + \sin^2 n^3}$$

to converge, since the nth term of the series is practically $1/2^n$ for large n, and we would expect the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$$

to diverge, since $(n+1)/(n^2+1)$ is practically $1/n$ for large n. These facts can be derived immediately from the following theorem, another kind of "comparison test."

THEOREM 2 (THE LIMIT COMPARISON TEST) If $a_n, b_n > 0$ and $\lim_{n \to \infty} a_n/b_n = c \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

PROOF Suppose $\sum_{n=1}^{\infty} b_n$ converges. Since $\lim_{n\to\infty} a_n/b_n = c$, there is some N such that

$$a_n \leq 2cb_n \qquad \text{for } n \geq N.$$

But the sequence $2c\sum_{n=N}^{\infty} b_n$ certainly converges. Then Theorem 1 shows that $\sum_{n=N}^{\infty} a_n$ converges, and this implies convergence of the whole series $\sum_{n=1}^{\infty} a_n$, which has only finitely many additional terms.

The converse follows immediately, since we also have $\lim_{n\to\infty} b_n/a_n = 1/c \neq 0$. ▮

The comparison test yields other important tests when we use previously analyzed series as catalysts. Choosing the geometric series $\sum_{n=0}^{\infty} r^n$, the convergent series *par excellence*, we obtain the most important of all tests for summability.

THEOREM 3 (THE RATIO TEST) Let $a_n > 0$ for all n, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$. On the other hand, if $r > 1$, then the terms a_n are unbounded, so $\sum_{n=1}^{\infty} a_n$ diverges. (Notice that it is therefore essential to compute $\lim_{n\to\infty} a_{n+1}/a_n$ and not $\lim_{n\to\infty} a_n/a_{n+1}$!)

PROOF Suppose first that $r < 1$. Choose any number s with $r < s < 1$. The hypothesis

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$$

implies that there is some N such that

$$\frac{a_{n+1}}{a_n} \leq s \qquad \text{for } n \geq N.$$

This can be written

$$a_{n+1} \leq sa_n \qquad \text{for } n \geq N.$$

Thus

$$\begin{aligned} a_{N+1} &\leq sa_N, \\ a_{N+2} &\leq sa_{N+1} \leq s^2 a_N, \\ &\;\;\vdots \\ a_{N+k} &\leq s^k a_N. \end{aligned}$$

Since $\sum_{k=0}^{\infty} a_N s^k = a_N \sum_{k=0}^{\infty} s^k$ converges, the comparison test shows that

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} a_{N+k}$$

converges. This implies the convergence of the whole series $\sum_{n=1}^{\infty} a_n$.

The case $r > 1$ is even easier. If $1 < s < r$, then there is a number N such that

$$\frac{a_{n+1}}{a_n} \geq s \qquad \text{for } n \geq N,$$

which means that

$$a_{N+k} \geq a_N s^k \qquad k = 0, 1, \ldots,$$

so that the terms are unbounded. ▌

As a simple application of the ratio test, consider the series $\sum_{n=1}^{\infty} 1/n!$. Letting $a_n = 1/n!$ we obtain

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{1}{(n+1)!}}{\dfrac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Thus

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0,$$

which shows that the series $\sum_{n=1}^{\infty} 1/n!$ converges. If we consider instead the series $\sum_{n=1}^{\infty} r^n/n!$, where r is some fixed positive number, then

$$\lim_{n \to \infty} \frac{\dfrac{r^{n+1}}{(n+1)!}}{\dfrac{r^n}{n!}} = \lim_{n \to \infty} \frac{r}{n+1} = 0,$$

so $\sum_{n=1}^{\infty} r^n/n!$ converges. It follows that

$$\lim_{n \to \infty} \frac{r^n}{n!} = 0,$$

a result already proved in Chapter 16 (the proof given there was based on the same ideas as those used in the ratio test). Finally, if we consider the series $\sum_{n=1}^{\infty} nr^n$ we have

$$\lim_{n\to\infty} \frac{(n+1)r^{n+1}}{nr^n} = \lim_{n\to\infty} r \cdot \frac{n+1}{n} = r,$$

since $\lim_{n\to\infty} (n+1)/n = 1$. This proves that if $0 \le r < 1$, then $\sum_{n=1}^{\infty} nr^n$ converges, and consequently

$$\lim_{n\to\infty} nr^n = 0.$$

(This result clearly holds for $-1 < r \le 0$, also.) It is a useful exercise to provide a direct proof of this limit, without using the ratio test as an intermediary.

Although the ratio test will be of the utmost theoretical importance, as a practical tool it will frequently be found disappointing. One drawback of the ratio test is the fact that $\lim_{n\to\infty} a_{n+1}/a_n$ may be quite difficult to determine, and may not even exist. A more serious deficiency, which appears with maddening regularity, is the fact that the limit might equal 1. The case $\lim_{n\to\infty} a_{n+1}/a_n = 1$ is precisely the one which is inconclusive: $\{a_n\}$ might not be summable (for example, if $a_n = 1/n$), but then again it might be. In fact, our very next test will show that $\sum_{n=1}^{\infty}(1/n)^2$ converges, even though

$$\lim_{n\to\infty} \frac{\left(\dfrac{1}{n+1}\right)^2}{\left(\dfrac{1}{n}\right)^2} = 1.$$

This test provides a quite different method for determining convergence or divergence of infinite series— like the ratio test, it is an immediate consequence of the comparison test, but the series chosen for comparison is quite novel.

THEOREM 4 (THE INTEGRAL TEST) Suppose that f is positive and decreasing on $[1, \infty)$, and that $f(n) = a_n$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the limit

$$\int_1^{\infty} f = \lim_{A\to\infty} \int_1^A f$$

exists.

PROOF The existence of $\lim_{A\to\infty} \int_1^A f$ is equivalent to convergence of the series

$$\int_1^2 f + \int_2^3 f + \int_3^4 f + \cdots.$$

Now, since f is decreasing we have (Figure 2)

$$f(n+1) < \int_n^{n+1} f < f(n).$$

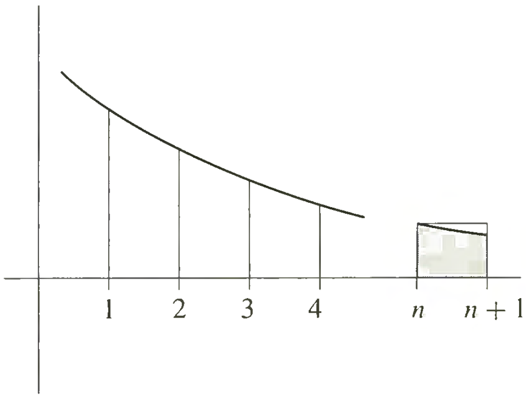


FIGURE 2

The first half of this double inequality shows that the series $\sum_{n=1}^{\infty} a_{n+1}$ may be compared to the series $\sum_{n=1}^{\infty} \int_n^{n+1} f$, proving that $\sum_{n=1}^{\infty} a_{n+1}$ (and hence $\sum_{n=1}^{\infty} a_n$) converges if $\lim_{A\to\infty} \int_1^A$ exists.

The second half of the inequality shows that the series $\sum_{n=1}^{\infty} \int_n^{n+1} f$ may be compared to the series $\sum_{n=1}^{\infty} a_n$, proving that $\lim_{A\to\infty} \int_1^A f$ must exist if $\sum_{n=1}^{\infty} a_n$ converges. ▮

Only one example using the integral test will be given here, but it settles the question of convergence for infinitely many series at once. If $p > 0$, the convergence of $\sum_{n=1}^{\infty} 1/n^p$ is equivalent, by the integral test, to the existence of

$$\int_1^{\infty} \frac{1}{x^p}\, dx.$$

Now

$$\int_1^A \frac{1}{x^p}\, dx = \begin{cases} -\dfrac{1}{(p-1)} \cdot \dfrac{1}{A^{p-1}} + \dfrac{1}{p-1}, & p \neq 1 \\ \log A, & p = 1. \end{cases}$$

This shows that $\lim_{A\to\infty} \int_1^A 1/x^p\, dx$ exists if $p > 1$, but not if $p \leq 1$. Thus $\sum_{n=1}^{\infty} 1/n^p$ converges precisely for $p > 1$. In particular, $\sum_{n=1}^{\infty} 1/n$ diverges.

The tests considered so far apply only to nonnegative sequences, but nonpositive sequences may be handled in precisely the same way. In fact, since

$$\sum_{n=1}^{\infty} a_n = -\left(\sum_{n=1}^{\infty} -a_n\right),$$

all considerations about nonpositive sequences can be reduced to questions involving nonnegative sequences. Sequences which contain both positive and negative terms are quite another story.

If $\sum_{n=1}^{\infty} a_n$ is a sequence with both positive and negative terms, one can consider instead the sequence $\sum_{n=1}^{\infty} |a_n|$, all of whose terms are nonnegative. Cheerfully ignoring the possibility that we may have thrown away all the interesting information about the original sequence, we proceed to eulogize those sequences which are converted by this procedure into convergent sequences.

DEFINITION

The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (In more formal language, the sequence $\{a_n\}$ is **absolutely summable** if the sequence $\{|a_n|\}$ is summable.)

Although we have no right to expect this definition to be of any interest, it turns out to be exceedingly important. The following theorem shows that the definition is at least not entirely useless.

THEOREM 5

Every absolutely convergent series is convergent. Moreover, a series is absolutely convergent if and only if the series formed from its positive terms and the series formed from its negative terms both converge.

PROOF

If $\sum_{n=1}^{\infty} |a_n|$ converges, then, by the Cauchy criterion,

$$\lim_{m,n\to\infty} |a_{n+1}| + \cdots + |a_m| = 0.$$

Since

$$|a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots + |a_m|,$$

it follows that

$$\lim_{m,n\to\infty} a_{n+1} + \cdots + a_m = 0,$$

which shows that $\sum_{n=1}^{\infty} a_n$ converges.

To prove the second part of the theorem, let

$$a_n{}^+ = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n \leq 0, \end{cases}$$

$$a_n{}^- = \begin{cases} a_n, & \text{if } a_n \leq 0 \\ 0, & \text{if } a_n \geq 0, \end{cases}$$

so that $\sum_{n=1}^{\infty} a_n{}^+$ is the series formed from the positive terms of $\sum_{n=1}^{\infty} a_n$, and $\sum_{n=1}^{\infty} a_n{}^-$ is the series formed from the negative terms.

If $\sum_{n=1}^{\infty} a_n{}^+$ and $\sum_{n=1}^{\infty} a_n{}^-$ both converge, then

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} [a_n{}^+ - (a_n{}^-)] = \sum_{n=1}^{\infty} a_n{}^+ - \sum_{n=1}^{\infty} a_n{}^-$$

also converges, so $\sum_{n=1}^{\infty} a_n$ converges absolutely.

On the other hand, if $\sum_{n=1}^{\infty} |a_n|$ converges, then, as we have just shown, $\sum_{n=1}^{\infty} a_n$ also converges. Therefore

$$\sum_{n=1}^{\infty} a_n{}^{+} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} |a_n|\right)$$

and

$$\sum_{n=1}^{\infty} a_n{}^{-} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} |a_n|\right)$$

both converge. ▮

It follows from Theorem 5 that every convergent series with positive terms can be used to obtain infinitely many other convergent series, simply by putting in minus signs at random. Not every convergent series can be obtained in this way, however—there are series which are convergent but not absolutely convergent (such series are called **conditionally convergent**). In order to prove this statement we need a test for convergence which applies specifically to series with positive and negative terms.

THEOREM 6 (LEIBNIZ'S THEOREM) Suppose that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0,$$

and that

$$\lim_{n\to\infty} a_n = 0.$$

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

converges.

PROOF Figure 3 illustrates relationships between the partial sums which we will establish:

(1) $s_2 \leq s_4 \leq s_6 \leq \cdots,$
(2) $s_1 \geq s_3 \geq s_5 \geq \cdots,$
(3) $s_k \leq s_l$ if k is even and l is odd.

FIGURE 3

To prove the first two inequalities, observe that

$$\begin{aligned}
(1)\quad s_{2n+2} &= s_{2n} + a_{2n+1} - a_{2n+2} \\
&\geq s_{2n}, \qquad \text{since } a_{2n+1} \geq a_{2n+2} \\
(2)\quad s_{2n+3} &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\
&\leq s_{2n+1}, \qquad \text{since } a_{2n+2} \geq a_{2n+3}.
\end{aligned}$$

To prove the third inequality, notice first that

$$\begin{aligned}
s_{2n} &= s_{2n-1} - a_{2n} \\
&\leq s_{2n-1} \qquad \text{since } a_{2n} \geq 0.
\end{aligned}$$

This proves only a special case of (3), but in conjunction with (1) and (2) the general case is easy: if k is even and l is odd, choose n such that

$$2n \geq k \quad \text{and} \quad 2n - 1 \geq l;$$

then

$$s_k \leq s_{2n} \leq s_{2n-1} \leq s_l,$$

which proves (3).

Now, the sequence $\{s_{2n}\}$ converges, because it is nondecreasing and is bounded above (by s_l for any odd l). Let

$$\alpha = \sup\{s_{2n}\} = \lim_{n\to\infty} s_{2n}.$$

Similarly, let

$$\beta = \inf\{s_{2n+1}\} = \lim_{n\to\infty} s_{2n+1}.$$

It follows from (3) that $\alpha \leq \beta$; since

$$s_{2n+1} - s_{2n} = a_{2n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0$$

it is actually the case that $\alpha = \beta$. This proves that $\alpha = \beta = \lim_{n\to\infty} s_n$. ▮

The standard example derived from Theorem 6 is the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots,$$

which is convergent, but *not* absolutely convergent (since $\sum_{n=1}^{\infty} 1/n$ does not converge). If the sum of this series is denoted by x, the following manipulations lead to quite a paradoxical result:

$$\begin{aligned}
x &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\
&= 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{7} - \tfrac{1}{14} - \tfrac{1}{16} + \cdots \\
&\quad \text{(the pattern here is one positive term followed by two negative ones)} \\
&= (1 - \tfrac{1}{2}) - \tfrac{1}{4} + (\tfrac{1}{3} - \tfrac{1}{6}) - \tfrac{1}{8} + (\tfrac{1}{5} - \tfrac{1}{10}) - \tfrac{1}{12} + (\tfrac{1}{7} - \tfrac{1}{14}) - \tfrac{1}{16} + \cdots \\
&= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{14} - \tfrac{1}{16} + \cdots \\
&= \tfrac{1}{2}(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots) \\
&= \tfrac{1}{2}x,
\end{aligned}$$

so $x = x/2$, implying that $x = 0$. On the other hand, it is easy to see that $x \neq 0$: the partial sum s_2 equals $\frac{1}{2}$, and the proof of Leibniz's Theorem shows that $x \geq s_2$.

This contradiction depends on a step which takes for granted that operations valid for finite sums necessarily have analogues for infinite sums. It is true that the sequence

$$\{b_n\} = 1,\ -\frac{1}{2},\ -\frac{1}{4},\ \frac{1}{3},\ -\frac{1}{6},\ -\frac{1}{8},\ \frac{1}{5},\ -\frac{1}{10},\ -\frac{1}{12},\ \dots$$

contains all the numbers in the sequence

$$\{a_n\} = 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ -\frac{1}{6},\ \frac{1}{7},\ -\frac{1}{8},\ \frac{1}{9},\ -\frac{1}{10},\ \frac{1}{11},\ -\frac{1}{12},\ \dots.$$

In fact, $\{b_n\}$ is a **rearrangement** of $\{a_n\}$ in the following precise sense: each $b_n = a_{f(n)}$ where f is a certain function which "permutes" the natural numbers, that is, every natural number m is $f(n)$ for precisely one n. In our example

$f(2m+1) = 3m+1$		(the terms $1, \frac{1}{3}, \frac{1}{5}, \dots$ go into the 1st, 4th, 7th, ... places),
$f(4m)$	$= 3m$	(the terms $-\frac{1}{4}, -\frac{1}{8}, -\frac{1}{12}, \dots$ go into the 3rd, 6th, 9th, ... places),
$f(4m+2) = 3m+2$		(the terms $-\frac{1}{2}, -\frac{1}{6}, -\frac{1}{10}, \dots$ go into the 2nd, 5th, 8th, ... places).

Nevertheless, there is no reason to assume that $\sum_{n=1}^{\infty} b_n$ should equal $\sum_{n=1}^{\infty} a_n$: these sums are, by definition, $\lim_{n\to\infty} b_1 + \cdots + b_n$ and $\lim_{n\to\infty} a_1 + \cdots + a_n$, so the particular order of the terms can quite conceivably matter. The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is not special in this regard; indeed, its behavior is typical of series which are not absolutely convergent—the following result (really more of a grand counterexample than a theorem) shows how bad conditionally convergent series are.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, but does not converge absolutely, then for any number α there is a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{n=1}^{\infty} b_n = \alpha$.

PROOF Let $\sum_{n=1}^{\infty} p_n$ denote the series formed from the positive terms of $\{a_n\}$ and let $\sum_{n=1}^{\infty} q_n$ denote the series of negative terms. It follows from Theorem 5 that at least one of these series does not converge. As a matter of fact, both must fail to converge, for if one had bounded partial sums, and the other had unbounded partial sums, then the original series $\sum_{n=1}^{\infty} a_n$ would also have unbounded partial sums, contradicting the assumption that it converges.

Now let α be any number. Assume, for simplicity, that $\alpha > 0$ (the proof for $\alpha < 0$ will be a simple modification). Since the series $\sum_{n=1}^{\infty} p_n$ is not convergent, there is a number N such that

$$\sum_{n=1}^{N} p_n > \alpha.$$

We will choose N_1 to be the *smallest* N with this property. This means that

$$(1) \qquad \sum_{n=1}^{N_1-1} p_n \leq \alpha,$$

$$\text{but} \quad (2) \qquad \sum_{n=1}^{N_1} p_n > \alpha.$$

Then if

$$S_1 = \sum_{n=1}^{N_1} p_n,$$

we have

$$S_1 - \alpha \leq p_{N_1}.$$

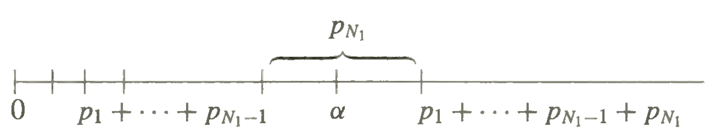


FIGURE 4

This relation, which is clear from Figure 4, follows immediately from equation (1):

$$S_1 - \alpha \leq S_1 - \sum_{n=1}^{N_1-1} p_n = p_{N_1}.$$

To the sum S_1 we now add on just enough negative terms to obtain a new sum T_1 which is less than α. In other words, we choose the smallest integer M_1 for which

$$T_1 = S_1 + \sum_{n=1}^{M_1} q_n < \alpha.$$

As before, we have

$$\alpha - T_1 \leq -q_{M_1}.$$

We now continue this procedure indefinitely, obtaining sums alternately larger and smaller than α, each time choosing the smallest N_k or M_k possible. The sequence

$$p_1, \ldots, p_{N_1}, q_1, \ldots, q_{M_1}, p_{N_1+1}, \ldots, p_{N_2}, \ldots$$

is a rearrangement of $\{a_n\}$. The partial sums of this rearrangement increase to S_1, then decrease to T_1, then increase to S_2, then decrease to T_2, etc. To complete the proof we simply note that $|S_k - \alpha|$ and $|T_k - \alpha|$ are less than or equal to p_{N_k} or $-q_{M_k}$, respectively, and that these terms, being members of the original sequence $\{a_n\}$, must decrease to 0, since $\sum_{n=1}^{\infty} a_n$ converges. ▌

Together with Theorem 7, the next theorem establishes conclusively the distinction between conditionally convergent and absolutely convergent series.

THEOREM 8 If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\{b_n\}$ is any rearrangement of $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ also converges (absolutely), and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

PROOF Let us denote the partial sums of $\{a_n\}$ by s_n, and the partial sums of $\{b_n\}$ by t_n.

Suppose that $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges, there is some N such that

$$\left| \sum_{n=1}^{\infty} a_n - s_N \right| < \varepsilon.$$

Moreover, since $\sum_{n=1}^{\infty} |a_n|$ converges, we can also choose N so that

$$\sum_{n=1}^{\infty} |a_n| - (|a_1| + \cdots + |a_N|) < \varepsilon,$$

i.e., so that

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots < \varepsilon.$$

Now choose M so large that each of $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$. Then whenever $m > M$, the difference $t_m - s_N$ is the sum of certain a_i, *where* $a_1, \ldots, a_N$ *are definitely excluded.* Consequently,

$$|t_m - s_N| \leq |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots .$$

Thus, if $m > M$, then

$$\begin{aligned}\left|\sum_{n=1}^{\infty} a_n - t_m\right| &= \left|\sum_{n=1}^{\infty} a_n - s_N - (t_m - s_N)\right| \\ &\leq \left|\sum_{n=1}^{\infty} a_n - s_N\right| + |t_m - s_N| \\ &< \varepsilon + \varepsilon.\end{aligned}$$

Since this is true for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} b_n$ converges to $\sum_{n=1}^{\infty} a_n$.

To show that $\sum_{n=1}^{\infty} b_n$ converges absolutely, note that $\{|b_n|\}$ is a rearrangement of $\{|a_n|\}$; since $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, $\sum_{n=1}^{\infty} |b_n|$ converges by the first part of the theorem. ▮

Absolute convergence is also important when we want to multiply two infinite series. Unlike the situation for addition, where we have the simple formula

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n),$$

there isn't quite so obvious a candidate for the product

$$\left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right) = (a_1 + a_2 + \cdots) \cdot (b_1 + b_2 + \cdots).$$

It would seem that we ought to sum all the products $a_i b_j$. The trouble is that these form a two-dimensional array, rather than a sequence:

$$\begin{array}{cccc} a_1 b_1 & a_1 b_2 & a_1 b_3 & \ldots \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & \ldots \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Nevertheless, all the elements of this array can be arranged in a sequence. The picture below shows one way of doing this, and of course, there are (infinitely) many other ways.

$$\begin{array}{cccc} a_1 b_1 & a_1 b_2 & a_1 b_3 & \ldots \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & \ldots \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & \ldots \end{array}$$

Suppose that $\{c_n\}$ is some sequence of this sort, containing each product $a_i b_j$ just once. Then we might naively expect to have

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

But this *isn't* true (see Problem 10), nor is this really so surprising, since we've said nothing about the specific arrangement of the terms. The next theorem shows that the result does hold when the arrangement of terms is irrelevant.

THEOREM 9 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, and $\{c_n\}$ is any sequence containing the products $a_i b_j$ for each pair (i, j), then

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

PROOF Notice first that the sequence

$$p_L = \sum_{i=1}^{L} |a_i| \cdot \sum_{j=1}^{L} |b_j|$$

converges, since $\{a_n\}$ and $\{b_n\}$ are absolutely convergent, and since the limit of a product is the product of the limits. So $\{p_L\}$ is a Cauchy sequence, which means that for any $\varepsilon > 0$, if L and L' are large enough, then

$$\left| \sum_{i=1}^{L'} |a_i| \cdot \sum_{j=1}^{L'} |b_j| - \sum_{i=1}^{L} |a_i| \cdot \sum_{j=1}^{L} |b_j| \right| < \frac{\varepsilon}{2}.$$

It follows that

$$(1) \qquad \sum_{i \text{ or } j > L} |a_i| \cdot |b_j| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now suppose that N is any number so large that the terms c_n for $n \leq N$ include every term $a_i b_j$ for $i, j \leq L$. Then the difference

$$\sum_{n=1}^{N} c_n - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j$$

consists of terms $a_i b_j$ with $i > L$ or $j > L$, so

$$(2) \qquad \left| \sum_{n=1}^{N} c_n - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j \right| \leq \sum_{i \text{ or } j > L} |a_i| \cdot |b_j|$$
$$< \varepsilon \qquad \text{by (1).}$$

But since the limit of a product is the product of the limits, we also have

$$(3) \qquad \left| \sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j \right| < \varepsilon$$

for large enough L. Consequently, if we choose L, and then N, large enough, we will have

$$\begin{aligned}\left|\sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{N} c_n\right| &\leq \left|\sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j\right| \\ &\quad + \left|\sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j - \sum_{n=1}^{N} c_n\right| \\ &< 2\varepsilon \qquad \text{by (2) and (3),}\end{aligned}$$

which proves the theorem. ▮

Unlike our previous theorems, which were merely concerned with summability, this result says something about the actual sums. Generally speaking, there is no reason to presume that a given infinite sum can be "evaluated" in any simpler terms. However, many simple expressions can be equated to infinite sums by using Taylor's Theorem. Chapter 20 provides many examples of functions for which

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + R_{n,a}(x),$$

where $\lim_{n\to\infty} R_{n,a}(x) = 0$. This is precisely equivalent to

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

which means, in turn, that

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

As particular examples we have

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| \leq 1, \\ \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots, \qquad -1 < x \leq 1.\end{aligned}$$

(Notice that the series for $\arctan x$ and $\log(1+x)$ do not even converge for $|x| > 1$; in addition, when $x = -1$, the series for $\log(1+x)$ becomes

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \cdots$$

which does not converge.)

Some pretty impressive results are obtained with particular values of x:

$$\begin{aligned}
0 &= \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots, \\
e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots, \\
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \\
\log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.
\end{aligned}$$

More significant developments may be anticipated if we compare the series for $\sin x$ and $\cos x$ a little more carefully. The series for $\cos x$ is just the one we would have obtained if we had enthusiastically differentiated both sides of the equation

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

term-by-term, ignoring the fact that we have never proved anything about the derivatives of infinite sums. Likewise, if we differentiate both sides of the formula for $\cos x$ formally (i.e., without justification) we obtain the formula $\cos'(x) = -\sin x$, and if we differentiate the formula for e^x we obtain $\exp'(x) = \exp(x)$. In the next chapter we shall see that such term-by-term differentiation of infinite sums is indeed valid in certain important cases.

PROBLEMS

1. Decide whether each of the following infinite series is convergent or divergent. The tools which you will need are Leibniz's Theorem and the comparison, ratio, and integral tests. A few examples have been picked with malice aforethought; two series which look quite similar may require different tests (and then again, they may not). The hint below indicates which tests may be used.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$.

(ii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(iii) $1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.

(v) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-1}}$. (The summation begins with $n = 2$ simply to avoid the meaningless term obtained for $n = 1$).

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$.

(vii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

(viii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$.

(ix) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log n}$.

(x) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$.

(xi) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

(xii) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{(\log n)^n}$.

(xiii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

(xiv) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

(xv) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\log n}$.

(xvi) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

(xvii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2(\log n)}$.

(xviii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

(xix) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

(xx) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$.

Hint: Use the comparison test for (i), (v), (vi), (ix), (x), (xi), (xiii), (xiv), (xvii); the ratio test for (vii), (xviii), (xix), (xx); the integral test for (viii), (xv), (xvi).

The next two problems examine, with hints, some infinite series that require more delicate analysis than those in Problem 1.

***2.** (a) If you have successfully solved examples (xix) and (xx) from Problem 1, it should be clear that $\displaystyle\sum_{n=1}^{\infty} a^n n!/n^n$ converges for $a < e$ and diverges for $a > e$. For $a = e$ the ratio test fails; show that $\displaystyle\sum_{n=1}^{\infty} e^n n!/n^n$ actually diverges, by using Problem 22-13.

(b) Decide when $\displaystyle\sum_{n=1}^{\infty} n^n/a^n n!$ converges, again resorting to Problem 22-13 when the ratio test fails.

***3.** Problem 1 presented the two series $\displaystyle\sum_{n=2}^{\infty} (\log n)^{-k}$ and $\displaystyle\sum_{n=2}^{\infty} (\log n)^{-n}$, of which the first diverges while the second converges. The series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}},$$

which lies between these two, is analyzed in parts (a) and (b).

(a) Show that $\int_1^{\infty} e^y/y^y\, dy$ exists, by considering the series $\displaystyle\sum_{n=1}^{\infty} (e/n)^n$.

(b) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$$

converges, by using the integral test. Hint: Use an appropriate substitution and part (a).

(c) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log(\log n)}}$$

diverges, by using the integral test. Hint: Use the same substitution as in part (b), and show directly that the resulting integral diverges.

4. Decide whether or not $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ converges.

5. (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then so does $\sum_{n=1}^{\infty} a_n{}^3$.

*(b) Show that this does not hold for conditional convergence.

6. Let f be a continuous function on an interval around 0, and let $a_n = f(1/n)$ (for large enough n).

(a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $f(0) = 0$.

(b) Prove that if $f'(0)$ exists and $\sum_{n=1}^{\infty} a_n$ converges, then $f'(0) = 0$.

(c) Prove that if $f''(0)$ exists and $f(0) = f'(0) = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) Suppose $\sum_{n=1}^{\infty} a_n$ converges. Must $f'(0)$ exist?

(e) Suppose $f(0) = f'(0) = 0$. Must $\sum_{n=1}^{\infty} a_n$ converge?

7. (a) Let $\{a_n\}$ be a sequence of integers with $0 \le a_n \le 9$. Prove that $\sum_{n=1}^{\infty} a_n 10^{-n}$ exists (and lies between 0 and 1). (This, of course, is the number which we usually denote by $0.a_1a_2a_3a_4\ldots$.)

(b) Suppose that $0 \le x \le 1$. Prove that there is a sequence of integers $\{a_n\}$ with $0 \le a_n \le 9$ and $\sum_{n=1}^{\infty} a_n 10^{-n} = x$. Hint: For example, $a_1 = [10x]$ (where $[y]$ denotes the greatest integer which is $\le y$).

(c) Show that if $\{a_n\}$ is repeating, i.e., is of the form $a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, a_1, a_2, \ldots$, then $\sum_{n=1}^{\infty} a_n 10^{-n}$ is a rational number (and find it). The same result naturally holds if $\{a_n\}$ is eventually repeating, i.e., if the sequence $\{a_{N+k}\}$ is repeating for some N.

(d) Prove that if $x = \sum_{n=1}^{\infty} a_n 10^{-n}$ is rational, then $\{a_n\}$ is eventually repeating. (Just look at the process of finding the decimal expansion of p/q dividing q into p by long division.)

8. Suppose that $\{a_n\}$ satisfies the hypothesis of Leibniz's Theorem. Use the proof of Leibniz's Theorem to obtain the following estimate:

$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} a_n - [a_1 - a_2 + \cdots \pm a_N] \right| < a_{N+1}.$$

9. (a) Prove that if $a_n \geq 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = r$, then $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$, and diverges if $r > 1$. (The proof is very similar to that of the ratio test.) This result is known as the "root test." More generally, $\sum_{n=1}^{\infty} a_n$ converges if there is some $s < 1$ such that all but finitely many $\sqrt[n]{a_n}$ are $\leq s$, and $\sum_{n=1}^{\infty} a_n$ diverges if infinitely many $\sqrt[n]{a_n}$ are ≥ 1. This result is known as the "delicate root test" (there is a similar delicate ratio test). It follows, using the notation of Problem 22-27, that $\sum_{n=1}^{\infty} a_n$ converges if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} < 1$ and diverges if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} > 1$; no conclusion is possible if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = 1$.

(b) Prove that if the ratio test works, the root test will also. Hint: Use a problem from the previous chapter.

It is easy to construct series for which the ratio test fails, while the root test works. For example, the root test shows that the series

$$\tfrac{1}{2} + \tfrac{1}{3} + (\tfrac{1}{2})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{3})^3 + \cdots$$

converges, even though the ratios of successive terms do not approach a limit. Most examples are of this rather artificial nature, but the root test is nevertheless quite an important theoretical tool.

10. For two sequences $\{a_n\}$ and $\{b_n\}$, let $c_n = \sum_{k=1}^{n} a_k b_{n+1-k}$. (Then c_n is the sum of the terms on the nth diagonal in the picture on page 486.) The series $\sum_{n=1}^{\infty} c_n$ is called the *Cauchy product* of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. If $a_n = b_n = (-1)^n/\sqrt{n}$, show that $|c_n| \geq 1$, so that the Cauchy product does not converge.

11. (a) Consider the collection A of natural numbers that do *not* contain a 9 in their usual (base 10) representation. Show that the sum of the reciprocals of the numbers in A *converges*. Hint: How many numbers between 1 and 9 are in A?; how many between 10 and 99?; etc.

(b) If B is the collection of natural numbers that do not have *all 10 digits* $0, \dots, 9$ in their usual representation, then the sum of the reciprocals of the numbers in B converges. (So "most" integers must have all ten digits in their representation.)

12. A sequence $\{a_n\}$ is called **Cesaro summable**, with Cesaro sum l, if

$$\lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} = l$$

(where $s_k = a_1 + \cdots + a_k$). Problem 22-16 shows that a summable sequence is automatically Cesaro summable, with sum equal to its Cesaro sum. Find a sequence which is *not* summable, but which *is* Cesaro summable.

13. Suppose that $a_n > 0$ and $\{a_n\}$ is Cesaro summable. Suppose also that the sequence $\{na_n\}$ is bounded. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges. Hint: If $s_n = \sum_{i=1}^{n} a_i$ and $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$, prove that $s_n - \frac{n}{n+1}\sigma_n$ is bounded.

14. This problem outlines an alternative proof of Theorem 8 which does not rely on the Cauchy criterion.

(a) Suppose that $a_n \geq 0$ for each n. Let $\{b_n\}$ be a rearrangement of $\{a_n\}$, and let $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Show that for each n there is some m with $s_n \leq t_m$.

(b) Show that $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ if $\sum_{n=1}^{\infty} b_n$ exists.

(c) Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

(d) Now replace the condition $a_n \geq 0$ by the hypothesis that $\sum_{n=1}^{\infty} a_n$ converges absolutely, using the second part of Theorem 5.

15. (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\{b_n\}$ is any subsequence of $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges (absolutely).

(b) Show that this is false if $\sum_{n=1}^{\infty} a_n$ does not converge absolutely.

*(c) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\sum_{n=1}^{\infty} a_n = (a_1 + a_3 + a_5 + \cdots) + (a_2 + a_4 + a_6 + \cdots).$$

16. Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

***17.** Problem 19-43 shows that the improper integral $\int_0^{\infty} (\sin x)/x\,dx$ converges. Prove that $\int_0^{\infty} |(\sin x)/x|\,dx$ diverges.

***18.** Find a continuous function f with $f(x) \geq 0$ for all x such that $\int_0^{\infty} f(x)\,dx$ exists, but $\lim_{x\to\infty} f(x)$ does not exist.

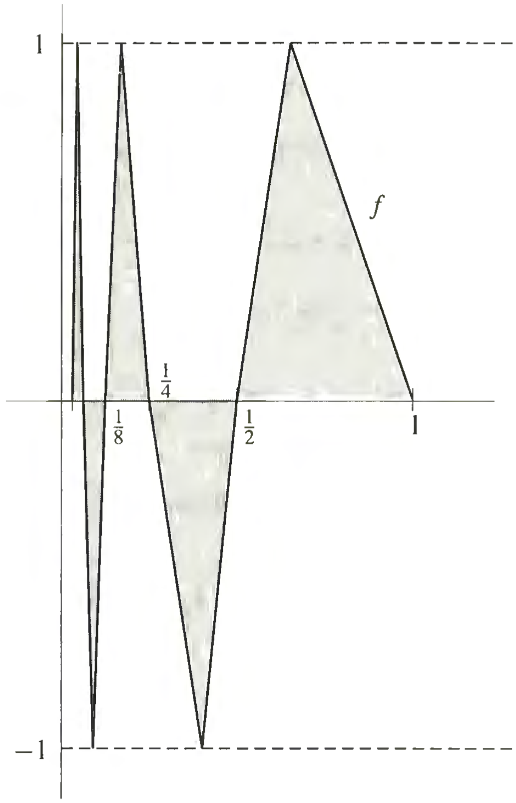


FIGURE 5

***19.** Let $f(x) = x \sin 1/x$ for $0 < x \le 1$, and let $f(0) = 0$. Recall the definition of $\ell(f, P)$ from Problem 13-25. Show that the set of all $\ell(f, P)$ for P a partition of $[0, 1]$ is not bounded (thus f has "infinite length"). Hint: Try partitions of the form

$$P = \left\{0, \frac{2}{(2n+1)\pi}, \dots, \frac{2}{7\pi}, \frac{2}{5\pi}, \frac{2}{3\pi}, \frac{2}{\pi}, 1\right\}.$$

20. Let f be the function shown in Figure 5. Find $\int_0^1 f$, and also the area of the shaded region in Figure 5.

***21.** In this problem we will establish the "binomial series"

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \qquad |x| < 1,$$

for any α, by showing that $\lim_{n\to\infty} R_{n,0}(x) = 0$. The proof is in several steps, and uses the Cauchy and Lagrange forms as found in Problem 20-21.

(a) Use the ratio test to show that the series $\sum_{k=0}^{\infty} \binom{\alpha}{k} r^k$ does indeed converge for $|r| < 1$ (this is not to say that it necessarily converges to $(1+r)^\alpha$). It follows in particular that $\lim_{n\to\infty} \binom{\alpha}{n} r^n = 0$ for $|r| < 1$.

(b) Suppose first that $0 \le x < 1$. Show that $\lim_{n\to\infty} R_{n,0}(x) = 0$, by using Lagrange's form of the remainder, noticing that $(1+t)^{\alpha-n-1} \le 1$ for $n + 1 > \alpha$.

(c) Now suppose that $-1 < x < 0$; the number t in Cauchy's form of the remainder satisfies $-1 < x < t \le 0$. Show that

$$|x(1+t)^{\alpha-1}| \le |x|M, \qquad \text{where } M = \max(1, (1+x)^{\alpha-1}),$$

and

$$\left|\frac{x-t}{1+t}\right| = |x|\left(\frac{1-t/x}{1+t}\right) \le |x|.$$

Using Cauchy's form of the remainder, and the fact that

$$(n+1)\binom{\alpha}{n+1} = \alpha\binom{\alpha-1}{n},$$

show that $\lim_{n\to\infty} R_{n,0}(x) = 0$.

22. (a) Suppose that the partial sums of the sequence $\{a_n\}$ are bounded and that $\{b_n\}$ is a sequence with $b_n \ge b_{n+1}$ and $\lim_{n\to\infty} b_n = 0$. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges. This is known as *Dirichlet's test*. Hint: Use Abel's Lemma (Problem 19-36) to check the Cauchy criterion.

(b) Derive Leibniz's Theorem from this result.

(c) Prove, using Problem 15-33, that the series $\sum_{n=1}^{\infty}(\cos nx)/n$ converges if x is not of the form $2k\pi$ for any integer k (in which case it clearly diverges).

(d) Prove *Abel's test*: If $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is a sequence which is either nondecreasing or nonincreasing and which is bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges. Hint: Consider $b_n - b$, where $b = \lim_{n\to\infty} b_n$.

***23.** Suppose $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} 2^n a_{2^n}$ also converges (the "Cauchy Condensation Theorem"). Notice that the divergence of $\sum_{n=1}^{\infty} 1/n$ is a special case, for if $\sum_{n=1}^{\infty} 1/n$ converged, then $\sum_{n=1}^{\infty} 2^n(1/2^n)$ would also converge; this remark may serve as a hint.

***24.** (a) Prove that if $\sum_{n=1}^{\infty} a_n{}^2$ and $\sum_{n=1}^{\infty} b_n{}^2$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(b) Prove that if $\sum_{n=1}^{\infty} a_n{}^2$ converges, then $\sum_{n=1}^{\infty} a_n/n^{\alpha}$ converges for any $\alpha > \frac{1}{2}$.

***25.** Suppose $\{a_n\}$ is decreasing and each $a_n > 0$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Hint: Write down the Cauchy criterion and be sure to use the fact that $\{a_n\}$ is decreasing.

***26.** If $\sum_{n=1}^{\infty} a_n$ converges, then the partial sums s_n are bounded, and $\lim_{n\to\infty} a_n = 0$. It is tempting to conjecture that boundedness of the partial sums, together with the condition $\lim_{n\to\infty} a_n = 0$, implies convergence of the series $\sum_{n=1}^{\infty} a_n$. Find a counterexample to show that this is *not* true. Hint: Notice that some *subsequence* of the partial sums will have to converge; you must somehow allow this to happen, without letting the sequence of partial sums itself converge.

27. Prove that if $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges. Hint: Compare the partial sums. Does the converse hold?

28. For $b_n > 0$ we say that the infinite product $\prod_{n=1}^{\infty} b_n$ *converges* if the sequence $p_n = \prod_{i=1}^{n} b_i$ converges, and also $\lim_{n\to\infty} p_n \neq 0$.

(a) Prove that if $\prod_{n=1}^{\infty} b_n$ converges, then b_n approaches 1.

(b) Prove that $\prod_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \log b_n$ converges.

(c) For $a_n \geq 0$, prove that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Hint: Use Problem 27 for one implication, and a simple estimate for $\log(1+a)$ for the reverse implication.

The remaining parts of this Problem show that the hypothesis $a_n \geq 0$ is needed.

(d) Use the Taylor series for $\log(1+x)$ to show that for sufficiently small x we have

$$\tfrac{1}{4}x^2 \leq x - \log(1+x) \leq \tfrac{3}{4}x^2.$$

Conclude that if all $a_n > -1$ and $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} \log(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} {a_n}^2$ converges. Similarly, if all $a_n > -1$ and $\sum_{n=1}^{\infty} {a_n}^2$ converges, then $\sum_{n=1}^{\infty} \log(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Hint: Use the Cauchy criterion.

(e) Show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges, but

$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

diverges.

(f) Consider the sequence

$$\{a_n\} = \underbrace{1, -\tfrac{1}{2},}_{1\text{ pair}} \underbrace{\tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{3}, -\tfrac{1}{4},}_{3\text{ pairs}} \underbrace{\tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{5}, -\tfrac{1}{6}, \ldots}_{5\text{ pairs}} \quad \ldots$$

(compare Problem 26). Show that $\sum_{n=1}^{\infty} a_n$ diverges, but

$$\prod_{n=1}^{\infty}(1+a_n)=1.$$

29. (a) Compute $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

(b) Compute $\displaystyle\prod_{n=1}^{\infty}(1+x^{2^n})$ for $|x|<1$.

30. The divergence of $\sum_{n=1}^{\infty} 1/n$ is related to the following remarkable fact: Any positive rational number x can be written as a *finite* sum of *distinct* numbers of the form $1/n$. The idea of the proof is shown by the following calculation for $\frac{27}{31}$: Since

$$\begin{gathered}
\tfrac{27}{31}-\tfrac{1}{2}=\tfrac{23}{62}\\
\tfrac{23}{62}-\tfrac{1}{3}=\tfrac{7}{186}\\
\tfrac{7}{186}<\tfrac{1}{4},\ldots,\tfrac{1}{26}\\
\tfrac{7}{186}-\tfrac{1}{27}=\tfrac{1}{1674}
\end{gathered}$$

we have

$$\tfrac{27}{31}=\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{27}+\tfrac{1}{1674}.$$

Notice that the numerators 23, 7, 1 of the differences are decreasing.

(a) Prove that if $1/(n+1)<x<1/n$ for some n, then the numerator in this sort of calculation must always decrease; conclude that x can be written as a finite sum of distinct numbers $1/k$.

(b) Now prove the result for all x by using the divergence of $\sum_{n=1}^{\infty} 1/n$.

CHAPTER 24 UNIFORM CONVERGENCE AND POWER SERIES

The considerations at the end of the previous chapter suggest an entirely new way of looking at infinite series. Our attention will shift from particular infinite sums to equations like

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

which concern sums of quantities that depend on x. In other words, we are interested in *functions* defined by equations of the form

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

(in the previous example $f_n(x) = x^{n-1}/(n-1)!$). In such a situation $\{f_n\}$ will be some sequence of functions; for each x we obtain a sequence of numbers $\{f_n(x)\}$, and $f(x)$ is the sum of this sequence. In order to analyze such functions it will certainly be necessary to remember that each sum

$$f_1(x) + f_2(x) + f_3(x) + \cdots$$

is, by definition, the limit of the sequence

$$f_1(x),\ f_1(x) + f_2(x),\ f_1(x) + f_2(x) + f_3(x),\ \ldots.$$

If we define a new sequence of functions $\{s_n\}$ by

$$s_n = f_1 + \cdots + f_n,$$

then we can express this fact more succinctly by writing

$$f(x) = \lim_{n\to\infty} s_n(x).$$

For some time we shall therefore concentrate on functions defined as limits,

$$f(x) = \lim_{n\to\infty} f_n(x),$$

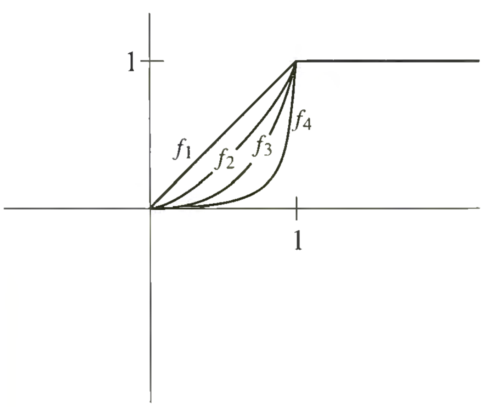


FIGURE 1

rather than on functions defined as infinite sums. The total body of results about such functions can be summed up very easily: nothing one would hope to be true actually is—instead we have a splendid collection of counterexamples. The first of these shows that even if each f_n is continuous, the function f may not be! Contrary to what you may expect, the functions f_n will be very simple. Figure 1 shows the graphs of the functions

$$f_n(x) = \begin{cases} x^n, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

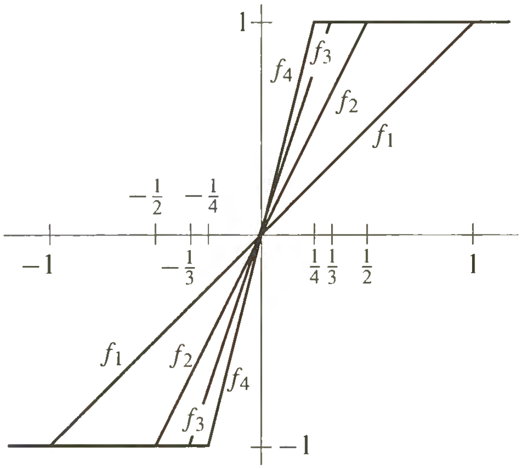


FIGURE 2

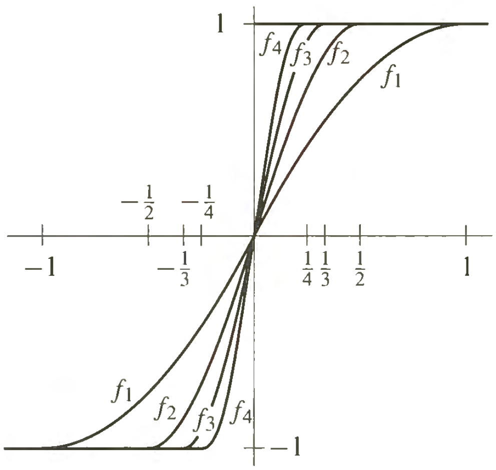


FIGURE 3

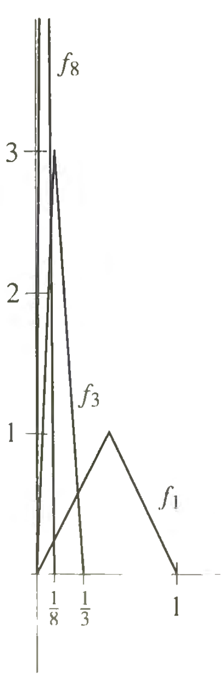


FIGURE 4

These functions are all continuous, but the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous; in fact,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x \geq 1. \end{cases}$$

Another example of this same phenomenon is illustrated in Figure 2; the functions f_n are defined by

$$f_n(x) = \begin{cases} -1, & x \leq -\dfrac{1}{n} \\ nx, & -\dfrac{1}{n} \leq x \leq \dfrac{1}{n} \\ 1, & \dfrac{1}{n} \leq x. \end{cases}$$

In this case, if $x < 0$, then $f_n(x)$ is eventually (i.e., for large enough n) equal to -1, and if $x > 0$, then $f_n(x)$ is eventually 1, while $f_n(0) = 0$ for all n. Thus

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0; \end{cases}$$

so, once again, the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous.

By rounding off the corners in the previous examples it is even possible to produce a sequence of *differentiable* functions $\{f_n\}$ for which the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous. One such sequence is easy to define explicitly:

$$f_n(x) = \begin{cases} -1, & x \leq -\dfrac{1}{n} \\ \sin\left(\dfrac{n\pi x}{2}\right), & -\dfrac{1}{n} \leq x \leq \dfrac{1}{n} \\ 1, & \dfrac{1}{n} \leq x. \end{cases}$$

These functions are differentiable (Figure 3), but we still have

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Continuity and differentiability are, moreover, not the only properties for which problems arise. Another difficulty is illustrated by the sequence $\{f_n\}$ shown in Figure 4; on the interval $[0, 1/n]$ the graph of f_n forms an isosceles triangle of altitude n, while $f_n(x) = 0$ for $x \geq 1/n$. These functions may be defined explicitly as follows:

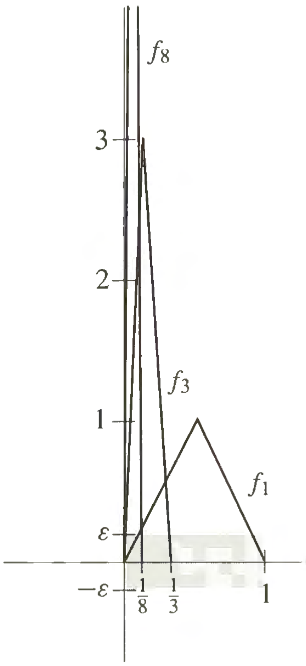


FIGURE 5

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \dfrac{1}{2n} \\ 2n - 2n^2x, & \dfrac{1}{2n} \le x \le \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \le x \le 1. \end{cases}$$

Because this sequence varies so erratically near 0, our primitive mathematical instincts might suggest that $\lim_{n\to\infty} f_n(x)$ does not always exist. Nevertheless, this limit does exist for all x, and the function $f(x) = \lim_{n\to\infty} f_n(x)$ is even continuous. In fact, if $x > 0$, then $f_n(x)$ is eventually 0, so $\lim_{n\to\infty} f_n(x) = 0$; moreover, $f_n(0) = 0$ for all n, so that we certainly have $\lim_{n\to\infty} f_n(0) = 0$. In other words, $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all x. On the other hand, the integral quickly reveals the strange behavior of this sequence; we have

$$\int_0^1 f_n(x)\,dx = \tfrac{1}{2},$$

but

$$\int_0^1 f(x)\,dx = 0.$$

Thus,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

This particular sequence of functions behaves in a way that we really never imagined when we first considered functions defined by limits. Although it is true that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for each } x \text{ in } [0, 1],$$

the graphs of the functions f_n do not "approach" the graph of f in the sense of lying close to it—if, as in Figure 5, we draw a strip around f of total width 2ε (allowing a width of ε above and below), then the graphs of f_n do not lie completely within this strip, no matter how large an n we choose. Of course, for each x there is some N such that the point $(x, f_n(x))$ lies in this strip for $n > N$; this assertion just amounts to the fact that $\lim_{n\to\infty} f_n(x) = f(x)$. But it is necessary to choose larger and larger N's as x is chosen closer and closer to 0, and no one N will work for all x at once.

