ENGINEERING MATHEMATICS - I

APARNA B. S
Department of Science and Humanities

Class Content

- Integral test
- The p-series test
- Examples

Integral Test

TEST (Integral Test)

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a positive decreasing function, and for* $n \in \mathbb{N}$*, put* $a_n = f(n)$*, where* $f(n)$ *is a continuous function with* $f(n) > 0$*. Then*

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ converges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx < \infty$.

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx = \infty$.

Integral Test

TEST (Integral Test)

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a positive decreasing function, and for* $n \in \mathbb{N}$, *put* $a_n = f(n)$, *where* $f(n)$ *is a continuous function with* $f(n) > 0$. *Then*

the series $\sum_{i=1}^{\infty} a_i$ converges **iff the improper integral** $\int_1^{\infty} f(x)dx < \infty$.

the series $\sum_{i=1}^{\infty} a_i$ diverges **iff the improper integral** $\int_1^{\infty} f(x)dx = \infty$.

Integral Test

TEST (Integral Test)

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a positive decreasing function, and for* $n \in \mathbb{N}$, *put* $a_n = f(n)$, *where* $f(n)$ *is a continuous function with* $f(n) > 0$. *Then*

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ converges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx < \infty$.

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx = \infty$.

Integral Test

TEST (Integral Test)

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a positive decreasing function, and for* $n \in \mathbb{N}$, *put* $a_n = f(n)$, *where* $f(n)$ *is a continuous function with* $f(n) > 0$. *Then*

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ converges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx < \infty$.

the series $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges **iff the improper integral** $\displaystyle\int_1^{\infty} f(x)\,dx = \infty$.

Integral Test

Example : Use the Integral Test to determine the set of all possible values of $p > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

Proof: By Integral test, $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)dx < \infty$.

$$a_n = \frac{1}{n^p} = f(n)$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$$

$$= \frac{x^{-p+1}}{-p+1}\Bigg\}_{x=1}^{x=\infty}.$$

Integral Test

Example : Use the Integral Test to determine the set of all possible values of $p > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

Proof: By Integral test, $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)dx < \infty$.

$$a_n = \frac{1}{n^p} = f(n)$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$$

$$= \frac{x^{-p+1}}{-p+1} \Bigg\}_{x=1}^{x=\infty}.$$

Integral Test

Example : Use the Integral Test to determine the set of all possible values of $p > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

Proof: By Integral test, $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)dx < \infty$.

$$a_n = \frac{1}{n^p} = f(n)$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$$

$$= \frac{x^{-p+1}}{-p+1} \Bigg\}_{x=1}^{x=\infty}.$$

Integral Test

Example : Use the Integral Test to determine the set of all possible values of $p > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

Proof: By Integral test, $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)dx < \infty$.

$$a_n = \frac{1}{n^p} = f(n)$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}dx$$

$$= \frac{x^{-p+1}}{-p+1}\Bigg\}_{x=1}^{x=\infty}.$$

Integral Test

Upper limit is zero if $1 - p < 0 \Leftrightarrow p > 1$

Upper limit is ∞ if $1 - p > 0 \Leftrightarrow p < 1$

When $p = 1$, $\int_1^\infty \frac{1}{x} = log(x) \Big\}_{x=1}^{x=\infty} = \infty$

Therefore, $\int_1^\infty \frac{1}{x^p} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

So the p - series is convergent whenever $p > 1$.

When $p < 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p < 1$.

When $p = 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p = 1$.

Integral Test

Upper limit is zero if $1 - p < 0 \Leftrightarrow p > 1$

Upper limit is ∞ if $1 - p > 0 \Leftrightarrow p < 1$

When $p = 1$, $\int_1^\infty \frac{1}{x} = log(x) \Big\}_{x=1}^{x=\infty} = \infty$

Therefore, $\int_1^\infty \frac{1}{x^p} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

So the p - series is convergent whenever $p > 1$.

When $p < 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p < 1$.

When $p = 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p = 1$.

Integral Test

Upper limit is zero if $1 - p < 0 \Leftrightarrow p > 1$

Upper limit is ∞ if $1 - p > 0 \Leftrightarrow p < 1$

When $p = 1$, $\int_1^\infty \frac{1}{x} = log(x) \Big\}_{x=1}^{x=\infty} = \infty$

Therefore, $\int_1^\infty \frac{1}{x^p} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

So the p - series is convergent whenever $p > 1$.

When $p < 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p < 1$.

When $p = 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p = 1$.

