(1) To define the distribution of X, we assign a probability to every possible value that X can take on. For, $Pr(X = 1)$, there are 11 different ways to roll two dice such that the value 1 is the smaller of the two, namely: $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}$. Since there are a total of 6^2 different ways to roll two dice, the final probability is $11/36$. For $Pr(X = 2)$, there are 9 different ways, hence, $9/36 = 1/4$ probability. If you continue to write out all of these probabilities, you will see a pattern emerge: $Pr(X = k) = (13 - 2k)/36$. You can plug in the remaining values of k to obtain those probabilities.

(2) In class, we talked about a common scenario in which (1) you repeat some task until you finally succeed, and (2) there is a a success probability of p on each try. Then, the expected number of tries before succeeding will be $1/p$. In this problem, each toss has a $p = 1/6$ chance of succeeding (i.e. seeing the first six). Then, the expected number of tosses before actually succeeding is $1/p = 1/(1/6) = 6$.

(3) Similar to problem 2, the probability of succeeding (i.e. all events happening together in one day) is $(0.8)(0.5)(0.25) = 0.1$. This means on each day, we have a 0.1 chance of succeeding, and thus, the expected number of days it will take is $1/0.1 = 10$.

(4)

a. Let X_i denote the event that exactly one person gets off on the ith floor. Then,

$$Pr(X_i) = n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}$$

since there are n ways to choose the person that gets off on the ith floor, a $1/10$ probability that the ith floor is chosen, and a $(9/10)^{n-1}$ probability that the other 9 floors are chosen from each of the $n-1$ people. (Note: It might be easier to think of this problem as n tosses of a 10-sided die, and asking $Pr(X_i)$, the probability that the ith number shows up exactly once out of the n tosses.)

b. Let's be more explicit with our definition of X_i. Let $X_i = 1$ when exactly one person gets off at the ith floor, and $X_i = 0$ otherwise. Then, $\mathbb{E}(X_i) = Pr(X_i = 1)$, which equals to the formula we found in part a. Thus, to get $\mathbb{E}(X)$, we can simply sum over $\mathbb{E}(X_i)$ for all i floors:

$$\mathbb{E}(X) = \sum_{i=1}^{10} \mathbb{E}(X_i) = 10n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1} = n\left(\frac{9}{10}\right)^{n-1}$$

(5)

a. Since we want to know about all m balls and whether or not each falls into bin 1, we simply add together X_i for each ball i. Thus, $X = \sum_{i=1}^{m} X_i$.

b. Let us first compute the expected value for whether or not the ith ball falls into bin 1: $\mathbb{E}(X_i) = (1)(Pr(X_i = 1) + (0)(Pr(X_i = 0) = 1/n$. Then we know that $\mathbb{E}(X) = \sum_{i=1}^{m} \mathbb{E}(X_i) = \sum_{i=1}^{m} 1/n = m/n$.

(6) This problem is identical to the "Fixed points of a permutation" section covered in lecture. We define a random variable X_i to be 1 if the ith student ends up in his/her own bed, and 0 otherwise. The probability of this happening is $Pr(X_i = 1) = 1/n$, since there are n beds to choose from for the ith student. The expected value, $\mathbb{E}(X_i)$ is also $(1/n)$ since X is a binary random variable. Then, we can use linearity to find the expected value of the total number of students who end up in their own bed.

This is simply:

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

(7)

a. Because we are drawing numbers from a set without replacement, this is not independent.

b. Again, we are drawing words out of a sentence without replacement, and thus, is not independent.

c. Since there are all possible combinations of suits and values in a deck of cards, the suit does not affect the value of a chosen card (or vice versa). Thus, this is independent.

d. Putting constraints on the suit still doesn't have any effect on the value of the card. Again, this is independent.

(8) Let us define the string S = "a rose is a rose", and let $X_i = 1$ if S begins at position i in the string. At any point in the string, S has equal probability of appearing, and so $Pr(X_i = 1) = (1/4)^5$ for any i. Since X_i is binary, it follows that $\mathbb{E}(X_i) = (1/4)^5$ as well. Now, let X = number of times S appears in a string of length n. The expected value $\mathbb{E}(X)$ should be the sum of the expected values of S appearing at each point in the string. Since there are $(n-4)$ possible positions that S can begin at, we can use the linearity of expectation to arrive at:

$$\mathbb{E}(X) = \sum_{i=1}^{n-4} \mathbb{E}(X_i) = \sum_{i=1}^{n-4} \left(\frac{1}{4}\right)^5 = \frac{(n-4)}{1024}$$

