ENGINEERING MATHEMATICS - I

Partial Differentiation

Dr. Anitha
Department of Science and Humanities

ENGINEERING MATHEMATICS - I

UNIT 2: Partial Differentiation
Session : 4
Sub Topic: Composite Functions, Total Derivative & Chain Rule

Dr. Anitha
Department of Science and Humanities

TOPICS

- *COMPOSITE FUNCTIONS*
- *TOTAL DERIVATIVE RULE*
- *CHAIN RULE*

Total derivative rule, Chain Rule and Composite functions

If $y = e^{\sin(x^2)}$ *then,* we can find the derivative of the function by using chain rule.

$$\frac{dy}{dx} = d\left(e^{sinx^2}\right).\, d(\sin(x^2))\, .\, d(x^2)$$

$$\frac{dy}{dx} = e^{sinx^2} \cos(x^2)\ \ 2x$$

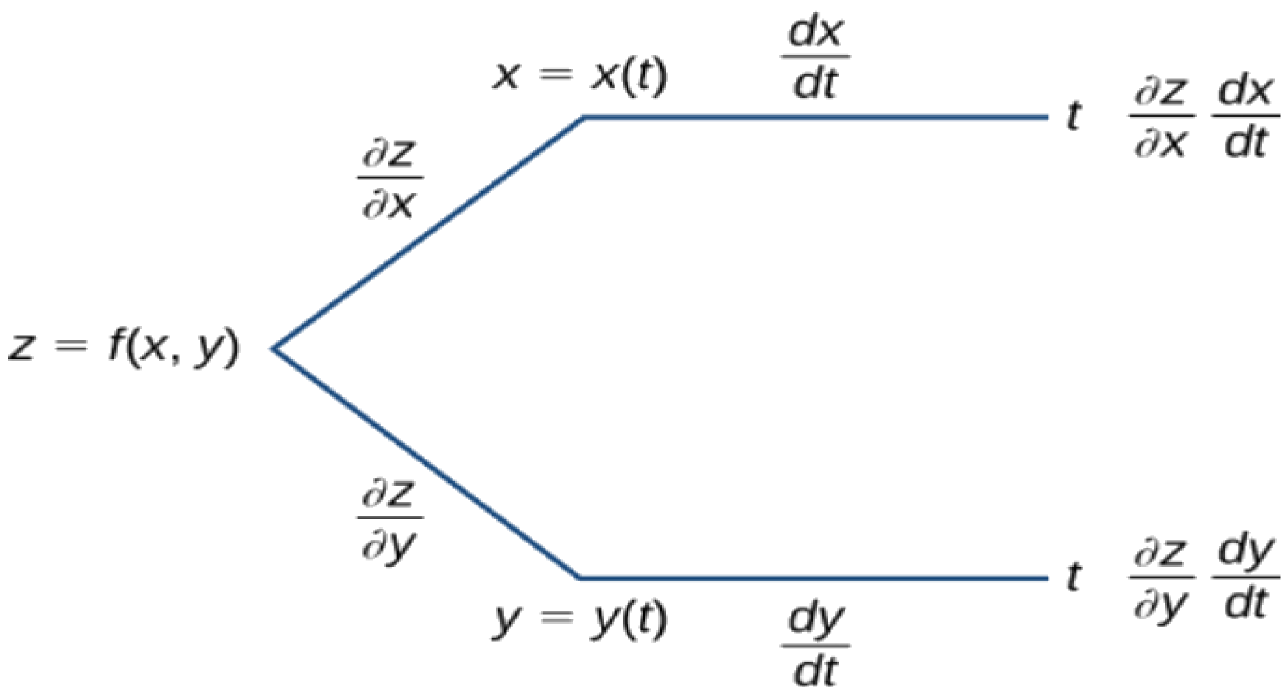
PES UNIVERSITY ONLINE

Total derivative rule:

If $z = f(x, y)$ where x and y are functions of one independent variable 't' then we can find the derivative of z with respect t as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Tree Diagram -Total derivative rule



Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

Total Derivative Rule

Example:

