(1) We have X denote the number of accidents that occur on Sunday.

a. Let $Y_i = 1$ if the ith accident occurs on Sunday, and 0 otherwise. Since we know $Pr(Y_i = 1) = 0.05$, then we have the following:

$$\begin{aligned}
\mathbb{E}(Y_i) &= Pr(Y_i = 1) = 0.05 \\
\mathbb{E}(X) &= \sum_{i=1}^{200} \mathbb{E}(Y_i) = (200)(0.05) = 10 \\
var(Y_i) &= \mathbb{E}(Y_i^2) - [\mathbb{E}(Y_i)]^2 = 0.05 - 0.05^2 = 0.0475 \\
var(X) &= \sum_{i=1}^{200} var(Y_i) = (200)(0.0475) = 9.5
\end{aligned}$$

b. We want to know $Pr(X = \mathbb{E}(X))$ or $Pr(X = 10)$. Think of this as 200 tosses of a die that has 7 faces (Monday, Tuesday, Wednesday, etc.). We want to know the probably of getting exactly 10 Sundays. The answer is:

$$\binom{200}{10}(0.05)^{10}(1-0.05)^{190}$$

The first term chooses the number of ways to place the 10 Sundays out of 200 tosses. The second term is the probability of flipping exactly 10 Sundays. The third term is the probability that all 190 other tosses are *not* Sundays.

(2)

a. To keep the ith bin empty, each of the m balls can only go into $n-1$ out of the n possible bins. Thus, $Pr(X_i = 0) = \left(\frac{n-1}{n}\right)^m$

b. This is similar to problem 1b. The answer is:

$$\binom{m}{1}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{m-1}$$

The first term corresponds to the m possible tosses for which the lone ball in the ith bin could be from. The second term is the probability of tossing that one ball into the ith bin. The third term is the probability that all other tosses do not go into the ith bin.

c. Let us introduce a new random variable: $B_j = 1$ if the jth ball falls into bin i. Then it should follow that $X_i = B_1 + B_2 + \cdots + B_m$ and $Pr(B_j = 1) = (1/n)$. We can compute the following:

$$\begin{aligned}
\mathbb{E}(B_j) &= Pr(B_j = 1) = \frac{1}{n} \\
\mathbb{E}(X_i) &= \sum_{j=1}^{m} \mathbb{E}(B_j) = \frac{m}{n}
\end{aligned}$$

d. Continuing from above, we have:

$$\begin{aligned}
var(B_j) &= \mathbb{E}(B_j^2) - [\mathbb{E}(B_j)]^2 = \frac{1}{n} - \frac{1}{n^2} \\
var(X_i) &= \sum_{j=1}^{m} var(B_j) = m\left(\frac{1}{n} - \frac{1}{n^2}\right)
\end{aligned}$$

(3) One example is if $X \in \{-1, 0, 1\}$ with uniform probability, and $Y = 2X$. So, X and Y are definitely not independent, and $Y \in \{-2, 0, 2\}$ also with uniform probability. Then, we have $\mathbb{E}(X) = \frac{1}{3}(-1+0+1)$ and $\mathbb{E}(Y) = \frac{1}{3}(-2+0+2)$. If $Z = XY$, Then $\mathbb{E}(Z) = \mathbb{E}(XY) = 2 \cdot \mathbb{E}(X^2) = \frac{2}{3}(-1^2 + 0^2 + 1^2) = 0$. Thus, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = 0$.

(4) Since the property $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$ only holds when X and Y are independent, it is quite easy to find a counterexample; most cases in which X and Y are not independent will satisfy $\text{var}(X+Y) \neq \text{var}(X) + \text{var}(Y)$; for example, if $X = Y$.

(5)

a. $\mathbb{E}(X_1)$ and $\mathbb{E}(X_{n-k+1})$ are the first and last possible starting points of a run of length exactly k. For each of the k tosses, there is a 1/2 probability that we get what we want (either heads if a run of heads, or tails if a run of tails), giving us a probability of $\frac{1}{2^k}$.

b. The expected values for starting points in the middle of the sequence, i.e. $1 < i < n-k+1$, are a bit different. Let's say the $i-1^{th}$ toss is tails. Then beginning at i, we want k heads, which has a probability of $\frac{1}{2^k}$. To ensure that the run is exactly k long, we need to make sure that the toss at $k+1$ is the opposite of whatever run we have (tails in our case). Thus, we need to multiply by an additional (1/2), giving us $\frac{1}{2^{k+1}}$

c. To get $\mathbb{E}(R_k)$, we can sum over all $\mathbb{E}(X_i)$ for all possible values of i:

$$\begin{aligned}
\mathbb{E}(R_k) &= \sum_{i=1}^{n-k+1} \mathbb{E}(X_i) = \frac{1}{2^k} + \frac{1}{2^k} + \sum_{i=2}^{n-k} \frac{1}{2^{k+1}} \\
&= \frac{2}{2^k} + \frac{n-k-1}{2^{k+1}} = \frac{2^2+n-k-1}{2^{k+1}} = \frac{n-k+3}{2^{k+1}}
\end{aligned}$$

(6)

a. First, let us establish the framework of this social network: each person p_i may only choose a friend p_j such that $j < i$. This means that each person p_i has $i-1$ friends to choose from. In our scenario, the probability of p_i having exactly one friend, is equivalent to finding the probability that no one after p_i chooses p_i as a friend.

