統計模擬HW4

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1

(a)

(a) $Y_i \overset{iid}{\sim} N(\mu,\sigma^2)$ $\quad \mathbf{y}=(y_1,y_2,\cdots,y_n)$

$$\Rightarrow f(y_i|\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y_i-\mu)^2}{2\sigma^2}\right\} \Rightarrow f(y|\mu,\sigma^2)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y_i-\mu)^2}{2\sigma^2}\right\}$$

$$=(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}\right\}$$

prior $\pi(\mu,\sigma^2)\propto\exp\left\{-\frac{(\mu-\theta_0)^2}{2\tau_0^2}\right\}\times\frac{1}{\sigma^2}\times 1_{\{\sigma^2>0\}}(\sigma^2)$

$\theta_0=11$ $\quad \tau_0^2=8$

joint posterior distribution of (μ,σ^2)

$$\pi(\mu,\sigma^2|\mathbf{y})=\frac{f(\mu,\sigma^2,y)}{\pi(\mu,\sigma^2)}\propto f(\mu,\sigma^2,\mathbf{y})=f(y|\mu,\sigma^2)\,\pi(\mu,\sigma^2)$$

$$\propto(\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}\right\}\exp\left\{-\frac{(\mu-\theta_0)^2}{2\tau_0^2}\right\}\times\frac{1}{\sigma^2}$$

$-\infty<\mu<\infty$

$\sigma^2>0$

(b)

(b) full conditional distribution for μ

$$\pi(\mu \mid \sigma^2, y) \propto \exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma^2} + \frac{(\mu-\theta_0)^2}{\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} y_i\mu + n\mu^2}{\sigma^2} + \frac{\mu^2 - 2\mu\theta_0 + \theta_0^2}{\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{n\mu^2 - 2\sum y_i\mu}{\sigma^2} + \frac{\mu^2 - 2\mu\theta_0}{\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{\tau_0^2 n\mu^2 - 2\tau_0^2\sum_{i=1}^{n} y_i\mu + \mu^2\sigma^2 - 2\sigma^2\mu\theta_0}{\sigma^2\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{1}{\frac{\sigma^2\tau_0^2}{n\tau_0^2+\sigma^2}}\left[\mu^2 - 2\frac{(\tau_0^2\sum_{i=1}^{n} y_i + \sigma^2\theta_0)}{n\tau_0^2+\sigma^2}\mu + \left(\frac{\tau_0^2\sum_{i=1}^{n} y_i + \sigma^2\theta_0}{n\tau_0^2+\sigma^2}\right)^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{(\mu-A)^2}{B}\right\} \qquad A = \frac{\tau_0^2\sum_{i=1}^{n} y_i + \sigma^2\theta_0}{n\tau_0^2+\sigma^2}$$

$$\mu \mid \sigma^2, y \sim N(A, B) \qquad B = \frac{\sigma^2\tau_0^2}{n\tau_0^2+\sigma^2}$$

full conditional distribution for σ^2

$$\pi(\sigma^2 \mid \mu, y) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma^2}\right\}\frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-(\frac{n}{2}+1)} \exp\left\{-\frac{1}{2}\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma^2}\right\}$$

$$\sigma^2 \mid \mu, y \sim \text{Inv Gamma}\left(\frac{n}{2}, \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right)$$

