PART I

THE APPROACH TO MATHEMATICS

CHAPTER 1

THE DREAD OF MATHEMATICS

'The greatest evil is fear.'
Epicurean Philosophy

THE main object of this book is to dispel the fear of mathematics. Many people regard mathematicians as a race apart, possessed of almost supernatural powers. While this is very flattering for successful mathematicians, it is very bad for those who, for one reason or another, are attempting to learn the subject.

Very many students feel that they will never be able to understand mathematics, but that they may learn enough to fool examiners into thinking they do. They are like a messenger who has to repeat a sentence in a language of which he is ignorant – full of anxiety to get the message delivered before memory fails, capable of making the most absurd mistakes in consequence.

It is clear that such study is a waste of time. Mathematical thinking is a tool. There is no point in acquiring it unless you mean to use it. It would be far better to spend time in physical exercise, which would at least promote health of body.

Further, it is extremely bad for human beings to acquire the habit of cowardice in any field. The ideal of mental health is to be ready to face any problem which life may bring – not to rush hastily, with averted eyes, past places where difficulties are found.

Why should such fear of mathematics be felt? Does it lie in the nature of the subject itself? Are great mathematicians essentially different from other people? Or does the fault lie mainly in the methods by which it is taught?

Quite certainly the cause does *not* lie in the nature of the subject itself. The most convincing proof of this is the fact that people in their everyday occupations – when they are making something – do, as a matter of fact, reason along lines *which are essentially the* *same as those used in mathematics:* but they are unconscious of this fact, and would be appalled if anyone suggested that they should take a course in mathematics. Illustrations of this will be given later.

The fear of mathematics is a tradition handed down from days when the majority of teachers knew little about human nature, and nothing at all about the nature of mathematics itself. What they did teach was an imitation.

Imitation Subjects

Nearly every subject has a shadow, or imitation. It would, I suppose, be quite possible to teach a deaf and dumb child to play the piano. When it played a wrong note, it would see the frown of its teacher, and try again. But it would obviously have no idea of what it was doing, or why anyone should devote hours to such an extraordinary exercise. It would have learnt an imitation of music. And it would fear the piano exactly as most students fear what is supposed to be mathematics.

What is true of music is also true of other subjects. One can learn imitation history – kings and dates, but not the slightest idea of the motives behind it all; imitation literature – stacks of notes of Shakespeare's phrases, and a complete destruction of the power to enjoy Shakespeare. Two students of law once provided a good illustration: one learnt by heart long lists of clauses; the other imagined himself to be a farmer, with wife and children, and he related everything to this farm. If he had to draw up a will, he would say, 'I must not forget to provide for Minnie's education, and something will have to be arranged about that mortgage.' One moved in a world of half-meaningless words; the other lived in the world of real things.

The danger of parrot-learning is illustrated by the famous howler, 'The abdomen contains the stomach and the bowels, which are A, E, I, O and U.' What image was in the mind of the child who wrote this? Large metal letters in the intestines? Or no image at all? Probably it had heard so many incomprehensible statements from the teacher, that the bowels being A, E, I, O and

U seemed no more mysterious than other things heard in school.

A large proportion of examination papers contain mathematical errors which are at least as absurd as this howler, and the reason is the same – words which convey no picture, the lack of realistic thinking.

Parrot-learning always involves this danger. The deaf child at the piano, whatever discord it may produce, remains unaware of it. Real education makes howlers impossible, but this is the least of its advantages. Much more important is the saving of unnecessary strain, the achievement of security and confidence in mind. It is far easier to learn the real subject properly, than to learn the imitation badly. And the real subject is interesting. So long as a subject seems dull, you can be sure that you are approaching it from the wrong angle. All discoveries, all great achievements, have been made by men who delighted in their work. And these men were normal, they were no freaks or high-brows. Edison felt compelled to make scientific experiments in just the same way that other boys feel compelled to mess about with motor bicycles or to make wireless sets. It is easy to see this in the case of great scientists, great engineers, great explorers. But it is equally true of all other subjects.

To master anything – from football to relativity – requires effort. But it does *not* require *unpleasant* effort, drudgery. The main task of any teacher is to make a subject interesting. If a child left school at ten, knowing nothing of detailed information, but knowing the pleasure that comes from agreeable music, from reading, from making things, from finding things out, it would be better off than a man who left university at twenty-two, full of facts but without any desire to inquire further into such dry domains. Right at the beginning of any course there should be painted a vivid picture of the benefits that can be expected from mastering the subject, and at every step there should be some appeal to curiosity or to interest which will make that step worth while.

Bad teaching is almost entirely responsible for the dislike which is shown in such words as 'high-brow'. Children want to know things, they want to do things. Teachers do not have to put life into them: the life is there, waiting for an outlet. All that is needed is to preserve and to direct its flow.

Too often, unfortunately, teaching seems to proceed on the philosophy that adults have to do dull jobs, and that children should get used to dull work as quickly as possible. The result is an entirely justified hatred and contempt for all kinds of learning and intellectual life.

Many members of the teaching profession are already in revolt against the tradition of dull education. Some excellent teaching has been heard over the wireless. The same ideas, the same methods are being developed independently in all parts of the country. No claim for originality is therefore made in respect of this book. It is no more than an individual expression of a feeling shared by thousands.

In the following chapters I shall try to show what mathematics is about, how mathematicians think, when mathematics can be of some use. In such a short space it is impossible to go into details. If you want to master any special department of mathematics, you will certainly need text-books. But most text-books contain vast masses of information, the object of which is not always obvious. It would be useless to burden your memory with all this purposeless information. It would be like having a hammer so heavy that you could not lift it. Mathematics is like a chest of tools: before studying the tools in detail, a good workman should know the object of each, when it is used, how it is used, what it is used for.

CHAPTER 2

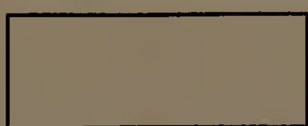
GEOMETRY – THE SCIENCE OF FURNITURE AND WALLS

'*So the Doctor buckled to his task again with renewed energy; to Euclid, Latin, grammar and fractions. Sam's good memory enabled him to make light of the grammar, and the fractions too were no* *great difficulty, but the Euclid was an awful trial. He could not make out what it was all about. He got on very well until he came nearly to the end of the first book and then getting among the parallelogram 'props' as we used to call them (may their fathers' graves be defiled!) he stuck dead. For a whole evening did he pore patiently over one of them till* AB, *setting to* CD, *crossed hands, pousetted and whirled round ' in Sahara waltz' through his throbbing head. Bed-time, but no rest! Who could sleep with that long-bodied ill-tempered looking parallelogram* AH *standing on the bedclothes, and crying out in tones loud enough to waken the house, that it never had been, nor ever would be equal to the fat jolly square* CK*?'*

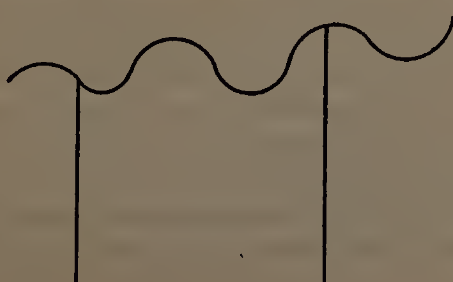
Henry Kingsley, *Geoffrey Hamlyn.*

IN the previous chapter it was mentioned that people, in their everyday life, used the same methods of reasoning as mathematicians, but that they did not realize this.

For instance, many people who would be paralysed if you said to them, 'Kindly explain to me the geometrical construction for a rectangle' would have no difficulty at all if you said, 'Please tell me a good way to make a table.' A 'rectangle' means the shape below –



and no one could make much of a table unless he understood well what this shape was. Suppose for instance you had a table like this



All the plates and tea-pots and milk-jugs would slide down into the hollows, or fall over, and altogether it would be very inconvenient. People who make tables are unanimous that the tops ought to be *straight*, not curved. Even if the top is straight, it may not be level; the table may look like this . And when the top is right, the legs may still look queer, such as or . In such cases the weight of the table-top would tend to break the joints. To avoid this, legs are usually made upright, and the table stands on the floor like this .

Anyone who understands what a table should look like understands what a rectangle is. You will find a lot about rectangles in books on geometry, because this shape is so important in practical life – though the older geometry books give no hint of this reason *why* we study rectangles.

Another craft which uses rectangles is bricklaying. An ordinary brick has a rectangle on top, below, at the ends and sides. Why? It is easy to guess. The bricks have to be laid level, if they are not to slide. (Even in making walls from rough stone, such as the Yorkshire dry walls, one tries to build with level layers.) So that the bricks must fit in between two level lines. But it would still be possible to have fancy shapes for the ends –

But this looks more like a jig-saw puzzle than a wall: the poor bricklayer would spend half his life looking for a brick that would fit. We want all the bricks to have the same shape. This can be done in several ways – /////// or ((((((. These would make ragged ends to the wall, and if two walls met there would be open spaces to fill. By having the ordinary shape of brick, all these complications are avoided.

No one will have any difficulty in following such an argument. Why, then, do people dislike geometry? Partly because it is a mystery to them: they do not realize (and are not told) how close it lies to everyday life. Secondly, because mathematics is supposed to be *perfect*. There is nothing in the geometry book about shapes being 'nearly triangles' or 'almost rectangles', while it is quite common for a door or table to be just a little out of true. This perfection puts people off. You can have several tries at making a table, and each attempt may be an improvement on the last. You learn as you go along. By insisting on 'mathematical exactness', it is easy to close this great road of advance, Trial and Error. If you remember how close geometry is to carpentry, you will not fall into this mistake. If you have a problem which puzzles you, the first thing to do is to try a few experiments: when you have found a method that seems to work, you may be able to find a logical, 'exact', 'perfect' justification for your method: you may be able to prove that it is right. But this perfection comes at the *end*: experiment comes at the beginning.

The first mathematicians, then, were practical men, carpenters and builders. This fact has left its mark on the very words used in the subject. What is a 'straight line'? If you look up 'straight' in the dictionary, you will find that it comes from the Old English word for 'stretched', while 'line' is the same word as 'linen', or 'linen thread'. A straight line, then, is a stretched linen thread – as anyone who is digging potatoes or laying bricks knows.

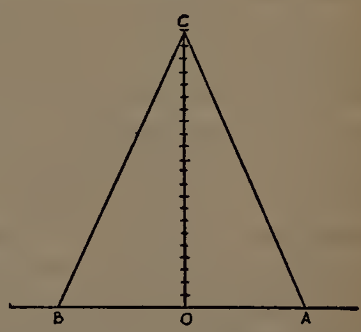
Euclid puts it rather differently. He says a straight line is the shortest distance between two points. But how do you find the shortest distance? If you take a tape-measure from one point to the other, and then pull one end as hard as you can, so that as little as possible of the tape-measure is left between the two points, you will have found the shortest path from one to the other. And the tape-measure will be 'stretched' in exactly the same way as the builder's or gardener's 'line'.

If you are told to define something, ask yourself, 'How would I make such a thing in practice?'

For instance, you might be asked to define a 'right angle'. A 'right angle' (in case the expression is new to you) means the figure formed when two lines meet as in a capital L, thus: |__. You will find a right angle at every corner of this sheet of paper. On the other hand, /__ and __/ are not right angles.

How would you make a right angle? Suppose you want to tear a sheet of note-paper into two neat halves: what do you do? You fold it over, and tear along the crease, which you know stands at the 'right' angle to the edge. If you fold it very carelessly, you do not get the 'right' angle, but something like __/__: too much paper is left on one side, too little on the other. We now see the special feature of a right angle – both sides of the crease *look the same*. If we had blots of ink on one side of the crease, we should get 'reflections' of these on the other when we unfolded the paper. The crease acts like a mirror. And the reflection of the edge of the paper – if we have the right angle – lies along the edge on the other side of the crease.

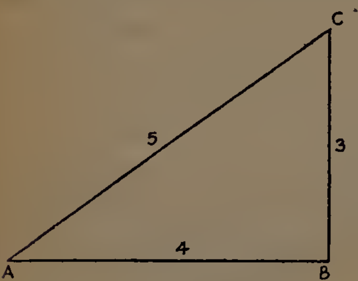
You can try this with a ruler or walking-stick. There is a position in which a stick can be held so that its reflection seems to be a continuation of the stick: you can look along the stick and its reflection, exactly as if you were squinting down the barrel of a rifle. The stick is then 'at right angles' to the mirror.



But suppose you are laying out a football field, and want to get a right angle. You cannot fold the touch-line over on to itself and notice where the crease comes! But this idea of a mirror shows a way of getting round the difficulty.

Suppose O is the point on the touch-line where you want to draw a line at right angles to the touch-line, OA. We know that a mirror, OC, in the correct position, would reflect the point A so that it appeared at B, also on the touch-line. If we folded the paper over the line OC, A would come on top of B. The line OA would just cover OB, and the line AC would just cover BC, after such folding.

But this suggests a way of finding the line OC. If we start at O, and measure OA, OB the same distance on opposite sides of O, we have A and its reflection, B. Since BC is the reflection of AC, both must be the same length. Take a rope of any convenient length, fasten one end to A, and walk round, scraping the other end of the rope in the ground. All the points on this 'scrape' will be a rope's length from A. Untie the rope from A, and fix it to B instead, and make another, similar scrape in the ground. Where the two scrapes cross, we have a point which is the same distance from B as it is from A. This will do for C. We drive a peg in here, stretch a line from C to O, and whitewash along it.



You can easily see how the above method, suitable for football fields, can be translated into a method for drawing right angles on paper with ruler and compass.

But there is another, very remarkable, way, which is actually used for marking out football fields.

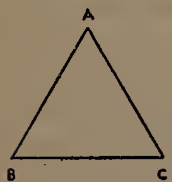
If you take three rods of lengths 3, 4, and 5 yards, and fit them together as shown in the figure, you will find that the angle at B turns out to be a right angle. No one could have guessed that this would be so. It seems to have been discovered about five thousand years ago, more or less by accident. It is not known who discovered it, but the discoverer was almost certainly someone engaged in the building trade – a workman or an architect. This way of making a right angle was used as part of the builder's craft: people did not ask why it was so, any more than a housewife asks why you use baking-powder. It was just known that you got good results if you used this method, and the Egyptians used it to make temples and pyramids with great success.

It is not known how far learned Egyptians bothered their heads trying to find an explanation of this fact, but certainly Greek travellers, who visited Egypt, found it a very intriguing and mysterious thing. Egyptian workmen saw nothing remarkable in it: if the Greeks asked them about it, they probably answered, 'Lor' bless you, it's always been done that way. How else would you do it?'

So the Greeks would go away still wondering, 'Why?' Why

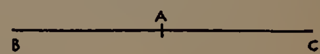
3, 4, and 5? Why not 7, 8, and 9? Anyhow, what does happen if you try 7, 8, and 9? Or any other three numbers?

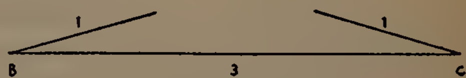
It would therefore be quite natural to start with fairly small numbers, and try making triangles, such as (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2), (2, 2, 2), etc. The Greeks had no Meccano – with Meccano it is easy to make such triangles quickly. How do they look?



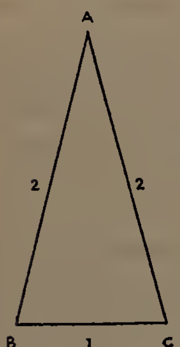
(1, 1, 1)

(1, 1, 2)

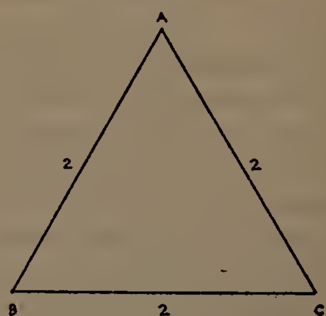




(1, 1, 3) cannot be done – sides will not meet



(1, 2, 2)



(2, 2, 2)

As soon as you start experimenting in this way, you begin to discover things. You sometimes find that it is impossible to make the triangle at all; e.g., (1, 1, 3), (1, 1, 4) and so on: in fact whenever one side (e. g., 3) is bigger than the other two sides (1 and 1) put together.

You may notice that doubling the sides of a triangle does not alter its shape: (2, 2, 2) looks much like (1, 1, 1).

Again the triangle (1, 2, 2) has a pleasing balanced appearance: if you turned it over, so that B and C changed places, it would still look just the same.

The more you experimented with drawing or making triangles, the more things you would notice about them. Not all these discoveries would be really new. For instance, we saw above that, in any triangle, AB plus AC must be bigger than BC. But this is not new. We know that the straight line BC is the shortest way from B to C, so naturally it is longer if we go from B to C via A, a distance equal to the sum of AB and AC. So that this particular result *could* have been found by reasoning: it follows from the fact that the straight line gives the shortest path between two points.

So that we can do two things in our study of the shapes of things. (i) We can discover a large number of facts. (ii) We can arrange them in a system, showing what follows from what.

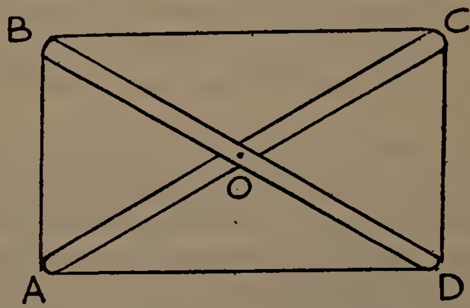
Actually, these two things were done by the Greeks, and by 300 B.C. Euclid had written his famous book on geometry, putting all the facts known into the form of a system. In this book you will find why (3, 4, 5) gives a right-angled triangle; and it is shown that other triangles, such as (5, 12, 13) or (24, 25, 7) or (33, 56, 65), do the same.

But all this took time. The Great Pyramid was built in 3900 B.C., by rules based on practical experience: Euclid's system did not appear until *3,600 years later*. It is quite unfair to expect children to start studying geometry in the form that Euclid gave it. One cannot leap 3,600 years of human effort so lightly! The best way to learn geometry is to follow the road which the human race originally followed: *Do* things, *make* things, *notice* things, *arrange* things, and only then – *reason* about things.

Above all, do not try to hurry. Mathematics, as you can see, does not advance rapidly. The important thing is to be sure that you know what you are talking about: to have a clear picture in your mind. Keep turning things over in your mind until you have a vivid realization of each idea. Once you have learnt how to think in clear pictures, you will advance quickly, without strain. But it is fatal to advance and to leave the enemy – confused thought – in your rear. Rather than this, start again at the multiplication tables!

SOME EXPERIMENTS CONNECTED WITH GEOMETRY

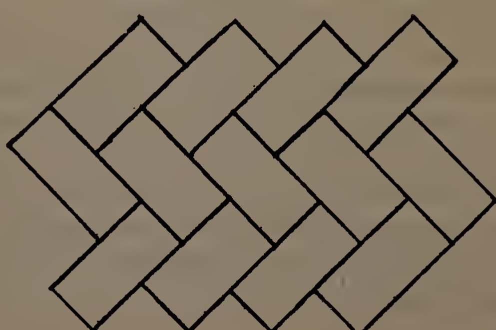
1. A boy has a strip of wood, AC, which is 4 feet long. He wishes to join a second strip to it, as in the figure, so that a string, ABCD, passed round the outside, will form a rectangle.



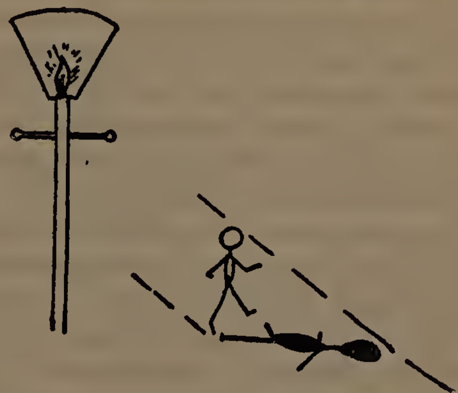
How long must he make the strip BD? At what point (O) must he nail the two strips together? Does it matter at what angle he places the two strips?

2. A flat piece of ground is to be covered with tiles. All the tiles must be the same shape and size, but it does not matter if there is a jagged edge on the outside border. Design as many different ways of doing this as you can. One example is shown in the drawing below.

3. A street lamp is 12 feet above the ground. A child, 3 feet in height, amuses itself by walking in such a way that the shadow of its head moves along lines chalked on the ground. How will it



walk if the chalked line is (i) straight, (ii) a circle, (iii) a square? What is the rule connecting the size and shape of the child's track with those of the chalked line? (*Note* – Do not quarrel with anyone about the answer to this, until you have actually made an experiment. A convenient form of experiment is to take an indoor lamp instead of the street lamp, and a pencil to represent the child. The pencil will record its own track as it moves.)



4. What difference would it make to the last question if the light came from the sun instead of from a lamp?

5. A man 6 feet tall stands at a distance of 10 feet from a lamp-post. The lamp is 12 feet above the ground. How long will the man's shadow be?

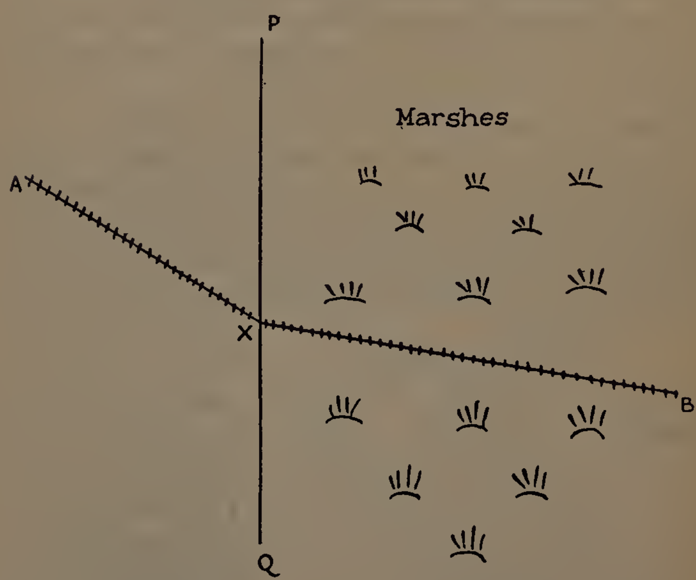
6. A hiker can see two church spires. One is straight in front of him. The other is directly to the left of him. He has a map on which the two churches are marked, but he has not the least idea of the direction in which he is facing, whether it is North or South, or any other point of the compass. What can he tell about where he is on the map? (Suggested method. Drive two nails into a flat piece of wood. Let these represent the churches. Take a piece of cardboard, one corner of which is a right angle, and fit this between the nails, as in the figure. Then P is a possible position for the hiker. For if he is

looking in the direction PA, B will be directly on his left. Mark the position P on the wood. Slide the cardboard, and mark other possible positions in the same way. These marks all lie on a certain curve. What is the curve?)

7. In a miniature rifle-range, 25 yards long, it is desired to construct a moving target, to represent a lorry, 20 feet long and 15 feet high, half-a-mile away, moving at 20 miles an hour. The marksman is supposed to be in such a position that he sees one side of the lorry. How large should the model be, and how quickly should it move across the screen?

8. A spider wishes to crawl from one corner of a brick, A, to the opposite corner, B, by the shortest possible way. Which path should it take? The spider of course crawls over the surface of the brick – it cannot burrow through the brick.

