(1) There are 4! total ways to return the hats to the women, but only 1 way for all women to receive the correct hat back. So the probability is $1/4! = 1/24$.

(2) The sample space is 2^6 since each of the six children can have two possible genders. We can think of the event space as finding all permutations that contain three females and three males (i.e. FFFMMM, FFMFMM, FMMFFM, etc.). Since we are basically finding all possible positions to place the three 'F's (as the remaining three 'M's will fall into place thereafter), the number of permutations is simply $\binom{6}{3}$. The total probability is thus $\binom{6}{3}/2^6 = \frac{5}{16}$

(3) The total number of ways to answer all 10 questions (i.e., the sample space) is 2^{10}.

a. There is only 1 way to answer all 10 questions correctly. Thus the probability is $1/2^{10} = 1/1024$.

b. Let's first find the number of ways that a student can answer exactly 8 questions (80%) correctly. There are $\binom{10}{8}$ ways to choose the 8 questions the student gets correct. Once we fix the questions that the student gets correct, the answers of the remaining questions are fixed as well (i.e. they must be the opposite of the correct answer). Thus, the probability of a student answering exactly 8 questions correctly is $\binom{10}{8}/2^{10}$. Since we must find the probability of a student getting *at least* 80% correct, we must also compute the probability of a student answering exactly 9 and 10 questions correctly. The final probability is as follows:

$$\frac{\binom{10}{8} + \binom{10}{9} + 1}{2^{10}} = \frac{7}{128}$$

(4)

a. There are $\binom{26}{3}$ ways to pick 3 distinct letters and $\binom{10}{3}$ ways to pick 3 distinct digits (0 through 9). Since order matters here, we must also enumerate all possible ways to order a set of six chosen symbols, which is 6!. Thus the sample space is: $\binom{26}{3}\binom{10}{3}6!$

b. There are still $\binom{10}{3}$ and $\binom{26}{3}$ ways to choose 3 distinct digits and 3 distinct letters, respectively. However, since the digits and letters cannot be interleaved anymore, we need only to compute the total number of ways to order within the 3 digits and the 3 letters. The probability, therefore, is:

$$\frac{3!\binom{26}{3}3!\binom{10}{3}}{\binom{26}{3}\binom{10}{3}6!} = \frac{1}{20}$$

c. There are still $\binom{10}{3}$ and $\binom{26}{3}$ ways to choose 3 distinct digits and 3 distinct letters, respectively. But this time, there is only one way to order within the digits and letters, since they must be increasing. However, there are still multiple ways to interleave the digits within the letters, while preserving the order within each group of symbols, for example: $(L_1, L_2, L_3, D_1, D_2, D_3)$ or $(L_1, D_1, L_2, D_2, L3, D_3)$. Here, we simply have to specify the 3 positions (out of the 6) in which we'd like to insert the letters (or vice versa); after that, the ordering within the letters, as well as the ordering and positions of the digits, are all fixed. Thus, the total number of ways to interleave the set of letters into the set of digits is $\binom{6}{3}$, and the final probability is:

$$\frac{\binom{26}{3}\binom{10}{3}\binom{6}{3}}{\binom{26}{3}\binom{10}{3}6!} = \frac{1}{36}$$

(5) The number of ways that Snow White can ask three of seven dwarfs is $\binom{7}{3}$, hence, our sample space.

a. Since we know that Dopey is in the group, we only have to choose 2 more dwarves out of the remaining 6 dwarves. This gives us a probability of $\binom{6}{2}/\binom{7}{3} = \frac{3}{7}$ that a random group of three dwarves contains Dopey. Alternatively, we can compute $1 - P(\text{Dopey not in group})$ sequentially. To compute this, we multiply the probability of choosing each of the three dwarves that is not Dopey, i.e. $\frac{6}{7}$ is the chance that the first dwarf is not Dopey, $\frac{5}{6}$ is the chance that the second dwarf is not Dopey, and so on. The final probability we get is $1 - \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} = \frac{3}{7}$.