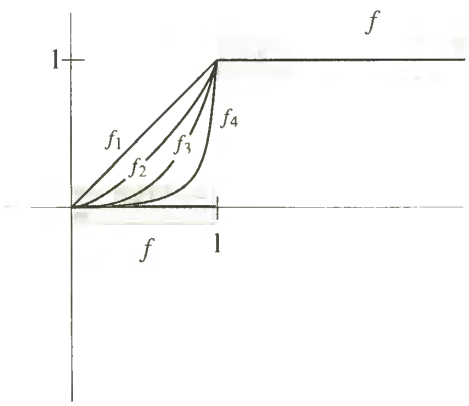


FIGURE 6

The same situation actually occurs, though less blatantly, for each of the other examples given previously. Figure 6 illustrates this point for the sequence

$$f_n(x) = \begin{cases} x^n, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

A strip of total width 2ε has been drawn around the graph of $f(x) = \lim_{n\to\infty} f_n(x)$. If $\varepsilon < \frac{1}{2}$, this strip consists of two pieces, which contain no points with second

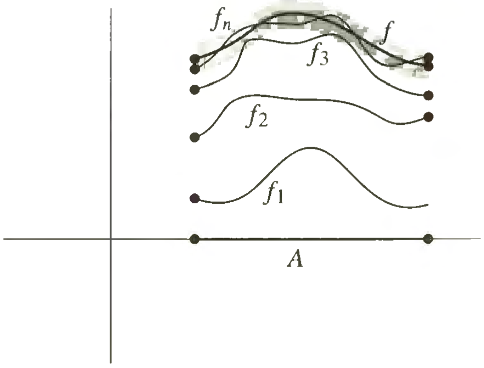


FIGURE 7

coordinate equal to $\frac{1}{2}$; since each function f_n takes on the value $\frac{1}{2}$, the graph of each f_n fails to lie within this strip. Once again, for each point x there is some N such that $(x, f_n(x))$ lies in the strip for $n > N$; but it is not possible to pick one N which works for all x at once.

It is easy to check that precisely the same situation occurs for each of the other examples. In each case we have a function f, and a sequence of functions $\{f_n\}$, all defined on some set A, such that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \text{ in } A.$$

This means that

> for all $\varepsilon > 0$, and for all x in A, there is some N such that if $n > N$, then $|f(x) - f_n(x)| < \varepsilon$.

But in each case different N's must be chosen for different x's, and in each case it is *not* true that

> for all $\varepsilon > 0$ there is some N such that for all x in A, if $n > N$, then $|f(x) - f_n(x)| < \varepsilon$.

Although this condition differs from the first only by a minor displacement of the phrase "for all x in A," it has a totally different significance. If a sequence $\{f_n\}$ satisfies this second condition, then the graphs of f_n eventually lie close to the graph of f, as illustrated in Figure 7. This condition turns out to be just the one which makes the study of limit functions feasible.

DEFINITION

Let $\{f_n\}$ be a sequence of functions defined on A, and let f be a function which is also defined on A. Then f is called the **uniform limit of** $\{f_n\}$ **on** A if for every $\varepsilon > 0$ there is some N such that for all x in A,

$$\text{if } n > N, \text{ then } |f(x) - f_n(x)| < \varepsilon.$$

We also say that $\{f_n\}$ **converges uniformly to f on A**, or that f_n **approaches f uniformly on A**.

As a contrast to this definition, if we know only that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for each } x \text{ in } A,$$

then we say that $\{f_n\}$ **converges pointwise to f on A**. Clearly, uniform convergence implies pointwise convergence (but not conversely!).

Evidence for the usefulness of uniform convergence is not at all difficult to amass. Integrals represent a particularly easy topic; Figure 7 makes it almost obvious that if $\{f_n\}$ converges uniformly to f, then the integral of f_n can be made as close to the integral of f as desired. Expressed more precisely, we have the following theorem.

THEOREM 1 Suppose that $\{f_n\}$ is a sequence of functions which are integrable on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f which is integrable on $[a, b]$. Then

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

PROOF Let $\varepsilon > 0$. There is some N such that for all $n > N$ we have

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{for all } x \text{ in } [a, b].$$

Thus, if $n > N$ we have

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| &= \left| \int_a^b [f(x) - f_n(x)]\,dx \right| \\ &\le \int_a^b |f(x) - f_n(x)|\,dx \\ &\le \int_a^b \varepsilon\,dx \\ &= \varepsilon(b-a). \end{aligned}$$

Since this is true for any $\varepsilon > 0$, it follows that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. \blacksquare$$

The treatment of continuity is only a little bit more difficult, involving an "$\varepsilon/3$-argument," a three-step estimate of $|f(x) - f(x+h)|$. If $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f, then there is some n such that

$$(1) \qquad |f(x) - f_n(x)| < \frac{\varepsilon}{3},$$

$$(2) \qquad |f(x+h) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Moreover, since f_n is continuous, for sufficiently small h we have

$$(3) \qquad |f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

It will follow from (1), (2), and (3) that $|f(x) - f(x+h)| < \varepsilon$. In order to obtain (3), however, we must restrict the size of $|h|$ in a way that cannot be predicted until n has already been chosen; it is therefore quite essential that there be some fixed n which makes (2) true, no matter how small $|h|$ may be—it is precisely at this point that uniform convergence enters the proof.

THEOREM 2 Suppose that $\{f_n\}$ is a sequence of functions which are continuous on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to f. Then f is also continuous on $[a, b]$.

PROOF For each x in $[a, b]$ we must prove that f is continuous at x. We will deal only with x in (a, b); the cases $x = a$ and $x = b$ require the usual simple modifications.

Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly to f on $[a, b]$, there is some n such that

$$|f(y) - f_n(y)| < \frac{\varepsilon}{3} \qquad \text{for all } y \text{ in } [a, b].$$

In particular, for all h such that $x + h$ is in $[a, b]$, we have

$$(1) \qquad |f(x) - f_n(x)| < \frac{\varepsilon}{3},$$

$$(2) \qquad |f(x+h) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Now f_n is continuous, so there is some $\delta > 0$ such that for $|h| < \delta$ we have

$$(3) \qquad |f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Thus, if $|h| < \delta$, then

$$\begin{aligned}
&|f(x+h) - f(x)| \\
&\qquad = |f(x+h) - f_n(x+h) + f_n(x+h) - f_n(x) + f_n(x) - f(x)| \\
&\qquad \leq |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| + |f_n(x) - f(x)| \\
&\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\qquad = \varepsilon.
\end{aligned}$$

This proves that f is continuous at x. ▮

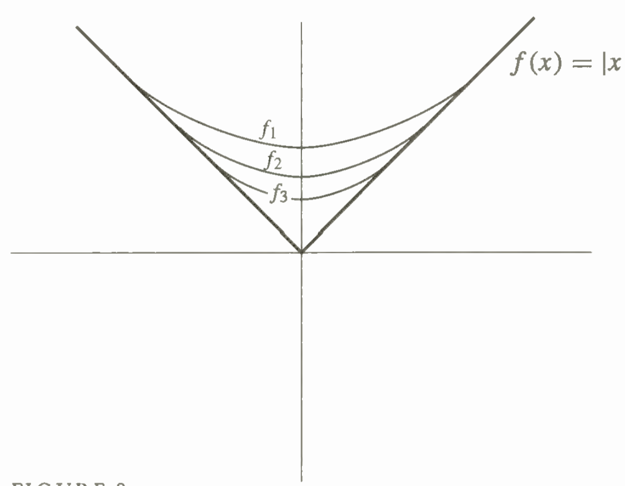


FIGURE 8

After the two noteworthy successes provided by Theorem 1 and Theorem 2, the situation for differentiability turns out to be very disappointing. If each f_n is differentiable, and if $\{f_n\}$ converges uniformly to f, it is still not necessarily true that f is differentiable. For example, Figure 8 shows that there is a sequence of differentiable functions $\{f_n\}$ which converges uniformly to the function $f(x) = |x|$.

Even if f *is* differentiable, it may not be true that

$$f'(x) = \lim_{n\to\infty} f_n'(x);$$

this is not at all surprising if we reflect that a smooth function can be approximated by very rapidly oscillating functions. For example (Figure 9), if

$$f_n(x) = \frac{1}{n}\sin(n^2x),$$

then $\{f_n\}$ converges uniformly to the function $f(x) = 0$, but

$$f_n'(x) = n\cos(n^2x),$$

and $\lim_{n\to\infty} n\cos(n^2x)$ does not always exist (for example, it does not exist if $x = 0$).

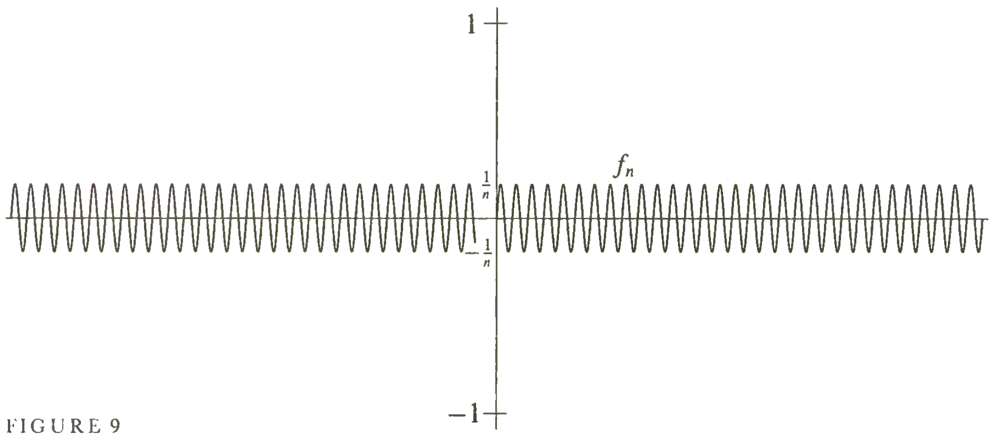


FIGURE 9

Despite such examples, the Fundamental Theorem of Calculus practically guarantees that some sort of theorem about derivatives will be a consequence of Theorem 1; the crucial hypothesis is that $\{f_n'\}$ converges uniformly (to *some* continuous function).

THEOREM 3 Suppose that $\{f_n\}$ is a sequence of functions which are differentiable on $[a,b]$, with integrable derivatives f_n', and that $\{f_n\}$ converges (pointwise) to f. Suppose, moreover, that $\{f_n'\}$ converges uniformly on $[a,b]$ to some continuous function g. Then f is differentiable and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

PROOF Applying Theorem 1 to the interval $[a,x]$, we see that for each x we have

$$\begin{aligned}\int_a^x g &= \lim_{n\to\infty}\int_a^x f_n' \\ &= \lim_{n\to\infty}[f_n(x) - f_n(a)] \\ &= f(x) - f(a).\end{aligned}$$

Since g is continuous, it follows that $f'(x) = g(x) = \lim_{n\to\infty} f_n{}'(x)$ for all x in the interval $[a, b]$. ▮

Now that the basic facts about uniform limits have been established, it is clear how to treat functions defined as infinite sums,

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots.$$

This equation means that

$$f(x) = \lim_{n\to\infty} f_1(x) + \cdots + f_n(x);$$

our previous theorems apply when the new sequence

$$f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3,\ \ldots$$

converges uniformly to f. Since this is the only case we shall ever be interested in, we single it out with a definition.

DEFINITION

The series $\sum_{n=1}^{\infty} f_n$ **converges uniformly** (more formally: the sequence $\{f_n\}$ is **uniformly summable**) **to f on A**, if the sequence

$$f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3,\ \ldots$$

converges uniformly to f on A.

We can now apply each of Theorems 1, 2, and 3 to uniformly convergent series; the results may be stated in one common corollary.

COROLLARY

Let $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a, b]$.

(1) If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.
(2) If f and each f_n is integrable on $[a, b]$, then

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n.$$

Moreover, if $\sum_{n=1}^{\infty} f_n$ converges (pointwise) to f on $[a, b]$, each f_n has an integrable derivative $f_n{}'$ and $\sum_{n=1}^{\infty} f_n{}'$ converges uniformly on $[a, b]$ to some continuous function, then

(3) $f'(x) = \sum_{n=1}^{\infty} f_n{}'(x)$ for all x in $[a, b]$.

PROOF (1) If each f_n is continuous, then so is each $f_1 + \cdots + f_n$, and f is the uniform limit of the sequence $f_1, f_1 + f_2, f_1 + f_2 + f_3, \ldots$, so f is continuous by Theorem 2.

(2) Since $f_1, f_1 + f_2, f_1 + f_2 + f_3, \ldots$ converges uniformly to f, it follows from Theorem 1 that

$$\begin{aligned}
\int_a^b f &= \lim_{n\to\infty} \int_a^b (f_1 + \cdots + f_n) \\
&= \lim_{n\to\infty} \left(\int_a^b f_1 + \cdots + \int_a^b f_n \right) \\
&= \sum_{n=1}^{\infty} \int_a^b f_n .
\end{aligned}$$

(3) Each function $f_1 + \cdots + f_n$ is differentiable, with derivative $f_1' + \cdots + f_n'$, and $f_1', f_1' + f_2', f_1' + f_2' + f_3', \ldots$ converges uniformly to a continuous function, by hypothesis. It follows from Theorem 3 that

$$\begin{aligned}
f'(x) &= \lim_{n\to\infty} [f_1'(x) + \cdots + f_n'(x)] \\
&= \sum_{n=1}^{\infty} f_n'(x). \ \blacksquare
\end{aligned}$$

At the moment this corollary is not very useful, since it seems quite difficult to predict when the sequence $f_1, f_1+f_2, f_1+f_2+f_3, \ldots$ will converge uniformly. The most important condition which ensures such uniform convergence is provided by the following theorem; the proof is almost a triviality because of the cleverness with which the very simple hypotheses have been chosen.

THEOREM 4 (THE WEIERSTRASS M-TEST) Let $\{f_n\}$ be a sequence of functions defined on A, and suppose that $\{M_n\}$ is a sequence of numbers such that

$$|f_n(x)| \le M_n \qquad \text{for all } x \text{ in } A.$$

Suppose moreover that $\sum_{n=1}^{\infty} M_n$ converges. Then for each x in A the series $\sum_{n=1}^{\infty} f_n(x)$ converges (in fact, it converges absolutely), and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

PROOF For each x in A the series $\sum_{n=1}^{\infty} |f_n(x)|$ converges, by the comparison test; consequently $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely). Moreover, for all x in A we have

$$\begin{aligned} \left| f(x) - [f_1(x) + \cdots + f_N(x)] \right| &= \left| \sum_{n=N+1}^{\infty} f_n(x) \right| \\ &\le \sum_{n=N+1}^{\infty} |f_n(x)| \\ &\le \sum_{n=N+1}^{\infty} M_n. \end{aligned}$$

Since $\sum_{n=1}^{\infty} M_n$ converges, the number $\sum_{n=N+1}^{\infty} M_n$ can be made as small as desired, by choosing N sufficiently large. ▌

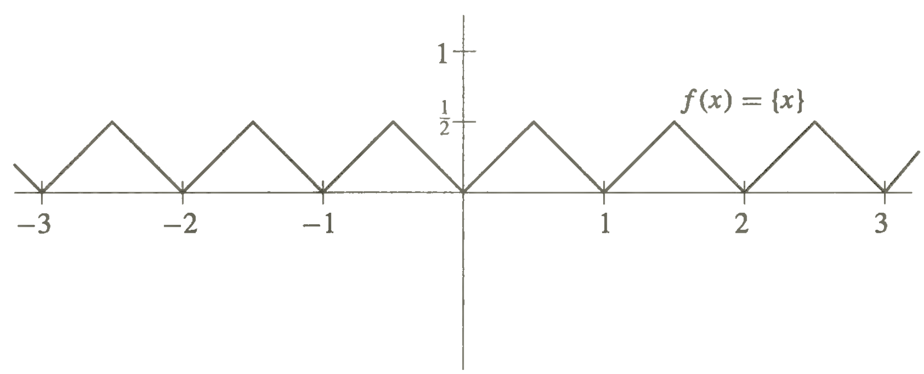


FIGURE 10

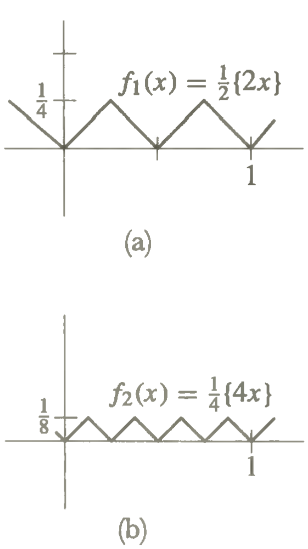


FIGURE 11

The following sequence $\{f_n\}$ illustrates a simple application of the Weierstrass M-test. Let $\{x\}$ denote the distance from x to the nearest integer (the graph of $f(x) = \{x\}$ is illustrated in Figure 10). Now define

$$f_n(x) = \frac{1}{10^n}\{10^n x\}.$$

The functions f_1 and f_2 are shown in Figure 11 (but to make the drawings simpler, 10^n has been replaced by 2^n). This sequence of functions has been defined so that the Weierstrass M-test automatically applies: clearly

$$|f_n(x)| \le \frac{1}{10^n} \qquad \text{for all } x,$$

and $\sum_{n=1}^{\infty} 1/10^n$ converges. Thus $\sum_{n=1}^{\infty} f_n$ converges uniformly; since each f_n is continuous, the corollary implies that the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{10^n}\{10^n x\}$$

is also continuous. Figure 12 shows the graph of the first few partial sums $f_1 + \cdots + f_n$. As n increases, the graphs become harder and harder to draw, and the infinite sum $\sum_{n=1}^{\infty} f_n$ is quite undrawable, as shown by the following theorem (included mainly as an interesting sidelight, to be skipped if you find the going too rough).

THEOREM 5 The function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{10^n}\{10^n x\}$$

is continuous everywhere and differentiable nowhere!

PROOF We have just shown that f is continuous; this is the only part of the proof which uses uniform convergence. We will prove that f is not differentiable at a, for any a, by the straightforward method of exhibiting a particular sequence $\{h_m\}$ approaching 0 for which

$$\lim_{m\to\infty} \frac{f(a+h_m) - f(a)}{h_m}$$

does not exist. It obviously suffices to consider only those numbers a satisfying $0 < a \le 1$.

Suppose that the decimal expansion of a is

$$a = 0.a_1a_2a_3a_4 \ldots .$$

Let $h_m = 10^{-m}$ if $a_m \ne 4$ or 9, but let $h_m = -10^{-m}$ if $a_m = 4$ or 9 (the reason for these two exceptions will appear soon). Then

$$\begin{aligned}\frac{f(a+h_m) - f(a)}{h_m} &= \sum_{n=1}^{\infty} \frac{1}{10^n} \cdot \frac{\{10^n(a+h_m)\} - \{10^n a\}}{\pm 10^{-m}} \\ &= \sum_{n=1}^{\infty} \pm 10^{m-n}[\{10^n(a+h_m)\} - \{10^n a\}].\end{aligned}$$

This infinite series is really a finite sum, because if $n \ge m$, then $10^n h_m$ is an integer, so

$$\{10^n(a+h_m)\} - \{10^n a\} = 0.$$

On the other hand, for $n < m$ we can write

$$\begin{aligned}10^n a &= \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3}\ldots a_m \ldots \\ 10^n(a+h_m) &= \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3}\ldots(a_m \pm 1)\ldots\end{aligned}$$

(in order for the second equation to be true it is essential that we choose $h_m = -10^{-m}$ when $a_m = 9$). Now suppose that

$$0.a_{n+1}a_{n+2}a_{n+3}\ldots a_m \cdots \le \tfrac{1}{2}.$$

Then we also have

$$0.a_{n+1}a_{n+2}a_{n+3}\ldots(a_m \pm 1)\cdots \le \tfrac{1}{2}$$

(in the special case $m = n+1$ the second equation is true because we chose $h_m = -10^{-m}$ when $a_m = 4$). This means that

$$\{10^n(a+h_m)\} - \{10^n a\} = \pm 10^{n-m},$$

and exactly the same equation can be derived when $0.a_{n+1}a_{n+2}a_{n+3}\ldots > \tfrac{1}{2}$. Thus, for $n < m$ we have

$$10^{m-n}[\{10^n(a+h_m)\} - \{10^n a\}] = \pm 1.$$

In other words,

$$\frac{f(a+h_m)-f(a)}{h_m}$$

is the sum of $m-1$ numbers, each of which is ± 1. Now adding $+1$ or -1 to a number changes it from odd to even, and vice versa. The sum of $m-1$ numbers each ± 1 is therefore an *even integer* if m is odd, and an *odd integer* if m is even. Consequently the sequence of ratios

$$\frac{f(a+h_m)-f(a)}{h_m}$$

cannot possibly converge, since it is a sequence of integers which are alternately odd and even. ∎

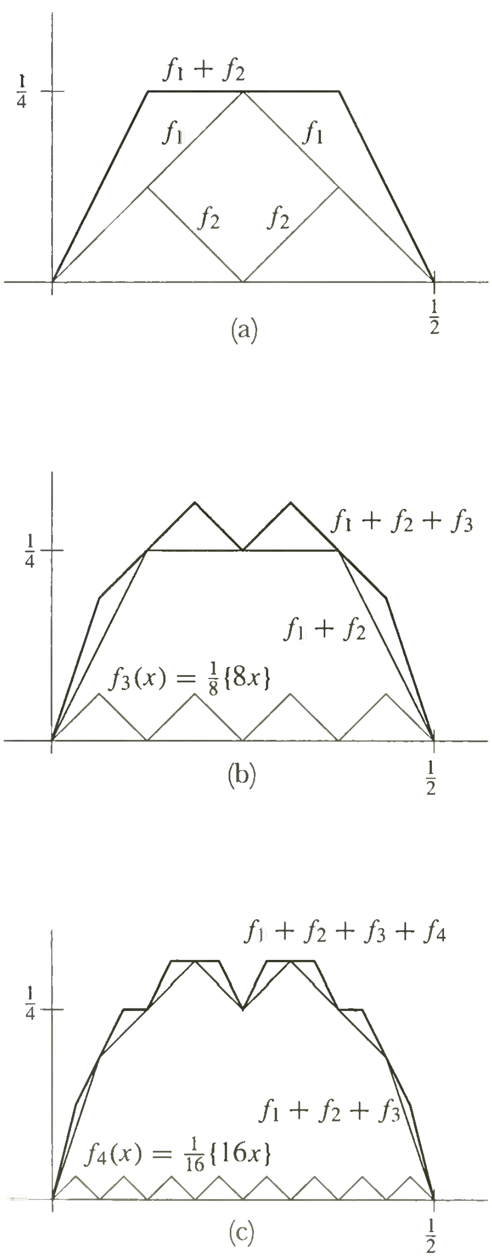


FIGURE 12

In addition to its role in the previous theorem, the Weierstrass M-test is an ideal tool for analyzing functions which are very well behaved. We will give special attention to functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n,$$

which can also be described by the equation

$$f(x) = \sum_{n=0}^{\infty} f_n(x),$$

for $f_n(x) = a_n(x-a)^n$. Such an infinite sum, of functions which depend only on powers of $(x-a)$, is called a **power series centered at a**. For the sake of simplicity, we will usually concentrate on power series centered at 0,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

One especially important group of power series are those of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where f is some function which has derivatives of all orders at a; this series is called the **Taylor series for f at a**. Of course, it is not necessarily true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

this equation holds only when the remainder terms satisfy $\lim_{n\to\infty} R_{n,a}(x) = 0$.

We already know that a power series $\sum_{n=0}^{\infty} a_n x^n$ does not necessarily converge for all x. For example, the power series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

converges only for $|x| \le 1$, while the power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

converges only for $-1 < x \le 1$. It is even possible to produce a power series which converges only for $x = 0$. For example, the power series

$$\sum_{n=0}^{\infty} n!\, x^n$$

does not converge for $x \ne 0$; indeed, the ratios

$$\frac{(n+1)!\,(x^{n+1})}{n!\,x^n} = (n+1)x$$

are unbounded for any $x \ne 0$. If a power series $\sum_{n=0}^{\infty} a_n x^n$ does converge for some $x_0 \ne 0$ however, then a great deal can be said about the series $\sum_{n=0}^{\infty} a_n x^n$ for $|x| < |x_0|$.

THEOREM 6 Suppose that the series

$$f(x_0) = \sum_{n=0}^{\infty} a_n {x_0}^n$$

converges, and let a be any number with $0 < a < |x_0|$. Then on $[-a, a]$ the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges uniformly (and absolutely). Moreover, the same is true for the series

$$g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}.$$

Finally, f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

for all x with $|x| < |x_0|$.

PROOF Since $\sum_{n=0}^{\infty} a_n x_0{}^n$ converges, the terms $a_n x_0{}^n$ approach 0. Hence they are surely bounded: there is some number M such that

$$|a_n x_0{}^n| = |a_n| \cdot |x_0{}^n| \le M \qquad \text{for all } n.$$

Now if x is in $[-a, a]$, then $|x| \le |a|$, so

$$\begin{aligned} |a_n x^n| &= |a_n| \cdot |x^n| \\ &\le |a_n| \cdot |a^n| \\ &= |a_n| \cdot |x_0|^n \cdot \left|\frac{a}{x_0}\right|^n \qquad \text{(this is the clever step)} \\ &\le M \left|\frac{a}{x_0}\right|^n. \end{aligned}$$

But $|a/x_0| < 1$, so the (geometric) series

$$\sum_{n=0}^{\infty} M \left|\frac{a}{x_0}\right|^n = M \sum_{n=0}^{\infty} \left|\frac{a}{x_0}\right|^n$$

converges. Choosing $M \cdot |a/x_0|^n$ as the number M_n in the Weierstrass M-test, it follows that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-a, a]$.

To prove the same assertion for $g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ notice that

$$\begin{aligned} |na_n x^{n-1}| &= n|a_n| \cdot |x^{n-1}| \\ &\le n|a_n| \cdot |a^{n-1}| \\ &= \frac{|a_n|}{|a|} \cdot |x_0|^n \, n \left|\frac{a}{x_0}\right|^n \\ &\le \frac{M}{|a|} n \left|\frac{a}{x_0}\right|^n. \end{aligned}$$

Since $|a/x_0| < 1$, the series

$$\sum_{n=1}^{\infty} \frac{M}{|a|} n \left|\frac{a}{x_0}\right|^n = \frac{M}{|a|} \sum_{n=1}^{\infty} n \left|\frac{a}{x_0}\right|^n$$

converges (this fact was proved in Chapter 23 as an application of the ratio test).

Another appeal to the Weierstrass M-test proves that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges uniformly on $[-a, a]$.

Finally, our corollary proves, first that g is continuous, and then that

$$f'(x) = g(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \qquad \text{for } x \text{ in } [-a, a].$$

Since we could have chosen any number a with $0 < a < |x_0|$, this result holds for all x with $|x| < |x_0|$. ▮

We are now in a position to manipulate power series with ease. Most algebraic manipulations are fairly straightforward consequences of general theorems about infinite series. For example, suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where the two power series both converge for some x_0. Then for $|x| < |x_0|$ we have

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n x^n + b_n x^n) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

So the series $h(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ also converges for $|x| < |x_0|$, and $h = f + g$ for these x.

The treatment of products is just a little more involved. If $|x| < |x_0|$, then we know that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge *absolutely*. So it follows from Theorem 23-9 that the product $\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n$ is given by

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i x^i b_j x^j,$$

where the elements $a_i x^i b_j x^j$ are arranged in any order. In particular, we can choose the arrangement

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

which can be written as

$$\sum_{n=0}^{\infty} c_n x^n \qquad \text{for } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This is the "Cauchy product" that was introduced in Problem 23-10. Thus, the Cauchy product $h(x) = \sum_{n=0}^{\infty} c_n x^n$ also converges for $|x| < |x_0|$ and $h = fg$ for these x.

Finally, suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 \neq 0$, so that $f(0) = a_0 \neq 0$. Then we can try to find a power series $\sum_{n=0}^{\infty} b_n x^n$ which represents $1/f$. This means that we want to have

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots.$$

Since the left side of this equation will be given by the Cauchy product, we want to have

$$\begin{aligned}
&a_0 b_0 = 1 \\
&a_0 b_1 + a_1 b_0 = 0 \\
&a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\
&\cdots
\end{aligned}$$

Since $a_0 \neq 0$, we can solve the first of these equations for b_0. Then we can solve the second for b_1, etc. Of course, we still have to prove that the new series $\sum_{n=0}^{\infty} b_n x^n$ does converge for some $x \neq 0$. This is left as an exercise (Problem 18).

For derivatives, Theorem 6 gives us all the information we need. In particular, when we apply Theorem 6 to the infinite series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,
\end{aligned}$$

we get precisely the results which are expected. Each of these converges for any x_0, hence the conclusions of Theorem 6 apply for any x:

$$\begin{aligned}
\sin'(x) &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots = \cos x, \\
\cos'(x) &= -\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -\sin x, \\
\exp'(x) &= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = \exp(x).
\end{aligned}$$

For the functions arctan and $f(x) = \log(1+x)$ the situation is only slightly more complicated. Since the series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

converges for $x_0 = 1$, it also converges for $|x| < 1$, and

$$\arctan'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2} \qquad \text{for } |x| < 1.$$

In this case, the series happens to converge for $x = -1$ also. However, the formula for the derivative is not correct for $x = 1$ or $x = -1$; indeed the series

$$1 - x^2 + x^4 - x^6 + \cdots$$

diverges for $x = 1$ and $x = -1$. Notice that this does not contradict Theorem 6, which proves that the derivative is given by the expected formula only for $|x| < |x_0|$.

Since the series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

converges for $x_0 = 1$, it also converges for $|x| < 1$, and

$$\frac{1}{1+x} = \log'(1 + x) = 1 - x + x^2 - x^3 + \cdots \qquad \text{for } |x| < 1.$$

In this case, the original series does not converge for $x = -1$; moreover, the differentiated series does not converge for $x = 1$.

All the considerations which apply to a power series will automatically apply to its derivative, at the points where the derivative is represented by a power series. If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all x in some interval $(-R, R)$, then Theorem 6 implies that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

for all x in $(-R, R)$. Applying Theorem 6 once again we find that

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and proceeding by induction we find that

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdot \ldots \cdot (n-k+1) a_n x^{n-k}.$$

Thus, a function defined by a power series which converges in some interval $(-R, R)$ is automatically infinitely differentiable in that interval. Moreover, the previous equation implies that

$$f^{(k)}(0) = k!\, a_k,$$

so that

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

In other words, *a convergent power series centered at 0 is always the Taylor series at 0 of the function which it defines.*

On this happy note we could easily end our study of power series and Taylor series. A careful assessment of our situation will reveal some unexplained facts, however.

The Taylor series of sin, cos, and exp are as satisfactory as we could desire; they converge for all x, and can be differentiated term-by-term for all x. The Taylor series of the function $f(x) = \log(1+x)$ is slightly less pleasing, because it converges only for $-1 < x \le 1$, but this deficiency is a necessary consequence of the basic nature of power series. If the Taylor series for f converged for any x_0 with $|x_0| > 1$, then it would converge on the interval $(-|x_0|, |x_0|)$, and on this interval the function which it defines would be differentiable, and thus continuous. But this is impossible, since it is unbounded on the interval $(-1, 1)$, where it equals $\log(1+x)$.

The Taylor series for arctan is more difficult to comprehend—there seems to be no possible excuse for the refusal of this series to converge when $|x| > 1$. This mysterious behavior is exemplified even more strikingly by the function $f(x) = 1/(1+x^2)$, an infinitely differentiable function which is the next best thing to a polynomial function. The Taylor series of f is given by

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots .$$

If $|x| \ge 1$ the Taylor series does not converge at all. Why? What unseen obstacle prevents the Taylor series from extending past 1 and -1? Asking this sort of question is always dangerous, since we may have to settle for an unsympathetic answer: it happens because it happens—that's the way things are! In this case there does happen to be an explanation, but this explanation is impossible to give at the present time; although the question is about real numbers, it can be answered intelligently only when placed in a broader context. It will therefore be necessary to devote two chapters to quite new material before completing our discussion of Taylor series in Chapter 27.

PROBLEMS

1. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and decide whether $\{f_n\}$ converges uniformly to this function.

(i) $f_n(x) = \sqrt[n]{x}$, on $[0, 1]$.

(ii) $f_n(x) = \begin{cases} 0, & x \le n \\ x - n, & x \ge n, \end{cases}$ on $[a, b]$, and on $\mathbf{R}$.

(iii) $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$.

(iv) $f_n(x) = e^{-nx^2}$, on $[-1, 1]$.

(v) $f_n(x) = \frac{e^{-x^2}}{n}$, on $\mathbf{R}$.

2. This problem asks for the same information as in Problem 1, but the functions are not so easy to analyze. Some hints are given at the end.

(i) $f_n(x) = x^n - x^{2n}$ on $[0, 1]$.

(ii) $f_n(x) = \dfrac{nx}{1 + n + x}$ on $[0, \infty)$.

(iii) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $[a, \infty)$, $a > 0$.

(iv) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $\mathbf{R}$.

(v) $f_n(x) = \sqrt{x + \dfrac{1}{n}} - \sqrt{x}$ on $[a, \infty)$, $a > 0$.

(vi) $f_n(x) = \sqrt{x + \dfrac{1}{n}} - \sqrt{x}$ on $[0, \infty)$.

(vii) $f_n(x) = n\left(\sqrt{x + \dfrac{1}{n}} - \sqrt{x}\right)$ on $[a, \infty)$, $a > 0$.

(viii) $f_n(x) = n\left(\sqrt{x + \dfrac{1}{n}} - \sqrt{x}\right)$ on $[0, \infty)$ and on $(0, \infty)$.

Hints: (i) For each n, find the maximum of $|f - f_n|$ on $[0, 1]$. (ii) For each n, consider $|f(x) - f_n(x)|$ for x large. (iii) Mean Value Theorem. (iv) Give a separate estimate of $|f(x) - f_n(x)|$ for small $|x|$. (vii) Use (v).

3. Find the Taylor series at 0 for each of the following functions.

(i) $f(x) = \dfrac{1}{x - a}$, $a \neq 0$.

(ii) $f(x) = \log(x - a)$, $a < 0$.

(iii) $f(x) = \dfrac{1}{\sqrt{1 - x}} = (1 - x)^{-1/2}$. (Use Problem 20-21.)

(iv) $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(v) $f(x) = \arcsin x$.

4. Find each of the following infinite sums.

(i) $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots.$

(ii) $1 - x^3 + x^6 - x^9 + \cdots$ for $|x| < 1$.
Hint: What is $1 - x + x^2 - x^3 + \cdots$?

(iii) $\dfrac{x^2}{2} - \dfrac{x^3}{3 \cdot 2} + \dfrac{x^4}{4 \cdot 3} - \dfrac{x^5}{5 \cdot 4} + \cdots$ for $|x| < 1$.
Hint: Differentiate.

5. Evaluate the following infinite sums. (In most cases they are $f(a)$ where a is some obvious number and $f(x)$ is given by some power series. To evaluate the various power series, manipulate them until some well-known power series emerge.)

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}.$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}.$

(iii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1}$

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}.$

(v) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n(n+1)}.$

(vi) $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2^n n!}.$

6. If $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$, find $f^{(k)}(0)$. Hint: Find the power series for f.

7. In this problem we deduce the binomial series $(1+x)^\alpha = \displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, $|x| < 1$ without all the work of Problem 23-21, although we will use a fact established in part (a) of that problem—the series $f(x) = \displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ does converge for $|x| < 1$.

(a) Prove that $(1+x)f'(x) = \alpha f(x)$ for $|x| < 1$.

(b) Now show that any function f satisfying part (a) is of the form $f(x) = c(1+x)^\alpha$ for some constant c, and use this fact to establish the binomial series. Hint: Consider $g(x) = f(x)/(1+x)^\alpha$.

8. Suppose that f_n are nonnegative bounded functions on A and let $M_n = \sup f_n$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, does it follow that $\sum_{n=1}^{\infty} M_n$ converges (a converse to the Weierstrass M-test)?

9. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

converges uniformly on $\mathbf{R}$.

10. (a) Prove that the series

$$\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$$

converges uniformly on $[a, \infty)$ for $a > 0$. Hint: $\lim_{h \to 0} (\sin h)/h = 1$.

(b) By considering the sum from N to ∞ for $x = 2/(\pi 3^N)$, show that the series does not converge uniformly on $(0, \infty)$.

11. (a) Prove that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2}$$

converges uniformly on $[a, \infty)$ for $a > 0$. Hint: First find the maximum of $nx/(1+n^4x^2)$ on $[0, \infty)$.

(b) Show that

$$f\left(\frac{1}{N}\right) \geq \frac{N}{2} \sum_{n \geq \sqrt{N}} \frac{1}{n^3},$$

and by using an integral to estimate the sum, show that $f(1/N^2) \geq 1/4$. Conclude that the series does not converge uniformly on $\mathbf{R}$.

(c) What about the series

$$\sum_{n=0}^{\infty} \frac{nx}{1+n^5x^2}?$$

12. (a) Use Problem 15-33 and Abel's Lemma (Problem 19-36) to obtain a "uniform Cauchy condition", showing that for any $\varepsilon > 0$,

$$\left|\sum_{k=m}^{n} \frac{\sin kx}{k}\right|$$

can be made arbitrarily small on the whole interval $[\varepsilon, 2\pi - \varepsilon]$ by choosing m (and n) large enough. Conclude that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges uniformly on $[\varepsilon, 2\pi - \varepsilon]$ for $\varepsilon > 0$.

(b) For $x = \pi/N$, with N large, show that

$$\left|\sum_{k=N}^{2N} \sin kx\right| = \left|\sum_{k=0}^{N} \sin kx\right| \geq \frac{N}{\pi}.$$

Conclude that

$$\left|\sum_{k=N}^{2N} \frac{\sin kx}{k}\right| \geq \frac{1}{2\pi},$$

and that the series does not converge uniformly on $[0, 2\pi]$.

13. (a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in some interval $(-R, R)$ and that $f(x) = 0$ for all x in $(-R, R)$. Prove that each $a_n = 0$. (If you remember the formula for a_n this is easy.)

(b) Suppose we know only that $f(x_n) = 0$ for some sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = 0$. Prove again that each $a_n = 0$. Hint: First show that $f(0) = a_0 = 0$; then that $f'(0) = a_1 = 0$, etc.

This result shows that if $f(x) = e^{-1/x^2} \sin 1/x$ for $x \neq 0$, then f cannot possibly be written as a power series. It also shows that a function defined by a power series cannot be 0 for $x \leq 0$ but nonzero for $x > 0$—thus a power series cannot describe the motion of a particle which has remained at rest until time 0, and then begins to move!

(c) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge for all x in some interval containing 0 and that $f(t_m) = g(t_m)$ for some sequence $\{t_m\}$ converging to 0. Show that $a_n = b_n$ for each n.

14. Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function, then $a_n = 0$ for n odd, and if f is an odd function, then $a_n = 0$ for n even.

15. Show that the power series for $f(x) = \log(1-x)$ converges only for $-1 \le x < 1$, and that the power series for $g(x) = \log[(1+x)/(1-x)]$ converges only for x in $(-1, 1)$.

***16.** Recall that the Fibonacci sequence $\{a_n\}$ is defined by $a_1 = a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$.

(a) Show that $a_{n+1}/a_n \le 2$.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^{n-1} = 1 + x + 2x^2 + 3x^3 + \cdots.$$

Use the ratio test to prove that $f(x)$ converges if $|x| < 1/2$.

(c) Prove that if $|x| < 1/2$, then

$$f(x) = \frac{-1}{x^2 + x - 1}.$$

Hint: This equation can be written $f(x) - xf(x) - x^2 f(x) = 1$.

(d) Use the partial fraction decomposition for $1/(x^2+x-1)$, and the power series for $1/(x-a)$, to obtain another power series for f.

(e) Since the two power series obtained for f must be the same (they are both the Taylor series of the function), conclude that

$$a_n = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

17. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Suppose we merely knew that $f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$ for some c_n, but we didn't know how to multiply series in general. Use Leibniz's formula (Problem 10-20) to show directly that this series for fg must indeed be the Cauchy product of the series for f and g.

18. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for some x_0, and that $a_0 \ne 0$; for simplicity, we'll assume that $a_0 = 1$. Let $\{b_n\}$ be the sequence defined recursively by

$$b_0 = 1$$

$$b_n = -\sum_{k=0}^{n-1} b_k a_{n-k}.$$

The aim of this problem is to show that $\sum_{n=0}^{\infty} b_n x^n$ also converges for some $x \neq 0$, so that it represents $1/f$ for small enough $|x|$.

(a) If all $|a_n x_0{}^n| \leq M$, show that

$$|b_n x_0{}^n| \leq M \sum_{k=0}^{n-1} |b_k x_0{}^k|.$$

(b) Choose M so that $|a_n x_0{}^n| \leq M$, and also so that $M/(M^2 - 1) \leq 1$. Show that

$$|b_n x_0{}^n| \leq M^{2n}.$$

(c) Conclude that $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x|$ sufficiently small.

***19.** Suppose that $\sum_{n=0}^{\infty} a_n$ converges. We know that the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ must converge uniformly on $[-a, a]$ for $0 < a < 1$, but it may not converge uniformly on $[-1, 1]$; in fact, it may not even converge at the point -1 (for example, if $f(x) = \log(1 + x)$). However, a beautiful theorem of Abel shows that the series *does* converge uniformly on $[0, 1]$. Consequently, f is continuous on $[0, 1]$ and, in particular, $\sum_{n=0}^{\infty} a_n = \lim_{x \to 1} \sum_{n=0}^{\infty} a_n x^n$. Prove Abel's Theorem by noticing that if $|a_m + \cdots + a_n| < \varepsilon$, then $|a_m x^m + \cdots + a_n x^n| < \varepsilon$, by Abel's Lemma (Problem 19-36).

20. A sequence $\{a_n\}$ is called **Abel summable** if $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$ exists; Problem 19 shows that a summable sequence is necessarily Abel summable. Find a sequence which is Abel summable, but which is not summable. Hint: Look over the list of Taylor series until you find one which does not converge at 1, even though the function it represents is continuous at 1.

21. (a) Using Problem 19, find the following infinite sums.

(i) $\dfrac{1}{2 \cdot 1} - \dfrac{1}{3 \cdot 2} + \dfrac{1}{4 \cdot 3} - \dfrac{1}{5 \cdot 4} + \cdots.$

(ii) $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots.$

(b) Let $\sum_{n=0}^{\infty} c_n$ be the Cauchy product of two convergent power series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, and suppose merely that $\sum_{n=0}^{\infty} c_n$ converges. Prove that, in fact, it converges to the product $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n$.

22. (a) Show that the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}$$

converges uniformly to $\frac{1}{2}\log(x+1)$ on $[-a, a]$ for $0 < a < 1$, but that at 1 it converges to $\log 2$. (Why doesn't this contradict Abel's Theorem (Problem 19)?)

23. (a) Suppose that $\{f_n\}$ is a sequence of bounded (not necessarily continuous) functions on $[a, b]$ which converge uniformly to f on $[a, b]$. Prove that f is bounded on $[a, b]$.

(b) Find a sequence of continuous functions on $[a, b]$ which converge pointwise to an unbounded function on $[a, b]$.

24. Suppose that f is differentiable. Prove that the function f' is the pointwise limit of a sequence of continuous functions. (Since we already know examples of discontinuous derivatives, this provides another example where the pointwise limit of continuous functions is not continuous.)

25. Find a sequence of integrable functions $\{f_n\}$ which converges to the (nonintegrable) function f that is 1 on the rationals and 0 on the irrationals. Hint: Each f_n will be 0 except at a few points.

26. (a) Prove that if f is the uniform limit of $\{f_n\}$ on $[a, b]$ and each f_n is integrable on $[a, b]$, then so is f. (So one of the hypotheses in Theorem 1 was unnecessary.)

(b) In Theorem 3 we assumed only that $\{f_n\}$ converges pointwise to f. Show that the remaining hypotheses ensure that $\{f_n\}$ actually converges uniformly to f.

(c) Suppose that in Theorem 3 we do not assume $\{f_n\}$ converges to a function f, but instead assume only that $f_n(x_0)$ converges for some x_0 in $[a, b]$. Show that f_n does converge (uniformly) to some f (with $f' = g$).

(d) Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$$

converges uniformly on $[0, \infty)$.

27. Suppose that f_n are continuous functions on $[0, 1]$ that converge uniformly to f. Prove that

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f.$$

Is this true if the convergence isn't uniform?

28. (a) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ which approaches 0 pointwise. Suppose moreover that we have $f_n(x) \geq f_{n+1}(x) \geq 0$ for all n and all x in $[a, b]$. Prove that $\{f_n\}$ actually approaches 0 uniformly on $[a, b]$. Hint: Suppose not, choose an appropriate sequence of points x_n in $[a, b]$, and apply the Bolzano-Weierstrass theorem.

(b) Prove Dini's Theorem: If $\{f_n\}$ is a nonincreasing sequence of continuous functions on $[a, b]$ which approaches the continuous function f pointwise, then $\{f_n\}$ also approaches f uniformly on $[a, b]$. (The same result holds if $\{f_n\}$ is a nondecreasing sequence.)

(c) Does Dini's Theorem hold if f isn't continuous? How about if $[a, b]$ is replaced by the open interval (a, b)?

29. (a) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ that converges uniformly to f. Prove that if x_n approaches x, then $f_n(x_n)$ approaches $f(x)$.

(b) Is this statement true without assuming that the f_n are continuous?

(c) Prove the converse of part (a): If f is continuous on $[a, b]$ and $\{f_n\}$ is a sequence with the property that $f_n(x_n)$ approaches $f(x)$ whenever x_n approaches x, then f_n converges uniformly to f on $[a, b]$. Hint: If not, there is an $\varepsilon > 0$ and a sequence x_n with $|f_n(x_n) - f(x_n)| > \varepsilon$ for infinitely many distinct x_n. Then use the Bolzano-Weierstrass theorem.

30. This problem outlines a completely different approach to the integral; consequently, it is unfair to use any facts about integrals learned previously.

(a) Let s be a step function on $[a, b]$, so that s is constant on (t_{i-1}, t_i) for some partition $\{t_0, \dots, t_n\}$ of $[a, b]$. Define $\displaystyle\int_a^b s$ as $\displaystyle\sum_{i=1}^{n} s_i \cdot (t_i - t_{i-1})$ where s_i is the (constant) value of s on (t_{i-1}, t_i). Show that this definition does not depend on the partition $\{t_0, \dots, t_n\}$.

(b) A function f is called a **regulated** function on $[a, b]$ if it is the uniform limit of a sequence of step functions $\{s_n\}$ on $[a, b]$. Show that in this case there is, for every $\varepsilon > 0$, some N such that for $m, n > N$ we have $|s_n(x) - s_m(x)| < \varepsilon$ for all x in $[a, b]$.

(c) Show that the sequence of numbers $\left\{\displaystyle\int_a^b s_n\right\}$ will be a Cauchy sequence.

(d) Suppose that $\{t_n\}$ is another sequence of step functions on $[a, b]$ which converges uniformly to f. Show that for every $\varepsilon > 0$ there is an N such that for $n > N$ we have $|s_n(x) - t_n(x)| < \varepsilon$ for x in $[a, b]$.

(e) Conclude that $\displaystyle\lim_{n\to\infty} \int_a^b s_n = \lim_{n\to\infty} \int_a^b t_n$. This means that we can *define* $\displaystyle\int_a^b f$ to be $\displaystyle\lim_{n\to\infty} s_n$ for any sequence of step functions $\{s_n\}$ converging uniformly to f. The only remaining question is: Which functions are regulated?

*(f) Prove that a continuous function is regulated. Hint: To find a step function s on $[a, b]$ with $|f(x) - s(x)| < \varepsilon$ for all x in $[a, b]$, consider all y for which there is such a step function on $[a, y]$.

(g) Every step function s has the property that $\lim_{x \to a^+} s(x)$ and $\lim_{x \to a^-} s(x)$ exist for all a. Conclude that every regulated function has the same property, and find an integrable function that is not regulated. (It is also true that, conversely, every function f with the property that $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist for all a is regulated.)

***31.** Find a sequence $\{f_n\}$ approaching f uniformly on $[0, 1]$ for which we have $\lim_{n \to \infty}$ (length of f_n on $[0, 1]$) $\neq$ length of f on $[0, 1]$. (Length is defined in Problem 13-25, but the simplest example will involve functions the length of whose graphs will be obvious.)

CHAPTER 25 COMPLEX NUMBERS

With the exception of the last few paragraphs of the previous chapter, this book has presented unremitting propaganda for the real numbers. Nevertheless, the real numbers do have a great deficiency—not every polynomial function has a root. The simplest and most notable example is the fact that no number x can satisfy $x^2 + 1 = 0$. This deficiency is so severe that long ago mathematicians felt the need to "invent" a number i with the property that $i^2 + 1 = 0$. For a long time the status of the "number" i was quite mysterious: since there is no number x satisfying $x^2 + 1 = 0$, it is nonsensical to say "let i be the number satisfying $i^2 + 1 = 0$." Nevertheless, admission of the "imaginary" number i to the family of numbers seemed to simplify greatly many algebraic computations, especially when "complex numbers" $a + bi$ (for a and b in $\mathbf{R}$) were allowed, and all the laws of arithmetical computation enumerated in Chapter 1 were assumed to be valid. For example, every quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

can be solved formally to give

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac \geq 0$, these formulas give correct solutions; when complex numbers are allowed the formulas seem to make sense in all cases. For example, the equation

$$x^2 + x + 1 = 0$$

has no real root, since

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 + \tfrac{3}{4} > 0, \qquad \text{for all } x.$$

But the formula for the roots of a quadratic equation suggest the "solutions"

$$x = \frac{-1 + \sqrt{-3}}{2} \quad \text{and} \quad x = \frac{-1 - \sqrt{-3}}{2};$$

if we understand $\sqrt{-3}$ to mean $\sqrt{3 \cdot (-1)} = \sqrt{3} \cdot \sqrt{-1} = \sqrt{3}\, i$, then these numbers would be

$$-\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

It is not hard to check that these, as yet purely formal, numbers do indeed satisfy the equation

$$x^2 + x + 1 = 0.$$

It is even possible to "solve" quadratic equations whose coefficients are themselves complex numbers. For example, the equation

$$x^2 + x + 1 + i = 0$$

ought to have the solutions

$$x = \frac{-1 \pm \sqrt{1 - 4(1 + i)}}{2} = \frac{-1 \pm \sqrt{-3 - 4i}}{2},$$

where the symbol $\sqrt{-3 - 4i}$ means a complex number $\alpha + \beta i$ whose square is $-3 - 4i$. In order to have

$$(\alpha + \beta i)^2 = \alpha^2 - \beta^2 + 2\alpha\beta i = -3 - 4i$$

we need

$$\begin{aligned} \alpha^2 - \beta^2 &= -3, \\ 2\alpha\beta &= -4. \end{aligned}$$

These two equations can easily be solved for real α and β; in fact, there are two possible solutions:

$$\begin{aligned} \alpha &= 1 \\ \beta &= -2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha &= -1 \\ \beta &= 2. \end{aligned}$$

Thus the two "square roots" of $-3 - 4i$ are $1 - 2i$ and $-1 + 2i$. There is no reasonable way to decide which one of these should be called $\sqrt{-3 - 4i}$, and which $-\sqrt{-3 - 4i}$; the conventional usage of $\sqrt{x}$ makes sense only for real $x \geq 0$, in which case $\sqrt{x}$ denotes the (real) nonnegative root. For this reason, the solution

$$x = \frac{-1 \pm \sqrt{-3 - 4i}}{2}$$

must be understood as an abbreviation for:

$$x = \frac{-1 + r}{2}, \qquad \text{where } r \text{ is one of the square roots of } -3 - 4i.$$

With this understanding we arrive at the solutions

$$\begin{aligned} x &= \frac{-1 + 1 - 2i}{2} = -i, \\ x &= \frac{-1 - 1 + 2i}{2} = -1 + i; \end{aligned}$$

as you can easily check, these numbers do provide formal solutions for the equation

$$x^2 + x + 1 + i = 0.$$

For cubic equations complex numbers are equally useful. Every cubic equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (a \neq 0)$$

with real coefficients a, b, c, and d, has, as we know, a real root α, and if we divide $ax^3 + bx^2 + cx + d$ by $x - \alpha$ we obtain a second-degree polynomial whose roots are the other roots of $ax^3 + bx^2 + cx + d = 0$; the roots of this second-degree polynomial may be complex numbers. Thus a cubic equation will have either three real roots or one real root and 2 complex roots. The existence of the real root is guaranteed by our theorem that every odd degree equation has a real root, but it is not really necessary to appeal to this theorem (which is of no use at all if the coefficients are complex); in the case of a cubic equation we can, with sufficient cleverness, actually find a formula for all the roots. The following derivation is presented not only as an interesting illustration of the ingenuity of early mathematicians, but as further evidence for the importance of complex numbers (whatever they may be).