(Material required – a number of bricks, having different shapes, and a piece of string to stretch from A to B. It is useful to make the bricks out of cardboard, by folding. After the

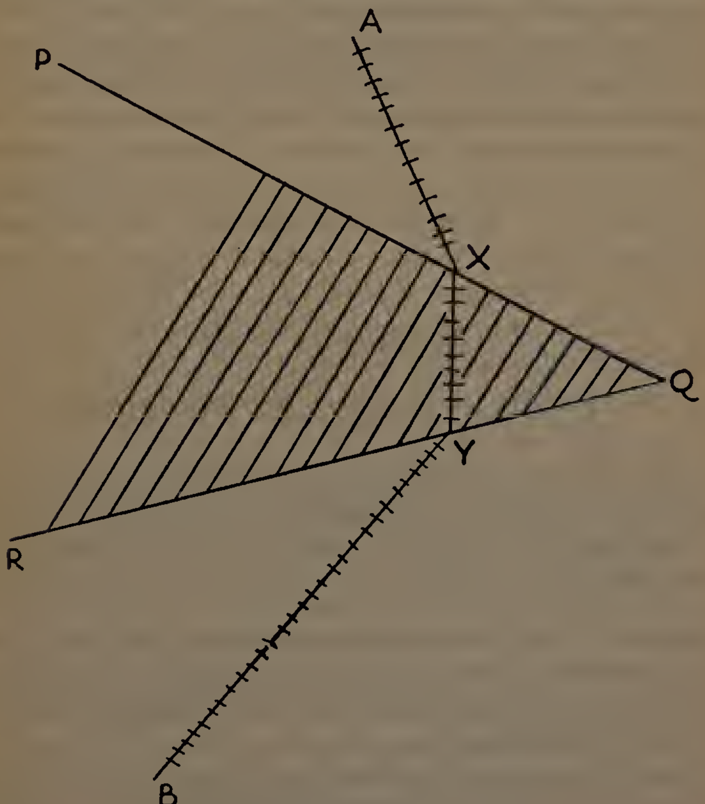


A to X, £10,000 a mile. X to B, £20,000 a mile.
Find the best position for X.

shortest path has been found and marked on the cardboard, the cardboard can again be flattened out, and the form the path then takes should be noted.)

9. Get a globe of the world. Stretch a thread between two places. Make a note of the places over which the thread passes. Mark these places on a map of the world in an atlas (Mercator's projection). Notice how different the curve joining them is from a straight line drawn on the map. This fact is important for sailors, and airmen flying long distances ('Great Circle Navigation').

10. A railway is to be built joining two towns, A and B. The ground to the right of the line PQ is marshy, and as a result it



A to X £10,000 a mile. X to Y, £20,000 a mile. Y to B, £10,000 a mile.
Find the best positions for X and Y.

costs twice as much to build one mile of railway here as it costs to build one mile on the ground to the left of PQ. Draw a number of possible routes for the railway from A to B, work out the costs of construction, and find, as nearly as you can, the route which makes the construction as cheap as possible. (See Fig., p. 20.)

(*Note* – You are *not* expected to answer this question purely by calculating. Draw a plan for yourself, put the towns A and B wherever you like, measure the lengths of any lines you want, assume that one mile of railway on the left of PQ costs £10,000. In real life, we want to get the answer by hook or by crook – by calculation, or by experiment, or by a mixture of the two. Never mind what Euclid said in 300 B.C.)

11. This question is similar to the last, except that the shape of the obstacle is different. A railway has to be built between A and B, but a wedge of difficult ground, PQR, lies between them. Find the best route for the railway. (This type of problem does occur in practice, when hilly country lies between towns. In this case, the extra expense would be due to the need to excavate cuttings.) (See Fig., p. 21.)

CHAPTER 3

THE NATURE OF REASONING

'*From my observations of men and boys I am inclined to think that my way of study is the common way, the natural way, and that the schoolmasters destroy it and replace it by something that conduces to mere learning.*' John Perry, 1901.

BERNARD SHAW once made an unkind remark to the effect that people who knew how to do anything went and did it; while those who did not know how to do anything were obliged to earn a living by teaching.

Actually, teaching is far harder than doing. You will find a hundred men who are brilliant footballers for every one that can teach you how to play the game well. You will find hundreds of clever children and hundreds of dull children, but you rarely find a child who was dull to begin with but became clever through the help of a teacher. But that is the test of a teacher. Most teachers who are honest with themselves are forced to admit that, in the main, the class would make the same progress if the teacher were not there at all, the clever remaining clever and the dull remaining dull.

Are there in fact two races of men, those who are born to succeed and those who are born to fail? Have the 'great men' some special way of thinking which ordinary people lack, and cannot understand?

There are, of course, certain differences between the bodies and brains which children inherit from their parents. There are cases of mental deficiency, where important glands fail to do their job, and the children have to be kept in special homes. It may be that glands, or other factors, set a natural limit to the powers of each one of us, and that it is foolish to strive for skill beyond this limit. This may be so. It is certain that not one person in a thousand makes full use of the glands and the brains actually possessed, or comes anywhere near the natural limit of intelligence. We certainly cannot explain it by glands if a person is bright and resourceful outside the classroom, and dull in everything connected with school. The reason must be sought elsewhere.

It is extremely interesting to inquire just what it is that the 'great man' or the successful performer does, that others fail to do. Just what qualities are necessary in order to play a game well, to be a painter, a musician, an engineer, a farmer or a mathematician? Can these qualities be developed by suitable exercises? Is it possible for an ordinary person, with determination, to acquire these qualities? When teachers are in a position to answer these questions, when everybody reaches the limit of his powers, the time will have come to talk about inherited differences in intelligence. But that will be a few centuries hence.

At present there are books which do really teach. It is better to spend hours searching a big library for such a book, rather than to read hundreds of books by second-rate authors. It is very unlikely that you or I possess any really original ideas. People seem to belong to certain types. If you feel strongly on any subject, the chances are that you will find some other individual has been concerned with exactly the same question, and you will find your own views worked out in his or her writings. You can then start to study the subject where that earlier worker left off.

One can often get help in teaching or learning a subject by reading books written on other subjects. In a library I came across a book, *Swimming for All*, by R. C. Venner (G. Bell & Sons, 1933). This book follows a method which could probably be applied to most other subjects. First the author explains the principles of swimming. He explains the difference between the movements which are needed in the water and those which we instinctively make as a result of living on dry land. Then he gives a series of experiments and exercises, by which one can convince oneself of the truth of his remarks, so that one not merely knows these facts, but comes to *feel* their truth and to do the right thing instinctively.

Writers on tennis make a remark which may serve as a parable. They say that you should not begin by trying to strike the ball into the court, but that you should begin by hitting it hard, and with a good style. Gradually you will find that the ball begins to land in the court. If you start by worrying about where the ball goes, you will always be a feeble player. Much the same holds in mathematics. The important thing is to learn how to strike out for yourself. Any mistakes you make can be corrected later. If you start by trying to be perfect, you will get nowhere. The road to perfection is by way of making mistakes.

This rather reminds me of a book on drawing, which I read several years ago. Unfortunately I did not make a note of the author's name.* He tried to teach drawing in such a way that his readers would be able to sit on the top of an omnibus and to record on paper the fleeting expressions of people's faces. According to him, you should use odd scraps of paper, and never alter a drawing. Throw it away if it goes wrong, and begin again. Do not bother if things are the right shape or not. Jot down

*Probably *Drawing for Children* by Vernon Blake. One remark quoted, however, is from L. Doust, *How to Sketch from Life.*

what you actually see, especially shadows. Do not draw lines unless you actually see them standing out. Regard your first drawings merely as sketches, noting what you actually see. Gradually you will find that the shapes of your drawings become more true to life, but even your earlier drawings, which have the wrong proportions, will suggest something solid and real. He gave very rough sketches to illustrate this fact. I know nothing about drawing, but if I wanted to learn, I should certainly learn this way.

In all subjects, it seems, there is a way of approach which is both interesting and encouraging. The 'great men' are often those who felt a strong interest in a subject and by accident, by experiment or through the influence of good teachers, hit on the correct approach. It is *ignorance of the way in which a subject is tackled* that causes geniuses to be regarded as a race apart. The more one studies the methods of the great, the more commonplace do these methods appear.

Very often anecdotes give us a false impression. There is a story about Newton and an apple: Newton saw an apple fall, and wondered *why* it fell – so we are told. It is extremely unlikely that Newton did anything of the sort. To this day we do not know *why* an apple falls. It is more likely that Newton thought rather as follows. What would happen if an apple were dropped from a very great height? Presumably it would still fall, however high, however far away from the earth it was taken. If not, there would be some height at which you suddenly found that the apple did not fall. This is possible, but not very likely. It seems probable, then, that, even if you went as far as the moon or the sun, you would still feel the pull of the earth, though perhaps not so strongly as you do here. Perhaps it is this pull which keeps the moon close to the earth, and keeps the earth circling round the sun? That at any rate was the conclusion to which the apple led Newton – that every piece of matter in the universe exerts a pull on every other piece, however far away it may be. No one denies the greatness of Newton. Equally, no one can say that there was anything superhuman about the type of argument he used.

Reasoning in Mathematics

Mathematics teaches us how to solve puzzles. Everyone knows that it is easy to do a puzzle if someone has told you the answer. That is simply a test of memory. You can claim to be a mathematician if, and only if, you feel that you will be able to solve a puzzle that neither you, nor anyone else, has studied before. That is the test of reasoning.

What exactly is this power of reasoning? Is it something separate from the other powers of our minds? Is it something fixed, or something that can be trained and encouraged? How do we come to possess such a power?

Mathematical reasoning does at first sight seem to be in a class by itself. It seems to find a place neither in the experimental sciences, nor in the creative arts.

Some subjects are clearly the result of experiment, or of experience. Chemistry deals with what happens when metals are dropped into liquids, or when the contents of one pot are mixed with those of another. Mechanics deals with the motion of solid objects. History records the actions of men. The study of languages deals with the words used by nations in different parts of the world. It is easy to see how the information contained in a book on chemistry, mechanics, history or French is obtained.

On the other hand there are the subjects (beloved by some people on which everybody disagrees. These are the subjects which do not depend on evidence at all – what you like, what you think ought to be done, the kind of person you admire, the political party you vote for: these are things for which you yourself take responsibility, they show what sort of person you are. You may be ready to fight to secure the type of world you think best: indeed, you should be. But you do not change your basic ideas of what is desirable as the result of argument and evidence. I suppose microbes have a vision of a world made safe for smallpox. We cannot *prove* that the world was not made for the benefit of microbes. All we can do is to use plenty of disinfectant.

Mathematics seems a peculiar subject. It is not a matter of taste. In it, more than in any other science, there is an answer which is right and an answer which is wrong. But, on the other hand, it does not seem to deal with anything definite. A great and important part of mathematics, for instance, deals with the square root of minus one – something which no one has ever seen or felt or tasted. Yet there is no sort of doubt about its properties.

In ancient times philosophers found it hard to explain man's powers of reasoning, and were led to more or less fantastic explanations. One such theory was that we lived in another world before we were born, and in that world were acquainted with the laws of arithmetic and geometry (how far the syllabus went I do not know). The object of education, in this world, was simply to awaken in us the memory of this knowledge.

One should not sneer at this ancient theory. It at least makes clear that education consists in co-operating with what is already inside a child's mind. A good teacher will nearly always be able to make his points simply by asking a class questions, by making the class realize clearly what they already know 'at the back of their minds'.

The clue which we possess today, which the old philosophers could hardly have guessed, comes from biology. It is now generally accepted that life has been on the earth for millions of years, and that we are born with instincts tested and tried in an age-long struggle for survival. On top of these instincts we have a training, given to us especially in the first five years of life, and based on traditions, some of which go back to the experience of thousands of years ago. By the time we are five years old we are, so to speak, a highly manufactured article, and it is generally after this age that we become aware of our ability to argue things out for ourselves.

If therefore we find in ourselves a strong desire to do a certain thing, or to believe a certain thing, it is at least possible that this desire exists in man, because it has enabled him to survive, because it has proved its worth in generations of struggle with the actual world. Those animals, those races survive, which *do* the correct thing. We should therefore expect human brains, human minds to be made on the whole in such a way as to produce *correct action* in any situation. We should *not* expect them to be made in such a way as to produce perfect, logical thought. In fact we often do find people doing the right thing for the wrong reason. A certain place is a centre of infection: savages know nothing of the causes of disease: they say that it is unlucky to go there, it is the abode of an evil spirit.

It was shown in Chapter 2 that geometry did, as a matter of fact, pass through the stage in which workmen *did* the right thing, but had no theory to explain why. Both geometry and arithmetic are closely connected with everyday life – geometry with building, arithmetic with money payment. If you give a tram-conductor threepence for two twopenny tickets, he is not prepared to believe that twice two being four is purely a university fad. He regards it as a fact, well established by the experience of everyday life.

It is now beginning to seem that mathematics is, like chemistry, something that we learn through experience of the real world. Some people will object strongly to this. They will say, 'I can imagine zinc being dropped into sulphuric acid and nothing happening. But can you imagine twice two being five?'

I certainly cannot imagine twice two being five. If a man claimed to perform miracles, and could turn twice two into five, I should give him full marks. It would impress me far more than any other miracle.

But that is not the point. The point is, *why* can we not imagine two twos making five?

There are two possible explanations. (i) We possess some mysterious faculty, given us in a previous life, or by other means. (ii) We cannot imagine twice two being five because in the whole of human history, twice two always has been four, and there has been no need for our minds to imagine it any other way.

The first explanation does not agree with the experience of most teachers. There may be people who possess a faculty of reasoning so perfect as to support this view: it would be interesting to know which schools and colleges they attend. In the works of the greatest mathematicians one finds evidence of blunders, of misunderstanding, of painful groping towards the truth.

Perhaps the greatest blow to the 'mysterious faculty' theory is the fact that mathematicians of today reject as untrue just those beliefs which earlier mathematicians most firmly believed. It was a custom at one time to say that some doctrine was true as certainly as the angles of a triangle added up to two right angles. If Einstein is right, the angles of a triangle do *not* add up to two right angles. Both the theory of relativity and quantum mechanics have destroyed long-cherished beliefs, and forced us to examine again the foundations of our belief.

If you accept Euclid's geometry because it agrees with what you can see of the shapes of things, you will not be unduly alarmed if someone suggests that Euclid may be wrong by a few millionths of an inch in certain places. For you cannot see a millionth of an inch, and Einstein's geometry only differs from Euclid's by millionths. But if you believe that Euclid represents absolute truth – then you are in a mess. Actually, Euclid himself only said, '*If* you admit certain things, then you must admit that the angles of a triangle add up to two right angles.'

On the other hand, there are interesting signs of the way in which human thought has been built up through daily experience. One such sign is to be found in the words we use. Try to imagine, if you can, a cave-man (or whoever it was that first developed language) trying to say to a friend, 'What this writer says about the square root of minus one does not agree with my philosophy at all.' How would he manage to make his friend understand what he meant by such abstract words as 'philosophy', 'minus one', 'agree', and so forth? Every child, in learning to speak, is faced by the same problem. How does it ever come to know the meaning of words, apart from the names of people and objects it can see?

It is instructive to take a dictionary, and to look up such words. Almost always, one finds that abstract words, the names of things which cannot be seen, come from words for actual objects or actions. Take, for instance, the word 'understand'. Both in German and English it is connected with the words 'to stand under'. In French, 'do you understand?' is 'comprenez-vous?', which means 'Can you take hold of that?' rather like the English phrase, 'Can you grasp that?' Still today, people make such remarks as 'Try to get that into your head.'

In learning to speak, a child follows much the same road. It learns the names of its parents and of household objects. It also learns words which describe its feelings, 'Are you hungry?' 'Are you tired?' 'He looks happy.' 'Don't be frightened.' 'Can't you remember?' 'Say you are sorry.'

Every philosopher, every professor, every school-teacher that ever lived began in this way – with words to describe things seen, or things felt. *And all the complicated ideas that have ever been thought of, rest upon this foundation.* Every writer or speaker that ever invented a new word had to explain its meaning by means of other words which people already knew and understood. It would be possible to draw a huge figure representing the English language, in which each word was represented by a block, resting on other blocks – the words used to explain it. At the bottom we should have blocks which did not rest on anything. These would be the words which we can understand directly from our own experience – what we see, what we feel, what we do.

For example, *philosophy*. Philosophy is what a *philosopher* does. Philosopher means '*a lover of wisdom*'. The meaning of *love*, and of being *wise*, we have to learn from everyday life.

What is true of philosophy is equally true of mathematics: its roots lie in the common experiences of daily life. If you can trace the way in which mathematical terms were gradually developed from everyday words, you can understand what mathematics is.

The essential point to grasp is that *mathematical reasoning* is not separate from the other powers of the mind, nor is mathematics separate from the rest of life. Quite the opposite: mathematics has grown from the rest of life, and reasoning has grown from experience.

Another sign of the way our minds are made is to be found in the law, well known to psychologists, 'There is nothing in imagination which was not previously in sense.' For instance, try to imagine a new colour. You will find that you are simply combining the effect of colours that you have already seen. Or try to imagine heaven, or a perfect world. You will find yourself putting together memories of your happiest moments, or turning upside down the things which have aroused your indignation. In a Scottish school, the children (sitting on hard seats) wrote an essay on 'The Perfect School'. Ninety per cent started by stating that the perfect school had cushions on the seats: they then described how the teachers were kept in fear and trembling by the stern rule of the pupils.

Reasoning and Imagination

Earlier we considered the argument, 'Twice two must be four, because we cannot imagine it otherwise.' This argument brings out clearly the connexion between reason and imagination: reason is in fact neither more nor less than *an experiment carried out in the imagination.* In any good detective story, the detective tries to imagine as clearly as possible the background of a crime, and to see how the statements of various witnesses fit into the picture. We are able to follow the story and the reasoning simply by using our own imagination. (*The Mystery of Marie Roget*, by Edgar Allan Poe, is a good example of imaginative reasoning applied to a crime in real life.)*

It is by no means necessary that reasoning should proceed by clearly stated steps. If you hear a rumour which means that your friend Smith has been involved in some particularly dirty business, you may say, 'I do not believe this story. Smith would never do a thing like that.' You may not be able to quote stories of heroic deeds performed by Smith, or to give evidence of any definite kind at all. You just feel that Smith is a decent person. Yet this is a perfectly good example of reasoning. Whether your conclusion is correct or not will depend on how long, and how well, you have actually known Smith. You will find it very hard to make the public share your faith in Smith. They have not your experience of Smith: therefore they cannot imagine him as you do: therefore they reason differently about him.

It is said that European explorers who told tropical peoples of

*See notes and introduction to Dorothy L. Sayers, *Great Short Stories of Detection, Mystery and Horror*, Part I (Gollancz).

the northern winter, when water became like a stone and men could walk on it, were met with polite disbelief. The natives looked at the warm sea waves rolling beneath the palm-trees, and refused to believe in ice. It was outside their experience: they were familiar with the tales of travellers.

People often make mistakes when they reason about things they have never seen. Children imagine kings wearing crowns; in real life, the odds are that a king wears a military cap or a bowler hat. Before the first locomotives were made, people refused to believe they would work. It was thought that the wheels would slip, and the train would remain motionless. A certain Mr Blenkinsop went so far as to invent a locomotive with spiked wheels to overcome this purely imaginary difficulty.

If anyone in the year 1700 had prophesied what the world would be like today, he would surely have been considered mad.

Imagination does not always give us the correct answer. We can only argue correctly about things of which we have experience or which are reasonably like the things we know well. If our reasoning leads us to an untrue conclusion, the fault lies with our reasoning. We must revise the picture in our minds, and learn to imagine things as they are.

When we find ourselves unable to reason (as one often does when presented with, say, a problem in algebra) it is because our imagination is not touched. One can begin to reason only when a clear picture has been formed in the imagination. Bad teaching is teaching which presents an endless procession of meaningless signs, words and rules, and fails to arouse the imagination.

The main aim of this book is not to explain how problems are solved: it is to show what the problems of mathematics *are*.

Abstraction

Let us now consider an example of reasoning applied to objects which everyone has seen and can therefore imagine correctly. Two railway stations, A and B, are connected by a single-track line. Owing to some mistake, a train leaves A for B at the same time as another train leaves B for A. There are no signals or safety devices on the line. Apart from exceptional happenings (such as a storm tearing up the railway track) one expects that there will be a collision.

You will agree that you could have reached this conclusion by the use of your own imagination. But now notice what a dim picture your imagination gave you. In what sort of country did you imagine the line to lie? Among woods, or through towns, or

1.

2.

STAGES OF ABSTRACTION

1. The rough impression of a scene as it might exist in a person's imagination.

2. A diagram, which leaves out all details except those needed for a particular purpose – namely, to show that the two trains are about to collide.

Most diagrams in mathematics are like '2'. All the details except those needed for a particular purpose are left out. But behind every diagram is a picture, like '1'. If you can discover what the picture is, you will find the diagram much easier to understand.

on the top of a precipice? Did you see clearly in your mind the movements of the pistons and the crowd of little devices on the wheels of the engines? How did you imagine the expression on the faces of the drivers, the colour of their hair, the build of their bodies? Did you imagine goods trains or passenger trains? One could continue thus for ever. It is certain, however vivid your imagination, there were points which you overlooked. But this did not in the least affect your answer to the question: will there be a collision? If you had thought of the two trains as two beads threaded on a wire (and the movement of trains might well be shown in this way when a timetable was being planned), you would still come to the correct conclusion. *For the purpose of this question* the trains and the railway-line might just as well be two beads and a wire. Of course for other purposes – if you had to provide ambulances for the wounded, or if you wanted to paint a picture of the event – it would be necessary to know further details.

It is impossible to imagine any event in perfect detail. In attacking any problem, we simplify the situation to a certain extent. We do not bother about those facts which seem unimportant. The result of our reasoning will be correct if the picture in our imagination is, *not* exactly correct, but *sufficiently correct for the purpose in hand.*

This process of forgetting unimportant details is known as *abstraction.* Without abstraction, thought is impossible. We should spend all our lives collecting information if we tried to make a *perfect* picture even of a simple event. Some mis-educated people continually interrupt sensible discussion by wailing, 'But you have not defined exactly what you mean by this word.' The great majority of words cannot be defined exactly (for instance, the word *red.*) The important thing is not exact definition: it is to know what you are talking about.

Serious misunderstanding can arise if one forgets the nature of abstraction, and tries to apply a picture of the world, which is entirely sufficient for some purpose, to another purpose for which it is entirely insufficient. Two examples of difficulties which arise in this connexion may be mentioned.

The Mechanical View of Life

At one time there was a great craze for explaining everything in terms of machinery. It had been discovered that many facts of nature, in particular the movements of the planets, the tides, and of solid objects on the earth's surface, could be explained by supposing the universe to be made up of hard little balls, attracting each other according to certain definite laws. Instead of saying, 'We have a theory sufficiently correct for certain purposes', philosophers and scientists leapt to the conclusion that they had the whole truth about the universe. Not only the sun and moon, but our brains also, were made out of these hard little balls, and everything we did was a consequence of the way they pulled each other about. Thought and feeling must therefore be pure illusions – this in spite of the fact that the theory itself was the result of thought!

The whole procedure was entirely unscientific. It is obvious to anyone that courage, loyalty, determination, affection are *facts*, just as much as pound weights or spring balances. Without these qualities, it is very unlikely that any race of men or animals could long survive. The scientific conclusion would have been: our theory gives us true results about the movement of the moon and the planets, therefore there is some truth in it, but it does not lead us to foresee the possibility of atoms coming together and being organized into living creatures, *therefore* it is incomplete, *therefore* it overlooks some of the things which atoms actually do.

The root of the matter is perhaps a superstitious feeling that results obtained by looking through a microscope or a telescope are in some mysterious way superior to the knowledge we get in everyday life. We have at times come near to the worship of scientists, to believing that men who work in laboratories can solve all our problems for us. The views of a great scientist on his own science are indeed worthy of respect, for they are based on facts. But by the very act of shutting himself inside a laboratory, a scientist shuts himself out from much of the daily life of human beings. If a scientist realizes this, if he tries to overcome his isolation by paying special attention to current events and by learning the history of mankind, he may be able to apply his scientific training to other departments of life. But if he rushes straight out of his laboratory, full, like any other human being, of prejudice and ignorance, he is likely to make a rare fool of himself.