b. We now only need to choose 1 more dwarf out of the remaining 5 dwarves. This gives us a probability of $5/\binom{7}{3} = \frac{1}{7}$.

c. Using the technique from parts a and b, we now want to choose 3 dwarves out of the remaining 5 dwarves (since Dopey and Sneezy are "off-limits"). This gives us a probability of $\binom{5}{3}/\binom{7}{3} = \frac{2}{7}$. Alternatively, we can compute its complement:

$$\begin{aligned}
Pr(\text{Neither Dopey nor Sneezy}) &= 1 - Pr(\text{Dopey or Sneezy}) \\
&= 1 - [Pr(\text{Dopey}) + Pr(\text{Sneezy}) - Pr(\text{Dopey and Sneezy})] \\
&= 1 - [3/7 + 3/7 - 1/7] = 2/7
\end{aligned}$$

(6) This problem is more easily viewed as a bins and balls problem. Let each digit $0-9$ be represented by a bin, and each of the appearing digits as a ball. This gives us 10 bins, and m balls. For example, if the first ball is thrown into bin 4, this indicates that the first digit in our sequence is 4. To find the probability of the digit 7 appearing at least once, it is easiest to compute $1 - Pr(\text{No ball lands in bin 7})$, which gives us $1 - \left(\frac{9}{10}\right)^m$. Next, we simply want to find the value m for which: $1 - \left(\frac{9}{10}\right)^m \geq 0.90$. After some trial and error, you should find this number to be $m = 22$.

(7) Let's solve the complement of this problem first: What is the probability that everyone's birthday is in a different month? It's easiest to solve this problem sequentially: The first person can have a birthday on any month, thus $\frac{12}{12} = 1$. The second person, to be unique, can only choose from the remaining 11 months, and thus has $\frac{11}{12}$ chance. This continues until the nth person, who only has $\frac{12-n+1}{12}$ chance of having a birthday in a month different from everyone else before him/her. Because this is only the complement of what we are trying to solve, we must subtract all of this from 1, giving the answer:

$$1 - \left(\frac{12}{12} \times \frac{11}{12} \times \cdots \times \frac{12-n+1}{12}\right) = 1 - \frac{12!}{12^n(12-n)!}$$

(8) Because each student must choose exactly two of three items, we can express the sample space as:

$$P(A \cap F) + P(A \cap M) + P(F \cap M) = 1$$

a. Using the summation rule, we know that $P(M) = P(M \cap A) + P(M \cap F)$, and subsequently, from the equation above, $P(M) = 1 - P(A \cap F) = 1 - 1/4 = 3/4$.

b. Art or French is a subset of all events in the sample space. Thus, the probability is 1.

(9) The general equation here is $P(S \cap H) = P(S) + P(H) - P(S \cup H)$. All of these values are provided to us in the problem, giving a final answer of $80\% + 60\% - 90\% = 50\%$.

(10) Since we are trying to lower-bound the intersection of all of the probabilities, we begin by assuming the lowest possible probabilities of all individual events (i.e. only 70% of soldiers lost an eye, etc.). Next, we can start lower-bounding the intersection of just the first two events:

$$\begin{aligned}
Pr(\text{Lose Eye} \cup \text{Lose Ear}) &\leq 100\% \\
Pr(\text{Lose Eye}) + Pr(\text{Lose Ear}) - P(\text{Lose Eye} \cap \text{Lose Ear}) &\leq 100\% \\
145\% - P(\text{Lose Eye} \cap \text{Lose Ear}) &\leq 100\%
\end{aligned}$$

Thus, the lower bound on $P(\text{Lose Eye} \cap \text{Lose Ear})$ must be 45%. We can use this same technique recursively to lower-bound $P((\text{Lose Eye} \cap \text{Lose Ear}) \cap \text{Lose Hand})$ using the following equation:

$$45\% + 80\% - P((\text{Lose Eye } \cap \text{ Lose Ear}) \cap \text{ Lose Hand}) \leq 100\%$$

Applying the result of this lower bound and the same technique to the probability of losing a leg gives us the final answer of 10%.