Integral Test

Upper limit is zero if $1 - p < 0 \Leftrightarrow p > 1$

Upper limit is ∞ if $1 - p > 0 \Leftrightarrow p < 1$

When $p = 1$, $\int_1^\infty \frac{1}{x} = log(x) \Big\}_{x=1}^{x=\infty} = \infty$

Therefore, $\int_1^\infty \frac{1}{x^p} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

So the p - series is convergent whenever $p > 1$.

When $p < 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p < 1$.

When $p = 1$, $\int_1^\infty \frac{1}{x^p} = \infty$.

So the p - series is divergent whenever $p = 1$.

The p - series

Thus, the Hyper Harmonic series or the p- series:

$$\sum_{n=1}^{n=\infty} \frac{1}{n^p},$$

The series $\displaystyle\sum_{n=1}^{n=\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

1. Converges if $p > 1$
2. Diverges if $p \leq 1$.

The p - series

Thus, the Hyper Harmonic series or the p- series:

$$\sum_{n=1}^{n=\infty} \frac{1}{n^p},$$

The series $\displaystyle\sum_{n=1}^{n=\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

1. Converges if $p > 1$
2. Diverges if $p \leq 1$.

The p - series

Thus, the Hyper Harmonic series or the p- series:

$$\sum_{n=1}^{n=\infty} \frac{1}{n^p},$$

The series $\displaystyle\sum_{n=1}^{n=\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

1. Converges if $p > 1$
2. Diverges if $p \leq 1$.

The p - series

Thus, the Hyper Harmonic series or the p- series:

$$\sum_{n=1}^{n=\infty} \frac{1}{n^p},$$

The series $\displaystyle\sum_{n=1}^{n=\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

1. Converges if $p > 1$
2. Diverges if $p \leq 1$.

Examples

1. Use Integral test to test the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \text{ to } \infty$$

To do so, we determine the convergence or divergence of the integral, $\displaystyle\int_{x=1}^{\infty} \frac{1}{x^2}$.

$$\int_{x=1}^{\infty} \frac{1}{x^2} = 1. \text{ which is convergent.}$$

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

REMARK

The validity of the above result can be verified using the p - series test also.

Examples

1. Use Integral test to test the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \text{ to } \infty$$

To do so, we determine the convergence or divergence of the integral, $\int_{x=1}^{\infty} \frac{1}{x^2}$.

$\int_{x=1}^{\infty} \frac{1}{x^2} = 1.$ which is convergent.

Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

REMARK

The validity of the above result can be verified using the p - series test also.

Examples

1. Use Integral test to test the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \text{ to } \infty$$

To do so, we determine the convergence or divergence of the integral, $\int_{x=1}^{\infty} \frac{1}{x^2}$.

$$\int_{x=1}^{\infty} \frac{1}{x^2} = 1. \text{ which is convergent.}$$

Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

REMARK

The validity of the above result can be verified using the p - series test also.

Examples

1. Use Integral test to test the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \text{ to } \infty$$

To do so, we determine the convergence or divergence of the integral, $\displaystyle\int_{x=1}^{\infty} \frac{1}{x^2}$.

$$\int_{x=1}^{\infty} \frac{1}{x^2} = 1. \text{ which is convergent.}$$

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

REMARK

The validity of the above result can be verified using the p - series test also.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{8tan^{-1}n}{1+n^2}$ converges or diverges.

Solution: Choose $f(x) = \frac{8tan^{-1}x}{1+x^2}$

Clearly, f is continuous, positive and a decreasing function on the interval $[1, \infty)$

Subustituting $y = tan^{-1}x$ we get, $\int_{x=1}^{\infty} \frac{8tan^{-1}x}{1+x^2} = \int_{\pi/4}^{\pi/2} 8ydy$

$= 4\left\{\frac{\pi^2}{4} - \frac{\pi^2}{16}\right\} = \frac{3\pi^2}{4}$. Therefore, the series converges by Integral test.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{8tan^{-1}n}{1+n^2}$ converges or diverges.

Solution: Choose $f(x) = \frac{8tan^{-1}x}{1+x^2}$

Clearly, f is continuous, positive and a decreasing function on the interval $[1, \infty)$

Subustituting $y = tan^{-1}x$ we get, $\int_{x=1}^{\infty} \frac{8tan^{-1}x}{1+x^2} = \int_{\pi/4}^{\pi/2} 8ydy$

$= 4\left\{\frac{\pi^2}{4} - \frac{\pi^2}{16}\right\} = \frac{3\pi^2}{4}$. Therefore, the series converges by Integral test.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{8tan^{-1}n}{1+n^2}$ converges or diverges.