Euclid's Straight Lines

Beginners in geometry are sometimes puzzled by being told that straight lines have no thickness. We shall never, we are told, meet a straight line in real life, because every real object has a certain thickness. One of Euclid's lines, however, has no thickness. Two lines meet in a point, and a point has no size at all, only position. We shall never meet a point in real life, either, for all real objects have a certain size, as well as a certain position. It is not surprising that pupils wonder how we know anything about objects which no one has ever seen or ever can see.

This difficulty is a good example of the confusion which can come from misunderstanding the methods of abstract thought. We have seen that Euclid's geometry grew out of the methods used for building, surveying and other work in ancient Egypt. This work was done with actual ropes or strings, real 'linen threads' with actual thickness. What does Euclid mean when he says that a line has no thickness, although he is using results suggested by the use of thick ropes? He means that in laying out a football field or in building a house you are *not interested in the size or the shape of the knots* made where one rope is joined to another. If you allowed for the fact that ropes possess a definite thickness, if you carefully described all the knots used by a bricklayer, you would make the subject extremely complicated, and no advantage would be gained. Euclid therefore says, if an actual rope has thickness, *neglect this* in order to keep the subject reasonably simple.

The position is not that Euclid's straight lines represent a perfect ideal which ropes and strings strive in vain to copy. It is the other way round. Euclid's straight lines represent a rough, simplified account of the complicated way in which actual ropes behave. For some purposes, this rough idea is sufficient. But for other purposes – such as teaching Boy Scouts how to tie knots – it is essential to remember that rope has thickness: for such purposes you will obtain wrong results if you think of ropes as Euclidean straight lines.

Similar illustrations might be drawn from any science. Scientific laws are true, within certain limits, for certain types of object. For other types the truth becomes doubtful, and after a certain point, positively misleading.

Some books on mechanics give a very good explanation of the word *particle*. They say: an object is spoken of as a particle when its size is small compared to the distances in which we are interested. The earth sweeps round the sun at a distance of about 90,000,000 miles. Compared with this, the diameter of the earth, say 8,000 miles, does not amount to much, and we might speak of the earth as if it were a point. On a map of the world London might be regarded as a point. According to the atlas, London is 51½ degrees north of the equator. It is not necessary to say whether this refers to Hampstead or the Isle of Dogs.

Readers will find some statements in this book very puzzling if they treat every sentence as being true (so to speak) to ten places of decimals. Particularly within the limits of a small booklet, it is impossible to hedge every sentence round with the remarks and cautions that would be necessary in a text-book. In any case, I do not wish to present people with ready-made opinions. It is up to the reader to approach this book in a mood of sturdy commonsense, with a readiness to criticize and reject anything which, through stupidity, carelessness or lack of space, or through the essential difficulty of saying anything with 100% truth, is misleading. My aim has been to convey a general impression, sufficient to show how mathematicians think.

The Mathematicians on the Second Floor

At this stage there will be a protest from the pure mathematicians, who will say that engineers and other practical mathematicians may think in this rough instinctive way, but that my statements neglect the very important body of people who work on mathematics itself, and neither know nor care what practical applications their work may have.

It is true that pure mathematicians, working in this way under the inner compulsion of an artistic urge, have not only enriched mathematics with many interesting discoveries, but have also created methods of the utmost value for practical men. It is very shortsighted (from the practical point of view) to discourage all work which has no *immediate* practical aim. It pays humanity to encourage the artist, even if the artist does not care in the least about humanity.

What is pure mathematics *about*? It does not seem to deal with any definite thing, yet there is no doubt about its truth, and its discoveries can be used with confidence for practical tasks.

Perhaps the square root of minus one will serve as an illustration. How do we come to study such an unearthly idea? The history of this idea is briefly sketched at the beginning of Chapter 15. Mathematicians first used the sign $\sqrt{-1}$, without in the least knowing what it could mean, *because it shortened work and led to correct results.* People naturally tried to find out *why* this happened and what $\sqrt{-1}$, really meant. After two hundred years they succeeded.

This suggests that pure mathematics first appears as *the study of methods.* Pure mathematicians do not appear on the scene until late in human history: they represent a high level of civilization. The first comers are the practical men, who study the world at first hand, and discover methods which work in practice. Pure mathematicians do not study the natural world. They sit, as it were, upstairs in the library, and study the writings of the practical men. Sometimes the practical men get taken in by a method which usually gives the correct result, but not always (see Chapter 14). The job of the pure mathematicians is then to sort out the methods which are logical (that is, which give correct results) from those which are not.

Pure mathematicians are in touch with the real world, but at second hand. They do not sit by themselves and think. The material they study consists of the books in the libraries of the world. These books do not consist solely of the writings of engineers. The chain is often very long. An engineer consults an 'applied mathematician' (one who studies the applications of mathematics to everyday problems): the applied mathematician consults a pure mathematician: the pure mathematician writes a paper on the question: another pure mathematician points out that the question could be solved if only we knew the solution to some more general problem, and so it goes on. A great literature arises, showing the connexion between different problems. A subject becomes so large that it is impossible to remember all that has been written about it: it becomes an urgent necessity to boil all the various results down into a few general rules. After a century or two, problems are being discussed which seem to have no connexion with the worries of the original engineer. But the connexion is there, even if it is not easy to see.

Is pure mathematics, then, merely the study of how mathematicians think? It certainly is not. Pure mathematicians take very little account of how people actually think. If all the applied mathematicians in the world suddenly went mad, pure mathematics would remain unchanged. Pure mathematics is the study of how people *ought* to think in order to get the correct results. It takes no account of human weaknesses. It would perhaps be more true to say that pure mathematics is the study of how we should have to construct calculating machines, if we decided to do without human mathematicians altogether.

Pure mathematics appeals to those who, like Rupert Brooke, appreciate

> 'the keen
> Unpassioned beauty of a great machine,'

but this is a taste that comes late both in the history of the human race and in the life of most individuals. For the purpose of teaching, it is essential to master the primitive methods of practical mathematicians before attempting to introduce the strict methods of the pure mathematician. The emphasis in this book is laid on practical mathematics, not because practical mathematicians can claim any superiority over pure mathematicians but simply because teaching experience shows that it is necessary to do so.

Nor would I claim any infallibility for the view I have suggested as to how it has come about that men can reason. I have had less time than I would have wished to study the history of mathematics, and of mankind generally. These views I merely believe to be on the right lines. But that most human beings think, and need to be taught, as this theory would lead one to expect – this, from direct experience, I know to be true.

Practical Conclusions

To sum up – successful reasoning is possible only when we have a clear picture in our minds of what we are studying. Imagination is developed, and is made reliable, through practical contact with the real world. Mathematics is difficult when it is presented as something quite apart from everyday life. Mathematical reasoning can grow gradually and naturally, through practical work with real objects. This holds both for elementary and 'higher' mathematics. Only for the 'highest' pure mathematics is the connexion with daily life rather indirect.

CHAPTER 4

THE STRATEGY AND TACTICS OF STUDY

'*I have taught mathematics and applied science or engineering to almost every kind of boy and man . . . In my experience there is hardly any man who may not become a discoverer, an advancer of knowledge, and the earlier the age at which you give him chances of exercising his individuality, the better.*' – John Perry, 1901.

THE two main conditions for success in any sort of work are interest and confidence. People usually pay little attention to these two factors, because they feel (quite rightly) that they cannot make themselves confident or interested by an effort of will.

It is quite true that you cannot increase confidence by an act of will. But neither can you increase the size of your muscles or make your heart beat more vigorously by sitting in a chair and willing it. This does not mean that it is impossible to change your muscular strength or the rate of your heart-beat. If you skip for half an hour you will do both.

Confidence and interest can also be changed *by taking the proper measures.*

The proper measures do not consist in rushing at work like a bull at a gate. It is well known that the effect of too intensive physical training is to destroy the body, not to build it up. The same is true of the mind.

In physical training, some of the vital organs lie beyond the control of our conscious minds. We cannot send direct orders to our heart, our liver, our glands. We have to find exercises depending on the movement of our limbs, on the efforts of muscles which we can control, that will produce the desired effect on the other organs. After a few months of proper training, we do not know what changes have taken place in our bodies, but we feel the benefit and know that changes must have occurred.

In mental training also the decisive changes take place outside consciousness. The test of any system of coaching is not whether it turns out students capable of doing certain tricks, like performing dogs. Such a method is futile and fundamentally degrading. It merely enables people to pass examinations in subjects which they do not understand, and to qualify for posts in which they will be unhappy and inefficient. The real test of any teaching method lies far deeper. With the correct approach, a student finds his whole feeling about the subject changing. He begins to understand what the subject is about, he feels confident that he can master it, he begins to take pleasure in it and to think about it outside working hours. Only when such an attitude has been created does the mind really *grasp* the subject. People show a greater degree of intelligence and knowledge in connexion with their hobbies than in any other department of life.

Lack of Interest

Is it possible to transfer the kind of interest we feel for a hobby, and to use it for the purpose of work? It depends on the reason for your lack of interest.

There are people whose interest is concentrated on one subject. If you feel that you have one single purpose in life, whether it be to paint pictures or to find a cure for cancer – if you feel that this one thing alone matters for you, that everything else – comfort, wealth, respectability, safety, family ties or social obligations – are without significance compared to it – then obviously you have no doubt what you are to do.

Only a few people are thus clear cut in their aims. Most men and women are prepared to fit in, more or less, with the customs they find around them, to work at any job by which they can earn a reasonable living.

There are probably some who fall between two stools – people who could be happy and efficient in some particular type of life, but who lack the self-knowledge or the courage or the determination needed to break away from the life which other people expect them to lead. The war has produced many cases in which people, who had previously been making a rather half-hearted effort to qualify for learned professions, found themselves doing practical work, putting out fires, driving lorries and so on. It was obvious that they had found the type of work for which nature designed them. In a perfect world they would be encouraged to do such work, without a war being necessary. For such people the question is not how to learn mathematics, but how to drop mathematics at the first possible moment.

This then is your first question: to which type do you belong? Are you a person with such a keen interest in some special type of activity, that you can afford to drop other subjects (including mathematics) and succeed as a specialist expert? Or do you belong to the more usual type, that is ready to tackle whatever comes along?

You must decide definitely one way or the other. Either your interests are so far from mathematics that you could never have any use for or amusement from mathematics, or there is something which you accept as worth doing, for which mathematical knowledge is necessary. In answering this question you must make allowance for the fact already mentioned – that the educational system seems specially designed to take all life and interest out of the subjects taught. By mathematics is meant the living subject, not what is taught in many schools.

In some cases, then, the lack of interest goes right down to the roots of personality. But the vast majority of people who hate mathematics do not come under this heading. By far the commonest cause of dislike is the way mathematics has been presented. You can test this for yourself. Do you like puzzles? Do you listen to the Brains Trust, or do crosswords? Do you play bridge, or chess, or draughts? Do you take part in the heated arguments one sometimes hears, such as the question what would happen if passengers in a motor-car threw a cricket ball straight up into the air – would it fall into the car again? Do you take an interest in any sort of scientific or mechanical development, such as radar or aeroplane working? If so, your basic interests are not very different from those of the mathematician. I know a family (by no means high-brow) that was divided into warring factions one Christmas by the car and cricket-ball question. At school, it was the most normal boys who felt most strongly about their own solutions to such problems. This interest in what would happen is close to the interest a scientist feels, and science soon leads to mathematics.

The Removal of Fear

Probably most people would be interested in mathematics, as most people would be interested in music, if they were not afraid of it. Interest and confidence are closely connected. If you find that you can do something, you are pleased. You like the feeling of having mastered nature and the feeling that other people will admire you. You want to do some more of it, and the more you do, the better you become. On the other hand, if you start off with a defeat, the effect is the opposite. Nobody likes to appear a fool. You avoid the subject, or try to make out that you do not bother about it. You decide that you never will be any good, so why waste energy? In any case, you convince yourself, it is no use. All of this has nothing to do with the facts of the case: it is the desperate attempt of a human mind to keep its balance and its self-respect. You probably concentrate on some other subject, or play hard at some game, and say to yourself, 'Well, I may not be able to do algebra, but I am pretty hot at cricket and chemistry.'

In some schools, the excellent custom is followed, when a boy is a complete failure at lessons, to put him on to some useful activity such as carpentry or ploughing. He then becomes sure that he can do *something* well, and he no longer needs to deceive himself about his lessons. He can take the risk of really trying to succeed, since his self-confidence will not be destroyed should he fail.

It is essential, if you are trying to overcome your dread of a subject, to realize what is your first objective. Your first job is *not* to learn any particular result. It is to get rid of fear. You must go back a certain way, and start with work which you are absolutely sure you can do. In learning a foreign language, for instance, it is helpful to get a book written in that language for children just learning to read. However badly you have been taught, you will almost certainly be able to read it. This is your first victory – you have read a book genuinely written for the use of someone speaking a foreign language.

In mathematics it is even more important to go back to an early stage. It is impossible to understand algebra if you have not mastered arithmetic: it is impossible to understand calculus if you have not mastered algebra. If you attempt the impossible, without realizing what you are doing, your morale will suffer.

Apart from this logical necessity, there is also a psychological reason. The chances are that you are still carrying around with you all the feelings of uncertainty that have troubled you during all the different stages of your education. You are still *feeling* the setbacks that you had when you were eight or nine. This feeling will immediately disappear if you go right back to the beginning, and read again the text-books that you then had. You will often find that the difficulties have vanished without your realizing it.

It is for this reason that there are chapters in this book dealing with such things as the multiplication table. You will read these chapters without difficulty. At some stage of the book you will find yourself again puzzled. This means that you have reached the stage where your knowledge of the subject begins to show gaps – at this point, or at some earlier point, your revision must begin. It is nothing unusual to be puzzled by the things you have just learnt. If you keep revising, and are perfectly clear about everything which you have done more than a year or more than six months ago, you need not worry.

A good way to revise is to take a text-book, and look through the examples in it. If you can do them easily, you need not read the book. The examples on some chapters may give you difficulty. If the text-book is one which you first read several years ago, you will probably know whether the results of these particular chapters are much used in later work. If so, you have found out the source of your difficulty in the later work. If they are not important, you may leave them for the time being.

In mathematics it is often necessary to work backwards. If you find a difficulty on page 157 of some book, try to find out why. See if page 157 uses the results of other, earlier pages in the book, or if it uses some fact explained in an earlier text-book. If page 157 depends on pages 9, 32, and 128, read these pages again and make sure that you understand them. If you do not understand them, you cannot possibly understand page 157.

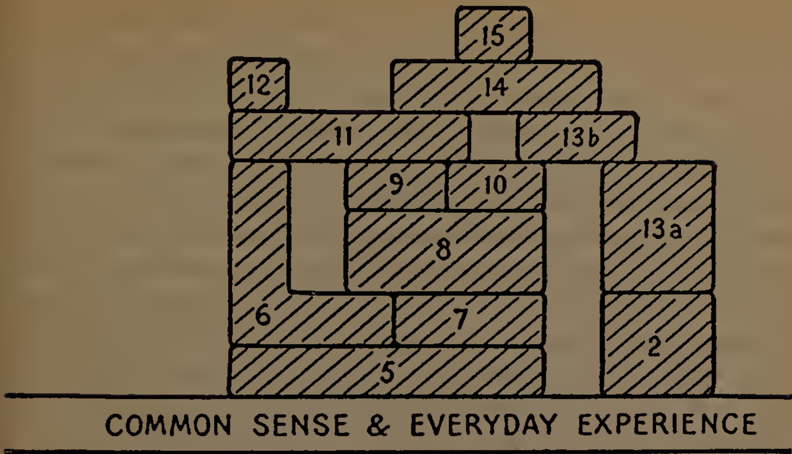
If you still have difficulty, ask someone else to explain the page to you. Notice very carefully if he uses any word, any sign, or any method which is strange to you. If so, ask where this word, sign, or method is explained.

If you can find out what your difficulty is, you are half-way to overcoming it. People often go about with a fog of small difficulties in their heads: they are not quite sure what the words mean, they are not quite sure what has gone before, they are not quite sure what is the object of the work. All these difficulties can be dealt with easily, if they are taken one at a time. Provided the book is written in reasonably simple language, a few minutes with a dictionary should clear up that difficulty.* The next thing is to find out what knowledge you are expected to have before you attempt to understand the proof of a new result. It is possible to make a diagram showing how a book hangs together, how one section depends on previous sections. One should learn a book both backwards and forwards: one should know that the result on page 50 is proved by the result on page 29, and that it is used to prove the result on page 144. (Of course no sane person will learn the numbers of the actual pages on which results occur. But it may be worth while to write in the margin of page 50, 'See p. 29; used, p. 144') Many people learn separate results, but never link them together in this way.

In this book it has not been possible, in every single sentence, to give references to all the remarks, made earlier in the book, that may help towards understanding. If you cannot understand some sentence, underline it. The chances are that somewhere earlier in the chapter, or in the book, a remark has been made that was especially intended to prepare for the difficult sentence. At the first reading you may not have noticed this remark at all. It seemed pointless. Look back for such remarks. If you succeed in finding them, put a note in the margin, 'This explains sentence underlined on page ...'

It may seem to you that this advice does not amount to much, that it is obvious. It may be obvious – but people need a lot of persuading before they do it. As a rule, someone who has difficulty with calculus or trigonometry is not prepared to believe that the real trouble is ignorance of algebra or arithmetic. There is always an examination coming in six weeks, or a year, or whatever it is, and this examination is on calculus or trigonometry – not on algebra and arithmetic. Trying to learn higher mathematics without a firm grasp of the earlier part is like trying to invent an aeroplane without knowing anything about motor-car engines. Until the motor industry had been developed, all attempts at aeroplanes were complete failures.

*I have tried to keep words in this book as short as possible. One or two words may not be known to everyone. It is only fair that readers should take the trouble to look these up.



THE GENERAL PLAN OF THIS BOOK

In this diagram each block represents a chapter. Chapters 1, 3, and 4 are of a general nature, and are not included in the diagram.

Each block depends upon the blocks below. Thus, it is impossible to understand Chapter 11 without first having read Chapters 6, 9, and 10, and Chapters 9 and 10 in turn cannot be understood without Chapter 8, etc.

In some cases, the upper block depends only on a small part of the lower one. For instance, Chapter 8 can be understood without understanding the whole of Chapter 6. In fact, it is only the part of Chapter 6 explaining the meaning of the signs 4^3, 10^6, etc., that is needed for Chapter 8. It is not possible to show this on the diagram.

Chapter 13 is split into two parts. 13*a* represents the greater part of the chapter, which is quite elementary. 13*b* represents the end of the chapter, which is more advanced.

If a reader finds difficulty, say in Chapter 10, he may find it worth while to leave Chapters 10, 11, and 12, for the time being, and to read the easier part of Chapter 13.

To revise elementary mathematics takes much less time than people imagine. How many text-books has a student of eighteen used? One on arithmetic, one on algebra, several perhaps on geometry, an elementary trigonometry, perhaps a book on calculus. Geometry we may leave on one side for the moment. How long does it take to look through an arithmetic book and one on algebra, and find out if there is any important result which you missed at school? How long does it take to write down on a sheet of paper a list of the contents of these books, and to put a tick against the results which you thoroughly understand? Not very long. The advantage of doing this is that you begin to see how much (or how little) you have to learn. One tends to think of algebra as a vast jungle of confusion, in the midst of which one wanders without map or compass. It is much better to think of algebra (or that part of algebra which you need to know) as being half a dozen methods, and twenty or so results, of which you probably already know 60%. Nor need you revise the whole of this at once. Suppose for instance you are finding difficulty with calculus because you do not properly know the Binomial Theorem. Get down your book of algebra, and look up *Binomial Theorem*. Never mind about the proof for the moment. First get quite clear what the Binomial Theorem is. It is full of signs such as ${}^nC_r$ or $\binom{n}{r}$ – different signs are used in different books. These signs are explained in the chapter on Permutations and Combinations. Again, do not bother about proof. See what these signs mean. Work out a few examples – ${}^4C_1$ ${}^4C_2$ and ${}^4C_3$, for instance. Work these right out, as numbers. Come back to the Binomial Theorem, and take particular examples of it. Put $n = 4$, for instance.* The binomial theorem deals with the expression $(x + a)^n$. Put $x = 10$ and $a = 1$. Work out 11^2, 11^3, 11^4. What is the connexion between 11^4 and the numbers worked out above? Work out 101×101 and $101 \times 101 \times 101$. What do you notice about 11×11 and 101×101? What do you notice about $11 \times 11 \times 11$ and $101 \times 101 \times 101$? The same numbers turn up both times? Do you think the same numbers will turn up in 1001×1001 as in 11×11? In $1001 \times 1001 \times 1001$ as in $11 \times 11 \times 11$? If so, you are not far from discovering the Binomial Theorem for yourself. (If you are not clear what 11^4 stands for, the remarks above will be meaningless to you. What 11^4 means is explained in Chapter 6.)

*If you are in the fortunate position of having never been taught algebra, and therefore having no mistaken ideas about it, take no notice of this paragraph. The meaning of algebraic signs is explained in Chapter 7.

In this way, tracing back and back, you get to know the parts of algebra which are useful for calculus. You know at least what the Binomial Theorem is, and how it helps you to write down $(1001)^4$, even if you cannot prove it. When a book or a lecturer refers to the Binomial Theorem, you will be able to follow the use that is made of it. When you are thoroughly familiar with the usefulness and the meaning of the Binomial Theorem, it *may* be worth your while to study the proof. (Some books contain very dull proofs. Look for a book with a proof that is short and that appeals to you.)

Reading with a Purpose

We have just been using an algebra book in a special way – with a purpose. We have not tried to read the whole book. We have taken no notice of any chapters, except those which are necessary for an understanding of the Binomial Theorem. You may not think this is much of a purpose, but it is better than none at all. You will be surprised how much more sensible a text-book becomes if you use it in this way. You have a definite interest in getting this information – it will save you from getting any further behind with your work. You are not littering your mind with all the information in the book. You are learning only things which you need to learn.

All mathematics grew rather in this way. Someone wanted to do or make something: it was impossible to do it without mathematics: so mathematics was studied, and the *purpose* gave meaning and unity to the work done. A very simple example – try to make a model mansion with gables and attics in the roof, by cutting out pieces of cardboard or paper and sticking them together. You will find it is not so easy as it looks to draw the shapes that will be required. From such a problem, scientifically investigated, can arise geometry and spherical trigonometry. By working and experimenting with this problem of the toy-maker and the architect, you will unconsciously acquire the type of imagination necessary for studying geometry, trigonometry and solid geometry.

Interest is a peculiar thing. There are hundreds of things in which you feel you *ought* to be interested – but for which (to be honest for once) you do not give a hang. There are hundreds of other things – odd remarks, pointless little stories, tricks with matches, stray pieces of information – which seem to have no use in life, but which stay in your memory for years. At school we read a history book by Warner and Marten. No one remembered the history (this was no fault of the authors). But there were certain footnotes in it: one about a curate who grew crops in the churchyard and said it would be turnips next year; a lady who blacked out a picture and said, 'She is blacker within'; a verse about someone longing to be at 'em and waiting for the Earl of Chatham – everyone knew these years after they left school. These were the things that really interested us.

If you want to remember a subject and enjoy it, you must somehow find a way of linking it up with something in which you are *really* interested. It is very unlikely that you will find much entertainment in text-books. If you read only the text-books, you will find the subject dull. *Text-books are written for people who already possess a strong desire to study mathematics: they are not written to create such a desire.* Do not begin by reading the subject: begin by reading *round* the subject – books about real life, which somehow bring in the subject, which show how the subject came to be needed.

In any reasonably large town, a public library gives an easy way of finding good books. Nearly all libraries use the same method of indexing books, the Dewey Decimal System. Take a look at the books between 501 and 531. On the open shelves of Manchester Central Reference Library, inside an hour and a half, I found the following books, and glanced through their contents. I give them just as I jotted them down – pass rapidly over any book which does not appeal to you. The reference numbers are given.

