

# **Q Notes**

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# Chapter 1

## Real and Complex Vector Spaces

All of quantum computing occurs in one vector space or another. In fact, the qubit, the fundamental unit, of quantum computing is a vector quantity.

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

The definition of a vector space is called its axioms. Axioms defines the objects and the rules (operations on the objects) of the vector space.

### 1.1 Objects of the Vector Space

Every vector space is made up of two fundamental sets: scalars and vectors.

**Scalars** The underlying field (number system) of the vector space

Examples of scalars sets:  $\mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers).

Examples of scalars:  $a = -\pi$ ,  $b = e^{i\frac{\pi}{4}} = \sqrt{2} + \sqrt{2}i$

**Vectors** The objects (ordered n-tuples) of the vector space.

Examples of vector set (space):  $\mathbb{R}^2$  (2-d real vectors),  $\mathbb{C}^3$  (3-d complex vectors).

Examples of vectors:  $\vec{v} = \begin{pmatrix} -e \\ \sqrt{\pi} \end{pmatrix}$ ,  $\vec{u} = \begin{pmatrix} 5 - 9i \\ -6 \\ 4i \end{pmatrix}$

### 1.2 Rules

These are the valid operations with scalars and vectors, assuming that the objects of the vector space are already defined.

**Vector Addition:**  $\vec{v} + \vec{w} \Rightarrow \vec{u}$

$$\begin{pmatrix} 16 + 20i \\ -6 + 7i \end{pmatrix} + \begin{pmatrix} 5 - 20i \\ -9 - 10i \end{pmatrix} = \begin{pmatrix} 21 \\ -15 - 3i \end{pmatrix}$$

Some properties of vector addition:

- Zero Vector:  $\vec{v} + 0 = 0 + \vec{v} = \vec{v}$

By definition, every vector space contains a zero vector. This vector is an n-tuples made up of all 0.

$$\begin{pmatrix} -13 - 20i \\ 3 + 14i \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -13 - 20i \\ 3 + 14i \end{pmatrix} = \begin{pmatrix} -13 - 20i \\ 3 + 14i \end{pmatrix}$$

- Additive Inverses:  $\vec{v} + (-\vec{v}) = 0$

$$\begin{pmatrix} 15 - 8i \\ -2 + 7i \end{pmatrix} + -\begin{pmatrix} 15 - 8i \\ -2 + 7i \end{pmatrix} = \begin{pmatrix} 15 - 8i \\ -2 + 7i \end{pmatrix} + \begin{pmatrix} -15 + 8i \\ 2 - 7i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Commutativity:  $\vec{v} + \vec{t} = \vec{t} + \vec{v}$

$$\begin{pmatrix} 9 - 2i \\ -19 + i \end{pmatrix} + \begin{pmatrix} 8i \\ 15 - 6i \end{pmatrix} = \begin{pmatrix} 8i \\ 15 - 6i \end{pmatrix} + \begin{pmatrix} 9 - 2i \\ -19 + i \end{pmatrix} = \begin{pmatrix} 9 + 6i \\ -4 - 5i \end{pmatrix}$$

- Associativity:  $(\vec{v} + \vec{t}) + \vec{w} = \vec{v} + (\vec{t} + \vec{w})$

$$\left[ \begin{pmatrix} 2 + 9i \\ -2 + 3i \end{pmatrix} + \begin{pmatrix} 1 - 12i \\ -2 + 3i \end{pmatrix} \right] + \begin{pmatrix} 4 + 6i \\ -5 - 2i \end{pmatrix} = \begin{pmatrix} 2 + 9i \\ -2 + 3i \end{pmatrix} + \left[ \begin{pmatrix} 1 - 12i \\ -2 + 3i \end{pmatrix} + \begin{pmatrix} 4 + 6i \\ -5 - 2i \end{pmatrix} \right]$$

