ENGINEERING MATHEMATICS - I

Partial Differentiation

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UNIT 2 : Partial Differentiation
Session : 1
Sub Topic : Continuity of a Function of Two Variables, Definition of Partial differentiation, Second and Mixed Order Partial Derivaives

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Continuity of a Function of Two Variables

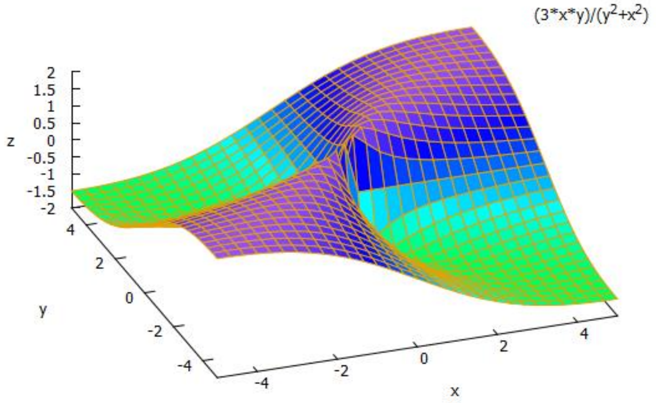
- Remember with functions of one variable, if the function was continuous then $\lim_{x \to c} f(x) = f(c)$
- The same is true for functions of two variables. If the function is continuous then $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$
- A function is continuous at a point if the graph doesn't have any holes or breaks at that point. Graphing the function is helpful.
- If a function is not continuous at (a,b), this does not mean the limit does not exist. We will need to consider the approaching (a,b) from different paths.

Problems on Continuity of a function of two Variables

1. Consider the function

$$f(x,y) = \frac{3x^2y}{x^2+y^2}$$

The function is not continuous at (0,0) as f(0,0) is not defined. However we can easily verify that limit (x,y) approching to (0,0) exists.

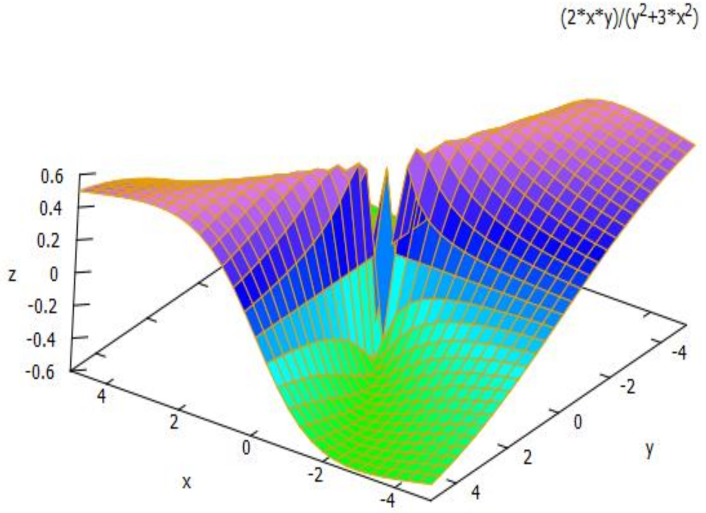


Problems on Continuity of a function of two Variables

2. Consider the function

$$f(x,y) = \frac{2xy}{3x^2 + y^2}$$

The function is not continuous at the origin as the limit of the function as (x,y) approaches to (0,0) does not exists.



History of Partial Differentiation

- Partial differentiation occurs even in ordinary processes of the calculus. The simplest example of partial differentiation is used by Leibnitz for the first time while defining product rule of differentiation.

- The Modern Partial derivative symbol was created by Adrien-Marie Legendre (1786) but later abandoned it.
- Carl Gustav Jacob Jacobi re introduced the symbol in 1841.

Definition of Partial Differentiation

- A partial derivative of a function of several variables is the derivative with respect to one of those variables, with the others held constant.

- Let $z = f(x, y)$ be a function of two variables $x \& y$ then the partial derivative of z with respect to x treating y as constant is denoted by $\frac{\partial z}{\partial x}$ $or\ z_x\ or$ $\frac{\partial f}{\partial x}$ $or\ f_x$ and with respect to y by treating x as constant is denoted by $\frac{\partial z}{\partial y}$ $or\ z_y\ or$ $\frac{\partial f}{\partial y}$ $or\ f_y$ and is defined as

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Second Order Partial Derivatives

Second order Partial derivatives of $z = f(x, y)$

With respect to x: $z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

With respect to y: $z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Mixed Partial derivatives of $z = f(x, y)$

$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$$

$$z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}$$

Mixed Second Order Partial Derivatives

Note 1: The crossed or mixed partial derivatives are in general equal.(Clairaut's theorem)

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

That is, the order of differentiation is immaterial if the second order derivatives involved are continuous.

Note 2: In the subscript notation , the subscripts are written in same order in which the differentiation is carried out, while in the ∂ notation the order is opposite. For example,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}$$

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