ENGINEERING MATHEMATICS - I

APARNA B. S
Department of Science and Humanities

Class Content

- Raabe's test
- Examples

Raabe's Test

Let $\sum a_n$ be a series of positive terms. Then,

If $\lim\limits_{n\to\infty} n\left(\dfrac{a_n}{a_{n+1}} - 1\right) = l$

$\sum\limits_{n=1}^{\infty} a_n$ is convergent if $l > 1$

$\sum\limits_{n=1}^{\infty} a_n$ is divergent if $l < 1$

Raabe's test fails whenever $l = 1$.

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Example for Raabe's Test

Problem 1: Discuss the convergence of the series $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\dots$

Soln: Here,

$$a_n=\frac{1.3.5.\dots(2n-1)}{2.4.6.\dots 2n},\qquad a_{n+1}=\frac{1.3.5.\dots(2n-1)(2n+1)}{2.4.6.\dots(2n)(2n+2)}$$

$$\frac{a_n}{a_{n+1}}=\frac{1.3.5.\dots(2n-1)}{2.4.6.\dots 2n}.\frac{2.4.6.\dots(2n)(2n+2)}{1.3.5.\dots(2n-1)(2n+1)}$$

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Example for Raabe's Test

$\frac{a_n}{a_{n+1}} = \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \to 1$ as $n \to \infty$.

$\therefore$ D'Alembert's Ratio test fails

Now, $n\left[\frac{a_n}{a_{n+1}} - 1\right] = n\left[\frac{2n+2}{2n+1} - 1\right] = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$

$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \frac{1}{2} < 1,$

$\therefore$ by Raabe's test, $\sum a_n$ diverges.

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Example for Raabe's Test

Problem 2: Discuss the convergence of the series

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots\ldots(x>0)$$

Soln: Neglecting the first term, we have,

$$a_n=\frac{1.3.5....(2n-1)}{2.4.6....(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$

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$$\frac{a_n}{a_{n+1}}=\frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2}$$

$$\frac{a_n}{a_{n+1}}=\frac{2n\left(1+\frac{1}{n}\right).2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right).2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

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$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2} . \frac{1}{x^2} = \frac{1}{x^2}$ $\therefore$ by Ratio test, $\sum a_n$ is convergent if

$\frac{1}{x^2} > 1$. i.e. $x^2 < 1$ and
divergent if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$.

If $x^2 = 1$, then Ratio test fails.

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Example for Raabe's Test

When $x^2 = 1$, we have

$$\frac{a_n}{a_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$\frac{a_n}{a_{n+1}} = \frac{4n^2+10n+6}{4n^2+4n+1}.$$

In the limit as $n \to \infty$, $\frac{a_n}{a_{n+1}} \to 1$. Hence Ratio test fails.

We proceed to apply Raabe's test.

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$$\left\{ \lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{4n^2+10n+6}{4n^2+4n+1} - 1\right] \right\}$$

$$\left\{ \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty} \frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1. \right\}$$

$\therefore$ by Raabe's test the series converges.

Hence $\sum a_n$ is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

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Hence $\sum a_n$ is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Problems

1) Test the convergence of the series:

$$1+\frac{3}{7}x+\frac{3}{7}\cdot\frac{6}{10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\cdots\infty.$$

Soln: Neglect 1^{st} term,

n^{th} term : $a_n = \dfrac{3\cdot 6\cdot 9\cdots(3n)}{7\cdot 10\cdot 13\cdots(3n+4)}x^n$

$n \to n+1$; $a_{n+1} = \dfrac{3\cdot 6\cdot 9\cdots(3n)(3n+3)}{7\cdot 10\cdot 13\cdots(3n+4)(3n+7)}x^{n+1}$

$$\frac{a_n}{a_{n+1}} = \frac{3n+7}{3n+3}\cdot\frac{1}{x} = \frac{1+\frac{7}{3n}}{1+\frac{1}{n}}\cdot\frac{1}{x} \longrightarrow \frac{1}{x} \text{ as } n\to\infty$$

By ratio test, $\sum a_n$ converges for $\frac{1}{x}>1$, diverges if $\frac{1}{x}<1$

Problems

and, the ratio test fails for $x = 1$.

When $x = 1$, $\dfrac{a_n}{a_{n+1}} = \dfrac{3n+7}{3n+3}$

Consider

$$n\left[\frac{a_n}{a_{n+1}} - 1\right] = n\left[\frac{3n+7}{3n+3} - 1\right] = n\left[\frac{4}{3n+3}\right] = \frac{4n}{3n+3}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \lim_{n\to\infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

By Raabe's test, $\sum a_n$ converges.

Problems

2) Test the convergence of the series:

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \cdots \infty.$$

Soln: Neglect 1st term,

n^{th} term: $a_n = \dfrac{2^2\cdot 4^2\cdot 6^2\cdots\cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots(2n+1)(2n+2)}\, x^{2n+2}$

$$a_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2(2n+2)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots(2n+1)(2n+2)(2n+2)(2n+4)}\, x^{2n+4}$$

$$\frac{a_n}{a_{n+1}} = \frac{4 + \frac{14}{n} + \frac{12}{n^2}}{4 + \frac{8}{n} + \frac{4}{n^2}}\cdot\frac{1}{x^2} \longrightarrow \frac{1}{x^2} \text{ as } n\to\infty$$

Problems

So the convergence/divergence of the series depends on values of $\frac{1}{x^2}$.

$\sum a_n$ converges if $x^2 < 1$, diverges if $x^2 > 1$.

When $x = 1$, ratio test fails, we use Raabe's test.

So when $x = 1$,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{6n^2 + 8n}{4n^2 + 8n + 4} \longrightarrow \frac{6}{4} > 1 \quad \text{as } n \to \infty$$

$\therefore$ By Raabe's test $\sum a_n$ converges for $x = 1$

$\sum a_n$ is convergent for $x^2 \leq 1$, diverges for $x^2 > 1$.

THANK YOU

Aparna B. S
Department of Science and Humanities
aparnabs@pes.edu