**Exercise:** Evaluate

**Scalar Multiplication:**  $c * \vec{v} \Rightarrow \vec{w}$  (\* represents multiplication)

$$(6 + 5i) * \begin{pmatrix} 8 - 3i \\ -6 + 5i \end{pmatrix} = \begin{pmatrix} 63 + 22i \\ -61 \end{pmatrix}$$

Some properties of scalar multiplication:

- Scalar Identity:  $1 * \vec{v} = \vec{v}, \quad \forall \vec{v}$

$$1 * \begin{pmatrix} -19 + 16i \\ 10 - 11i \end{pmatrix} = \begin{pmatrix} -19 + 16i \\ 10 - 11i \end{pmatrix}$$

- Associativity:  $c * \vec{v} = \vec{v} * c$

$$2 * \begin{pmatrix} 6 - 3i \\ -8 + 2i \end{pmatrix} = \begin{pmatrix} 6 - 3i \\ -8 + 2i \end{pmatrix} * 2 = \begin{pmatrix} 12 - 6i \\ -16 + 4i \end{pmatrix}$$

- Distributivity:  $c * (\vec{v} + \vec{t}) = (c * \vec{v}) + (c * \vec{t})$

$$(-15 - 2i) * \left[ \begin{pmatrix} 5 \\ 2 - 10i \end{pmatrix} + \begin{pmatrix} -6 + i \\ -17i \end{pmatrix} \right]$$

**Exercise:** Expand the above expression by distributivity and evaluate

**Inner Product:**  $\vec{v} \cdot \vec{w} \Rightarrow c$

This is also called dot product.

To see how to calculate real and complex inner (dot) product, see Chapter 1, Complex Inner Product

$$\begin{pmatrix} 2i \\ 9 - 5i \end{pmatrix} \cdot \begin{pmatrix} -12 - 6i \\ 5 + 6i \end{pmatrix} = -106 - 60i$$

The complex inner product can be either real or complex.

Some properties of inner products:

- Commutativity:  $\vec{v} \cdot \vec{t} = \vec{t} \cdot \vec{v}$

Recall that for complex inner product,

$$\vec{v} \cdot \vec{t} = (\vec{v})^\dagger \cdot \vec{t}$$

Commutativity only holds if the Hermitian conjugate is the same vector

$$(\vec{v})^\dagger \cdot \vec{t} = \vec{t} \cdot (\vec{v})^\dagger$$

$$\begin{pmatrix} -2 + i \\ 1 + 5i \end{pmatrix} \cdot \begin{pmatrix} -2 - 6i \\ 15 + 16i \end{pmatrix} = \begin{pmatrix} -2 - 6i \\ 15 + 16i \end{pmatrix} \cdot \begin{pmatrix} -2 + i \\ 1 + 5i \end{pmatrix} = -123 + 47i$$

- Distributivity:  $\vec{v} \cdot (\vec{t} + \vec{w}) = \vec{v} \cdot \vec{t} + \vec{v} \cdot \vec{w}$

$$\begin{pmatrix} 8 - 9i \\ 12 - 13i \end{pmatrix} \cdot \left[ \begin{pmatrix} 19 - i \\ -5i \end{pmatrix} + \begin{pmatrix} 16 \\ -12 - 7i \end{pmatrix} \right]$$

**Exercise:** Let the left vector be the Hermitian conjugate, expand the above expression by distributivity and evaluate

- Associativity with Scalar Multiplication:  $c * (\vec{v} \cdot \vec{t}) = (c * \vec{v}) \cdot \vec{t} = \vec{v} \cdot (c * \vec{t})$

However, for complex inner product, there are some minor modifications:

$$c * (\vec{v} \cdot \vec{t}) = (c^\dagger * \vec{v}) \cdot \vec{t} = \vec{v} \cdot (c * \vec{t})$$

**Exercise:** Explain why the  $(c^\dagger * \vec{v})$  in the middle expression is necessary

**Length (Modulus or Norm):**  $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$ ,  $||\vec{v}|| \geq 0$

$$\left\| \begin{pmatrix} 5 - 8i \\ 6 + 3i \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} 5 - 8i \\ 6 + 3i \end{pmatrix} \cdot \begin{pmatrix} 5 - 8i \\ 6 + 3i \end{pmatrix}} = \sqrt{134}$$

**Orthogonality:**  $\vec{v} \cdot \vec{t} = 0$

Any two vectors that are not multiples of one another will form a plane. On that plane, if the two vectors are perpendicular with one another, they are orthogonal.