510.8 Horsburgh: *Modern Instruments of Calculation.* Do not try to read this straight through. Photographs of calculating machines, round about page 26. If you hate arithmetic, why not make a calculating machine for yourself?

510.2 Mellor. *Higher Mathematics for Students of Chemistry* *or Physics*. An excellent book, but do not try to read it until you are ready for it.

515. Abbott. *Practical Geometry and Engineering Graphics*. A book full of illustrations. It covers problems rather like that of the model house. How do you cut a flat sheet of metal to make a stove-pipe with a bend in it? What curve is best for making gear wheels? Glance through the whole book, see what subjects interest you, then trace backwards, and see what type of mathematics is needed for each. Fairly technical language. Beginners should rest content with a general impression.

523. Serviss. *Pleasures of the Telescope*. Very simple indeed. Well illustrated. Contains star maps. Will appeal to artistic people. Useful for airmen and sailors, who may steer by the stars in emergencies.

522.2 Bell. *The Telescope*. Intended for the makers of telescopes and field-glasses. Only a small part of the book consists of mathematics. Best method – read through the book: note anything you cannot understand; then consult an elementary book on Optics (under the number 535). Try your hand at designing a telescope, a microscope, a magic lantern or cinema projector, an epidiascope, a camera obscura. The advantage of doing your own design is that you can use any 'scrap' – old spectacles, magnifying glasses, etc. – that you may have. Very simple geometry is sufficient for this purpose, if you find the right method.

526.8 Hinks. *Maps and Survey*. Chapter 2 explains why it is necessary to have maps. Chapter 8 deals with the kind of map made by an explorer, and Chapter 10 with the rough survey made by the first settlers in a frontier town. Chapter 12 shows how maps are made from aerial photography. By reading *the right parts* of this book, a beginner in trigonometry can get a useful background – how to make a rough plan of a field, etc. The book also brings out some unexpected connexions between practical life and scientific questions: the exact shape of the earth and observations of stars are needed to make a map of a big country like Africa; it is hard to make a proper map of India, because the Himalayas are heavy enough to exert a noticeable pull on a plumb-line, and cause it not to point straight at the centre of the earth.

While on the subject of maps, *A Key to Maps*, by Brigadier H. St J. L. Winterbotham, may be mentioned. Among other things, it tells hikers how to see from a map what the view from any place will be like. Many libraries have, or can obtain, this book.

530.2 Saunders. *A Survey of Physics*. In the words of the author, 'The reader will be introduced to some of the mysteries of nature, as well as to many ingenious inventions of mankind.'

531. Goodman. *Mechanics Applied to Engineering*. Contains a great amount of information. I am not sure how it will appear to beginners. As with all other books, look through it, learn anything you can, but do not be distressed if there are parts of the book you cannot follow at all.

You may find something to interest you under 385, Railways; 620.9, History of Engineering; 626, Canals. If you are particularly interested in any subject, the library assistants will tell you where to look. Look right through the catalogue, under any section that interests you. It pays to spend a long time searching for an interesting book on the subject, rather than to read half a dozen books that will bore you.

It is often good policy to read a book of which nine-tenths merely reminds you of things you already knew, while the remaining one-tenth is new. Your mind will then have plenty of energy to learn the new facts. Do not make a great effort to remember every detail. Anything that interests you will stick in your mind. If you find some useful piece of information that may be needed later, write it down in a note-book kept for the purpose. Your aim should be to have in your mind a general view of the subject, in your desk a collection of exact facts that you can use for any particular problem.

Books on the History and Teaching of Mathematics

If you find these suggestions of any use, if by browsing in a library or by looking about you in the street you hit on anything which you genuinely like and want to know more about (where there's no will, there's no way), you will soon find yourself becoming a specialist in this. It may be anything from radar to how drains are fitted, provided it beckons you on. As you get to know more about this question, you will get impatient with popular introductions, you will find yourself wanting a complete answer to questions, the professional way of dealing with the subject. You will find yourself pulling down bulky volumes that seemed infinitely dry a year ago. You will not read them from cover to cover. You will search with a skilful eye for the paragraph or two that deals with what you want to know at the moment. And you will realize that, while you may not now be interested in other subjects, if you were to become interested in anything, however complicated, you could deal with that too in the same professional way. This confidence, this freedom from fear, is the main thing that distinguishes the expert. An expert does not need to know much. He must know *how* and *where* information can be found.

As the subject you have chosen for your hobby becomes better known to you, you will begin to realize how much like yourself were the men who worked at it and discovered it. When you reach this stage, you may find it useful to have some idea of the dates at which these men lived. There are various reasons for this. (i) By noticing the dates you can get an idea of how much of the subject you know. For instance, if you find that the mathematics you know was all discovered before 1800, you will realize that there is much yet to learn. The nineteenth century saw tremendous mathematical activity. You will not make the mistake of trying to research on questions for yourself without first making some effort to find out if the problem that puzzles you has already been solved. (ii) If you know how much of the subject was known at any time, it is often much easier to see how particular discoveries were suggested by things already known. This helps you to understand the subject. (iii) If you are baffled by something, reading the history of that discovery may help you. The life of the actual discoverer is often very helpful: the attempts he made, the experiments he carried out may supply the clue. In this way you can outflank your difficulty by reading round it – much better than battering your head against it. Far too little use is made of history in teaching mathematics.

In choosing an historical book, as with any other book, look for one which appeals to you, and do not be worried if you cannot read the whole book. Read what you can.

It may also help you to read a book on the teaching of mathematics. There is no ideal way of teaching. What suits one student is useless to another. A teacher who has to deal with a class of fifty has an almost impossible job. If you read a book on teaching, you will find that there are several entirely different ways of tackling the subject. You may feel that you would have done much better if you had been taught by one of these methods. Note the names of the people who developed this method, and see if there are any books by them in your library. *The Teaching of Mathematics*, by J. W. A. Young (1911), contains a description of several movements in mathematics teaching, and the author is human and enlightened. In it will be found a great number of references, on which further reading can be based. One of the reformers mentioned by Young is Professor John Perry, whose *Address to the British Association*, 1901, is the source of the quotation at the head of this chapter. It is well worth reading, both for Perry's speech and for the remarks (mainly approving) of the leading mathematicians of the day. Anything by Perry is worth reading. His book *Calculus for Engineers* may be mentioned. It is forty years since Perry gave this lead. If parents, teachers, and teaching authorities today were fully aware of what was said in 1901, much mental suffering among children would be prevented. The tide is undoubtedly flowing in that direction. It still has a long way to flow.

PART II

ON CERTAIN PARTS OF MATHEMATICS

CHAPTER 5

ARITHMETIC

'One, two, plenty.'

– Tasmanian Method of Counting.

ARITHMETIC plays a very small part in mathematics, especially in the higher mathematics. Geometry, as we have already seen, can be studied direct from diagrams, in which simple numbers – 3, 4, 5, etc. – occasionally occur. The higher one goes, the less arithmetic is likely to be used. This is why there are so many stories about famous mathematicians quarrelling with tram conductors about their change, and being wrong.

Arithmetic does depend on certain things which have to be learnt by heart, such as the multiplication table, addition and subtraction tables. These operations can be carried out by machines, and to a certain extent anyone who learns arithmetic has to become a machine. For instance, a clerk adding up long sums of figures does not need to think deep thoughts about the nature of a number. It is sufficient for him if the sight of 7 and 8 immediately bring the number 15 into his mind.

Whilst it is possible to teach arithmetic in a purely mechanical manner, it is certainly not desirable to do so. Even for the simplest operations, it is easier to remember what has to be done if one knows the reason. For anyone who wants to go on to the other branches of mathematics, mechanical learning is fatal. No one has yet invented a machine that will think for itself. It is a pity that there are still schools (especially girls' schools) where arithmetic is still taught on the lines of 'You do this, then you do that' – as though the subject were some form of religious ritual.

Arithmetic is not a difficult subject to discover for oneself. There are many things which lie on the edge of arithmetic. For instance, when an operation is being carried out in hospital, a nurse has a board with hooks, holding all the things that will go into the patient and must come out again. Before the patient is sewn up, the nurse must see that no hook is empty. This procedure is not counting, but it is very near to it. When we count on our fingers (the original method!) we are simply using fingers instead of hooks.

Counting with fingers (or fingers and toes) will do only for numbers up to ten (or twenty).* Team-work is needed to go further. If a friend is willing to be nudged each time you reach ten, and to count the nudges on his or her fingers, it is possible to reach a hundred. With six people, a million can be counted thus, though the sixth person can go to sleep most of the time. (I do not see why counting by this method should not actually be carried out in classes for young children, in order to explain what is meant by a number such as 243. In playing hide-and-seek children of their own free will count up to quite large numbers, and seem to enjoy it.)

The same idea, in essentials, is used in the devices which measure how far a motor-car or bicycle has gone. Each wheel, on reaching ten, 'nudges' the next. Adding machines are made on similar lines.

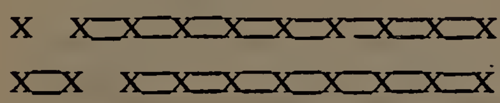
The imagination can be further aided if actual objects (say matches) are being counted. The first person ties the matches into bundles of ten. The second person takes ten bundles, and puts them into a box. Ten boxes go into a bag, ten bags into a sack, ten sacks into a truck, ten trucks form a train – the latter stages in imagination only! At the same time, the progress of the work could be exhibited, as on a cricket score-board. Quite soon, the symbol, 127: the sound, 'one hundred and twenty seven': the picture, one box, two bundles, seven matches: would be welded together in the mind of each child.

All the operations, such as adding together 14 and 28, subtracting 17 from 21, dividing 84 into three equal parts, can be

* Some entertaining details of primitive methods of counting may be found in E. B. Tylor, *Primitive Culture*, Chapter 7; Tobias Dantzig, *Number, the Language of Science*, Chapters 1, 2.

carried out, first by experiments with the actual objects: secondly with the objects and with 'score-boards' simultaneously: finally, by written work alone.

In the Montessori method, the tables of addition are taught in some such way. The children have sticks, representing the numbers one to nine, and have to arrange these so as to get ten units in each row: thus –



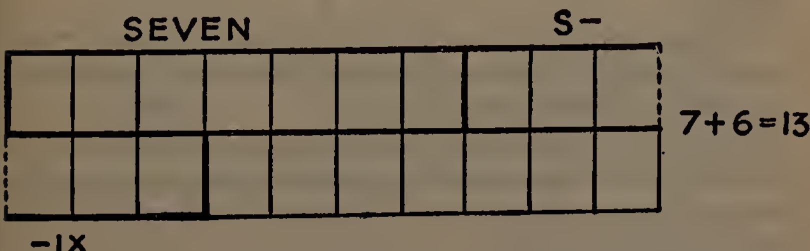
$1 + 9 = 10,$

$2 + 8 = 10,$ etc.

It is quite useful to have squared paper, cut into strips, ten squares in breadth. The squares are numbered –

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20 etc.

The addition table can then be worked out, by pasting on strips of correct length. If two numbers, such as 7 and 6, require more than one complete row, the extra squares are cut off and go to the next row.



I have heard successful mathematicians say that in adding 7 and 6 the thought was present 'in the back of their minds' that 3 of the 6 units were needed to bring 7 up to 10, so that 3 was left over to provide the odd units.

This method can be extended to the multiplication table, by repeatedly sticking on strips containing the same number of squares. Quite striking patterns emerge. How crude the table for two times

	2		4		6		8		10
	12		14		16		18		20

is, compared to 6 times! –

					6				
	12						18		
			24						30
					36				
	42						48		
			54						60

The crudest of all (and the easiest to learn) is 10 times: then 5 and 2; then 9 and 3; then 4, 6, 8; the most subtle is 7 – it would make a good wall-paper.

The appearance of pattern appeals to the artistic side, which is strong in children. Good mathematicians are very sensitive to patterns.

Patterns also suggest questions. Why does '3 times' have a pattern rather like that of '9 times'? Why are '5 times' and '2 times' arranged in upright lines?

It was said of Ramanujan that every number seemed to be his personal friend. One should try to present arithmetic to children in such a way that they come to realize the 'personality' which each number possesses.

Owing to the accidental fact that we possess ten fingers, the multiplication tables depend on this number 10. If we had eight or twelve fingers, the patterns would be different.

One can, however, represent multiplication, quite apart from the question of fingers, by means of rectangles. A piece of linoleum, 3 yards by 2, costs six times as much as a piece 1 yard square. We can, if we like, think of 2 times 3 as the area of a rectangle 2 by 3.

$2 \times 3 = 6$

This idea may be helpful in connexion with fractions. We may explain the meaning of $\frac{2}{3} \times \frac{5}{7}$ by saying that it represents the area of a piece of linoleum that measures $\frac{2}{3}$ yard by $\frac{5}{7}$ yard.

Multiplication of fractions seems to cause trouble. Children are often puzzled why two-thirds *of* five-sevenths should be the same as two-thirds *multiplied* by five-sevenths. They see no connexion between 'of' and 'times'. This difficulty is very largely one of language. It is quite natural to say that one field is 3 times or 4 times or $3\frac{7}{8}$ times as large as another. It is perhaps less usual to say a field is $\frac{7}{8}$ times as large as another, more usual to say it is $\frac{7}{8}$ of the size of the other. It is at any rate clear that to draw an area $3\frac{7}{8}$ *times* as large as this page one would take 3 pages, and add this to $\frac{7}{8}$ *of* a page.

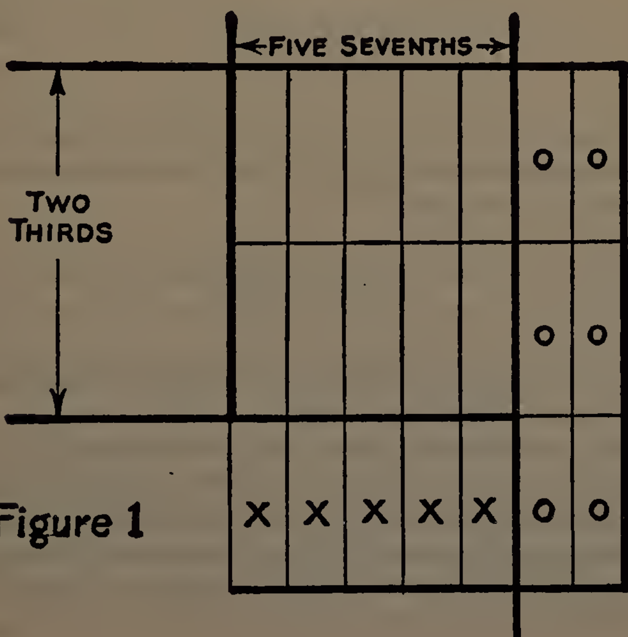


Figure 1

Multiplication of fractions is often taught purely by rule, but it is easy to show why the rule works. Consider for example, $\frac{2}{3}$ of $\frac{5}{7}$. Let us take 1 square yard of linoleum and see what two-thirds of five-sevenths of a square yard looks like. To obtain five-sevenths we must first divide the linoleum into seven equal pieces – by the upright lines in Fig. 1 – and take five of these pieces. If we cut along the heavy upright line, the piece to the left contains five-sevenths. We now require two-thirds *of this.* The level lines divide the whole figure into three equal parts. By cutting along the heavy flat line we shall obtain a piece which is two-thirds of five-sevenths. After the first cut the pieces marked with noughts are removed: after the second cut, those marked with crosses.

This figure shows how to represent $\frac{2}{3}$ times $\frac{5}{7}$ as a single fraction We have divided our square yard into 21 pieces – all the same size and shape. The rectangle, $\frac{2}{3}$ by $\frac{5}{7}$, contains 10 of these little pieces. Each piece is $\frac{1}{21}$ of a square yard, so our answer is $\frac{10}{21}$. In fact, we have found the rule for multiplying fractions –

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7}$$

A common mistake found in examination papers is due to pupils mixing up the rule for adding and for multiplying fractions. They write, for instance –

$$\frac{1}{3} + \frac{3}{5} = \frac{1 + 3}{3 + 5}$$

which is complete nonsense, since the answer thus obtained, $\frac{4}{8}$, reduces to $\frac{1}{2}$, which is *less* than $\frac{3}{5}$.

With parrot learning, such a mistake is quite natural. '$\times$' has been turned into '$+$': that is all. Such a mistake is much less likely in a pupil who has made experiments with $\times$ and $+$, and has come to *feel* the entirely different meanings of these two signs.

The reader may care to work out a diagram which illustrates the correct way of adding $\frac{1}{3}$ and $\frac{3}{5}$.

Decimals

No difficulty at all should be found in teaching or learning decimals. Decimals can be demonstrated by exactly the same 'team work' as was suggested for whole numbers.

The measurement of a line is a convenient illustration. A metre is a French measure, not much different from a yard. A decimetre is one-tenth of a metre; a centimetre one-tenth of a decimetre; a millimetre one-tenth of a centimetre. A line whose length is 1 metre, 3 decimetres, 2 centimetres, and 5 millimetres is written for short as 1·325 metres.

While in English measure it is not simple to turn 2 yards 1 foot and 3 inches into inches, it is at once obvious in French measure that 1·325 metres is 1325 millimetres, or 132·5 centimetres, or 13·25 decimetres.

An ordinary school ruler has millimetres, centimetres, and decimetres marked on it. It is therefore quite easy to build up the length mentioned above – one strip a metre long, three strips of a decimetre, two of a centimetre, five of a millimetre.

Addition of decimals is the same as addition of whole numbers. Multiplication of decimals can be illustrated by means of rectangles, just as was done with ordinary fractions.

Negative Numbers

A picture in *Punch* during the 1914–18 war showed an official saying to a farmer, 'My dear sir, you cannot kill a whole sheep at once!'

This absurd remark illustrates the fact that *fractions* have no meaning for certain things: you cannot have half a live sheep: you cannot tear a sheet of paper into $3\frac{1}{2}$ pieces. But fractions have a meaning in other connexions: it is quite easy to have $3\frac{1}{2}$ feet of lead piping.

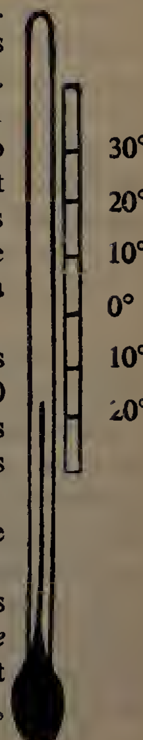
In the same way, there are times when you cannot speak of numbers less than 0: there are other times when you can.

A man may have no children, but he cannot have less than none. A box may have nothing in it: it cannot have less than nothing.

But there are examples where we can go below 0. For instance, in the Fahrenheit system of temperatures, water freezes at 32 degrees, a mixture of water and salt freezes at 0 degrees, and it is possible to have temperatures much colder than this. These temperatures are written with a *minus* sign. Thus —10 degrees means the temperature 10 degrees colder than 0 degrees. A temperature of —22 degrees is met with in refrigerators using ammonia. Note that —22 degrees is *colder* than —10 degrees.

In the same way, we may deal with heights and depths. If a bomb falls into the sea from a height of 50 feet we can trace its descent from 50 feet, to 40, 30, 20, 10, and 0 feet above sea-level. But the bomb need not stop at sea-level. It may descend 10 feet below sea-level, and we may speak of this as a *height of* —10 *feet.*

A man who is in debt to the extent of £1 is worse off than a man who has no money and no debts (like a tramp). The tramp at least is free. If we call the tramp's fortune £0, we may call that of the other man £(—1). When you own £(—1), someone has to give you £1 before you reach the position of owning nothing. To own £(—100) means to be bankrupt to the extent of £100. Again —100 is *worse* than —1. If a minus sign is in front of a number, the order of the numbers is turned right round. £(—1) represents a *better* fortune than £(–10,000).



In the same way, an army retreating at 10 miles an hour might be spoken of as 'advancing at —10 miles an hour'. If the army is moving at '—1 miles an hour', that is *better* than moving at '—10 miles an hour'.

A minus sign turns everything upside down, like the reflection of trees and houses in a river.

For a long time, mathematicians felt that it was unfair to use minus numbers (also called *negative numbers*), but it was found in the course of time that minus numbers could be used, added, subtracted, multiplied and divided, and useful results obtained.

Working With Negative Numbers

We may see how to use minus numbers, if we think of ordinary numbers as meaning something *given*, minus numbers as something *taken away*. We might think of 5, for instance as a five-pound note, or as something given five times: —5 would then mean a bill for £5, or something taken away five times.

Very often we put brackets round minus numbers. For instance, if we want to say 'add —4 to —3', it looks rather queer if we write simply —4 + —3. So we write (—4) + (—3). This means that the thing in the first bracket, —4, has to be added to the thing in the second bracket, —3. (—4) — (—3) would mean that we had to take —3 away from —4.

What would these things mean in practice? We might say that —4 added to —3 meant that a man already owed £4, and then he got a further bill for £3, so that he would be altogether £7 in debt. Or an army might have lost 4 miles of territory, and then lost another 3 miles. The second *loss* has been *added* to the *first*. In either case, we see that a *loss of* 4 together with a *loss of* 3 is the same as a single *loss of* 7. In the signs of arithmetic, (—4) + (—3) = (—7).

In the same way, if we have to add 4 and —3, this means a *gain* of 4 followed by a *loss* of 3, which is clearly the same as a single gain of 1. In short, 4 + (—3) = 1. In fact, 4 + (—3) means exactly the same as 4 —3.

There is nothing new in this, except the signs, and these signs are often used in ordinary life, to show the changes in trade, in unemployment, in the state of parties at elections, + for increases, — for decreases.

Subtracting minus numbers is sometimes a little confusing at first. It is well first to be clear what subtraction means. 7 —3 = 4 means that a man with £7 as compared with a man having £3 is £4 better off. Subtraction means comparing two things. And we can compare losses as well as gains. An army which has lost 200 men is better off than an army which has lost 1,000 men to the extent of 800 men's lives saved. A *loss* of 200 is written for short —200. A *loss* of 1,000 is written —1,000. To *compare* the two we subtract. $(-200) - (-1{,}000) = 800$. Note that there is no *minus* sign with the 800. If two opposing armies begin by being equal, the one that loses 200 men is *stronger* than its opponent, who loses 1,000, by 800 living men.

We could instead interpret $(-200) - (-1{,}000) = 800$ as meaning that a man bankrupt for £200 is better off than a man bankrupt for £1,000 by £800. Or we could say that a wreck 200 feet below sea level is higher up than one 1,000 feet down by the amount of 800 feet. It would be correspondingly easier to salvage.

Multiplication? We can only mention this briefly. We may think of 4×5 as meaning, 'Give someone four £5 notes.' This is the same as giving £20, and $4 \times 5 = 20$.

What would $4 \times (-5)$ mean? -5 stands for taking away £5, or for a bill for £5. $4 \times (-5)$ means 'Four bills for £5', the same then as 'A bill for £20'. So $4 \times (-5) = -20$.

$(-4) \times 5$ comes to much the same thing. It would correspond to 'Take away four £5 notes', that is, 'Take away £20.' So $(-4) \times 5 = -20$.

The trickiest case is $(-4) \times (-5)$. Treating -5 as meaning 'A bill for £5' and -4 as 'Take away 4 times', $(-4) \times (-5)$ would mean 'Take away 4 bills for £5'. If the postman comes to you and says, 'I think you have four bills for £5. They should have been delivered to the family next door,' you find yourself £20 better off than you would have been, had the bills really been meant for you. 'Better off' means $+$. So the effect of two minuses, *multiplied together*, is to give $+$. We conclude that $(-4) \times (-5) = 20$.

