## DSE 210: Probability and statistics

Winter 2015

## Worksheet 4 — Random variable, expectation, and variance

- 1. A die is thrown twice. Let  $X_1$  and  $X_2$  denote the outcomes, and define random variable X to be the minimum of  $X_1$  and  $X_2$ . Determine the distribution of X.
- 2. A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?
- 3. On any given day, the probability it will be sunny is 0.8, the probability you will have a nice dinner is 0.25, and the probability that you will get to bed early is 0.5. Assume these three events are independent. What is the expected number of days before all three of them happen together?
- 4. An elevator operates in a building with 10 floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to 10.
  - (a) What is the probability that exactly one person gets out at the ith floor? Give your answer in terms of n.
  - (b) What is the expected number of floors in which exactly one person gets out? Hint: let  $X_i$  be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.
- 5. You throw m balls into n bins, each independently at random. Let X be the number of balls that end up in bin 1.
  - (a) Let  $X_i$  be the event that the *i*th ball falls in bin 1. Write X as a function of the  $X_i$ .
  - (b) What is the expected value of X?
- 6. There is a dormitory with n beds for n students. One night the power goes out, and because it is dark, each student gets into a bed chosen uniformly at random. What is the expected number of students who end up in their own bed?
- 7. In each of the following cases, say whether X and Y are independent.
  - (a) You randomly permute (1, 2, ..., n). X is the number in the first position and Y is the number in the second position.
  - (b) You randomly pick a sentence out of Hamlet. X is the first word in the sentence and Y is the second word.
  - (c) You randomly pick a card from a pack of 52 cards. X is 1 if the card is a nine, and is 0 otherwise. Y is 1 if the card is a heart, and is 0 otherwise.
  - (d) You randomly deal a ten-card hand from a pack of 52 cards. X is 1 if the hand contains a nine, and is 0 otherwise. Y is 1 if all cards in the hand are hearts, and is 0 otherwise.
- 8. A die has six sides that come up with different probabilities:

$$Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8, Pr(5) = Pr(6) = 1/4.$$

(a) You roll the die; let Z be the outcome. What is  $\mathbb{E}(Z)$  and var(Z)?

- (b) You roll the die 10 times, independently; let X be the *sum* of all the rolls. What is  $\mathbb{E}(X)$  and var(X)?
- (c) You roll the die n times and take the average of all the rolls; call this A. What is  $\mathbb{E}(A)$ ? What is var(A)?
- 9. Let  $X_1, X_2, \dots, X_{100}$  be the outcomes of 100 independent rolls of a fair die.
  - (a) What are  $\mathbb{E}(X_1)$  and  $\text{var}(X_1)$ ?
  - (b) Define the random variable X to be  $X_1 X_2$ . What are  $\mathbb{E}(X)$  and var(X)?
  - (c) Define the random variable Y to be  $X_1 2X_2 + X_3$ . What is  $\mathbb{E}(Y)$  and var(Y)?
  - (d) Define the random variable  $Z = X_1 X_2 + X_3 X_4 + \cdots + X_{99} X_{100}$ . What are  $\mathbb{E}(Z)$  and var(Z)?
- 10. Suppose you throw m balls into n bins, where  $m \ge n$ . For the following questions, give answers in terms of m and n.
  - (a) Let  $X_i$  be the number of balls that fall into bin i. What is  $Pr(X_i = 0)$ ?
  - (b) What is  $Pr(X_i = 1)$ ?
  - (c) What is  $\mathbb{E}(X_i)$ ?
  - (d) What is  $var(X_i)$ ?
- 11. Give an example of random variables X and Y such that  $var(X+Y) \neq var(X) + var(Y)$ .
- 12. Suppose a fair coin is tossed repeatedly until the same outcome occurs twice in a row (that is, two heads in a row or two tails in a row). What is the expected number of tosses?
- 13. In a sequence of coin tosses, a run is a series of consecutive heads or consecutive tails. For instance, the longest run in HTHHHTTHHTHH consists of three heads. We are interested in the following question: when a fair coin is tossed n times, how long a run is the resulting sequence likely to contain? To study this, pick any k between 1 and n, and let  $R_k$  denote the number of runs of length exactly k (for instance, a run of length k+1 doesn't count). In order to figure out  $\mathbb{E}(R_k)$ , we define the following random variables:  $X_i = 1$  if a run of length exactly k begins at position i, where  $i \leq n k + 1$ .
  - (a) What are  $\mathbb{E}(X_1)$  and  $\mathbb{E}(X_{n-k+1})$ ?
  - (b) What is  $\mathbb{E}(X_i)$  for 1 < i < n k + 1?
  - (c) What is  $\mathbb{E}(R_k)$ ?
  - (d) What is, roughly, the largest k for which  $\mathbb{E}(R_k) \geq 1$ ?