ENGINEERING MATHEMATICS - I

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Class Content

- Cauchy’s root test
- Examples

Cauchy's Root test

Let $\sum a_n$ be a series of positive terms.

If $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges

If $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} \sqrt[n]{a_n} = L = 1 \implies$ Cauchy's root test fails.

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Examples on Cauchy's Root Test

Problem 1: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$

Soln: Here $a_n = \left(\frac{1}{(logn)}\right)^n$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{logn} = 0 < 1$$

$\therefore$ by Cauchy's Root test $\sum a_n$ is convergent.

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Problem 1: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$

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Problem 2: Test the convergence of the series $\sum \left(\frac{n+1}{3n}\right)^n$

Soln:

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right) = \frac{1}{3} < 1$$

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Examples for Cauchy's Root Test

Problem 3: Test the convergence of the series $\sum\left(\frac{n^2}{3^n}\right)$

$$\lim_{n\to\infty}(a_n)^{1/n}$$
$$= \lim_{n\to\infty}\frac{\left[n^{1/n}\right]^2}{3} = \frac{1^2}{3} < 1$$

$\therefore$ by Cauchy's Root test $\sum a_n$ is convergent.

REMARK

$$\lim_{n\to\infty}\left\{n^{1/n}\right\} = 1.$$

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Examples on Cauchy's Root Test

Problem 4: Test the convergence of the series $\sum \left(\frac{n+1}{n+2}x\right)^n$

$\lim_{n\to\infty}(a_n)^{1/n}$
$= \lim_{n\to\infty}\left[\frac{n+1}{n+2}^n x^n\right]^{1/n} = x$

$\therefore$ by Cauchy's Root test ,
the convergence or divergence of the series depends on x.
$\sum a_n$ is convergent whenever $x < 1$. $\sum a_n$ is divergent whenever $x > 1$.

The test fails if $x = 1$.

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Examples on Cauchy's Root Test

When $x = 1$,

the series becomes, $\sum \left\{ \frac{n+1}{n+2} \right\}^n$,

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left\{1 - \frac{1}{n+2}\right\}^n$$

$$\lim_{n\to\infty}\left\{\left(1 - \frac{1}{n+2}\right)^{-(n+2)}\right\}^{\left(\frac{-n}{n+2}\right)}$$

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Examples On Cauchy's Root Test

$$\lim_{n\to\infty}\left\{\left(1-\frac{1}{n+2}\right)^{-(n+2)}\right\}^{\left(\dfrac{-n}{n+2}\right)} = \frac{1}{e} \neq 0.$$

By necessary condition for the convergence of a

series of positive terms, the given series is divergent.

Thus, the given series is convergent whenever $x < 1$,

and divergent if $x \geq 1$.

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Problems

1) Test the convergence of the series : $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ or $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

Soln: n^{th} term : $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

Consider $(a_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^{n} = \left[\frac{n+1}{n}\right]^{-n}$

$$= \left\{\left[1+\frac{1}{n}\right]^{n}\right\}^{-1}$$

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left\{\underbrace{\left[1+\frac{1}{n}\right]^{n}}_{e}\right\}^{-1} = e^{-1} = \frac{1}{e} < 1$$

$\sum a_n$ is convergent by Cauchy's root test.

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Problems

2) $\sum n^n \cdot x^n$, $(x > 0)$

Soln: $a_n = n^n \cdot x^n$; $(a_n)^{1/n} = (n^n \cdot x^n)^{1/n} = nx$

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} nx = \infty > 1$, By Cauchy's n^{th} root test,

$\sum a_n$ is ~~convergent~~ divergent

3) $\sum 5^{[-n-(-1)^n]}$

Soln: $a_n = 5^{[-n-(-1)^n]}$, $(a_n)^{1/n} = 5^{-1-\frac{(-1)^n}{n}}$

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} 5^{-1-\frac{(-1)^n}{n}} = 5^{-1} = \frac{1}{5} < 1$, $\rightarrow$ Convergent.

Problems

4) $\sum q^{n^2} \cdot r^n$; $(r > 0)$

Soln : $a_n = q^{n^2} \cdot r^n$; $(a_n)^{1/n} = q^n \cdot r$.

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} q^n \cdot r = \begin{cases} 0 & \text{if } 0 < q < 1 \\ \infty & \text{if } q > 1 \end{cases}$$

$\therefore$ By Cauchy's root test, $\sum a_n$ is convergent if $0 < q < 1$

Divergent if $q > 1$; If $q = 1$, root test fails,

If $q = 1$, $\sum a_n = \sum r^n \longrightarrow$ Geometric Series with common ratio $r > 0$

$\left. 0 < r < 1, \sum a_n \text{ is convergent}, r \geq 1, \sum a_n \text{ is divergent} \right\}$ $q = 1$

Problems

We say : Given series $\sum a_n$ is convergent if

(i) $0 < q < 1$ or

(ii) $q = 1$ and $0 < r < 1$

$\sum a_n$ is divergent if

(i) $q > 1$ or

(ii) $q = 1$ and $r \geq 1$.

THANK YOU

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