You may very likely feel that this is a tremendous song-and-dance about nothing at all. Everyone knows that you are better off if your creditors destroy your I.O.U.s. Why make all this fuss about $+$ and $-$ signs? The answer is that we are not going to use minus signs simply to find out what happens to people in debt. Rather we are going to be concerned with formulae, such as $y = x^2 - 3x$, or $y = (x-1)(x-2)$, in which $-$ signs may occur. That is why we have to know how to handle minus signs. What formulae are, and what uses can be made of them, will appear in later chapters.

Imaginary Numbers, or Operators

You will notice that 3×3 is 9, and -3×-3 is also 9. There is no ordinary number (either + or —) which, when multiplied by itself, gives —9. 'Two minuses make a plus.'

It is usual to call 3×3 'the square of 3': 9 is the square of 3. 9 is also the square of —3. 3 and —3 are called the 'square roots' of 9.

Every possible number has two square roots: one +, one —. The square roots of 4 are +2 and —2. The square roots of 10 are (very nearly) 3·16 and —3·16.

But negative numbers do not seem to have any square roots. —9 has not, nor has —4 or —10. So far as square roots are concerned, negative numbers are the Cinderellas of mathematics. But mathematicians have succeeded in finding a sort of substitute. If they have not found a man for Cinderella, they have at least found a robot. These robots are called Operators. Operators are not Numbers, but they can do many things which real numbers do, just as robots do *some* things that men do. For instance, you can multiply operators. And a particular operator, called '3i', is such that 3i times 3i is —9, while another, called 'i', is such that $i \times i = -1$.

This 'i' sounds like a mathematical fairy story. The interesting thing is that for many very practical purposes – such as wireless or electric lighting – 'i' is very useful indeed. We shall later on explain what 'i' is, and show that there is nothing mysterious about it at all.

EXERCISES

Questions of Pattern

In questions 1–4, squared paper can be used, as was done in connexion with the multiplication tables. For each question, the reader should consider what is the best number of squares to have in each row. For instance, in the first question, if we take 9 squares in each row (as illustrated) what is happening becomes clear.

1. Albert Smith and his wife Betty are both in the forces. Albert is off duty every ninth evening: his wife is off duty every sixth evening. Albert is off duty this evening: Betty is off duty tomorrow evening. When (if ever) will they be off duty the same evening?

A	B						B	
A				B				
A	B						B	
A				B				
A	B						B	
A				B				

In the pattern, each square represents an evening. It is marked A when Albert is free, B when Betty is. It will be seen that the Bs always come in the 2nd, 5th or 8th column – never in the 1st, where the As are. The answer is: They are never free together.

2. In Question 1, would it have made any difference, if Betty had been free every fifth night, instead of every sixth?

3. On a fire-watching rota Alf watches every third night, Bill every fourth night, Charlie every fifth night, Dave every sixth, and Edward every seventh night. All the men begin their duties on the same night, a Friday.

How long will it be before Alf and Bill are again on duty together? Alf and Charlie? Bill and Charlie?

Will Alf and Charlie ever be on duty together without Dave being there?

On Fridays, when not fire-watching, the men attend a club. How often does Alf miss club night? How often do the others miss it?

Is there any night of the week for which Alf can make a regular appointment? Or does he, sooner or later, do duty on every day of the week? How are the others placed in this respect?

(Use squared paper, with 7 in a row, so that all the Sundays come in one column, etc.)

4. Can you see any principle underlying the answers to Question 3? Can you answer the question – How long will it be before Alf, Bill, Charlie, Dave, and Edward are all again on duty the same night? What night of the week will this be?

5. Two men are walking side by side. One takes four steps in the same time that the other takes three. At the beginning they step off together. In what order will the sound of their footsteps be heard?

(Draw a line, representing the passing of time, and mark on it the moments when the feet of the two men strike the ground.)

6. Question 5 may be varied as much as desired – 5 steps against 4, 7 steps against 5, etc. – and the diagrams drawn.

7. A box has to be made, which can be exactly filled, *either* with packages 6 inches long, *or* with packages 8 inches long, placed end to end. What is the smallest length that the box can be made?

Two Questions for Research

A mathematician does not usually just solve a problem and then forget all about it. If he has solved a problem, he starts altering the conditions of the problem, and sees if he can still solve the problem. He wants to make sure that he will be able to answer any question *of that type* that may confront him in future. He wants to discover if there is any simple principle or rule underlying the problem. You can see from the two examples below how a mathematician goes to work.

The Cup-Tie Problem. If 7 teams enter for a knock-out competition, how many matches will have to be played? (It may be assumed that there are no draws or replays.)

In the first round, one team must be given a bye, and 3 matches will be played. This will leave 4 teams for the second round, the semi-final. There will be 2 matches in the semi-final. There is 1 match in the final round. The total number of matches is $3 + 2 + 1$. The answer is therefore 6.

The particular problem is easily answered. But suppose, instead of 7 teams, 70 or 700 had entered. How many matches would be necessary then? It would take rather a long time to work out directly. It would help us a lot if we could find a simple rule that saves working out all the rounds separately.

To see if there is any simple rule, a mathematician would start working out the simplest possible cases. If only 1 team entered, no matches at all would be necessary. 2 teams could settle the question by playing 1 match. Work out how many matches would be necessary in the cases where 3, 4, 5, 6, etc., teams entered. You will soon see that there is a simple rule connecting the number of teams with the number of matches.

Lastly, can you see the reason *why* there is this simple rule?

How many matches would have to be played if 2,176,893 teams entered a competition?

The Income Problem. There is a well-known puzzle, as follows.

Two clerks are appointed in an office. Smith is to be paid yearly, starting at £105, and rising by £10 each year. Jones is to be paid half-yearly, starting with £50, and rising £5 each half-year. Which has made the better bargain?

Most people are rather surprised when they see how this works out. All one needs to do is to write down what each receives, as follows –

	SMITH	JONES		
		January–June	July–December	Total
	£	£	£	£
1st Year	105	50	55	105
2nd Year	115	60	65	125
3rd Year	125	70	75	145
4th Year	135	80	85	165

One naturally thinks that a rise of £5 every half-year is the same as a rise of £10 every year. But it is not The yearly salary of Jones rises by £20 each year. He has made a much better bargain than Smith.

This question naturally suggests others. A rise of £5 every six months is as good as a rise of £20 a year. What would have happened if the payment had been quarterly? What is a rise of £5 a quarter worth, in terms of yearly salary? Or monthly pay? What is £1 a month rise worth, in yearly pay? What is 1*s.* a week's rise in weekly pay equal to?

Or the other way round – what rise every six months is the same as a rise of £10 each year? Every quarter? Every month? Every week?

What is the principle involved? Why do things work out this way?

CHAPTER 6

HOW TO FORGET THE MULTIPLICATION TABLE

'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you so found out, I wonder nobody found it out before, when now known it is so easy.'

– Briggs to Napier (From F. Cajori, *History of Mathematics*)

If you ask an engineer, 'What is 3 times 4?' he does not answer at once. He fishes a contraption known as a slide-rule out of his pocket, fiddles with it for a moment, and then says, 'Oh, about 12'. This may not impress you very much. But if you say to him, 'What is 371 times 422?' he will give you the answer to this in just about the same time, and without needing to write down any figures.

What is a slide-rule? How is it made? How was it invented? How is it used?

A slide-rule consists of two scales, on each of which can be seen the numbers 1, 2, 3, 4, 5, etc. These numbers are not spaced evenly, like the numbers on a 12-inch ruler. The distance between

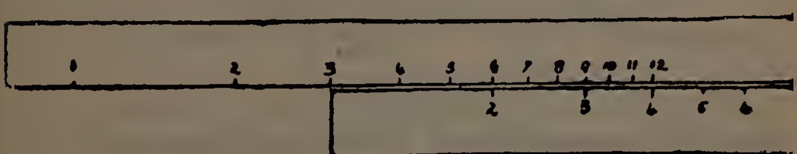


Fig. 2

2 and 3 is less than that between 1 and 2, and the further you go the more closely the numbers are crowded together.

Fig. 2 shows how the engineer would set his slide-rule to find 3×4. He pushes the lower scale along, until the 1 on it is opposite the 3 on the upper scale. Now notice how the numbers stand opposite each other. Above 2, there stands 6: above 3, 9: above 4, 12. Above every number on the lower scale, one finds three times that number on the upper scale. So we read off the number above 4, and that gives us our answer 12.

What principle lies behind the working of this instrument? How could anyone have been led to invent it? Why is it possible to make a multiplying machine at all?

We are all familiar with machines which man uses to *multiply* his own strength – pulleys, levers, gears, etc. Suppose you are fire-watching on the roof of a house, and have to lower an injured comrade by means of a rope. It would be natural to pass the rope round some object, such as a post, so that the friction of the rope on the post would assist you in checking the speed of your friend's descent. In breaking-in horses the same idea is used: a rope passes round a post, one end being held by a man, the other fastened to the horse. To get away, the horse would have to pull many times harder than the man.

The effect of such an arrangement depends on the roughness of the rope. Let us suppose that we have a rope and a post which multiply one's strength by ten, when the rope makes one complete turn.

What will be the effect if we have a series of such posts? A pull of 1 lb. at *A* is sufficient to hold 10 lb. at *B*, and this will hold 100 lb. at *C*, or 1,000 lb. at *D* (Fig. 3).

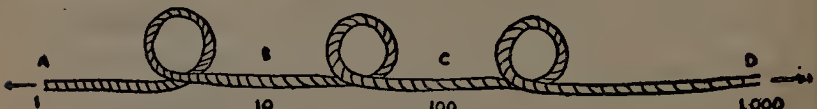


Fig. 3

Each extra post multiplies by 10. One post magnifies by 10: two by 10×10: three by $10 \times 10 \times 10$.

As it takes too much space to write long rows of tens, an abbreviation is usually written. 10^2 is written for 10×10. 10^3 for $10 \times 10 \times 10$, and so on. (In the same way, 8^5 would mean $8 \times 8 \times 8 \times 8 \times 8$.)

Thus 10^8 *will represent the effect of* 8 *posts*, and 10^{11} the effect of 11. This is a *multiplying* effect. If we pass a rope round 8 posts and then round a further 11 posts, the effect will be $10^8 \times 10^{11}$. But 8 posts and 11 posts add up to 19, so that this must be exactly the same thing as 10^{19}.

The number of turns required to get any number is called the logarithm of the number. For instance, you need 6 posts to multiply your strength by 1,000,000. So 6 is the logarithm of 1,000,000. In the same way, 4 is the logarithm of 10,000.

So far we have spoken of whole turns. But the same idea would apply to incomplete turns. If you gradually wind a rope round a post, the effect also increases gradually. At first you must bear the entire weight yourself: as the rope winds on to the post, friction comes to your aid, and there will be stages at which you can hold twice, three times, four times the amount of your pull. When one complete turn is on, you will have reached ten times.

Accordingly, $10^{\frac{1}{2}}$ will mean the magnifying effect of half a turn, $10^{2\frac{3}{8}}$ will mean the effect of $2\frac{3}{8}$ turns. And so for any number.

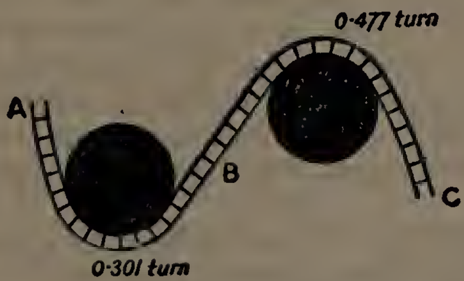


Fig. 4

The logarithm of 2 will be that *fraction of a turn* which is necessary to magnify your pull 2 times. This number is usually called 'log 2' for short. Actually, 0·301 of a turn is required to magnify 2 times. 0·477 of a turn magnifies 3 times: so that log $3=0{\cdot}477$. (These numbers could be found by experiment.)

We can put this another way round. 2 is the effect of 0·301 of a turn. So $2=10^{0{\cdot}301}$. In the same way, $3=10^{0{\cdot}477}$.

Now what will happen if we wind 0·301 of a turn on one post, and then 0·477 on the next?

We know that the effect of the first post is to double our effort. If we pull A with a force of 1 lb. it will be sufficient to balance a tension of 2 lb. in the rope at B (Fig. 4). The second post multiplies by 3: 2 lb. at B will balance 6 lb. at C. So 1 lb. at A will hold 6 lb. at C. And, on the two posts together, we have $0{\cdot}301+0{\cdot}477 = 0{\cdot}778$ of a turn.

0·778 of a turn is needed to multiply 6 times. 0·778 is the logarithm of 6. The logarithm of 3×2 has been found by *adding together* log 3 and log 2.

It is not necessary to use separate posts. We can economize in timber by winding the rope again and again round the same post. The only thing that matters is the length of rope *in contact* with the wood. (The post itself must be round. Corners would cause complications.)

If we were given two pieces of rope, and knew that one piece was sufficient to multiply 7 times, and the other sufficient to multiply 8 times, we should only have to join these pieces end to end, to get a piece that would multiply by 7×8.

It is exactly this principle of joining *end to end* that is used in the slide rule. On the slide rule, the distance between 1 and 3 is equal to the length of rope required to multiply 3 times; the distance between 1 and 4 is equal to the length of rope required to multiply 4 times; and in finding 3×4 we place these lengths end to end.

1 of course comes at the end of the scale, as you do not need any rope at all to multiply your strength by 1.

You will now be able to see why numbers crowd together on the slide-rule as we go farther along. 1 corresponds to no rope; 10 to 1 turn; 100 to 2 turns; 1,000 to 3 turns. The distance on the slide-rule from 1 to 10 is the same as that from 10 to 100, or from 100 to 1,000: each of these is equal to 'one complete turn'. But we have only 9 numbers to fit between 1 and 10: there are 90 between 10 and 100, 900 between 100 and 1000. This accounts for the overcrowding of the larger numbers.

If we want to get a set of evenly spaced numbers we have to take a set like 1, 10, 100, 1000 . . . or 1, 2, 4, 8, 16, 32 . . . In the first set, each number is 10 times the previous one: there is 1 turn between each and the next. In the second set, each number is twice the previous one: at each step we add a length of rope equal to 0.301 of a complete turn.

How Logarithms are Calculated

We have explained what a logarithm is, but we have not shown how to calculate it. We have said that, on a slide-rule, the distance between 1 and 7 is the length of rope needed to magnify a pull 7 times – *i.e.*, log 7. But to make an actual slide-rule we should need to know log 2, log 3, log 4, etc., so that we could mark 2, 3, 4 . . . at the corresponding distances.

The only logarithms that have been found so far are those of 10, 100, 1000, etc. We know that these are 1, 2, 3 . . . All this tells us about log 70 is that it must lie somewhere between 1 and 2: for we need more than one turn, but less than 2, to produce any numbers between 10 and 100.

There is one other thing that is vague. We have spoken all along of so many complete 'turns'. But the size of the post has not been specified. We could in fact take a circular post of any size, and pass a rope round it. This arrangement might multiply our pull by less than 10: we could correct this by making the post more rough. If it magnified our pull too much, we could correct this by polishing the post. So that we may suppose 'one turn' to represent any length we like. A slide-rule can be made any size we like. We could, for instance, mark 1 at the end of the scale, and 10 at a distance of one foot. 100 would have to come 2 feet from 1, 1000 3 feet – and by this time we should feel that the whole thing had got quite large enough. Notice that our simple argument has only helped us to mark four points on a yard stick – 1, 10, 100, 1000.

But we could go about the question another way. If we start from 1 and keep *doubling*, we shall also get a set of evenly spaced points: the distance between each point and the next is log 2. (Earlier, we stated that log 2 was 0·301. But *no reason* was given for this statement.) Instead of fixing our scale by taking 10 at 1 foot, suppose we fix it by taking 2 at a convenient distance. We might choose it at an inch from 1. 4, being 2×2, must now come at 2 inches; 8, being 2×4, will come at 3 inches; 16 at 4 inches; 32 at 5; 64 at 6; 128 at 7; 256 at 8; 512 at 9; 1024 at 10. This slide-rule has turned out to be smaller than the last one: 1000 this time has come just below 10 inches from 1. But that is not the important point. The chief thing to notice is that the first slide-rule had only four points on it – 1, 10, 100, 1000. But our second attempt has given us *eleven points* – 1, 2, 4, 8, 16, 32, 64, 128 256, 512, 1024. This suggests that we shall get still better results by taking, instead of 10 or 2, some number closer to 1, such as $1\frac{1}{8}$, or 1·1, or 1·01. It will take more work, with the smaller numbers, to get from 1 to 1000; but once the slide-rule has been made, we shall be able to use it whenever multiplication has to be done, and the work is thus repaid.

Before we leave our slide-rule with eleven points, we may notice that it enables us to get a rough idea of the value of log 2. We saw that 1024 was marked at a distance of 10 inches, so that 10 inches of rope around the post multiply our effort by 1024. But we know that three complete turns multiply by 1000. So that 10 inches must be slightly more than three complete turns. One inch must be slightly more than $\frac{3}{10}$, or 0·3, of a turn. But the figure 2 is marked at a distance of 1 inch. So that 2 corresponds to slightly more than 0·3 of a turn, which is the same as saying that log 2 is just over 0·3. So that our earlier statement that log 2 was 0·301 was at least near to the truth.

How Logarithms were Invented

We made our second slide-rule by a process of continual doubling. We shall now make a better one, using 1·1 instead of 2.

Suppose, then, that we mark on our scale two points, to represent 1 and 1·1. The distance between them could be, say, $\frac{1}{10}$ inch. We then know that moving $\frac{1}{10}$ inch down the scale (if you prefer, adding $\frac{1}{10}$ inch to the length of the rope) represents multiplication by 1·1. We shall thus be able to mark the points 1, 1·1, 1·21, 1·331, etc., each number being one and one-tenth times the previous one.

This set of numbers is exactly the set that you get if you allow £1 to accumulate at compound interest of 10% over a number of years. Each year that passes increases the sum invested by one-tenth – that is, every year multiplies the amount by 1·1. It was probably the study of tables of compound interest that originally suggested the idea of logarithms to their inventor, Napier.

Some of the numbers found in this way, and the distances at which they have to be marked, are shown in the following table.

Number	*Distance (inches)*
1·948	0·7
2·143	0·8
2·852	1·1
3·137	1·2
5·053	1·7
6·725	2·0
7·397	2·1
9·846	2·4
10·831	2·5

This shows us that the figure 2 has to be marked somewhere between 0·7 and 0·8 inch; 5 just below 1·7 inches; 7 somewhere between 2·0 and 2·1 inches; 10 a little above 2·4 inches. So 'one turn' corresponds to a little more than 2·4 inches.

This information is still not sufficient for making a really good slide-rule. For instance, we cannot find an accurate position for the figure 7. Our table contains no number between 6·725 and 7·397. We can only guess where 7 lies, between these two numbers. Our slide-rule would, in fact, be liable to errors of about 10%: owing to the fact that the numbers are obtained by adding on 10% at a time – we cannot expect any higher degree of accuracy.

We can use this table to find the logarithms of the numbers 2, 3, 4, etc., but the results for this also are likely to be crude. 'One turn' is the distance corresponding to 10: we guess it to be 2·42 inches. To 2 corresponds a distance between 0·7 and 0·8 inch – perhaps 0·73. If we express 0·73 as a fraction of a 'turn', we shall have an estimate of log 2. 0·73 divided by 2·42 gives 0·3016 – a suspiciously good result for a guess!

The reader will easily see how accurate slide-rules and tables of logarithms can be made, by using a number such as 1·000001 for repeated multiplication. Napier, in making the first tables of logarithms, used 1·0000001.

It is not, of course, necessary for us to make our own tables of logarithms. This work has been done once and for all. The only advantage to be gained from making your own table of logarithms and your own slide-rule is the insight this gives into the underlying principles.

The method of making logarithm tables, described above, shows clearly why such tables can be used for multiplication. We found, for instance, that multiplying by 1·1 seven times was the same as multiplying once by 1·948; while multiplying seventeen times by 1·1 was the same as multiplying once by 5·053 (see the table above). So 1·948 $\times$ 5·053 corresponds to 7 multiplications by 1·1, followed by 17 more – that is, it corresponds to 24 multiplications. And this (from our table) corresponds to 9·846. So 1·948 $\times$ 5·053 = 9·846.

The method is clear. 1·948 is the 7th number; 5·053 is the 17th; $7 + 17 = 24$; the 24th number in the table gives the answer.

In making our slide-rule, we put 1·1 a tenth of an inch from 1, 1·948 seven times as far away, and so on. It does not matter how far 1·1 is from 1, so long as 1·948 is seven times that distance, 5·053 seventeen times as far, etc. The argument will still hold.

In ordinary logarithm tables, log 10 is 1. We saw, on our slide-rule, that 10 came between 24 and 25 times the distance of 1·1. If we choose a distance for 1·1 which lies somewhere between $\frac{1}{24}$ and $\frac{1}{25}$ inch, we shall get 10 coming at a distance of 1 inch. The distance corresponding to any number will be its logarithm.

This change of scale rather camouflages the simple relation,

$7 + 17 = 24$. In the logarithm tables, 1·1 corresponds to ·0414; 1·948 to ·2896; 5·053 to ·7036. In the surface, there is nothing simple about these numbers. But notice these facts. (i) log 1·1 lies between $\frac{1}{24}$ and $\frac{1}{25}$, as we expected, (ii) log 1·948 is seven times log 1·1, and log 5·053 is seventeen times log 1·1. The simple relations are still there. The change of scale does not in any way alter the method: to multiply numbers we add their logarithms.

If we are asked to calculate an expression such as 12^{35} – i.e., the effect of multiplying by 12 thirty-five times – this is easily done. To multiply by 12, one has to add log 12 to the logarithm of the number being multiplied. If one multiplies by 12 thirty-five times, one will thus add log 12 thirty-five times; log 12 is 1·0792. Thirty-five times this is 37·772. 37·772 is the logarithm of 5916 followed by 34 noughts! So this, roughly, is the effect of multiplying by 12 thirty-five times. It would take rather long to find this result by any other method.

A Musical Slide-Rule

One well-known object is in effect a slide-rule – a piano keyboard. The strings at the bottom end of a piano vibrate slowly: as one goes up the keyboard, the rate increases. An octave corresponds to doubling the rate of vibration. Each note vibrates about 6% more rapidly than the note immediately below it. Every time one goes a certain distance along the keyboard, one multiplies the rate by a corresponding amount. This is just the same thing as happens on a slide-rule.

EXERCISES

1. If you can get hold of a slide-rule and a book of Logarithm Tables, verify the statement made in the text, that every number is marked on the slide-rule at a distance proportional to its logarithm.
2. Make a slide-rule for yourself, using tables of logarithms to tell you at what distance each number should be marked.
3. Make a slide-rule by the method explained in Chapter 6.

4. Where is the square root of 10 marked on a slide-rule?

5. Check the accuracy of your slide-rule by finding on it 2×2, 2×3, 4×5 and other simple multiplications.

6. The logarithm of 2 is 0·301. The logarithm of 1·05 is 0·0212. How many years will money invested at 5% require to double itself?

7. An Eastern monarch sends 10,000 golden vessels to a brother monarch, whose kingdom is many days march distant. The gift is carried on camels. Each merchant, who supplies camels for some part of the journey, demands as commission 10% of what passes through his hands. Thus the first merchant hands over to the second not 10,000, but 9,000 golden vessels. Altogether, the vessels pass through the hands of twenty merchants. How many vessels does the brother monarch receive at the end?

CHAPTER 7

ALGEBRA – THE SHORTHAND OF MATHEMATICS

'*Mathematics* is *a language*.' – J. Willard Gibbs.

ALGEBRA plays a part in mathematics which may be compared to that of writing or of shorthand in ordinary life. It can be used either to make a statement or to give instructions, in a concise form.

Shorthand, of itself, does not make new discoveries possible. In the same way, most problems that can be solved by algebra can also be solved by common sense. Statements in algebra can be translated into ordinary speech, and vice versa. The statement in algebra is much shorter: some facts or instructions, which are easily written in algebraic form, are too long and complicated in ordinary speech. This is the advantage of algebra: while results *could* be got without it, it is unlikely that they *would*.