To solve the most general cubic equation, it obviously suffices to consider only equations of the form

$$x^3 + bx^2 + cx + d = 0.$$

It is even possible to eliminate the term involving x^2, by a fairly straight-forward manipulation. If we let

$$x = y - \frac{b}{3},$$

then

$$\begin{aligned} x^3 &= y^3 - by^2 + \frac{b^2y}{3} - \frac{b^3}{27}, \\ x^2 &= y^2 - \frac{2by}{3} + \frac{b^2}{9}, \end{aligned}$$

so

$$\begin{aligned} 0 &= x^3 + bx^2 + cx + d \\ &= \left(y^3 - by^2 + \frac{b^2y}{3} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d \\ &= y^3 + \left(\frac{b^2}{3} - \frac{2b^2}{3} + c\right)y + \left(\frac{b^3}{9} - \frac{b^3}{27} - \frac{bc}{3} + d\right). \end{aligned}$$

The right-hand side now contains no term with y^2. If we can solve the equation for y we can find x; this shows that it suffices to consider in the first place only equations of the form

$$x^3 + px + q = 0.$$

In the special case $p = 0$ we obtain the equation $x^3 = -q$. We shall see later on that every complex number does have a cube root, in fact it has three, so that this equation has three solutions. The case $p \neq 0$, on the other hand, requires quite an ingenious step. Let

$$(*) \qquad x = w - \frac{p}{3w}$$

Then

$$\begin{aligned} 0 = x^3 + px + q &= \left(w - \frac{p}{3w}\right)^3 + p\left(w - \frac{p}{3w}\right) + q \\ &= w^3 - \frac{3w^2p}{3w} + \frac{3wp^2}{9w^2} - \frac{p^3}{27w^3} + pw - \frac{p^2}{3w} + q \\ &= w^3 - \frac{p^3}{27w^3} + q. \end{aligned}$$

This equation can be written

$$27(w^3)^2 + 27q(w^3) - p^3 = 0,$$

which is a quadratic equation in w^3 (!!).

Thus

$$\begin{aligned} w^3 &= \frac{-27q \pm \sqrt{(27)^2q^2 + 4 \cdot 27p^3}}{2 \cdot 27} \\ &= -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \end{aligned}$$

Remember that this really means:

$$w^3 = -\frac{q}{2} + r, \quad \text{where } r \text{ is a square root of } \frac{q^2}{4} + \frac{p^3}{27}.$$

We can therefore write

$$w = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}};$$

this equation means that w is some cube root of $-q/2+r$, where r is some square root of $q^2/4 + p^3/27$. This allows six possibilities for w, but when these are substituted into (∗), yielding

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3 \cdot \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}},$$

it turns out that only 3 different values for x will be obtained! An even more surprising feature of this solution arises when we consider a cubic equation all of whose roots are real; the formula derived above may still involve complex numbers in an essential way. For example, the roots of

$$x^3 - 15x - 4 = 0$$

are 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$. On the other hand, the formula derived above (with $p=-15$, $q=-4$) gives as one solution

$$\begin{aligned} x &= \sqrt[3]{2+\sqrt{4-125}} - \frac{-15}{3\cdot\sqrt[3]{2+\sqrt{4-125}}} \\ &= \sqrt[3]{2+11i} + \frac{15}{3\cdot\sqrt[3]{2+11i}}. \end{aligned}$$

Now,

$$\begin{aligned} (2+i)^3 &= 2^3 + 3\cdot 2^2 i + 3\cdot 2\cdot i^2 + i^3 \\ &= 8+12i-6-i \\ &= 2+11i, \end{aligned}$$

so one of the cube roots of $2+11i$ is $2+i$. Thus, for one solution of the equation we obtain

$$\begin{aligned} x &= 2+i+\frac{15}{6+3i} \\ &= 2+i+\frac{15}{6+3i}\cdot\frac{6-3i}{6-3i} \\ &= 2+i+\frac{90-45i}{36+9} \\ &= 4\ (!). \end{aligned}$$

The other roots can also be found if the other cube roots of $2+11i$ are known. The fact that even one of these real roots is obtained from an expression which depends on complex numbers is impressive enough to suggest that the use of complex numbers cannot be entirely nonsense. As a matter of fact, the formulas for the solutions of the quadratic and cubic equations can be interpreted entirely in terms of real numbers.

Suppose we agree, for the moment, to write all complex numbers as $a+bi$, writing the real number a as $a+0i$ and the number i as $0+1i$. The laws of ordinary arithmetic and the relation $i^2=-1$ show that

$$\begin{aligned} (a+bi)+(c+di) &= (a+c)+(b+d)i \\ (a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i. \end{aligned}$$

Thus, an equation like

$$(1+2i)\cdot(3+1i) = 1+7i$$

may be regarded simply as an abbreviation for the *two* equations

$$\begin{aligned} 1\cdot 3 - 2\cdot 1 &= 1, \\ 1\cdot 1 + 2\cdot 3 &= 7. \end{aligned}$$

The solution of the quadratic equation $ax^2 + bx + c = 0$ with real coefficients could be paraphrased as follows:

$$\text{If}\begin{cases} u^2 - v^2 = b^2 - 4ac, \\ uv = 0, \end{cases}$$
$$\text{(i.e., if } (u + vi)^2 = b^2 - 4ac\text{)},$$

$$\text{then}\begin{cases} a\left[\left(\dfrac{-b+u}{2a}\right)^2 - \left(\dfrac{v}{2a}\right)^2\right] + b\left[\dfrac{-b+u}{2a}\right] + c = 0, \\ a\left[2\left(\dfrac{-b+u}{2a}\right)\left(\dfrac{v}{2a}\right)\right] + b\left[\dfrac{v}{2a}\right] = 0, \end{cases}$$

$$\left(\text{i.e., then } a\left(\frac{-b+u+vi}{2a}\right)^2 + b\left(\frac{-b+u+vi}{2a}\right) + c = 0\right).$$

It is not very hard to check this assertion about real numbers without writing down a single "i," but the complications of the statement itself should convince you that equations about complex numbers are worthwhile as abbreviations for pairs of equations about real numbers. (If you are still not convinced, try paraphrasing the solution of the cubic equation.) If we really intend to use complex numbers consistently, however, it is going to be necessary to present some reasonable definition.

One possibility has been implicit in this whole discussion. All mathematical properties of a complex number $a + bi$ are determined completely by the real numbers a and b; any mathematical object with this same property may reasonably be used to define a complex number. The obvious candidate is the ordered pair (a, b) of real numbers; we shall accordingly *define* a complex number to be a pair of real numbers, and likewise *define* what addition and multiplication of complex numbers is to mean.

DEFINITION

A **complex number** is an ordered pair of real numbers; if $z = (a, b)$ is a complex number, then a is called the **real part** of z, and b is called the **imaginary part** of z. The set of all complex numbers is denoted by $\mathbf{C}$. If (a, b) and (c, d) are two complex numbers we define

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (a \cdot c - b \cdot d, a \cdot d + b \cdot c).$$

(The $+$ and $\cdot$ appearing on the left side are new symbols being defined, while the $+$ and $\cdot$ appearing on the right side are the familiar addition and multiplication for real numbers.)

When complex numbers were first introduced, it was understood that real numbers were, in particular, complex numbers; if our definition is taken seriously this is not true — a real number is not a pair of real numbers, after all. This difficulty is only a minor annoyance, however. Notice that

$$\begin{aligned}(a,0)+(b,0)&=(a+b,0+0)=(a+b,0),\\ (a,0)\cdot(b,0)&=(a\cdot b-0\cdot 0,a\cdot 0+0\cdot b)=(a\cdot b,0);\end{aligned}$$

this shows that the complex numbers of the form $(a,0)$ behave precisely the same with respect to addition and multiplication of complex numbers as real numbers do with their own addition and multiplication. For this reason we will adopt the convention that $(a,0)$ will be denoted simply by a. The familiar $a+bi$ notation for complex numbers can now be recovered if one more definition is made.

DEFINITION

$$\boxed{i=(0,1).}$$

Notice that $i^2=(0,1)\cdot(0,1)=(-1,0)=-1$ (the last equality sign depends on our convention). Moreover

$$\begin{aligned}(a,b)&=(a,0)+(0,b)\\ &=(a,0)+(b,0)\cdot(0,1)\\ &=a+bi.\end{aligned}$$

You may feel that our definition was merely an elaborate device for defining complex numbers as "expressions of the form $a+bi$." That is essentially correct; it is a firmly established prejudice of modern mathematics that new objects must be defined as something specific, not as "expressions." Nevertheless, it is interesting to note that mathematicians were sincerely worried about using complex numbers until the modern definition was proposed. Moreover, the precise definition emphasizes one important point. Our aim in introducing complex numbers was to avoid the necessity of paraphrasing statements about complex numbers in terms of their real and imaginary parts. This means that we wish to work with complex numbers in the same way that we worked with rational or real numbers. For example, the solution of the cubic equation required writing $x=w-p/3w$, so we want to know that $1/w$ makes sense. Moreover, w^3 was found by solving a quadratic equation, which requires numerous other algebraic manipulations. In short, we are likely to use, at some time or other, any manipulations performed on real numbers. We certainly do not want to stop each time and justify every step. Fortunately this is not necessary. Since all algebraic manipulations performed on real numbers can be justified by the properties listed in Chapter 1, it is only necessary to check that these properties are also true for complex numbers. In most cases this is quite easy, and these facts will not be listed as formal theorems. For example, the proof of P1,

$$[(a,b)+(c,d)]+(e,f)=(a,b)+[(c,d)+(e,f)]$$

requires only the application of the definition of addition for complex numbers. The left side becomes

$$([a+c]+e,[b+d]+f),$$

and the right side becomes

$$(a + [c + e], b + [d + f]);$$

these two are equal because P1 is true for real numbers. It is a good idea to check P2–P6 and P8 and P9. Notice that the complex numbers playing the role of 0 and 1 in P2 and P6 are $(0, 0)$ and $(1, 0)$, respectively. It is not hard to figure out what $-(a, b)$ is, but the multiplicative inverse for (a, b) required in P7 is a little trickier: if $(a, b) \neq (0, 0)$, then $a^2 + b^2 \neq 0$ and

$$(a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = (1, 0).$$

This fact could have been guessed in two ways. To find (x, y) with

$$(a, b) \cdot (x, y) = (1, 0)$$

it is only necessary to solve the equations

$$\begin{aligned} ax - by &= 1, \\ bx + ay &= 0. \end{aligned}$$

The solutions are $x = a/(a^2 + b^2)$, $y = -b/(a^2 + b^2)$. It is also possible to reason that if $1/(a + bi)$ means anything, then it should be true that

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2}.$$

Once the existence of inverses has actually been proved (after guessing the inverse by some method), it follows that this manipulation is really valid; it is the easiest one to remember when the inverse of a complex number is actually being sought—it was precisely this trick which we used to evaluate

$$\begin{aligned} \frac{15}{6 + 3i} &= \frac{15}{6 + 3i} \cdot \frac{6 - 3i}{6 - 3i} \\ &= \frac{90 - 45i}{36 + 9}. \end{aligned}$$

Unlike P1–P9, the rules P10–P12 do not have analogues: it is easy to prove that there is *no* set P of *complex* numbers such that P10–P12 are satisfied for all *complex* numbers. In fact, if there were, then P would have to contain 1 (since $1 = 1^2$) and also -1 (since $-1 = i^2$), and this would contradict P10. The absence of P10–P12 will not have disastrous consequences, but it does mean that we cannot define $z < w$ for complex z and w. Also, you may remember that for the real numbers, P10–P12 were used to prove that $1 + 1 \neq 0$. Fortunately, the corresponding fact for complex numbers can be reduced to this one: clearly $(1, 0) + (1, 0) \neq (0, 0)$.

Although we will usually write complex numbers in the form $a + bi$, it is worth remembering that the set of all complex numbers $\mathbf{C}$ is just the collection of all pairs of real numbers. Long ago this collection was identified with the plane, and for this reason the plane is often called the "complex plane." The horizontal axis, which consists of all points $(a, 0)$ for a in $\mathbf{R}$, is often called the *real axis*, and the vertical axis is called the *imaginary axis.* Two important definitions are also related to this geometric picture.

DEFINITION

> If $z = x + iy$ is a complex number (with x and y real), then the **conjugate** $\bar{z}$ of z is defined as
> $$\bar{z} = x - iy,$$
> and the **absolute value** or **modulus** $|z|$ of z is defined as
> $$|z| = \sqrt{x^2 + y^2}.$$
> (Notice that $x^2 + y^2 \geq 0$, so that $\sqrt{x^2 + y^2}$ is defined unambiguously; it denotes the nonnegative real square root of $x^2 + y^2$.)

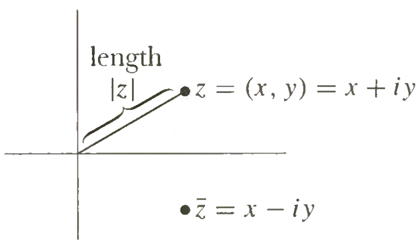


FIGURE 1

Geometrically, $\bar{z}$ is simply the reflection of z in the real axis, while $|z|$ is the distance from z to $(0, 0)$ (Figure 1). Notice that the absolute value notation for complex numbers is consistent with that for real numbers. The **distance** between two complex numbers z and w can be defined quite easily as $|z - w|$. The following theorem lists all the important properties of conjugates and absolute values.

THEOREM 1 Let z and w be complex numbers. Then

(1) $\bar{\bar{z}} = z$.
(2) $\bar{z} = z$ if and only if z is real (i.e., is of the form $a + 0i$, for some real number a).
(3) $\overline{z + w} = \bar{z} + \bar{w}$.
(4) $\overline{-z} = -\bar{z}$.
(5) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.
(6) $\overline{z^{-1}} = (\bar{z})^{-1}$, if $z \neq 0$.
(7) $|z|^2 = z \cdot \bar{z}$.
(8) $|z \cdot w| = |z| \cdot |w|$.
(9) $|z + w| \leq |z| + |w|$.

PROOF Assertions (1) and (2) are obvious. Equations (3) and (5) may be checked by straightforward calculations and (4) and (6) may then be proved by a trick:

$$0 = \bar{0} = \overline{z + (-z)} = \bar{z} + \overline{-z}, \qquad \text{so } \overline{-z} = -\bar{z},$$
$$1 = \bar{1} = \overline{z \cdot (z^{-1})} = \bar{z} \cdot \overline{z^{-1}}, \qquad \text{so } \overline{z^{-1}} = (\bar{z})^{-1}.$$

Equations (7) and (8) may also be proved by a straightforward calculation. The only difficult part of the theorem is (9). This inequality has, in fact, already occurred (Problem 4-9), but the proof will be repeated here, using slightly different terminology.

It is clear that equality holds in (9) if $z = 0$ or $w = 0$. It is also easy to see that (9) is true if $z = \lambda w$ for any real number λ (consider separately the cases $\lambda > 0$ and $\lambda < 0$). Suppose, on the other hand, that $z \neq \lambda w$ for any real number λ, and that

$w \neq 0$. Then, for all real numbers λ,

$$
(*) \qquad \begin{aligned}
0 < |z - \lambda w|^2 &= (z - \lambda w) \cdot \overline{(z - \lambda w)} \\
&= (z - \lambda w) \cdot (\bar{z} - \lambda \bar{w}) \\
&= z\bar{z} + \lambda^2 w\bar{w} - \lambda(w\bar{z} + z\bar{w}) \\
&= \lambda^2 |w|^2 + |z|^2 - \lambda(w\bar{z} + z\bar{w}).
\end{aligned}
$$

Notice that $w\bar{z} + z\bar{w}$ is real, since

$$\overline{w\bar{z} + z\bar{w}} = \bar{w}\bar{\bar{z}} + \bar{z}\bar{\bar{w}} = \bar{w}z + \bar{z}w = w\bar{z} + z\bar{w}.$$

Thus the right side of $(*)$ is a quadratic equation in λ with real coefficients and no real solutions; its discriminant must therefore be negative. Thus

$$(w\bar{z} + z\bar{w})^2 - 4|w|^2 \cdot |z|^2 < 0;$$

it follows, since $w\bar{z} + \bar{z}w$ and $|w| \cdot |z|$ are real numbers, and $|w| \cdot |z| \geq 0$, that

$$(w\bar{z} + z\bar{w}) < 2|w| \cdot |z|.$$

From this inequality it follows that

$$
\begin{aligned}
|z + w|^2 &= (z + w) \cdot (\bar{z} + \bar{w}) \\
&= |z|^2 + |w|^2 + (w\bar{z} + z\bar{w}) \\
&< |z|^2 + |w|^2 + 2|w| \cdot |z| \\
&= (|z| + |w|)^2,
\end{aligned}
$$

which implies that

$$|z + w| < |z| + |w|. \blacksquare$$

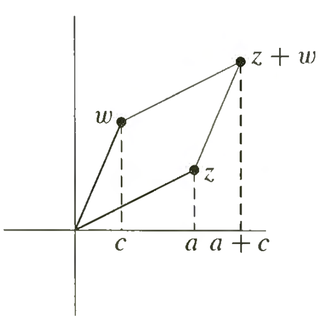


FIGURE 2

The operations of addition and multiplication of complex numbers both have important geometric interpretations. The picture for addition is very simple (Figure 2). Two complex numbers $z = (a, b)$ and $w = (c, d)$ determine a parallelogram having for two of its sides the line segment from $(0, 0)$ to z, and the line segment from $(0, 0)$ to w; the vertex opposite $(0, 0)$ is $z + w$ (a proof of this geometric fact is left to you [compare Appendix 1 to Chapter 4]).

The interpretation of multiplication is more involved. If $z = 0$ or $w = 0$, then $z \cdot w = 0$ (a one-line computational proof can be given, but even this is unnecessary—the assertion has already been shown to follow from P1–P9), so we may restrict our attention to nonzero complex numbers. We begin by putting every nonzero complex number into a special form (compare Appendix 3 to Chapter 4).

For any complex number $z \neq 0$ we can write

$$z = |z| \frac{z}{|z|};$$

in this expression, $|z|$ is a positive real number, while

$$\left| \frac{z}{|z|} \right| = \frac{|z|}{|z|} = 1,$$

so that $z/|z|$ is a complex number of absolute value 1. Now any complex number $a = x + iy$ with $1 = |a| = x^2 + y^2$ can be written in the form

$$a = (\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$$

for some number θ. Thus every nonzero complex number z can be written

$$z = r(\cos\theta + i\sin\theta)$$

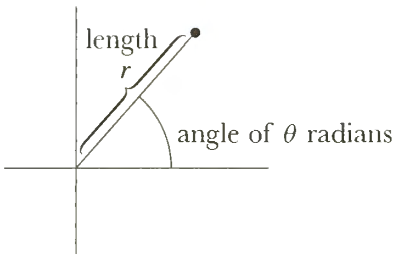


FIGURE 3

for some $r > 0$ and some number θ. The number r is unique (it equals $|z|$), but θ is not unique; if θ_0 is one possibility, then the others are $\theta_0 + 2k\pi$ for k in $\mathbf{Z}$—any one of these numbers is called an **argument** of z. Figure 3 shows z in terms of r and θ. (To find an argument θ for $z = x + iy$ we may note that the equation

$$x + iy = z = |z|(\cos\theta + i\sin\theta)$$

means that

$$\begin{aligned} x &= |z|\ \cos\theta, \\ y &= |z|\ \sin\theta. \end{aligned}$$

So, for example, if $x > 0$ we can take $\theta = \arctan y/x$; if $x = 0$, we can take $\theta = \pi/2$ when $y > 0$ and $\theta = 3\pi/2$ when $y < 0$.)

Now the product of two nonzero complex numbers

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta), \\ w &= s(\cos\phi + i\sin\phi), \end{aligned}$$

is

$$\begin{aligned} z \cdot w &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)] \\ &= rs[\cos(\theta + \phi) + i\sin(\theta + \phi)]. \end{aligned}$$

Thus, the absolute value of a product is the product of the absolute values of the factors, while the sum of any argument for each of the factors will be an argument for the product. For a nonzero complex number

$$z = r(\cos\theta + i\sin\theta)$$

it is now an easy matter to prove by induction the following very important formula (sometimes known as De Moivre's Theorem):

$$z^n = |z|^n(\cos n\theta + i\sin n\theta), \text{ for any argument } \theta \text{ of } z.$$

This formula describes z^n so explicitly that it is easy to decide just when $z^n = w$:

THEOREM 2 Every nonzero complex number has exactly n complex nth roots.

More precisely, for any complex number $w \neq 0$, and any natural number n, there are precisely n different complex numbers z satisfying $z^n = w$.

PROOF Let

$$w = s(\cos\phi + i\sin\phi)$$

for $s = |w|$ and some number ϕ. Then a complex number

$$z = r(\cos\theta + i\sin\theta)$$

satisfies $z^n = w$ if and only if

$$r^n(\cos n\theta + i\sin n\theta) = s(\cos\phi + i\sin\phi),$$

which happens if and only if

$$r^n = s,$$
$$\cos n\theta + i\sin n\theta = \cos\phi + i\sin\phi.$$

From the first equation it follows that

$$r = \sqrt[n]{s},$$

where $\sqrt[n]{s}$ denotes the positive real nth root of s. From the second equation it follows that for some integer k we have

$$\theta = \theta_k = \frac{\phi}{n} + \frac{2k\pi}{n}.$$

Conversely, if we choose $r = \sqrt[n]{s}$ and $\theta = \theta_k$ for some k, then the number $z = r(\cos\theta + i\sin\theta)$ will satisfy $z^n = w$. To determine the number of nth roots of w, it is therefore only necessary to determine which such z are distinct. Now any integer k can be written

$$k = nq + k'$$

for some integer q, and some integer k' between 0 and $n-1$. Then

$$\cos\theta_k + i\sin\theta_k = \cos\theta_{k'} + i\sin\theta_{k'}.$$

This shows that every z satisfying $z^n = w$ can be written

$$z = \sqrt[n]{s}\,(\cos\theta_k + i\sin\theta_k) \quad k = 0, \dots, n-1.$$

Moreover, it is easy to see that these numbers are all different, since any two θ_k for $k = 0, \dots, n-1$ differ by less than 2π. ▮

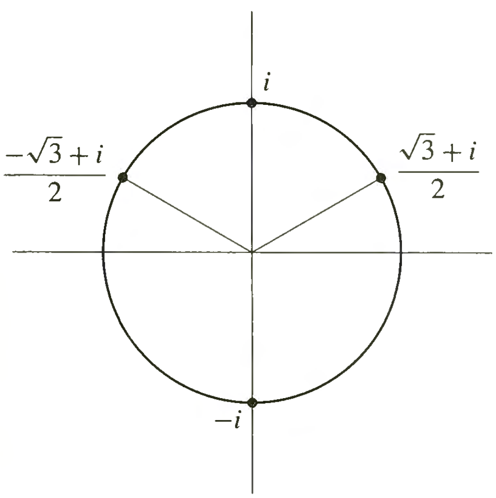


FIGURE 4

In the course of proving Theorem 2, we have actually developed a method for finding the nth roots of a complex number. For example, to find the cube roots of i (Figure 4) note that $|i| = 1$ and that $\pi/2$ is an argument for i. The cube roots of i are therefore

$$1 \cdot \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right],$$

$$1 \cdot \left[\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right] = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6},$$

$$1 \cdot \left[\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right] = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}.$$

Since

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2},$$
$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \qquad \sin\frac{5\pi}{6} = \frac{1}{2},$$
$$\cos\frac{3\pi}{2} = 0, \qquad \sin\frac{3\pi}{2} = -1,$$

the cube roots of i are

$$\frac{\sqrt{3}+i}{2}, \qquad \frac{-\sqrt{3}+i}{2}, \qquad -i.$$

In general, we cannot expect to obtain such simple results. For example, to find the cube roots of $2+11i$, note that $|2+11i| = \sqrt{2^2+11^2} = \sqrt{125}$ and that $\arctan\frac{11}{2}$ is an argument for $2+11i$. One of the cube roots of $2+11i$ is therefore

$$\sqrt[6]{125}\left[\cos\left(\frac{\arctan\frac{11}{2}}{3}\right) + i\sin\left(\frac{\arctan\frac{11}{2}}{3}\right)\right]$$
$$= \sqrt{5}\left[\cos\left(\frac{\arctan\frac{11}{2}}{3}\right) + i\sin\left(\frac{\arctan\frac{11}{2}}{3}\right)\right].$$

Previously we noted that $2+i$ is also a cube root of $2+11i$. Since $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$, and since $\arctan\frac{1}{2}$ is an argument of $2+i$, we can write this cube root as

$$2+i = \sqrt{5}(\cos\arctan\tfrac{1}{2} + i\sin\arctan\tfrac{1}{2}).$$

These two cube roots are actually the same number, because

$$\frac{\arctan\frac{11}{2}}{3} = \arctan\frac{1}{2}$$

(you can check this by using the formula in Problem 15-9), but this is hardly the sort of thing one might notice!

The fact that every complex number has an nth root for all n is just a special case of a very important theorem. The number i was originally introduced in order to provide a solution for the equation $x^2+1=0$. The *Fundamental Theorem of Algebra* states the remarkable fact that this one addition automatically provides solutions for all other polynomial equations: every equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0 \qquad a_0, \dots, a_{n-1} \text{ in } \mathbf{C}$$

has a complex root!

In the next chapter we shall give an almost complete proof of the Fundamental Theorem of Algebra; the slight gap left in the text can be filled in as an exercise (Problem 26-5). The proof of the theorem will rely on several new concepts which come up quite naturally in a more thorough investigation of complex numbers.

PROBLEMS

1. Find the absolute value and argument(s) of each of the following.

(i) $3 + 4i$.
(ii) $(3 + 4i)^{-1}$.
(iii) $(1 + i)^5$.
(iv) $\sqrt[7]{3 + 4i}$.
(v) $|3 + 4i|$.

2. Solve the following equations.

(i) $x^2 + ix + 1 = 0$.
(ii) $x^4 + x^2 + 1 = 0$.
(iii) $x^2 + 2ix - 1 = 0$.
(iv) $\begin{cases} ix - (1 + i)y = 3, \\ (2 + i)x + iy = 4 \end{cases}.$
(v) $x^3 - x^2 - x - 2 = 0$.

3. Describe the set of all complex numbers z such that

(i) $\bar{z} = -z$.
(ii) $\bar{z} = z^{-1}$.
(iii) $|z - a| = |z - b|$.
(iv) $|z - a| + |z - b| = c$.
(v) $|z| < 1 -$ real part of z.

4. Prove that $|z| = |\bar{z}|$, and that the real part of z is $(z+\bar{z})/2$, while the imaginary part is $(z - \bar{z})/2i$.

5. Prove that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$, and interpret this statement geometrically.

6. What is the pictorial relation between z and $\sqrt{i} \cdot \overline{z\sqrt{-i}}$? (Note that there may be more than one answer, because $\sqrt{i}$ and $\sqrt{-i}$ both have two different possible values.) Hint: Which line goes into the real axis under multiplication by $\sqrt{-i}$?

7. (a) Prove that if $a_0, \dots, a_{n-1}$ are *real* and $a + bi$ (for a and b real satisfies the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then $a - bi$ also satisfies this equation. (Thus the nonreal roots of such an equation always occur in pairs, and the number of such roots is even.)
(b) Conclude that $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is divisible by $z^2 - 2az + (a^2 + b^2)$ (whose coefficients are real).

***8.** (a) Let c be an integer which is not the square of another integer. If a and b are integers we define the **conjugate** of $a + b\sqrt{c}$, denoted by $\overline{a + b\sqrt{c}}$, as $a - b\sqrt{c}$. Show that the conjugate is well defined by showing that a number can be written $a + b\sqrt{c}$, for integers a and b, in only one way.

(b) Show that for all α and β of the form $a+b\sqrt{c}$, we have $\bar{\bar{\alpha}}=\alpha$; $\bar{\alpha}=\alpha$ if and only if α is an integer; $\overline{\alpha+\beta}=\bar{\alpha}+\bar{\beta}$; $\overline{-\alpha}=-\bar{\alpha}$; $\overline{\alpha\cdot\beta}=\bar{\alpha}\cdot\bar{\beta}$; and $\overline{\alpha^{-1}}=(\bar{\alpha})^{-1}$ if $\alpha\neq 0$.

(c) Prove that if $a_0, \dots, a_{n-1}$ are *integers* and $z=a+b\sqrt{c}$ satisfies the equation $z^n+a_{n-1}z^{n-1}+\dots+a_0=0$, then $\bar{z}=a-b\sqrt{c}$ also satisfies this equation.

9. Find all the 4th roots of i; express the one having smallest argument in a form that does not involve any trigonometric functions.

***10.** (a) Prove that if ω is an nth root of 1, then so is ω^k.

(b) A number ω is called a **primitive nth root** of 1 if $\{1,\omega,\omega^2,\dots,\omega^{n-1}\}$ is the set of all nth roots of 1. How many primitive nth roots of 1 are there for $n=3, 4, 5, 9$?

(c) Let ω be an nth root of 1, with $\omega\neq 1$. Prove that $\sum_{k=0}^{n-1}\omega^k=0$.

***11.** (a) Prove that if $z_1, \dots, z_k$ lie on one side of some straight line through 0, then $z_1+\dots+z_k\neq 0$. Hint: This is obvious from the geometric interpretation of addition, but an analytic proof is also easy: the assertion is clear if the line is the real axis, and a trick will reduce the general case to this one.

(b) Show further that $z_1^{-1}, \dots, z_k^{-1}$ all lie on one side of a straight line through 0, so that $z_1^{-1}+\dots+z_k^{-1}\neq 0$.

***12.** Prove that if $|z_1|=|z_2|=|z_3|$ and $z_1+z_2+z_3=0$, then z_1, z_2, and z_3 are the vertices of an equilateral triangle. Hint: It will help to assume that z_1 is real, and this can be done with no loss of generality. Why?

CHAPTER 26 COMPLEX FUNCTIONS

You will probably not be surprised to learn that a deeper investigation of complex numbers depends on the notion of functions. Until now a function was (intuitively) a rule which assigned real numbers to certain other real numbers. But there is no reason why this concept should not be extended; we might just as well consider a rule which assigns complex numbers to certain other complex numbers. A rigorous definition presents no problems (we will not even accord it the full honors of a formal definition): a function is a collection of pairs of complex numbers which does not contain two distinct pairs with the same first element. Since we consider real numbers to be certain complex numbers, the old definition is really a special case of the new one. Nevertheless, we will sometimes resort to special terminology in order to clarify the context in which a function is being considered. A function f is called **real-valued** if $f(z)$ is a real number for all z in the domain of f, and **complex-valued** to emphasize that it is not necessarily real-valued. Similarly, we will usually state explicitly that a function f is defined on [a subset of] $\mathbf{R}$ in those cases where the domain of f is [a subset of] $\mathbf{R}$; in other cases we sometimes mention that f is defined on [a subset of] $\mathbf{C}$ to emphasize that $f(z)$ is defined for complex z as well as real z.

Among the multitude of functions defined on $\mathbf{C}$, certain ones are particularly important. Foremost among these are the functions of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, \ldots, a_n$ are complex numbers. These functions are called, as in the real case, polynomial functions; they include the function $f(z) = z$ (the "identity function") and functions of the form $f(z) = a$ for some complex number a ("constant functions"). Another important generalization of a familiar function is the "absolute value function" $f(z) = |z|$ for all z in $\mathbf{C}$.

Two functions of particular importance for complex numbers are Re (the "real part function") and Im (the "imaginary part function"), defined by

$$\begin{aligned} \operatorname{Re}(x+iy) &= x, \\ \operatorname{Im}(x+iy) &= y, \end{aligned} \qquad \text{for } x \text{ and } y \text{ real.}$$

The "conjugate function" is defined by

$$f(z) = \bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z).$$

Familiar real-valued functions defined on $\mathbf{R}$ may be combined in many ways to produce new complex-valued functions defined on $\mathbf{C}$—an example is the function

$$f(x+iy) = e^y \sin(x-y) + ix^3 \cos y.$$

The formula for this particular function illustrates a decomposition which is always possible. Any complex-valued function f can be written in the form

$$f = u + iv$$

for some real-valued functions u and v—simply define $u(z)$ as the real part of $f(z)$, and $v(z)$ as the imaginary part. This decomposition is often very useful, but not always; for example, it would be inconvenient to describe a polynomial function in this way.

One other function will play an important role in this chapter. Recall that an *argument* of a nonzero complex number z is a (real) number θ such that

$$z = |z|(\cos\theta + i\sin\theta).$$

There are infinitely many arguments for z, but just one which satisfies $0 \le \theta < 2\pi$. If we call this unique argument $\theta(z)$, then θ is a (real-valued) function (the "argument function") on $\{z \text{ in } \mathbf{C} : z \ne 0\}$.

"Graphs" of complex-valued functions defined on $\mathbf{C}$, since they lie in 4-dimensional space, are presumably not very useful for visualization. The alternative picture of a function mentioned in Chapter 4 can be used instead: we draw two copies of $\mathbf{C}$, and arrows from z in one copy, to $f(z)$ in the other (Figure 1).

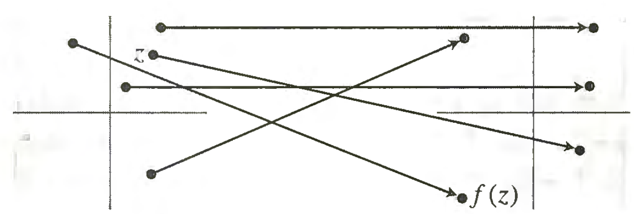


FIGURE 1

The most common pictorial representation of a complex-valued function is produced by labeling a point in the plane with the value $f(z)$, instead of with z (which can be estimated from the position of the point in the picture). Figure 2 shows this sort of picture for several different functions. Certain features of the function are illustrated very clearly by such a "graph." For example, the absolute value function is constant on concentric circles around 0, the functions Re and Im are constant on the vertical and horizontal lines, respectively, and the function $f(z) = z^2$ wraps the circle of radius r twice around the circle of radius r^2.

Despite the problems involved in visualizing complex-valued functions in general, it is still possible to define analogues of important properties previously defined for real-valued functions on $\mathbf{R}$, and in some cases these properties may be easier to visualize in the complex case. For example, the notion of limit can be defined as follows:

> $\lim_{z \to a} f(z) = l$ means that for every (real) number $\varepsilon > 0$ there is a (real) number $\delta > 0$ such that, for all z, if $0 < |z - a| < \delta$, then $|f(z) - l| < \varepsilon$.

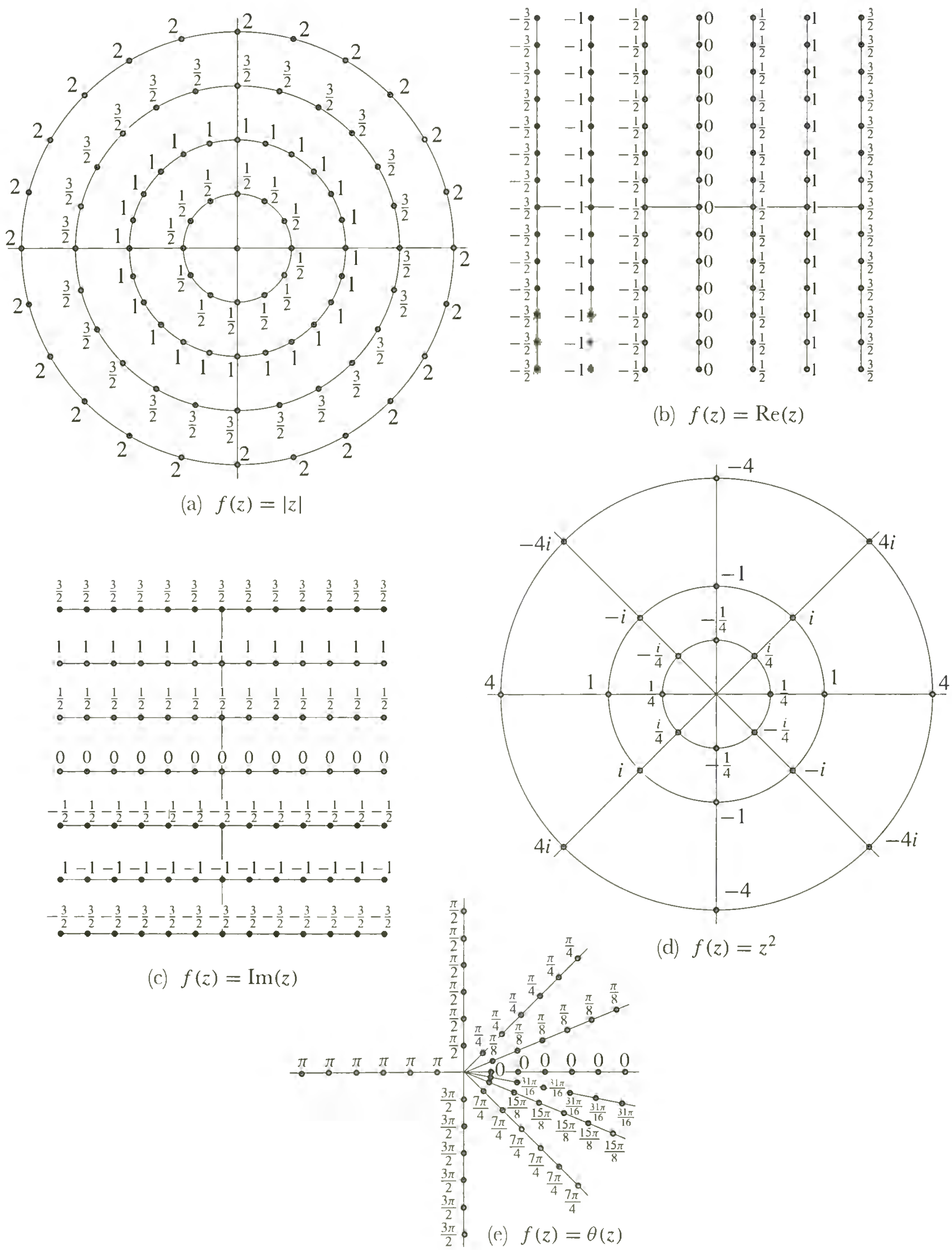


FIGURE 2

Although the definition reads precisely as before, the interpretation is slightly different. Since $|z - w|$ is the distance between the complex numbers z and w, the equation $\lim_{z \to a} f(z) = l$ means that the values of $f(z)$ can be made to lie inside any given circle around l, provided that z is restricted to lie inside a sufficiently small circle around a. This assertion is particularly easy to visualize using the "two copy" picture of a function (Figure 3).

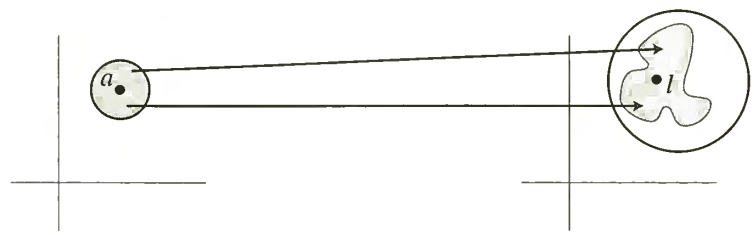


FIGURE 3

Certain facts about limits can be proved exactly as in the real case. In particular,

$$\lim_{z \to a} c = c,$$

$$\lim_{z \to a} z = a,$$

$$\lim_{z \to a} [f(z) + g(z)] = \lim_{z \to a} f(z) + \lim_{z \to a} g(z),$$

$$\lim_{z \to a} f(z) \cdot g(z) = \lim_{z \to a} f(z) \cdot \lim_{z \to a} g(z),$$

$$\lim_{z \to a} \frac{1}{g(z)} = \frac{1}{\lim_{z \to a} g(z)}, \quad \text{if } \lim_{z \to a} g(z) \neq 0.$$

The essential property of absolute values upon which these results are based is the inequality $|z + w| \leq |z| + |w|$, and this inequality holds for complex numbers as well as for real numbers. These facts already provide quite a few limits, but many more can be obtained from the following theorem.

THEOREM 1 Let $f(z) = u(z) + iv(z)$ for real-valued functions u and v, and let $l = \alpha + i\beta$ for real numbers α and β. Then $\lim_{z \to a} f(z) = l$ if and only if

$$\lim_{z \to a} u(z) = \alpha,$$

$$\lim_{z \to a} v(z) = \beta.$$

PROOF Suppose first that $\lim_{z \to a} f(z) = l$. If $\varepsilon > 0$, there is $\delta > 0$ such that, for all z,

$$\text{if } 0 < |z - a| < \delta, \text{ then } |f(z) - l| < \varepsilon.$$

The second inequality can be written

$$\big|[u(z) - \alpha] + i[v(z) - \beta]\big| < \varepsilon,$$

or

$$[u(z) - \alpha]^2 + [v(z) - \beta]^2 < \varepsilon^2.$$

Since $u(z) - \alpha$ and $v(z) - \beta$ are both real numbers, their squares are positive; this inequality therefore implies that

$$[u(z) - \alpha]^2 < \varepsilon^2 \quad \text{and} \quad [v(z) - \beta]^2 < \varepsilon^2,$$

which implies that

$$|u(z) - \alpha| < \varepsilon \quad \text{and} \quad |v(z) - \beta| < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that

$$\lim_{z \to a} u(z) = \alpha \quad \text{and} \quad \lim_{z \to a} v(z) = \beta.$$

Now suppose that these two equations hold. If $\varepsilon > 0$, there is a $\delta > 0$ such that, for all z, if $0 < |z - a| < \delta$, then

$$|u(z) - \alpha| < \frac{\varepsilon}{2} \quad \text{and} \quad |v(z) - \alpha| < \frac{\varepsilon}{2},$$

which implies that

$$\begin{aligned} |f(z) - l| &= \big|[u(z) - \alpha] + i[v(z) - \beta]\big| \\ &\leq |u(z) - \alpha| + |i| \cdot |v(z) - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that $\lim_{z \to a} f(z) = l$. ▮

In order to apply Theorem 1 fruitfully, notice that since we already know the limit $\lim_{z \to a} z = a$, we can conclude that

$$\lim_{z \to a} \operatorname{Re}(z) = \operatorname{Re}(a),$$
$$\lim_{z \to a} \operatorname{Im}(z) = \operatorname{Im}(a).$$

A limit like

$$\lim_{z \to a} \sin(\operatorname{Re}(z)) = \sin(\operatorname{Re}(a))$$

follows easily, using continuity of sin. Many applications of these principles prove such limits as the following:

$$\lim_{z \to a} \bar{z} = \bar{a},$$
$$\lim_{z \to a} |z| = |a|,$$
$$\lim_{(x+iy) \to a+bi} e^y \sin x + ix^3 \cos y = e^b \sin a + ia^3 \cos b.$$

Now that the notion of limit has been extended to complex functions, the notion of continuity can also be extended: f is **continuous at a** if $\lim_{z \to a} f(z) = f(a)$, and

f is **continuous** if f is continuous at a for all a in the domain of f. The previous work on limits shows that all the following functions are continuous:

$$\begin{aligned} f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \\ f(z) &= \bar{z}, \\ f(z) &= |z|, \\ f(x+iy) &= e^y \sin x + i x^3 \cos y. \end{aligned}$$

Examples of discontinuous functions are easy to produce, and certain ones come up very naturally. One particularly frustrating example is the "argument function" θ, which is discontinuous at all nonnegative real numbers (see the "graph" in Figure 2). By suitably redefining θ it is possible to change the discontinuities; for example (Figure 4), if $\theta'(z)$ denotes the unique argument of z with $\pi/2 \le \theta'(z) < 5\pi/2$, then θ' is discontinuous at ai for every nonnegative real number a. But, no matter how θ is redefined, some discontinuities will always occur.

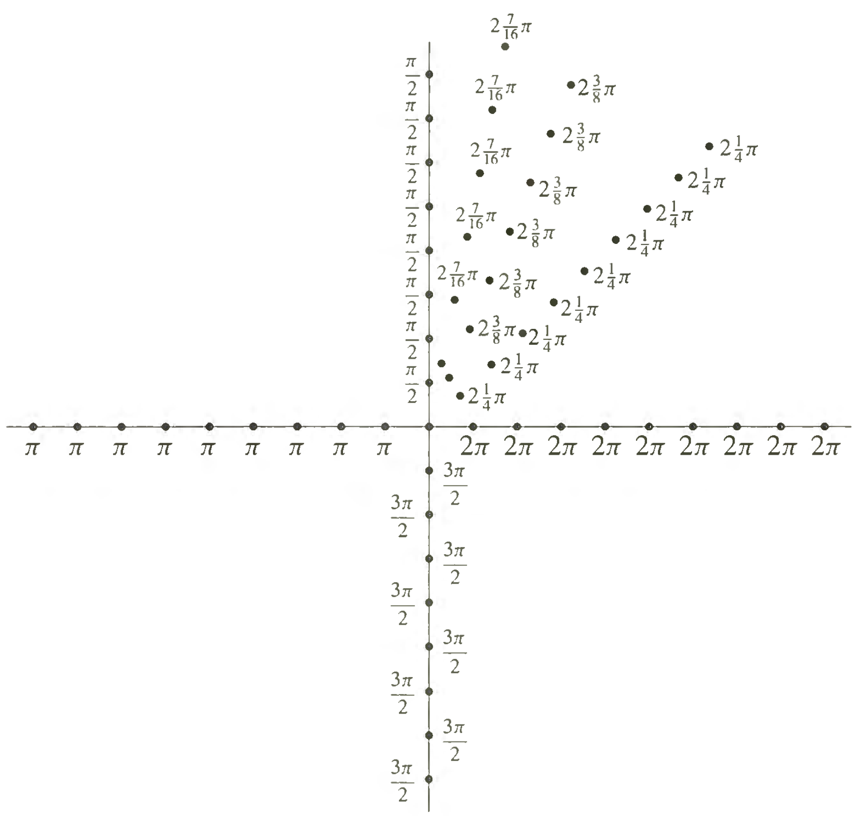


FIGURE 4 $f(z) = \theta'(z)$

The discontinuity of θ has an important bearing on the problem of defining a "square-root function," that is, a function f such that $(f(z))^2 = z$ for all z. For real numbers the function $\sqrt{}$ had as domain only the nonnegative real numbers. If complex numbers are allowed, then every number has two square roots (except 0, which has only one). Although this situation may seem better, it is in some ways worse; since the square roots of z are complex numbers, there is no clear criterion for selecting one root to be $f(z)$, in preference to the other.

One way to define f is the following. We set $f(0) = 0$, and for $z \neq 0$ we set

$$f(z) = \sqrt{|z|}\left(\cos\frac{\theta(z)}{2} + i\sin\frac{\theta(z)}{2}\right).$$

Clearly $(f(z))^2 = z$, but the function f is discontinuous, since θ is discontinuous. As a matter of fact, it is impossible to find a continuous f such that $(f(z))^2 = z$ for all z. In fact, it is even impossible for $f(z)$ to be defined for all z with $|z| = 1$. To prove this by contradiction, we can assume that $f(1) = 1$ (since we could always replace f by $-f$). Then we claim that for all θ with $0 \leq \theta < 2\pi$ we have

$$(*) \qquad f(\cos\theta + i\sin\theta) = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}.$$

The argument for this is left to you (it is a standard type of least upper bound argument). But $(*)$ implies that

$$\begin{aligned}\lim_{\theta\to 2\pi} f(\cos\theta + i\sin\theta) &= \cos\pi + i\sin\pi \\ &= -1 \\ &\neq f(1),\end{aligned}$$

even though $\cos\theta + i\sin\theta \to 1$ as $\theta \to 2\pi$. Thus, we have our contradiction. A similar argument shows that it is impossible to define continuous "nth-root functions" for any $n \geq 2$.

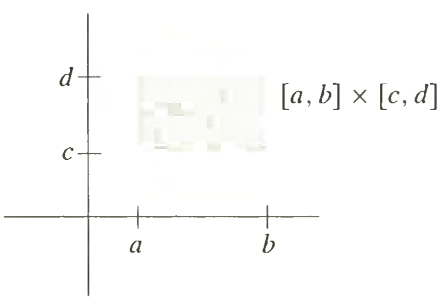


FIGURE 5

For continuous complex functions there are important analogues of certain theorems which describe the behavior of real-valued functions on closed intervals. A natural analogue of the interval $[a, b]$ is the set of all complex numbers $z = x + iy$ with $a \leq x \leq b$ and $c \leq y \leq d$ (Figure 5). This set is called a **closed rectangle**, and is denoted by $[a, b] \times [c, d]$.

If f is a continuous complex-valued function whose domain is $[a, b] \times [c, d]$, then it seems reasonable, and is indeed true, that f is bounded on $[a, b] \times [c, d]$. That is, there is some real number M such that

$$|f(z)| \leq M \quad \text{for all } z \text{ in } [a, b] \times [c, d].$$

It does not make sense to say that f has a maximum and a minimum value on $[a, b] \times [c, d]$, since there is no notion of order for complex numbers. If f is a real-valued function, however, then this assertion does make sense, and is true. In particular, if f is any complex-valued continuous function on $[a, b] \times [c, d]$, then $|f|$ is also continuous, so there is some z_0 in $[a, b] \times [c, d]$ such that

$$|f(z_0)| \leq |f(z)| \quad \text{for all } z \text{ in } [a, b] \times [c, d];$$

a similar statement is true with the inequality reversed. It is sometimes said that "f attains its maximum and minimum modulus on $[a, b] \times [c, d]$."

