ENGINEERING MATHEMATICS - I

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Class Content

- Monotonic sequences
- Examples on Monotonic sequences
- Bounded above sequence
- Bounded below sequence
- Bounded sequence
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Bounded above Sequence

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DEFINITION (Bounded above sequence)

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Bounded Sequence *A sequence is said to be bounded if it is both bounded above and bounded below.*

Examples :

1. The sequence $\{a_n\}$ defined by $a_n = \dfrac{1}{n}$, $\forall n \in \mathbb{N}$, since, the sequence is bounded below by 0 and bounded above by 1 i.e every term of the sequence a_n is such that $0 \leq a_n \leq 1$.

2.The sequence $\{a_n\}$ defined by $a_n = n$, $\forall n \in \mathbb{N}$, is bounded below by 1 and is not bounded above since $\exists$ no K such that every term of the sequence is $\leq K$.

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The above definition asserts that there is neither a $'K'$ *nor a* $'k'$*, such that* $a_n \nleq K$ *or* $k \nleq a_n$*.*

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Examples on Monotonic Sequences

Example: Prove that the following sequence, whose n^{th} term is given, is monotonic.
Find out whether it is decreasing or increasing.
$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n}$
Solution:

$$a_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$$

$$a_{n+1} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$a_{n+1} - a_n = \frac{1}{2^{n+1}} > 0, \ \forall n \in \mathbb{N}.$$

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Subsequence (Statement Only)

(Subsequence). Let $\{a_n\}$ be a given sequence.
If n_k is a strictly increasing sequence of natural numbers i.e $n_1 < n_2 < n_3 \cdots$,
then a_{n_k} is called a subsequence of $\{a_n\}$.
Examples:

- $\{a_{2n}\}$. $\{a_{2n-1}\}$,$\{a_{n^2}\}$ are all subsequences of $\{a_n\}$.
- $\{2, 3, 6, \cdots\}$, $\{1, 3, 5, \cdots\}$, $\{1, 4, 9, 16, \cdots\}$ are all subsequences of the sequence $\{n\}$.
- Every sequence is a subsequence of itself.

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(Cauchy Sequence)
A sequence $\{a_n\}$ is said to be a Cauchy sequence
if given $\epsilon > 0$,
however small, $\exists$ a positive integer M
(depending on the choice of ϵ)
such that $|a_p - a_q| < \epsilon \ \forall \ p, q \geq M$.

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Problems:

Discuss the boundedness of the sequences whose n^{th} term is as given:

1. $a_n = 3$

Every finite sequence is bounded

Range of the sequence $\{a_n\}$ is $\{3\}$

its a finite set. Hence Bounded.

2. $a_n = (-1)^n . 5 = \begin{cases} 5 & ; n \text{ is even} \\ -5 & ; n \text{ is odd} \end{cases}$

Range of sequence $\{a_n\}$ is $\{-5, 5\}$

its a finite set. Hence Bounded.

3. $a_n = n^{th}$ Prime

$a_1 = 2, \ a_2 = 3, \ a_3 = 5, \ \ldots$

Every prime $\geq 2 \ \forall \ n$

$\therefore \ \{a_n\}$ is bounded below by '2'

$\{a_n\}$ is unbounded above

$\{a_n\}$ is not bounded

4. $a_n = n^2$

$\{1^2, 2^2, 3^2, 4^2 \ldots\ldots \infty\} \rightarrow$ Not bounded

$a_n \geq 1 \Rightarrow \{a_n\}$ bounded below by '1'

unbounded above

Prove that sequences whose n^{th} terms are given are monotonic, Find Out whether they are increasing or decreasing?

1. $a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}$ → n^{th} term not the series

Replace n by $n+1$

$$a_{n+1} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1}$$

$$a_{n+1} - a_n = \frac{1}{2n} + \frac{1}{2n+1} + \left(-\frac{1}{n}\right)$$

$$= -\frac{1}{2n(2n+1)} \quad \forall n < 0$$

$\therefore$ $a_{n+1} < a_n$

$\langle a_n \rangle = \{a_n\}$ is monotonic decreasing

Hence Monotonic

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Consider $a_n = \dfrac{3n+7}{4n+8} = \dfrac{3n+7}{4(n+2)}$

$$a_{n+1} = \frac{3n+10}{4(n+3)}$$

$$\therefore \quad a_{n+1} - a_n = \left(\frac{3n+10}{4(n+3)}\right) - \left(\frac{3n+7}{4(n+2)}\right)$$

$$= \frac{-1}{4(n+3)(n+2)} < 0$$

$$a_{n+1} < a_n$$

$\Rightarrow$ Monotonic decreasing

$a_n = \dfrac{n}{n^2+1} \Rightarrow a_{n+1} = \dfrac{n+1}{(n+1)^2+1}$

$$\therefore \quad a_{n+1} - a_n = \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \ \forall\ n$$

$a_{n+1} < a_n \quad \therefore \{a_n\}$ is monotonic decreasing

$$a_n = \frac{n}{n^2+1} > 0 \ \forall\ n$$

$\{a_n\}$ is bounded below by '0'

Sequence which is monotonic and bounded is convergent. $\therefore$ Given sequence is convergent.

THANK YOU

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