We shall consider some simple questions – perhaps not very useful ones – to illustrate the form given to common-sense arguments, when the symbols of algebra are used.

The Cakes and Buns Problem

Most books on algebra, in a chapter headed 'Simultaneous Equations', deal with some such question as this. 'I visit a tea-shop on two occasions. The first time I order two buns and a cake; my bill is for 4*d.* The second time I order three buns and two cakes; the bill is for 7*d.* What are the prices of buns and cakes?'

I have tried this problem on people who know nothing about algebra, and they usually solve it. They argue: the second bill is 3*d.* more than the first. So 3*d.* represents the cost of the extra bun and cake. A bun and a cake cost 3*d.* But two buns and a cake cost 4*d.* So the difference, 1*d.*, is the cost of a bun. A cake must cost 2*d.*

This problem may not sound important, yet in one form or another it repeatedly occurs in mathematical investigations of a very practical type. We shall ourselves need to solve such a problem in Chapter 8.

Mathematicians have therefore been forced to apply the argument outlined above on many different occasions. And – like other people – they have gradually introduced abbreviations to shorten the work. One can imagine the argument soon being written –

	2 buns & 1 cake	4*d.*
	3 buns & 2 cakes	7*d.*
So	1 bun & 1 cake	3*d.*
But	2 buns & 1 cake	4*d.*
So	1 bun	1*d.*
And	1 cake	2*d.*

Later one might begin to write '*b*' where 'bun' comes, '*c*' for 'cake'. If we replace '&' by '+', we have the modern form

$$2b + c = 4$$
$$3b + 2c = 7$$

So $b + c = 3$
But $2b + c = 4$
So $b = 1$
And $c = 2$

In this, b stands for the number of pence paid for a bun, c the number paid for a cake. You will notice that we write $2b$ for twice the number b. We do not write any multiplication sign between the 2 and the b. It is no use arguing whether a multiplication sign ought to be written here. If you feel happier with $2 \times b$, by all means write it that way. It is open to the objection that we often use x to represent a number, and $\times$ might easily be confused with x.

Of course $12b$ means *twelve* times b, not $1 \times 2 \times b$. You may feel it is confusing to have this distinction – but every shorthand system has its faults. In algebra, numbers such as 123 placed together have the same meaning as in arithmetic, but $2bc$ mean $2 \times b \times c$.

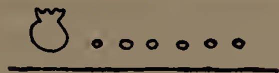
Try translating into ordinary language the following statements:

$$b + c + t = 6$$
$$2b + 3c + t = 11$$
$$4b + 8c + t = 23$$

b and c here have the same meanings as before, but each meal now contains a pot of tea costing t pence. This problem is also quite simple to solve. If you consider the difference in cost between the first meal and the second, you will obtain an equation which contains only b and c. Comparing the second meal with the third, you will get another statement, in which the price of tea does not appear. You now have two statements about buns and cakes:

$$b + 2c = 5$$
$$2b + 5c = 12.$$

If a bun and 2 cakes cost 5*d*., 2 buns and 4 cakes – twice as much – must cost 10*d*. So $2b + 4c = 10$. But we have above $2b + 5c = 12$. Comparing these, we see that $c = 2$. So $b = 1$. Going back to the first meal, we see that $t = 3$.

WORDS	PICTURE	ALGEBRA
I. Think of a number.		n
II. Add 6 to it.		$n+6$
III. Multiply by 2.		$2(n+6)$ or $2n+12$.
IV. Take away 8.		$2(n+6)-8$ or $2n+4$
V. Divide by 2.		$\frac{2(n+6)-8}{2}$ or $n+2$
VI. Take away the number you first thought of.		$\frac{2(n+6)-8}{2}-n$ or 2.
VII. The answer is 2.		$\frac{2(n+6)-8}{2}-n=2$.

Each bag is supposed to contain as many marbles as the number you thought of, whatever that was.

The same picture can often be described in different ways. Thus we might describe the picture III as 'A bag and six marbles, twice,' or simply as 'Two bags and twelve marbles.' In algebraic shorthand, these descriptions are $2(n+6)$ and $2n+12$.

As a rule, there is no difficulty is solving problems of this type.

This shorthand can also be used to state truths. An old trick runs as follows. 'Think of a number. Add 6 to it. Multiply by 2. Take away 8. Divide by 2. Take away the number you first thought of.' Whatever number you think of, the answer is always 2. Why?

We could deal with this by thinking in pictures. You think of a number – any number. We will think of this as marbles placed in a bag. 'Add 6 to it.' This gives us one bag and six loose marbles. 'Multiply by two' – two bags and twelve marbles. 'Take away 8' – two bags and four marbles. 'Divide by 2' – one bag and two marbles. 'Take away the number you first thought of' – that is, take away the bag. Two marbles remain – whatever the number in the bag.

In algebra, we need not talk about bags of marbles. We say, let n stand for the number you think of. 'Add 6'; we get $n + 6$. 'Multiply by 2', $2n + 12$. 'Subtract 8', $2n + 4$. 'Divide by 2', $n + 2$. 'Take away n', 2 is the answer.

We can express the whole process, and the fact that the answer is always 2, by writing the single equation

$$\frac{2(n + 6) - 8}{2} - n = 2.$$

The expression on the left-hand side indicates that you take twice $(n + 6)$, subtract 8 and divide by 2, finally subtract n. One line of symbols replaces a paragraph of talk.

This example shows that two apparently different expressions may in fact represent the same thing. An important part of algebra therefore consists in learning how to express any result in the simplest possible way: this is known as Simplifying.

It is sometimes possible for a question to have two answers which at first sight appear different, but which are actually both correct.

Suppose, for instance, you were asked to discover the rule by which the following numbers have been chosen: 0, 3, 8, 15, 24, 35, 48, 63. You might notice that these numbers are given by the rule 1^2-1, 2^2-1, 3^2-1, 4^2-1, 5^2-1, 6^2-1, 7^2-1, 8^2-1. (You will remember from Chapter 6 that 5^2 is short for 5×5.) In short, the n^{th} number is n^2-1.

But you might also notice that 63 is 7×9, that 48 is 6×8, and so on. The eighth number is the number before 8 (that is, 7) multiplied by the number after 8 (that is, 9). The first number, 0, is the number before 1 (i.e., 0) multiplied by the number after 1 (i.e., 2). This suggests the rule for the n^{th} number: multiply the number before n (which is $n - 1$) by the number after n (which is $n + 1$). This gives us the formula $(n - 1)(n + 1)$.

Both these rules are correct. Whatever number you may choose for n, you will always find that $(n - 1)(n + 1)$ is the same as $n^2 - 1$.

Here we have used algebraic signs as a shorthand for writing *instructions*, how to find the numbers in a certain set. This use of algebra is very common. The person who uses a formula need not understand why a formula is right. For instance, a sapper who has to blow up a railway bridge will work out how much explosive to use by means of a formula: he does not need to know how the formula is obtained in the first place. In the same way, there is a rule which says, if you wish to see n miles out to sea, you must have your eyes $\frac{2n^2}{3}$ feet above sea-level. This formula is found by means of geometry: but without knowing any geometry you can use this formula, and discover that a tall man on the beach can see nearly 3 miles out to sea (since $\frac{2 \times 3^2}{3}$ gives 6 feet as the height needed), while to see 12 miles, you need a cliff 96 feet high. Such a formula could be used in designing a battleship, to tell how high an observer must be to see the effect of the ship's guns.

Most formulae contain several different letters. For instance, we might wish to know how much metal is required to make a circular tube. We must be told how long the tube is, how thick the wall is, and the measurement around the tube. Let us call the *length* L inches, the *thickness* T inches, the outside *measurement* around the tube M inches. A formula tells us the tube will then contain $LT(M - 3{\cdot}14T)$ cubic inches of metal. Thus a tube 10 inches long, $\frac{1}{2}$ inch thick, and 15 inches round will contain $10 \times \frac{1}{2} \times (15 - 3{\cdot}14 \times \frac{1}{2}) = 5 \times 13{\cdot}43 = 67{\cdot}15$ cubic inches. A rule such as this could be stated in words, but would be much longer. The shorthand is so simple – L for length, T for thickness, *M* for measurement – that no difficulty can arise in learning it. Yet many people are terrified by the sight of a page of algebraic symbols, and others have a reputation for immense intelligence because they understand algebra.

In bygone ages, a man who could read and write counted as a scholar. Today we think nothing of reading and writing. Algebra too is a language – neither more nor less mysterious than ordinary print, once its alphabet and its grammar have been learned.

EXAMPLES

1. We may express the instructions, 'Think of a number (n), double it, and add 5' by the shorthand sign $2n + 5$. Translate into algebraic shorthand the following sentences –

(i) Think of a number, add 5 to it, and double the result.
(ii) Think of a number, multiply by 3, and add 2.
(iii) Think of a number. Write down the number after it. Add the two numbers together.
(iv) Multiply a number by the number after it.
(v) Think of a number, and multiply it by itself.

2. Translate the following shorthand signs back into sentences such as those of Question 1.

(i) $4n + 4$. (iii) $3(n + 1)$. (v) $\frac{1}{2}n(n - 1)$.
(ii) $n - 1$. (iv) $\frac{1}{2}(4n + 8)$.

Work out what these signs would give if the number thought of, n, happened to be 6. Also what they would give if the number n was 3.

3. To see n miles out to sea you must have your eye $\dfrac{2n^2}{3}$ feet above sea-level. Make a table showing how high your eye would have to be, in order to see 0, 1, 2, 3, ... 10 miles.

4. We have seen that $(n - 1)(n + 1)$ gives exactly the same number as $n^2 - 1$. We tested this by putting n in turn equal to 1, 2, 3, ... 8. This did not prove that the two expressions were

always the same, but it made it likely. By this test you can show that some of the statements below are *probably true*, while others are *certainly untrue*. Which are which? Here n means 'any number.'

(i) $(n + 1) + (n - 1) = 2n$.

(ii) $n = 2n$.

(iii) $2(n + 3) = 2n + 6$.

(iv) $n^3 - 1 = (n - 1)(n^2 + n + 1)$.

(v) $4n + 2$ is always an even number (n being a whole number).

(vi) $n(n + 1)$ is always an even number (n being a whole number).

(vii) $(n + 1)(n + 1) = n^2 + 1$.

(viii) $\dfrac{2n}{n^2 - 1} = \dfrac{1}{n + 1} + \dfrac{1}{n - 1}$.

5. If it takes a minutes to set up a certain machine, and b minutes to make a single article on the machine, how long does it take to set up the machine and make 10 articles?

How long would it take to set up the machine and make n articles?

Write down in turn the time needed to set up the machine; the time needed to set it up and make 1 article; the time to set it up and make 2 articles, etc.

6. In question 5, the time taken to make n articles after setting up the machine is $a + bn$ minutes. The last part of the question requires us to put n in turn equal to 0, 1, 2 ... in the formula $a + bn$, giving a, $a + b$, $a + 2b$, ... etc. Putting a definite value for n in the formula is called *substituting* for n.

Thus, $a + 2b$ is the result of substituting $n = 2$ in the formula $a + bn$.

Again, if in the formula $n^2 - 1$ we substitute $n = 7$, we get $7^2 - 1$. In question 3, we substituted in turn 0, 1, 2 ... for n in the formula $\dfrac{2n^2}{3}$. You will be able to follow the argument of Chapter 8 only if you are familiar with this idea. That is the point of the following examples.

(i) If a train travels at v miles an hour, it goes nv miles in n hours.
Make a table showing how far it goes in 0, 1, 2, 3, 4 and 5 hours.

(ii) What does this table become if $v = 30$, i.e. if the train travels at 30 m.p.h.?

(iii) What does it become if $v = 10$?

(iv) What is the result of substituting $n = 4$ in the expression $5n - 1$?

(v) What is the result of substituting $n = 1$ in $2n^2+3n+5$?

(vi) If you substitute in turn, 0, 1, 2, 3, ... etc., for n in the formula $n^2 + 4n + 4$, what do you notice about the answers?

(vii) What is the result of substituting in turn $n = 0, 1, 2, \ldots$ in the expression $an^2 + bn + c$?

Note – This last question seems to trouble students, but we need the answer to it for Chapter 8. Compare it with section (v). Putting $n = 1$ in $2n^2 + 3n + 5$ gives 10. For when $n = 1$, n^2 becomes 1^2, that is, 1, and the expression $2n^2 + 3n + 5$ becomes $2 \times 1 + 3 \times 1 + 5$, that is, $2 + 3 + 5$. So that when $n = 1$, the three numbers that occur in the expression (2, 3, and 5) *are simply added together*. This would happen whatever these numbers were. If you put $n = 1$ in the expression (say) $10n^2 + 17n + 35$, you would get the result $10 + 17 + 35$, which adds up to 62. If you put $n = 1$ in any expression of this type (so much n^2, so much n, and a number added together), you get the answer by adding together the three numbers that occur in the expression. So, if you put $n = 1$ in $an^2 + bn + c$ you get the answer $a + b + c$.

In the same way, you will find that the result of putting $n = 0$ is simply to give you the third number.

For instance, when $n = 0$ the expression $4n^2 + 173n + 45$ becomes 45.

When we substitute $n = 0$, $an^2 + bn + c$ becomes c.

You will also find that the result of substituting $n = 2$ is to give you

4 times the first number in the formula.
+2 times the second number in the formula.
+ the third number in the formula.

In shorthand, this result is $4a + 2b + c$.

Find for yourself the rules for what you get when you put $n = 3$, or $n = 4$, or $n = 5$.

If you find great difficulty with this chapter, try to get hold of an engineer, who will tell you just what he does when he uses a formula for a practical problem. The earlier part of Chapter 8 may also help you to see how a formula is used in practical life. It is only towards the end of Chapter 8 – 'The Calculus of Finite Differences' – that we have to do problems similar to Question 6, above.

CHAPTER 8

WAYS OF GROWING

'*When it is considered how essential is their use in a vast range of trades and professions – from plumbing to* Dreadnought *building – it is hardly extravagant to say that facility in the working, interpretation and application of formulae is one of the most important objects at which early mathematical studies can aim.*' – T. Percy Nunn, *The Teaching of Algebra.*

WE are very often interested in knowing how quickly a thing grows. If one skyscraper is twice as high as another, it must have a stronger framework to support the extra weight. It will cost more than twice as much to build. How much more? Four times? Eight times?

As an army advances into hostile territory, the difficulty of bringing up supplies and maintaining communications increases. To cover 1,000 miles requires more than ten times the lorries necessary for 100 miles: how many, then?

If you are fire-watching on a high building, you may need to jump off in an emergency. Is it twice as dangerous to jump from 40 feet as from 20? Or more than twice? Or less than twice?

If a housewife is buying firewood, and pays *6d.* for a bundle measuring 1 foot around, how much should she pay for one measuring 6 inches?

In all these questions, we are interested in the different ways in which something grows. The answer may be quite important for practical purposes. Anyone who tries to provide housing economically has to know the answer to the first question: otherwise he may find that a saving in ground-rent has been altogether swallowed up in extra costs for building material.

Again, many would-be inventors have been disappointed through ignorance of the effects of change in scale. Many people have, in the past, invented flying-machines and successfully made small models which flew very well. They then enlarged their model, and built a full-size machine – and it would not fly at all. The reason was that the weight of a machine and the lifting-power vary in entirely different ways. If one constructed a flea, enlarged to the size of an elephant, its performance would be entirely different from that of an actual flea – as you can well imagine.

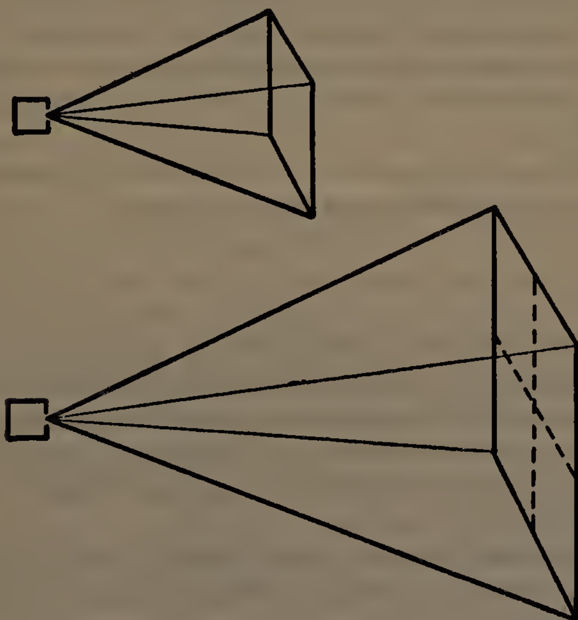
It is therefore quite natural that engineers and scientists expect mathematicians to supply them with a simple way of writing down the manner in which any quantity grows.

Of course, things grow in a great variety of ways. If, for instance, one considers the population of Manchester during the last 150 years, this is a quantity which has changed in a very complicated way. A mathematician can study it, and describe it, but you must not expect the description to be very short or simple. Many things will enter into it – the Industrial Revolution, the varying fortunes of the cotton trade, evacuation of children in war-time, and so on. On the other hand, some things vary in a very simple way. We shall give examples below. In between these two extremes come a great variety of types of growth, which can be studied and written down with greater or less difficulty. You must not expect mathematics to make a complicated question simple. Mathematics may help us to discover the underlying causes of things, but if the causes are many and intricate, then the mathematical description too will be far from simple. We shall here study simple cases only: do not make the mistake of supposing that every problem, however profound, can be forced into these simple forms.

The Simplest Form of Growth

An example of a simple relation is the cost of any article bought by the yard. If one yard of lath costs 2*d.*, 2 yards will cost 4*d.*, x yards will cost $2x$ pence. If p stands for the price in pence of x yards, we have $p = 2x$.

We can make this formula more general. A yard of lath need not



THE SIZE OF CINEMA SCREENS

In the lower diagram the screen is twice as far from the projector as in the upper diagram. The lower screen is twice as broad and twice as high as the upper one. The dotted lines show that the lower screen could be cut into four screens, each as large as the upper screen.

cost 2*d*. Suppose it costs a pence, where a may stand for any number. Then the price of x yards is given by $p = ax$. We assume that the price of each yard does not depend on the number of yards bought: there is to be no reduction for quantity. In mathematical language this is stated by saying that a is *constant*.

Such relations are common. For instance, the circumference and diameter of a circle – C and D – are connected by the relation $C = 3{\cdot}14\ D$. Again, in a spring-balance the spring stretches by a distance proportional to the weight hung on it. If 1 lb. causes the spring to stretch k inches, than x lb. will cause it to stretch kx inches. This fact was discovered by Hooke, about 1660. Hooke was led to study the properties of springs by his work on clock-making. His invention of the balance-wheel – by which a hair-spring replaces the pendulum of a clock – was a practical consequence of his investigation. Hooke's Law is true only for reasonably small weights. A very heavy weight will stretch a spring too far: when the weight is removed the spring does not return to its original length.

Other formulae of the type ax will be found in almost every branch of mathematics, engineering or science.

Powers of x

Another type of growth occurs when a cinema projector, or magic lantern, is moved further away from the screen. If you double the distance of a magic lantern from the screen, the picture will take up – not twice, but – four times as much space. If you treble the distance, the amount of material needed for the screen will be nine times as much.

The rule is clear: 4 is 2×2, or 2^2, 9 is 3×3, or 3^2. If we put the lantern n times as far away, we need n^2 times as much material for the screen.

In the same way, if we enlarge a photograph or a map n times, n^2 times as much paper will be necessary.

We have here the answer to our earlier question about firewood. The bundle tied with a 12-inch string contains four times as much firewood as the 6-inch bundle. You can see that the large

Fig. 5

bundle must contain more than twice the amount of the small one, by looking at Fig. 5. The black part represents two small bundles: the large circle represents one large bundle.

If a stone is dropped from the top of a cliff, you will find that it falls about 16 feet in the first second; after two seconds it has fallen 64 feet; after three seconds, 144 feet. These results can be expressed by the formula that, in x seconds, the stone falls about $16x^2$ feet. Thus in 5 seconds the stone would fall 16 times $5^2 = 16$ times $25 = 400$ feet. If you can remember this formula, and have a watch with you, it is easy to find the height of a cliff, or the depth of a well, by dropping a stone, and noting how many seconds it takes to reach the bottom.

Another formula gives the speed with which it strikes the bottom. This is $v^2 = 64\,h$. In this formula h is the height of the cliff in feet, and v is the velocity, in feet per second, with which the stone lands. If the cliff is 100 feet high, h is 100, so $v^2 = 6{,}400$, and $v = 80$. To produce a speed twice as big, the cliff would have to be *four* times as high. (This answers the question about fire-watching.) It is easily worked out that a speed of 3 feet a second is roughly the same as 2 miles an hour. So 80 feet a second is about 53 miles an hour. Jumping off a house 100 feet high is as dangerous as a car crash at 53 miles an hour.

In the same way, we could discuss the way of growing represented by x^3. You would need 8 cubes of sugar to make 1 cube with twice the normal measurements. To make a cube x times the usual size, you would need x^3 ordinary cubes. If you enlarge any solid object x times – it need not be a cube – you multiply the material contained by x^3. For instance, if you double all the measurements of a drawer, or a box, or a trunk, you multiply by 8 the amount that can be contained.

When a model of any object is enlarged, all these different kinds of growing may be involved. Suppose, for simplicity, we have a box in the form of a cube, made out of cardboard. If the side of the cube is 1 foot, the box will hold 1 cubic foot, 6 square feet of cardboard will be sufficient to make it, and a string 4 feet long could extend round it. If we make instead a box in the form of a cube, but with each side 2 feet long, we find that it holds eight times as much, but that it needs only four times as much cardboard, and a piece of string only twice as long can extend round it. It is cheaper to pack goods in large boxes than in small ones – so long as the cardboard does not burst.

It is easy to see why the early aeroplane inventors had disappointing results, when they tried to enlarge the scale of their models. If the scale is doubled, the weight is multiplied by 8, but the wing-surface is multiplied by only 4.

x, x^2, and x^3 thus turn up naturally enough, when we consider changes in the scale of models and plans. In other applications, we make use of x^4, x^5, x^6 and so on.

For instance, a common device for storing energy is a flywheel. Suppose we make two flywheels by cutting circular pieces out of a sheet of metal – one circle having twice the radius of the other. Suppose both wheels are turning at the same rate – say, one revolution a second. Will the large wheel have twice, or four times, or eight times the energy of the small one? No. Experiment shows that it has *sixteen* times as much, 2^4 as much, that is. If we enlarge the radius x times, we increase the energy x^4 times. Aeroplane engines can be started by means of a flywheel. The flywheel is made to rotate by hand, and is then suddenly connected to the aeroplane engine. If you were using the larger flywheel, mentioned above, you would have sixteen times the energy at your disposal, as compared with a man using the small one: as it would take you sixteen times as long to get your flywheel spinning, you would obtain a vivid picture of the meaning of 2^4!

If you double not only the radius of the flywheel, but also the thickness of the metal, you multiply the energy by 2^5, or 32. In this case, the bigger flywheel is twice as large *in every direction* as the smaller one. The effect of enlarging a flywheel x times in every direction, is to increase its energy (at a given speed or rotation) x^5 times.

x, x^2, x^3, x^4, x^5 are called the first, second, third, fourth and fifth powers of x. Instead of 2, 3, 4 or 5, we could have any number. Using n as an abbreviation for 'any number', we may say that x^n is called the n^{th} power of x.

Powers of x may occur mixed with each other, and with constants. For instance, a tennis club might charge 5*s*. entrance fee, and 1*s*. for every afternoon on which a member actually played during a season. The cost of one afternoon's play would thus be 6*s*., of two would be 7*s*., of x would be $(5 + x)$ shillings.