The various facts listed in the previous paragraph will not be proved here, although proofs are outlined in Problem 5. Assuming these facts, however, we can now give a proof of the Fundamental Theorem of Algebra, which is really quite surprising, since we have not yet said much to distinguish polynomial functions from other continuous functions.

THEOREM 2 (THE FUNDAMENTAL THEOREM OF ALGEBRA) Let $a_0, \dots, a_{n-1}$ be any complex numbers. Then there is a complex number z such that

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 = 0.$$

PROOF Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Then f is continuous, and so is the function $|f|$ defined by

$$|f|(z) = |f(z)| = |z^n + a_{n-1}z^{n-1} + \cdots + a_0|.$$

Our proof is based on the observation that a point z_0 with $f(z_0) = 0$ would clearly be a minimum point for $|f|$. To prove the theorem we will first show that $|f|$ does indeed have a smallest value on the *whole complex plane*. The proof will be almost identical to the proof, in Chapter 7, that a polynomial function of even degree (with real coefficients) has a smallest value on all of $\mathbf{R}$; both proofs depend on the fact that if $|z|$ is large, then $|f(z)|$ is large.

We begin by writing, for $z \neq 0$,

$$f(z) = z^n\left(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right),$$

so that

$$|f(z)| = |z|^n \cdot \left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|.$$

Let

$$M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|).$$

Then for all z with $|z| \geq M$, we have $|z^k| \geq |z|$ and

$$\frac{|a_{n-k}|}{|z^k|} \leq \frac{|a_{n-k}|}{|z|} \leq \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n},$$

so

$$\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \leq \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| \leq \frac{1}{2},$$

which implies that

$$\left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq 1 - \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq \frac{1}{2}.$$

This means that

$$|f(z)| \geq \frac{|z|^n}{2} \quad \text{for } |z| \geq M.$$

In particular, if $|z| \geq M$ and also $|z| \geq \sqrt[n]{2|f(0)|}$, then

$$|f(z)| \geq |f(0)|.$$

Now let $[a, b] \times [c, d]$ be a closed rectangle (Figure 6) which contains $\{z : |z| \leq \max(M, \sqrt[n]{2|f(0)|})\}$, and suppose that the minimum of $|f|$ on $[a, b] \times [c, d]$ is attained at z_0, so that

$$(1) \quad |f(z_0)| \leq |f(z)| \quad \text{for } z \text{ in } [a, b] \times [c, d].$$

It follows, in particular, that $|f(z_0)| \leq |f(0)|$. Thus

$$(2) \quad \text{if } |z| \geq \max(M, \sqrt[n]{2|f(0)|}), \text{ then } |f(z)| \geq |f(0)| \geq |f(z_0)|.$$

Combining (1) and (2) we see that $|f(z_0)| \leq |f(z)|$ for all z, so that $|f|$ attains its minimum value on the whole complex plane at z_0.

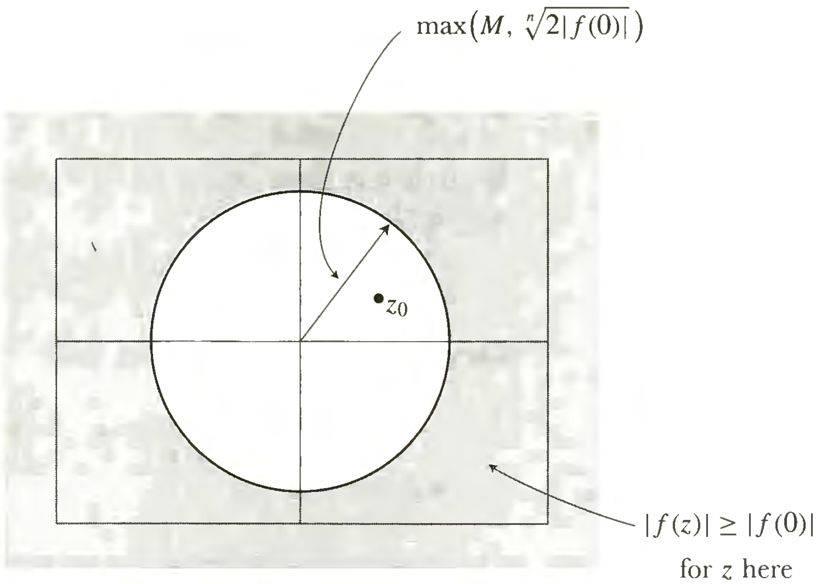


FIGURE 6

To complete the proof of the theorem we now show that $f(z_0) = 0$. It is convenient to introduce the function g defined by

$$g(z) = f(z + z_0).$$

Then g is a polynomial function of degree n, whose minimum absolute value occurs at 0. We want to show that $g(0) = 0$.

Suppose instead that $g(0) = \alpha \neq 0$. If m is the smallest positive power of z which occurs in the expression for g, we can write

$$g(z) = \alpha + \beta z^m + c_{m+1}z^{m+1} + \cdots + c_n z^n,$$

where $\beta \neq 0$. Now, according to Theorem 25-2 there is a complex number γ such that

$$\gamma^m = -\frac{\alpha}{\beta}.$$

Then, setting $d_k = c_k\gamma^k$, we have

$$\begin{aligned}
|g(\gamma z)| &= |\alpha + \beta\gamma^m z^m + d_{m+1}z^{m+1} + \cdots + d_n z^n| \\
&= |\alpha - \alpha z^m + d_{m+1}z^{m+1} + \cdots| \\
&= \left|\alpha\left(1 - z^m + \frac{d_{m+1}}{\alpha}z^{m+1} + \cdots\right)\right| \\
&= \left|\alpha\left(1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right)\right| \\
&= |\alpha| \cdot \left|1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right|.
\end{aligned}$$

This expression, so tortuously arrived at, will enable us to reach a quick contradiction. Notice first that if $|z|$ is chosen small enough, we will have

$$\left|\frac{d_{m+1}}{\alpha}z + \cdots\right| < 1.$$

If we choose, from among all z for which this inequality holds, some z which is *real and positive*, then

$$\left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| < |z^m| = z^m.$$

Consequently, if $0 < z < 1$ we have

$$\begin{aligned}
\left|1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| &\le |1 - z^m| + \left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| \\
&= 1 - z^m + \left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| \\
&< 1 - z^m + z^m \\
&= 1.
\end{aligned}$$

This is the desired contradiction: for such a number z we have

$$|g(\gamma z)| < |\alpha|,$$

contradicting the fact that $|\alpha|$ is the minimum of $|g|$ on the whole plane. Hence, the original assumption must be incorrect, and $g(0) = 0$. This implies, finally, that $f(z_0) = 0$. ∎

Even taking into account our omission of the proofs for the basic facts about continuous complex functions, this proof verified a deep fact with surprisingly little work. It is only natural to hope that other interesting developments will arise if we pursue further the analogues of properties of real functions. The next obvious step is to define derivatives: a function f is **differentiable at *a*** if

$$\lim_{z\to 0} \frac{f(a+z) - f(a)}{z} \text{ exists,}$$

in which case the limit is denoted by $f'(a)$. It is easy to prove that

$$\begin{aligned} f'(a) &= 0 \qquad \text{if } f(z) = c, \\ f'(a) &= 1 \qquad \text{if } f(z) = z, \\ (f+g)'(a) &= f'(a) + g'(a), \\ (f \cdot g)'(a) &= f'(a)g(a) + f(a)g'(a), \\ \left(\frac{1}{g}\right)'(a) &= \frac{-g'(a)}{[g(a)]^2} \qquad \text{if } g(a) \neq 0, \\ (f \circ g)'(a) &= f'(g(a)) \cdot g'(a); \end{aligned}$$

the proofs of all these formulas are exactly the same as before. It follows, in particular, that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. These formulas only prove the differentiability of rational functions however. Many other obvious candidates are *not* differentiable. Suppose, for example, that

$$f(x+iy) = x - iy \quad (\text{i.e., } f(z) = \bar{z}).$$

If f is to be differentiable at 0, then the limit

$$\lim_{(x+iy)\to 0} \frac{f(x+iy) - f(0)}{x+iy} = \lim_{(x+iy)\to 0} \frac{x-iy}{x+iy}$$

must exist. Notice however, that

$$\text{if } y = 0, \text{ then } \frac{x-iy}{x+iy} = 1,$$

and

$$\text{if } x = 0, \text{ then } \frac{x-iy}{x+iy} = -1;$$

therefore this limit cannot possibly exist, since the quotient has both the values 1 and -1 for $x+iy$ arbitrarily close to 0.

In view of this example, it is not at all clear where other differentiable functions are to come from. If you recall the definitions of sin and exp, you will see that there is no hope at all of generalizing these definitions to complex numbers. At the moment the outlook is bleak, but all our problems will soon be solved.

PROBLEMS

1. (a) For any real number y, define $\alpha(x) = x + iy$ (so that α is a complex-valued function defined on $\mathbf{R}$). Show that α is continuous. (This follows immediately from a theorem in this chapter.) Show similarly that $\beta(y) = x + iy$ is continuous.

(b) Let f be a continuous function defined on $\mathbf{C}$. For fixed y, let $g(x) = f(x+iy)$. Show that g is a continuous function (defined on $\mathbf{R}$). Show similarly that $h(y) = f(x+iy)$ is continuous. Hint: Use part (a).

2. (a) Suppose that f is a continuous real-valued function defined on a closed rectangle $[a,b] \times [c,d]$. Prove that if f takes on the values $f(z)$ and $f(w)$

for z and w in $[a, b] \times [c, d]$, then f also takes all values between $f(z)$ and $f(w)$. Hint: Consider $g(t) = f(tz + (1 - t)w)$ for t in $[0, 1]$.

*(b) If f is a continuous complex-valued function defined on $[a, b] \times [c, d]$, the assertion in part (a) no longer makes any sense, since we cannot talk of complex numbers between $f(z)$ and $f(w)$. We might conjecture that f takes on all values on the line segment between $f(z)$ and $f(w)$, but even this is false. Find an example which shows this.

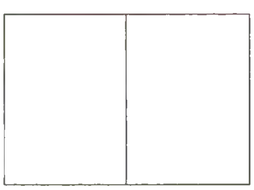
3. (a) Prove that if $a_0, \dots, a_{n-1}$ are any complex numbers, then there are complex numbers $z_1, \dots, z_n$ (not necessarily distinct) such that

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod_{i=1}^{n}(z - z_i).$$

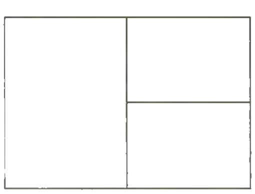
(b) Prove that if $a_0, \dots, a_{n-1}$ are *real*, then $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ can be written as a product of linear factors $z+a$ and quadratic factors z^2+az+b all of whose coefficients are real. (Use Problem 25-7.)

4. In this problem we will consider only polynomials with real coefficients. Such a polynomial is called a **sum of squares** if it can be written as $h_1{}^2+\cdots+h_n{}^2$ for polynomials h_i with real coefficients.

(a) Prove that if f is a sum of squares, then $f(x) \geq 0$ for all x.
(b) Prove that if f and g are sums of squares, then so is $f \cdot g$.
(c) Suppose that $f(x) \geq 0$ for all x. Show that f is a sum of squares. Hint: First write $f(x) = \prod_{i=1}^{k}(x - a_i)^2 g(x)$, where $g(x) > 0$ for all x. Then use Problem 3(b).



(a)



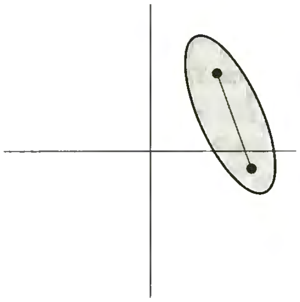
(b)

FIGURE 7

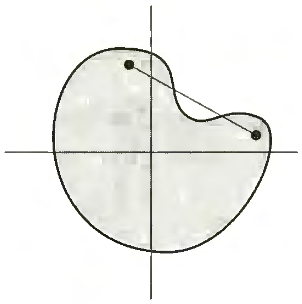
5. (a) Let A be a set of complex numbers. A number z is called, as in the real case, a **limit point** of the set A if for every (real) $\varepsilon > 0$, there is a point a in A with $|z - a| < \varepsilon$ but $z \neq a$. Prove the two-dimensional version of the Bolzano-Weierstrass Theorem: If A is an infinite subset of $[a, b] \times [c, d]$, then A has a limit point in $[a, b] \times [c, d]$. Hint: First divide $[a, b] \times [c, d]$ in half by a vertical line as in Figure 7(a). Since A is infinite, at least one half contains infinitely many points of A. Divide this in half by a horizontal line, as in Figure 7(b). Continue in this way, alternately dividing by vertical and horizontal lines.

(The two-dimensional bisection argument outlined in this hint is so standard that the title "Bolzano-Weierstrass" often serves to describe the method of proof, in addition to the theorem itself. See, for example, H. Petard, "A Contribution to the Mathematical Theory of Big Game Hunting," *Amer. Math. Monthly*, 45 (1938), 446–447.)

(b) Prove that a continuous (complex-valued) function on $[a, b] \times [c, d]$ is bounded on $[a, b] \times [c, d]$. (Imitate Problem 22-31.)
(c) Prove that if f is a real-valued continuous function on $[a, b] \times [c, d]$, then f takes on a maximum and minimum value on $[a, b] \times [c, d]$. (You can use the same trick that works for Theorem 7-3.)



(a) a convex subset of the plane



(b) a nonconvex subset of the plane

FIGURE 8

***6.** The proof of Theorem 2 cannot be considered to be completely elementary because the possibility of choosing γ with $\gamma^m = -\alpha/\beta$ depends on Theorem 25-2, and thus on the trigonometric functions. It is therefore of some interest to provide an elementary proof that there is a solution for the equation $z^n - c = 0$.

(a) Make an explicit computation to show that solutions of $z^2 - c = 0$ can be found for any complex number c.

(b) Explain why the solution of $z^n - c = 0$ can be reduced to the case where n is odd.

(c) Let z_0 be the point where the function $f(z) = z^n - c$ has its minimum absolute value. If $z_0 \neq 0$, show that the integer m in the proof of Theorem 2 is equal to 1; since we can certainly find γ with $\gamma^1 = -\alpha/\beta$, the remainder of the proof works for f. It therefore suffices to show that the minimum absolute value of f does not occur at 0.

(d) Suppose instead that f has its minimum absolute value at 0. Since n is odd, the points $\pm\delta, \pm\delta i$ go under f into $-c\pm\delta^n, -c\pm\delta^n i$. Show that for small δ at least one of these points has smaller absolute value than $-c$, thereby obtaining a contradiction.

7. Let $f(z) = (z-z_1)^{m_1} \cdot \ldots \cdot (z-z_k)^{m_k}$ for $m_1, \ldots, m_k > 0$.

(a) Show that $f'(z) = (z-z_1)^{m_1} \cdot \ldots \cdot (z-z_k)^{m_k} \cdot \displaystyle\sum_{\alpha=1}^{k} m_\alpha (z-z_\alpha)^{-1}$.

(b) Let $g(z) = \displaystyle\sum_{\alpha=1}^{k} m_\alpha (z-z_\alpha)^{-1}$. Show that if $g(z) = 0$, then $z_1, \ldots, z_k$ cannot all lie on the same side of a straight line through z. Hint: Use Problem 25-11.

(c) A subset K of the plane is **convex** if K contains the line segment joining any two points in it (Figure 8). For any set A, there is a smallest convex set containing it, which is called the **convex hull** of A (Figure 9); if a

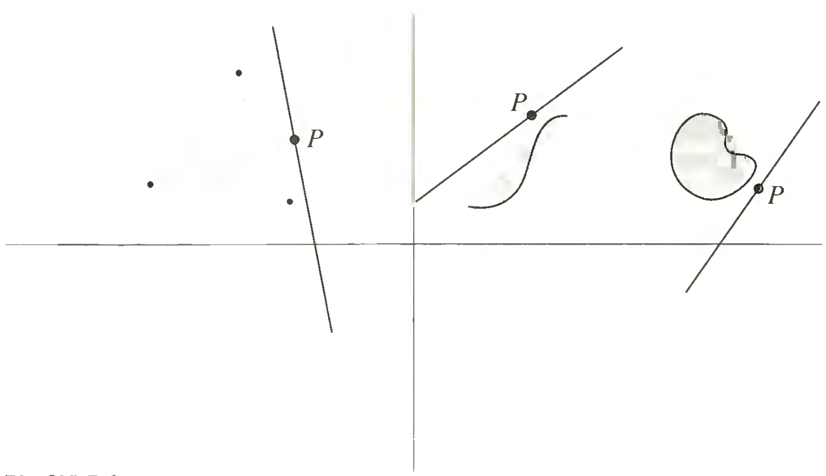


FIGURE 9

point P is not in the convex hull of A, then all of A is contained on one side of some straight line through P. Using this information, prove that the roots of $f'(z) = 0$ lie within the convex hull of the set $\{z_1, \ldots, z_k\}$. Further information on convex sets will be found in reference [18] of the Suggested Reading.

8. Prove that if f is differentiable at z, then f is continuous at z.

***9.** Suppose that $f = u + iv$ where u and v are real-valued functions.

(a) For fixed y_0 let $g(x) = u(x + iy_0)$ and $h(x) = v(x + iy_0)$. Show that if $f'(x_0 + iy_0) = \alpha + i\beta$ for real α and β, then $g'(x_0) = \alpha$ and $h'(x_0) = \beta$.

(b) On the other hand, suppose that $k(y) = u(x_0 + iy)$ and $l(y) = v(x_0 + iy)$. Show that $l'(y_0) = \alpha$ and $k'(y_0) = -\beta$.

(c) Suppose that $f'(z) = 0$ for all z. Show that f is a constant function.

10. (a) Using the expression

$$f(x) = \frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right),$$

find $f^{(k)}(x)$ for all k.

(b) Use this result to find $\arctan^{(k)}(0)$ for all k.

CHAPTER 27 COMPLEX POWER SERIES

If you have not already guessed where differentiable complex functions are going to come from, the title of this chapter should give the secret away: we intend to define functions by means of infinite series. This will necessitate a discussion of infinite sequences of complex numbers, and sums of such sequences, but (as was the case with limits and continuity) the basic definitions are almost exactly the same as for real sequences and series.

An **infinite sequence** of complex numbers is, formally, a complex-valued function whose domain is $\mathbf{N}$; the convenient subscript notation for sequences of real numbers will also be used for sequences of complex numbers. A sequence $\{a_n\}$ of complex numbers is most conveniently pictured by labeling the points a_n in the plane (Figure 1).

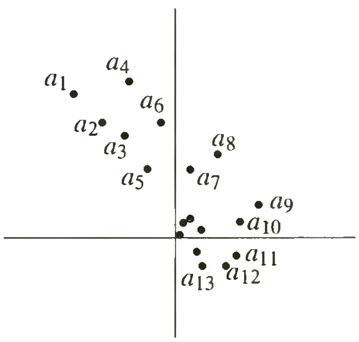


FIGURE 1

The sequence shown in Figure 1 converges to 0, "convergence" of complex sequences being defined precisely as for real sequences: the sequence $\{a_n\}$ **converges** to l, in symbols

$$\lim_{n\to\infty} a_n = l,$$

if for every $\varepsilon > 0$ there is a natural number N such that, for all n,

$$\text{if } n > N, \text{ then } |a_n - l| < \varepsilon.$$

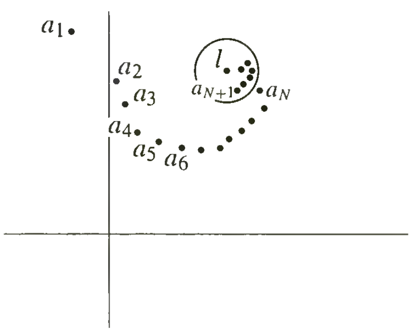


FIGURE 2

This condition means that any circle drawn around l will contain a_n for all sufficiently large n (Figure 2); expressed more colloquially, the sequence is eventually inside any circle drawn around l.

Convergence of complex sequences is not only defined precisely as for real sequences, but can even be reduced to this familiar case.

THEOREM 1 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n,$$

and let

$$l = \beta + i\gamma \qquad \text{for real } \beta \text{ and } \gamma.$$

Then $\lim_{n\to\infty} a_n = l$ if and only if

$$\lim_{n\to\infty} b_n = \beta \qquad \text{and} \qquad \lim_{n\to\infty} c_n = \gamma.$$

PROOF The proof is left as an easy exercise. If there is any doubt as to how to proceed, consult the similar Theorem 1 of Chapter 26. ▌

The **sum** of a sequence $\{a_n\}$ is defined, once again, as $\lim_{n\to\infty} s_n$, where

$$s_n = a_1 + \cdots + a_n.$$

Sequences for which this limit exists are **summable**; alternatively, we may say that the infinite series $\sum_{n=1}^{\infty} a_n$ **converges** if this limit exists, and **diverges** otherwise. It is unnecessary to develop any new tests for convergence of infinite series, because of the following theorem.

THEOREM 2 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, and in this case

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + i\left(\sum_{n=1}^{\infty} c_n\right).$$

PROOF This is an immediate consequence of Theorem 1 applied to the sequence of partial sums of $\{a_n\}$. ∎

There is also a notion of absolute convergence for complex series: the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges (this is a series of real numbers, and consequently one to which our earlier tests may be applied). The following theorem is not quite so easy as the preceding two.

THEOREM 3 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n.$$

Then $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge absolutely.

PROOF Suppose first that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge absolutely, i.e., that $\sum_{n=1}^{\infty} |b_n|$ and $\sum_{n=1}^{\infty} |c_n|$ both converge. It follows that $\sum_{n=1}^{\infty} |b_n| + |c_n|$ converges. Now,

$$|a_n| = |b_n + ic_n| \leq |b_n| + |c_n|.$$

It follows from the comparison test that $\sum_{n=1}^{\infty} |a_n|$ converges (the numbers $|a_n|$ and $|b_n| + |c_n|$ are real and nonnegative). Thus $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Now suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. Since

$$|a_n| = \sqrt{b_n{}^2 + c_n{}^2},$$

it is clear that

$$|b_n| \le |a_n| \quad \text{and} \quad |c_n| \le |a_n|.$$

Once again, the comparison test shows that $\sum_{n=1}^{\infty} |b_n|$ and $\sum_{n=1}^{\infty} |c_n|$ converge. ▮

Two consequences of Theorem 3 are particularly noteworthy. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ also converge absolutely; consequently $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ converge, by Theorem 23-5, so $\sum_{n=1}^{\infty} a_n$ converges by Theorem 2. In other words, absolute convergence implies convergence. Similar reasoning shows that any rearrangement of an absolutely convergent series has the same sum. These facts can also be proved directly, without using the corresponding theorems for real numbers, by first establishing an analogue of the Cauchy criterion (see Problem 13).

With these preliminaries safely disposed of, we can now consider **complex power series**, that is, functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots .$$

Here the numbers a and a_n are allowed to be complex, and we are naturally interested in the behavior of f for complex z. As in the real case, we shall usually consider power series centered at 0,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n;$$

in this case, if $f(z_0)$ converges, then $f(z)$ will also converge for $|z| < |z_0|$. The proof of this fact will be similar to the proof of Theorem 24-6, but, for reasons that will soon become clear, we will not use all the paraphernalia of uniform convergence and the Weierstrass M-test, even though they have complex analogues. Our next theorem consequently generalizes only a small part of Theorem 24-6.

THEOREM 4 Suppose that

$$\sum_{n=0}^{\infty} a_n z_0{}^n = a_0 + a_1 z_0 + a_2 z_0{}^2 + \cdots$$

converges for some $z_0 \neq 0$. Then if $|z| < |z_0|$, the two series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

both converge absolutely.

PROOF As in the proof of Theorem 24-6, we will need only the fact that the set of numbers $a_n z_0{}^n$ is bounded: there is a number M such that

$$|a_n z_0{}^n| \leq M \qquad \text{for all } n.$$

We then have

$$\begin{aligned} |a_n z^n| &= |a_n z_0{}^n| \cdot \left|\frac{z}{z_0}\right|^n \\ &\leq M \left|\frac{z}{z_0}\right|^n, \end{aligned}$$

and, for $z \neq 0$,

$$\begin{aligned} |n a_n z^{n-1}| &= \frac{1}{|z|} n |a_n z_0{}^n| \cdot \left|\frac{z}{z_0}\right|^n \\ &\leq \frac{M}{|z|} n \left|\frac{z}{z_0}\right|^n. \end{aligned}$$

Since the series $\sum_{n=0}^{\infty} |z/z_0|^n$ and $\sum_{n=1}^{\infty} n\,|z/z_0|^n$ converge, this shows that both $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converge absolutely (the argument for $\sum_{n=1}^{\infty} n a_n z^{n-1}$ assumed that $z \neq 0$, but this series certainly converges for $z = 0$ also). ▮

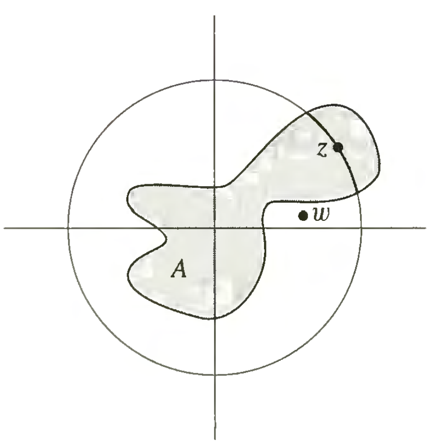


FIGURE 3

Theorem 4 evidently restricts greatly the possibilities for the set

$$\left\{ z : \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}.$$

For example, the shaded set A in Figure 3 cannot be the set of all z where $\sum_{n=0}^{\infty} a_n z^n$ converges, since it contains z, but not the number w satisfying $|w| < |z|$.

It seems quite unlikely that the set of points where a power series converges could be anything except the set of points inside a circle. If we allow "circles of radius 0" (when the power series converges only at 0) and "circles of radius ∞" (when the power series converges at all points), then this assertion is true (with one complication which we will soon mention); the proof requires only Theorem 4 and a knack for good organization.

THEOREM 5 For any power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

one of the following three possibilities must be true:

(1) $\sum_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$.

(2) $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z in $\mathbf{C}$.

(3) There is a number $R > 0$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < R$ and diverges if $|z| > R$. (Notice that we do not mention what happens when $|z| = R$.)

PROOF Let

$$S = \left\{ x \text{ in } \mathbf{R} : \sum_{n=0}^{\infty} a_n w^n \text{ converges for some } w \text{ with } |w| = x \right\}.$$

Suppose first that S is unbounded. Then for any complex number z, there is a number x in S such that $|z| < x$. By definition of S, this means that $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|w| = x > |z|$. It follows from Theorem 4 that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Thus, in this case possibility (2) is true.

Now suppose that S is bounded, and let R be the least upper bound of S. If $R = 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$, so possibility (1) is true. Suppose, on the other hand, that $R > 0$. Then if z is a complex number with $|z| < R$, there is a number x in S with $|z| < x$. Once again, this means that $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|z| < |w|$, so that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Moreover, if $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge, since $|z|$ is not in S. ▌

The number R which occurs in case (3) is called the **radius of convergence** of $\sum_{n=0}^{\infty} a_n z^n$. In cases (1) and (2) it is customary to say that the radius of convergence is 0 and ∞, respectively. When $0 < R < \infty$, the circle $\{z : |z| = R\}$ is called the **circle of convergence** of $\sum_{n=0}^{\infty} a_n z^n$. If z is outside the circle, then, of course, $\sum_{n=0}^{\infty} a_n z^n$ does not converge, but actually a much stronger statement can be made: the terms $a_n z^n$ are not even bounded. To prove this, let w be any number with $|z| > |w| > R$; if the terms $a_n z^n$ were bounded, then the proof of Theorem 4 would show that $\sum_{n=0}^{\infty} a_n w^n$ converges, which is false. Thus (Figure 4), inside the circle of convergence the series $\sum_{n=0}^{\infty} a_n z^n$ converges in the best possible way (absolutely) and outside the circle the series diverges in the worst possible way (the terms $a_n z^n$ are not bounded).

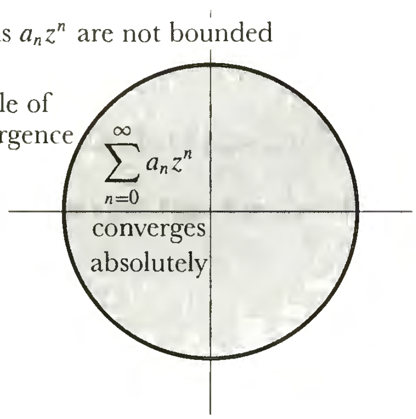


FIGURE 4

What happens *on* the circle of convergence is a much more difficult question. We will not consider that question at all, except to mention that there are power series which converge everywhere on the circle of convergence, power series which converge nowhere on the circle of convergence, and power series that do just about anything in between. (See Problem 5.)

Algebraic manipulations on complex power series can be justified just as in the real case. Thus, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ both have radius of convergence $\geq R$, then $h(z) = \sum_{n=0}^{\infty}(a_n + b_n)z^n$ also has radius of convergence $\geq R$ and $h = f + g$ inside the circle of radius R. Similarly, the Cauchy product $h(z) = \sum_{n=0}^{\infty} c_n z^n$, for $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, has radius of convergence $\geq R$ and $h = fg$ inside the circle of radius R. And if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence > 0 and $a_0 \neq 0$, then we can find a power series $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence > 0 which represents $1/f$ inside its circle of convergence.

But our real goal in this chapter is to produce differentiable functions. We therefore want to generalize the result proved for real power series in Chapter 24, that a function defined by a power series can be differentiated term-by-term inside the circle of convergence. At this point we can no longer imitate the proof of Chapter 24, even if we were willing to introduce uniform convergence, because no analogue of Theorem 24-3 seems available. Instead we will use a direct argument (which could also have been used in Chapter 24). Before beginning the proof, we notice that at least there is no problem about the convergence of the series produced by term-by-term differentiation. If the series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then Theorem 4 immediately implies that the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ also converges for $|z| < R$. Moreover, if $|z| > R$, so that the terms $a_n z^n$ are unbounded, then the terms $na_n z^{n-1}$ are surely unbounded, so $\sum_{n=1}^{\infty} na_n z^{n-1}$ does not converge.

This shows that the radius of convergence of $\sum_{n=1}^{\infty} na_n z^{n-1}$ is also exactly R.

THEOREM 6 If the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$, then f is differentiable at z for all z with $|z| < R$, and

$$f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}.$$

PROOF We will use another "$\varepsilon/3$ argument." The fact that the theorem is clearly true for polynomial functions suggests writing

$$(*) \quad \left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right| = \left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right|$$
$$\leq \left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} \right|$$
$$+ \left| \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=1}^{N} na_n z^{n-1} \right|$$
$$+ \left| \sum_{n=1}^{N} na_n z^{n-1} - \sum_{n=1}^{\infty} na_n z^{n-1} \right|.$$

We will show that for any $\varepsilon > 0$, each absolute value on the right side can be made $< \varepsilon/3$ by choosing N sufficiently large and h sufficiently small. This will clearly prove the theorem.

Only the first term in the right side of $(*)$ will present any difficulties. To begin with, choose some z_0 with $|z| < |z_0| < R$; henceforth we will consider only h with $|z + h| \leq |z_0|$. The expression $((z+h)^n - z^n)/h$ can be written in a more convenient way if we remember that

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}.$$

Applying this to

$$\frac{(z+h)^n - z^n}{h} = \frac{(z+h)^n - z^n}{(z+h) - z},$$

we obtain

$$\frac{(z+h)^n - z^n}{h} = (z+h)^{n-1} + z(z+h)^{n-2} + \cdots + z^{n-1}.$$

Since

$$|(z+h)^{n-1}+z(z+h)^{n-2}+\cdots+z^{n-1}|\le n|z_0|^{n-1},$$

we have

$$\left|a_n\frac{((z+h)^n-z^n)}{h}\right|\le n|a_n|\cdot|z_0|^{n-1}.$$

But the series $\sum_{n=1}^{\infty} n|a_n|\cdot|z_0|^{n-1}$ converges, so if N is sufficiently large, then

$$\sum_{n=N+1}^{\infty} n|a_n|\cdot|z_0|^{n-1}<\frac{\varepsilon}{3}.$$

This means that

$$\begin{aligned}&\left|\sum_{n=0}^{\infty} a_n\frac{((z+h)^n-z^n)}{h}-\sum_{n=0}^{N} a_n\frac{((z+h)^n-z^n)}{h}\right|\\ &\qquad=\left|\sum_{n=N+1}^{\infty} a_n\frac{((z+h)^n-z^n)}{h}\right|\le\sum_{n=N+1}^{\infty}\left|a_n\frac{((z+h)^n-z^n)}{h}\right|\\ &\qquad\le\sum_{n=N+1}^{\infty} n|a_n|\cdot|z_0|^{n-1}<\frac{\varepsilon}{3}.\end{aligned}$$

In short, if N is sufficiently large, then

$$(1)\qquad \left|\sum_{n=0}^{\infty} a_n\frac{((z+h)^n-z^n)}{h}-\sum_{n=0}^{N} a_n\frac{((z+h)^n-z^n)}{h}\right|<\frac{\varepsilon}{3},$$

for *all* h with $|z+h|\le|z_0|$.

It is easy to deal with the third term on the right side of (*): Since $\sum_{n=1}^{\infty} na_nz^{n-1}$ converges, it follows that if N is sufficiently large, then

$$(2)\qquad \left|\sum_{n=1}^{\infty} na_nz^{n-1}-\sum_{n=1}^{N} na_nz^{n-1}\right|<\frac{\varepsilon}{3}.$$

Finally, choosing an N such that (1) and (2) are true, we note that

$$\lim_{h\to 0}\sum_{n=0}^{N} a_n\frac{((z+h)^n-z^n)}{h}=\sum_{n=1}^{N} na_nz^{n-1},$$

since the polynomial function $g(z)=\sum_{n=0}^{N} a_nz^n$ is certainly differentiable. Therefore

$$(3)\qquad \left|\sum_{n=0}^{N}\frac{a_n((z+h)^n-z^n)}{h}-\sum_{n=1}^{N} na_nz^{n-1}\right|<\frac{\varepsilon}{3}.$$

for sufficiently small h.

As we have already indicated, (1), (2), and (3) prove the theorem. ▮

Theorem 6 has an obvious corollary: a function represented by a power series is infinitely differentiable inside the circle of convergence, and the power series is its Taylor series at 0. It follows, in particular, that f is continuous inside the circle of convergence, since a function differentiable at z is continuous at z (Problem 26-8).

The continuity of a power series inside its circle of convergence helps explain the behavior of certain Taylor series obtained for real functions, and gives the promised answers to the questions raised at the end of Chapter 24. We have already seen that the Taylor series for the function $f(z) = 1/(1+z^2)$, namely,

$$1 - z^2 + z^4 - z^6 + \cdots,$$

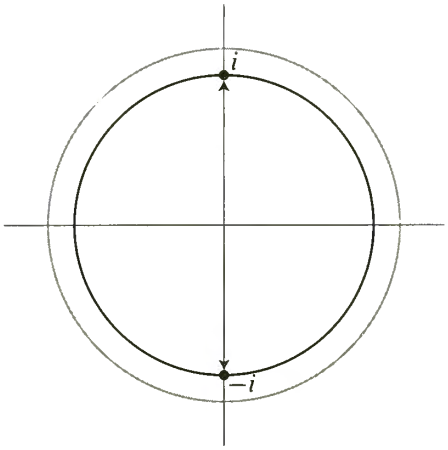


FIGURE 5

converges for real z only when $|z| < 1$, and consequently has radius of convergence 1. It is no accident that the circle of convergence contains the two points i and $-i$ at which f is undefined. If this power series converged in a circle of radius greater than 1, then (Figure 5) it would represent a function which was continuous in that circle, in particular at i and $-i$. But this is impossible, since it equals $1/(1+z^2)$ inside the unit circle, and $1/(1+z^2)$ does not approach a limit as z s approaches i or $-i$ from inside the unit circle.

The use of complex numbers also sheds some light on the strange behavior of the Taylor series for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Although we have not yet defined e^z for complex z, it will presumably be true that if y is real and unequal to 0, then

$$f(iy) = e^{-1/(iy)^2} = e^{1/y^2}.$$

The interesting fact about this expression is that it becomes large as y becomes small. Thus f will not even be continuous at 0 when defined for complex numbers, so it is hardly surprising that it is equal to its Taylor series only for $z = 0$.

The method by which we will actually define e^z (as well as $\sin z$ and $\cos z$) for complex z should by now be clear. For real x we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

For complex z we therefore *define*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots,$$

$$\exp(z) = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

Then $\sin'(z) = \cos z$, $\cos'(z) = -\sin z$, and $\exp'(z) = \exp(z)$ by Theorem 6. Moreover, if we replace z by iz in the series for e^z, and make a rearrangement of the terms (justified by absolute convergence), something particularly interesting happens:

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{iz^4}{4!} + \frac{iz^5}{5!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right), \end{aligned}$$

so

$$e^{iz} = \cos z + i \sin z.$$

It is clear from the definitions (i.e., the power series) that

$$\begin{aligned} \sin(-z) &= -\sin z, \\ \cos(-z) &= \cos z, \end{aligned}$$

so we also have

$$e^{-iz} = \cos z - i \sin z.$$

From the equations for e^{iz} and e^{-iz} we can derive the formulas

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2}. \end{aligned}$$

The development of complex power series thus places the exponential function at the very core of the development of the elementary functions—it reveals a connection between the trigonometric and exponential functions which was never imagined when these functions were first defined, and which could never have been discovered without the use of complex numbers. As a by-product of this relationship, we obtain a hitherto unsuspected connection between the numbers e and π: if in the formula

$$e^{iz} = \cos z + i \sin z$$

we take $z = \pi$, we obtain the remarkable result

$$e^{i\pi} = -1.$$

(More generally, $e^{2\pi i/n}$ is an nth root of 1.)

With these remarks we will bring to a close our investigation of complex functions. And yet there are still several basic facts about power series which have not been mentioned. Thus far, we have seldom considered power series centered at a,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n,$$

except for $a = 0$. This omission was adopted partly to simplify the exposition. For power series centered at a there are obvious versions of all the theorems in this chapter (the proofs require only trivial modifications): there is a number R (possibly 0 or "∞") such that the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely for z with $|z-a| < R$, and has unbounded terms for z with $|z-a| > R$; moreover, for all z with $|z-a| < R$ the function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

has derivative

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}.$$

It is less straightforward to investigate the possibility of representing a function as a power series centered at b, if it is already written as a power series centered at a. If

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

has radius of convergence R, and b is a point with $|b-a| < R$ (Figure 6), then it is true that $f(z)$ can also be written as a power series centered at b,

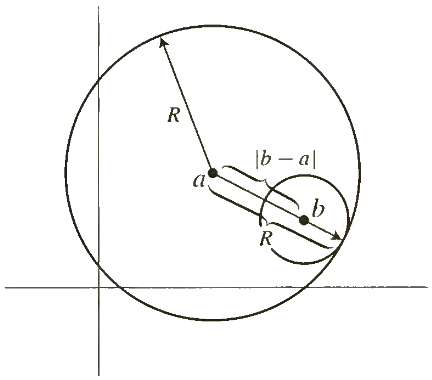


FIGURE 6

$$f(z) = \sum_{n=0}^{\infty} b_n(z-b)^n$$

(the numbers b_n are necessarily $f^{(n)}(b)/n!$); moreover, this series has radius of convergence at least $R - |b-a|$ (*it may be larger*).

We will *not* prove the facts mentioned in the previous paragraph, and there are several other important facts we shall not prove. For example, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n(z-b)^n,$$

and $g(b) = a$, then we would expect that $f \circ g$ can be written as a power series centered at b. All such facts could be proved now without introducing any basic new ideas, but the proofs would not be as easy as the proofs about sums, products and reciprocals of power series. The possibility of changing a power series centered at a into one centered at b is quite a bit more involved, and the treatment of $f \circ g$ requires still more skill. Rather than end this section with a *tour de force* of computations, we will instead give a preview of "complex analysis," one of the most beautiful branches of mathematics, where all these facts are derived as straightforward consequences of some fundamental results.

Power series were introduced in this chapter in order to provide complex functions which are differentiable. Since these functions are actually infinitely differentiable, it is natural to suppose that we have therefore selected only a very special collection of differentiable complex functions. The basic theorems of complex analysis show that this is not at all true:

> *If a complex function is defined in some region A of the plane and is differentiable in A, then it is automatically infinitely differentiable in A. Moreover, for each point a in A the Taylor series for f at a will converge to f in any circle contained in A (Figure 7).*

FIGURE 7

These facts are among the first to be proved in complex analysis. It is impossible to give any idea of the proofs themselves—the methods used are quite different from anything in elementary calculus. If these facts are granted, however, then the facts mentioned before can be proved very easily.

Suppose, for example, that f and g are functions which can be written as power series. Then, as we have shown, f and g are differentiable—it then follows from easy general theorems that $f+g$, $f \cdot g$, $1/g$ and $f \circ g$ are also differentiable. Appealing to the results from complex analysis, it follows that they can be written as power series.

We already know how to compute the power series for $f+g$, $f \cdot g$ and $1/g$ from those for f and g. It is also easy to guess how one would compute an expression for $f \circ g$ as a power series in $(z-b)$ when we are given the power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

$$g(z) = \sum_{k=0}^{\infty} b_k(z-b)^k,$$

with $a = g(b) = b_0$, so that

$$g(z) - a = \sum_{k=1}^{\infty} b_k(z-b)^k.$$

First of all, we know how to compute the power series

$$(g(z)-a)^l = \left(\sum_{k=1}^{\infty} b_k(z-b)^k\right)^l,$$

and this power series will begin with $(z-b)^l$. Consequently, the coefficient of z^n in

$$f(g(z)) = \sum_{l=0}^{\infty} a_l(g(z)-a)^l$$

can be calculated as a finite sum, involving only coefficients arising from the first n powers of $g(z)-a$.

Similarly, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

has radius of convergence R, then f is differentiable in the region $A = \{z : |z-a| < R\}$. Thus, if b is in A, it is possible to write f as a power series centered at b, which will converge in the circle of radius $R - |b - a|$. The coefficient of z^n will be $f^{(n)}(b)/n!$. This series may actually converge in a larger circle, because $\sum_{n=0}^{\infty} a_n(z-a)^n$ may be the series for a function differentiable in a larger region than A. For example, suppose that $f(z) = 1/(1+z^2)$. Then f is differentiable, except at i and $-i$, where it is not defined. Thus $f(z)$ can be written as a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 1 (as a matter of fact, we know that $a_{2n} = (-1)^n$ and $a_k = 0$ if k is odd). It is also possible to write

$$f(z) = \sum_{n=0}^{\infty} b_n(z - \tfrac{1}{2})^n,$$

where $b_n = f^{(n)}(\frac{1}{2})/n!$. We can easily predict the radius of convergence of this series: it is $\sqrt{1+(\frac{1}{2})^2}$, the distance from $\frac{1}{2}$ to i or $-i$ (Figure 8).

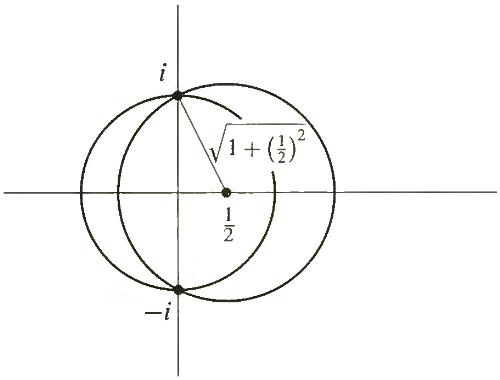


FIGURE 8

As an added incentive to investigate complex analysis further, one more result will be mentioned, which lies quite near the surface, and which will be found in any treatment of the subject.

For real z the values of $\sin z$ always lie between -1 and 1, but for complex z this is not at all true. In fact, if $z = iy$, for y real, then

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i}.$$

If y is large, then $\sin iy$ is also large in absolute value. This behavior of sin is typical of functions which are defined and differentiable on the whole complex plane (such functions are called *entire*). A result which comes quite early in complex analysis is the following:

Liouville's Theorem: The only bounded entire functions are the constant functions.

As a simple application of Liouville's Theorem, consider a polynomial function

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

where $n > 1$, so that f is not a constant. We already know that $f(z)$ is large for large z, so Liouville's Theorem tells us nothing interesting about f. But consider the function

$$g(z) = \frac{1}{f(z)}.$$

If $f(z)$ were never 0, then g would be entire; since $f(z)$ becomes large for large z, the function g would also be bounded, contradicting Liouville's Theorem. Thus $f(z) = 0$ for some z, and we have proved the Fundamental Theorem of Algebra.

PROBLEMS

1. Decide whether each of the following series converges, and whether it converges absolutely.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+i)^n}{n!}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2i}{2^n}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (\tfrac{1}{2}+\tfrac{1}{2}i)^n$.

(v) $\displaystyle\sum_{n=2}^{\infty} \frac{\log n}{n} + i^n \frac{\log n}{n}$.

2. Use the ratio test to show that the radius of convergence of each of the following power series is 1. (In each case the ratios of successive terms will approach a limit < 1 if $|z| < 1$, but for $|z| > 1$ the ratios will tend to ∞ or to a limit > 1.)

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} z^n$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (n+2^{-n})z^n$.

(v) $\displaystyle\sum_{n=1}^{\infty} 2^n z^{n!}$.

3. Use the root test (Problem 23-9) to find the radius of convergence of each of the following power series. (In some cases, you will need limits derived in the problems to Chapter 22.)

(i) $\displaystyle \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{2^2} + \frac{z^4}{3^2} + \frac{z^5}{2^3} + \frac{z^6}{3^3} + \cdots$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} z^n$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,z^n}{n^n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} z^n$.

(v) $\displaystyle\sum_{n=1}^{\infty} 2^n z^{n!}$.

4. The root test can always be used, in theory at least, to find the radius of convergence of a power series; in fact, a close analysis of the situation leads to a formula for the radius of convergence, known as the "Cauchy-Hadamard formula." Suppose first that the set of numbers $\sqrt[n]{|a_n|}$ is bounded.

(a) Use Problem 23-9 to show that if $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\,|z| < 1$, then $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges.

(b) Also show that if $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\,|z| > 1$, then $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has unbounded terms.

(c) Parts (a) and (b) show that the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is $1/\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$ (where "$1/0$" means "∞"). To complete the formula, define $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \infty$ if the set of all $\sqrt[n]{|a_n|}$ is unbounded. Prove that in this case, $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ diverges for $z \neq 0$, so that the radius of convergence is 0 (which may be considered as "$1/\infty$").

5. Consider the following three series from Problem 2:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} z^n.$$

Prove that the first series converges everywhere on the unit circle; that the third series converges nowhere on the unit circle; and that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

6. (a) Prove that $e^z \cdot e^w = e^{z+w}$ for all complex numbers z and w by showing that the infinite series for e^{z+w} is the Cauchy product of the series for e^z and e^w.

(b) Show that $\sin(z + w) = \sin z \cos w + \cos z \sin w$ and $\cos(z + w) = \cos z \cos w - \sin z \sin w$ for all complex z and w.

7. (a) Prove that every complex number of absolute value 1 can be written e^{iy} for some real number y.

(b) Prove that $|e^{x+iy}| = e^x$ for real x and y.

8. (a) Prove that exp takes on every complex value except 0.

(b) Prove that sin takes on every complex value.

9. For each of the following functions, compute the first three nonzero terms of the Taylor series centered at 0 by manipulating power series.

(i) $f(z) = \tan z$.

(ii) $f(z) = z(1 - z)^{-1/2}$.

(iii) $f(z) = \dfrac{e^{\sin z} - 1}{z}$.

(iv) $f(z) = \log(1 - z^2)$.

(v) $f(z) = \dfrac{\sin^2 z}{z^2}$.

(vi) $f(z) = \dfrac{\sin(z^2)}{z \cos^2 z}$.

(vii) $f(z) = \dfrac{1}{z^4 - 2z^2 + 3}$.

(viii) $f(z) = \dfrac{1}{z}[e^{(\sqrt{1+z}-1)} - 1]$.

10. (a) Suppose that we write a differentiable complex function f as $f = u + iv$, where u and v are real-valued. Let $\bar{u}$ and $\bar{v}$ denote the restrictions of u and v to the real numbers. In other words, $\bar{u}(x) = u(x)$ for real numbers x (but $\bar{u}$ is not defined for other x). Using Problem 26-9, show that for real x we have

$$f'(x) = \bar{u}'(x) + i\bar{v}'(x),$$

where f' denotes the complex derivative, while $\bar{u}'$ and $\bar{v}'$ denote the ordinary derivatives of these real-valued functions on $\mathbf{R}$.

(b) Show, more generally, that

$$f^{(k)}(x) = \bar{u}^{(k)}(x) + i\bar{v}^{(k)}(x).$$

(c) Suppose that f satisfies the equation

$$(*) \qquad f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f = 0,$$

where the a_i are real numbers, and where the $f^{(k)}$ denote higher-order complex derivatives. Show that $\bar{u}$ and $\bar{v}$ satisfy the same equation, where $\bar{u}^{(k)}$ and $\bar{v}^{(k)}$ now denote higher-order derivatives of real-valued functions on $\mathbf{R}$.

(d) Show that if $a = b + ci$ is a complex root of the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then $f(x) = e^{bx}\sin cx$ and $f(x) = e^{bx}\cos cx$ are both solutions of $(*)$.

11. (a) Show that exp is *not* one-one on $\mathbf{C}$.

(b) Given $w \neq 0$, show that $e^z = w$ if and only if $z = x + iy$ with $x = \log|w|$ (here log denotes the real logarithm function), and y an argument of w.

*(c) Show that there does not exist a continuous function log defined for nonzero complex numbers, such that $\exp(\log(z)) = z$ for all $z \neq 0$. (Show that log cannot even be defined continuously for $|z| = 1$.)

Since there is no way to define a continuous logarithm function we cannot speak of *the* logarithm of a complex number, but only of "a logarithm for w," meaning one of the infinitely many numbers z with $e^z = w$. And for complex numbers a and b we define a^b to be a *set* of complex numbers, namely the set of all numbers $e^{b \log a}$ or, more precisely, the set of all numbers e^{bz} where z is a logarithm for a.

(d) If m is an integer, then a^m consists of only one number, the one given by the usual elementary definition of a^m.

(e) If m and n are integers, then the set $a^{m/n}$ coincides with the set of values given by the usual elementary definition, namely the set of all b^m where b is an nth root of a.

(f) If a and b are real and b is irrational, then a^b contains infinitely many members, even for $a > 0$.

(g) Find all logarithms of i, and find all values of i^i.

(h) By $(a^b)^c$ we mean the set of all numbers of the form z^c for some number z in the set a^b. Show that $(1^i)^i$ has infinitely many values, while $1^{i \cdot i}$ has only one.