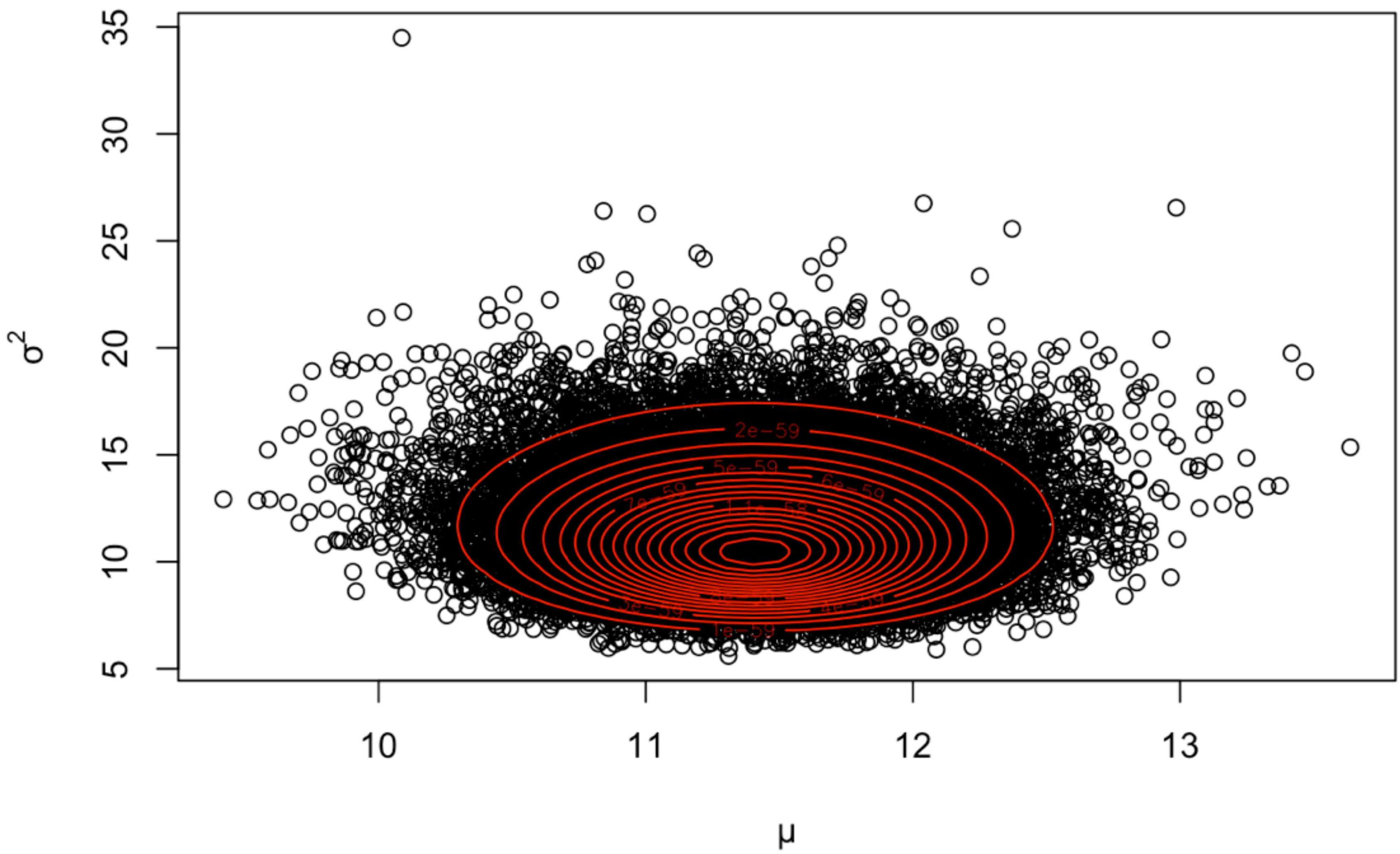
rate

(c)

```r
data.y <- c(14.52, 8.49, 11.86, 6.42, 8.41, 7.66, 10.4, 9.99, 16.49,
              6.55, 16.54, 15.53, 5.66, 14.67, 9.06, 13.69, 8.49, 12.72,
              7.86, 13.03, 13.06, 5.67, 8.18, 18.74, 7.63, 14.76, 18.28,
              15.82, 12.67, 11.72, 16.13, 11.5, 11.88, 9.3, 12.67, 10.61,
              12.35, 8.41, 11.17, 14.91, 5.58, 7.74, 12.78, 11.32, 11.12,
              12.01, 13.75, 11.36, 11.63, 10.22)



#(c)
gibbs1=function(freq,data.y,theta,tau_square){
  n=length(data.y)
  par=matrix(rep(NA,freq*2),nrow=2,ncol = freq)
  par[,1]=c(mean(data.y),var(data.y))
  for(i in 2:freq){
    A=(tau_square*sum(data.y)+theta*par[2,i-1])/(n*tau_square+par[2,i-1])
    B=tau_square*par[2,i-1]/(n*tau_square+par[2,i-1])
    par[1,i]=rnorm(1,A,sqrt(B))
    par[2,i]=1/rgamma(1,n/2,sum((data.y-par[1,i])^2)/2)
  }
  return(par)
}
```

(d)

```r
#(d)
freq=30000
theta=11
tau_square=8
res=gibbs1(freq,data.y,theta,tau_square)

plot(res[1,seq(0.1*freq,freq)],res[2,seq(0.1*freq,freq)],xlab = "μ",ylab=expressio
n(σ^2))



pdf <- function(data.y,mu, sigma_square,theta,tau_square){
  fxy <- matrix(NA, length(mu), length(sigma_square))
  n=length(data.y)
  for(i in 1:length(mu)){
    for(j in 1:length(sigma_square)){
      fxy[i, j] =(2*3.14*sigma_square[j])^(-n/2)*exp(-sum((data.y-mu[i])^2)/(2*sig
ma_square[j]))*exp(-(mu[i]-theta)^2/(2*tau_square))/sigma_square[j]
    }
  }
  return(fxy)
}
mu=seq(9,14,0.1)
sigma_square=seq(5,30,0.1)
contour(mu, sigma_square, pdf(data.y,mu,sigma_square,theta,tau_square),nlevels = 1
5, col = 2, add = TRUE)
```