Solution: Choose $f(x) = \frac{8tan^{-1}x}{1+x^2}$

Clearly, f is continuous, positive and a decreasing function on the interval $[1, \infty)$

Subustituting $y = tan^{-1}x$ we get, $\int_{x=1}^{\infty} \frac{8tan^{-1}x}{1+x^2} = \int_{\pi/4}^{\pi/2} 8ydy$

$= 4\left\{\frac{\pi^2}{4} - \frac{\pi^2}{16}\right\} = \frac{3\pi^2}{4}$. Therefore, the series converges by Integral test.

Examples

Determine if the series $\sum_{n=1}^{\infty.} \frac{8tan^{-1}n}{1+n^2}$ converges or diverges.

Solution: Choose $f(x) = \frac{8tan^{-1}x}{1+x^2}$

Clearly, f is continuous, positive and a decreasing function on the interval $[1, \infty)$

Subustituting $y = tan^{-1}x$ we get, $\int_{x=1}^{\infty} \frac{8tan^{-1}x}{1+x^2} = \int_{\pi/4}^{\pi/2} 8ydy$

$= 4\left\{\frac{\pi^2}{4} - \frac{\pi^2}{16}\right\} = \frac{3\pi^2}{4}$. Therefore, the series converges by Integral test.

Examples

- 1. The series $\sum \dfrac{1}{\sqrt{n}}$ is divergent as $n = \dfrac{1}{2} < 1$.
- 2. The series $\sum \dfrac{1}{n\sqrt{n}}$ is convergent as $n = \dfrac{3}{2} > 1$.
- 3. The series $\sum \dfrac{1}{n}$ is divergent as $n = 1$.

Examples

- 1. The series $\sum \frac{1}{\sqrt{n}}$ is divergent as $n = \frac{1}{2} < 1$.
- 2. The series $\sum \frac{1}{n\sqrt{n}}$ is convergent as $n = \frac{3}{2} > 1$.
- 3. The series $\sum \frac{1}{n}$ is divergent as $n = 1$.

Examples

- 1. The series $\sum \frac{1}{\sqrt{n}}$ is divergent as $n = \frac{1}{2} < 1$.
- 2. The series $\sum \frac{1}{n\sqrt{n}}$ is convergent as $n = \frac{3}{2} > 1$.
- 3. The series $\sum \frac{1}{n}$ is divergent as $n = 1$.

Examples

- 1. The series $\sum \frac{1}{\sqrt{n}}$ is divergent as $n = \frac{1}{2} < 1$.
- 2. The series $\sum \frac{1}{n\sqrt{n}}$ is convergent as $n = \frac{3}{2} > 1$.
- 3. The series $\sum \frac{1}{n}$ is divergent as $n = 1$.

Examples

Determine if the series $\sum_{n=1}^{\infty.} \frac{1}{n^{\pi}}$ converges or diverges.

Solution: This is a p - series, with $p = \pi > 1$.
Hence the given series is convergent.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges or diverges.

Solution: This is a p - series, with $p = \pi > 1$.
Hence the given series is convergent.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges or diverges.

Solution: This is a p - series, with $p = \pi > 1$.
Hence the given series is convergent.

Examples

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges or diverges.

Solution: This is a p - series, with $p = \pi > 1$.
Hence the given series is convergent.

Examples

Determine if the series $\displaystyle\sum_{n=1}^{\infty.} \frac{2}{3+5n}$ converges or diverges.

Solution: Here, $a_n = \dfrac{2}{3+5n} > \dfrac{2}{3+5(n+1)}$.

So the terms of the series are decreaisng.

$$\int_{x=1}^{\infty} \frac{2}{3+5x}$$

$$= \lim_{t\to\infty} \int_{x=1}^{t} \frac{2}{3+5x}dx = \lim_{t\to\infty} \frac{2}{5} ln(3+5x)\Big\}_{0}^{t} = \infty$$

So, the given series is divergent.

Examples

Determine if the series $\sum_{n=1}^{\infty.} \frac{2}{3+5n}$ converges or diverges.

Solution: Here, $a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)}$.

So the terms of the series are decreaisng.

$$\int_{x=1}^{\infty} \frac{2}{3+5x}$$

$$= \lim_{t\to\infty} \int_{x=1}^{t} \frac{2}{3+5x} dx = \lim_{t\to\infty} \frac{2}{5} ln(3+5x) \Big\}_{0}^{t} = \infty$$

So, the given series is divergent.