This can be represented by their dot product being 0.

$$\begin{pmatrix} 5 + 6i \\ 9 - 3i \end{pmatrix} \cdot \begin{pmatrix} -15 - 15i \\ 16 - 7i \end{pmatrix} = 0 \therefore \text{orthogonal}$$

If a set of vectors are all orthogonal to one another and all has length of 1, they are called an orthonormal.

### 1.3 Positive Definiteness of Inner Product

More generally, the inner product of a space is any operation that satisfies positive definiteness. In theory, a vector space can have more than one inner products or it can have none.

- The inner product of any vectors with itself is non-negative scalar value.

$$\vec{v} \cdot \vec{v} \geq 0$$

- The length of any vector that is not the zero vector is non-zero.

$$\vec{v} \neq 0 \Rightarrow ||\vec{v}|| > 0$$

If an operation does not satisfies positive definiteness, it's called a paring.

### 1.4 Unit Vector

A unit vector has length of 1 unit. To turn a vector into a unit vector, simply scale the magnitude of the vector. This is done to conserve the direction of the vector.

$$\vec{v} \longrightarrow \frac{\vec{v}}{||\vec{v}||}$$

## 1.5 Linear Combination

The sums of the scalar products of vectors are called linear combinations in maths & superpositions in physics.

$$\vec{t} = \alpha\vec{v} + \beta\vec{w} + \gamma\vec{u} + \dots$$

Or in a more concise notation, where  $c_k$  are coefficients and  $\vec{v}_k$  are vectors

$$\vec{t} = \sum_{k=0}^{n-1} c_k \vec{v}_k$$

## 1.6 Linear Independence

Set  $\mathcal{S}$  is linearly independent if the span of set  $\mathcal{S}$  with all non-zero coefficients does not contain the zero vector.

$$\sum_{k=0}^{n-1} c_k \vec{v}_k \neq \vec{0} \text{ if } c_k \text{ are non-zero}$$

The consequences of this are

1. The vectors in set  $\mathcal{S}$  are not scaled version of one another.

$$\vec{v}_{k1} \neq \alpha \vec{v}_{k2}$$

2. One vector cannot be represented as the linear combination of another.

$$\vec{v}_t \neq \sum_{\substack{\text{all } k \\ \text{except } t}} -\frac{c_k}{c_t} \vec{v}_k = \sum_{\substack{\text{all } k \\ \text{except } t}} d_k \vec{v}_k, \quad t \in \{0, \dots, n-1\}$$

If this is not the case, then set  $\mathcal{S}$  linearly dependent.

For example,

$$\text{Let } \mathcal{L} = \left\{ \begin{pmatrix} 8-5i \\ 6+3i \end{pmatrix}, \begin{pmatrix} -34-i \\ -12-21i \end{pmatrix}, \begin{pmatrix} 9+7i \\ 2i \end{pmatrix} \right\}$$

After doing some arithmetic, we can see that the second vector is  $-3-2i$  times the first vector.

This simple can tell us right off the bat that set  $\mathcal{L}$  is linearly dependent. However, if we remove either one of these vectors,  $\mathcal{L}$  will be linearly independent.

The process of determining the linear independence of a set is by solving for the constant  $c_k$  in the simultaneous equations. If at least one  $c_k$  is 0, then the set is linearly independent.