(i) Show that all values of $a^{b \cdot c}$ are also values of $(a^b)^c$. Is $a^{b \cdot c} = (a^b)^c \cap (a^c)^b$?

12. (a) For real x show that we can choose $\log(x+i)$ and $\log(x-i)$ to be

$$\log(x+i) = \log\left(\sqrt{1+x^2}\right) + i\left(\frac{\pi}{2} - \arctan x\right),$$

$$\log(x-i) = \log\left(\sqrt{1+x^2}\right) - i\left(\frac{\pi}{2} - \arctan x\right).$$

(It will help to note that $\pi/2 - \arctan x = \arctan 1/x$ for $x > 0$.)

(b) The expression

$$\frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

yields, formally,

$$\int \frac{dx}{1+x^2} = \frac{1}{2i}[\log(x-i) - \log(x+i)].$$

Use part (a) to check that this answer agrees with the usual one.

13. (a) A sequence $\{a_n\}$ of complex numbers is called a **Cauchy sequence** if $\lim_{m,n\to\infty} |a_m - a_n| = 0$. Suppose that $a_n = b_n + ic_n$, where b_n and c_n are real. Prove that $\{a_n\}$ is a Cauchy sequence if and only if $\{b_n\}$ and $\{c_n\}$ are Cauchy sequences.

(b) Prove that every Cauchy sequence of complex numbers converges.

(c) Give direct proofs, without using theorems about real series, that an absolutely convergent series is convergent and that any rearrangement has the same sum. (It is permitted, and in fact advisable, to use the *proofs* of the corresponding theorems for real series.)

14. (a) Prove that

$$\sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{1-e^{inx}}{1-e^{ix}} = \frac{\sin\left(\dfrac{n}{2}x\right)}{\sin\dfrac{x}{2}} e^{i(n+1)x/2}.$$

(b) Deduce the formulas for $\sum_{k=1}^{n} \cos kx$ and $\sum_{k=1}^{n} \sin kx$ that are given in Problem 15-33.

15. Let $\{a_n\}$ be the Fibonacci sequence, $a_1 = a_2 = 1, a_{n+2} = a_n + a_{n+1}$.

(a) If $r_n = a_{n+1}/a_n$, show that $r_{n+1} = 1 + 1/r_n$.

(b) Show that if $r = \lim_{n\to\infty} r_n$ exists, then $r = 1+1/r$, so that $r = \left(1+\sqrt{5}\right)/2$.

(c) Prove that the limit does exist. Hint: If $r_n < \left(1+\sqrt{5}\right)/2$, then $r_n{}^2 - r_n - 1 < 0$ and $r_n < r_{n+2}$.

(d) Show that $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence $2/\left(1+\sqrt{5}\right)$. (Using the unproved theorems in this chapter and the fact that $\sum_{n=1}^{\infty} a_n z^n = -1/(z^2+z-1)$ from Problem 24-16 we could have predicted that the radius of convergence is the smallest absolute value of the roots of $z^2 + z - 1 = 0$; since the roots are $\left(-1 \pm \sqrt{5}\right)/2$, the radius of convergence should be $\left(-1+\sqrt{5}\right)/2$. Notice that this number is indeed equal to $2/\left(1+\sqrt{5}\right)$.)

16. Since $(e^z - 1)/z$ can be written as the power series $1 + z/2! + z^2/3! + \cdots$ which is nonzero at 0, it follows that there is a power series

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$$

with nonzero radius of convergence. Using the unproved theorems in this chapter, we can even predict the radius of convergence; it is 2π, since this is the smallest absolute value of the non-zero numbers $z = 2k\pi i$ for which $e^z - 1 = 0$. The numbers b_n appearing here are called the **Bernoulli numbers.***

(a) Clearly $b_0 = 1$. Now show that

$$\frac{z}{e^z-1} = -\frac{z}{2} + \frac{z}{2} \cdot \frac{e^z+1}{e^z-1},$$

$$\frac{e^{-z}+1}{e^{-z}-1} = -\frac{e^z+1}{e^z-1},$$

and deduce that

$$b_1 = -\tfrac{1}{2}, \quad b_n = 0 \quad \text{if } n \text{ is odd and } n > 1.$$

*Sometimes the numbers $B_n = (-1)^{n-1} b_{2n}$ are called the Bernoulli numbers, because $b_n = 0$ if n is odd and > 1 (see part (a)) and because the numbers b_{2n} alternate in sign, although we will not prove this. Other modifications of this nomenclature are also in use.

(b) By finding the coefficient of z^n in the right side of the equation

$$z = \left(\sum_{k=0}^{\infty} \frac{b_k}{k!} z^k\right)\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right),$$

show that

$$\sum_{i=0}^{n-1} \binom{n}{i} b_i = 0 \quad \text{for } n > 1.$$

This formula allows us to compute any b_k in terms of previous ones, and shows that each is rational. Calculate two or three of the following:

$$b_2 = \tfrac{1}{6}, \quad b_4 = -\tfrac{1}{30}, \quad b_6 = \tfrac{1}{42}, \quad b_8 = -\tfrac{1}{30}.$$

*(c) Part (a) shows that

$$\sum_{n=0}^{\infty} \frac{b_{2n}}{(2n)!} z^{2n} = \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1} = \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}.$$

Replace z by $2iz$ and show that

$$z \cot z = \sum_{n=0}^{\infty} \frac{b_{2n}}{(2n)!} (-1)^n 2^{2n} z^{2n}.$$

*(d) Show that

$$\tan z = \cot z - 2 \cot 2z.$$

*(e) Show that

$$\tan z = \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!} (-1)^{n-1} 2^{2n} (2^{2n} - 1) z^{2n-1}.$$

(This series converges for $|z| < \pi/2$.)

17. The Bernoulli numbers play an important role in a theorem which is best introduced by some notational nonsense. Let us use D to denote the "differentiation operator," so that Df denotes f'. Then $D^k f$ will mean $f^{(k)}$ and $e^D f$ will mean $\sum_{n=0}^{\infty} f^{(n)}/n!$ (of course this series makes no sense in general, but it will make sense if f is a polynomial function, for example). Finally, let Δ denote the "difference operator" for which $\Delta f(x) = f(x+1) - f(x)$. Now Taylor's Theorem implies, disregarding questions of convergence, that

$$f(x+1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!},$$

or

$$(*) \quad f(x+1) - f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!};$$

we may write this symbolically as $\Delta f = (e^D - 1)f$, where 1 stands for the "identity operator." Even more symbolically this can be written $\Delta = e^D - 1$, which suggests that

$$D = \frac{D}{e^D - 1}\Delta.$$

Thus we obviously ought to have

$$D = \sum_{k=0}^{\infty} \frac{b_k}{k!} D^k \Delta,$$

i.e.,

$$(**) \quad f'(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(x+1) - f^{(k)}(x)].$$

The beautiful thing about all this nonsense is that it works!

(a) Prove that (**) is literally true if f is a polynomial function (in which case the infinite sum is really a finite sum). Hint: By applying (*) to $f^{(k)}$, find a formula for $f^{(k)}(x+1) - f^{(k)}(x)$; then use the formula in Problem 16(b) to find the coefficient of $f^{(j)}(x)$ in the right side of (**).

(b) Deduce from (**) that

$$f'(0) + \cdots + f'(n) = \sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(n+1) - f^{(k)}(0)].$$

(c) Show that for any polynomial function g we have

$$g(0) + \cdots + g(n) = \int_0^{n+1} g(t)\,dt + \sum_{k=1}^{\infty} \frac{b_k}{k!}[g^{(k-1)}(n+1) - g^{(k-1)}(0)].$$

(d) Apply this to $g(x) = x^p$ to show that

$$\sum_{k=1}^{n-1} k^p = \frac{n^{p+1}}{p+1} + \sum_{k=1}^{p} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

Using the fact that $b_1 = -\frac{1}{2}$, show that

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=2}^{p} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

The first ten instances of this formula were written out in Problem 2-7, which offered as a challenge the discovery of the general pattern. This may now seem to be a preposterous suggestion, but the Bernoulli numbers were actually discovered in precisely this way! After writing out these 10 formulas, Bernoulli claims (in his posthumously printed work *Ars Conjectandi*, 1713): "Whoever will examine the series as to their regularity may be able to continue the table." He then writes down the above formula, offering no proof at all, merely noting that the coefficients b_k (which he denoted simply by $A, B, C, \dots$) satisfy the equation in Problem 16(b). The relation between these numbers and the coefficients in the power series for $z/(e^z - 1)$ was discovered by Euler.

***18.** The formula in Problem 17(c) can be generalized to the case where g is not a polynomial function; the infinite sum must be replaced by a finite sum plus a remainder term. In order to find an expression for the remainder, it is useful to introduce some new functions.

(a) The *Bernoulli polynomials* φ_n are defined by

$$\varphi_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k.$$

The first three are

$$\varphi_1(x) = x - \frac{1}{2},$$
$$\varphi_2(x) = x^2 - x + \frac{1}{6},$$
$$\varphi_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}.$$

Show that

$$\begin{aligned} \varphi_n(0) &= b_n, \\ \varphi_n(1) &= b_n \qquad \text{if } n > 1, \\ \varphi_n{}'(x) &= n\varphi_{n-1}(x), \\ \varphi_n(x) &= (-1)^n \varphi_n(1-x). \end{aligned}$$

Hint: Prove the last equation by induction on n.

(b) Let $R_N{}^k(x)$ be the remainder term in Taylor's Theorem for $f^{(k)}$, on the interval $[x, x+1]$, so that

$$(*) \qquad f^{(k)}(x+1) - f^{(k)}(x) = \sum_{n=1}^{N} \frac{f^{(k+n)}(x)}{n!} + R_N{}^k(x).$$

Prove that

$$f'(x) = \sum_{k=0}^{N} \frac{b_k}{k!} [f^{(k)}(x+1) - f^{(k)}(x)] - \sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^k(x).$$

Hint: Imitate Problem 17(a). Notice the subscript $N - k$ on R.

(c) Use the integral form of the remainder to show that

$$\sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^k(x) = \int_x^{x+1} \frac{\varphi_N(x+1-t)}{N!} f^{(N+1)}(t)\, dt.$$

(d) Deduce the "Euler-Maclaurin Summation Formula":

$$g(x) + g(x+1) + \cdots + g(x+n)$$
$$= \int_x^{x+n+1} g(t)\,dt + \sum_{k=1}^{N} \frac{b_k}{k!}[g^{(k-1)}(x+n+1) - g^{(k-1)}(x)] + S_N(x,n),$$

where

$$S_N(x,n) = -\sum_{j=0}^{n} \int_{x+j}^{x+j+1} \frac{\varphi_N(x+j+1-t)}{N!} g^{(N)}(t)\,dt.$$

(e) Let ψ_n be the periodic function, with period 1, which satisfies $\psi_n(t) = \varphi_n(t)$ for $0 \le t < 1$. (Part (a) implies that if $n > 1$, then ψ_n is continuous, since $\varphi_n(1) = \varphi_n(0)$, and also that ψ_n is even if n is even and odd if n is odd.) Show that

$$S_N(x,n) = -\int_x^{x+n+1} \frac{\psi_N(x-t)}{N!} g^{(N)}(t)\,dt$$
$$\left(= (-1)^{N+1} \int_x^{x+n+1} \frac{\psi_N(t)}{N!} g^{(N)}(t)\,dt \qquad \text{if } x \text{ is an integer}\right).$$

Unlike the remainder in Taylor's Theorem, the remainder $S_N(x,n)$ usually does not satisfy $\lim_{N\to\infty} S_N(x,n) = 0$, because the Bernoulli numbers and functions become large very rapidly (although the first few examples do not suggest this). Nevertheless, important information can often be obtained from the summation formula. The general situation is best discussed within the context of a specialized study ("asymptotic series"), but the next problem shows one particularly important example.

****19.** (a) Use the Euler-Maclaurin Formula, with $N = 2$, to show that

$$\log 1 + \cdots + \log(n-1)$$
$$= \int_1^n \log t\,dt - \frac{1}{2}\log n + \frac{1}{12}\left(\frac{1}{n} - 1\right) + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt.$$

(b) Show that

$$\log\left(\frac{n!}{n^{n+1/2}e^{-n+1/(12n)}}\right) = \frac{11}{12} + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt.$$

(c) Explain why the improper integral $\beta = \int_1^\infty \psi_2(t)/2t^2\,dt$ exists, and show that if $\alpha = \exp(\beta + 11/12)$, then

$$\log\left(\frac{n!}{\alpha n^{n+1/2}e^{-n+1/(12n)}}\right) = -\int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt.$$

(d) Problem 19-41(d) shows that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}.$$

Use part (c) to show that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{\alpha^2 n^{2n+1} e^{-2n} 2^{2n}}{\alpha (2n)^{2n+1/2} e^{-2n}\sqrt{n}},$$

and conclude that $\alpha = \sqrt{2\pi}$.

(e) Show that

$$\int_0^{1/2} \varphi_2(t)\,dt = \int_0^1 \varphi_2(t)\,dt = 0.$$

(You can do the computations explicitly, but the result also follows immediately from Problem 18(a).) Conclude that

$$\bar{\psi}(x) = \int_0^x \psi_2(t)\,dt \quad \begin{cases} \geq 0 & \text{for } 0 \leq x \leq 1/2 \\ \leq 0 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

with $\bar{\psi}(n) = 0$ for all n. Hint: Graph $\bar{\psi}$ on $[0, 1]$, paying particular attention to its values at x_0, $\frac{1}{2}$, and x_1, where x_0 and x_1 are the roots of φ_2 (Figure 9).

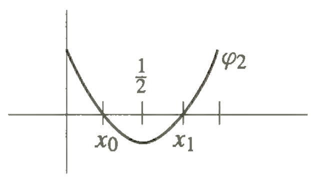


FIGURE 9

(f) Noting that $\bar{\psi}(x) = -\bar{\psi}(1-x)$, show that

$$\bar{\bar{\psi}}(x) = \int_0^x \bar{\psi}(t)\,dt \geq 0 \quad \text{on } [0, 1],$$

and hence everywhere, with $\bar{\bar{\psi}}(n) = 0$ for all n.

(g) Finally, use this information and integration by parts to show that

$$\int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt > 0.$$

(h) Using the fact that the maximum value of $|\varphi_2(x)|$ for x in $[0, 1]$ is $\frac{1}{6}$, conclude that

$$0 < \int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt < \frac{1}{12n}.$$

(i) Finally, conclude that

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} < n! < \sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n)}.$$

The final result of Problem 19, a strong form of Stirling's Formula, shows that $n!$ is approximately $\sqrt{2\pi}\, n^{n+1/2}e^{-n}$, in the sense that this expression differs from $n!$ by an amount which is small compared to n when n is large. For example, for $n = 10$ we obtain 3598696 instead of 3628800, with an error $< 1\%$.

A more general form of Stirling's Formula illustrates the "asymptotic" nature of the summation formula. The same argument which was used in Problem 19 can now be used to show that for $N \geq 2$ we have

$$\log\left(\frac{n!}{\sqrt{2\pi}\, n^{n+1/2}e^{-n}}\right) = \sum_{k=2}^{N} \frac{b_k}{k(k-1)n^{k-1}} \pm \int_n^\infty \frac{\psi_N(t)}{Nt^N}\, dt.$$

Since ψ_N is bounded, we can obtain estimates of the form

$$\left| \int_n^\infty \frac{\psi_N(t)}{Nt^N}\, dt \right| \leq \frac{M_N}{n^{N-1}}.$$

If N is large, the constant M_N will also be large; but for very large n the factor n^{1-N} will make the product very small. Thus, the expression

$$\sqrt{2\pi}\, n^{n+1/2}e^{-n} \cdot \exp\left(\sum_{k=2}^{N} \frac{b_k}{k(k-1)n^{k-1}}\right)$$

may be a very bad approximation for $n!$ when n is small, but for large n (*how* large depends on N) it will be an extremely good one (*how* good depends on N).

PART 5
EPILOGUE

There was a most ingenious Architect
who had contrived a new Method
for building Houses,
by beginning at the Roof, and working
downwards to the Foundation.

JONATHAN SWIFT

CHAPTER 28 FIELDS

Throughout this book a conscientious attempt has been made to define all important concepts, even terms like "function," for which an intuitive definition is often considered sufficient. But **Q** and **R**, the two main protagonists of this story, have only been named, never defined. What has never been defined can never be analyzed thoroughly, and "properties" P1–P13 must be considered assumptions, not theorems, about numbers. Nevertheless, the term "axiom" has been purposely avoided, and in this chapter the logical status of P1–P13 will be scrutinized more carefully.

Like **Q** and **R**, the sets **N** and **Z** have also remained undefined. True, some talk about all four was inserted in Chapter 2, but those rough descriptions are far from a definition. To say, for example, that **N** consists of 1, 2, 3, etc., merely names some elements of **N** without identifying them (and the "etc." is useless). The natural numbers *can* be defined, but the procedure is involved and not quite pertinent to the rest of the book. The Suggested Reading list contains references to this problem, as well as to the other steps that are required if one wishes to develop calculus from its basic logical starting point. The further development of this program would proceed with the definition of **Z**, in terms of **N**, and the definition of **Q** in terms of **Z**. This program results in a certain well-defined set **Q**, certain explicitly defined operations $+$ and $\cdot$, and properties P1–P12 as *theorems*. The final step in this program is the construction of **R**, in terms of **Q**. It is this last construction which concerns us. Assuming that **Q** has been defined, and that P1–P12 have been proved for **Q**, we shall ultimately *define* **R** and *prove* all of P1–P13 for **R**.

Our intention of proving P1–P13 means that we must define not only real numbers, but also addition and multiplication of real numbers. Indeed, the real numbers are of interest only as a set together with these operations: how the real numbers behave with respect to addition and multiplication is crucial; what the real numbers may actually be is quite irrelevant. This assertion can be expressed in a meaningful mathematical way, by using the concept of a "field," which includes as special cases the three important number systems of this book. This extraordinarily important abstraction of modern mathematics incorporates the properties P1–P9 common to **Q**, **R**, and **C**. A **field** is a set F (of objects of any sort whatsoever), together with two "binary operations" $\mathbf{+}$ and $\boldsymbol{\cdot}$ defined on F (that is, two rules which associate to elements a and b in F, other elements $a \mathbf{+} b$ and $a \boldsymbol{\cdot} b$ in F) for which the following conditions are satisfied:

(1) $(a \mathbf{+} b) \mathbf{+} c = a \mathbf{+} (b \mathbf{+} c)$ for all a, b, and c in F.

(2) There is some element $\mathbf{0}$ in F such that

  (i) $a \mathbf{+} \mathbf{0} = a$ for all a in F,

  (ii) for every a in F, there is some element b in F such that $a \mathbf{+} b = \mathbf{0}$.

(3) $a + b = b + a$ for all a and b in F.
(4) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, and c in F.
(5) There is some element $\mathbf{1}$ in F such that $\mathbf{1} \neq \mathbf{0}$ and

  (i) $a \cdot \mathbf{1} = a$ for all a in F,
  (ii) For every a in F with $a \neq \mathbf{0}$, there is some element b in F such that $a \cdot b = \mathbf{1}$.

(6) $a \cdot b = b \cdot a$ for all a and b in F.
(7) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, and c in F.

The familiar examples of fields are, as already indicated, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, with $+$ and $\cdot$ being the familiar operations of $+$ and $\cdot$. It is probably unnecessary to explain why these are fields, but the explanation is, at any rate, quite brief. When $+$ and $\cdot$ are understood to mean the ordinary $+$ and $\cdot$, the rules (1), (3), (4), (6), (7) are simply restatements of P1, P4, P5, P8, P9; the elements which play the role of $\mathbf{0}$ and $\mathbf{1}$ are the numbers 0 and 1 (which accounts for the choice of the symbols $\mathbf{0}$, $\mathbf{1}$); and the number b in (2) or (5) is $-a$ or a^{-1}, respectively. (For this reason, in an arbitrary field F we denote by $-a$ the element such that $a + (-a) = \mathbf{0}$, and by a^{-1} the element such that $a \cdot a^{-1} = \mathbf{1}$, for $a \neq \mathbf{0}$.)

In addition to $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, there are several other fields which can be described easily. One example is the collection F_1 of all numbers $a + b\sqrt{2}$ for a, b in $\mathbf{Q}$. The operations $+$ and $\cdot$ will, once again, be the usual $+$ and $\cdot$ for real numbers. It is necessary to point out that these operations really do produce new elements of F_1:

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}, \quad \text{which is in } F_1;$$
$$(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}, \quad \text{which is in } F_1.$$

Conditions (1), (3), (4), (6), (7) for a field are obvious for F_1: since these hold for all real numbers, they certainly hold for all real numbers of the form $a + b\sqrt{2}$. Condition (2) holds because the number $0 = 0 + 0\sqrt{2}$ is in F_1 and, for $\alpha = a + b\sqrt{2}$ in F_1 the number $\beta = (-a) + (-b)\sqrt{2}$ in F_1 satisfies $\alpha + \beta = 0$. Similarly, $1 = 1 + 0\sqrt{2}$ is in F_1, so (5i) is satisfied. The verification of (5ii) is the only slightly difficult point. If $a + b\sqrt{2} \neq 0$, then

$$a + b\sqrt{2} \cdot \frac{1}{a + b\sqrt{2}} = 1;$$

it is therefore necessary to show that $1/(a + b\sqrt{2})$ is in F_1. This is true because

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a - b\sqrt{2})(a + b\sqrt{2})} = \frac{a}{a^2 - 2b^2} + \frac{(-b)}{a^2 - 2b^2}\sqrt{2}.$$

(The division by $a - b\sqrt{2}$ is valid because the relation $a - b\sqrt{2} = 0$ could be true only if $a = b = 0$ (since $\sqrt{2}$ is irrational) which is ruled out by the hypothesis $a + b\sqrt{2} \neq 0$.)

The next example of a field, F_2, is considerably simpler in one respect: it contains only two elements, which we might as well denote by $\mathbf{0}$ and $\mathbf{1}$. The operations

+ and **·** are described by the following tables.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

The verification of conditions (1)–(7) are straightforward, case-by-case checks. For example, condition (1) may be proved by checking the 8 equations obtained by setting a, b, $c = \mathbf{0}$ or $\mathbf{1}$. Notice that in this field $\mathbf{1} \mathbin{\boldsymbol{+}} \mathbf{1} = \mathbf{0}$; this equation may also be written $\mathbf{1} = \mathbf{-1}$.

Our final example of a field is rather silly: F_3 consists of all pairs (a, a) for a in $\mathbf{R}$, and **+** and **·** are defined by

$$(a, a) \mathbin{\boldsymbol{+}} (b, b) = (a + b, a + b),$$
$$(a, a) \mathbin{\boldsymbol{\cdot}} (b, b) = (a \cdot b, a \cdot b).$$

(The $+$ and $\cdot$ appearing on the right side are ordinary addition and multiplication for **R**.) The verification that F_3 is a field is left to you as a simple exercise.

A detailed investigation of the properties of fields is a study in itself, but for our purposes, fields provide an ideal framework in which to discuss the properties of numbers in the most economical way. For example, the consequences of P1–P9 which were derived for "numbers" in Chapter 1 actually hold for any field; in particular, they are true for the fields **Q**, **R**, and **C**.

Notice that certain common properties of **Q**, **R**, and **C** do not hold for all fields. For example, it is possible for the equation $\mathbf{1} \mathbin{\boldsymbol{+}} \mathbf{1} = \mathbf{0}$ to hold in some fields, and consequently $a \mathbin{\boldsymbol{-}} b = b \mathbin{\boldsymbol{-}} a$ does not necessarily imply that $a = b$. For the field **C** the assertion $1 + 1 \neq 0$ was derived from the explicit description of **C**; for the fields **Q** and **R**, however, this assertion was derived from further properties which do not have analogues in the conditions for a field. There is a related concept which does use these properties. An **ordered field** is a field F (with operations **+** and **·**) together with a certain subset **P** of F (the "positive" elements) with the following properties:

(8) For all a in F, one and only one of the following is true:

- (i) $a = \mathbf{0}$,
- (ii) a is in **P**,
- (iii) $\mathbf{-}a$ is in **P**.

(9) If a and b are in **P**, then $a \mathbin{\boldsymbol{+}} b$ is in **P**.

(10) If a and b are in **P**, then $a \mathbin{\boldsymbol{\cdot}} b$ is in **P**.

We have already seen that the field **C** cannot be made into an ordered field. The field F_2, with only two elements, likewise cannot be made into an ordered field: in fact, condition (8), applied to $\mathbf{1} = \mathbf{-1}$, shows that $\mathbf{1}$ must be in **P**; then (9) implies that $\mathbf{1} \mathbin{\boldsymbol{+}} \mathbf{1} = \mathbf{0}$ is in **P**, contradicting (8). On the other hand, the field F_1, consisting of all numbers $a + b\sqrt{2}$ with a, b in **Q**, certainly can be made into an ordered field: let **P** be the set of all $a + b\sqrt{2}$ which are positive real numbers (in the ordinary sense). The field F_3 can also be made into an ordered field; the description of **P** is left to you.

It is natural to introduce notation for an arbitrary ordered field which corresponds to that used for **Q** and **R**: we define

$$\begin{aligned} a > b &\quad \text{if} \quad a - b \text{ is in } \mathbf{P}, \\ a < b &\quad \text{if} \quad b > a, \\ a \leq b &\quad \text{if} \quad a < b \text{ or } a = b, \\ a \geq b &\quad \text{if} \quad a > b \text{ or } a = b. \end{aligned}$$

Using these definitions we can reproduce, for an arbitrary ordered field F, the definitions of Chapter 7:

> A set A of elements of F is **bounded above** if there is some x in F such that $x \geq a$ for all a in A. Any such x is called an **upper bound** for A. An element x of F is a **least upper bound** for A if x is an upper bound for A and $x \leq y$ for every y in F which is an upper bound for A.

Finally, it is possible to state an analogue of property P13 for **R**; this leads to the last abstraction of this chapter:

> A **complete ordered field** is an ordered field in which every nonempty set which is bounded above has a least upper bound.

The consideration of fields may seem to have taken us far from the goal of constructing the real numbers. However, we are now provided with an intelligible means of formulating this goal. There are two questions which will be answered in the remaining two chapters:

1. Is there a complete ordered field?
2. Is there only one complete ordered field?

Our starting point for these considerations will be **Q**, assumed to be an ordered field, containing **N** and **Z** as certain subsets. At one crucial point it will be necessary to assume another fact about **Q**:

> Let x be an element of **Q** with $x > 0$. Then for any y in **Q** there is some n in **N** such that $nx > y$.

This assumption, which asserts that the rational numbers have the Archimedean property of the real numbers, does not follow from the other properties of an ordered field (for the example that demonstrates this conclusively see reference [14] of the Suggested Reading). The important point for us is that when **Q** is explicitly constructed, properties P1–P12 appear as theorems, and so does this additional assumption; if we really began from the beginning, no assumptions about $\mathbf{Q}$ would be necessary.

PROBLEMS

1. Let F be the set $\{0, 1, 2\}$ and define operations $+$ and $\cdot$ on F by the following tables. (The rule for constructing these tables is as follows: add or multiply in the usual way, and then subtract the highest possible multiple of 3; thus $2 \cdot 2 = 4 = 3 + 1$, so $2 \cdot 2 = 1$.)

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Show that F is a field, and prove that it cannot be made into an ordered field.

2. Suppose now that we try to construct a field F having elements 0, 1, 2, 3 with operations $+$ and $\cdot$ defined as in the previous example, by adding or multiplying in the usual way, and then subtracting the highest possible multiple of 4. Show that F will *not* be a field.

3. Let $F = \{0, 1, \alpha, \beta\}$ and define operations $+$ and $\cdot$ on F by the following tables.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\cdot$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

Show that F is a field.

4. (a) Let F be a field in which $\mathbf{1} + \mathbf{1} = \mathbf{0}$. Show that $a + a = \mathbf{0}$ for all a (this can also be written $a = -a$).

(b) Suppose that $a + a = \mathbf{0}$ for some $a \neq \mathbf{0}$. Show that $\mathbf{1} + \mathbf{1} = \mathbf{0}$ (and consequently $b + b = \mathbf{0}$ for all b).

5. (a) Show that in any field we have

$$\underbrace{(1+\cdots+1)}_{m \text{ times}}\cdot\underbrace{(1+\cdots+1)}_{n \text{ times}}=\underbrace{1+\cdots+1}_{mn \text{ times}}$$

for all natural numbers m and n.

(b) Suppose that in the field F we have

$$\underbrace{1+\cdots+1}_{m \text{ times}}=0$$

for some natural number m. Show that the smallest m with this property must be a prime number (this prime number is called the **characteristic** of F).

6. Let F be any field with only finitely many elements.

(a) Show that there must be distinct natural numbers m and n with

$$\underbrace{1+\cdots+1}_{m \text{ times}}=\underbrace{1+\cdots+1}_{n \text{ times}}.$$

(b) Conclude that there is some natural number k with

$$\underbrace{1+\cdots+1}_{k \text{ times}}=\mathbf{0}.$$

7. Let a, b, c, and d be elements of a field F with $a\cdot d-b\cdot c\neq\mathbf{0}$. Show that for any α and β in F the equations

$$\begin{aligned} a\cdot x+b\cdot y&=\alpha,\\ c\cdot x+d\cdot y&=\beta, \end{aligned}$$

can be solved for x and y in F.

8. Let a be an element of a field F. A "square root" of a is an element b of F with $b^2=b\cdot b=a$.

(a) How many square roots does $\mathbf{0}$ have?

(b) Suppose $a\neq\mathbf{0}$. Show that if a has a square root, then it has two square roots, unless $\mathbf{1}+\mathbf{1}=\mathbf{0}$, in which case a has only one.

9. (a) Consider an equation $x^2+b\cdot x+c=\mathbf{0}$, where b and c are elements of a field F. Suppose that $b^2-\mathbf{4}\cdot c$ has a square root r in F. Show that $(-b+r)/\mathbf{2}$ is a solution of this equation. (Here $\mathbf{2}=\mathbf{1}+\mathbf{1}$ and $\mathbf{4}=\mathbf{2}+\mathbf{2}$.)

(b) In the field F_2 of the text, both elements clearly have a square root. On the other hand, it is easy to check that neither element satisfies the equation $x^2+x+\mathbf{1}=\mathbf{0}$. Thus some detail in part (a) must be incorrect. What is it?

10. Let F be a field and a an element of F which does *not* have a square root. This problem shows how to construct a bigger field F', containing F, in which a does have a square root. (This construction has already been carried through in a special case, namely, $F = \mathbf{R}$ and $a = -1$; this special case should guide you through this example.)

Let F' consist of all pairs (x, y) with x and y in F. If the operations on F are $+$ and $\cdot$, define operations $\oplus$ and $\odot$ on F' as follows:

$$(x, y) \oplus (z, w) = (x + z, y + w),$$
$$(x, y) \odot (z, w) = (x \cdot z + a \cdot y \cdot w, y \cdot z + x \cdot w).$$

(a) Prove that F', with the operations $\oplus$ and $\odot$, is a field.

(b) Prove that

$$(x, \mathbf{0}) \oplus (y, \mathbf{0}) = (x + y, \mathbf{0}),$$
$$(x, \mathbf{0}) \odot (y, \mathbf{0}) = (x \cdot y, \mathbf{0}),$$

so that we may agree to abbreviate $(x, \mathbf{0})$ by x.

(c) Find a square root of $a = (a, \mathbf{0})$ in F'.

11. Let F be the set of all four-tuples (w, x, y, z) of real numbers. Define $+$ and $\cdot$ by

$$(s, t, u, v) + (w, x, y, z) = (s + w, t + x, u + y, v + z),$$
$$(s, t, u, v) \cdot (w, x, y, z) = (sw - tx - uy - vz, sx + tw + uz - vy,$$
$$sy + uw + vx - tz, sz + vw + ty - ux).$$

(a) Show that F satisfies all conditions for a field, except (6). At times the algebra will become quite ornate, but the existence of multiplicative inverses is the only point requiring any thought.

(b) It is customary to denote

$(0, 1, 0, 0)$ by i,
$(0, 0, 1, 0)$ by j,
$(0, 0, 0, 1)$ by k.

Find all 9 products of pairs i, j, and k. The results will show in particular that condition (6) is definitely false. This "skew field" F is known as the **quaternions**.

CHAPTER 29 CONSTRUCTION OF THE REAL NUMBERS

The mass of drudgery which this chapter necessarily contains is relieved by one truly first-rate idea. In order to prove that a complete ordered field exists we will have to explicitly describe one in detail; verifying conditions (1)–(10) for an ordered field will be a straightforward ordeal, but the description of the field itself, of the elements in it, is ingenious indeed.

At our disposal is the set of rational numbers, and from this raw material it is necessary to produce the field which will ultimately be called the real numbers. To the uninitiated this must seem utterly hopeless—if only the rational numbers are known, where are the others to come from? By now we have had enough experience to realize that the situation may not be quite so hopeless as that casual consideration suggests. The strategy to be adopted in our construction has already been used effectively for defining functions and complex numbers. Instead of trying to determine the "real nature" of these concepts, we settled for a definition that described enough about them to determine their mathematical properties completely.

A similar proposal for defining real numbers requires a description of real numbers in terms of rational numbers. The observation, that a real number ought to be determined completely by the set of rational numbers less than it, suggests a strikingly simple and quite attractive possibility: a real number might (and in fact eventually will) be described as a collection of rational numbers. In order to make this proposal effective, however, some means must be found for describing "the set of rational numbers less than a real number" without mentioning real numbers, which are still nothing more than heuristic figments of our mathematical imagination.

If A is to be regarded as the set of rational numbers which are less than the real number α, then A ought to have the following property: If x is in A and y is a rational number satisfying $y < x$, then y is in A. In addition to this property, the set A should have a few others. Since there should be some rational number $x < \alpha$, the set A should not be empty. Likewise, since there should be some rational number $x > \alpha$, the set A should not be all of $\mathbf{Q}$. Finally, if $x < \alpha$, then there should be another rational number y with $x < y < \alpha$, so A should not contain a greatest member.

If we temporarily regard the real numbers as known, then it is not hard to check (Problem 8-17) that a set A with these properties is indeed the set of rational numbers less than some real number α. Since the real numbers are presently in limbo, your proof, if you supply one, must be regarded only as an unofficial comment on these proceedings. It will serve to convince you, however, that we have not failed to notice any crucial property of the set A. There appears to be no reason for hesitating any longer.

DEFINITION

A **real number** is a set α, of rational numbers, with the following four properties:

(1) If x is in α and y is a rational number with $y < x$, then y is also in α.
(2) $\alpha \neq \emptyset$.
(3) $\alpha \neq \mathbf{Q}$.
(4) There is no greatest element in α; in other words, if x is in α, then there is some y in α with $y > x$.

The set of all real numbers is denoted by $\mathbf{R}$.

Just to remind you of the philosophy behind our definition, here is an explicit example of a real number:

$$\alpha = \{x \text{ in } \mathbf{Q} : x < 0 \text{ or } x^2 < 2\}.$$

It should be clear that α is the real number which will eventually be known as $\sqrt{2}$, but it is not an entirely trivial exercise to show that α actually is a real number. The whole point of such an exercise is to prove this using only facts about $\mathbf{Q}$; the hard part will be checking condition (4), but this has already appeared as a problem in a previous chapter (finding out which one is up to you). Notice that condition (4), although quite bothersome here, is really essential in order to avoid ambiguity; without it both

$$\{x \text{ in } \mathbf{Q} : x < 1\}$$

and

$$\{x \text{ in } \mathbf{Q} : x \leq 1\}$$

would be candidates for the "real number 1."

The shift from A to α in our definition indicates both a conceptual and a notational concern. Henceforth, a real number *is*, by definition, a set of rational numbers. This means, in particular, that a rational number (a member of $\mathbf{Q}$) is *not* a real number; instead every rational number x has a natural counterpart which is a real number, namely, $\{y \text{ in } \mathbf{Q} : y < x\}$. After completing the construction of the real numbers, we can mentally throw away the elements of $\mathbf{Q}$ and agree that $\mathbf{Q}$ will henceforth denote these special sets. For the moment, however, it will be necessary to work at the same time with rational numbers, real numbers (sets of rational numbers) and even sets of real numbers (sets of sets of rational numbers). Some confusion is perhaps inevitable, but proper notation should keep this to a minimum. Rational numbers will be denoted by lower case Roman letters (x, y, z, a, b, c) and real numbers by lower case Greek letters (α, β, γ); capital Roman letters (A, B, C) will be used to denote sets of real numbers.

The remainder of this chapter is devoted to the definition of $\mathbf{+}$, $\boldsymbol{\cdot}$, and $\mathbf{P}$ for $\mathbf{R}$, and a proof that with these structures $\mathbf{R}$ is indeed a complete ordered field.

We shall actually begin with the definition of $\mathbf{P}$, and even here we shall work backwards. We first define $\alpha \mathbf{<} \beta$; later, when $\mathbf{+}$, $\boldsymbol{\cdot}$, and $\mathbf{0}$ are available, we shall define $\mathbf{P}$ as the set of all α with $\mathbf{0} \mathbf{<} \alpha$, and prove the necessary properties for $\mathbf{P}$.

The reason for beginning with the definition of $<$ is the simplicity of this concept in our present setup:

Definition. If α and β are real numbers, then $\alpha < \beta$ means that α is contained in β (that is, every element of α is also an element of β), but $\alpha \neq \beta$.

A repetition of the definitions of $\leq, >, \geq$ would be stultifying, but it is interesting to note that $\leq$ can now be expressed more simply than $<$; if α and β are real numbers, then $\alpha \leq \beta$ if and only if α is contained in β.

If A is a bounded collection of real numbers, it is almost obvious that A should have a least upper bound. Each α in A is a collection of rational numbers; if these rational numbers are all put in one collection β, then β is presumably $\sup A$. In the proof of the following theorem we check all the little details which have not been mentioned, not least of which is the assertion that β *is* a real number. (We will not bother numbering theorems in this chapter, since they all add up to one big Theorem: There is a complete ordered field.)

THEOREM If A is a set of real numbers and $A \neq \emptyset$ and A is bounded above, then A has a least upper bound.

PROOF Let $\beta = \{x : x \text{ is in some } \alpha \text{ in } A\}$. Then β is certainly a collection of rational numbers; the proof that β is a real number requires checking four facts.

(1) Suppose that x is in β and $y < x$. The first condition means that x is in α for some α in A. Since α is a real number, the assumption $y < x$ implies that y is in α. Therefore it is certainly true that y is in β.
(2) Since $A \neq \emptyset$, there is some α in A. Since α is a real number, there is some x in α. This means that x is in β, so $\beta \neq \emptyset$.
(3) Since A is bounded above, there is some real number γ such that $\alpha < \gamma$ for every α in A. Since γ is a real number, there is some rational number x which is not in γ. Now $\alpha < \gamma$ means that α is contained in γ, so it is also true that x is not in α for any α in A. This means that x is not in β; so $\beta \neq \mathbf{Q}$.
(4) Suppose that x is in β. Then x is in α for some α in A. Since α does not have a greatest member, there is some rational number y with $x < y$ and y in α. But this means that y is in β; thus β does not have a greatest member.

These four observations prove that β is a real number. The proof that β is the least upper bound of A is easier. If α is in A, then clearly α is contained in β; this means that $\alpha \leq \beta$, so β is an upper bound for A. On the other hand, if γ is an upper bound for A, then $\alpha \leq \gamma$ for every α in A; this means that α is contained in γ, for every α in A, and this surely implies that β is contained in γ. *This,* in turn, means that $\beta \leq \gamma$; thus β is the least upper bound of A. ▮

The definition of $+$ is both obvious and easy, but is must be complemented with a proof that this "obvious" definition makes any sense at all.

Definition. If α and β are real numbers, then

$$\alpha \mathbf{+} \beta = \{x : x = y + z \text{ for some } y \text{ in } \alpha \text{ and some } z \text{ in } \beta\}.$$

THEOREM If α and β are real numbers, then $\alpha \mathbf{+} \beta$ is a real number.

PROOF Once again four facts must be verified.

(1) Suppose $w < x$ for some x in $\alpha \mathbf{+} \beta$. Then $x = y + z$ for some y in α and some z in β, which means that $w < y + z$, and consequently, $w - y < z$. This shows that $w - y$ is in β (since z is in β, and β is a real number). Since $w = y + (w - y)$, it follows that w is in $\alpha \mathbf{+} \beta$.
(2) It is clear that $\alpha \mathbf{+} \beta \neq \emptyset$, since $\alpha \neq \emptyset$ and $\beta \neq \emptyset$.
(3) Since $\alpha \neq \mathbf{Q}$ and $\beta \neq \mathbf{Q}$, there are rational numbers a and b with a not in α and b not in β. Any x in α satisfies $x < a$ (for if $a < x$, then condition (1) for a real number would imply that a is in α); similarly any y in β satisfies $y < b$. Thus $x + y < a + b$ for any x in α and y in β. This shows that $a + b$ is not in $\alpha \mathbf{+} \beta$, so $\alpha \mathbf{+} \beta \neq \mathbf{Q}$.
(4) If x is in $\alpha \mathbf{+} \beta$, then $x = y + z$ for y in α and z in β. There are y' in α and z' in β with $y < y'$ and $z < z'$; then $x < y' + z'$ and $y' + z'$ is in $\alpha \mathbf{+} \beta$. Thus $\alpha \mathbf{+} \beta$ has no greatest member. ▮

By now you can see how tiresome this whole procedure is going to be. Every time we mention a new real number, we must prove that it *is* a real number; this requires checking four conditions, and even when trivial they require concentration. There is really no help for this (except that it will be less boring if you check the four conditions for yourself). Fortunately, however, a few points of interest will arise now and then, and some of our theorems will be easy. In particular, two properties of $\mathbf{+}$ present no problems.

THEOREM If α, β, and γ are real numbers, then $(\alpha \mathbf{+} \beta) \mathbf{+} \gamma = \alpha \mathbf{+} (\beta \mathbf{+} \gamma)$.

PROOF Since $(x + y) + z = x + (y + z)$ for all rational numbers x, y, and z, every member of $(\alpha \mathbf{+} \beta) \mathbf{+} \gamma$ is also a member of $\alpha \mathbf{+} (\beta \mathbf{+} \gamma)$, and vice versa. ▮

THEOREM If α and β are real numbers, then $\alpha \mathbf{+} \beta = \beta \mathbf{+} \alpha$.

PROOF Left to you (even easier). ▮

To prove the other properties of $\mathbf{+}$ we first define $\mathbf{0}$.

Definition. $\mathbf{0} = \{x \text{ in } \mathbf{Q} : x < 0\}$.

It is, thank goodness, obvious that $\mathbf{0}$ is a real number, and the following theorem is also simple.

THEOREM If α is a real number, then $\alpha \mathbin{\mathbf{+}} \mathbf{0} = \alpha$.

PROOF If x is in α and y is in $\mathbf{0}$, then $y < 0$, so $x + y < x$. This implies that $x + y$ is in α. Thus every member of $\alpha \mathbin{\mathbf{+}} \mathbf{0}$ is also a member of α.

On the other hand, if x is in α, then there is a rational number y in α such that $y > x$. Since $x = y + (x - y)$, where y is in α, and $x - y < 0$ (so that $x - y$ is in $\mathbf{0}$), this shows that x is in $\alpha \mathbin{\mathbf{+}} \mathbf{0}$. Thus every member of α is also a member of $\alpha \mathbin{\mathbf{+}} \mathbf{0}$. ▮

The reasonable candidate for $\mathbf{-}\alpha$ would seem to be the set

$$\{x \text{ in } \mathbf{Q} : -x \text{ is not in } \alpha\}$$

(since $-x$ not in α means, intuitively, that $-x > \alpha$, so that $x < -\alpha$). But in certain cases this set will not even be a real number. Although a real number α does not have a greatest member, the set

$$\mathbf{Q} - \alpha = \{x \text{ in } \mathbf{Q} : x \text{ is not in } \alpha\}$$

may have a *least* element x_0; when α is a real number of this kind, the set $\{x : -x \text{ is not in } \alpha\}$ will have a greatest element $-x_0$. It is therefore necessary to introduce a slight modification into the definition of $\mathbf{-}\alpha$, which comes equipped with a theorem.

Definition. If α is a real number, then

$$\mathbf{-}\alpha = \{x \text{ in } \mathbf{Q} : -x \text{ is not in } \alpha, \text{ but } -x \text{ is not the least element of } \mathbf{Q} - \alpha\}.$$

THEOREM If α is a real number, then $\mathbf{-}\alpha$ is a real number.

PROOF

(1) Suppose that x is in $\mathbf{-}\alpha$ and $y < x$. Then $-y > -x$. Since $-x$ is not in α, it is also true that $-y$ is not in α. Moreover, it is clear that $-y$ is not the smallest element of $\mathbf{Q} - \alpha$, since $-x$ is a smaller element. This shows that y is in $\mathbf{-}\alpha$.

(2) Since $\alpha \neq \mathbf{Q}$, there is some rational number y which is not in α. We can assume that y is not the smallest rational number in $\mathbf{Q} - \alpha$ (since y can always be replaced by any $y' > y$). Then $-y$ is in $\mathbf{-}\alpha$. Thus $\mathbf{-}\alpha \neq \emptyset$.

(3) Since $\alpha \neq \emptyset$, there is some x in α. Then $-x$ cannot possibly be in $\mathbf{-}\alpha$, so $\mathbf{-}\alpha \neq \mathbf{Q}$.

(4) If x is in $\mathbf{-}\alpha$, then $-x$ is not in α, and there is a rational number $y < -x$ which is also not in α. Let z be a rational number with $y < z < -x$. Then z is also not in α, and z is clearly not the smallest element of $\mathbf{Q} - \alpha$. So $-z$ is in $\mathbf{-}\alpha$. Since $-z > x$, this shows that $\mathbf{-}\alpha$ does not have a greatest element. ▮

The proof that $\alpha \mathbin{\mathbf{+}} (\mathbf{-}\alpha) = \mathbf{0}$ is not entirely straightforward. The difficulties are not caused, as you might presume, by the finicky details in the definition of $-\alpha$. Rather, at this point we require the Archimedean property of $\mathbf{Q}$ stated on page 584, which does not follow from P1–P12. This property is needed to prove the following lemma, which plays a crucial role in the next theorem.

LEMMA Let α be a real number, and z a positive rational number. Then there are (Figure 1) rational numbers x in α, and y not in α, such that $y - x = z$. Moreover, we may assume that y is not the smallest element of $\mathbf{Q} - \alpha$.

PROOF Suppose first that z is in α. If the numbers

$$z, 2z, 3z, \ldots$$

were *all* in α, then *every* rational number would be in α, since every rational number w satisfies $w < nz$ for some n, by the additional assumption on page 584. This contradicts the fact that α is a real number, so there is some k such that $x = kz$ is in α and $y = (k+1)z$ is not in α. Clearly $y - x = z$.

Moreover, if y happens to be the smallest element of $\mathbf{Q} - \alpha$, let $x' > x$ be an element of α, and replace x by x', and y by $y + (x' - x)$.

If z is not in α, there is a similar proof, based on the fact that the numbers $(-n)z$ cannot all fail to be in α. ▌

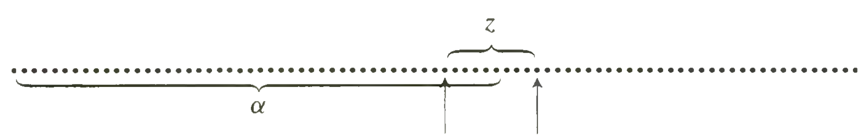


FIGURE 1

THEOREM If α is a real number, then

$$\alpha + (-\alpha) = \mathbf{0}.$$

PROOF Suppose x is in α and y is in $-\alpha$. Then $-y$ is not in α, so $-y > x$. Hence $x + y < 0$, so $x + y$ is in $\mathbf{0}$. Thus every member of $\alpha + (-\alpha)$ is in $\mathbf{0}$.

It is a little more difficult to go in the other direction. If z is in $\mathbf{0}$, then $-z > 0$. According to the lemma, there is some x in α, and some y not in α, with y not the smallest element of $\mathbf{Q} - \alpha$, such that $y - x = -z$. This equation can be written $x + (-y) = z$. Since x is in α, and $-y$ is in $-\alpha$, this proves that z is in $\alpha + (-\alpha)$. ▌

Before proceeding with multiplication, we define the "positive elements" and prove a basic property:

Definition. $\mathbf{P} = \{\alpha \text{ in } \mathbf{R} : \alpha > \mathbf{0}\}$.

Notice that $\alpha + \beta$ is clearly in $\mathbf{P}$ if α and β are.

THEOREM If α is a real number, then one and only one of the following conditions holds:

(i) $\alpha = \mathbf{0}$,
(ii) α is in $\mathbf{P}$,
(iii) $-\alpha$ is in $\mathbf{P}$.

PROOF If α contains any positive rational number, then α certainly contains all negative rational numbers, so α contains 0 and $\alpha \neq \mathbf{0}$, i.e., α is in $\mathbf{P}$. If α contains no positive rational numbers, then one of two possibilities must hold:

(1) α contains all negative rational numbers; then $\alpha = \mathbf{0}$.
(2) there is some negative rational number x which is not in α; it can be assumed that x is not the least element of $\mathbf{Q} - \alpha$ (since x could be replaced by $x/2 > x$); then $-\alpha$ contains the positive rational number $-x$, so, as we have just proved, $-\alpha$ is in $\mathbf{P}$.

This shows that *at least one* of (i)–(iii) must hold. If $\alpha = \mathbf{0}$, it is clearly impossible for condition (ii) or (iii) to hold. Moreover, it is impossible that $\alpha > \mathbf{0}$ and $-\alpha > \mathbf{0}$ both hold, since this would imply that $\mathbf{0} = \alpha + (-\alpha) > \mathbf{0}$. ▌

Recall that $\alpha > \beta$ was defined to mean that α contains β, but is unequal to β. This definition was fine for proving completeness, but now we have to show that it is equivalent to the definition which would be made in terms of $\mathbf{P}$. Thus, we must show that $\alpha - \beta > \mathbf{0}$ is equivalent to $\alpha > \beta$. This is clearly a consequence of the next theorem.

THEOREM If α, β, and γ are real numbers and $\alpha > \beta$, then $\alpha + \gamma > \beta + \gamma$.

PROOF The hypothesis $\alpha > \beta$ implies that β is contained in α; it follows immediately from the definition of $+$ that $\beta + \gamma$ is contained in $\alpha + \gamma$. This shows that $\alpha + \gamma \geq \beta + \gamma$. We can easily rule out the possibility of equality, for if

$$\alpha + \gamma = \beta + \gamma,$$

then

$$\alpha = (\alpha + \gamma) + (-\gamma) = (\beta + \gamma) + (-\gamma) = \beta,$$

which is false. Thus $\alpha + \gamma > \beta + \gamma$. ▌

Multiplication presents difficulties of its own. If $\alpha, \beta > \mathbf{0}$, then $\alpha \cdot \beta$ can be defined as follows.

