Statistical Simulation

Homework of MCMC

Due Date: 00:00, December 26, 2019

1. Let data $\boldsymbol{y}$ from the normal distribution (μ, σ^2).

$$Y_i \sim^{iid} N(\mu, \sigma^2) \text{ and } \boldsymbol{y} = (y_1, y_2, \ldots, y_n),$$

The joint prior distribution of the parameters of (μ, σ^2) is

$$\text{Prior: } \pi(\mu, \sigma^2) \propto \frac{e^{-(\mu-\theta_0)/s_0}}{s_0(1+e^{-(\mu-\theta_0)/s_0})^2} \times \frac{1}{\sigma^2}\mathbf{1}_{\{\sigma^2>0\}}(\sigma^2),$$

where $\theta_0 = 11$ and $s_0 = 2.5$. That is, the prior distribution of μ is a logistic distribution. The collected data $\boldsymbol{y}$ is as follows:

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data.y <- c(14.52, 8.49, 11.86, 6.42, 8.41, 7.66, 10.4, 9.99, 16.49,
            6.55, 16.54, 15.53, 5.66, 14.67, 9.06, 13.69, 8.49, 12.72,
            7.86, 13.03, 13.06, 5.67, 8.18, 18.74, 7.63, 14.76, 18.28,
            15.82, 12.67, 11.72,  16.13, 11.5, 11.88, 9.3, 12.67, 10.61,
            12.35, 8.41, 11.17, 14.91, 5.58, 7.74, 12.78, 11.32, 11.12,
            12.01, 13.75, 11.36, 11.63, 10.22)
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(a) What is the form of the joint posterior distribution of (μ, σ^2)?

(b) What is the full conditional distribution for each of μ and σ^2?

(c) Use (b) to develop the Gibbs algorithm for drawing samples from the joint posterior distribution (a)? Please specify the steps in the Gibbs sampling. (Just show the steps, no need to provide codes)

(d) Is there any step in (c) need the Metropolis(-Hastings) algorithm? If yes, develop the Metropolis-Hastings algorithm for each needed step in (c). (Just show the steps, no need to provide codes)

(e) Combine (c) and (d) to draw the samples from the joint distribution. It is necessary to check the average rate of acceptance and the autocorrelation function (ACF) plots to check if the proposal distribution is fine. (Need a combined codes of (c) and (d))

(f) Draw the samples and plot them on the figure with the contour plot.

(g) What is the posterior mean for μ and σ^2?

2. Draw samples from the Student's t-distribution with degrees of freedom ν. The probability density function (pdf) is

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty.$$

   (a) What is proposal distribution you choose?

   (b) Write the MCMC code in R for sampling from this distribution. Given $\nu = 1, 2, 3, 5, 10, 20, 30, 35$, show ACF plots to determine the lag you pick the samples for each degree of freedom.

   (c) Based on the lag you choose, draw the histograms of samples with the true pdf (red line) and the pdf the standard normal distribution (blue line) for $\nu = 1, 2, 3, 5, 10, 20, 30, 35$.

   (d) According to (c), what is the value of ν that makes the t-distribution is approaching to the standard normal distribution?

3. A random variable X is from the inverse Gaussian distribution and its pdf is

$$f(x) \propto x^{-3/2}\exp\left\{-\theta_1 x - \frac{\theta_2}{x} + 2\sqrt{\theta_1\theta_2} + \log\sqrt{2\theta_2}\right\}, \quad x > 0.$$

   It can be shown that

$$E(X) = \sqrt{\frac{\theta_2}{\theta_1}} \quad \text{and} \quad E\left(\frac{1}{X}\right) = \sqrt{\frac{\theta_1}{\theta_2}} + \frac{1}{2\theta_2}.$$

   Let $\theta_1 = 2$ and $\theta_2 = 3$.

   (a) A Gamma distribution is used to be the proposal distribution. Draw the samples by the Metropolis-Hastings algorithm.

   (b) What are the parameters used in the proposal distribution? What is the average rate of acceptance? Show the ACF of the draws.

   (c) Use the simulation way to estimate $E(X)$ and $E\left(\frac{1}{X}\right)$. Compare your answers to the analytical values.

4. Draw 2-dimensional random variables from the pdf:

$$f(x,y) = \frac{2}{(1-e^{-x-y})^3} e^{-x}(1-e^{-x})e^{-y}(1-e^{-y}),\ x > 0,\ y > 0.$$

   (a) Let $u = e^{-x}$ and $v = e^{-y}$. Show that

$$f(u,v) = 2(1-u)(1-v)(1-uv)^{-3}.$$

   (b) Use the Gibbs sampling and Metropolis-Hastings to draw the samples. Make a contour plot to demonstrate the draws of the function with respect to u and v.

   (c) Make a contour plot to demonstrate the draws of the function with respect to x and y.

5. Each member must write down what you learned and practiced from the homework and what the difficulty is in the methodology of MCMC in the report.