Examples

Determine if the series $\sum_{n=1}^{\infty.} \frac{2}{3+5n}$ converges or diverges.

Solution: Here, $a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)}$.

So the terms of the series are decreaisng.

$$\int_{x=1}^{\infty} \frac{2}{3+5x}$$

$$= \lim_{t\to\infty} \int_{x=1}^{t} \frac{2}{3+5x} dx = \lim_{t\to\infty} \frac{2}{5} ln(3+5x) \Big\}_{0}^{t} = \infty$$

So, the given series is divergent.

Examples

Determine if the series $\sum_{n=1}^{\infty.} \frac{2}{3+5n}$ converges or diverges.

Solution: Here, $a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)}$.

So the terms of the series are decreaisng.

$$\int_{x=1}^{\infty} \frac{2}{3+5x}$$

$$= \lim_{t\to\infty} \int_{x=1}^{t} \frac{2}{3+5x} dx = \lim_{t\to\infty} \frac{2}{5} ln(3+5x) \Big\}_{0}^{t} = \infty$$

So, the given series is divergent.

Problems

1) Test the convergence of the series : $\sum_{n=1}^{\infty} \frac{1}{2n+3}$, using Cauchy's integral test.

Soln : Here n^{th} term, $a_n = \frac{1}{2n+3} = f(n)$

$\Rightarrow f(x) = \frac{1}{2x+3}$. For $x \geq 1$, $f(x) \geq 0$ and $f(x)$ is a monotonically decreasing function. Hence. Cauchy's integral test is applicable.

Let $I_n = \int_1^n f(x)\,dx = \int_1^n \frac{dx}{2x+3} = \frac{1}{2} \log(2x+3)\Big]_1^n$

$= \frac{1}{2}\left[\log(2n+3) - \log 5\right]$. $\therefore \int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n$

Problems

$$= \lim_{n \to \infty} \frac{1}{2}\left[\log(2n+3) - \log 5\right] = \infty$$

$\therefore \int_1^\infty f(x)\,dx$ diverges and, hence, by Integral test,

$\sum_{n=1}^{\infty} a_n$ is a divergent series.

2) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}}$; $a_n = \frac{1}{n\sqrt{n^2-1}} = f(n)$, valid for $n \geq 2$

$\therefore f(x) = \frac{1}{x\sqrt{x^2-1}}$; $x \geq 2$

Problems

For $x \geq 2$, $f(x)$ is +ve and decreasing.

$\therefore$ Cauchy's integral test is applicable.

By Cauchy's integral test,

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx \text{ and } \sum_{n=2}^\infty a_n \text{ converges or}$$

diverges together.

Let $I_n = \int_2^{\cancel{\infty}\, n} f(x)\,dx = \int_2^n \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x)\Big|_2^{\cancel{\infty}\, n} = \sec^{-1} n - \sec^{-1} 2$

$$\therefore \int_2^\infty f(x)\,dx = \lim_{n\to\infty} \int_2^n f(x)\,dx = \lim_{n\to\infty}\left[\sec^{-1}(n) - \sec^{-1}(2)\right] = \sec^{-1}\infty - \sec^{-1}(2)$$

Problems

$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$, finete quantity.

$\therefore$ By integral test; $\sum a_n$ is convergent.

3) $\sum_{n=1}^{\infty} n \cdot e^{-n^2} = \sum f(n)$; $f(x) = x \cdot e^{-x^2}$.

For $x \geq 1$, $f(x)$ is positive and decreasing function.

$\therefore$ By Cauchy's integral test, $\int_1^{\infty} f(x)\,dx$ & $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

Problems

Let $I_n = \int_1^n f(x)\,dx = \int_1^n x e^{-x^2} dx = -\frac{1}{2}\int_1^n e^{-x^2}(-2x)\,dx$

$= -\frac{1}{2}\int_{-1}^{-n^2} e^t\,dt$, Here $t = -x^2$.

$= -\frac{1}{2}\left[e^t\right]_{-1}^{-n^2} = -\frac{1}{2}\left[e^{-n^2} - e^{-1}\right] = \frac{1}{2}\left(\frac{1}{e} - e^{-n^2}\right)$

$\therefore \int_1^\infty f(x)\,dx = \lim_{n\to\infty}\int_1^n f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\frac{1}{2}\left[\frac{1}{e} - e^{-n^2}\right]$

$= \frac{1}{2}\left[\frac{1}{e} - 0\right]$, a finite quantity. $\therefore$ By integral test Σa_n is convergent.

THANK YOU

Aparna B. S
Department of Science and Humanities
aparnabs@pes.edu