Definition. If α and β are real numbers and $\alpha, \beta > \mathbf{0}$, then

$$\alpha \cdot \beta = \{z : z \leq 0 \text{ or } z = x \cdot y \text{ for some } x \text{ in } \alpha \text{ and } y \text{ in } \beta \text{ with } x, y > 0\}.$$

THEOREM If α and β are real numbers with $\alpha, \beta > \mathbf{0}$, then $\alpha \cdot \beta$ is a real number.

PROOF As usual, we must check four conditions.

(1) Suppose $w < z$, where z is in $\alpha \cdot \beta$. If $w \leq 0$, then w is automatically in $\alpha \cdot \beta$. Suppose that $w > 0$. Then $z > 0$, so $z = x \cdot y$ for some positive x in α and positive y in β. Now

$$w = \frac{wz}{z} = \frac{wxy}{z} = \left(\frac{w}{z} \cdot x\right) \cdot y.$$

Since $0 < w < z$, we have $w/z < 1$, so $(w/z) \cdot x$ is in α. Thus w is in $\alpha \cdot \beta$.

(2) Clearly $\alpha \cdot \beta \neq \emptyset$.

(3) If x is not in α, and y is not in β, then $x > x'$ for all x' in α, and $y > y'$ for all y' in β. Hence $xy > x'y'$ for all such positive x' and y'. So xy is not in $\alpha \cdot \beta$; thus $\alpha \cdot \beta \neq \mathbf{Q}$.

(4) Suppose w is in $\alpha \cdot \beta$, and $w \leq 0$. There is some x in α with $x > 0$ and some y in β with $y > 0$. Then $z = xy$ is in $\alpha \cdot \beta$ and $z > w$. Now suppose $w > 0$. Then $w = xy$ for some positive x in α and some positive y in β. Moreover, α contains some $x' > x$; if $z = x'y$, then $z > xy = w$, and z is in $\alpha \cdot \beta$. Thus $\alpha \cdot \beta$ does not have a greatest element. ▮

Notice that $\alpha \cdot \beta$ is clearly in $\mathbf{P}$ if α and β are. This completes the verification of all properties of $\mathbf{P}$. To complete the definition of $\cdot$ we first define $|\alpha|$.

Definition. If α is a real number, then

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha \geq \mathbf{0} \\ -\alpha, & \text{if } \alpha \leq \mathbf{0}. \end{cases}$$

Definition. If α and β are real numbers, then

$$\alpha \cdot \beta = \begin{cases} \mathbf{0}, & \text{if } \alpha = \mathbf{0} \text{ or } \beta = \mathbf{0} \\ |\alpha| \cdot |\beta|, & \text{if } \alpha > \mathbf{0}, \beta > \mathbf{0} \text{ or } \alpha < \mathbf{0}, \beta < \mathbf{0} \\ -(|\alpha| \cdot |\beta|), & \text{if } \alpha > \mathbf{0}, \beta < \mathbf{0} \text{ or } \alpha < \mathbf{0}, \beta > \mathbf{0}. \end{cases}$$

As one might suspect, the proofs of the properties of multiplication usually involve reduction to the case of positive numbers.

THEOREM If α, β, and γ are real numbers, then $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

PROOF This is clear if $\alpha, \beta, \gamma > \mathbf{0}$. The proof for the general case requires considering separate cases (and is simplified slightly if one uses the following theorem). ▮

THEOREM If α and β are real numbers, then $\alpha \cdot \beta = \beta \cdot \alpha$.

PROOF This is clear if $\alpha, \beta > \mathbf{0}$, and the other cases are easily checked. ▮

Definition. $\mathbf{1} = \{x \text{ in } \mathbf{Q} : x < 1\}$.
(It is clear that $\mathbf{1}$ is a real number.)

THEOREM If α is a real number, then $\alpha \cdot \mathbf{1} = \alpha$.

PROOF Let $\alpha > \mathbf{0}$. It is easy to see that every member of $\alpha \cdot \mathbf{1}$ is also a member of α. On the other hand, suppose x is in α. If $x \leq 0$, then x is automatically in $\alpha \cdot \mathbf{1}$. If $x > 0$, then there is some rational number y in α such that $x < y$. Then $x = y \cdot (x/y)$, and x/y is in $\mathbf{1}$, so x is in $\alpha \cdot \mathbf{1}$. This proves that $\alpha \cdot \mathbf{1} = \alpha$ if $\alpha > \mathbf{0}$.

If $\alpha < \mathbf{0}$, then, applying the result just proved, we have

$$\alpha \cdot \mathbf{1} = -(|\alpha| \cdot |\mathbf{1}|) = -(|\alpha|) = \alpha.$$

Finally, the theorem is obvious when $\alpha = \mathbf{0}$. ▮

Definition. If α is a real number and $\alpha > \mathbf{0}$, then
$\alpha^{-1} = \{x \text{ in } \mathbf{Q} : x \leq 0, \text{ or } x > 0 \text{ and } 1/x \text{ is not in } \alpha, \text{ but } 1/x \text{ is not the smallest member of } \mathbf{Q} - \alpha\}$;
if $\alpha < \mathbf{0}$, then $\alpha^{-1} = -(|\alpha|)^{-1}$.

THEOREM If α is a real number unequal to $\mathbf{0}$, then α^{-1} is a real number.

PROOF Clearly it suffices to consider only $\alpha > \mathbf{0}$. Four conditions must be checked.

(1) Suppose $y < x$, and x is in α^{-1}. If $y \leq 0$, then y is in α^{-1}. If $y > 0$, then $x > 0$, so $1/x$ is not in α. Since $1/y > 1/x$, it follows that $1/y$ is not in α, and $1/y$ is clearly not the smallest element of $\mathbf{Q} - \alpha$, so y is in α^{-1}.
(2) Clearly $\alpha^{-1} \neq \emptyset$.
(3) Since $\alpha > \mathbf{0}$, there is some positive rational number x in α. Then $1/x$ is not in α^{-1}, so $\alpha^{-1} \neq \mathbf{Q}$.
(4) Suppose x is in α^{-1}. If $x \leq 0$, there is clearly some y in α^{-1} with $y > x$ because α^{-1} contains some positive rationals. If $x > 0$, then $1/x$ is not in α. Since $1/x$ is not the smallest member of $\mathbf{Q} - \alpha$, there is a rational number y not in α, with $y < 1/x$. Choose a rational number z with $y < z < 1/x$. Then $1/z$ is in α^{-1}, and $1/z > x$. Thus α^{-1} does not contain a largest member. ▮

In order to prove that α^{-1} is really the multiplicative inverse of α, it helps to have another lemma, which is the multiplicative analogue of our first lemma.

LEMMA Let α be a real number with $\alpha > \mathbf{0}$, and z a rational number with $z > 1$. Then there are rational numbers x in α, and y not in α, such that $y/x = z$. Moreover, we can assume that y is not the least element of $\mathbf{Q} - \alpha$.

PROOF Suppose first that z is in α. Since $z - 1 > 0$ and

$$z^n = (1 + (z - 1))^n \geq 1 + n(z - 1),$$

it follows that the numbers

$$z, z^2, z^3, \ldots$$

cannot all be in α. So there is some k such that $x = z^k$ is in α, and $y = z^{k+1}$ is not in α. Clearly $y/x = z$. Moreover, if y happens to be the least element of $\mathbf{Q} - \alpha$, let $x' > x$ be an element of α, and replace x by x' and y by yx'/x.

If z is not in α, there is a similar proof, based on the fact that the numbers $1/z^k$ cannot all fail to be in α. ▮

THEOREM If α is a real number and $\alpha \neq \mathbf{0}$, then $\alpha \bullet \alpha^{-1} = \mathbf{1}$.

PROOF It obviously suffices to consider only $\alpha > \mathbf{0}$, in which case $\alpha^{-1} > \mathbf{0}$. Suppose that x is a positive rational number in α, and y is a positive rational number in α^{-1}. Then $1/y$ is not in α, so $1/y > x$; consequently $xy < 1$, which means that xy is in $\mathbf{1}$. Since all rational numbers $x \leq 0$ are also in $\mathbf{1}$, this shows that every member of $\alpha \bullet \alpha^{-1}$ is in $\mathbf{1}$.

To prove the converse assertion, let z be in $\mathbf{1}$. If $z \leq 0$, then clearly z is in $\alpha \bullet \alpha^{-1}$. Suppose $0 < z < 1$. According to the lemma, there are positive rational numbers x in α, and y not in α, such that $y/x = 1/z$; and we can assume that y is not the smallest element of $\mathbf{Q} - \alpha$. But this means that $z = x \cdot (1/y)$, where x is in α, and $1/y$ is in α^{-1}. Consequently, z is in $\alpha \bullet \alpha^{-1}$. ▮

We are almost done! Only the proof of the distributive law remains. Once again we must consider many cases, but do not despair. The case when all numbers are positive contains an interesting point, and the other cases can all be taken care of very neatly.

THEOREM If α, β, and γ are real numbers, then $\alpha \bullet (\beta + \gamma) = \alpha \bullet \beta + \alpha \bullet \gamma$.

PROOF Assume first that $\alpha, \beta, \gamma > \mathbf{0}$. Then both numbers in the equation contain all rational numbers ≤ 0. A positive rational number in $\alpha \bullet (\beta + \gamma)$ is of the form $x \cdot (y + z)$ for positive x in α, y in β, and z in γ. Since $x \cdot (y + z) = x \cdot y + x \cdot z$, where $x \cdot y$ is a positive element of $\alpha \bullet \beta$, and $x \cdot z$ is a positive element of $\alpha \bullet \gamma$, this number is also in $\alpha \bullet \beta + \alpha \bullet \gamma$. Thus, every element of $\alpha \bullet (\beta + \gamma)$ is also in $\alpha \bullet \beta + \alpha \bullet \gamma$.

On the other hand, a positive rational number in $\alpha \bullet \beta + \alpha \bullet \gamma$ is of the form $x_1 \cdot y + x_2 \cdot z$ for positive x_1, x_2 in α, y in β, and z in γ. If $x_1 \leq x_2$, then $(x_1/x_2) \cdot y \leq y$, so $(x_1/x_2) \cdot y$ is in β. Thus

$$x_1 \cdot y + x_2 \cdot z = x_2[(x_1/x_2)y + z]$$

is in $\alpha \bullet (\beta + \gamma)$. Of course, the same trick works if $x_2 \leq x_1$.

To complete the proof it is necessary to consider the cases when α, β, and γ are not all $> \mathbf{0}$. If any one of the three equals $\mathbf{0}$, the proof is easy and the cases involving $\alpha < 0$ can be derived immediately once all the possibilities for β and γ have been accounted for. Thus we assume $\alpha > 0$ and consider three cases: $\beta, \gamma < 0$, and $\beta < 0, \gamma > 0$, and $\beta > 0, \gamma < 0$. The first follows immediately from the case already proved, and the third follows from the second by interchanging β and γ. Therefore we concentrate on the case $\beta < 0$, $\gamma > 0$. There are then two possibilities:

(1) $\beta + \gamma \geq 0$. Then

$$\alpha \cdot \gamma = \alpha \cdot ([\beta + \gamma] + |\beta|) = \alpha \cdot (\beta + \gamma) + \alpha \cdot |\beta|,$$

so

$$\begin{aligned} \alpha \cdot (\beta + \gamma) &= -(\alpha \cdot |\beta|) + \alpha \cdot \gamma \\ &= \alpha \cdot \beta + \alpha \cdot \gamma. \end{aligned}$$

(2) $\beta + \gamma \leq 0$. Then

$$\alpha \cdot |\beta| = \alpha \cdot (|\beta + \gamma| + \gamma) = \alpha \cdot |\beta + \gamma| + \alpha \cdot \gamma,$$

so

$$\alpha \cdot (\beta + \gamma) = -(\alpha \cdot |\beta + \gamma|) = -(\alpha \cdot |\beta|) + \alpha \cdot \gamma = \alpha \cdot \beta + \alpha \cdot \gamma. \ \blacksquare$$

This proof completes the work of the chapter. Although long and frequently tedious, this chapter contains results sufficiently important to be read in detail at least once (and preferably not more than once!). For the first time we know that we have not been operating in a vacuum—there is indeed a complete ordered field, the theorems of this book are not based on assumptions which can never be realized. One interesting and horrid possibility remains: there may be several complete ordered fields. If this is true, then the theorems of calculus are unexpectedly rich in content, but the properties P1–P13 are disappointingly incomplete. The last chapter disposes of this possibility; properties P1–P13 completely characterize the real numbers—anything that can be proved about real numbers can be proved on the basis of these properties alone.

PROBLEMS

There are only two problems in this set, but each asks for an entirely different construction of the real numbers! The detailed examination of another construction is recommended only for masochists, but the main idea behind these other constructions is worth knowing. The real numbers constructed in this chapter might be called "the algebraist's real numbers," since they were purposely defined so as to guarantee the least upper bound property, which involves the ordering <, an algebraic notion. The real number system constructed in the next problem might be called "the analyst's real numbers," since they are devised so that Cauchy sequences will always converge.

1. Since every real number ought to be the limit of some Cauchy sequence of rational numbers, we might try to *define* a real number to be a Cauchy sequence of rational numbers. Since two Cauchy sequences might converge to the same real number, however, this proposal requires some modifications.

(a) Define two Cauchy sequences of rational numbers $\{a_n\}$ and $\{b_n\}$ to be *equivalent* (denoted by $\{a_n\} \sim \{b_n\}$) if $\lim_{n\to\infty}(a_n - b_n) = 0$. Prove that $\{a_n\} \sim \{a_n\}$, that $\{b_n\} \sim \{a_n\}$ if $\{a_n\} \sim \{b_n\}$, and that $\{a_n\} \sim \{c_n\}$ if $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$.

(b) Suppose that α is the set of all sequences equivalent to $\{a_n\}$, and β is the set of all sequences equivalent to $\{b_n\}$. Prove that either $\alpha \cap \beta = \emptyset$ or $\alpha = \beta$. (If $\alpha \cap \beta \neq \emptyset$, then there is some $\{c_n\}$ in both α and β. Show that in this case α and β both consist precisely of those sequences equivalent to $\{c_n\}$.)

Part (b) shows that the collection of all Cauchy sequences can be split up into disjoint sets, each set consisting of all sequences equivalent to some fixed sequence. We define a real number to be such a collection, and denote the set of all real numbers by **R**.

(c) If α and β are real numbers, let $\{a_n\}$ be a sequence in α, and $\{b_n\}$ a sequence in β. Define $\alpha + \beta$ to be the collection of all sequences equivalent to the sequence $\{a_n + b_n\}$. Show that $\{a_n + b_n\}$ is a Cauchy sequence and also show that this definition does not depend on the particular sequences $\{a_n\}$ and $\{b_n\}$ chosen for α and β. Check also that the analogous definition of multiplication is well defined.

(d) Show that **R** is a field with these operations; existence of a multiplicative inverse is the only interesting point to check.

(e) Define the positive real numbers P so that **R** will be an ordered field.

(f) Prove that every Cauchy sequence of real numbers converges. Remember that if $\{\alpha_n\}$ is a sequence of real numbers, then each α_n is itself a collection of Cauchy sequences of rational numbers.

2. This problem outlines a construction of "the high-school student's real numbers." We define a real number to be a pair $(a, \{b_n\})$, where a is an integer and $\{b_n\}$ is a sequence of natural numbers from 0 to 9, with the proviso that the sequence is not eventually 9; intuitively, this pair represents $a + \sum_{n=1}^{\infty} b_n 10^{-n}$. With this definition, a real number is a very concrete object, but the difficulties involved in defining addition and multiplication are formidable (how do you add infinite decimals without worrying about carrying digits infinitely far out?). A reasonable approach is outlined below; the trick is to use least upper bounds right from the start.

(a) Define $(a, \{b_n\}) \prec (c, \{d_n\})$ if $a < c$, or if $a = c$ and for some n we have $b_n < d_n$ but $b_j = d_j$ for $1 \le j < n$. Using this definition, prove the least upper bound property.

(b) Given $\alpha = (a, \{b_n\})$, define $\alpha_k = a + \sum_{n=1}^{k} b_n 10^{-n}$; intuitively, α_k is the rational number obtained by changing all decimal places after the kth to 0. Conversely, given a rational number r of the form $a + \sum_{n=1}^{k} b_n 10^{-n}$, let r' denote the real number $(a, \{b_n{}'\})$, where $b_n{}' = b_n$ for $1 \le n \le k$ and $b_n{}' = 0$ for $n > k$. Now for $\alpha = (a, \{b_n\})$ and $\beta = (c, \{d_n\})$ define

$$\alpha \mathbin{\pmb{+}} \beta = \sup\{(\alpha_k + \beta_k)' : k \text{ a natural number}\}$$

(the least upper bound exists by part (a)). If multiplication is defined similarly, then the verification of all conditions for a field is a straightforward task, not highly recommended. Once more, however, existence of multiplicative inverses will be the hardest.

CHAPTER 30 UNIQUENESS OF THE REAL NUMBERS

We shall now revert to the usual notation for real numbers, reserving boldface symbols for other fields which may turn up. Moreover, we will regard integers and rational numbers as special kinds of real numbers, and forget about the specific way in which real numbers were defined. In this chapter we are interested in only one question: are there any complete ordered fields other than **R**? The answer to this question, if taken literally, is "yes." For example, the field F_3 introduced in Chapter 28 is a complete ordered field, and it is certainly not **R**. This field is a "silly" example because the pair (a, a) can be regarded as just another name for the real number a; the operations

$$\begin{aligned}(a, a) \mathbin{\mathbf{+}} (b, b) &= (a + b, a + b),\\ (a, a) \mathbin{\bullet} (b, b) &= (a \cdot b, a \cdot b),\end{aligned}$$

are consistent with this renaming. This sort of example shows that any intelligent consideration of the question requires some mathematical means of discussing such renaming procedures.

If the elements of a field F are going to be used to rename elements of **R**, then for each a in **R** there should correspond a "name" $f(a)$ in F. The notation $f(a)$ suggests that renaming can be formulated in terms of functions. In order to do this we will need a concept of function much more general than any which has occurred until now; in fact, we will require the most general notion of "function" used in mathematics. A function, in this general sense, is simply a rule which assigns to some things, other things. To be formal, a **function** is a collection of ordered pairs (of objects of any sort) which does not contain two distinct pairs with the same first element. The **domain** of a function f is the set A of all objects a such that (a, b) is in f for some b; this (unique) b is denoted by $\boldsymbol{f(a)}$. If $f(a)$ is in the set B for all a in A, then f is called a function **from** A **to** B. For example,

> if $f(x) = \sin x$ for all x in **R** (and f is defined only for x in **R**), then f is a function from **R** to **R**; it is also a function from **R** to $[-1, 1]$;
>
> if $f(z) = \sin z$ for all z in **C**, then f is a function from **C** to **C**;
>
> if $f(z) = e^z$ for all z in **C**, then f is a function from **C** to **C**; it is also a function from **C** to $\{z \text{ in } \mathbf{C} : z \neq 0\}$;
>
> θ is a function from $\{z \text{ in } \mathbf{C} : z \neq 0\}$ to $\{x \text{ in } \mathbf{R} : 0 \leq x < 2\pi\}$;
>
> if f is the collection of all pairs $(a, (a, a))$ for a in **R**, then f is a function from **R** to F_3.

Suppose that F_1 and F_2 are two fields; we will denote the operations in F_1 by $\oplus$, $\odot$, etc., and the operations in F_2 by $+$, $\cdot$, etc. If F_2 is going to be considered as a collection of new names for elements of F_1, then there should be a function from F_1 to F_2 with the following properties:

(1) The function f should be one-one, that is, if $x \neq y$, then we should have $f(x) \neq f(y)$; this means that no two elements of F_1 have the same name.
(2) The function f should be "onto," that is, for every element z in F_2 there should be some x in F_1 such that $z = f(x)$; this means that every element of F_2 is used to name some element of F_1.
(3) For all x and y in F_1 we should have

$$f(x \oplus y) = f(x) + f(y),$$
$$f(x \odot y) = f(x) \cdot f(y);$$

this means that the renaming procedure is consistent with the operations of the field.

If we are also considering F_1 and F_2 as ordered fields, we add one more requirement:

(4) If $x \otimes y$, then $f(x) < f(y)$.

A function with these properties is called an *isomorphism* from F_1 to F_2. This definition is so important that we restate it formally.

DEFINITION

If F_1 and F_2 are two fields, an **isomorphism** from F_1 to F_2 is a function f from F_1 to F_2 with the following properties:

(1) If $x \neq y$, then $f(x) \neq f(y)$.
(2) If z is in F_2, then $z = f(x)$ for some x in F_1.
(3) If x and y are in F_1, then

$$f(x \oplus y) = f(x) + f(y),$$
$$f(x \odot y) = f(x) \cdot f(y).$$

If F_1 and F_2 are ordered fields we also require:

(4) If $x \otimes y$, then $f(x) < f(y)$.

The fields F_1 and F_2 are called **isomorphic** if there is an isomorphism between them. Isomorphic fields may be regarded as essentially the same—any important property of one will automatically hold for the other. Therefore, we can, and should, reformulate the question asked at the beginning of the chapter; if F is a complete ordered field it is silly to expect F to equal $\mathbf{R}$—rather, we would like to know if F is isomorphic to $\mathbf{R}$. In the following theorem, F will be a field, with operations $+$ and $\cdot$, and "positive elements" $\mathbf{P}$; we write $a < b$ to mean that $b - a$ is in $\mathbf{P}$, and so forth.

THEOREM If F is a complete ordered field, then F is isomorphic to **R**.

PROOF Since two fields are defined to be isomorphic if there is an isomorphism between them, we must actually construct a function f from **R** to F which is an isomorphism. We begin by defining f on the integers as follows:

$$f(0) = \mathbf{0},$$
$$f(n) = \underbrace{\mathbf{1} + \ldots + \mathbf{1}}_{n \text{ times}} \qquad \text{for } n > 0,$$
$$f(n) = -(\underbrace{\mathbf{1} + \ldots + \mathbf{1}}_{|n| \text{ times}}) \qquad \text{for } n < 0.$$

It is easy to check that

$$f(m+n) = f(m) + f(n),$$
$$f(m \cdot n) = f(m) \cdot f(n),$$

for all integers m and n, and it is convenient to denote $f(n)$ by $\boldsymbol{n}$. We then define f on the rational numbers by

$$f(m/n) = \boldsymbol{m/n} = \boldsymbol{m} \cdot \boldsymbol{n}^{-1}$$

(notice that the n-fold sum $\mathbf{1} + \cdots + \mathbf{1} \neq \mathbf{0}$ if $n > 0$, since F is an ordered field). This definition makes sense because if $m/n = k/l$, then $ml = nk$, so $\boldsymbol{m} \cdot \boldsymbol{l} = \boldsymbol{k} \cdot \boldsymbol{n}$, so $\boldsymbol{m} \cdot \boldsymbol{n}^{-1} = \boldsymbol{k} \cdot \boldsymbol{l}^{-1}$. It is easy to check that

$$f(r_1 + r_2) = f(r_1) + f(r_2),$$
$$f(r_1 \cdot r_2) = f(r_1) \cdot f(r_2),$$

for all rational numbers r_1 and r_2, and that $f(r_1) < f(r_2)$ if $r_1 < r_2$.

The definition of $f(x)$ for arbitrary x is based on the now familiar idea that any real number is determined by the rational numbers less than it. For any x in **R**, let A_x be the subset of F consisting of all $f(r)$, for all rational numbers $r < x$. The set A_x is certainly not empty, and it is also bounded above, for if r_0 is a rational number with $r_0 > x$, then $f(r_0) > f(r)$ for all $f(r)$ in A_x. Since F is a complete ordered field, the set A_x has a least upper bound; we define $f(x)$ as $\sup A_x$.

We now have $f(x)$ defined in two different ways, first for rational x, and then for any x. Before proceeding further, it is necessary to show that these two definitions agree for rational x. In other words, if x is a rational number, we want to show that

$$\sup A_x = f(x),$$

where $f(x)$ here denotes $\boldsymbol{m/n}$, for $x = m/n$. This is not automatic, but depends on the completeness of F; a slight digression is thus required.

Since F is complete, the elements

$$\underbrace{\mathbf{1} + \ldots + \mathbf{1}}_{n \text{ times}} \qquad \text{for natural numbers } n$$

form a set which is not bounded above; the proof is exactly the same as the proof for $\mathbf{R}$ (Theorem 8-2). The consequences of this fact for $\mathbf{R}$ have exact analogues in F: in particular, if a and b are elements of F with $a \prec b$, then there is a rational number r such that

$$a \prec f(r) \prec b.$$

Having made this observation, we return to the proof that the two definitions of $f(x)$ agree for rational x. If y is a rational number with $y < x$, then we have already seen that $f(y) \prec f(x)$. Thus every element of A_x is $\prec f(x)$. Consequently,

$$\sup A_x \preceq f(x).$$

On the other hand, suppose that we had

$$\sup A_x \prec f(x).$$

Then there would be a rational number r such that

$$\sup A_x \prec f(r) \prec f(x).$$

But the condition $f(r) \prec f(x)$ means that $r < x$, which means that $f(r)$ is in the set A_x; this clearly contradicts the condition $\sup A_x \prec f(r)$. This shows that the original assumption is false, so

$$\sup A_x = f(x).$$

We thus have a certain well-defined function f from $\mathbf{R}$ to F. In order to show that f is an isomorphism we must verify conditions (1)–(4) of the definition. We will begin with (4).

If x and y are real numbers with $x < y$, then clearly A_x is contained in A_y. Thus

$$f(x) = \sup A_x \preceq \sup A_y = f(y).$$

To rule out the possibility of equality, notice that there are rational numbers r and s with

$$x < r < s < y.$$

We know that $f(r) \prec f(s)$. It follows that

$$f(x) \preceq f(r) \prec f(s) \preceq f(y).$$

This proves (4).

Condition (1) follows immediately from (4): If $x \neq y$, then either $x < y$ or $y < x$; in the first case $f(x) \prec f(y)$, and in the second case $f(y) \prec f(x)$; in either case $f(x) \neq f(y)$.

To prove (2), let a be an element of F, and let B be the set of all rational numbers r with $f(r) \prec a$. The set B is not empty, and it is also bounded above, because there is a rational number s with $f(s) \succ a$, so that $f(s) \succ f(r)$ for r in B, which implies that $s > r$. Let x be the least upper bound of B; we claim that $f(x) = a$. In order to prove this it suffices to eliminate the alternatives

$$f(x) \prec a,$$
$$a \prec f(x).$$

In the first case there would be a rational number r with

$$f(x) < f(r) < a.$$

But this means that $x < r$ and that r is in B, which contradicts the fact that $x = \sup B$. In the second case there would be a rational number r with

$$a < f(r) < f(x).$$

This implies that $r < x$. Since $x = \sup B$, this means that $r < s$ for some s in B. Hence

$$f(r) < f(s) < a,$$

again a contradiction. Thus $f(x) = a$, proving (2).

To check (3), let x and y be real numbers and suppose that $f(x + y) \neq f(x) + f(y)$. Then either

$$f(x + y) < f(x) + f(y) \qquad \text{or} \qquad f(x) + f(y) < f(x + y).$$

In the first case there would be a rational number r such that

$$f(x + y) < f(r) < f(x) + f(y).$$

But this would mean that

$$x + y < r.$$

Therefore r could be written as the sum of two rational numbers

$$r = r_1 + r_2, \qquad \text{where } x < r_1 \text{ and } y < r_2.$$

Then, using the facts checked about f for *rational* numbers, it would follow that

$$f(r) = f(r_1 + r_2) = f(r_1) + f(r_2) > f(x) + f(y),$$

a contradiction. The other case is handled similarly.

Finally, if x and y are positive real numbers, the same sort of reasoning shows that

$$f(x \cdot y) = f(x) \cdot f(y);$$

the general case is then a simple consequence. ∎

This theorem brings to an end our investigation of the real numbers, and resolves any doubts about them: There *is* a complete ordered field and, up to isomorphism, only one complete ordered field. It is an important part of a mathematical education to follow a construction of the real numbers in detail, but it is not necessary to refer ever again to this particular construction. It is utterly irrelevant that a real number happens to be a collection of rational numbers, and such a fact should never enter the proof of any important theorem about the real numbers. Reasonable proofs should use only the fact that the real numbers are a complete ordered field, because this property of the real numbers characterizes them up to isomorphism, and any significant mathematical property of the real numbers will be true for all isomorphic fields. To be candid I should admit that this last assertion is just a prejudice of the author, but it is one shared by almost all other mathematicians.

PROBLEMS

1. Let f be an isomorphism from F_1 to F_2.

(a) Show that $f(\mathbf{0}) = \mathbf{0}$ and $f(\mathbf{1}) = \mathbf{1}$. (Here $\mathbf{0}$ and $\mathbf{1}$ on the left denote elements in F_1, while $\mathbf{0}$ and $\mathbf{1}$ on the right denote elements of F_2.)

(b) Show that $f(-a) = -f(a)$ and $f(a^{-1}) = f(a)^{-1}$, for $a \neq \mathbf{0}$.

2. Here is an opportunity to convince yourself that any significant property of a field is shared by any field isomorphic to it. The point of this problem is to write out very formal proofs until you are certain that all statements of this sort are obvious. F_1 and F_2 will be two fields which are isomorphic; for simplicity we will denote the operations in both by $\mathbf{+}$ and $\boldsymbol{\cdot}$. Show that:

(a) If the equation $x^2 \mathbf{+} \mathbf{1} = \mathbf{0}$ has a solution in F_1, then it has a solution in F_2.

(b) If every polynomial equation $x^n \mathbf{+} a_{n-1} \boldsymbol{\cdot} x^{n-1} \mathbf{+} \cdots \mathbf{+} a_0 = \mathbf{0}$ with $a_0, \dots, a_{n-1}$ in F_1, has a root in F_1, then every polynomial equation $x^n \mathbf{+} b_{n-1} \boldsymbol{\cdot} x^{n-1} \mathbf{+} \cdots \mathbf{+} b_0 = \mathbf{0}$ with $b_0, \dots, b_{n-1}$ in F_2 has a root in F_2.

(c) If $1 \mathbf{+} \cdots \mathbf{+} 1$ (summed m times) $= \mathbf{0}$ in F_1, then the same is true in F_2.

(d) If F_1 and F_2 are ordered fields (and the isomorphism f satisfies $f(x) \mathbf{<} f(y)$ for $x \mathbf{<} y$) and F_1 is complete, then F_2 is complete.

3. Let f be an isomorphism from F_1 to F_2 and g an isomorphism from F_2 to F_3. Define the function $g \circ f$ from F_1 to F_3 by $(g \circ f)(x) = g(f(x))$. Show that $g \circ f$ is an isomorphism.

4. Suppose that F is a complete ordered field, so that there is an isomorphism f from $\mathbf{R}$ to F. Show that there is actually only *one* isomorphism from $\mathbf{R}$ to F. Hint: In case $F = \mathbf{R}$, this is Problem 3-17. Now if f and g are two isomorphisms from $\mathbf{R}$ to F consider $g^{-1} \circ f$.

5. Find an isomorphism from $\mathbf{C}$ to $\mathbf{C}$ other than the identity function.

SUGGESTED READING

A man ought to read
just as inclination leads him;
for what he reads as a task
will do him little good.

SAMUEL JOHNSON

One purpose of this bibliography is to guide the reader to other sources, but the most important function it can serve is to indicate the variety of mathematical reading available. Consequently, there is an attempt to achieve diversity, but no pretense of being complete. The present plethora of mathematics books would make such an undertaking almost hopeless in any case, and since I have tried to encourage independent reading, the more standard a text, the less likely it is to appear here. In some cases, this philosophy may seem to have been carried to extremes, as some entries in the list cannot be read by a student just finishing a first course of calculus until several years have elapsed. Nevertheless, there are many selections which can be read now, and I can't believe that it hurts to have some idea of what lies ahead.

For most references, only the title and author have been given, since so many of these books have gone through numerous editions and printings, often having gone out of print at some point only to be resurrected later on by a different publisher (often as an inexpensive paperback by the redoubtable Dover Publications or by the Mathematical Association of America). More exact information really isn't necessary, since it is now so easy to search for books on-line at Amazon.com and other sites.

† is used to indicate books whose availability, either new or used, is problematical. Author and title searches may turn up other intriguing books by the same author, or other books with similar titles. In addition, many of these books will still be found in well-stocked academic libraries, perhaps the best place of all to search; despite the convenience of the internet, nothing matches the experience of an actual (as opposed to a virtual) library, with books stacked according to subject, awaiting serendipitous discovery.

One of the most elementary unproved theorems mentioned in this book is the "Fundamental Theorem of Arithmetic", that every natural number can be written as a product of primes in only one way. This follows from the basic fact alluded to on page 444, a proof of which will be found near the beginning of almost any book on elementary number theory. Few books have won so enthusiastic an audience as

[1] *An Introduction to the Theory of Numbers*, by G. H. Hardy and E. M. Wright.

Two other recommended books are

† [2] *A Selection of Problems in the Theory of Numbers*, by W. Sierpinski.
[3] *Three Pearls of Number Theory*, by A. Khinchin.

The Fundamental Theorem also applies in more general algebraic settings, see references [33] and [34].

The subject of irrational numbers straddles the fields of number theory and analysis. An excellent introduction will be found in

[4] *Irrational Numbers*, by I. M. Niven.

Together with many historical notes, there are references to some fairly elementary articles in journals. There is also a proof that π is transcendental (see also [59]) and, finally, a proof of the "Gelfond-Schneider theorem": If a and b are algebraic, with $a \neq 0$ or 1, and b is irrational, then a^b is transcendental.

All the books listed so far begin with natural numbers, but whenever necessary take for granted the irrational numbers, not to mention the integers and rational numbers. Several books present a construction of the rational numbers from the natural numbers, but one of the most lucid treatments is still to be found in

[5] *Foundations of Analysis*, by E. Landau.

While many mathematicians are content to accept the natural numbers as a natural starting point, numbers can be defined in terms of sets, the most basic starting point of all. A charming exposition of set theory can be found in a sophisticated little book called

[6] *Naive Set Theory*, by P. R. Halmos.

Another very good introduction is

[7] *Theory of Sets*, by E. Kamke.

Perhaps it is necessary to assure some victims of the "new math" that set theory does have some mathematical content (in fact, some very deep theorems). Using these deep results, Kamke proves that there is a discontinuous function f such that $f(x+y) = f(x) + f(y)$ for all x and y.

Inequalities, which were treated as an elementary topic in Chapters 1 and 2, actually form a specialized field. A good elementary introduction is provided by

[8] *Analytic Inequalities*, by N. Kazarinoff.

Twelve different proofs that the geometric mean is less than or equal to the arithmetic mean, each based on a different principle, can be found in the beginning of the more advanced book

[9] *An Introduction to Inequalities*, by E. Beckenbach and R. Bellman.

The classic work on inequalities is

[10] *Inequalities*, by G. H. Hardy, J. E. Littlewood, and G. Polya.

Each of the authors of this triple collaboration has provided his own contribution to the sparse literature about the nature of mathematical thinking, written from a mathematician's point of view. My favorite is

[11] *A Mathematician's Apology*, by G. H. Hardy.

Littlewood's anecdotal selections are entitled

† [12] *A Mathematician's Miscellany*, by J. E. Littlewood.

Polya's contribution is pedagogy at the highest level:

[13] *Mathematics and Plausible Reasoning* (Vol. I: *Induction and Analogy in Mathematics;* † Vol. II: *Patterns of Plausible Inference*), by G. Polya.

Geometry is the other main field which can be considered as background for calculus. Though Euclid's *Elements* is still a masterful mathematical work, greater perspective is supplied by some more modern texts, which examine foundational questions, non-Euclidean geometry, the role of the "Archimedean axiom" in geometry, and further results from "classical geometry". Of the following three books, the first, listed in previous editions of this book, has probably been supplanted by the later ones, which cover some more advanced material, and perhaps require a little more sophistication on the part of the reader.

† [14] *Elementary Geometry from an Advanced Standpoint*, by E. Moise.
[15] *Euclidean and Non-Euclidean Geometries*, by M. J. Greenberg.
[16] *Geometry: Euclid and Beyond*, by R. Hartshorne.

In addition, all sorts of fascinating geometric things can be found in

[17] *Introduction to Geometry*, by H. S. Coxeter.

Almost all treatments of geometry at least mention convexity, which forms another specialized topic. I cannot imagine a better introduction to convexity, or a better mathematical experience in general, than reading and working through

† [18] *Convex Figures*, by I. M. Yaglom and W. G. Boltyanskii.

This book contains a carefully arranged sequence of definitions and *statements* of theorems, whose proofs are to be supplied by the reader (worked-out proofs are supplied in the back of the book). Its current unavailability is perhaps a testament to the lack of interest in working through exercises, which might also apply to another geometry book modeled on the same principle:

† [19] *Combinatorial Geometry in the Plane*, by H. Hadwiger and H. Debrunner.

Along with these two out-of-the-ordinary books, I might mention an extremely valuable little book, also of a specialized sort,

[20] *Counterexamples in Analysis*, by B. Gelbaum and J. Olmsted.

Many of the example in this book come from more advanced topics in analysis, but quite a few can be appreciated by someone who knows calculus.

Of the infinitude of calculus books, two are considered classics:

[21] *A Course of Pure Mathematics*, by G. H. Hardy.
[22] *Differential and Integral Calculus* (two volumes), by R. Courant.

Courant is especially strong on applications to physics. There is also a more modern update

[23] *Introduction to Calculus and Analysis*, by R. Courant and F. John.

Speaking of applications to physics, an elegant exposition of the material in Chapter 17, together with much further discussion, can be found in the article

[24] *On the geometry of the Kepler problem*, by John Milnor; in *The American Mathematical Monthly*, Volume 90 (1983), pp. 353–365.

(In this paper the curve c' of Chapter 17 is denoted by $\mathbf{v}$, and the derivative of the important composition $\mathbf{v} \circ \theta^{-1}$ (page 334) is introduced quite off-handedly as $d\mathbf{v}/d\theta$.) A "straight-forward" derivation of Kepler's laws, together with numerous references, can be found in another article in this same journal,

[25] *The mathematical relationship between Kepler's laws and Newton's laws*, by Andrew T. Hyman; in *The American Mathematical Monthly*, Volume 100 (1993), pp. 932–936.

The later parts of Volume I of Courant contain material usually found in advanced calculus, including differential equations and Fourier series. An introduction to Fourier series (requiring a little advanced calculus) will also be found in

[26] *An Introduction to Fourier Series and Integrals*, by R. Seeley.

The second volume of Courant (advanced calculus in earnest) contains additional material on differential equations, as well as an introduction to the calculus of variations. A widely admired book on differential equations is

[27] *Lectures on Ordinary Differential Equations*, by W. Hurewicz.

A good example of new approaches and new topics is provided by

[28] *Differential Equations, Dynamical Systems, and An Introduction to Chaos*, by M. Hirsch, S. Smale, and R. L. Devaney.

I will bypass the more or less standard advanced calculus books (which can easily be found by the reader) since nowadays the presentation of advanced calculus for mathematics students is based upon linear algebra. One of the first treatments of advanced calculus using linear algebra is the very nice book

† [29] *Calculus of Vector Functions*, by R. H. Crowell and R. E. Williamson.

More recent books to be recommended are

[30] *Advanced Calculus of Several Variables*, by C. H. Edwards, Jr.
[31] *Multivariable Mathematics*, by T. Shifrin.

And of course I am still partial to an older text

[32] *Calculus on Manifolds*, by M. Spivak.

There are three other topics which are somewhat out of place in this bibliography because they are gradually becoming established as part of a standard undergraduate curriculum. The purposeful study of fields and related systems is the domain of "algebra." Two excellent texts are

[33] *Algebra*, by Michael Artin.
[34] *Abstract Algebra*, by D. Dummit and R. Foote.

For "complex analysis", the promised land of Chapter 27, the classical text is

[35] *Complex Analysis*, by L. Ahlfors.

Rather revolutionary when it was first published, it might now be considered somewhat old-fashioned, and you might prefer the second in a series of books (3 and counting) that have appeared more recently:

[36] *Fourier Analysis: An Introduction*, by E. Stein and R. Shakarchi.
[37] *Complex Analysis*, by E. Stein and R. Shakarchi.
[38] *Real Analysis*, by E. Stein and R. Shakarchi.

And, since the topic of "real analysis" [high-octane Calculus] has been broached, two classics should be mentioned. The first, affectionately known as "baby Rudin", was the source of several problems that appear in this book.

[39] *Principles of Mathematical Analysis*, by W. Rudin.
[40] *Functional Analysis*, by W. Rudin.

The subject of "topology" has not been mentioned before, but it has really been in the background of many discussions, since it is the natural generalization of the ideas about limits and continuity which play such a prominent role in Part II of this book. The standard text is now

[41] *Topology*, by J. R. Munkres.

For the related field of "differential topology", see

[42] *Differential Topology*, by V. Guillemin and A. Pollack.

The next few topics, ranging from elementary to very difficult, are included in this bibliography because they have been alluded to in the text. The gamma function has an elegant little book devoted entirely to its properties, most of them proved by using the theorem of Bohr and Mollerup which was mentioned in Problem 19-40:

† [43] *The Gamma Function*, by E. Artin

The gamma function is only one of several important improper integrals in mathematics. In particular, the calculation of $\int_0^\infty e^{-x^2}\,dx$ (see Problem 19-42) is important in probability theory, where the "normal distribution function"

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$$

plays a fundamental role. A classic book on probability theory is

[44] *An Introduction to Probability Theory and Its Applications*, by W. Feller.

The impossibility of integrating certain functions in elementary terms (among them $f(x) = e^{-x^2}$) is a fairly esoteric topic. A discussion of the possibilities of integrating in elementary terms, with an outline of the impossibility proofs, and references to the original papers of Liouville, will be found in

[45] *The Integration of Functions of a Single Variable*, by G. H. Hardy.

The basically algebraic ideas behind the arguments were made much clearer over a hundred years after Liouville's work, in the paper

[46] *On Liousville's Theorem on functions with elementary integrals*, by M. Rosenlicht; in *Pacific Journal of Mathematics*, Volume 24, No. 1 (1968), pp. 153–161. (Also available on-line: go to `projecteuclid.org` and search for `Rosenlicht`.)

For a good overview of the subject, and some more recent developments, see

[47] *Integration in finite terms: the Liousville theory*, by T. Kasper; in *Mathematics Magazine*, Volume 53, No. 4 (1980), pp. 195–201.

Reference [46] makes use of the notions of "differential algebra", a field in which a related but seemingly more difficult problem had been solved earlier: There are simple differential equations ($y'' + xy = 0$ is a specific example) whose solutions cannot be expressed even in terms of indefinite integrals of elementary functions. This fact is proved on page 43 of the (60-page) book:

† [48] *An Introduction to Differential Algebra*, by I. Kaplansky

A few words should also be said in defense of the process of integrating in elementary terms, which many mathematicians look upon as an art (unlike differentiation, which is merely a skill). You are probably already aware that the process of integration can be expedited by tables of indefinite integrals. There are several books containing extensive tables of integrals (and also tables of series and products), but for most integrations it suffices to consult one of the fairly extensive tables of indefinite integrals that are available on-line, for example, at `sosmath.com`, and at `wikipedia.org`, with its ever-expanding source of generally definitive entries for mathematics and physics.

The remaining references are of a somewhat different sort. They fall into three categories, of which the first is historical.

For the history of calculus itself, an excellent comprehensive source, filled with detailed explicit examples, rather than generalized descriptions, is

[49] *The Historical Development of Calculus*, by C. H. Edwards, Jr.

Some historical remarks, and an attempt to incorporate them into the teaching of calculus, will be found in

[50] *The Calculus: A Genetic Approach*, by O. Toeplitz.

An admirable textbook on the history of mathematics in general is

[51] *An Introduction to the History of Mathematics*, by H. Eves.

As might be inferred from the quotation on page 39, the basic idea for constructing the real numbers is derived from Dedekind, whose contributions can be found in

[52] *Essays on the Theory of Numbers*, by R. Dedekind.

The most important notions of set theory, especially the proper treatment of infinite numbers, were first introduced by Cantor, whose work is reproduced in

[53] *Contributions to the Founding of the Theory of Transfinite Numbers*, by G. Cantor.

The letter of H. A. Schwarz referred to in Problem 11-69 will be found in

† [54] *Ways of Thought of Great Mathematicians*, by H. Meschkowski.

Finally, a great deal of interesting historical material may also be found on-line at the site `www-groups.dcs.st-and.ac.uk/~history/`

The second category in this final group of books might be described as "popularizations." There are a surprisingly large number of first-rate ones by real mathematicians:

[55] *What is Mathematics?*, by R. Courant and H. Robbins.
[56] *Geometry and the Imagination*, by D. Hilbert and S. Cohn-Vossen.
[57] *The Enjoyment of Mathematics*, by H. Rademacher and O. Toeplitz.
† [58] *Famous Problems of Mathematics*, by H. Tietze; Graylock Press, 1965.

One of the most renowned "popularizations" is especially concerned with the teaching of mathematics:

[59] *Elementary Mathematics from an Advanced Standpoint*, by F. Klein (vol. 1: *Arithmetic, Algebra, Analysis*; vol. 2: *Geometry*); Dover, 1948.

Volume 1 contains a proof of the transcendence of π which, although not so elementary as the one in [4], is a direct analogue of the proof that e is transcendental, replacing integrals with complex line integrals. It can be read as soon as the basic facts about complex analysis are known.

The third category is the very opposite extreme— original papers. The difficulties encountered here are formidable, and I have only had the courage to list one such paper, the source of the quotation for Part IV. It is not even in English, although you do have a choice of foreign languages. The article in the original French is in

[60] *Oeuvres Complètes d'Abel.*

It first appeared in a German translation in the *Journal für die reine und angewandte Mathematik*, Volume 1, 1826. To compound the difficulties, these references will usually be available only in university libraries. Yet the study of this paper will probably be as valuable as any other reading mentioned here. The reason is suggested by a remark of Abel himself, who attributed his profound knowledge of mathematics to the fact that he read the masters, rather than the pupils.

ANSWERS TO SELECTED PROBLEMS

CHAPTER 1

1. (i) $1 = a^{-1}a = a^{-1}(ax) = (a^{-1}a)x = 1 \cdot x = x$.
(iii) If $x^2 = y^2$, then $0 = x^2 - y^2 = (x - y)(x + y)$, so either $x - y = 0$ or $x + y = 0$, that is, either $x = -y$ or $x = y$.
(vi) Replace y by $-y$ in (iv).
2. One step requires dividing by $x - y = 0$.
3. (i) $a/b = ab^{-1} = (ac)(b^{-1}c^{-1}) = (ac)(bc)^{-1}$ (by (iii)) $= ac/bc$.
(ii) $(ad + bc)/(bd) = (ad + bc)(bd)^{-1} = (ad + bc)(b^{-1}d^{-1})$ (by (iii)) $= ab^{-1} + cd^{-1} = a/b + c/d$.
(iii) $ab(a^{-1}b^{-1}) = (a \cdot a^{-1})(b \cdot b^{-1}) = 1$, so $a^{-1} \cdot b^{-1} = (ab)^{-1}$.
(v) $(a/b)/(c/d) = (a/b)(c/d)^{-1} = (a \cdot b^{-1})(c \cdot d^{-1})^{-1} = (a \cdot b^{-1})(c^{-1} \cdot d) = ad(b^{-1} \cdot c^{-1}) = ad(bc)^{-1} = (ad)/(bc)$.
4. (i) $x < -1$.
(iii) $x > \sqrt{7}$ or $x < -\sqrt{7}$.
(v) All x, since $x^2 - 2x + 2 = (x - 1)^2 + 1$.
(vii) $x > 3$ or $x < -2$, since 3 and -2 are the roots of $x^2 - x - 6 = 0$.
(ix) $x > \pi$ or $-5 < x < 3$.
(xi) $x < 3$.
(xiii) $0 < x < 1$.
5. (i) $b - a$ and $d - c$ are in P, so $(b - a) + (d - c) = (b + d) - (a + c)$ is in P. Thus, $b + d > a + c$.
(iii) Using (ii), $-c < -d$; then (i) implies that $a + (-c) < b + (-d)$.
(v) $(b-a)$ and $-c$ are in P, so $-c(b-a) = ac-bc$ is in P, that is, $ac > bc$.
(vii) Using (iv), $a > 0$ and $a < 1$, so $a^2 < a$.
(ix) Substitute a for c and b for d in (viii).
9. (i) $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.
(iii) $|a + b| + |c| - |a + b + c|$.
(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$.
10. (i) a if $a \geq -b$ and $b \geq 0$;
$-a$ if $a \leq -b$ and $b \leq 0$;
$a + 2b$ if $a \geq -b$ and $b \leq 0$;
$-a - 2b$ if $a \leq -b$ and $b \geq 0$.
(iii) $x - x^2$ if $x \geq 0$;
$-x - x^2$ if $x \leq 0$.
11. (i) $x = 11, -5$.
(iii) $-6 < x < -2$.
(v) No x (the distance from x to 1 plus the distance from x to -1 is at least 2).
(vii) $x = 1, -1$.
12. (i) $(|xy|)^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 = (|x| \cdot |y|)^2$; since $|xy|$ and $|x| \cdot |y|$ are both ≥ 0, this proves that $|xy| = |x| \cdot |y|$.
(iii) $|x|/|y| = |x| \cdot |y|^{-1} = |x| \cdot |y^{-1}|$ (by (ii)) $= |xy^{-1}|$ (by (i)) $= |x/y|$.
(v) It follows from (iv) that $|x| = |y - (y - x)| \leq |y| + |y - x|$, so $|x| - |y| \leq |x - y|$.
(vii) $|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$. If equality holds, then $|x + y| = |x| + |y|$, so x and y have the same sign. Moreover, z must

have the same sign as $x+y$, so x, y, and z must all have the same sign (unless one is 0).

CHAPTER 2

1. (i) Since $1^2 = 1\cdot(2)\cdot(2\cdot 1+1)/6$, the formula is true for $n=1$. Suppose that the formula is true for k. Then

$$\begin{aligned}1^2+\cdots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\ &= \frac{(k+1)}{6}[k(2k+1)+6(k+1)]\\ &= \frac{(k+1)}{6}[(k+2)(2k+3)]\\ &= \frac{(k+1)(k+2)(2[k+1]+1)}{6},\end{aligned}$$

so the formula is true for $k+1$.