Again, if a ball is thrown straight up with a speed of 40 feet a second, after x quarter seconds its height is given by the formula $10x - x^2$ feet. (Quarter-seconds are used instead of seconds partly to give simpler numbers, partly because a ball spends such a short time in the air.) We could make a table as follows –

Number of quarter-seconds.	0	1	2	3	4	5	6	7	8	9	10
Height in feet.	0	9	16	21	24	25	24	21	16	9	0

Examine the set of numbers in the lower row of this table. You will notice that it reads the same backwards as forwards. Do you notice anything else about it?

Write down the change between each number and the next one. Thus –

Numbers.	0	9	16	21	24	25	24	21	16	9	0
Change.	9	7	5	3	1	—1	—3	—5	—7	—9	

There is a very simple rule to be seen in this set of numbers: each number is 2 less than the number before it. This is due to the steady downward drag which gravity exerts on the ball. In the first interval of a quarter-second the ball rises 9 feet. But it is being slowed down all the time. In the second interval it covers only 7 feet, in the third 5 feet, and so on. In the fifth it rises only 1 foot. In the sixth it descends 1 foot. (As usual, + 1 means one foot *up*, — 1 one foot *down*.) Now it comes down faster and faster: 3, 5, 7, 9 feet in successive intervals.

We can keep on writing down such rows of numbers – each row giving the changes in the row above. We should thus get the following table –

TABLE I

0	9	16	21	24	25	24	21	16	9	0
9	7	5	3	1	—1	—3	—5	—7	—9	
—2	—2	—2	—2	—2	—2	—2	—2	—2		
0	0	0	0	0	0	0	0			
0	0	0	0	0	0	0				

All the numbers in the third row are the same, —2. There is no change as we go from one to the next. So all the numbers in the fourth row are noughts. We can go as far as we like: we shall merely get more noughts, in the fifth, sixth and following rows.

Let us try this on some other expressions we have already had. If we write down the numbers corresponding to x^2, we obtain –

TABLE II

0	1	4	9	16	25	36	49	64	81
1	3	5	7	9	11	13	15	17	
2	2	2	2	2	2	2	2		
0	0	0	0	0	0	0			

If we try x^3 we find

TABLE III

0	1	8	27	64	125	216	343	512
1	7	19	37	61	91	127	169	
6	12	18	24	30	36	42		
6	6	6	6	6	6			
0	0	0	0	0				

Try for yourself x^4 and x^5. Make up expressions such as $2x + 3$, $x^2 + 5x + 7$, and try it on these. You will find that, after a certain number of rows, you always get noughts. What is the rule giving the number of rows that occur, before the noughts are reached? The answer to this question is given later. But try to guess it for yourself. Work out a large number of different examples: group together those which have one row, then noughts: in another group put those with two rows, and so on. The rule is quite a simple one.

Exponential Functions

We have just seen that any expression, made up by mixing together powers of x, will lead to rows of noughts at a certain stage of the process described above.

Not all ways of growing have this property: in fact, expressions formed by mixing powers of x are the only type that possess this characteristic.

If you try some other rule, you will soon see that this is true. Take, for instance, the set of numbers 1, 2, 4, 8, 16, 32, 64, 128, etc., where each number is formed by doubling the previous one. This set corresponds to the formula 2^x. (Remember that we start with $x = 0$. If a sum of money doubled itself each year, a high rate of interest, £1 would become £2 after 1 year, £4 after 2 years, etc. After x years it would be £2^x.) If we make a table for this set of numbers, using the same method as before, we get –

```
1   2   4   8   16   32   64   128  .....
  1   2   4   8   16   32   64  .....
    1   2   4   8   16   32  .....
```

Each row is exactly the same as the one before! However long we go on, we shall never find a row all noughts.

Try 3^x. This gives us the table –

```
1   3   9   27   81   243  .....
  2   6   18   54   162  .....
    4   12   36   108  .....
```

Here each row is *twice* the previous one. However long we go on, we shall never find a row all noughts.

2^x, 3^x are called *exponential functions*. If we use a as shorthand for 'any number', a^x is an exponential function.

The Calculus of Finite Differences

It very often happens that we want to know the rule by which a certain set of numbers has been formed. An engineer might find, by experiment, the pressure required to burst boilers made from sheet metal of various thicknesses. It would be helpful to other engineers if he could express his results in the form of a simple rule. A scientist might measure the size of a plant each day, and try to find the rule by which it grew.

A great part of science consists of the attempt to find rules by studying the results of experiments.

When one quantity depends on another, it is said to be a *function* of the latter quantity. Thus, the bursting pressure of a boiler depends on the thickness of the boiler walls. Calling the pressure p and the thickness t, we say that p is a function of t; the pressure needed to burst a wall of thickness t may be written $p(t)$. Thus $p(2)$ would mean the pressure needed to burst a boiler built with metal 2 inches thick, $p(\frac{3}{8})$ would mean the pressure to burst a boiler with $\frac{3}{8}$-inch walls. Naturally, we suppose that the design of the boiler is fixed, and that the same metal is used for all the experiments.

In the same way, if we denote 'the number of days' by x, and 'the size of a plant in inches' by y, y is a function of x. $y(17)$ will mean the size of the plant after 17 days: $y(x)$ the size after x days.

If we say, 'What function is y of x? we mean, 'By what particular rule is y connected with x?'

This question is used in intelligence tests. A child is shown the numbers 1, 2, 3, 4, 5 and is asked, 'What is the next number?' Of course 6 is the answer. An older child might be shown 2, 4, 6, 8 and expected to guess the next number as being 10.

Such simple cases can be guessed without any special method. But suppose you were shown the table below –

x	0	1	2	3	4	5	6	7	8	9	10
y	1	3	7	13	21	31	43	57	73	91	111

How is the number y in the second row found, corresponding to any number x in the first row? A child might be forgiven if, shown the numbers 1, 3, 7, 13, 21, 31 it failed to guess that the next number was 43! But how quickly our method – of writing down the change from one number to the next – supplies a clue. It gives the table –

TABLE IV

1	3	7	13	21	31	43	57	73	91	111
2	4	6	8	10	12	14	16	18	20	
2	2	2	2	2	2	2	2	2	2	
0	0	0	0	0	0	0	0	0		

We have noughts in the bottom row: the formula must be a simple one, containing only powers of x.

But how many powers of x shall we need? Shall we have to bring in x^5 as was necessary in the case of the flywheel? Or need we not go so far?

Perhaps you have already discovered the answer to the question asked earlier.* If not, here is the answer, Any formula, such as $2x + 3$, which contains no power higher than x, gives us two rows of numbers, and then noughts. If x^2 comes in – as in $5x^2 + 3x - 2$, for instance – we have three rows, then noughts. If x^3 comes into the formula, we have four rows, then noughts. And so on. If x^n occurs, we have $(n + 1)$ rows before the noughts come. This works also the other way round. If we have four rows before the noughts, it will be possible to find a formula which does not use any power above x^3. If we have $(n + 1)$ rows, the formula will contain powers up to x^n.

This helps us to find the formula which gives the numbers 1, 3, 7, 13, etc. This set of numbers, as we have just seen, leads to a table containing only three rows. The formula cannot contain any power of x higher than x^2. It will be sufficient to take a certain amount of x^2, together with a certain amount of x, and with some number added. In algebraic shorthand, our formula will be $ax^2 + bx + c$, where a stands for the number that goes with x^2, b for the number that goes with x, c for the number that is added. (Thus, in the formula $5x^2 + 3x - 2$, a is 5, b is 3, c is -2.) We do not yet know what a, b and c have to be. All we

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know is that it is possible to get the right formula by choosing the proper values for a, b and c. This of course is a great help. When we started this problem we had to be prepared for *any* formula: it might have been $x + 2^x$, or x^9, or even worse expressions.

Once we know that the formula is of the type $ax^2 + bx + c$, it is very easy to find a, b and c. We know that the numbers 1, 3, 7, 13, etc., result if, in the proper formula, we replace x by the numbers 0, 1, 2, 3, etc., in turn.

If in the formula $ax^2 + bx + c$ we replace x by 0, we get c: if we replace it by 1, we get $a + b + c$: if we replace it by 2, we get $4a + 2b + c$. (If you like, you can turn these results into words by reading $4a$ as 'four times the number that goes with x^2 in the formula', and so on.)

We can now compare these two sets of results. If y is given by the formula $ax^2 + bx + c$, $y(0) = c$. But $y(0)$, the value of y corresponding to $x = 0$, is 1. c must be 1. Again the formula gives $y(1) = a + b + c$. But $y(1)$ is 3. So we must choose a, b and c in such a way that $a + b + c = 3$. In the same way, comparing the formula for $y(2)$ and what it ought to give, we get the equation $4a + 2b + c = 7$. Altogether we have three equations:

$$\begin{aligned} c &= 1 \\ a + b + c &= 3 \\ 4a + 2b + c &= 7. \end{aligned}$$

This is exactly similar to the problem of the cakes and the buns and the pot of tea. It is easily solved by the method described in Chapter 7, and leads to the result $a = 1$, $b = 1$, $c = 1$. So that we have the formula $y = x^2 + x + 1$. This is the rule by which the numbers in the table were found.

In the subject known by the imposing name of the Calculus of Finite Differences, the method we have just used is further developed, and proofs of its correctness are given.

It has been found convenient to introduce certain abbreviations. We have had to keep on referring to 'the second row in the table', 'the third row', and so on. To avoid this, certain signs are used, as names for these rows. The first row (which, in our last example, contained the numbers 1, 3, 7, 13 ...) we have already called y. The second row (the numbers 2, 4, 6, 8. . .in that example) are called Δy. The sigh Δ is short for 'the change in'. As each row represents the changes that occur in the previous row, we get one more sign Δ every time we go down a row. The third row, for instance, represents the changes in Δy, and could be written $\Delta\Delta y$. Note that Δ does not stand for any number as did a, b and the other letters. Δ stands for '*the change in*' – that and nothing else. It can always be replaced by these words. Usually $\Delta\Delta y$ is still further shortened to $\Delta^2 y$. This is especially convenient when large numbers of Δ are involved. For instance $\Delta^5 y$ is much more convenient than $\Delta\Delta\Delta\Delta\Delta y$ as an abbreviation for the numbers in the sixth row.

Sometimes we want to refer briefly to a particular number in one of the rows. We have already used the sign $y(x)$ to describe the number in the first row which corresponds to the value x: so that the numbers of the first row are $y(0)$, $y(1)$, $y(2)$, $y(3)$ and so on. We use similar signs for the numbers in the following rows. The numbers in the second row are called $\Delta y(0)$, $\Delta y(1)$, $\Delta y(2)$ and so on; in the third row, $\Delta^2 y(0)$, $\Delta^2 y(1)$, $\Delta^2 y(2)$ etc.; and so on for any row.

You will find these signs in any book on the Calculus of Finite Differences. At first they may seem strange, but once you are accustomed to them, and have realized that $\Delta^2 y(1)$ means nothing more terrifying than 'the second number in the third row' of a table such as Table III or Table IV, you will find the subject quite a good one to experiment with. You may try your hand at the following problems.

(1) A car drives past a lamp-post. One second later it is 3 yards away from the lamp-post; after 2 seconds, 10 yards; after 3 seconds, 21 yards; after 4 seconds, 36 yards. How far away is it after $\frac{1}{2}$ second, $1\frac{1}{2}$ seconds, $2\frac{1}{2}$ seconds? Is it speeding up or slowing down?

(2) What is the missing number in the following set?

$$3, 4, \ldots 24, 43, 68.$$

If you succeed in guessing the right number, the table for Δy, $\Delta^2 y$, etc., will make it quite clear that you are right. You will not be in any doubt about it, once you have tried the right number. And it must be a number between 5 and 23. If the worst comes to the worst, you can try all of these in turn.

Binomial Coefficients

It is possible to work out a table similar to Table IV, if we suppose the first row, y, to contain *any* set of numbers. In fact, we may represent the numbers in the first row by algebraic symbols. Let a stand for the first number (whatever that is), b for the second, c for the third, d for the fourth and so on. The row y then reads –

$$a, b, c, d, e, f \ldots$$

How are we to form the next row Δy? The first number in it shows the change from a to b. It is obtained by subtracting a from b, and may therefore be written $b - a$. In the same way, the next number may be written $c - b$. (Check these statements for yourself. In Table IV, what numbers are a, b, and c? Is it true that the row Δy begins with numbers equal to $b - a$ and $c - b$?) The second row in fact is $b - a$, $c - b$, $d - c$, $e - d$, $f - e$, etc.

From the second row the third row can be found. The first number in it is $(c - b) - (b - a)$, which simple algebra shows to be equal to $c - 2b + a$. The second number in this row is $d - 2c + b$.

Continuing in this way, we obtain the expressions collected in Table V.

TABLE V

a		b		c		d		e
	$b-a$		$c-b$		$d-c$		$e-d$	
		$c-2b+a$		$d-2c+b$		$e-2d+c$		
			$d-3c+3b-a$		$e-3d+3c-b$			
				$e-4d+6c-4b+a$				

You will notice certain things about this table. A particular set of numbers appears in each row. In the row $\Delta^4 y$, for instance, we have the numbers 1, 4, 6, 4, 1. In the row $\Delta^3 y$ we find the numbers 1, 3, 3, 1. In the row $\Delta^2 y$ we find 1, 2, 1, and in the row Δy we find simply 1, 1. (No notice has been taken of whether these numbers appear with a $+$ or $-$ sign.) You will notice that these sets of numbers read the same backwards as forwards. For example, 1, 3, 3, 1 is the same, whether read backwards of forwards. You will notice that the first and last numbers are always 1. What else do you notice? What is the rule that gives the number next to the end? Can you find the formula for the number next to that? (You will need to work out a few more rows of Table V to do this.) Apply the method already explained, for finding the formula of a set of numbers.

These numbers are known as the *Binomial Coefficients*. Mathematicians came to know them in exactly the way you have done – by noticing that they turned up in the course of work. They turn up, for instance, if you work out 11^2, 11^3, and 11^4, which are, in fact, 121, 1331, 14641. (After this point *carrying* comes into the arithmetic, and the simple connexion does not hold. The numbers in $\Delta^5 y$ are 1, 5, 10, 10, 5, 1, and of course the number *ten* cannot appear as a single figure in 11^5. 11^5 is actually 161,051, which does not read the same backwards as forwards.) They appear, too, in $(x + 1)^2$, $(x + 1)^3$ and $(x + 1)^4$, etc. We may write them in a table, as below –

TABLE VI

1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Now you can measure your place among the great mathematicians. This table was known as early as 1544. Gradually people noticed all sorts of odd facts about it. But it was not until 1664, 120 years later, that the greatest of English mathematicians found *a formula* giving the numbers in any column of Table VI. The first column is obvious enough – always 1. The second column contains 1, 2, 3, 4, 5, 6, 7 – a simple law here. But what is the rule for the third column, 1, 3, 6, 10, 15, 21? You will find this problem quite easy if you use the method outlined earlier in this chapter – make a table on the lines of Tables I-IV, and then hunt for a formula.

The rule that Newton found (and that I hope you will find for yourself) is known as the *Binomial Theorem.* That is all the Binomial Theorem is – a rule for writing down the numbers in Table VI.

The object of explaining Δy, $\Delta^2 y$, etc., to you is that you may see how theorems are discovered, and may be able to discover results for yourself.

EXERCISES

1. If the temperature of L feet of steel is raised T degrees Fahrenheit, the length of the steel increases by an amount $0{\cdot}000006\ LT$.

If the temperature of a mile of steel (say, on a railway) rises 10 degrees Fahrenheit, how much extra room will it need?

2. In scientific work temperature is measured in degrees Centigrade. This can be changed to degrees Fahrenheit by means of the formula

$$F=\frac{9C}{5}+32,$$

where F degrees and C degrees stand for the temperature Fahrenheit and Centigrade respectively. What temperature Fahrenheit corresponds to 15 degrees Centigrade? What is the temperature 90 degrees Fahrenheit in terms of degrees Centigrade?

3. 10 feet of common 2-inch-bore lead piping weighs 50 lb. What is the formula for the weight of x feet of such lead piping?

4 One square foot of ordinary brickwork will support with

safety a weight of 6 tons. How many tons will a piece of brickwork x feet square support?

With particularly good bricks, 1 square foot of brickwork will support 54 tons. What weight will a block x feet square support?

A brickwork foundation capable of bearing a weight of 1,000 tons is required. The foundation is to be square in shape. How large must it be if it is made (i) of ordinary brick, (ii) of particularly good bricks?

5. When a train moves at x miles an hour, the total pressure on the front of the locomotive, due to air pressure (y lb.), is given by the following table.

x	0	20	40	60	80	100
y	0	79·5	318·0	715·5	1272·0	1987·5

Is there a simple formula connecting x and y? If so, what is it?

6. One sometimes sees on trams a table giving the fare between any two points on the route, as in the illustration.

From \ *To*	Blue Boar	Green Man	Black Market
Green Man	1*d.*		
Black Market	2*d.*	1*d.*	
Red Lion	3*d.*	2*d.*	1*d.*

It is clear that no diagram at all is necessary unless the tram has at least two stations to connect. With 3 stations, the diagram contains 3 squares. Work out the number of squares in the diagram when there are 4, 5, 6, etc., stopping stations. How many squares are needed if there are x stations? Is there any connexion with the numbers in Table VI?

7. If n teams enter for a knock-out competition, how many matches will be played? (See the question at the end of Chapter 5.)

8. In Britain and the United States the horse-power rating of a motor-car is found from the formula –

$$H = \frac{2Nd^2}{5}$$

where $H =$ horse-power rating
$N =$ number of cylinders
$d =$ bore of cylinder in inches.

What bore would be necessary, for a four-cylinder car, to give 40 H.P.? What for 10 H.P.? What would give a horse-power just below 23 H.P.?

9. The breaking strength of three-strand Manila rope is given by the formula –

$$L = 5000\, d(d+1)$$

where L lb. is the load needed to snap a rope d inches in diameter. How many lb. will be required to snap a rope $1\frac{1}{2}$ inches in diameter? What rope will just snap under a load of 60,000 lb.?

10. The safe load, S hundredweight, for a rope measuring C inches in circumference is given by the formula –

$$S = C^2.$$

How many hundredweight may safely be put on ropes of circumference 2, 3, and 4 inches? What size of rope is needed to carry safely $\frac{1}{4}$ ton? How many ropes 2 inches in circumference would be needed to support $\frac{1}{4}$ ton? (Fractions of ropes are not used.) How many ropes 3 inches around would do the same job?

Further formulae, on which to practise, can be found in Kempe's *Engineer's Yearbook*, from which many of the examples used here have been drawn. Formulae will be found covering subjects varying from the amount of sludge deposited by sewage to the amount of rainfall in Northern India.

CHAPTER 9

GRAPHS, OR THINKING IN PICTURES

'*Care is taken, not that the hearer may understand, if he will, but that he must understand, whether he will or not.*'

– Henry Bett, *Some Secrets of Style.*

THE great problem of every teacher is to present facts in such a way that the students cannot help *seeing* what is meant. A bald statement is soon forgotten. Vivid images remain in the memory. Many people must have noticed the difference between reading a history text-book and seeing an historical film. Whatever the relative accuracy of the book and the film, the film certainly makes one realize events more intensely, and remember them longer.

In films it is sometimes necessary to explain quite complicated ideas, not to a class of students, but to an audience which represents the whole population of a country. Cinema audiences, too, are in no mood for concentrated thought. They want to relax, to be amused. It is extremely instructive to examine how film directors go about the job. They rarely fail to make their point understood – a fact which should be seriously considered by those defeatists in education whose perpetual alibi is the stupidity of pupils.

In our last chapter we considered the different ways in which any quantity can grow. Suppose we wished to present this idea to a cinema audience. How could this be done? We might wish to represent the fact that a man's wealth began to grow – perhaps an inventor's first success. We might show him storing sovereigns in a safe. The first week he puts a sovereign in. Next week he adds two more sovereigns. Then he adds a pile of three sovereigns. The earnings of each week form a pile, and each pile contains one sovereign more than the previous pile.

Very good – in the x^{th} week he saves £x, and the steadily rising heaps of coin show at a glance the meaning of this fact.

But the plot need not end there. The inventor has a friend who lives by fraud and violence, a gangster. The gangster is determined to prove that it does not pay to be honest. He urges the inventor to come where the big money can be made. Scornfully, each week, he places a pile of coins behind the inventor's. The first week, £1; the second, £2; the third week, £4; the fourth week, £8 – doubling each week. The steady rise of the inventor's earnings sinks into insignificance. He may save £x – the gangster saves £2^{x-1}, each week. (Fig. 6).

These figures not only show that the incomes of the inventor and the gangster are growing. They bring out the significance of the different ways in which they are growing. Here we have the essential idea of a *graph* – that is, a diagram which will show to the eye the meaning of a mathematical expression such as x or 2^{x-1}.

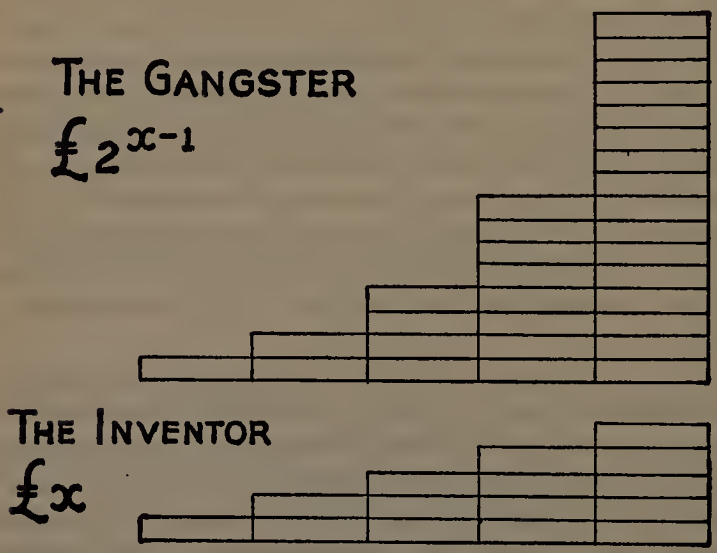


Fig. 6

In this particular illustration we have been considering something which grows by steps, by sudden jumps. For instance, the gangster's weekly savings rise from £4 to £8 without passing through the values between £5, £6 and £7. It is also possible for a thing to grow steadily, like a plant, without jumps. We can, and usually do, draw graphs to illustrate this steady growth. For instance, if a plant grows continuously according to the formula $y = x$ (where y means the height of the plant in inches after x weeks), this not only means that the plant is 1 inch higher after 1 week and 2 inches after 2 weeks. It means that it is $1\frac{1}{4}$ inches high after $1\frac{1}{4}$ weeks, $1\frac{1}{2}$ inches after $1\frac{1}{2}$ weeks, and so on. We show this by a diagram in which the steps have been smoothed out (Fig. 7.).

Whether one draws a continuous curve, or one which rises by steps, is decided by the nature of the process which the graph is intended to illustrate. The size of a plant, the distance travelled by a train, the weight of a child – these give continuous graphs. The number of children in a family, the seats held by a party in Parliament, the number of battleships in a navy – these change by steps.