$z = x^2y^3, x = t^2, y = 2t^3$

Differentiating z partially with respect to x we get

$$\frac{\partial z}{\partial x} = 2xy^3 = 2t^{11}$$

And the partial derivative of z with respect to y will be

$$\frac{\partial z}{\partial y} = 3x^2y^2 = 9t^{10}$$

Also,

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 6t^2$$

Substituting, we get $\frac{dz}{dt} = 60t^{12}$

Problems on Total derivatives

1. A point P is moving along the curve of intersection of surface whose cartesian equation is $\frac{x^2}{16} - \frac{y^2}{9} = z$ (A paraboloid) and the surface whose cartesian equation is $x^2 + y^2 = 5$ (A Cylinder). If x is increasing at the rate of 0.2cms/sec, how fast is z changing when $x = 2$?

Solution:

Given $\frac{dx}{dt} = 0.2$ cm\sec

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$x^2 + y^2 = 5, \qquad \frac{dy}{dx} = -\frac{x}{y}$

Problems on Total derivatives

Contd….

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -0.4\, cms \backslash \sec$$

$$\frac{dz}{dt} = 0.139\, cms/sec$$

Problems on Total derivatives

2. Given that $f(2,-3) = 6, f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1,-3.03)$.

Solution:

The total differential approximates how much f changes from the point $(2,-3)$ to $(2.1,-3.03)$. With $dx = 0.1$ and $dy = 0.03$, we have

$$\begin{aligned} dz &= f_x(2,-3)dx + f_y(2,-3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148 \end{aligned}$$

The change in z is approximately 0.148, so we approximate

$$f(2.1,-3.03) \approx 6.148$$

PES UNIVERSITY ONLINE

3. One side of a triangle is increasing at a rate of 3cm/s and second side is decreasing at a rate of 2cm/s. If the area of a triangle remains constant, at what rate does the angle between the sides change when the first side is 20cms long, the second side is 30cms and the angle is $\frac{\pi}{6}$.

Solution:

A$= \frac{1}{2}$(base *height)

$= \frac{1}{2} xysin\theta$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$$

Contd...

$$\frac{dA}{dt} = \frac{1}{2}\left(ysin\theta(3) + xsin\theta(-2) + xycos\theta\,\frac{d\theta}{dt}\right)$$

Here $x = 20cm\,; y = 30cm; \theta = \frac{\pi}{6}$

$$\frac{d\theta}{dt} = \frac{-25}{300\sqrt{3}} = 0.04831$$

θ should decrease at the rate of 0.0481 rad/sec

4. A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution:

A cylindrical solid with height *h* and radius *r* has volume

$V = \pi r^2 h.$

$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi rh$ and $\frac{\partial V}{\partial h} = V_h(r,h) = \pi r^2$

$dV = (2\pi rh)\, dr + (\pi r^2) dh.$

When $h = 10$ and $r = 2$, we have

$dV = 40\pi dr + 4\pi dh.$

Contd….

Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

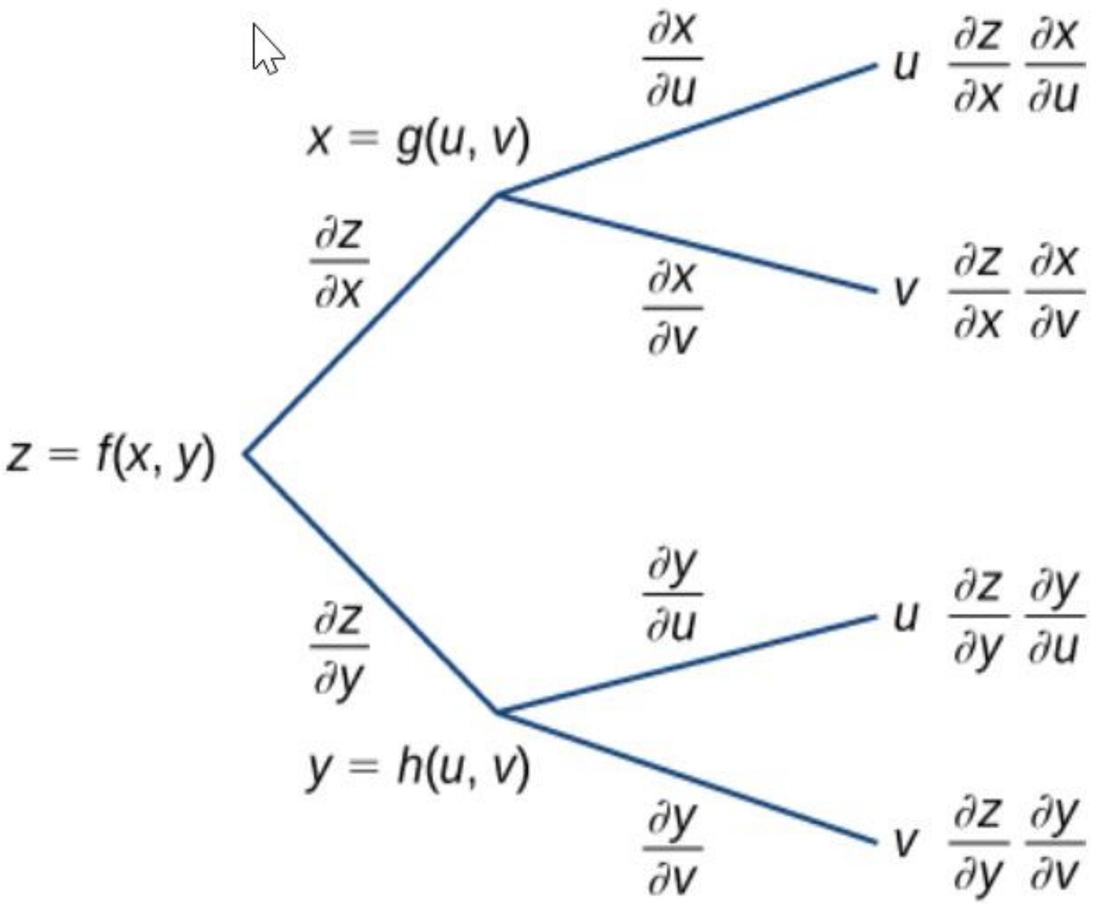
We can define chain rule for a function of two or more independent variable:

If $z = f(x, y)$ is a bivariate function and x, y are functions of two or more independent variables say u, v then we can find the partial derivative of z with respect to u and v, given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Tree diagram for Chain Rule



Tree diagram for $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial u}$ and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial v}$.

1. If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6v_x + 4v_y + 3v_z$=0.

Solution:

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}$$

$Let\ r = 2x - 3y, s = 3y - 4z\ and\ t = 4z - 2x$

$r_x = 2, s_x = 0, t_x = -2$

Similarly

$r_y = -3, s_y = 3, t_y = 0.$

Also

$r_z = 0, s_z = -4, t_z = 4$

$$\frac{\partial v}{\partial x} = 2\frac{\partial v}{\partial r} - 2\frac{\partial v}{\partial t}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}$$

$$= -3\frac{\partial v}{\partial r} + \frac{\partial v}{\partial s}$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial z}$$

$$= -4\frac{\partial v}{\partial s} + 4\frac{\partial v}{\partial t}$$

Therefore $6v_x + 4v_y + 3v_z = 0.$

2. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where $z = f(r, \theta)$ and $r = ts - t^2$, $\theta = \sqrt{s^2 + t^2}$

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$r_s = t, \theta_s = \frac{s}{\sqrt{s^2 + t^2}}$$

$$\frac{\partial z}{\partial s} = f_r(r, \theta)t + f_\theta(r, \theta)\frac{s}{\sqrt{s^2 + t^2}}$$

Similarly

$$r_t = s - 2t, \theta_t = \frac{t}{\sqrt{s^2+t^2}}$$

$$\frac{\partial z}{\partial t} = f_r(r, \theta)(s - 2t) + f_\theta(r, \theta)\frac{t}{\sqrt{s^2 + t^2}}$$

Dr. Anitha

Department of Science and Humanities

nanitha@pes.edu

Extn 730