2. (i)

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= 1+3+5+\cdots+(2n-1)\\ &= 1+2+3+\cdots+2n-2(1+\cdots+n)\\ &= \frac{(2n)(2n+1)}{2}-n(n+1)\\ &= n^2.\end{aligned}$$

5. (a) Since

$$1+r=\frac{1-r^2}{1-r},$$

the formula is true for $n=1$. Suppose that

$$1+r+\cdots+r^n=\frac{1-r^{n+1}}{1-r}.$$

Then

$$\begin{aligned}1+r+\cdots+r^n+r^{n+1} &= \frac{1-r^{n+1}}{1-r}+r^{n+1}\\ &= \frac{1-r^{n+1}+r^{n+1}(1-r)}{1-r}\\ &= \frac{1-r^{n+2}}{1-r}.\end{aligned}$$

(b)

$$\begin{aligned}S &= 1+r+\cdots+r^n\\ rS &= \quad\ r+\cdots+r^n+r^{n+1}.\end{aligned}$$

Thus

$$S(1-r)=S-rS=1-r^{n+1},$$

so

$$S = \frac{1 - r^{n+1}}{1 - r}.$$

6. (i) From

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1, \qquad k = 1, \dots, n$$

we obtain

$$(n+1)^4 - 1 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n,$$

so

$$\sum_{k=1}^{n} k^3 = \frac{(n+1)^4 - 1 - 6\dfrac{n(n+1)(2n+1)}{6} - 4\dfrac{n(n+1)}{2} - n}{4}$$

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

(iii) From

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}, \qquad k = 1, \dots, n$$

we obtain

$$1 - \frac{1}{n+1} = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

8. 1 is either even or odd, in fact it is odd. Suppose n is either even or odd; then n can be written either as $2k$ or $2k+1$. In the first case $n+1 = 2k+1$ is odd; in the second case $n+1 = 2k+1+1 = 2(k+1)$ is even. In either case, $n+1$ is either even or odd. (Admittedly, this looks fishy, but it is really correct.)

9. Let B be the set of all natural numbers l such that $n_0 - 1 + l$ is in A. Then 1 is in B, and $l+1$ is in B if l is in B, so B contains all natural numbers, which means that A contains all natural numbers $\geq n_0$.

12. (a) Yes, for if $a+b$ were rational, then $b = (a+b) - a$ would be rational. If a and b are irrational, then $a+b$ could be rational, for b could be $r - a$ for some rational number r.
(b) If $a = 0$, then ab is rational. But if $a \neq 0$, then ab could not be rational, for then $b = (ab) \cdot a^{-1}$ would be rational.
(c) Yes; for example, $\sqrt[4]{2}$.
(d) Yes; for example, $\sqrt{2}$ and $-\sqrt{2}$.

13. (a) Since

$$(3n+1)^2 = 9n^2 + 6n + 1 = 3(3n^2 + 2n) + 1,$$
$$(3n+2)^2 = 9n^2 + 12n + 4 = 3(3n^2 + 4n + 1) + 1,$$

it follows that if k^2 is divisible by 3, then k must also be divisible by 3. Now suppose that $\sqrt{3}$ were rational, and let $\sqrt{3} = p/q$ where p and

q have no common factor. Then $p^2 = 3q^2$, so p^2 is divisible by 3, so p must be. Thus, $p = 3p'$ for some natural number p', and consequently $(3p')^2 = 3q^2$, or $3(p')^2 = q^2$. Thus, q is also divisible by 3, a contradiction.

The same proofs work for $\sqrt{5}$ and $\sqrt{6}$, because the equations

$$\begin{aligned}
(5n+1)^2 &= 25n^2 + 10n + 1 = 5(5n^2 + 2n) + 1,\\
(5n+2)^2 &= 25n^2 + 20n + 4 = 5(5n^2 + 4n) + 4,\\
(5n+3)^2 &= 25n^2 + 30n + 9 = 5(5n^2 + 6n + 1) + 4,\\
(5n+4)^2 &= 25n^2 + 40n + 16 = 5(5n^2 + 8n + 3) + 1,
\end{aligned}$$

and the corresponding equations for numbers of the form $6n+m$, show that if k^2 is divisible by 5 or 6, then k must be. The proof fails for $\sqrt{4}$, because $(4n+2)^2$ is divisible by 4. (For precisely this reason this proof cannot be used to show that in general $\sqrt{a}$ is irrational if a is not a perfect square—we have no guarantee that $(an+m)^2$ might not be a multiple of a for some $m < a$. Actually, this assertion *is* true, but the proof requires the information in Problem 17.)

(b) Since

$$(2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1,$$

it follows that if k^3 is even, then k is even. If $\sqrt[3]{2} = p/q$ where p and q have no common factors, then $p^3 = 2q^3$, so p^3 is divisible by 2, so p must be. Thus, $p = 2p'$ for some natural number p', and consequently $(2p')^3 = 2q^3$, or $4(p')^3 = q^3$. Thus, q is also even, a contradiction.

The proof for $\sqrt[3]{3}$ is similar, using the equations

$$\begin{aligned}
(3n+1)^3 &= 27n^3 + 27n^2 + 9n + 1 = 3(9n^3 + 9n^2 + 3n) + 1,\\
(3n+2)^3 &= 27n^3 + 54n^2 + 36n + 8 = 3(9n^3 + 18n^2 + 12n + 2) + 2.
\end{aligned}$$

19. If $n = 1$, then $(1+h)^n = 1 + nh$. Suppose that $(1+h)^n \ge 1 + nh$. Then

$$\begin{aligned}
(1+h)^{n+1} &= (1+h)(1+h)^n \ge (1+h)(1+nh), \qquad \text{since } 1+h > 0\\
&= 1 + (n+1)h + nh^2 \ge 1 + (n+1)h.
\end{aligned}$$

For $h > 0$, the inequality follows directly from the binomial theorem, since all the other terms appearing in the expansion of $(1+h)^n$ are positive.

CHAPTER 3

1. (i) $(x+1)/(x+2)$; the expression $f(f(x))$ makes sense only when $x \ne -1$ and $x \ne -2$.
   (iii) $1/(1+cx)$ (for $x \ne -1/c$ if $c \ne 0$).
   (v) $(x+y+2)/(x+1)(y+1)$ (for $x, y \ne -1$).
   (vii) Only $c = 1$, since $f(x) = f(cx)$ implies that $x = cx$, and this must be true for at least one $x \ne 0$.
2. (i) $y \ge 0$ and rational, or $y \ge 1$.
   (iii) 0.
   (v) $-1, 0, 1$.

3. (i) $\{x : -1 \le x \le 1\}$.
   (iii) $\{x : x \ne 1 \text{ and } x \ne 2\}$.
   (v) $\emptyset$.
4. (i) 2^{2y}.
   (iii) $2^{2\sin t} + \sin(2^t)$.
5. (i) $P \circ s$.
   (iii) $s \circ S$.
   (v) $P \circ P$.
   (vii) $s \circ s \circ s \circ P \circ P \circ P \circ s$.
11. (a) y.
   (b) $H(y)$.
   (c) $H(y)$.
12. (a)

	even	odd
even	even	neither
odd	neither	odd

   (b)

	even	odd
even	even	odd
odd	odd	even

   (c)

	f even	f odd
g even	even	even
g odd	even	odd

   (d) Let $g(x) = f(x)$ for $x \ge 0$ and define g arbitrarily for $x < 0$.
21. (a) Let $g(x) = h(x) = 1$ and let f be a function for which $f(2) \ne f(1) + f(1)$. Then $f \circ (g + h) \ne f \circ g + f \circ h$.
   (b) $[(g+h) \circ f](x) = (g+h)(f(x)) = g(f(x)) + h(f(x)) = (g \circ f)(x) + (h \circ f)(x) = [(g \circ f) + (h \circ f)](x)$.
   (c) $\dfrac{1}{f \circ g}(x) = \dfrac{1}{f(g(x))} = \dfrac{1}{f}(g(x)) = \left(\dfrac{1}{f} \circ g\right)(x)$.
   (d) Let $g(x) = 2$ and let f be a function for which $f(\frac{1}{2}) \ne 1/f(2)$. Then $1/(f \circ g) \ne f \circ (1/g)$.

CHAPTER 4

1. (i) $(2, 4)$.
   (iii) $(a - \varepsilon, a + \varepsilon)$.
   (v) $[-2, 2]$.
   (vii) $(-\infty, 1] \cup [1, \infty)$.
3. (i) All points below the graph of $f(x) = x$.
   (iii) All points below the graph of $f(x) = x^2$.
   (v) All points between the graphs of $f(x) = x + 1$ and $f(x) = x - 1$.
   (vii) A collection of straight lines parallel to the graph of $f(x) = -x$, intersecting the horizontal axis at the points $(n, 0)$ for integers n.
   (ix) All points inside the circle of radius 1 and around $(1, 2)$.
4. (i) A square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
   (iii) The union of the graph of $f(x) = x$ and of $f(x) = 2 - x$.
   (v) The point $(0, 0)$.
   (vii) The circle of radius $\sqrt{5}$ around $(1, 0)$, since $x^2 - 2x + y^2 = (x - 1)^2 + y^2 - 1$.
6. (a) Simply observe that the graph of $f(x) = m(x - a) + b = mx + (b - ma)$ *is* a straight line with slope m, which goes through the point (a, b). (The important point about this exercise is simply to remember the point slope form.)
   (b) The straight line through (a, b) and (c, d) has slope $(d - b)/(c - a)$, so the equation follows from part (a).
   (c) When $m = m'$ and $b \neq b'$. In that case, there is clearly no number x with $f(x) = g(x)$, while such a number x always exists if $m \neq m'$, namely, $x = (b' - b)/(m - m')$.
7. (a) If $B = 0$ and $A \neq 0$, then the set is the vertical straight line formed by all points (x, y) with $x = -C/A$. If $B \neq 0$, the set is the graph of $f(x) = (-A/B)x + (-C/A)$.
   (b) The points (x, y) on the vertical line with $x = a$ are precisely the ones which satisfy $1 \cdot x + 0 \cdot y + (-a) = 0$. The points (x, y) on the graph of $f(x) = mx + b$ are precisely the ones which satisfy $(-m)x + 1 \cdot y + (-b) = 0$.
11. (i) The graph of f is symmetric with respect to the vertical axis.
    (ii) The graph of f is symmetric with respect to the origin. Equivalently, the part of the graph to the left of the vertical axis is obtained by reflecting first through the vertical axis, and then through the horizontal axis.
    (iii) The graph of f lies above or on the horizontal axis.
    (iv) The graph of f repeats the part between 0 and a over and over.
21. (a) The square of the distance from (x, x^2) to $(0, \frac{1}{4})$ is

$$\begin{aligned} x^2 + \left(x^2 - \frac{1}{4}\right)^2 &= x^2 + x^4 - \frac{x^2}{2} + \frac{1}{16} \\ &= x^4 + \frac{x^2}{2} + \frac{1}{16} \\ &= (x^2 + \tfrac{1}{4})^2. \end{aligned}$$

which is the square of the distance from (x, x^2) to the graph of g.

(b) The point (x, y) satisfies this condition if and only if

$$(x-\alpha)^2 + (y-\beta)^2 = (y-\gamma)^2,$$

or

$$x^2 - 2\alpha x + \alpha^2 + y^2 - 2\beta y + \beta^2 = y^2 - 2\gamma y + \gamma^2,$$

or

$$y = \left(\frac{1}{2\beta - 2\gamma}\right)x^2 + \left(\frac{\alpha}{\gamma - \beta}\right)x + \left(\frac{\alpha^2 + \beta^2 - \gamma^2}{2\beta - 2\gamma}\right).$$

If $\beta = \gamma$, so that P is on the line L, then the solution is the vertical line through P.

CHAPTER 5

1. (ii)

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12.$$

(iv)

$$\begin{aligned}\lim_{x\to y}\frac{x^n - y^n}{x-y} &= \lim_{x\to y} x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\\ &= y^{n-1} + y^{n-1} + \cdots + y^{n-1} = ny^{n-1}.\end{aligned}$$

(vi)

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h} &= \lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}\\ &= \frac{1}{2\sqrt{a}}.\end{aligned}$$

3. (i) $l = 0$. For all x we have $|\cos(x^2)| \le 1$, so $|3 - \cos(x^2)| \le 4$, and thus

$$|f(x) - 0| = |x|\cdot|3-\cos(x^2)| \le 4\cdot|x|.$$

So we can take $\delta = \varepsilon/4$.

(iii) $l = 100$. We have

$$\left|\frac{100}{x} - 100\right| = 100\cdot\left|\frac{1}{x} - 1\right| = 100\cdot\frac{1}{|x|}\cdot|x-1|.$$

The initial stipulation $|x-1| < \frac{1}{2}$ makes $x > \frac{1}{2}$, so $1/|x| < 2$, so we then have

$$|f(x) - 100| < 200\cdot|x-1|.$$

So we can take $\delta = \min(1/2, \varepsilon/200)$.

(v) $l = 2$. The same sort of argument that was used in the text and in number (iii) shows that

$$\left|\frac{1}{x} - 1\right| < \varepsilon \text{ for } 0 < |x-1| < \delta_1 = \min(1/2, \varepsilon/2),$$

so that

$$\left|\frac{1}{x} - 1\right| < \frac{\varepsilon}{2} \text{ for } 0 < |x-1| < \bar{\delta}_1 = \min(1/2, \varepsilon/4).$$

Similarly, the solution to number (iv) gives a δ_2 such that

$$|x^4 - 1| < \varepsilon \text{ for } 0 < |x-1| < \delta_2,$$

and we have a corresponding $\bar{\delta}_2$. Then we can take $\delta = \min(\bar{\delta}_1, \bar{\delta}_2)$.

(vii) $l = 0$. Let $\delta = \varepsilon^2$.

6. (i) We need $|f(x) - 2| < \varepsilon/2$ and $|g(x) - 4| < \varepsilon/2$, so we need

$$0 < |x-2| < \min\left(\sin^2\left(\frac{\varepsilon^2}{36}\right) + \frac{\varepsilon}{2}, \frac{\varepsilon^2}{4}\right) = \delta.$$

(iii) We need

$$|g(x) - 4| < \min\left(\frac{|4|}{2}, \frac{\varepsilon|4|^2}{2}\right),$$

so we need

$$0 < |x-2| < [\min(2, 8\varepsilon)]^2 = \delta.$$

9. Let $l = \lim_{x\to a} f(x)$ and define $g(h) = f(a+h)$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - l < \varepsilon|$. Now, if $0 < |h| < \delta$, then $0 < |(h+a) - a| < \delta$, so $|f(a+h) - l| < \varepsilon$. This inequality can be written $|g(h) - l| < \varepsilon$. Thus, $\lim_{h\to 0} g(h) = l$, which can also be written $\lim_{h\to 0} f(a+h) = l$. The same sort of argument shows that if $\lim_{h\to 0} f(a+h) = m$, then $\lim_{x\to a} f(x) = m$. So either limit exists if the other does, and in this case they are equal.

10. (a) Intuitively, we can get $f(x)$ as close to l as we like if and only if we can get $f(x) - l$ as close to 0 as we like. The formal proof is so trivial that it takes a bit of work to make it look like a proof at all. To be very precise, suppose $\lim_{x\to a} f(x) = l$ and let $g(x) = f(x) - l$. Then for all $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - l| < \varepsilon$. This last inequality can be written $|g(x) - 0| < \varepsilon$, so $\lim_{x\to a} g(x) = 0$. The argument in the other direction is similarly uninteresting.

(b) Intuitively, making x close to a is the same as making $x - a$ close to 0. Formally: Suppose that $\lim_{x\to a} f(x) = l$, and let $g(x) = f(x-a)$. Then for all $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x) - l| < \varepsilon$. Now, if $0 < |y| < \delta$, then $0 < |(y+a) - a| < \delta$, so

$|f(y+a)-l| < \varepsilon$. But this last inequality can be written $|g(y)-l| < \varepsilon$. So $\lim_{y\to 0} g(y) = l$. The argument in the reverse direction is similar.

(c) Intuitively, x is close to 0 if and only if x^3 is. Formally: Let $\lim_{x\to 0} f(x) = l$. For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x| < \delta$, then $|f(x)-l| < \varepsilon$. Then if $0 < |x| < \min(1,\delta)$, we have $0 < |x^3| < \delta$, so $|f(x^3)-l| < \varepsilon$. Thus, $\lim_{x\to 0} f(x) = l$. On the other hand, if we assume that $\lim_{x\to 0} f(x^3)$ exists, say $\lim_{x\to 0} f(x^3) = m$, then for all $\varepsilon > 0$ there is a δ such that if $0 < |x| < \delta$, then $|f(x^3)-m| < \varepsilon$. Then if $0 < |x| < \delta^3$, we have $0 < |\sqrt[3]{x}| < \delta$, so $|f([\sqrt[3]{x}]^3)-m| < \varepsilon$, or $|f(x)-m| < \varepsilon$. Thus $\lim_{x\to 0} f(x) = m$.

(d) Let $f(x) = 1$ for $x \geq 0$, and $f(x) = -1$ for $x < 0$. Then $\lim_{x\to 0} f(x^2) = 1$, but $\lim_{x\to 0} f(x)$ does not exist.

17. (a) The function $f(x) = 1/x$ cannot approach a limit at 0, since it becomes arbitrarily large near 0. In fact, no matter what $\delta > 0$ may be, there is some x satisfying $0 < |x| < \delta$, but $1/x > |l| + \varepsilon$, namely, any $x < \min(\delta, 1/(|l|+\varepsilon))$. Any such x does not satisfy $|(1/x) - l)| < \varepsilon$.

(b) No matter what $\delta > 0$ may be, there is some x satisfying $0 < |x-1| < \delta$, but $1/(x-1) > |l| + \varepsilon$, namely, any $x < \min(1+\delta, 1+1/(|l|+\varepsilon))$. Such an x does not satisfy $|1/(x-1)-l| < \varepsilon$. (It is also possible to apply Problem 10(b): $\lim_{x\to 0} 1/x = \lim_{x\to 1} 1/(x-1)$ if the latter exists, so this limit does not exist, because of part (a).)

25. (i) This is the usual definition, simply calling the numbers δ and ε, instead of ε and δ.

(ii) This is a minor modification of (i): if the condition is true for *all* $\delta > 0$, then it applies to $\delta/2$, so there is an $\varepsilon > 0$ such that if $0 < |x-a| < \varepsilon$, then $|f(x)-l| \leq \delta/2 < \delta$.

(iii) This is a similar modification: apply it to $\delta/5$ to obtain (i).

(iv) This is also a modification: it says the same thing as (i), since $\varepsilon/10 > 0$, and it is only the existence of *some* $\varepsilon > 0$ that is in question.

29. If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$, then for every $\varepsilon > 0$ there are $\delta_1, \delta_2 > 0$ such that, for all x,

$$\text{if } a < x < a+\delta_1, \text{ then } |f(x)-l| < \varepsilon,$$
$$\text{if } a-\delta_2 < x < a, \text{ then } |f(x)-l| < \varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x-a| < \delta$, then either $a-\delta_2 < a-\delta < x < a$ or else $a < x < a+\delta < a+\delta_1$, so $|f(x)-l| < \varepsilon$.

30. (i) If $l = \lim_{x\to 0^+} f(x)$, then for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x)-l| < \varepsilon$ for $0 < x < \delta$. If $-\delta < x < 0$, then $0 < -x < \delta$, so $|f(-x)-l| < \varepsilon$. Thus $\lim_{x\to 0^-} f(-x) = l$. Similarly, if $\lim_{x\to 0^-} f(x)$ exists, then $\lim_{x\to 0^+} f(x)$ exists and has the same value. (Intuitively, x is close to 0 and positive if and only if $-x$ is close to 0 and negative.)

(ii) If $l = \lim_{x \to 0^+} f(x)$, then for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for $0 < x < \delta$. So if $0 < |x| < \delta$, then $|f(|x|) - l| < \varepsilon$. Thus $\lim_{x \to 0} f(|x|) = l$. The reverse direction is similar. (Intuitively, if x is close to 0, then $|x|$ is close to 0 and positive.)

(iii) If $l = \lim_{x \to 0^+} f(x)$, then for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for $0 < x < \delta$. If $0 < |x| < \sqrt{\delta}$, then $0 < x^2 < \delta$, so $|f(x^2) - l| < \varepsilon$. Thus $\lim_{x \to 0} f(x^2) = l$. The reverse direction is similar. (Intuitively, if x is close to 0, then x^2 is close to 0 and positive.)

34. If $l = \lim_{x \to \infty} f(x)$, then for every $\varepsilon > 0$ there is some N such that $|f(x) - l| < \varepsilon$ for $x > N$, and we can clearly assume that $N > 0$. Now, if $0 < x < 1/N$, then $1/x > N$, so $|f(1/x) - l| < \varepsilon$. Thus $\lim_{x \to 0^+} f(1/x) = l$. The reverse direction is similar.

CHAPTER 6

1. (i) $F(x) = x + 2$ for all x.
   (iii) $F(x) = 0$ for all x.

CHAPTER 7

1. (i) Bounded above and below; minimum value 0; no maximum value.
   (iii) Bounded below but not above; minimum value 0.
   (v) Bounded above and below. If $a \le -1/2$, then $a \le -a - 1$, so $f(x) = a+2$ for all x in $(-a-1, a+1)$, so $a+2$ is the maximum and minimum value. If $-1/2 < a \le 0$, then f has the minimum value a^2, and if $a \ge 0$, then f has the minimum value 0. Since $a + 2 > (a+1)^2$ only for $(-1 - \sqrt{5})/2 < a < (-1 + \sqrt{5})/2$, when $a \ge -1/2$ the function f has a maximum value only for $a \le (-1 + \sqrt{5})/2$ (the maximum value being $a + 2$).
   (vii) Bounded above and below; maximum value 1; minimum value 0.
   (ix) Bounded above and below; maximum value 1; minimum value -1.
   (xi) f has a maximum and minimum value, since f is continuous.
2. (i) $n = -2$, since $f(-2) < 0 < f(-1)$.
   (iii) $n = -1$, since $f(-1) = -1 < 0 < f(0)$.
3. (i) If $f(x) = x^{179} + 163/(1 + x^2 + \sin^2 x) - 119$, then f is continuous on $\mathbf{R}$ and $f(2) > 0$, while $f(-2) < 0$, so $f(x) = 0$ for some x in $(-2, 2)$.
5. f is constant, for if f took on two different values, then f would take on all values in between, which would include irrational values.
7. (1) $f(x) = x$;
   (2) $f(x) = -x$;
   (3) $f(x) = |x|$;
   (4) $f(x) = -|x|$.
10. Apply Theorem 1 to $f - g$.

11. If $f(0) = 0$ or $f(1) = 1$, choose $x = 0$ or 1. If $f(0) > 0 = I(0)$ and $f(1) < 1 = I(1)$, then Problem 10 applied to f and I implies that $f(x) = x$ for some x.

CHAPTER 8

1. (i) 1 is the greatest element, and the greatest lower bound is 0, which is not in the set.
   (iii) 1 is the greatest element, and 0 is the least element.
   (v) Since $\{x : x^2+x+1 \geq 0\} = \mathbf{R}$, there is no least upper bound or greatest lower bound.
   (vii) Since $\{x : x < 0 \text{ and } x^2+x-1 < 0\} = ([-1-\sqrt{5}]/2, 0)$, the greatest lower bound is $[-1-\sqrt{5}]/2$, and the least upper bound is 0; neither belongs to the set.
2. (a) Since $A \neq \emptyset$, there is some x in A. Then $-x$ is in $-A$, so $-A \neq \emptyset$. Since A is bounded below, there is some y such that $y \leq x$ for all x in A. Then $-y \geq -x$ for all x in A, so $-y \geq z$ for all z in $-A$, so $-A$ is bounded above. Let $\alpha = \sup(-A)$. Then α is an upper bound for $-A$, so, reversing the argument just given, $-\alpha$ is a lower bound for A.

   Moreover, if β is any lower bound for A, then $-\beta$ is an upper bound for $-A$, so $-\beta \geq \alpha$, so $\beta \leq -\alpha$. Thus $-\alpha$ is the greatest lower bound for A.
5. (a) If l is the largest integer with $l \leq x$, then $l+1 > x$, but $l+1 \leq x+1 < y$. So we can let $k = l+1$. (Proof that a largest such integer l exists: Since $\mathbf{N}$ is not bounded above, there is some natural number n with $-n < x < n$. There are consequently only a finite number of integers l with $-n \leq l \leq x$. Pick the largest.)
   (b) Since $y - x > 0$, there is some natural number n with $1/n < y - x$. Since $ny - nx > 1$, there is, by part (a), an integer k with $nx < k < ny$, which means that $x < k/n < y$.
   (c) Choose $r + \sqrt{2}(s-r)/2$.
   (d) By part (b), there is a rational number r with $x < r < y$, and therefore a rational number s with $x < r < s < y$. Apply part (c) to $r < s$.
10. Let k be the largest integer $\leq x/\alpha$ (the solution to Problem 5 shows that such a k exists), and let $x' = x - k\alpha \geq 0$. If $x - k\alpha = x' \geq \alpha$, then $x \geq (k+1)\alpha$, so $k+1 \leq x/\alpha$, contradicting the choice of k. So $0 \leq x' < \alpha$.
12. (a) Since any y in B satisfies $y \geq x$ for all x in A, any y in B is an upper bound for A, so $y \geq \sup A$.
    (b) Part (a) shows that $\sup A$ is a lower bound for B, so $\sup A \leq \inf B$.
13. Since $x \leq \sup A$ and $y \leq \sup B$ for every x in A, and y in B, it follows that $x + y \leq \sup A + \sup B$. Thus, $\sup A + \sup B$ is an upper bound for $A + B$, so $\sup(A+B) \leq \sup A + \sup B$. If x and y are chosen in A and B, respectively, so that $\sup A - x < \varepsilon/2$ and $\sup B - y < \varepsilon/2$, then $\sup A + \sup B - (x+y) < \varepsilon$. Hence,

$$\sup(A+B) \geq x + y > \sup A + \sup B - \varepsilon.$$

CHAPTER 9

1. (a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$$
$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(b) The tangent line through $(a, 1/a)$ is the graph of

$$g(x) = \frac{-1}{a^2}(x-a) + \frac{1}{a}$$
$$= \frac{-x}{a^2} + \frac{2}{a}.$$

If $f(x) = g(x)$, then

$$\frac{1}{x} = -\frac{x}{a^2} + \frac{2}{a}$$

or

$$x^2 - 2ax + a^2 = 0,$$

so $x = a$.

2. (a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h}$$
$$= \lim_{h\to 0} \frac{(-2ah - h^2)}{ha^2(a+h)^2} = -\frac{2}{a^3}.$$

(b) The tangent line through $(a, 1/a^2)$ is the graph of

$$g(x) = -\frac{2}{a^3}(x-a) + \frac{1}{a^2}$$
$$= -\frac{2x}{a^3} + \frac{3}{a^2}.$$

If $f(x) = g(x)$, then

$$\frac{1}{x^2} = \frac{-2x}{a^3} + \frac{3}{a^2},$$

or

$$2x^3 - 3ax^2 + a^3 = 0,$$

or

$$0 = (x-a)(2x^2 - ax - a^2) = (x-a)(2x+a)(x-a).$$

So $x = a$ or $x = -a/2$; the point $(-a/2, 4/a^2)$ lies on the opposite side of the vertical axis from $(a, 1/a^2)$.

3.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}\\
&= \lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})} = \lim_{h\to 0}\frac{h}{h(\sqrt{a+h}+\sqrt{a})}\\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}
$$

4. Conjecture: $S_n{}'(x) = nx^{n-1}$. Proof:

$$
\begin{aligned}
S_n{}'(x) &= \lim_{h\to 0}\frac{S_n(x+h)-S_n(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}\\
&= \lim_{h\to 0}\frac{\displaystyle\sum_{j=0}^{n}\binom{n}{j}x^{n-j}h^j - x^n}{h}\\
&= \lim_{h\to 0}\sum_{j=1}^{n}\binom{n}{j}x^{n-j}h^{j-1}\\
&= \binom{n}{1}x^{n-1} = nx^{n-1}, \qquad \text{since } \lim_{h\to 0} h^{j-1} = 0 \text{ for } j > 1.
\end{aligned}
$$

5. $f'(x) = 0$ for x not an integer, and $f'(x)$ is not defined if x is an integer.

6. (a)

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+c]-[f(x)+c]}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x).
\end{aligned}
$$

(b)

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\
&= c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x).
\end{aligned}
$$

7. (a) $f'(9) = 3\cdot 9^2$; $f'(25) = 3\cdot(25)^2$; $f'(36) = 3\cdot(36)^2$.
   (b) $f'(3^2) = f'(9) = 3\cdot 9^2$; $f'(5^2) = f'(25) = 3\cdot(25)^2$; $f'(6^2) = f'(36) = 3\cdot(36)^2$.
   (c) $f'(a^2) = 3(a^2)^2 = 3a^4$; $f'(x^2) = 3(x^2)^2 = 3x^4$.
   (d) $f'(x^2) = 3x^4$; but $g(x) = x^6$, so $g'(x) = 6x^5$.

8. (a)

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{f(x+h+c)-f(x+c)}{h}\\
&= \lim_{h\to 0}\frac{f([x+c]+h)-f(x+c)}{h} = f'(x+c).
\end{aligned}
$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{f(cx+ch)-f(cx)}{h} \\ &= \lim_{h\to 0} \frac{c[f(cx+ch)-f(cx)]}{ch} = \lim_{k\to 0} \frac{c[f(cx+k)-f(cx)]}{k} \\ &= c\cdot \lim_{k\to 0} \frac{f(cx+k)-f(cx)}{k} = c\cdot f'(cx). \end{aligned}$$

(Compare the manipulations in this calculation with Problem 5-14.)

(c) If $g(x) = f(x+a)$, then $g'(x) = f'(x+a)$, by part (a). But $g = f$, so $f'(x) = g'(x) = f'(x+a)$ for all x, which means that f' is periodic, with period a.

9. (i) If $g(x) = x^5$, then $g'(x) = 5x^4$. Now $f(x) = g(x+3)$, so by Problem 8(a), $f'(x) = g'(x+3) = 5(x+3)^4$. And $f'(x+3) = 5(x+6)^4$.
(ii) $f(x) = (x-3)^5$, so $f'(x) = 5(x-3)^4$, as in part (i). And $f'(x+3) = 5x^4$.
(iii) $f(x) = (x+2)^7$, so $f'(x) = 7(x+2)^6$, as in part (i). And $f'(x+3) = 7(x+5)^6$.

10. If $f(x) = g(t+x)$, then $f'(x) = g'(t+x)$, by Problem 8(a). If $f(t) = g(t+x)$, then $f'(t) = g'(t+x)$, by Problem 8(a), so $f'(x) = g'(2x)$.

11. (a) If $s(t) = ct^2$, then $s'(t) = 2ct$, and there is no number k such that $s'(t) = ks(t)$ [that is, $2ct = kct^2$] for all t.
(By the way, at this point we do not know any nonzero function f for which f' is proportional to f. After Chapter 18 it might be amusing to determine what the world would be like if Galileo were correct.)
(b) (i) If $s(t) = (a/2)t^2$, then $s'(t) = at$, so $s''(t) = a$.
(ii) $[s'(t)]^2 = (at)^2 = 2a\cdot(a/2)t^2 = 2as(t)$.
(c) The chandelier falls $s(t) = 16t^2$ feet in t seconds, so it falls 400 feet in t seconds, if $400 = 16t^2$, or $t = 5$. After 5 seconds the velocity will be $s'(5) = 5a = 5\cdot 32 = 160$ feet per second. The speed was half this amount when $80 = s'(t) = 32t$, or $t = \frac{5}{2}$.

21. (a) This is another way of writing the definition (see Problem 5-9).
(b) This follows from Problem 5-11, applied to the functions $\alpha(h) = [f(a+h)-f(a)]/h$ and $\beta(h) = [g(a+h)-g(a)]/h$.

26. (i) $f''(x) = 6x$.
(iii) $f''(x) = 4x^3$.

30. (i) means that $f'(a) = na^{n-1}$ if $f(x) = x^n$.
(iii) means that $g'(a) = f'(a)$ if $g(x) = f(x)+c$.
(v) means the same as (iii).
(vii) means that $g'(b) = f'(b+a)$ if $g(x) = f(x+a)$.
(ix) means that $g'(b) = cf'(cb)$ if $g(x) = f(cx)$.

CHAPTER 10

1. (i) $(1+2x)\cdot\cos(x+x^2)$.
(iii) $(-\sin x)\cdot\cos(\cos x)$.
(v) $\cos\left(\dfrac{\cos x}{x}\right)\cdot\dfrac{-x\sin x-\cos x}{x^2}$.

(vii) $(\cos(x+\sin x))\cdot(1+\cos x)$.

2. (i) $(\cos((x+1)^2(x+2)))\cdot[2(x+1)(x+2)+(x+1)^2]$.

(iii) $[2\sin((x+\sin x)^2)\cos((x+\sin x)^2)]\cdot 2(x+\sin x)(1+\cos x)$.

(v) $(\cos(x\sin x))\cdot(\sin x+x\cos x)+(\cos(\sin x^2)(\cos x^2))\cdot 2x$.

(vii) $(2\sin x\cos x\sin x^2\sin^2 x^2)+(2x\cos x^2\sin^2 x\sin^2 x^2)$
$+(4x\sin x^2\cos x^2\sin^2 x\sin x^2)$.

(ix) $6(x+\sin^5 x)^5(1+5\sin^4 x\cos x)$.

(xi) $\cos(\sin^7 x^7+1)^7\cdot 7(\sin^7 x^7+1)^6\cdot(7\sin^6 x^7\cdot\cos x^7\cdot 7x^6)$.

(xiii) $\cos(x^2+\sin(x^2+\sin x^2))\cdot[(2x+\cos(x^2+\sin x^2)\cdot(2x+2x\cos x^2))]$.

(xv) $$\frac{(1+\sin x)(2x\cos x^2\cdot\sin^2 x+\sin x^2\cdot 2\sin x\cos x)-\cos x\sin x^2\sin^2 x}{(1+\sin x)^2}.$$

(xvii) $$\cos\left(\frac{x^3}{\sin\left(\dfrac{x^3}{\sin x}\right)}\right)\cdot$$
$$\frac{3x^2\sin\left(\dfrac{x^3}{\sin x}\right)-x^3\cos\left(\dfrac{x^3}{\sin x}\right)\cdot\left(\dfrac{3x^2\sin x-x^3\cos x}{\sin^2 x}\right)}{\sin^2\left(\dfrac{x^3}{\sin x}\right)}.$$

4. (i) $-\dfrac{(x+1)^2}{(x+2)^2}$.

(iii) $2x^2$.

5. (i) $-x^2$.

(iii) 17.

6. (i) $f'(x)=g'(x+g(a))$.

(iii) $f'(x)=g'(x+g(x))\cdot(1+g'(x))$.

(v) $f'(x)=g(a)$.

7. (a) $A'(t)=2\pi r(t)r'(t)$. Since $r'(t)=4$ for that t with $r(t)=6$, it follows that $A'(t)=2\pi\cdot 6\cdot 4=48\pi$ when $r(t)=6$.

(b) If $V(t)$ is the volume at time t, then $V(t)=4\pi r(t)^3/3$, so $V'(t)=4\pi r(t)^2 r'(t)=4\pi\cdot 6^2\cdot 4=576\pi$ when $r(t)=6$.

(c) First method: Since $A'(t)=2\pi r(t)r'(t)$, and $A'(t)=5$ for $r(t)=3$, it follows that
$$r'(t)=\frac{A'(t)}{2\pi r(t)}=\frac{5}{6\pi}\quad\text{when } r(t)=3.$$
Thus
$$\begin{aligned}V'(t)&=4\pi r(t)^2 r'(t)\\&=4\pi\cdot 9\cdot\frac{5}{6\pi}\\&=30\quad\text{when } r(t)=3.\end{aligned}$$

To apply the second method, we first note that if

$$f(t) = A(t)^{3/2} = \sqrt{A(t)^3},$$

then, using Problem 9-3 and the Chain Rule,

$$\begin{aligned} f'(t) &= \frac{1}{2\sqrt{A(t)^3}} \cdot 3A(t)^2 A'(t) \\ &= \frac{1}{2A(t)^{3/2}} \cdot 3A(t)^2 A'(t) \\ &= \frac{3}{2} A(t)^{1/2} A'(t) \quad \text{(just as we might have guessed).} \end{aligned}$$

Now

$$\begin{aligned} V(t) &= \frac{4\pi r(t)^3}{3} = \frac{4\pi [r(t)^2]^{3/2}}{3} \\ &= \frac{4[\pi r(t)^2]^{3/2}}{3\pi^{1/2}} \\ &= \frac{4A(t)^{3/2}}{3\pi^{1/2}}. \end{aligned}$$

So

$$\begin{aligned} V'(t) &= \frac{4}{3\pi^{1/2}} \cdot \frac{3}{2}\sqrt{A(t)}A'(t) \\ &= \frac{2}{\pi^{1/2}} \cdot \pi^{1/2} r(t) A'(t) \\ &= 2 \cdot 3 \cdot 5 = 30. \end{aligned}$$

10. (i) $(f \circ h)'(0) = f'(h(0)) \cdot h'(0) = f'(3) \cdot \sin^2(\sin 1) =$ $[6 \sin \frac{1}{3} - \cos \frac{1}{3}] \sin^2(\sin 1)$.
    (iii) $\alpha'(x^2) = h'(x^4) \cdot 2x^2 = \sin^2(\sin(x^4 + 1)) \cdot 2x^2$.

12. The Chain Rule implies that

$$\begin{aligned} \left(\frac{1}{g}\right)'(x) &= (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \\ &= -\frac{1}{g(x)^2} \cdot g'(x). \end{aligned}$$

35. (i) $\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx} = (\cos y) \cdot (1 + 2x) = (\cos(x + x^2)) \cdot (1 + 2x)$.
    (iii) $\dfrac{dz}{dx} = \dfrac{dz}{du} \cdot \dfrac{du}{dx} = (\cos u) \cdot (\cos x) = (\cos(\sin x)) \cdot (\cos x)$.

CHAPTER 11

1. (i) $0 = f'(x) = 3x^2 - 2x - 8$ for $x = 2$ and $x = -\frac{4}{3}$, both of which are in $[-2, 2]$;
   $f(-2) = 5$, $f(2) = -11$, $f(-\frac{4}{3}) = \frac{203}{27}$;
   maximum $= \frac{203}{27}$, minimum $= -11$.

(iii) $0 = f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1)$ for $x = 0$ and $x = 1$, of which only 0 is in $[-\frac{1}{2}, \frac{1}{2}]$;
$f(-\frac{1}{2}) = \frac{43}{16}$, $f(\frac{1}{2}) = \frac{11}{16}$, $f(0) = 0$;
maximum $= \frac{43}{16}$, minimum $= 0$.

(v) $0 = f'(x) =$

$$\frac{x^2 + 1 - (x+1)2x}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}$$

for $x = -1 + \sqrt{2}$ and $x = -1 - \sqrt{2}$, of which only $-1 + \sqrt{2}$ is in $[-1, \frac{1}{2}]$;
$f(-1) = 0$, $f(\frac{1}{2}) = \frac{6}{5}$, $f(-1 + \sqrt{2}) = (1 + \sqrt{2})/2$;
maximum $= (1 + \sqrt{2})/2$, minimum $= 0$.

2. (i) $-\frac{4}{3}$ is a local maximum point, and 2 is a local minimum point.
(iii) 0 is a local minimum point, and there are no local maximum points.
(v) $-1 + \sqrt{2}$ is a local maximum point, and $-1 - \sqrt{2}$ is a local minimum point.

4. (a) Notice that f actually has a minimum value, since f is a polynomial function of even degree. The minimum occurs at a point x with

$$0 = f'(x) = 2\sum_{i=1}^{n}(x - a_i),$$

so $x = (a_1 + \cdots + a_n)/n$.

5. (i) 3 and 7 are local maximum points, and 5 and 9 are local minimum points.
(iii) All irrational $x > 0$ are local minimum points, and all irrational $x < 0$ are local maximum points.
(v) x is a local minimum point if its decimal expansion does not contain a 5. It is a local maximum point if its decimal expansion contains exactly one 5 that is followed by an infinite string of 9's. In all other cases, x is both a local maximum point and a local minimum point.

7. If $f(x)$ is the total length of the path, then

$$f(x) = \sqrt{x^2 + a^2} + \sqrt{(1-x)^2 + b^2}.$$

The positive function f clearly has a minimum, since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, and f is differentiable everywhere, so the minimum occurs at a point x with $f'(x) = 0$. Now, $f'(x) = 0$ when

$$\frac{x}{\sqrt{x^2 + a^2}} - \frac{(1-x)}{\sqrt{(1-x)^2 + b^2}} = 0.$$

This equation says that $\cos\alpha = \cos\beta$.

It is also possible to notice that $f(x)$ is equal to the sum of the lengths of the dashed line segment and the line segment from $(x, 0)$ to $(1, b)$. This is shortest when the two line segments lie along a line (because of Problem 4-9(b), if a rigorous reason is required); a little plane geometry shows that this happens when $\alpha = \beta$.

10. If x is the length of one side of a rectangle of perimeter P, then the length of the other side is $(P-2x)/2$, so the area is

$$A(x) = \frac{x(P-2x)}{2}.$$

So the rectangle with greatest area occurs when x is the maximum point for f on $(0, P/2)$. Since A is continuous on $[0, P/2]$, and $A(0) = A(P/2) = 0$, and $A(x) > 0$ for x in $(0, P/2)$, the maximum exists. Since A is differentiable on $(0, P/2)$, the minimum point x satisfies

$$\begin{aligned} 0 = A'(x) &= \frac{P-2x}{2} - x \\ &= \frac{P-4x}{2}, \end{aligned}$$

so $x = P/4$.

11. Let $S(r)$ be the surface area of the right circular cylinder of volume V with radius r. Since

$$V = \pi r^2 h \quad \text{where } h \text{ is the height,}$$

we have $h = V/\pi r^2$, so

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2V}{r}. \end{aligned}$$

We want the minimum point of S on $(0, \infty)$; this exists, since $\lim\limits_{r\to 0} S(r) = \lim\limits_{r\to\infty} S(r) = \infty$. Since S is differentiable on $(0, \infty)$, the minimum point r satisfies

$$\begin{aligned} 0 = S'(r) &= 4\pi r - \frac{2V}{r^2} \\ &= \frac{4\pi r^3 - 2V}{r^2}, \end{aligned}$$

or

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

21. 1 is a local maximum point, and 3 is a local minimum point.

28. (a) We have

$$\begin{aligned} \frac{f(b)-f(a)}{b-a} &= f'(x) \quad \text{for some } x \text{ in } (a,b) \\ &\geq M, \end{aligned}$$

so $f(b) - f(a) \geq M(b-a)$.

(b) We have

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= f'(x) \quad \text{for some } x \text{ in } (a,b)\\ &\le m,\end{aligned}$$

so $f(b)-f(a) \le m(b-a)$.

(c) If $|f'(x)| \le M$ for all x in $[a,b]$, then $-M \le f(x) \le M$, so

$$f(a)-M(b-a) \le f(b) \le f(a)+M(b-a),$$

or

$$|f(b)-f(a)| \le M(b-a).$$

31. (a) $f(x) = -\cos x + a$ for some number a (because $f(x) = -\cos x$ is one such function, and any two such functions differ by a constant function).

(b) $f'(x) = x^4/4+a$ for some number a, so $f(x) = x^5/20+ax+b$ for some numbers a and b.

(c) $f''(x) = x^2+x^3/3+a$ for some a, so $f'(x) = x^3/6+x^4/12+ax+b$ for some a and b, so $f(x) = x^4/24+x^5/60+ax^2/2+bx+c$ for some numbers a, b, and c. Equivalently, and more simply, $f(x) = x^4/24+x^5/60+ax^2+bx+c$ for some numbers a, b, and c.

32. (a) Since $s''(t) = -32$, we have $s'(t) = -32t+\alpha$ for some α, so $s(t) = -16t^2+\alpha t+\beta$ for some α and β.

(b) Clearly, $s(0) = 0+0+\beta$ and $s'(0) = 0+\alpha$. Thus, $\alpha = v_0$ and $\beta = s_0$.

(c) In this case, $s_0 = 0$ and $v_0 = v$, so $s(t) = -16t^2+vt$. The maximum value of s occurs when $0 = s'(t) = -32t+v$, or $t = v/32$, so the maximum value is

$$\begin{aligned}s\left(\frac{v}{32}\right) &= -16\left(\frac{v}{32}\right)^2 + v\cdot\left(\frac{v}{32}\right)\\ &= \frac{-v^2}{64}+\frac{v^2}{32}\\ &= \frac{v^2}{64}.\end{aligned}$$

At that moment the velocity is clearly 0, but the acceleration is -32 (as at any time). The weight hits the ground at time $t > 0$ when

$$0 = s(t) = -16t^2+vt,$$

or $t = v/16$ (it takes as long to fall back down as it took to reach the top). The velocity is then

$$\begin{aligned}s'(v/16) &= -32\left(\frac{v}{16}\right)+v\\ &= -v\end{aligned}$$

(the same velocity with which it was initially moving upward).

47. Apply the Mean Value Theorem to $f(x) = \sqrt{x}$ on $[64, 66]$:

$$\frac{\sqrt{66} - \sqrt{64}}{66 - 64} = f'(x) = \frac{1}{2\sqrt{x}} \quad \text{for some } x \text{ in } [64, 66].$$

Since $64 < x < 81$, we have $8 < \sqrt{x} < 9$, so

$$\frac{1}{2 \cdot 9} < \frac{\sqrt{66} - 8}{2} < \frac{1}{2 \cdot 8}.$$

51. l'Hôpital's Rule does not lead to the equation

$$\lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2}$$

because $\lim\limits_{x \to 1} 3x^2 + 1 \neq 0$.

52. (i)

$$\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{1}{\sec^2 x} = \lim_{x \to 0} \cos^2 x = 1.$$

(ii)

$$\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2} = \lim_{x \to 0} \frac{-2 \sin x \cos x}{2x} = -1.$$

CHAPTER 12

1. (i) $f^{-1}(x) = (x - 1)^{1/3}$. (If $y = f^{-1}(x)$, then $x = f(y) = y^3 + 1$, so $y = (x - 1)^{1/3}$.)

(iii) $f^{-1} = f$. (If $y = f^{-1}(x)$, then

$$x = f(y) = \begin{cases} y, & y \text{ rational} \\ -y, & y \text{ irrational}; \end{cases}$$

since $\pm y$ is rational or irrational if and only if y is, we have $y = x$ if x is rational and $y = -x$ if x is irrational, so $y = f(x)$.)

(v)

$$f^{-1}(x) = \begin{cases} x, & x \neq a_1, \dots, a_n \\ a_{i-1}, & x = a_i, \quad i = 2, \dots, n \\ a_n, & x = a_1. \end{cases}$$

(vii) $f^{-1} = f$.

2. (i) f^{-1} is increasing and $f^{-1}(x)$ is not defined for $x \leq 0$.

(iii) f^{-1} is decreasing and $f^{-1}(x)$ is not defined for $x \leq 0$.

3. Suppose f is increasing. Let $a < b$. Then $f^{-1}(a) \neq f^{-1}(b)$, since f^{-1} is one-one. So either $f^{-1}(a) < f^{-1}(b)$ or $f^{-1}(a) > f^{-1}(b)$. But if $f^{-1}(a) > f^{-1}(b)$, then

$$b = f(f^{-1}(b)) < f(f^{-1}(a)) = a,$$

a contradiction. The proof is similar for decreasing f, or one can consider $-f$ instead.

4. Clearly, $f + g$ is increasing, for if $f(a) < f(b)$ and $g(a) < g(b)$, then $(f + g)(a) = f(a) + g(a) < f(b) + g(b) = (f + g)(b)$.
$f \cdot g$ is not necessarily increasing; for example, if $f(x) = g(x) = x$. (But $f \cdot g$

is increasing if $f(x), g(x) \geq 0$ for all x.)
$f \circ g$ is increasing, for if $a < b$, then $g(a) < g(b)$, so $f(g(a)) < f(g(b))$.

5. (a) If $(f \circ g)(x) = (f \circ g)(y)$, so that $f(g(x)) = f(g(y))$, then $g(x) = g(y)$, since f is one-one, so $x = y$, since g is one-one.
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$: for if $y = (f \circ g)^{-1}(x)$, then $x = (f \circ g)(y) = f(g(y))$, so $g(y) = f^{-1}(x)$, so $y = g^{-1}(f^{-1}(x))$.

6. If $f(x) = f(y)$, then
$$\frac{ax+b}{cx+d} = \frac{ay+b}{cy+d},$$
so
$$acxy + bcy + adx + bd = acxy + ady + bcx + bd,$$
or
$$ad(x-y) = bc(x-y).$$
If $ad \neq bc$, this implies that $x - y = 0$. (But if $ad = bc$, then $f(x) = f(y)$ for all x and y in the domain of f.)
If $y = f^{-1}(x)$, then $x = f(y)$, so
$$x = \frac{ay+b}{cy+d}$$
so
$$f^{-1}(x) = y = \frac{-dx+b}{cx-a} \qquad \text{for } x \neq a/c.$$

7. (i) Those intervals $[a, b]$ which are contained in $(-\infty, 0]$ or $[0, 2]$ or $[2, \infty)$, since f is increasing on $(-\infty, 0]$ and $[2, \infty)$, and decreasing on $[0, 2]$.
(iii) Those intervals $[a, b]$ which are contained in $(-\infty, 0]$ or $[0, \infty)$, since f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$.

11. Since
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$
we have
$$\begin{aligned}(f^{-1})''(x) &= \frac{-f''(f^{-1}(x)) \cdot (f^{-1})'(x)}{[f'(f^{-1}(x))]^2} \\ &= \frac{-f''(f^{-1}(x))}{[f'(f^{-1}(x))]^3}.\end{aligned}$$

20. The formula for the derivative reads:
$$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.$$
(In this formula, it is understood that dx/dy means $(f^{-1})'(y)$, while dy/dx is an "expression involving x," and in the final answer x must be replaced by y, by means of the equation $y = f(x)$.)

The computation in Problem 20, when completed, shows that

$$\frac{dx^{1/n}}{dx} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{nx^{1-(1/n)}}$$
$$= \frac{1}{n}x^{(1/n)-1}.$$

21.

$$\begin{aligned} G'(x) &= x(f^{-1})'(x) + f^{-1}(x) - F'(f^{-1}(x)) \cdot (f^{-1})'(x) \\ &= x(f^{-1})'(x) + f^{-1}(x) - f(f^{-1}(x)) \cdot (f^{-1})'(x) \\ &= x(f^{-1})'(x) + f^{-1}(x) - x(f^{-1})'(x) \\ &= f^{-1}(x). \end{aligned}$$

22. (i)

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))} = \frac{1}{h'(0)} = \frac{1}{\sin^2(\sin 1)}.$$

CHAPTER 13

1. If $P_n = \{t_0, \dots, t_n\}$ is the partition with $t_i = ib/n$, then

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (t_{i-1})^3 \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (i-1)^3 \cdot \frac{b3}{n^3} \cdot \frac{b}{n} \\ &= \frac{b^4}{n^4} \sum_{j=0}^{n-1} j^3 \\ &= \frac{b^4}{n^4}\left[\frac{(n-1)^4}{4} + \frac{(n-1)^3}{2} + \frac{(n-1)^2}{4}\right], \end{aligned}$$

and similarly

$$\begin{aligned} U(f, P_n) &= \frac{b^4}{n^4} \sum_{j=1}^{n} j^3 \\ &= \frac{b^4}{n^4}\left[\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right]. \end{aligned}$$

Clearly $L(f, P_n)$ and $U(f, P_n)$ can be made as close to $b^4/4$ as desired by choosing n sufficiently large, so $U(f, P_n) - L(f, P_n)$ can be made as small as desired, by choosing n large enough. This shows that f is integrable. Moreover, there is only one number a with $L(f, P_n) \le a \le U(f, P_n)$ for all n; since $\int_0^b x^3\,dx$ has this property, the proof that $\int_0^b x^3\,dx = b^4/4$ will be complete once we show that $L(f, P_n) \le b^4/4 \le U(f, P_n)$ for all n. This can be done by a straightforward computation, but it actually follows from the fact that $L(f, P_n)$ and $U(f, P_n)$ can be made as close to $b^4/4$ as desired by choosing n sufficiently large. In fact, if it were true that $b^4/4 < \int_0^b x^3\,dx$, then it would not be possible to make $U(f, P_n)$ as close as desired to $b^4/4$ by choosing n large enough, since each $U(f, P_n) \geq \int_0^b x^3\,dx$, and similarly we cannot have $b^4/4 > \int_0^b x^3\,dx$.