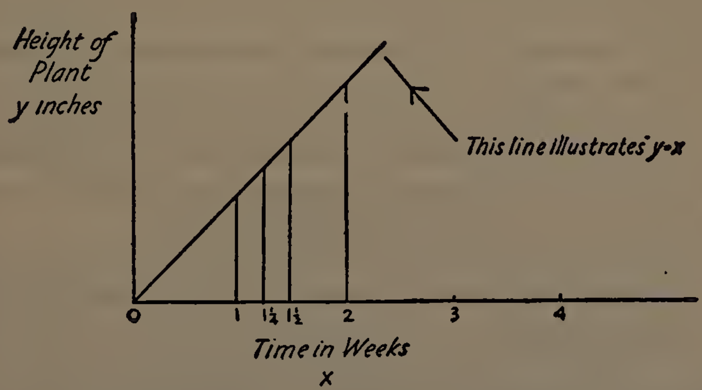


Fig. 7

There are certain cases in which one may use either a continuous curve or a graph with steps. Suppose, for instance, we wish to represent the growth of the population of Great Britain from 1800 to 1900. Strictly speaking, this changes by jumps – increases by one whenever a child is born and decreases by one at every death. But the population itself is measured in millions: to get our graph a reasonable size, we must take a scale such that a million people are represented by not more than an inch. Each individual birth or death corresponds to a change of not more than a millionth of an inch. This is far less than the thickness of a pencil line – even if we could draw each little step, we should be unable to see the effect. So the population curve would appear as a continuous curve – not as a staircase.

Graphs have become so much a part of everyday life that it is not really necessary to explain them. People entirely without mathematical training are usually able to see the significance of the temperature chart over a patient's bed, of the curves showing changes in unemployment or the history of Lancashire's cotton exports. Graphs are used to show the progress of a campaign to raise money, or the output of a factory. Business journals contain graphs showing the trend of prices. Wireless valves are marketed with graphs to show their characteristics. At some holiday resorts one can see instruments which record curves, showing how the barometer has risen and fallen, and charts of the rainfall and sunshine from day to day. The general idea of a graph is already widely understood.

It may be useful to explain exactly how a graph is drawn. A graph illustrates the connexion between two sets of numbers. For instance, we have already considered the possibility that a plant might grow as shown in the following table:

Number of weeks (x)	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Height in inches (y)	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2

We drew a level line, along which we marked the number of weeks. We then drew lines upwards, representing the height of the plant corresponding to any number of weeks. In Fig. 7 such lines have been drawn corresponding to 1, $1\frac{1}{4}$, $1\frac{1}{2}$ and 2 weeks. The upright line corresponding to 1 week is 1 inch high – the size of the plant after 1 week. The upright line corresponding to $1\frac{1}{4}$ weeks represents the height of the plant after $1\frac{1}{4}$ weeks. So one can go on, drawing as many upright lines as one likes. These upright lines show the growth of the plant, in the same way that the piles of coins show the growth of the inventor's weekly savings. After drawing a large number of these upright lines, we can see that the tops all lie on a certain straight line. (In other examples the tops lie on a curve.) Drawing the line (or curve) joining the tops of the upright lines, we obtain *the graph of the plant's growth.* As the plant grows according to the formula $y = x$, this line is also called *the graph of* $y = x$.

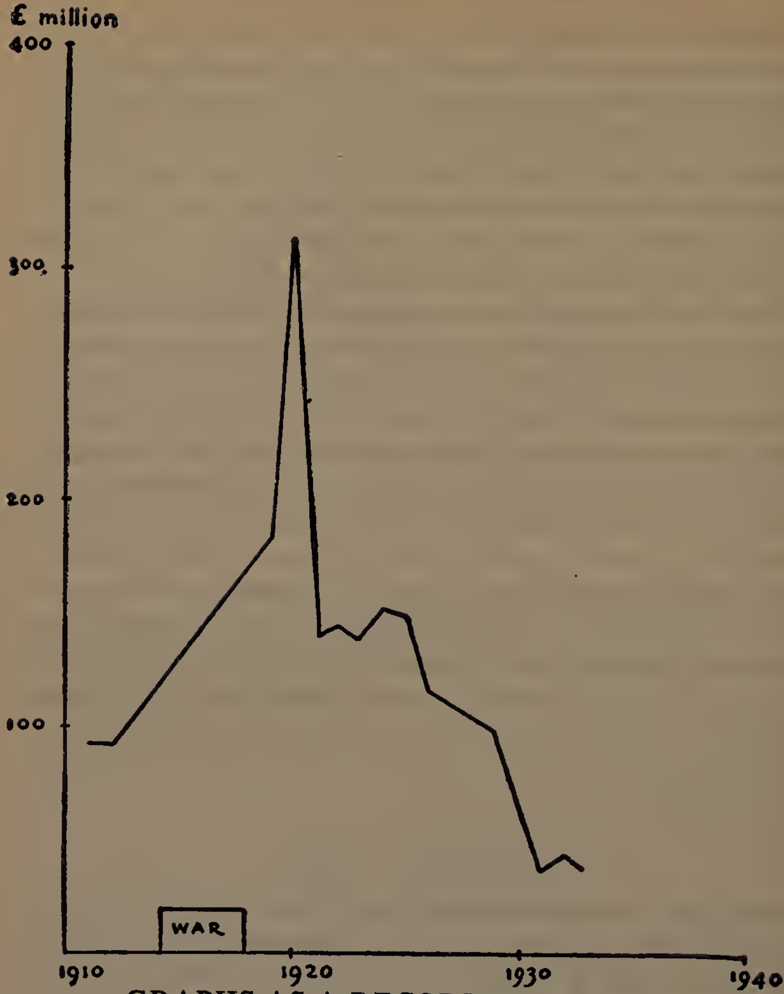
Any other process, or the mathematical formula which describes it, can be graphed by this method. On page 93 there is a table showing the motion of a ball thrown into the air. Draw for yourself a graph to illustrate this table. The tops of the upright lines will lie not on a straight line, but on a curve. Notice how this curve rises so long as the ball is going up, and descends as the ball comes down. What would the graph of a bouncing ball look like?

In both these examples, y has been a mathematical function of x. For the plant, $y = x$. For the ball, $y = 10x - x^2$. But do not suppose that graphs can be drawn only when a simple formula exists. One can graph the temperature of a patient or the price of milk: it is extremely unlikely that a simple formula can be found to fit either of these.

The Uses of Graphs

Graphs have a great advantage over tables of figures, when information has to be taken in at a glance. It is quite easy to run an eye down a row of figures, and fail to see that one number is much larger than the rest. On a graph, such a number would stand out like a mountain peak. A sudden bend in a graph is easily seen – a casual glance at the corresponding figures would certainly not reveal its existence. Graphs are particularly useful for busy men who want to know the general outlines of a situation but do not want to be bothered by going into every small detail.

This is the simplest use of a graph – to convey a general impression. An historian or an economist may simply want to know that Lancashire was prosperous in 1920 and that a sharp crash



GRAPHS AS A RECORD OF TRADE

The graph shows the exports of cotton cloth, in million pounds, during the years in question. One can see in a few seconds the general outline of Lancashire's fortunes in this period. One would grasp far less from seeing the actual figures. Try for yourself taking a column of figures from an encyclopedia or year-book. Glance at them for fifteen seconds, put them away and write down the things you have noticed – when the figures are largest, when they are small, when they are growing, when they are getting less, etc. You will not notice very much in a short time. Now make a graph of the figures, and notice how the graph brings out things you have missed.

came in 1921. One glance at a graph of cotton exports will remind him of this fact.

Again, graphs can be used to bring out the connexion between two events. Most books on Germany point out how the distress in Germany during the world slump created a mood of extremism and desperation, and helped the rise of the Nazi party. How far can we accept this view as being true? Let us draw, on the same piece of paper, two graphs, one showing the amount of unemployment in Germany, the other showing the number of Nazi M.P.s, between 1926 and 1933 (Fig. 8).

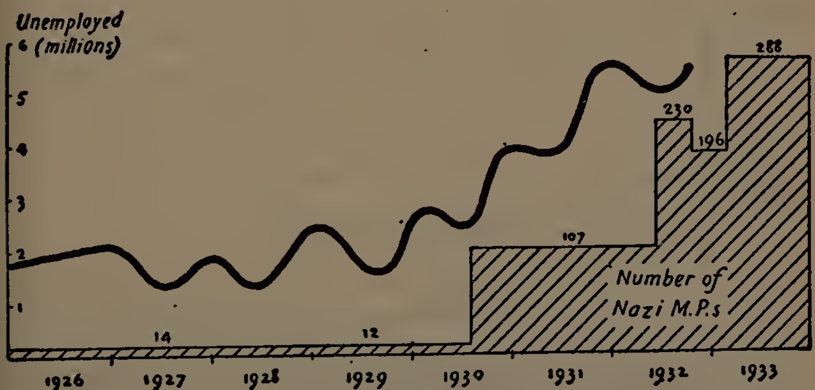


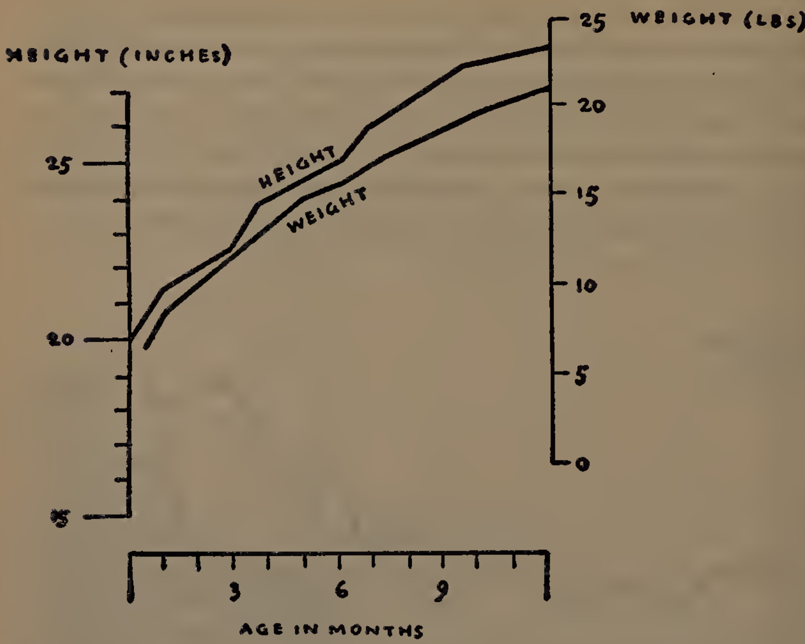
Fig. 8

This graph immediately shows that there is some truth in the idea. During the boom years, the elections returned negligible numbers of Nazi M.P.s: 14 and 12. The two curves, in the main, rise together.

It would, however, be absurd to try to find a mathematical formula connecting the two things. The number of M.P.s changes by steps, at every general election. Unemployment and insecurity are not the only causes acting. For instance, the setback to the Nazis in the second election of 1932 was due to political causes – quarrels among the Nazis, a belief that the Army would oppose Hitler, and so forth.

You will notice how a graph calls your attention to unexplained

GRAPHS USED TO DETECT UNDER-NOURISHMENT



The two graphs show the height and weight of a baby in the first year of its life. If the graph for weight does not keep pace with the graph for height, something is wrong.

facts, and urges you on to further enquiry. We drew the graph to see how far the slump explained the rise of Hitler. The graph not only gives us an indication of the probable answer to this question: it brings to our notice the fall in the Nazi vote at the end of 1932, which is in no way connected with the curve for unemployment, and sets us searching for further facts to explain the setback.

Again, one can hardly help noticing the wave-like shape of the unemployment curve, which rises every winter and falls every summer. This reminds us that there are some trades – such as building – which cease work during bad weather. The summer of 1926 seems to have been an exception. One wonders why. The longer one looks at a graph, the more questions it suggests and the more information it helps one to remember.

It is worth while to collect graphs on any subject in which one is interested. One often hears remarks, and wonders what evidence there can be for their truth. A visit to a public library will often establish the truth or expose the falsehood of a statement. If it is possible to illustrate the question by means of a graph, one has a way of recording much information in a little space. One need not look up the same facts again. As time passes, the collection of graphs is likely to contain some interesting facts.

Quiller-Couch, in *The Art of Reading*, mentions a girl who kept a graph of the attendance at a village church, and tried to account for every rise and fall that occurred. She must have gained an amazing knowledge of the qualities of the preachers and the habits of the village.

Graphs are used by doctors, to show whether children are being properly nourished. The weight of a child, and its height, are graphed on the same piece of paper. For a healthy child, the two curves go up together. If the child is not getting the food it needs, the curve for weight fails to keep pace with that for height. The doctor need not wait until there is a big gap between the two curves. If he notices that the curve for weight begins to bend downwards, this may be the first sign that something is going wrong. If, after special treatment or extra food, the curve begins to bend upwards again, the doctor knows that good effects are beginning to be felt. Part of the science of interpreting graphs consists in knowing how a graph looks when something is increasing, when it is increasing very fast, when it is increasing faster and faster, when it is increasing but increasing slower and slower. (Draw graphs, to illustrate these different possibilities.)

In all these examples the conclusions drawn from the graph have been of a rather general nature. The doctor sees that a child is getting healthier or less healthy, but he does not attempt to *measure* how healthy it is. He cannot say that it is 80% healthy, any more than we can say that someone is 80% happy or 80% honest. Such things as health, happiness and honesty can be measured only indirectly: statistics of deaths, suicides, thefts, may throw some light on these matters. But it is quite possible to know a lot about how healthy, happy or honest a person is, without being able to give a single figure of anything that can be measured.

There are some departments of life, on the other hand, in which measurement plays a large part. This is particularly true of such subjects as engineering, chemistry, physics. Quite a small bump on a railway track may be sufficient to derail a train: if a ball-bearing is one-thousandth of an inch too large, it may take all the weight that should be spread over several ball-bearings and wear out too fast. In such matters very exact calculations are frequently necessary. For this reason engineers and scientists cannot be content with rough statements. They sometimes want to say, not merely that a curve rises slowly, but that it rises at a rate of 1 in 100, or 1 in 87. Much of mathematics has developed in an attempt to satisfy such demands of engineers: mathematicians have been led to invent a whole set of numbers, by means of which one can not only describe, but measure, exactly what a curve is doing at any point. The next chapter – on the study of speed – explains how this is done.

Mathematicians and Graphs

Mathematicians use graphs for many different purposes, some of which are indicated in the following paragraphs.

Graphs may be used to help us to know what we are talking about. It often happens when long calculations are being made with algebraic symbols that we lose sight of the meaning of these symbols; we have at the end a formula, which has been obtained by using the rules of algebra, but we have no way of feeling what it means. It deepens our understanding of the subject if we do not rest content with having found a correct formula, but try to realize what this formula means.

For instance, the formula –

$$P = 364\,V - \frac{V^3}{270{,}000}$$

gives the power (P) transmitted when a shaft is driven by means of a leather belt. V represents the velocity in feet per second at which the leather belt is travelling. The formula holds in certain conditions which do not concern us at the moment.

What does this formula mean? It contains a quite striking result. It would be natural to suppose that, by turning the driving pulley sufficiently fast, one could transmit as much power as one wished. But draw the graph of P, taking V for values between 0 and 8,000. You will find that P rises until V is 5,700, after which it decreases. If you drive the belt faster than 5,700 feet a second, you do not transmit more power but *less*. One glance at the graph shows this. If, however, one did not draw the graph, but used the formula blindly, one might make serious mistakes, such as designing a plant which worked at a speed too high to be economical.*

Graphs can be very helpful to anyone learning mathematics. Many people can follow all the steps in the solution of a problem, when the solution is shown to them, but they are unable to discover the solution for themselves. They understand each separate step, but they do not know which series of steps will bring them out of the wood. This difficulty can be overcome only if one learns to *see* the meaning of mathematical formulae. Many mathematicians think about their problems all day, wherever they may be. They do not remember all the formulae: they remember a *picture* which the problem has created in their minds. They keep thinking about this picture, until a *method* of solving the problem occurs to them. Then they go home to their pens and paper and collections of formulae, and work out the solution in full. Graphs are one of the ways by which it is possible to form a picture of a problem.

It is a good practice to collect, and to become familiar with, the graphs of the more common functions, such as $y=x$, $y=2x+1$. $y = 3 - 2x$, $y = x^2$, $y = x^2 + 2x + 5$, $y = x^3$, $y = x^4$, $y = 2^x$, $y = \frac{1}{2}x$, and so on.

In scientific work one often obtains a set of results by

* Both the formula and the graph can be found in J. Goodman, *Mechanics Applied to Engineering*, Vol. 1, page 355, 9th edition.

experiment, and then tries to find a formula to fit these results. This can be very difficult, since there are many different types of formula, any one of which *might* be the correct one. It is often helpful to represent the experimental results by means of a graph. If one is familiar with the graphs of many functions, one may at once recognize the type of function which produces such a graph. For instance, all functions which have straight lines as graphs are of the type $y = ax + b$.

Of course, small errors always creep in, and one does not expect the points to lie exactly on a smooth curve. Such small errors in measurement are due to various causes – the thickness of the lines on a ruler when a length is being measured, for instance. Occasionally a big mistake occurs – for instance, one might copy 7197 as 7917, or forget to close a switch when an experiment was being done. Such big mistakes are easily detected on a graph. All the other readings cluster around a smooth curve, but the big mistake is far away from the curve, and one suspects it immediately.

This way of detecting errors is useful not for scientific work only, but also for mathematics itself. For instance, in calculating a set of numbers we may make slips in one or two of the numbers. By drawing a graph it is usually possible to see which numbers are incorrect. When all the numbers are correct, the graph will be a smooth curve – at any rate, this is true in the great majority of cases.

Graphs not only enable us to express a formula by a curve: they enable us to describe a curve by a formula. For instance, when there is no wind, a jet of water from a hose or a small pipe forms a simple curve. If a board is held near the jet of water, the curve can be traced. One can then study this curve, and try to find the formula of which it is the graph. The formula, once found, provides a sort of name for the curve. The part of mathematics known as *Analytical Geometry* is based on this idea of describing every line or curve by the formula corresponding to it. If you want to learn analytical geometry, but find the text-books difficult, the best thing to do is to experiment with graphs for yourself. Draw graphs of the type $y = ax + b$, taking all sorts of values for a and b, positive and negative, large and small. Verify the statement made earlier that all these graphs are straight lines. What do you notice about the graphs of $y = x$ and $y = x + 1$? Can you find a formula which gives a straight line at right angles to $y = x$? Experiment on these lines, record your experiments, and try to reach general conclusions: see how long it is before you can tell, simply by looking at the formulae, that two lines are at right angles. *Then* read the chapter in the text-book headed 'The Straight Line' or 'The Equation of the Straight Line', and you will find your own results, in another person's language. Since you already know what the author is trying to say, it will not be long before you come to understand his language.

EXAMPLES

1. Draw the following graphs. What do you notice about them? How would you describe in words the figure they form?

(i) $y=2x$. (ii) $y=2x+1$. (iii) $y=2x+2$. (iv) $y=5-\frac{1}{2}x$.

2. Draw and describe, as in question 1, the four graphs following.

(i) $y=3x$. (ii) $y=3x+1$. (iii) $y=3x+2$. (iv) $y=4-\frac{1}{3}x$.

3. What do you notice about the following two graphs?

(i) $y=x^2+2x$. (ii) $y=x^2+4x+3$.

4. Draw the graph $y=x(9-x)$. For what value of x is y largest, and what is the largest value of y?

5. What do you notice about the graphs following? –

(i) $y=25-x^2$. (ii) $y=x^2$?

Minus Numbers on Graphs

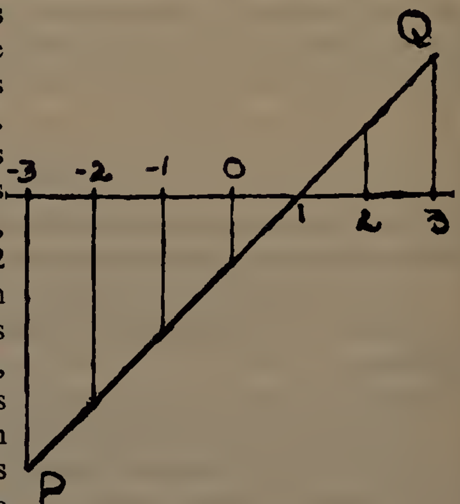
Often we want to draw a graph, in some part of which x or y or both are minus numbers. For instance, we may want to draw a graph showing the length of an iron rail for temperatures which are *below zero*. If x degrees is the temperature, this means that x is a minus number. If in our graph $x=1$ is one inch to the right, $x=-1$ will be one inch to the left. $x=-2$ will be two inches to the left, and so on.

In the same way, if $y=1$ is one inch up, $y=-1$ will be one inch down.

For instance, we may draw the graph of $y=x-1$, for values of x lying between -3 and 3. We first make the table –

x	-3	-2	-1	0	1	2	3
y	-4	-3	-2	-1	0	1	2

We mark the values of x on a level line. $x=3$ is marked 3 inches to the right of 0, -3 three inches to the left of 0, and so on. The corresponding values of y are then shown as upright lines. When $x=3$, $y=2$, so that a line 2 inches high is drawn from the point where $x=3$ is marked. When $x=-3$, $y=-4$, so a line 4 inches *downwards* is drawn from the point where $x=-3$ is marked. The ends of these lines give us the line PQ, which is the required graph of $y=x-1$.



One of the advantages of using minus numbers will be seen in the examples on the shape of bridges. Very often the formula is much simpler if we take $x=0$ at the middle of the bridge, than if we take it at the end.

6. Draw the graph of $y=x-2$ for values of x between 2 and 6.

This gives part of a straight line. With a ruler extend this straight line in the 'south-westerly' direction. Check that this line now passes through the points given by the table for x between -4 and 2.

7. Draw the graph $y=5-x$ for x between 0 and 5. This gives part of a straight line. Extend this line, using a ruler. Read off the

values of y corresponding to $x=6$ and $x=7$. For what values of x is y equal to 6 and 7? Does this agree with the way of finding $5-(-1)$ and $5-(-2)$ explained in Chapter 5?

8. *Graphs to Describe Shapes.* – The book *Building with Steel* by R. B. Way and N. D. Green contains sketches of various famous bridges. The curves there shown seem to agree with the graphs given below –

(i) The Langwies Viaduct, Switzerland. The central arch, of reinforced concrete, resembles the curve –

$$y=2-\frac{2x^2}{9}.$$

(ii) The long low arch of the Royal Tweed Bridge at Berwick,

$$y=1-\frac{2x^2}{37}, \text{ from } x=-4{\cdot}3 \text{ to } x=4{\cdot}3.$$

(iii) The lower cable on the suspension bridge at the side of Tower Bridge,

$$y=\frac{9x^2}{80}.$$

The Outline of Victoria Falls Bridge

(iv) The arch of Victoria Falls bridge,

$$y=\frac{116-21x^2}{120}.$$

The upright lines are to be marked at $x=2{\cdot}35$ and $x=-2{\cdot}35$. The level line is at the height $y=1{\cdot}25$.

9. In each of the following sets of numbers a mistake occurs. On a graph, all the numbers of each set ought to give a smooth curve. Which numbers are incorrect? What are the correct numbers that ought to be in place of the wrong ones? (Only a *rough* answer is expected to the second question.)

(i) 10, 13, 61, 19, 22.
(ii) 4, 11, 13, 19, 20, 19, 16.
(iii) 0, 0, 5, 9, 12, 14, 15, 15.
(iv) 23, 34, 41, 49, 50, 52, 53.
(v) 3610, 4000, 4410, 4640, 5290, 5760.

10. One of the following numbers is wrong, but not much. Can you find which one it is?

3844, 3969, 4096, 4252, 4356, 4489, 4624, 4761.

(Probably you will not be able to do this by the method of question 9. The point of this question is to show what *cannot* be done easily by graphs. The point of question 9 is to show what *can* be done with graphs. Somewhere in this book you will find a method that supplies a clue to the question just asked.)

11. Even if you answer question 10 by graphs, you will certainly need another method to find the wrong number in the set below –

6724, 6889, 7065, 7225, 7396, 7569.

CHAPTER 10

DIFFERENTIAL CALCULUS – THE STUDY OF SPEED

'*When I was an apprentice, I wanted to know engineering theory, and the only two books available contained unknown mathematical symbols ... Other boys might sigh for luxuries, but to me the one thing wanting was a knowledge of* $\frac{dy}{dx}$ *and* $\int$. *Looking back, I seem*