# Homework 3

Many of these questions are taken from Grinstead and Snell or from Feller; some have been lightly edited.

1. A particular car manufacturer has three factories $F_1$, $F_2$, $F_3$ making 25%, 35%, and 40%, respectively, of its cars. Of their output, 5%, 4%, and 2%, respectively, are defective. A car is chosen at random from the manufacturer's supply.

   (a) What is the probability that the car is defective?

   (b) Given that it is defective, what is the probability that it came from factory $F_1$?

2. Suppose that there are equal numbers of men and women in the world, and that 5% of men are colorblind whereas only 1% of women are colorblind. A person is chosen at random and found to be colorblind. What is the probability that the person is male?

3. A doctor assumes that his patients has one of the three diseases $d_1$, $d_2$, or $d_3$, each with probability 1/3. He carries out a test that will be positive with probability 0.8 if the patient has $d_1$, with probability 0.6 if the patient has $d_2$, and with probability 0.4 if the patient has $d_3$.

   (a) What is the probability that the test will be positive?

   (b) Suppose that the outcome of the test is positive. What probabilities should the doctor now assign to the three possible diseases?

4. One coin in a collection of 65 coins has two heads; the rest of the coins are fair. If a coin, chosen at random from the lot and then tossed, turns up heads six times in a row, what is the probability that it is the two-headed coin?

5. A fair coin is tossed $n$ times. What is the probability that the tenth toss comes up heads, given that the total number of heads is $k$?

6. A scientist discovers a fossil fragment that he believes is either some kind of tiger (with probability 1/3) or mammoth (with probability 2/3). To shed further light on this question, he conducts a test which has the property that for tigers, it will come out positive with probability 5/6 whereas for mammoths it will come out positive with probability just 1/3. Suppose the test comes out negative. What is the probability that the fossil comes from a tiger?

7. Sherlock Holmes finds paw prints at the scene of a murder, and thinks that they are either from a dog, with probability 3/4, or from a small bear, with probability 1/4. He then discovers some unusual scratches on a nearby tree. The probability that a dog would produce these scratches is 1/10, while the probability that a bear would is 3/5. What is the probability that the animal is a bear?

8. A coin is tossed three times. Consider the following five events:

   - $A$: Heads on the first toss
   - $B$: Tails on the second toss
   - $C$: Heads on the third toss

- $D$: All three outcomes the same
- $E$: Exactly one head

Which of the following pairs of events are independent?

(1) $A$ and $B$    (2) $A$ and $D$    (3) $A$ and $E$    (4) $D$ and $E$.

9. You randomly shuffle a standard deck and deal two cards. Which of the following pairs of events are independent?

   (1) $A = \{$first card is a heart$\}, B = \{$second card is a heart$\}$
   (2) $A = \{$first card is a heart$\}, B = \{$first card is a 10$\}$
   (3) $A = \{$first card is a 10$\}, B = \{$second card is a 9$\}$
   (4) $A = \{$first card is a heart$\}, B = \{$second card is a 10$\}$

10. A student applies to UCLA and UCSD. He estimates that he has a probability of 0.5 of being accepted at UCLA and a probability of 0.3 of being accepted at UCSD. He further estimates that the probability that he will be accepted by both is 0.2.

    (a) What is the probability that he is accepted at UCSD if he is accepted at UCLA?
    (b) Is the event "accepted at UCLA" independent of the event "accepted at UCSD"?

11. Each of the four engines on an airplane functions correctly on a given flight with probability 0.99, and the engines function independently of each other. Assume that the plane can make a safe landing if at least two of its engines are functioning correctly. What is the probability of a safe landing?

12. Here's a different approach to analyzing the birthday paradox. Suppose $m$ balls are thrown at random, one at a time, into $n$ bins, and let $E_i$ denote the event that bin $i$ gets two or more balls.

    (a) Fix $i$ and let $F_{jk}$ be the event that the $j$th and $k$th balls fall in bin $i$. What is $\Pr(F_{jk})$?
    (b) Using (a), show that $\Pr(E_i) \leq m^2/2n^2$.
    (c) Using (b), give an upper bound on the probability that some bin gets two or more balls (ie. that at least one bin gets two or more balls).
    (d) For what values of $m$ is this probability at most $1/2$?

13. **(Challenge)** Suppose $m$ balls are thrown into $n$ bins. It turns out that you can throw roughly $m = n^a$ balls, for some $0 < a < 1$, while being reasonably sure that no bin gets *three* or more balls. What is $a$?

14. **(Challenge)** In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time: the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted it won't rain, are both 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3.

    (a) What is the probability that the forecaster predicts rain? (Hint: find an equation containing this quantity and solve for it.)
    (b) What is the probability that Pickwick has no umbrella, given that it rains?
    (c) What is the probability that he brings his umbrella, given that it does not rain?