Statistical Simulation

Homework of Gibbs Sampling and Bayesian Analysis

Due Date: 00:00, December 14, 2019

1. Let data $\boldsymbol{y}$ from the normal distribution $(\mu,\, \sigma^2)$. The joint prior distribution of the parameters of $(\mu,\, \sigma^2)$ is

$$Y_i \sim^{iid} N(\mu, \sigma^2) \text{ and } \boldsymbol{y} = (y_1, y_2, \ldots, y_n),$$
$$\text{Prior: } \pi(\mu, \sigma^2) \propto \exp\left\{-\frac{(\mu-\theta_0)^2}{2\tau_0^2}\right\} \times \frac{1}{\sigma^2}\mathbf{1}_{\{\sigma^2>0\}}(\sigma^2),$$

where $\theta_0 = 11$ and $\tau_0^2 = 8$. The collected data $\boldsymbol{y}$ is as follows:

```
data.y <- c(14.52, 8.49, 11.86, 6.42, 8.41, 7.66, 10.4, 9.99, 16.49,
            6.55, 16.54, 15.53, 5.66, 14.67, 9.06, 13.69, 8.49, 12.72,
            7.86, 13.03, 13.06, 5.67, 8.18, 18.74, 7.63, 14.76, 18.28,
            15.82, 12.67, 11.72,  16.13, 11.5, 11.88, 9.3, 12.67, 10.61,
            12.35, 8.41, 11.17, 14.91, 5.58, 7.74, 12.78, 11.32, 11.12,
            12.01, 13.75, 11.36, 11.63, 10.22)
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   (a) What is the form of the joint posterior distribution of (μ, σ^2)?

   (b) What is the full conditional distribution for each of μ and σ^2?

   (c) Use (b) to develop an algorithm for drawing samples from the joint posterior distribution (a)?

   (d) Draw the samples and plot them on the figure with the contour plot.

   (e) What is the posterior mean for μ and σ^2?

2. Suppose the joint density of X, Y, Z is given by

$$f(x, y, z) = C\exp^{-(x+y+z+axy+bxz+cyz)},\ \ x > 0,\ \ y > 0,\ \ z > 0,$$

where a, b, c are specified non-negative constants, and C does not depend on x, y, z. Use the Gibbs sampling and prepare the steps to draw the samples, and run the simulation to estimate $E(XYZ)$ when $a = 2$, $b = 3$, and $c = 6$.

3. Let X and Y be random variables with the following joint pdf:

$$f(x, y) = \frac{n!}{x!(n-x)!}y^{(x+\alpha)}(1-y)^{(n-x+\beta)},\ x = 0, 1, \ldots, n,\ \text{and } 0 \le y \le 1.$$

Develop an algorithm to draw the samples from the joint pdf by setting $(n, \alpha, \beta) = (10, 3, 5)$.

4. Draw 2-dimensional random variables from the pdf:

$$f(\mathbf{x}) \propto \frac{\omega_1}{\sqrt{|\Sigma_1|}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu}_1)'\Sigma_1^{-1}(\mathbf{x}-\boldsymbol{\mu}_1)/2\right\} + \frac{\omega_2}{\sqrt{|\Sigma_2|}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu}_2)'\Sigma_2^{-1}(\mathbf{x}-\boldsymbol{\mu}_2)/2\right\},$$

where $\omega_1+\omega_2 = 1$, $\mathbf{x}' = [x_1, x_2]$, and $|\Sigma|$ is the determinant of Σ. It is a mixture bivariate normal distribution. Let $\omega_1 = 0.7$, $\omega_2 = 0.3$, $\boldsymbol{\mu}_1' = [-5, -7]$, $\boldsymbol{\mu}_2' = [5, 7]$, $\Sigma_1 = \begin{bmatrix} 1 & -0.7 \\ -0.7 & 1 \end{bmatrix}$, $\Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(a) Write a code to draw samples from the distribution.

(b) Give a plot of the samples with it contour plot.