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0}$$

## 1.7 Span

Let there be a set of vectors  $\mathcal{S} = \{\vec{v}_0, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}\}$

The span of set  $\mathcal{S}$  is another set of vectors  $\mathcal{T}$  that can be created by taking all the possible linear combinations of the vectors in set  $\mathcal{S}$ .

If linear combination was a function  $f$  with domain  $\mathcal{S}$ , then its range (span) would be  $\mathcal{T}$ .

$$f(\mathcal{S}) \longrightarrow \mathcal{T}$$

In other words, the span of a set  $\mathcal{S}$  is all the vectors linearly dependent to it.

## 1.8 Bases of a Vector Space

The basis of a space is the minimal set of vectors that completely span the vector space.

**Conjecture:** All elements of the basis set are linearly independent vectors.

To proof this conjecture, we have to show that these vectors satisfies:

1. Completeness (Spanning)
2. Set Minimality

### 1. Completeness (Spanning)

Let  $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{z-1}$  be a set of  $z$  vectors in the vector space  $\mathcal{V}$ .

To repeat, the span of  $\mathcal{A}$  is all the possible linear combinations of the vectors in  $\mathcal{A}$ .

$$\text{span}(\mathcal{A}) = \sum_{k=0}^{z-1} c_k \vec{v}_k, \quad c_k \in \text{field of } \mathcal{A}$$

We want to make the set  $\mathcal{A}$  spans completely over  $\mathcal{V}$ .

$$\text{span}(\mathcal{A}) = \mathcal{V}$$

Let there be an arbitrary vector  $\vec{w} \notin \text{span}(\mathcal{A})$ ,

From 1.7, this implies linearly independence with  $\mathcal{A}$ .

$$\vec{w} \neq \sum_{k=0}^{z-1} c_k \vec{v}_k$$

Add  $\vec{w}$  to set  $\mathcal{A}$  to extend the span of  $\mathcal{A}$ .



Repeat the process until  $\mathcal{A}$  spans completely over  $\mathcal{V}$ .

$$\mathcal{A}_f = \{\vec{w}_0, \vec{w}_1, \dots, \vec{v}_k, \}_{k=0}^{z-1}$$

## 2. Set Minimality

The cardinality of a set is a measure of its size, or the number of elements in that set.

For examples, set  $\mathcal{B} = \{0, 1, 2, 3\}$  has cardinality 4.

$$|\mathcal{B}| = 4$$

A minimal set contains a finite and is bounded away from zero number of elements that satisfies certain conditions. In this case, it's the smallest set that satisfies completeness.

$$0 < \# \text{ elements in a basis set} < \infty$$

A vector space can have more than one set of basis. However, the number of elements of each of them is the same.

### Proof of Lower Bound $> 0$

Assume set  $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1}$  from before is a random collection of vectors that span  $\mathcal{V}$ .

From 1.7, if a vector from set  $\mathcal{A}$  is linearly dependent to the set, this implies

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0} \text{ if } c_k \text{ are non-zero}$$

This means that at least one vector is made up of linear combinations of other vectors.

$$\vec{v}_t = \sum_{\substack{\text{all } k \\ \text{except } t}} -\frac{c_k}{c_t} \vec{v}_k = \sum_{\substack{\text{all } k \\ \text{except } t}} d_k \vec{v}_k, \quad t \in \{0, \dots, n-1\}$$

Substituting  $\vec{v}_t$  back into the first equation, we get

$$\sum_{\substack{\text{all } k \\ \text{excluding } t}} c_k \vec{v}_k + \sum_{\substack{\text{all } k \\ \text{excluding } t}} d_k \vec{v}_k = \sum_{\substack{\text{all } k \\ \text{excluding } t}} (c_k + d_k) \vec{v}_k = \vec{0}$$

The initial sum has more elements than the final sum.

$$\sum_{k=0}^{n-1} c_k \vec{v}_k = \vec{0} = \sum_{\substack{\text{all } k \\ \text{excluding } t}} (c_k + d_k) \vec{v}_k$$

Follow this process, we can reduce the number of elements in the set until we have at least only one element, which give us the lower bound of 1.

$$\underline{1} < |\mathcal{A}| < \infty$$

Because we eliminated all the dependent vectors, we have have left are linearly independent vectors.