(9)

a. We compute all relevant statistics of Z below:

$$\begin{aligned} \mathbb{E}(Z) &= \frac{1}{8}(1+2+3+4) + \frac{1}{4}(5+6) = 4 \\ \mathbb{E}(Z^2) &= \frac{1}{8}(1^2+2^2+3^2+4^2) + \frac{1}{4}(5^2+6^2) = 19 \\ var(Z) &= \mathbb{E}(Z^2) - [\mathbb{E}(Z)]^2 = 19 - 4^2 = 3 \end{aligned}$$

b. Let X be the sum of all 10 rolls, and Z_i be the outcome of the ith roll. Then, $X = \sum_{i=1}^{10} Z_i$ and

$$\mathbb{E}(X) = \mathbb{E}\left[\sum_{i=1}^{10} Z_i\right] = \sum_{i=1}^{10} \mathbb{E}(Z_i) = (10)(4) = 40.$$

To obtain var(X), we must first compute $\mathbb{E}(X^2)$

$$\mathbb{E}(X^2) = \mathbb{E}\left[\left(\sum_{i=1}^{10} Z_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{10}(Z_i)^2 + \sum_{i \neq j} Z_i Z_j\right] = \sum_{i=1}^{10} \mathbb{E}(Z_i^2) + \sum_{i \neq j} \mathbb{E}(Z_i Z_j)$$

The first component of the sum uses what we computed for part a: $\sum_{i=1}^{10} \mathbb{E}(Z_i^2) = (10)(19) = 190$. The second component, $\mathbb{E}(Z_i Z_j)$ looks a bit trickier. However, it becomes substantially simpler when we realize that rolls Z_i and Z_j are independent. Thus, $\mathbb{E}(Z_i Z_j) = \mathbb{E}(Z_i)\mathbb{E}(Z_j) = (4)(4) = 16$.

Since there are (10)(9) different pairs for which $i \neq j$, the final result is:

$$\mathbb{E}(X^2) = \sum_{i=1}^{10} \mathbb{E}(Z_i^2) + \sum_{i \neq j} \mathbb{E}(Z_i)\mathbb{E}(Z_j) = (19)(10) + (10)(9)(4)(4) = 1630$$

And finally, to compute the variance:

$$var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1630 - (40)^2 = 30$$

c. There are $\binom{10}{5}$ ways to choose the order of rolls that result in sixes. Then, the probability of rolling exactly five sixes is $\left(\frac{1}{4}\right)^5$, and the probability of rolling exactly five numbers that are not sixes is $\left(1 - \frac{1}{4}\right)^5$. The final probability is thus:

$$\binom{10}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^5$$

d. Similar to problem 2, we have a $p = (1/4)$ chance of succeeding (i.e. rolling a six) on each try. Then the expected number of rolls before seeing a six should be $1/p = 1/(1/4) = 4$.

e. Let X_1 = number of rolls before getting first six, and X_2 = number of rolls before getting second six. Since the rolls are independent, $\mathbb{E}(X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_1) = 4 + 4 = 8$.

f. Again, let Z_i = the outcome of the ith roll. We can express

$$\begin{aligned} A &= \frac{1}{n} \sum_{i=1}^{n} Z_i \\ \mathbb{E}(A) &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i\right] = \frac{1}{n}\,\mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(Z_i) = \left(\frac{1}{n}\right)(4n) = 4 \end{aligned}$$

The last component of the equation draws on our knowledge that $\mathbb{E}(Z_i) = 4$, from part a. To compute the variance, we have to figure out var(A^2) first:

$$\begin{aligned} \mathbb{E}(A^2) &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2\right] = \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} Z_i\right)^2\right] \\ &= \frac{1}{n^2}\left[\sum_{i=1}^{n}\mathbb{E}(Z_i^2) + \sum_{i \neq j}\mathbb{E}(Z_i)\mathbb{E}(Z_j)\right] \\ &= \frac{1}{n^2}\left[19n + (n)(n-1)(4)(4)\right] = \frac{3 + 16n}{n} \end{aligned}$$

We see that the expectation computations in the above equation are almost identical to the ones in part b. The only difference is that instead of 10 tosses, we now have n tosses. Now, to get the variance:

$$var(A^2) = \mathbb{E}(A^2) - [\mathbb{E}(A)]^2 = \frac{3 + 16n}{n} - 4^2 = \frac{3}{n}$$