ENGINEERING MATHEMATICS - I

APARNA B. S
Department of Science and Humanities

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SEQUENCES AND SERIES

Aparna B. S
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Class Content

- Basic concepts and definitions
- Sequences of real numbers
- Limit of a sequence
- Convergence and Divergence of a sequence
- Examples using the definition of Limit

Basic Concepts and Definitions

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5. Helps in deciding the convergence or divergence of a series.

Sequence of real numbers

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** The sequence of real numbers $\{a_n\}$ need not begin with the index $n = 1$.

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Examples :

- $$\left\{\frac{1}{n}\right\}_{n=1}^{n=\infty} = \left\{ \underbrace{1}_{\text{when n=1}}, \underbrace{\frac{1}{2}}_{\text{when n=2}}, \underbrace{\frac{1}{3}}_{\text{when n=3}}, \underbrace{\frac{1}{4}}_{\text{when n=4}}, \cdots \right\}.$$

- $$\left\{\frac{n^2+1}{2n+3}\right\}_{n=0}^{n=\infty} = \left\{ \underbrace{\frac{1}{3}}_{\text{when n=0}}, \underbrace{\frac{2}{5}}_{\text{when n=1}}, \underbrace{\frac{5}{7}}_{\text{when n=2}}, \underbrace{\frac{10}{9}}_{\text{when n=3}}, \cdots \right\}.$$

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3. Can we say that the terms of a sequence
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Is the sequence increasing / decreasing ?

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Alternatively,

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DEFINITION (Sequence)

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n^{th} term of a sequence is specified

$\Rightarrow$ Any number of terms of the sequence can be generated

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DEFINITION (**Range of a Sequence**)

The set of all distinct terms of a sequence is called its range.

For example:

1. If $x_n = (-1)^n, n \in \mathbb{N}$, then $x_n = \{-1, 1, -1, 1, -1, \cdots\}$
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Limit of a Real Sequence

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A sequence $\{a_n\}$ *is said to converge to a limit* l *if for every* $\epsilon > 0$ *there exists a natural number* N *such that* $|\, a_n - l \,| < \epsilon$ *for all* $n > N$.

* We then write $\lim\limits_{n\to\infty} a_n = l$ or simply $\{a_n\} \to l$ as $n \to \infty$.

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REMARKS:

1. Asserts that given any $\epsilon > 0$, however small,
all the terms of the se~~e~~quence, except the first $N - 1$ terms, lie in the interval $(l - \epsilon, l + \epsilon)$.

2. The first $N - 1$ terms of the sequence,
may be scattered anywhere.

3. The choice of $\epsilon > 0$ decides the number of terms
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REMARK : Every convergent sequence has a unique limit.

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DEFINITION (Divergence of a sequence)

If $\lim\limits_{n\to\infty} a_n = +\infty$ *or* $-\infty$, *we say that the sequence* $\{a_n\}$ *diverges.*

Examples of Divergent sequences:

- The sequence $\{n\}$ diverges to ∞.
  For $\lim\limits_{n\to\infty} n = +\infty$
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- The sequence $\left\{\frac{1}{n}\right\}$ is a convergent sequence.
  For, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, a finite quantity.
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Problems :

1) $\{1, 2, 3, 4 \cdots\cdots \infty\} = \{n\}$

n^{th} term : 'n'

2) $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\cdots \infty\} = \{\frac{1}{n}\}$

n^{th} term : '$\frac{1}{n}$'

3) $\{2n-1\}$ sequence ...? $n = 1, 2 \cdots \infty$

$\{1, 3, 5, 7, \cdots\cdots \infty\}$

4) $\{\frac{n+1}{n}\} \rightarrow \{a_n\}$ $a_1 = \frac{1+1}{1} = 2$, $a_2 = \frac{2+1}{2} = \frac{3}{2}$, $a_3 = \frac{4}{3}$

$\hookrightarrow$ n^{th} term a_n or v_n or U_n

$\hookrightarrow$ $\{2, \frac{3}{2}, \frac{4}{3}, \cdots\cdots \infty\}$

5) $\left\{ \frac{-1}{1}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots \infty \right\}$

n^{th} term $= \left\{ (-1)^n \cdot \frac{1}{n} \right\} \rightarrow$ sequence

6) $\{2, 2, 2 \dots \infty\}$ constant sequence

n^{th} term $= 2$

Sequence $\{a_n\} = \{2\}$

sequence is convergent, divergent or oscillatory....

$\left\{ \frac{1}{n} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3} \dots \infty \right\} = \{a_n\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

sequence $\{a_n\}$ converges to zero

2) $\left\{\frac{n+1}{n}\right\}$ ... sequence ..? convergent or divergent

n^{th} term $\{a_n\} \rightarrow a_n$ is the n^{th} term

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

$\Rightarrow$ Given sequence converges to '1'

3) $\left\{\dfrac{3n-4}{7n+3}\right\}$; $a_n = \dfrac{3n-4}{7n+3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-4}{7n+3} = \cancel{\frac{-4}{3}}\ \frac{3}{7}$$

$\Rightarrow$ Given sequence $\{a_n\}$ converges to $\cancel{-4/3}\ 3/7$

4) $\{n\}$ $\quad a_n = n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

$\Rightarrow$ Given sequence diverges to $+\infty$

5) $\left\{\dfrac{-n}{2}\right\}$; $a_n = \dfrac{-n}{2} \to \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{-n}{2} = -\infty$

$\Rightarrow$ Given sequence diverges to $-\infty$

THANK YOU

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