Now we only have to confirm the upper bound.

### **Proof of Upper Bound $< \infty$**

The vector space  $\mathcal{V}$ , in essence, is a set of n-tuples vectors in the field  $\mathbb{C}$ .

The famous mathematician Georg Cantor has proven three results that are relevant to our proof:

1. The cardinality of real numbers  $\mathbb{R}$ , called  $\beth_1$  (beth-one), is bigger than the cardinality of natural numbers  $\mathbb{N}$ , called  $\aleph_0$  (aleph-null) or  $\beth_0$  (beth-null).

$$|\mathbb{R}| > |\mathbb{N}| \text{ or } \beth_1 > \beth_0$$

In other words, what this means is that there are more real numbers than natural numbers.

Specifically,  $|\mathbb{R}|$ , or  $\beth_1$ , is equal to the cardinality of the power set of  $|\mathbb{N}|$ , or  $\beth_0$ .

$$\beth_1 = P(\beth_0)$$

The power set of a set  $\mathcal{B}$  is the set of all subsets of  $\mathcal{B}$ , including the empty set and  $\mathcal{B}$  itself.

For example, let  $\mathcal{B} = \{a, b, c\}$

$$P(\mathcal{B}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$$

The cardinality of the power set of a set with size  $n$  is  $2^n$ .

$$|\mathcal{S}| = n \rightarrow |P(\mathcal{S})| = \sum_{k=0}^{n-1} \binom{n}{k} = 2^n$$

The cardinality of the power set of is larger than that of the set.

$$P(\mathcal{S}) = 2^{|\mathcal{S}|} > |\mathcal{S}|$$

For the case of real numbers, we can conclude that the following

$$\beth_1 = P(\beth_0) = 2^{\beth_0} > \beth_0$$

2. The cardinality of the real numbers  $\mathbb{R}$  is equal to that of any n-dimensional Euclidean space  $\mathbb{R}^n$  (see space-filling curve).

$$|\mathbb{R}| = |\mathbb{R}^n| = \beth_1$$

3. The cardinality of the real numbers  $\mathbb{R}$  is equal to that of complex numbers  $\mathbb{C}$ .

$$|\mathbb{R}| = |\mathbb{C}| = \beth_1$$

From these three results, we can reason that the cardinality of any  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is equal to that of any  $n$ -dimensional complex Euclidean space  $\mathbb{C}^n$ .

$$|\mathbb{R}^n| = |\mathbb{C}^n| = \beth_1$$

Now, we have all the necessary tools to proceed with our proof.

Returning to  $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{z-1}$  after we have eliminated all the linearly dependent vectors, repeat the process of adding linearly independent vectors not in the span of  $\mathcal{A}$  to it.

The power set,  $P(\mathcal{A})$ , has  $2^z$  elements. Each element in this  $P(\mathcal{A})$  is a subset of  $\mathcal{A}$ , containing a different combination of  $\vec{v}_k$ .

Now take the sum of each subset.

$$\text{sum}(\{\vec{v}_\alpha, \vec{v}_\beta, \vec{v}_\gamma\}) = \vec{v}_\alpha + \vec{v}_\beta + \vec{v}_\gamma$$

If we don't consider the coefficients,  $c_k$ , we can produce at least  $2^z$  linear combinations from a set of  $z$  linearly independent vectors

The results of these linear combinations are vectors that are linearly dependent to  $\mathcal{A}$

Notice that if we pick a random vector  $\vec{u}$  out of  $\mathcal{V}$ , there are only two possibilities,  $\vec{u}$  is linearly independent to  $\mathcal{A}$  or not.

If  $\vec{u}$  is linearly independent to  $\mathcal{A}$ , we add it to  $\mathcal{A}$ .