2. We have

$$L(f, P_n) = \frac{b^5}{n^5}\left[\frac{(n-1)^5}{5} + \frac{(n-1)^4}{2} + \frac{(n-1)^3}{3} - \frac{(n-1)}{30}\right],$$

$$U(f, P_n) = \frac{b^5}{n^5}\left[\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right].$$

Clearly $L(f, P_n)$ and $U(f, P_n)$ can be made as close to $b^5/5$ as desired by choosing n large enough. As in Problem 1, this implies that $\int_0^b x^4\,dx = b^5/5$.

7. (i) $\int_0^2 f = 0$.
(iii) $\int_0^2 f = 3$.
(v) f is not integrable.
(vii) $\int_0^2 f = 1$.
(For a rigorous proof that the functions in (i), (iii), and (vii) are integrable, see Problem 19. The values of the integrals, which are clear from the geometric picture, can also be deduced rigorously by using the ideas in the proof of Problem 19, together with known integrals.)

8. (i)

$$\int_{-2}^{2}\left[\left(\frac{x^2}{2}+2\right) - x^2\right] dx = \frac{16}{3}.$$

(iii)

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} [(1-x^2) - x^2]\,dx = \frac{2\sqrt{2}}{3}.$$

(v)

$$\int_0^2 [(x^2 - 2x + 4) - x^2]\,dx = 4.$$

9.

$$\begin{aligned}\int_a^b \left(\int_c^d f(x)g(y)\,dy\right) dx &= \int_a^b \left(f(x)\int_c^d g(y)\,dy\right) dx \quad \text{(here } f(x) \text{ is the constant)}\\ &= \int_c^d g(y)\,dy \cdot \int_a^b f(x)\,dx\\ &\qquad \text{(here } \int_c^d g(y)\,dy \text{ is the constant).}\end{aligned}$$

13. (a) Clearly $L(f, P) \geq 0$ for every partition P.

(b) Apply part (a) to $f-g$, and use the fact that

$$\int_a^b (f-g) = \int_a^b f - \int_a^b g.$$

23. (a) Clearly

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

for all partitions P of $[a,b]$. Consequently,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Thus

$$\mu = \frac{\displaystyle\int_a^b f(x)\,dx}{b-a}$$

satisfies $m \le \mu \le M$.

(b) Let m and M be the minimum and maximum values of f on $[a,b]$. Since f is continuous, it takes on the values m and M, and consequently the number μ of part (a).

33. (a) 0.
(b) $\frac{1}{2}$.

37. Since

$$-|f| \le f \le |f|,$$

we have

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

so

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

(Problem 36 implies that $\displaystyle\int_a^b |f|$ makes sense.)

CHAPTER 14

1. (i) $(\sin^3 x^3)\cdot 3x^2$.

(iii) $\displaystyle\int_8^x \frac{1}{1+t^2+\sin^2 t}\,dt$.

(v) $\displaystyle\int_a^b \frac{1}{1+t^2+\sin^2 t}\,dt$.

(vii) $\displaystyle(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} = F^{-1}(x)$.

2. (i) All $x \ne 1$.
(iii) All $x \ne 1$.
(v) All x.
(vii) All $x \ne 0$. (F is not differentiable at 0 because $F(x)=0$ for $x \le 0$, but there are $x > 0$ arbitrarily close to 0 with $\frac{F(x)}{x} = \frac{1}{2}$.)

4. (i)

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{1+\sin(\sin(f^{-1}(0)))}$$
$$= \frac{1}{1+\sin(\sin 0)} = 1.$$

8. $F(x) = x\int_0^x f(t)\,dt$, so

$$F'(x) = xf(x) + \int_0^x f(t)\,dt.$$

11.

$$f(x) = \int_0^x \left(\int_0^y \left(\int_0^z \frac{1}{\sqrt{1+\sin^2 t}}\,dt\right) dz\right) dy.$$

13. We can choose

$$f(x) = \frac{x^{(1/n)+1}}{\dfrac{1}{n}+1}.$$

Then

$$\int_0^b \sqrt[n]{x}\,dx = f(b) - f(0) = \frac{b^{(1/n)+1}}{\dfrac{1}{n}+1}.$$

CHAPTER 15

1. (i)

$$\frac{1}{1+\arctan^2(\arctan x)} \cdot \frac{1}{1+\arctan^2 x} \cdot \frac{1}{1+x^2}.$$

(iii)

$$\frac{1}{1+(\tan x \arctan x)^2} \cdot \left(\sec^2 x \arctan x + \frac{\tan x}{1+x^2}\right).$$

2. (i) 0.
(iii) 0.
(v) 0.

7. (a)

$$\begin{aligned}
\sin 2x &= \sin(x+x) = \sin x\cos x + \cos x \sin x = 2\sin x\cos x.\\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.\\
\sin 3x &= \sin(2x+x) = \sin 2x\cos x + \cos 2x \sin x\\
&= 2\sin x\cos^2 x + (\cos^2 x - \sin^2 x)\sin x\\
&= 3\sin x\cos^2 x - \sin^3 x.\\
\cos 3x &= \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x\\
&\qquad = (\cos^2 x - \sin^2 x)\cos x = 2\sin^2 x\cos x\\
&\qquad = \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x\\
&\qquad = \cos^3 x - 3\sin^2 x\cos x\\
&\qquad = 4\cos^3 x - 3\cos x.
\end{aligned}$$

(b) Since $\cos \pi/4 > 0$ and

$$0 = \cos \frac{\pi}{2} = \cos 2 \cdot \frac{\pi}{4} = 2\cos^2 \frac{\pi}{4} - 1,$$

we have $\cos \pi/4 = \sqrt{2}/2$. It follows, since $\sin \pi/4 > 0$ and $\sin^2 + \cos^2 = 1$, that $\sin \pi/4 = \sqrt{2}/2$, and consequently $\tan \pi/4 = 1$. Similarly, since $\cos \pi/6 > 0$ and

$$0 = \cos \frac{\pi}{2} = \cos 3 \cdot \frac{\pi}{6} = 4\cos^3 \frac{\pi}{6} - 3\cos \frac{\pi}{6},$$

we have $\cos \pi/6 = \sqrt{3}/2$. It follows, since $\sin \pi/6 > 0$, that $\sin \pi/6 = \sqrt{1 - (\sqrt{3}/2)^2} = \frac{1}{2}$.

9. (a)

$$\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} \\
&= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} \\
&= \frac{\tan x + \tan y}{1 - \tan x \tan y}.
\end{aligned}$$

(b) From part (a) we have

$$\begin{aligned}
\tan(\arctan x + \arctan y) &= \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} \\
&= \frac{x+y}{1-xy},
\end{aligned}$$

provided that $\arctan x$, $\arctan y$, and $\arctan x + \arctan y \neq k\pi + \pi/2$. Since $-\pi/2 < \arctan x, \arctan y < \pi/2$, this is always the case except when $\arctan x + \arctan y = \pm\pi/2$, which is equivalent to $xy = 1$. From this equation we can conclude that

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

provided that $\arctan x + \arctan y$ lies in $(-\pi/2, \pi/2)$, which is true whenever $xy < 1$. (If $x, y > 0$ and $xy > 1$, so that $\arctan x + \arctan y > \pi/2$, then we must add π to the right side, and if $x, y < 0$ and $xy > 1$, so that $\arctan x + \arctan y < -\pi/2$, then we must subtract π.)

11. The first formula is derived by subtracting the second of the following two equations from the first:

$$\begin{aligned}\cos(m-n)x &= \cos(mx-nx) = \cos mx\cos(-nx) - \sin mx\sin(-nx)\\ &= \cos mx\cos nx + \sin mx\sin nx,\\ \cos(m+n)x &= \cos mx\cos nx - \sin mx\sin nx.\end{aligned}$$

The other formulas are derived similarly.

12. It follows from Problem 11 that if $m \neq n$, then

$$\begin{aligned}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x - \cos(m+n)x]\,dx\\ &= \frac{1}{2}\left\{\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n}\right]\right.\\ &\qquad\left.-\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n}\right]\right\}\\ &= 0.\end{aligned}$$

(Note that $\sin(m-n)(-\pi) = \sin(m-n)\pi$ since $m-n$ is an integer.) But if $m = n$, then

$$\begin{aligned}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}1-\cos(m+n)x\,dx\\ &= \frac{1}{2}\left\{\left[\pi - \frac{\sin(m+n)\pi}{m+n}\right] - \left[-\pi - \frac{\sin(m+n)\pi}{m+n}\right]\right\}\\ &= \pi.\end{aligned}$$

The other formulas are proved similarly.

15. (a) We have

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1.\end{aligned}$$

So

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2},\\ \cos^2 x &= \frac{1+\cos 2x}{2}.\end{aligned}$$

(b) These formulas follow from part (a), because $\cos x/2 \geq 0$ and $\sin x/2 \geq 0$ (since $0 \leq x \leq \pi/2$).

(c)

$$\int_a^b \sin^2 x\,dx = \int_a^b \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}(b-a) - \frac{1}{4}(\sin 2b - \sin 2a).$$

$$\int_a^b \cos^2 x\,dx = \int_a^b \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}(b-a) + \frac{1}{4}(\sin 2b - \sin 2a).$$

19. (a) $\arctan 1 - \arctan 0 = \pi/4.$

    (b) $\lim\limits_{x\to\infty} \arctan x - \arctan 0 = \pi/2.$

20. $\lim\limits_{x\to\infty} x \sin\dfrac{1}{x} = \lim\limits_{x\to 0^+} \dfrac{1}{x}\sin x = 1.$

21. (a)

$$(\sin^\circ)'(x) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos^\circ(x).$$
$$(\cos^\circ)'(x) = \frac{\pi}{180}\cdot -\sin\left(\frac{\pi x}{180}\right) = \frac{-\pi}{180}\sin^\circ(x).$$

(b) $\lim\limits_{x\to 0}\dfrac{\sin^\circ x}{x} = \lim\limits_{x\to 0}\dfrac{\sin(\pi x/180)}{x} = \lim\limits_{x\to 0}\dfrac{\pi}{180}\cdot\dfrac{\sin(\pi x/180)}{\pi x/180} = \dfrac{\pi}{180}.$

$\lim\limits_{x\to\infty} x\sin^\circ\dfrac{1}{x} = \lim\limits_{x\to 0^+}\dfrac{\sin^\circ x}{x} = \dfrac{\pi}{180}.$

CHAPTER 18

1. (i) $e^{e^{e^{e^x}}}\cdot e^{e^{e^x}}\cdot e^{e^x}\cdot e^x.$

   (iii) $(\sin x)^{\sin(\sin x)}[(\log(\sin x))\cdot\cos(\sin x)\cdot\cos x$
   $+ (\cos x/\sin x)\cdot\sin(\sin x)]\,.$

   (v) $(\sin x)^{(\sin x)^{\sin x}}\left[(\sin x)^{\sin x}\cdot\log(\sin x)\left\{\cos x\cdot\log(\sin x) + \sin x\dfrac{\cos x}{\sin x}\right\}\right.$
   $\left. + (\sin x)^{\sin x}\cdot\dfrac{\cos x}{\sin x}\right].$

   (vii) $\left[\arcsin\left(\dfrac{x}{\sin x}\right)\right]^{\log(\sin e^x)}\left[\left(\log\left(\arcsin\left(\dfrac{x}{\sin x}\right)\right)\right)\cdot\dfrac{(\cos e^x)e^x}{\sin e^x}\right.$
   $\left. + \log(\sin e^x)\cdot\dfrac{\sin x - x\cos x}{\arcsin\left(\dfrac{x}{\sin x}\right)\sqrt{1-\left(\dfrac{x}{\sin x}\right)^2}\cdot\sin^2 x}\right].$

   (ix) $(\log x)^{\log x}\cdot\left[\log(\log x)\cdot\dfrac{1}{x} + \log x\cdot\dfrac{1}{\log x}\cdot\dfrac{1}{x}\right].$

   (xi) $\cos\left(x^{\sin(x^{\sin x})}\right)\cdot x^{\sin(x^{\sin x})}\cdot\left[\cos\left(x^{\sin x}\right)\cdot x^{\sin x}\cdot\log x\right.$
   $\left.\left\{\cos x\cdot\log x + \dfrac{\cos x}{x}\right\} + \dfrac{\sin(x^{\sin x})}{x}\right].$

5. (i) 0.

   (iii) $\frac{1}{6}$.

   (v) $\frac{1}{3}$.

8. (a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \left[\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}\right] - \left[\frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}\right]$$
$$= 1.$$

(c)
$$\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$
$$= \left[\frac{e^{x+y}}{4} - \frac{e^{-x-y}}{4} - \frac{e^{-x+y}}{4} + \frac{e^{x-y}}{4}\right] + \left[\frac{e^{x+y}}{4} - \frac{e^{-x-y}}{4} + \frac{e^{-x+y}}{4} - \frac{e^{x-y}}{4}\right]$$
$$= \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y).$$

(e) Since
$$\sinh x = \frac{e^x + e^{-x}}{2},$$
we have
$$\sinh'(x) = \frac{e^x - e^{-x}}{2} = \cosh x.$$

(g) Since
$$\tanh x = \frac{\sinh x}{\cosh x},$$
we have
$$\tanh'(x) = \frac{(\cosh x)^2 - (\sinh x)^2}{\cosh^2 x}$$
$$= \frac{1}{\cosh^2 x} \quad \text{by part (a).}$$

9. (a) If $y = \cosh^{-1} x$, then $y \geq 0$ and
$$x = \cosh y = \sqrt{1 + \sinh^2 y} \quad \text{by Problem 7(a).}$$
So
$$\sinh(\cosh^{-1} x) = \sinh y = \sqrt{x^2 - 1} \quad \text{since } \sinh y \geq 0 \text{ for } y \geq 0.$$

(c)
$$(\sinh^{-1})'(x) = \frac{1}{\sinh'(\sinh^{-1}(x))}$$
$$= \frac{1}{\cosh(\sinh^{-1}(x))}$$
$$= \frac{1}{\sqrt{1 + x^2}} \quad \text{by part (b).}$$

(e)

$$\begin{aligned}(\tanh^{-1})'(x) &= \frac{1}{\tanh'(\tanh^{-1}(x))},\\ &= \cosh^2(\tanh^{-1}(x)).\end{aligned}$$

Now,

$$\tanh^2 y + \frac{1}{\cosh^2 y} = 1 \quad \text{by Problem 8(b)},$$

so

$$\tanh^2(\tanh^{-1}(x)) + \frac{1}{\cosh^2(\tanh^{-1}(x))} = 1,$$

or

$$\cosh^2(\tanh^{-1}(x)) = \frac{1}{1-x^2}.$$

10. (a) If $y = \sinh^{-1} x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so

$$\begin{aligned} e^y - e^{-y} &= 2x,\\ e^{2y} - 2xe^y - 1 &= 0,\\ e^y &= \frac{2x \pm \sqrt{4x^2+4}}{2}\end{aligned}$$

so

$$e^y = x + \sqrt{1+x^2} \quad \text{since } e^y > 0$$

or

$$y = \sinh^{-1} x = \log(x + \sqrt{1+x^2}).$$

Similarly,

$$\begin{aligned}\cosh^{-1} x &= \log(x + \sqrt{x^2-1}),\\ \tanh^{-1} x &= \tfrac{1}{2}\log(1+x) - \tfrac{1}{2}\log(1-x).\end{aligned}$$

(b)

$$\begin{aligned}\int_a^b \frac{1}{\sqrt{1+x^2}}\,dx &= \sinh^{-1} b - \sinh^{-1} a \quad \text{by Problem 9(c)}\\ &= \log(b+\sqrt{1+b^2}) - \log(a+\sqrt{1+a^2}).\end{aligned}$$

$$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx = \begin{cases} \log(b+\sqrt{b^2-1}) - \log(a+\sqrt{a^2-1}) & a,b>1\\ -\log(-b+\sqrt{b^2-1}) + \log(-a+\sqrt{a^2-1}) & a,b<-1\end{cases}$$

$$\int_a^b \frac{1}{1-x^2}\,dx = \frac{1}{2}[\log(1+b) - \log(1-b) - \log(1+a) + \log(1-a)].$$

13. (a) $\lim_{x\to\infty} a^x = \lim_{x\to\infty} e^{x\log a}$. Since $\log a < 0$, we have $\lim_{x\to\infty} x\log a = -\infty$, so $\lim_{x\to\infty} e^{x\log a} = 0$.

(c) $\lim_{x\to\infty} \dfrac{(\log x)^n}{x} = \lim_{y\to\infty} \dfrac{y^n}{e^y} = 0$.

(e) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x}$. Now, $\lim_{x\to 0^+} x\log x = 0$ by part (d), so $\lim_{x\to 0^+} x^x = 1$.

17. (a) $\lim_{y\to 0} \log(1+y)/y = \log'(1) = 1$.

(b) $\lim_{x\to\infty} x\log(1+1/x) = \lim_{y\to 0^+} \log(1+y)/y = 1$.

(c)

$$
\begin{aligned}
e = \exp(1) &= \exp(\lim_{x\to\infty} x\log(1+1/x)) \\
(*) &= \lim_{x\to\infty} \exp(x\log(1+1/x)) \\
&= \lim_{x\to\infty} (1+1/x)^x.
\end{aligned}
$$

(The starred equality depends on the continuity of exp at 1, and can be justified as follows. For every $\varepsilon > 0$ there is some $\delta > 0$ such that $|e - \exp y| < \varepsilon$ for $|y-1| < \delta$. Moreover, there is some N such that $|x\log(1+1/x) - 1| < \delta$ for $x > N$. So $|e - \exp(x\log(1+1/x))| < \varepsilon$ for $x > N$.

(d)

$$
\begin{aligned}
e^a &= [\lim_{x\to\infty} (1+1/x)^x]^a = \lim_{x\to\infty} (1+1/x)^{ax} \\
&= \lim_{ax\to\infty} (1+1/x)^{ax} \\
&= \lim_{y\to\infty} (1+a/y)^y.
\end{aligned}
$$

19. After one year the number of dollars yielded by an initial investment of one dollar will be

$$\lim_{x\to\infty} (1+a/100x)^x = e^{a/100}.$$

20. (a) Clearly $f'(x) = 1/x$ for $x > 0$. If $x < 0$, then $f(x) = \log(-x)$, so $f'(x) = (-1)\cdot 1/(-x) = 1/x$.

(b) We can write $\log|f|$ as $g\circ f$ where $g(x) = \log|x|$ is the function of part (a). So $(\log|f|)' = (g'\circ f)\cdot f' = 1/f\cdot f'$.

21. (c) Let $g(x) = f(x)/e^{cx}$. Then

$$g'(x) = \frac{e^{cx}f'(x) - f(x)ce^{cx}}{e^{2cx}} = 0,$$

so there is some number k such that $g(x) = k$ for all x.

22. (a) According to Problem 21, there is some k such that $A(t) = ke^{ct}$. Then $k = ke^{0\cdot t} = A_0$. So $A(t) = A_0e^{ct}$.

(b) If $A(t+\tau) = A(t)/2$, then

$$A_0e^{ct+c\tau} = \frac{A_0e^{ct}}{2},$$

so $e^{ct}e^{c\tau} = e^{ct}/2$ or $e^{c\tau} = \frac{1}{2}$, so $\tau = -(\log 2)/c$. It is easy to check that this τ does work.

23. Newton's law states that, for a certain (positive) number c,

$$T'(t) = c(T - M),$$

which can be written

$$(T - M)' = c(T - M).$$

So by Problem 21 there is some number k such that

$$T(t) - M = ke^{ct},$$

and $k = ke^{0\cdot t} = T(0) - M = T_0 - M$. So $T(t) = M + (T_0 - M)e^{ct}$.

CHAPTER 19

1. (i) $(\sqrt[5]{x^3} + \sqrt[6]{x})/\sqrt{x} = x^{1/10} + x^{-1/3}$.

(ii) $\dfrac{1}{\sqrt{x-1} + \sqrt{x+1}} = \dfrac{\sqrt{x-1} - \sqrt{x+1}}{-2}$.

(iii) $(e^x + e^{2x} + e^{3x})/e^{4x} = e^{-3x} + e^{-2x} + e^{-x}$.

(iv) $a^x/b^x = (a/b)^x = e^{x\log(a/b)}$.

(v) $\tan^2 x = \sec^2 x - 1$.

(vi) $\dfrac{1}{a^2 + x^2} = \dfrac{1/a^2}{1 + \left(\dfrac{x}{a}\right)^2}$.

(vii) $\dfrac{1}{\sqrt{a^2 - x^2}} = \dfrac{1/a}{\sqrt{1 - (x/a)^2}}$.

(viii) $\dfrac{1}{1 + \sin x} = \dfrac{1 - \sin x}{1 - \sin^2 x} = \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$.

(ix) $\dfrac{8x^2 + 6x + 4}{x + 1} = 8x - 2 + \dfrac{6}{x + 1}$.

(x) $\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x-1)^2}}$.

2. (i) $-\cos e^x$. (Let $u = e^x$.)

(iii) $(\log x)^2/2$. (Let $u = \log x$.)

(v) e^{e^x}. (Let $u = e^{e^x}$.)

(vii) $2e^{\sqrt{x}}$. (Let $u = \sqrt{x}$.)

(ix) $-(\log(\cos x))^2/2$. (Let $u = \log(\cos x)$.)

3. (i)
$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx = x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$$
$$= x^2e^x - 2xe^x + 2e^x.$$

(iii) We have

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx$$
$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} - \frac{b}{a}\int e^{ax}(-\sin bx)\,dx\right],$$

so

$$\int e^{ax}\sin bx\,dx = \frac{a}{a^2+b^2}e^{ax}\sin bx - \frac{b}{a^2+b^2}e^{ax}\cos bx.$$

(v) Using the result $\int (\log x)^2\,dx = x(\log x)^2 - 2x(\log x) + 2x$ from the text, we have

$$\begin{aligned}\int (\log x)^3\,dx &= [x(\log x)^2 - 2x(\log x) + 2x]\log x \\ &\qquad - \int \frac{1}{x}[x(\log x)^2 - 2x(\log x) + 2x]\,dx \\ &= x(\log x)^3 - 2x(\log x)^2 + 2x\log x \\ &\qquad - \int (\log x)^2\,dx + 2[x\log x - x] - 2x \\ &= x(\log x)^3 - 2x(\log x)^2 + 2x\log x \\ &\qquad - [x(\log x)^2 - 2x(\log x) + 2x] + 2[x\log x - x] - 2x \\ &= x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x.\end{aligned}$$

(vii)

$$\begin{aligned}\int \sec^3 x\,dx = \int (\sec^2 x)(\sec x)\,dx &= \tan x\sec x - \int (\tan x)(\sec x\tan x)\,dx \\ &= \tan x\sec x - \int \sec x(\sec^2 x - 1)\,dx \\ &= \tan x\sec x - \int \sec^3 x\,dx + \int \sec x\,dx,\end{aligned}$$

so

$$\int \sec^3 x\,dx = \tfrac{1}{2}[\tan x\sec x + \log(\sec x + \tan x)].$$

(ix)

$$\begin{aligned}\int \sqrt{x}\log x\,dx &= \frac{2x^{3/2}}{3}\log x - \frac{2}{3}\int x^{3/2}\cdot\frac{1}{x}\,dx \\ &= \frac{2x^{3/2}}{3}\log x - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2x^{3/2}}{3}\log x - \frac{4}{9}x^{3/2}.\end{aligned}$$

4. (i) Let $x = \sin u$, $dx = \cos u\,du$. The integral becomes

$$\int \frac{\cos u\,du}{\sqrt{1-\sin^2 u}} = \int 1\,du = u = \arcsin x.$$

(iii) Let $x = \sec u$, $dx = \sec u \tan u\, du$. The integral becomes

$$\int \frac{\sec u \tan u\, du}{\sqrt{\sec^2 u - 1}} = \int \sec u\, du = \log(\sec u + \tan u)$$
$$= \log(x + \sqrt{x^2 - 1}).$$

(v) Let $x = \sin u$, $dx = \cos u\, du$. The integral becomes

$$\int \frac{\cos u\, du}{\sin u \sqrt{1 - \sin^2 u}} = \int \csc u\, du = -\log(\csc u + \cot u)$$
$$= -\log\left(\frac{1}{x} + \frac{\sqrt{1 - x^2}}{x}\right).$$

(vii) Let $x = \sin u$, $dx = \cos u\, du$. The integral becomes

$$\int (\sin^3 u \cos u) \cos u\, du = \int \sin^3 u \cos^2 u\, du = \int (\sin u)(1 - \cos^2 u) \cos^2 u\, du$$
$$= \int (\sin u)(\cos^2 u - \cos^4 u)\, du = -\frac{\cos^3 u}{3} + \frac{\cos^5 u}{5}$$
$$= -\frac{(1 - x^2)^{3/2}}{3} + \frac{(1 - x^2)^{5/2}}{5}.$$

(ix) Let $x = \tan u$, $dx = \sec^2 u\, du$. The integral becomes

$$\int \sec u \sec^2 u\, du = \int \sec^3 u\, du$$
$$= \tfrac{1}{2}[\tan u \sec u + \log(\sec u + \tan u)] \quad \text{by Problem 3(vii)}$$
$$= \tfrac{1}{2}[x\sqrt{1 + x^2} + \log(x + \sqrt{1 + x^2})].$$

5. (i) Let $u = \sqrt{x + 1}$, $x = u^2 - 1$, $dx = 2u\, du$. The integral becomes

$$\int \frac{2u\, du}{1 + u} = \int \left(2 + \frac{-2}{1 + u}\right) du$$
$$= 2u - 2\log(1 + u) = 2\sqrt{x + 1} - 2\log(1 + \sqrt{x + 1}).$$

(iii) Let $u = x^{1/6}$, $x = u^6$, $dx = 6u^5\, du$. The integral becomes

$$\int \frac{6u^5\, du}{u^3 + u^2} = 6\int \left(u^2 - u + 1 - \frac{1}{u + 1}\right) du = 2u^3 - 3u^2 + 6u - 6\log(u + 1)$$
$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log(\sqrt[6]{x} + 1).$$

(v) Let $u = \tan x$, $x = \arctan u$, $dx = du/(1+u^2)$. The integral becomes

$$\begin{aligned}\int \frac{du}{(1+u^2)(2+u)} &= \frac{1}{5}\int\left(\frac{1}{2+u} - \frac{u-2}{1+u^2}\right) du\\ &= \frac{1}{5}\int \frac{du}{2+u} - \frac{1}{10}\int \frac{2u}{1+u^2}\,du + \frac{2}{5}\int \frac{du}{1+u^2}\\ &= \frac{1}{5}\log(2+u) - \frac{1}{10}\log(1+u^2) + \frac{2}{5}\arctan u\\ &= \frac{1}{5}\log(2+\tan x) - \frac{1}{10}\log(1+\tan^2 x) + \frac{2}{5}x.\end{aligned}$$

(vii) Let $u = 2^x$, $x = (\log u)/(\log 2)$, $dx = du/(u\log 2)$. The integral becomes

$$\begin{aligned}\frac{1}{\log 2}\int \frac{u^2+1}{(u+1)u}\,du &= \frac{1}{\log 2}\int\left(1 + \frac{1-u}{u(u+1)}\right) du\\ &= \frac{1}{\log 2}\int\left(1 + \frac{1}{u} - \frac{2}{u+1}\right) du\\ &= \frac{1}{\log 2}[u + \log u - 2\log(u+1)]\\ &= \frac{1}{\log 2}[2^x + x\log 2 - 2\log(2^x+1)].\end{aligned}$$

(ix) Let $u = \sqrt{x}$, $x = u^2$, $dx = 2u$. The integral becomes

$$\int \frac{\sqrt{1-u^2}\,2u\,du}{1-u}.$$

Now let $u = \sin y$, $du = \cos y\,dy$. The integral becomes

$$\begin{aligned}\int \frac{2\cos y \sin y \cos y}{1-\sin y}\,dy &= 2\int \frac{(1-\sin^2 y)\sin y}{1-\sin y}\,dy\\ &= 2\int (1+\sin y)\sin y\,dy\\ &= 2\int \sin y\,dy + \int 1 - \cos 2y\,dy\\ &= -2\cos y + y - \frac{\sin 2y}{2} = -2\cos y + y - \sin y\cos y\\ &= -2\sqrt{1-u^2} + \arcsin u - u\sqrt{1-u^2}\\ &= -2\sqrt{1-x} + \arcsin\sqrt{x} - \sqrt{x}\sqrt{1-x}.\end{aligned}$$

The substitution $u = \sqrt{1-x}$, $x = 1-u^2$, $dx = -2u\,du$ leads to

$$\int \frac{-2u^2\,du}{1-\sqrt{1-u^2}}$$

and the substitution $u = \sin y$ then leads to

$$\begin{aligned}\int \frac{-2\sin^2 y \cos y\, dy}{1-\cos y} &= -2\sin y - y - \sin y \cos y\\ &= -2u - \arcsin u - u\sqrt{1-u^2}\\ &= -2\sqrt{1-x} - \arcsin\sqrt{1-x} - \sqrt{1-x}\sqrt{x}.\end{aligned}$$

These answers agree, since

$$\arcsin\sqrt{x} = \frac{\pi}{2} - \arcsin\sqrt{1-x}$$

(check this by comparing their derivatives and their values for $x = 0$).

6. In these problems I will denote the original integral.

(i)

$$\begin{aligned}I &= \int \frac{2}{x-1}\,dx + \int \frac{3}{(x+1)^2}\,dx\\ &= 2\log(x-1) - \frac{3}{x+1}.\end{aligned}$$

(iii)

$$\begin{aligned}I &= \int \frac{1}{(x-1)^2}\,dx + \int \frac{4}{(x+1)^3}\,dx\\ &= -\frac{1}{(x-1)} - \frac{2}{(x+1)^2}.\end{aligned}$$

(v)

$$\begin{aligned}I &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx\\ &= \tfrac{1}{2}\log(x^2+1) + 4\arctan x.\end{aligned}$$

(vii)

$$\begin{aligned}I &= \int \frac{1}{(x+1)}\,dx + \int \frac{2x}{(x^2+x+1)}\,dx\\ &= \int \frac{1}{x+1}\,dx + \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{1}{x^2+x+1}\,dx.\end{aligned}$$

Now

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,dx &= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx\\ &= \frac{4}{3}\int \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}\,dx\\ &= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\\ &= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right),\end{aligned}$$

so

$$I = \log(x+1) + \log(x^2+x+1) - \frac{2\sqrt{3}}{3}\arctan\left(\tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right).$$

(ix)

$$\begin{aligned} I &= \int \frac{2x+1}{(x^2+x+1)^2}\,dx - \int \frac{1}{(x^2+x+1)^2}\,dx \\ &= \int \frac{2x+1}{(x^2+x+1)^2}\,dx - \frac{16}{9}\int \frac{1}{\left(\left[\tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right]^2+1\right)^2}\,dx. \end{aligned}$$

Now the substitution

$$u = \tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right), \qquad dx = \tfrac{\sqrt{3}}{2}\,du$$

changes the second integral to

$$-\frac{16}{9}\cdot\frac{\sqrt{3}}{2}\int \frac{du}{(u^2+1)^2}.$$

Using the reduction formula, this can be written

$$-\frac{8\sqrt{3}}{9}\left[\frac{u}{2(u^2+1)} + \frac{1}{2}\int \frac{du}{u^2+1}\right],$$

so

$$I = -\frac{1}{x^2+x+1} - \frac{\sqrt{3}(x+\tfrac{1}{2})}{4(x^2+x+1)} - \frac{4\sqrt{3}}{9}\arctan\left(\tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right).$$

14. The equation $\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$ means that any function F with $F'(x) = e^x \sin x$ can be written $F(x) = e^x \sin x - e^x \cos x - G(x)$ where G is another function with $G'(x) = e^x \sin x$. Of course, $G = F + c$ for some number c, but it is not necessarily true that $F = G$.

16. (a)

$$\begin{aligned} \int \arcsin x\,dx = \int 1\cdot \arcsin x\,dx &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= x\arcsin x + \sqrt{1-x^2}. \end{aligned}$$

17. (a)

$$\begin{aligned}\int \sin^4 x\,dx &= -\frac{\sin^3 x\cos x}{4}+\frac{3}{4}\int \sin^2 x\,dx\\ &= -\frac{\sin^3 x\cos x}{4}+\frac{3}{4}\left[-\frac{\sin x\cos x}{2}+\frac{1}{2}\int 1\,dx\right]\\ &= -\frac{\sin^3 x\cos x}{4}-\frac{3\sin x\cos x}{8}+\frac{3}{8}x.\end{aligned}$$

$$\begin{aligned}\int \sin^4 x\,dx &= \int\left(\frac{1-\cos 2x}{2}\right)^2 dx=\int\left(\frac{1}{4}-\frac{\cos 2x}{2}+\frac{\cos^2 2x}{4}\right)dx\\ &= \frac{x}{4}-\frac{\sin 2x}{4}+\frac{1}{4}\int\frac{1+\cos 4x}{2}\,dx\\ &= \frac{x}{4}-\frac{\sin 2x}{4}+\frac{1}{4}\left[\frac{x}{2}+\frac{\sin 4x}{8}\right]\\ &= \frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}.\end{aligned}$$

(b) It follows that these two answers are the same, since they have the same value for $x=0$.

21. (a)

$$\begin{aligned}\sin^n x\,dx &= \int(\sin x)(\sin^{n-1}x)\,dx\\ &= -\cos x\sin^{n-1}x+(n-1)\int \cos x(\sin^{n-2}x)\cos x\,dx\\ &= -\cos x\sin^{n-1}x+(n-1)\int(\sin^{n-2}x-\sin^n x)\,dx,\end{aligned}$$

so

$$\int \sin^n x\,dx=-\frac{1}{n}\cos x\sin^{n-1}x+\frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

(b)

$$\begin{aligned}\int \cos^n x\,dx &= \int(\cos x)(\cos^{n-1}x)\,dx\\ &= \sin x\cos^{n-1}x+(n-1)\int \sin x(\cos^{n-2}x)\sin x\,dx\\ &= \sin x\cos^{n-1}x+(n-1)\int(\cos^{n-2}x-\cos^n x)\,dx,\end{aligned}$$

so

$$\int \cos^n x\,dx=\frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

(c)

$$\begin{aligned}\int \frac{dx}{(x^2+1)^n} &= \int \frac{dx}{(x^2+1)^{n-1}} - \int \frac{x^2\,dx}{(x^2+1)^n}\\ &= \int \frac{dx}{(x^2+1)^{n-1}} - \int x \cdot \frac{x}{(x^2+1)^n}\,dx\\ &= \int \frac{dx}{(x^2+1)^{n-1}} - \left[\frac{x}{2(1-n)(x^2+1)^{n-1}}\right.\\ &\qquad \left. - \int \frac{dx}{2(1-n)(x^2+1)^{n-1}}\right]\end{aligned}$$

so

$$\int \frac{dx}{(x^2+1)^n} = \frac{1}{2(n-1)}\,\frac{x}{(x^2+1)^{n-1}} - \frac{2n-3}{2(n-1)}\int \frac{1}{(x^2+1)^{n-1}}\,dx.$$

We can also use the substitution $x = \tan u$, $dx = \sec^2 u\,du$, which changes the integral to

$$\begin{aligned}\int \frac{\sec^2 u\,du}{\sec^{2n} u} &= \int \cos^{2n-2} u\,du\\ &= \frac{1}{2n-2}\cos^{2n-3} u \sin u + \frac{2n-3}{2n-2}\int \cos^{2n-4} u\,du\\ &= \frac{1}{2n-2} \cdot \frac{1}{(\sqrt{x^2+1}\,)^{2n-3}} \cdot \frac{x}{\sqrt{x^2+1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(x^2+1)^{n-1}}\\ &= \frac{1}{2(n-1)}\,\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(x^2+1)^{n-1}}.\end{aligned}$$

CHAPTER 20

1. (i) $P_{3,0}(x) = e + ex + ex^2 + (5e/3!)x^3$.

   (iii) $P_{2n,\pi/2}(x) = 1 - \dfrac{(x-\pi/2)^2}{2!} + \dfrac{(x-\pi/2)^4}{4!} - \cdots + \dfrac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}$.

   (v) $P_{n,1}(x) = e + e(x-1) + \dfrac{e(x-1)^2}{2!} + \cdots + \dfrac{e(x-1)^n}{n!}$.

   (vii) $P_{4,0}(x) = x + x^3$.

   (ix) $P_{2n+1,0}(x) = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}$.

2. If f is a polynomial function of degree n, then $f^{(n+1)} = 0$. It follows from Taylor's Theorem that $R_{n,a}(x) = 0$, so $f(x) = P_{n,a}(x)$.

   (i) $-12 + 2(x-3) + (x-3)^2$.

   (iii) $243 + 405(x-3) + 270(x-3)^2 + 90(x-3)^3 + 15(x-3)^4 + (x-3)^5$.

3. (i) $\displaystyle\sum_{i=0}^{9} \frac{(-1)^i}{(2i+1)!}$ $\left(\text{since } \dfrac{1}{(2n+2)!} < 10^{-17} \text{ for } 2n+2 \geq 19, \text{ or } n \geq 9\right)$.

   (iii) $\displaystyle\sum_{i=0}^{8} \frac{(-1)^i}{2^i(2i+1)!}$ $\left(\text{since } \dfrac{1}{2^{2n+2}(2n+2)!} < 10^{-20} \text{ for } 2n+2 \geq 18, \text{ or } n \geq 8\right)$.

(v) $\displaystyle\sum_{i=0}^{13} \frac{2^i}{i!} \left(\text{since } \frac{3^2 \cdot 2^{n+1}}{(n+1)!} < 10^{-5} \text{ for } n+1 \geq 14, \text{ or } n \geq 13 \right).$

7. (i) $c_i = a_i + b_i$.
(iii) $c_i = (i+1)a_i$.
(v) $c_0 = \displaystyle\int_0^a f(t)\,dt$; $c_i = a_{i-1}/i$ for $i > 0$.

CHAPTER 22

1. (i) $1 - n/(n+1) = 1/(n+1) < \varepsilon$ for $n+1 > 1/\varepsilon$.
(iii) $\displaystyle\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = \lim_{n\to\infty} \left(\sqrt[8]{n^2+1} - \sqrt[8]{n^2}\right) + \lim_{n\to\infty} \left(\sqrt[4]{n} - \sqrt[4]{n+1}\right)$ $= 0 + 0 = 0$. (Each of these two limits can be proved in the same way that $\displaystyle\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$ was proved in the text.)
(v) Clearly $\displaystyle\lim_{n\to\infty} (\log a)/n = 0$. So $\displaystyle\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} e^{(\log a)/n} = e^0$ (by Theorem 1) $= 1$.
(vii) $\sqrt[n]{n^2} \leq \sqrt[n]{n^2+n} \leq \sqrt[n]{2n^2}$, so $(\sqrt[n]{n})^2 \leq \sqrt[n]{n^2+n} \leq \sqrt[n]{2}(\sqrt[n]{n})^2$, and $\displaystyle\lim_{n\to\infty} (\sqrt[n]{n})^2 = \lim_{n\to\infty} \sqrt[n]{2}(\sqrt[n]{n})^2 = 1$ by parts (v) and (vi).
(ix) Clearly $\alpha(n) \leq \log_2 n$, and $\displaystyle\lim_{n\to\infty} (\log_2 n)/n = 0$.

5. (a) If $0 < a < 2$, then $a^2 < 2a < 4$, so $a < \sqrt{2a} < 2$.
(b) Part (a) shows that
$$\sqrt{2} < \sqrt{2\sqrt{2}} < \sqrt{2\sqrt{2\sqrt{2}}} < \cdots < 2,$$
so the sequence converges by Theorem 2.
(c) If this sequence is denoted by $\{a_n\}$, then the sequence $\left\{\sqrt{2a_n}\right\}$ is the same as $\{a_{n+1}\}$. So the hint shows that $l = \sqrt{2l}$, or $l = 2$.

8. If x is rational, then $n!\pi x$ is a multiple of π for sufficiently large n, so $(\cos n!\pi x)^{2k} = 1$ for all such n, so $\displaystyle\lim_{n\to\infty}\left(\lim_{k\to\infty} (\cos n!\pi x)^{2k}\right) = 1$. If x is irrational, then $n!\pi x$ is not a multiple of π for any n, so $|\cos n!\pi x| < 1$, so $\displaystyle\lim_{k\to\infty} (\cos n!\pi x)^{2k} = 0$, so $f(x) = 0$.

9. (i) $\displaystyle\int_0^1 e^x\,dx = e - 1$. (Use partitions of $[0, 1]$ into n equal parts.)
(iii) $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \log 2$.
(v) $\displaystyle\int_0^1 \frac{1}{(1+x)^2}\,dx = \frac{1}{2}$.

CHAPTER 23

1. (i) (Absolutely) convergent, since $|(\sin n\theta)/n^2| \leq 1/n^2$.
(iii) Divergent, since the first $2n$ terms have sum $\frac{1}{2} + \cdots + 1/n$. (Leibniz's Theorem does not apply since the terms are not decreasing in absolute value.)

(v) Divergent, since

$$\frac{1}{\sqrt[3]{n^2-1}} \geq \frac{1}{n^{2/3}}.$$

(vii) Convergent, since

$$\lim_{n\to\infty} \frac{(n+1)^2/(n+1)!}{n^2/n!} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{n+1} = 0.$$

(ix) Divergent, since $1/(\log n) > 1/n$.

(xi) Convergent, since $1/(\log n)^n < \dfrac{1}{2^n}$ for $n > 9$.

(xiii) Divergent, since

$$\frac{n^2}{n^3+1} > \frac{1}{2n}$$

for large enough n.

(xv) Divergent, since

$$\int_2^N \frac{1}{x \log x}\, dx = \log(\log N) - \log(\log 2) \to \infty \text{ as } N \to \infty.$$

(Notice that $f(x) = 1/(x \log x)$ is decreasing on $[2, \infty)$, since

$$f'(x) = \frac{-[1 + \log x]}{(x \log x)^2} < 0 \qquad \text{for } x > 1.$$

(xvii) Convergent, since $1/n^2(\log n) < 1/n^2$ for $n > 2$.

(xix) Convergent, since

$$\begin{aligned} \lim_{n\to\infty} \frac{2^{n+1}(n+1)!/(n+1)^{n+1}}{2^n n!/n^n} &= \lim_{n\to\infty} \frac{2(n+1)n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} \frac{2}{\left(1+\dfrac{1}{n}\right)^n} = \frac{2}{e}, \end{aligned}$$

by Problem 18-17.

7. (a) For each N we clearly have

$$0 \leq \sum_{n=1}^{N} a_n 10^{-n} < 9 \sum_{n=1}^{\infty} 10^{-n} = 1,$$

so $\displaystyle\sum_{n=1}^{\infty} a_n 10^{-n}$ converges by the boundedness criterion, and lies between 0 and 1. (Actually, this number is denoted by $0.a_1a_2a_3a_4\ldots$ only when the sequence $\{a_n\}$ is not eventually 0.)

20. The area of the shaded region is $\frac{1}{2}$. The integral is

$$\begin{aligned}
&\tfrac{1}{2}([1-\tfrac{1}{2}]+[\tfrac{1}{4}-\tfrac{1}{8}]+[\tfrac{1}{16}-\tfrac{1}{32}]+\cdots)-\tfrac{1}{2}([\tfrac{1}{2}-\tfrac{1}{4}]+[\tfrac{1}{8}-\tfrac{1}{16}]+\cdots)\\
&\quad=\tfrac{1}{2}(\tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{32}+\tfrac{1}{128}+\cdots)-\tfrac{1}{2}(\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\tfrac{1}{256}+\cdots)\\
&\quad=\tfrac{1}{4}(1+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\cdots)-\tfrac{1}{8}(1+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\cdots)\\
&\quad=\frac{1}{8}\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right)\\
&\quad=\frac{1}{8}\cdot\frac{1}{1-\frac{1}{4}}\\
&\quad=\frac{1}{6}.
\end{aligned}$$

CHAPTER 24

1. (i)

$$f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}0, & x=0\\ 1, & 0<x\le 1.\end{cases}$$

$\{f_n\}$ does not converge uniformly to f.

(iii) $f(x)=\lim_{n\to\infty}f_n(x)=0$ (since $\lim_{n\to\infty}x^n=\infty$ for $x>1$). The sequence $\{f_n\}$ does not converge uniformly to f; in fact, for any n we have $f_n(x)$ large for sufficiently large x.

(v) $f(x)=\lim_{n\to\infty}f_n(x)=0$, and $\{f_n\}$ converges uniformly to f, since $|f_n(x)|\le 1/n$ for all x.

3. (i) $-\dfrac{1}{a}-\dfrac{x}{a^2}-\dfrac{x^2}{a^3}-\cdots.$

(iii) $\displaystyle\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}x^k.$

(v) $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\binom{-\frac{1}{2}}{k}}{2k+1}x^{2k+1}.$

4. (i) e^{-x}.

(iii) If

$$f(x)=\frac{x^2}{2}-\frac{x^3}{3\cdot 2}+\frac{x^4}{4\cdot 3}-\cdots,\qquad |x|\le 1$$

then

$$\begin{aligned}f'(x)&=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\\&=\log(1+x)\qquad |x|<1,\end{aligned}$$

so for $|x|<1$ we have $f(x)=(1+x)\log(1+x)-(1+x)+c$ for some number c. Since $f(0)=0$, we have $c=1$, so $f(x)=(1+x)\cdot\log(1+x)-x$ for $|x|<1$.

6. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

we have

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

(notice that the right side is 1 for $x = 0$). So

$$f^{(k)}(0) = \begin{cases} \dfrac{(-1)^n}{2n+1}, & k = 2n \\ 0, & k \text{ odd.} \end{cases}$$

CHAPTER 25

1. (i) $|3 + 4i| = 5$; $\theta = \arctan \frac{4}{3}$.
   (iii) $|(1+i)^5| = (|1+i|)^5 = (\sqrt{2})^5$; since $\pi/4 = \arctan 1/1$ is an argument for $1+i$, an argument for $(1+i)^5$ is $5\pi/4$.
   (v) $|(|3+4i|)| = |5| = 5$; $\theta = 0$.
2. (i)

$$\begin{aligned} x &= \frac{-i \pm \sqrt{-1-4}}{2} \\ &= \frac{-i \pm \sqrt{5}\,i}{2} \\ &= \frac{(-1+\sqrt{5})i}{2} \quad \text{or} \quad \frac{(-1-\sqrt{5})i}{2}. \end{aligned}$$

   (iii) $x^2 + 2ix - 1 = (x+i)^2$, so the only solution is $x = -i$.
   (v) $x^3 - x^2 - x - 2 = (x-2)(x^2+x+1)$. The solutions are

$$2, \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

3. (i) All $z = iy$ with y real.
   (iii) All z on the perpendicular bisector of the line segment between a and b.
   (v) For $z = x + iy$, we need $\sqrt{x^2+y^2} < 1 - x$. This requires that $1 - x > 0$, and then our inequality is equivalent to $x^2 + y^2 < (1-x)^2$, or $x < (1-y^2)/2$ (and conversely this inequality implies that $x < \frac{1}{2}$, so that $1 - x > 0$ holds). The set of points $x+iy$ with $x = (1-y^2)/2$ is the parabola pointing along the second axis, with the point $\frac{1}{2} + 0i$ closest to the origin, and which passes through the points $0+i$ and $0-i$; the desired set of complex numbers is the set of points inside this parabola.
4. $|x+iy|^2 = x^2 + y^2 = x^2 + (-y)^2 = |x - iy|^2$.
   $(z+\bar{z})/2 = [(x+iy) + (x-iy)]/2 = x$.
   $(z-\bar{z})/2 = [(x+iy) - (x-iy)]/2i = y$.
5. $|z+w|^2 + |z-w|^2 = (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w}) = 2z\bar{z} + 2w\bar{w} = 2(|z|^2 + |w|^2)$. Geometrically, this says that the sum of the squares of the diagonals of a parallelogram equal the sum of the squares of the sides.

CHAPTER 27

1. (i) Converges absolutely, since $|(1+i)^n/n!| = (\sqrt{2})^n/n!$, and $\sum_{n=1}^{\infty}(\sqrt{2})^n/n!$ converges.

   (iii) Converges, but not absolutely, since the real terms form the series
$$-\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{8} - \cdots$$
and the imaginary terms form the series
$$i(\tfrac{1}{1} - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots).$$

   (v) Diverges, since the real terms form the series
$$\frac{\log 3}{3} + 2\frac{\log 4}{4} + \frac{\log 5}{5} + \frac{\log 7}{7} + 2\frac{\log 8}{8} + \frac{\log 9}{9} + \cdots.$$

2. (i) The limit
$$\lim_{n\to\infty} \frac{|z|^{n+1}/(n+1)^2}{|z|^n/n^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 |z| = |z|$$
is < 1 for $|z| < 1$, but > 1 for $|z| > 1$.

   (iii) The limit
$$\lim_{n\to\infty} \frac{|z|^{n+1}}{|z|^n} = |z|$$
is < 1 for $|z| < 1$ but > 1 for $|z| > 1$.

   (v) The limit
$$\lim_{n\to\infty} \frac{2^{n+1}|z|^{(n+1)!}}{2^n|z|^{n!}} = \lim_{n\to\infty} 2|z|^{(n+1)!-n!}$$
is 0 for $|z| < 1$, but ∞ for $|z| > 1$.

3. (i) The limits
$$\lim_{n\to\infty} \sqrt[2n]{\frac{|z|^{2n}}{3^n}} = \frac{|z|}{\sqrt{3}} \qquad \text{and} \qquad \lim_{n\to\infty} \sqrt[2n+1]{\frac{|z|^{2n+1}}{2^{n+1}}} = \frac{|z|}{\sqrt{2}}$$
are < 1 for $|z| < \sqrt{2}$, so the series converges absolutely for $|z| < \sqrt{2}$. But the series does not converge absolutely for $|z| > \sqrt{2}$, so the radius of convergence is $\sqrt{2}$.

   (iii) Since
$$\lim_{n\to\infty} \sqrt[n]{\frac{n|z|^n}{2^n}} = \lim_{n\to\infty} \frac{|z|}{2}\sqrt[n]{n} = \frac{|z|}{2} \quad \text{by Problem 22-1(vi),}$$
the radius of convergence is 2.

   (v) The limit
$$\lim_{n\to\infty} \sqrt[n]{2^n z^{n!}} = 2 \lim_{n\to\infty} z^{(n-1)!}$$
is 0 for $|z| < 1$, but ∞ for $|z| > 1$, so the radius of convergence is 1.