CSE 103: Probability and statistics **Fall 2010**

Homework 4

Many of these questions are taken from Grinstead and Snell or from Feller; some have been lightly edited.

1. A die is thrown twice. Let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2. Determine the distribution of X.

2. A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?

3. On any given day, the probability it will be sunny is 0.8, the probability you will have a nice dinner is 0.25, and the probability that you will get to bed early is 0.5. Assume these three events are independent. What is the expected number of days before all three of them happen together?

4. An elevator operates in a building with 10 floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to 10.

   (a) What is the probability that exactly one person gets out at the ith floor? Give your answer in terms of n.

   (b) What is the expected number of floors in which exactly one person gets out? *Hint:* let X_i be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.

5. You throw m balls into n bins, each independently at random. Let X be the number of balls that end up in bin 1.

   (a) Let X_i be the event that the ith ball falls in bin 1. Write X as a function of the X_i.

   (b) What is the expected value of X?

6. There is a dormitory with n beds for n students. One night the power goes out, and because it is dark, each student gets into a bed chosen uniformly at random. What is the expected number of students who end up in their own bed?

7. In each of the following cases, say whether X and Y are independent.

   (a) You randomly permute $(1, 2, \ldots, n)$. X is the number in the first position and Y is the number in the second position.

   (b) You randomly pick a sentence out of *Hamlet*. X is the first word in the sentence and Y is the second word.

   (c) You randomly pick a card from a pack of 52 cards. X is 1 if the card is a nine, and is 0 otherwise. Y is 1 if the card is a heart, and is 0 otherwise.

   (d) You randomly deal a ten-card hand from a pack of 52 cards. X is 1 if the hand contains a nine, and is 0 otherwise. Y is 1 if *all* cards in the hand are hearts, and is 0 otherwise.

8. You create a document of length n by repeatedly picking words at random from the small vocabulary {a, is, not, rose} (each word is equally likely to be picked). What is the expected number of times the phrase "a rose is a rose" will appear in the document? *Hint:* Let X_i be 1 if the phrase appears starting at position i, and 0 otherwise. Then use linearity of expectation.

9. A die has six sides that come up with different probabilities:

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = 1/8,\ \Pr(5) = \Pr(6) = 1/4.$$

   (a) You roll the die; let Z be the outcome. What is $\mathbb{E}(Z)$ and $\mathrm{var}(Z)$?

   (b) You roll the die 10 times, independently; let X be the *sum* of all the rolls. What is $\mathbb{E}(X)$ and $\mathrm{var}(X)$?

   (c) You roll the die 10 times. What is the probability that *exactly five* of the rolls are sixes?

   (d) You keep rolling the die until you get a six. What is expected number of rolls?

   (e) You keep rolling until you get a second six (that is, roll until the first six, and then keep going until you get another six; they don't have to be consecutive). What is the expected number of rolls?

   (f) You roll the die n times and take the average of all the rolls; call this A. What is $\mathbb{E}(A)$? What is $\mathrm{var}(A)$?

Answer key 1. $\Pr(1) = 11/36$, $\Pr(2) = 1/4$, $\Pr(3) = 7/36$, $\Pr(4) = 5/36$, $\Pr(5) = 1/12$, $\Pr(6) = 1/36$. 2. 6. 3. 10. 4 (a) $(n/10)(9/10)^{n-1}$ (b) $n(9/10)^{n-1}$. 5 (a) $X = X_1 + \cdots + X_m$ (b) m/n. 6. 1. 7 (a) No (b) No (c) Yes (d) Yes. 8. $(n-4)/1024$. 9 (a) $\mathbb{E}(Z) = 4$, $\mathrm{var}(Z) = 3$ (b) $\mathbb{E}(X) = 40$, $\mathrm{var}(X) = 30$ (c) $\binom{10}{5}(\frac{1}{4})^5(\frac{3}{4})^5$ (d) 4 (e) 8 (f) $\mathbb{E}(A) = 4$, $\mathrm{var}(A) = 3/n$.