$\mathcal{A}$  now has  $z + 1$  elements. Follow the same process as above, we can show that the vectors in  $\mathcal{A}$  can produce at least  $2^{z+1}$  vectors linearly dependent to  $\mathcal{A}$ .

Do this long enough, we should get a  $\mathcal{A}$  that spans completely over  $\mathcal{V}$ , and we called this  $\mathcal{A}_f$ .

**Supposition:** The cardinality  $\mathcal{A}_f$  is infinite.

More rigorously, the cardinality of  $\mathcal{A}_f$  is equal to that of  $\mathcal{V}$ .

$$|\mathcal{A}_f| = |\mathcal{V}|$$

From  $\mathcal{A}_f$ , we can produce at least  $2^{|\mathcal{A}_f|}$  linearly combinations that are vectors linearly dependent to  $\mathcal{A}_f$ .

The set of these vectors, call it  $\mathcal{D}$ , has the following cardinality

$$|\mathcal{D}| = 2^{|\mathcal{A}_f|} = P(\mathcal{A}_f)$$

We have already shown that the cardinality of a power set is larger than that of the set. Therefore,

$$|\mathcal{D}| = P(\mathcal{A}_f) > |\mathcal{A}_f|$$

This means that the cardinality of the linearly dependent vectors set is larger than that of the vector space itself!

$$|\mathcal{A}_f| = |\mathcal{V}| \longrightarrow |\mathcal{D}| > |\mathcal{V}|$$

This obviously cannot be true, thus the supposition is refuted.

This means that  $\mathcal{A}_f$  is a finite set.

$$1 < |\mathcal{A}| < \underline{\infty}$$

Because all the vectors independent to  $\mathcal{A}$  in  $\mathcal{V}$  is already in  $\mathcal{A}$ , we know that all other vectors in  $\mathcal{V}$  are linearly dependent to  $\mathcal{A}$ . Therefore,

$$\text{span}(\mathcal{A}) = \mathcal{V}$$

This implies that all elements in  $\mathcal{A}$  is linearly independent.

Finally, we have to proof that  $|\mathcal{A}|$  well-defined for all  $\mathbb{C}^n$  vector space.

**Can someone help with this?**

To summarize, the basis of a space is a minimal set of linearly independent vectors that span the vector space completely.

## 1.9 Natural (Standard) Basis

The natural basis set are the set of vectors in  $\mathbb{C}^n$  is  $\mathcal{S} = \{\hat{e}_j\}_{j=0}^{n-1}$ , where

$$\hat{e}_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ n \text{ elements} \end{pmatrix} \text{ at } j\text{th index}$$

\*hat denotes unit length

Notice that this is a unit vector because it is 1 at the  $j$ th index and 0 everywhere.

$$\sqrt{0^2 + 0^2 + 1^2 + \dots + 0^2} = 1$$

Also, because the position of 1 is different for all  $\hat{e}_j \in \mathcal{S}$ , we know that they are orthogonal to each other.

$$\hat{e}_a \cdot \hat{e}_b = 0$$

Natural basis set is a special set of orthonormal bases.

For example, the natural basis of  $\mathbb{C}^2$  is

$$\mathcal{S} = \{\hat{x}, \hat{y}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

**Exercise:** Prove Orthonormality (Unit length and Orthogonality)

From this we can build any vectors in  $\mathbb{C}^2$  by taking linear combinations of the natural basis vectors.

$$\begin{pmatrix} 8 + \pi i \\ -\sqrt{3} - 5i \end{pmatrix} = (8 + \pi i) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-\sqrt{3} - 5i) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

## 1.10 Orthonormal Bases

To iterate, orthonormal bases are set of basis vectors that are of unit length and are mutually orthogonal to each other.