(11) All of the problems below use the formula for conditional probability: $Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$.

a. $Pr(\text{Two H} \mid \text{First is H}) = \frac{Pr(\{HHT,HTH\})}{Pr(\{H\})} = \frac{2/2^3}{1/2} = \frac{1}{2}$

b. $Pr(\text{Two H} \mid \text{First is T}) = \frac{Pr(\{THH\})}{Pr(\{T\})} = \frac{1/2^3}{1/2} = \frac{1}{4}$

c. $Pr(\text{Two H} \mid \text{First two are H}) = \frac{Pr(\{HHT\})}{Pr(\{HH\})} = \frac{1/2^3}{1/2^2} = \frac{1}{2}$

d. $Pr(\text{Two H} \mid \text{First two are T}) = \frac{Pr(\{\})}{Pr(\{TT\})} = \frac{0}{1/2^2} = 0$

e. $Pr(\text{Two H} \mid \text{First is H, Third is T}) = \frac{Pr(\{HHT\})}{Pr(\{HHT,HTT\})} = \frac{1/2^3}{2/2^3} = \frac{1}{2}$

(12)

a. $Pr(\text{Heart} \mid \text{Red}) = \frac{Pr(\{\heartsuit 1,\heartsuit 2,\ldots,\heartsuit J,\heartsuit Q,\heartsuit K\})}{Pr(\{Red\})} = \frac{13/52}{1/2} = \frac{1}{2}$

b. $Pr(\text{J, Q, K, A} \mid \text{Red}) = \frac{Pr(\{\heartsuit J,\heartsuit Q,\heartsuit K,\heartsuit A,\diamondsuit J,\diamondsuit Q,\diamondsuit K,\diamondsuit A\})}{Pr(\{Red\})} = \frac{8/52}{1/2} = \frac{4}{13}$

c. $Pr(\text{J} \mid \text{J, Q, K, A}) = \frac{Pr(\{J\})}{Pr(\{J,Q,K,A\})} = \frac{1/13}{4/13} = \frac{1}{4}$

(13) Re-arranging the formula for conditional probability gives us $P(A \cap B) = Pr(A|B) \times Pr(B) = Pr(A|B) \times (1 - Pr(B^c)) = (1/2) \times (1 - 1/4) = 3/8$.

(14)

a. $Pr(\text{Sum} > 7) \mid \text{First is 4}) = \frac{Pr(\{(4,4),(4,5),(4,6)\})}{Pr(\{4\})} = \frac{3/6^2}{1/6} = \frac{1}{2}$

b. $Pr(\text{Sum} > 7) \mid \text{First is 1}) = \frac{Pr(\{\})}{Pr(\{4\})} = \frac{0}{1/6} = 0$

c. $Pr(\text{Sum} > 7) \mid \text{First is} > 3) = \frac{Pr(\{(4,4),(4,5),(4,6),(5,3),\ldots,(5,6),(6,2),\ldots,(6,6)\})}{Pr(\{4,5,6\})} = \frac{(3+4+5)/6^2}{3/6} = \frac{2}{3}$

d. $Pr(\text{Sum} > 7) \mid \text{First is} < 5) = \frac{Pr(\{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6)\ldots,(6,6)\})}{Pr(\{1,2,3,4\})} = \frac{6/6^2}{4/6} = \frac{1}{4}$

(15) There are three sets for which these numbers can add up to 12: $\{2,4,6\}, \{2,2,8\}, \{4,4,4\}$. For each set, there are multiple orderings for which they could be drawn: $\{2,4,6\}$ can be drawn $3! = 6$ different ways, $\{2,2,8\}$ can be drawn $\binom{3}{2} = 3$ different ways, and $\{4,4,4\}$ can only be drawn 1 way. Thus, the sample space is $6 + 3 + 1 = 10$. Only the set $\{2,2,8\}$ gives exactly two 2's, and there are exactly 3 ways to draw this set. The final probability is therefore $\frac{3}{10}$.