The probability of this happening for person p_{i+1} is $\frac{i-1}{i}$ since this person has $(i+1)-1 = i$ friends to choose from; and the probability that he/she doesn't pick person p_i is simply one less than that: $(i+1)-2 = i-1$. This continues for the next person: the probability that person p_{i+2} does not choose p_i is $\frac{i}{i+1}$, and so on. We end up with a probability that looks like this:

$$\left(\frac{i-1}{i}\right) \times \left(\frac{i}{i+1}\right) \times \left(\frac{i+1}{i+2}\right) \cdots \times \left(\frac{n-2}{n-1}\right)$$

All terms will eventually cancel out except for $(i-1)$ on top and $(n-1)$ on the bottom. Thus, we end up with $Pr(F_i = 1) = \frac{i-1}{n-1}$

b. Let $Y_i = 1$ if p_i has exactly one friend. From part a, we know that $Pr(Y_i = 1) = \frac{(i-1)}{(n-1)}$. To compute X = total number of people with exactly one friend, we simply sum over all Y_i's.

$$\begin{aligned}
\mathbb{E}(Y_i) &= Pr(Y_i = 1) = \frac{i-1}{n-1} \\
\mathbb{E}(X) &= \sum_{i=2}^{n} \mathbb{E}(Y_i) = \sum_{i=2}^{n} \frac{(i-1)}{(n-1)} = \frac{1}{n-1}\sum_{i=2}^{n}(i-1) = \left[\frac{1}{n-1}\right]\left[\frac{(n-1)(1+n-1)}{2}\right] = \frac{n}{2}
\end{aligned}$$

(7)

a. $Pr(E_{t,i})$ is the probability that at time t, process p_i flips a head while all other processes p_j where $i \neq j$ flips tails. This probability can be expressed as $Pr(E_{t,i}) = \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1}$. To use the approximation $1-x \sim e^{-x}$, we get:

$$Pr(E_{t,i}) = \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1} \approx \left(\frac{1}{n}\right)\left(e^{-\frac{n-1}{n}}\right) = \frac{1}{ne^{\frac{n-1}{n}}} \approx \frac{1}{ne}$$

The last approximation can be used when n is large since the exponent on e will be close to 1.

b. Following the hint, the probability that one of the k processes will run is $k\left(\frac{1}{ne}\right)$ since at each time, every processor has an equal chance of getting to run (despite whether or not it has gone yet). Now, to find the expected amount of additional time of wait, we can view this as a "repeated trials with probability of success of p" problem. At each time t, there is a $\frac{k}{ne}$ chance of success (here, success = one of the k jobs gets to run). Then the expected wait time is simply $\frac{1}{\frac{k}{ne}} = \frac{ne}{k}$.

c. The expected time by which all processes gets to run (let's call this T) can be obtained by summing over the expected wait times for each processor that has not yet gone:

$$\mathbb{E}(T) = \sum_{k=1}^{n} \frac{ne}{k} = ne \sum_{k=1}^{n} \frac{1}{k} \approx ne \ln n$$

The last term is a finite harmonic series. It is estimated to be $\ln n$ since $\int_1^n \frac{1}{x} dx = \ln n$.

(8)

a. The probability that exactly k balls fall into the i^{th} bin, or $Pr(X_i = k)$ is

$$\binom{m}{k} p_i^k (1-p_i)^{m-k}$$

The first term is the number of ways that the k balls can be thrown (out of m tosses), the second term is the probability that exactly k balls land in the same bin, the third term is the probability that all other balls land in other bins.

b. Let $B_j = 1$ if the ball from the j^{th} toss lands in bin i, and 0 otherwise. Then, $\mathbb{E}(B_j) = Pr(B_j = 1) = p_i$, and $\text{var}(B_j) = p_i - p_i^2$. Since $X_i = B_1 + B_2 + \cdots + B_m$, then

$$\mathbb{E}(X_i) = \sum_{j=1}^{m} \mathbb{E}(B_j) = mp_i$$

$$var(X_i) = \sum_{j=1}^{m} var(B_j) = m(p_i - p_i^2) = mp_i(1-p_i)$$

c. Let $A_i = 1$ if bin i is empty. Then what we want to find is the union of all A_i's. This can be expressed with the following upperbound: $Pr(A_1) \cup Pr(A_2) \cup \cdots \cup Pr(A_n) \leq Pr(A_1) + Pr(A_2) + \cdots + Pr(A_n)$. Since $Pr(A_i) = (1-p_i)^m$, an upperbound on the probability that there is an empty bin is $\sum_{i=1}^{n} (1-p_i)^m$.

d. Here, let $A_i = 1$ if bin i has at least two balls. Again, we want to find an upperbound on the union of all A_i's. We see that $Pr(A_i) = \binom{m}{2} p_i^2$. Note that this is very similar to the answer in part a, except that we are missing the last term, since we are not requiring that all other balls land in other bins. Then, summing over all i's, we get $\sum_{i=1}^{n} \binom{m}{2} p_i^2 = \binom{m}{2} \sum_{i=1}^{n} p_i^2$.