The basis set  $\mathcal{B} = \{\vec{b}_k\}$  is orthonormal if  $\vec{b}_k \cdot \vec{b}_j = \delta_{kj}$ ,  $\forall \vec{b}_j, \vec{b}_k \in \mathcal{B}$

This means that  $\vec{b}_k$  is orthogonal with all other basis vectors except itself.

Turning an arbitrary vector into a unit vector requires normalization,

$$\vec{v} \rightarrow \frac{\vec{v}}{\|\vec{v}\|}$$

The Gram-Schmidt process turns a random set of basis vectors in to an orthonormal basis.

$$\{\vec{v}_j\}_{j=0}^{n-1} \rightarrow \left\{ \frac{\vec{v}_j - \sum_{i=1}^{j-1} (\vec{v}_j \cdot \hat{e}_i) \hat{e}_i}{\left\| \vec{v}_j - \sum_{i=1}^{j-1} (\vec{v}_j \cdot \hat{e}_i) \hat{e}_i \right\|} \right\}_{j=0}^{n-1} \quad \text{where } \hat{e}_0 = \frac{\vec{v}_0}{\|\vec{v}_0\|}$$

## 1.11 Expansion Coefficients Of Orthonormal Basis

For a linear combination of an arbitrary set of basis vectors  $\{\vec{v}_k\}_{k=0}^{n-1}$ ,

$$v_f = c_0 \vec{v}_0 + c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots c_k \vec{v}_k$$

The scalar constants  $c_k$  are called expansion coefficients.

Finding these coefficients for an arbitrary set of basis vectors, requires solving systems of equations,

$$x + 3y = 5$$

$$6x - 3y = -7$$

However, the set of basis vectors is orthonormal, we can simply take the dot product of the resultant vectors with each of the basis to get the corresponding coefficient.

For  $\mathbf{v} = \sum_{k=1}^n \alpha_k \mathbf{b}_k$ , if we want to extract the coefficient  $\alpha_j$ ,

$$\mathbf{b}_j \cdot \mathbf{v} = \mathbf{b}_j \cdot \sum_{k=0}^{n-1} (\alpha_k \mathbf{b}_k) = \sum_{k=0}^{n-1} \mathbf{b}_j \cdot (\alpha_k \mathbf{b}_k) = \sum_{k=0}^{n-1} \alpha_k (\mathbf{b}_j \cdot \mathbf{b}_k)$$

From the definition of orthonormal basis above,

$$\sum_{k=0}^{n-1} \alpha_k (\mathbf{b}_j \cdot \mathbf{b}_k) = \sum_{k=1}^n \alpha_k \delta_{kj} = \alpha_j$$

When the basis set is orthonormal, the dot product with one of the basis vector is the coordinate representation expanded along that basis.

**Can someone help visualize this?**

Let  $\vec{t}$  be the linear combination of vectors in the orthonormal basis set  $\mathcal{A} = \{\vec{v}_k\}_{k=0}^{n-1}$ ,

$$\vec{t} = \sum_{k=0}^{n-1} c_k \vec{v}_k$$

$\vec{t}$  can be represented a vector of coefficients  $c_k$  under the basis  $\mathcal{A}$

$$\vec{t} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix}_{\mathcal{A}}$$

More about this in the next chapter.

## 1.12 Subspace

A subspace of a (parent) vector space is a subset of the parent vectors that is closed under the vector/scalar operations.

For example, let  $\mathcal{A}$  the basis of the space  $\mathcal{V}$ , the subset of  $\mathcal{A}$  spans a subspace of  $\mathcal{V}$

Take an arbitrary basis in  $\mathbb{R}^3$

$$\mathcal{A} = \left\{ \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \right\}$$

The span of these vectors is the 3-D real space.

Remove one of the vectors so that  $\mathcal{B} \subset \mathcal{A}$ ,

$$\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} \right\}$$

Now, the linear combination of vectors in  $\mathcal{B}$  only span a 2-D plane.

**Can someone help draw this? A diagram of the 3-d space and the resulting 2-d subspace**