Sets and counting

DSE 210

Sets

$A = \{a, b, c, \ldots, z\}$ $\quad |A| = 26$

$B = \{0, 1\}$ $\quad |B| = 2$

$E = \{\text{all even integers}\}$ $\quad |E| = \infty$

$S = \{x \in E : x \text{ is a multiple of } 3\}$

$I = [0, 1] = \{x : 0 \leq x \leq 1\}$

In a set, the *order* of elements doesn't matter:

$$\{0, 1, 2\} = \{2, 0, 1\}$$

and there are no duplicates.

Tuples

Let $C = \{H, T\}$.

All pairs of elements from C:

$$\{(H, H), (H, T), (T, H), (T, T)\} = C \times C = C^2$$

All triples of elements of C:

$$\{(H, H, H), (H, H, T), (H, T, H), \ldots\} = C \times C \times C = C^3$$

All sequences of k elements from C: denoted $C^k = C \times C \times \cdots \times C$.

How many sequences of length k are there? $|C^k| = |C|^k = 2^k$.

In a sequence, the order of elements matters:

$$(H, T) \neq (T, H).$$

Let $A = \{a, b, c, \ldots, z\}$.

How many sequences of length 2? 26^2

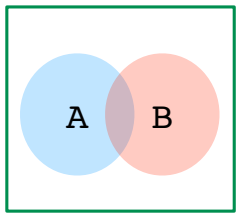
How many sequences of length 10? 26^{10}

How many sequences of length n? 26^n

An alien language has an alphabet of size 10. Every sequence of ≤ 5 of these characters is a valid word. How many words are there in this language?

$10^1 + 10^2 + 10^3 + 10^4 + 10^5 = 10 + 100 + 1000 + 10000 + 100000 = 111110.$

Union and intersection



$A \cup B = \{\text{any element in } A \text{ or in } B \text{ or in both}\}$
$A \cap B = \{\text{any element in } A \text{ and in } B\}$

$M = \{2, 3, 5, 7, 11\}$ and $N = \{1, 3, 5, 7, 9\}$

$M \cup N = \{1, 2, 3, 5, 7, 9, 11\}$

$M \cap N = \{3, 5, 7\}$

$S = \{\text{all even integers}\}$ and $T = \{\text{all odd integers}\}$

$S \cup T = \{\text{all integers}\}$

$S \cap T = \emptyset$

Permutations

How many ways to order the three letters A, B, C?

$ABC, ACB, BAC, BCA, CAB, CBA$
3 choices for the first, 2 choices for the second, 1 choice for the third
$3 \times 2 \times 1 = 6$. Call this 3!

How many ways to order A, B, C, D, E?

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

How many ways to place 6 men in a line-up?

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

How many possible outcomes of shuffling a deck of cards?

52!

General rule: The number of ways to order n distinct items is:

$$n! = n(n-1)(n-2)\cdots 1.$$

Combinations

An ice-cream parlor has flavors $\{\text{chocolate}, \text{vanilla}, \text{strawberry}, \text{pecan}\}$.
You are allowed to pick two of them. How many options do you have?

$$CV, CS, CP, VS, VP, SP$$

In general, the number of ways to pick k items out of n is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

For instance, $\binom{4}{2} = \frac{4 \cdot 3}{2!} = 6$.

How many ways to pick three ice-cream flavors?

$$\binom{4}{3} = 4$$

Pick any 4 of your favorite 100 songs. How many ways to do this?

$$\binom{100}{4} = \frac{100 \cdot 99 \cdot 98 \cdot 97}{4 \cdot 3 \cdot 2 \cdot 1}$$