ENGINEERING MATHEMATICS - I

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Class Content

- D'Alembert's Ratio test
- Examples

Ratio Test

Ratio Test:

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges.

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L = 1, \implies$ Ratio test fails

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If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L = 1, \implies$ Ratio test fails

Example for Ratio Test

Problem 1: Test the convergence of the series $\frac{1}{3} + \frac{8}{9} + \frac{27}{27} + \frac{64}{81} + \frac{125}{243} + \ldots$

Soln: The given series is

$$\frac{1^3}{3^1} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \ldots$$

$$\text{Here } a_n = \frac{n^3}{3^n}, \quad \therefore a_{n+1} = \frac{(n+1)^3}{3^{n+1}}$$

$$\frac{a_n}{a_{n+1}} = \frac{n^3}{(n+1)^3} . \frac{3^{n+1}}{3^n} = 3 . \frac{n^3}{n^3(1+\frac{1}{n})^3}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^3}{3} = \frac{1}{3} < 1$$

$\therefore$ by D'Alembert's Ratio test $\sum a_n$ is convergent.

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Example for Ratio Test

Problem 2: Test the convergence of the series $\frac{1}{2} + \frac{2!}{8} + \frac{3!}{32} + \frac{4!}{128} + \ldots$

Soln: The given series is

$$\frac{1}{2^1} + \frac{2!}{2^3} + \frac{3!}{2^5} + \frac{4!}{2^7} + \ldots$$

$$\text{Here } a_n = \frac{n!}{2^{2n-1}}, \quad \therefore a_{n+1} = \frac{(n+1)!}{2^{2n+1}}$$

$$\frac{a_n}{a_{n+1}} = \frac{n!}{(n+1)!} \cdot \frac{2^{2n+1}}{2^{2n-1}} = \frac{4}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{4} = \infty > 1$$

$\therefore$ by D'Alembert's Ratio test $\sum a_n$ is divergent.

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$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n+1}{4}=\infty>1$$

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$\therefore$ by D'Alembert's Ratio test $\sum a_n$ is divergent.

Problems

1) Test the convergence of the following series:

$$\sum \frac{n^3+a}{2^n+a}.$$

Solution: Here $a_n = \dfrac{n^3+a}{2^n+a}$, $a_{n+1} = \dfrac{(n+1)^3+a}{2^{n+1}+a}$

$$\frac{a_n}{a_{n+1}} = \frac{n^3+a}{2^n+a} \cdot \frac{2^{n+1}+a}{(n+1)^3+a} = \frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}} \cdot \frac{2\left(1+\frac{a}{2^{n+1}}\right)}{1+\frac{a}{2^n}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1+0}{1+0} \cdot 2\left(\frac{1+0}{1+0}\right) = 2 > 1$$

$\therefore$ By Ratio test, $\sum v_n$ is convergent.

Problems

2) $\sum \frac{x^n}{3^n \cdot n^2}$, $x > 0$

Soln : $a_n = \frac{x^n}{3^n \cdot n^2}$, $a_{n+1} = \frac{x^{n+1}}{3^{n+1} \cdot (n+1)^2}$

$$\frac{a_n}{a_{n+1}} = \frac{x^n}{3^n \cdot n^2} \cdot \frac{3^{n+1} \cdot (n+1)^2}{x^{n+1}} = \frac{3}{x} \cdot \left(1+\frac{1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{3}{x}\left(1+\frac{1}{n}\right)^2 = \frac{3}{x}$$

This series is convergent if $\frac{3}{x} > 1$, divergent if $\frac{3}{x} < 1$

When $x = 3$, $\frac{3}{x} = 1$, the test fails.

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Problems

When $x = 3$, $a_n = \dfrac{3^n}{3^n \cdot n^2} = \dfrac{1}{n^2}$

$\sum a_n = \sum \frac{1}{n^2}$, p-series with $p = 2 > 1$

$\sum a_n$ is convergent.

Given series is convergent for $x \leq 3$

divergent for $x > 3$.

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Problems

3) Examine the convergence / divergence of the series :

Ignore.

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots + \frac{2^{n+1}-2}{2^{n+1}+1}x^n + \cdots \infty$$

Soln : $a_n = \dfrac{2^{n+1}-2}{2^{n+1}+1}x^n$, $a_{n+1} = \dfrac{2^{n+2}-2}{2^{n+2}+1} \cdot x^{n+1}$

$$\frac{a_n}{a_{n+1}} = \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^n}} \cdot \frac{1+\frac{1}{2^{n+1}}}{1-\frac{1}{2^{n+1}}} \cdot \frac{1}{x}, \quad \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}$$

nature of the series depends on $\frac{1}{x}$

By Ratio test,

$\sum a_n$ converges for $\frac{1}{x} > 1$, $\sum a_n$ diverges for $\frac{1}{x} < 1$

Test fails for $\frac{1}{x} = 1 \Rightarrow x = 1$.

Problems

When $x = 1$, $a_n = \dfrac{2^{n+1} - 2}{2^{n+1} + 2} = \dfrac{2^{n+1}\left[1 - \frac{2}{2^{n+1}}\right]}{2^{n+1}\left[1 + \frac{2}{2^{n+1}}\right]} = \dfrac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^n}}$

$\therefore \lim_{n \to \infty} a_n = 1 \neq 0 \Rightarrow \sum a_n$ does not converge.

Being a series of +ve terms, $\sum a_n$ must diverge.

$\sum a_n$ converges for $x < 1$

diverges for $x \geq 1$

Problems

Problems

THANK YOU

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