(e)

```
#(e)
mu=mean(res[1,seq(0.1*freq,freq)])
sigma_square=mean(res[2,seq(0.1*freq,freq)])
mu
```

```
## [1] 11.40761
```

```
sigma_square
```

```
## [1] 11.53395
```

posterior mean for μ =11.4076137

posterior mean for $\sigma 2$=11.5339529

2

$$f(x,y,z) = C\ \exp\{-(x+y+z+axy+bxz+cyz)\}, \quad x>0,\ y>0\ z>0$$

$$f(x|y,z) = \frac{f(x,y,z)}{f(y,z)} = \frac{\exp\{-(x+y+z+axy+bxz+cyz)\}}{\int_0^\infty \exp\{-(x+y+z+axy+bxz+cyz)\}dx}$$

$$= \frac{\exp\{-(x+y+z+axy+bxz+cyz)\}}{\frac{1}{ay+bz+1}\exp\{-(y+z+cyz)\}} = (ay+bz+1)\exp\{-(x+axy+bxz)\}$$

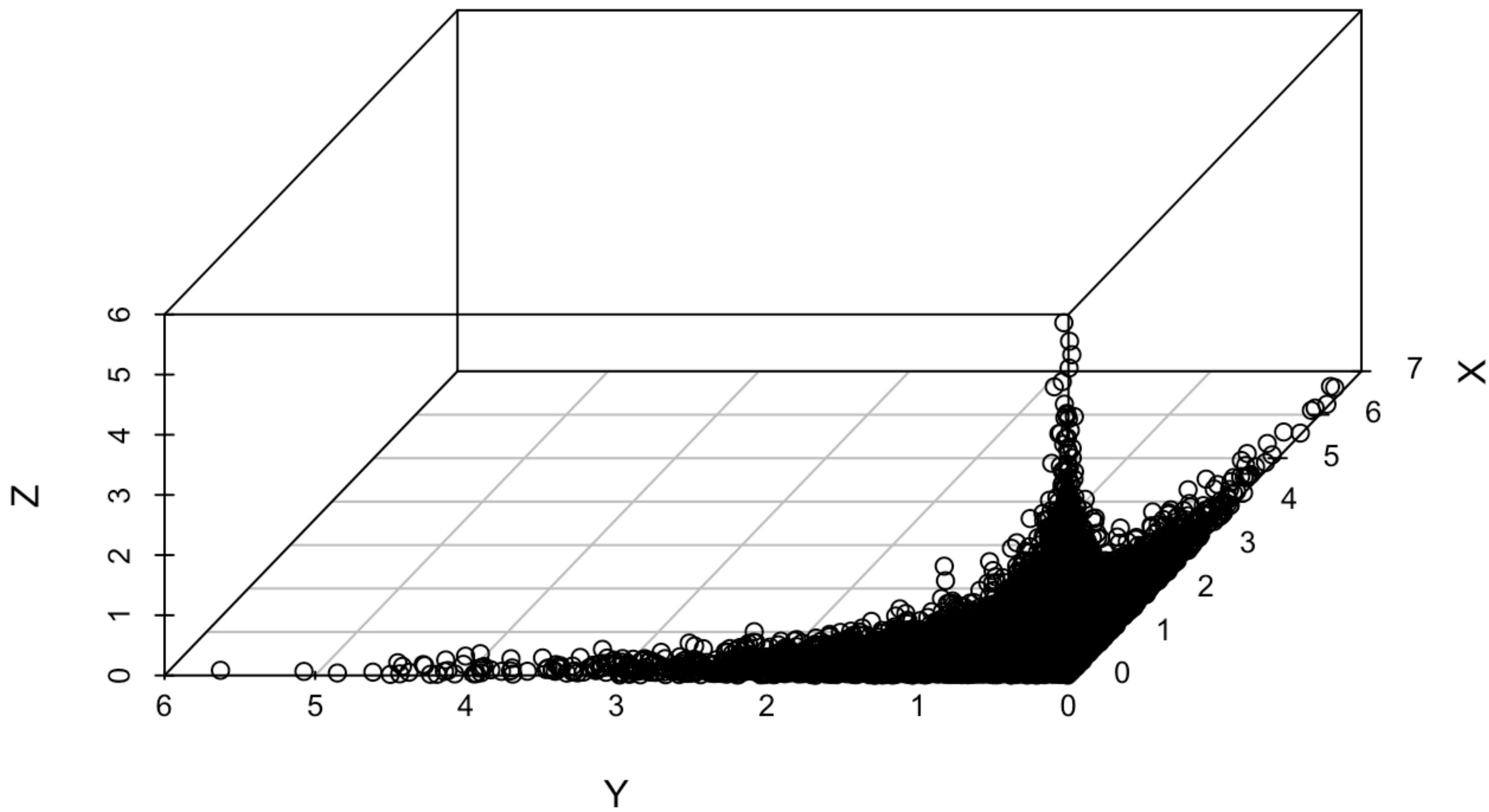
$$= (ay+bz+1)\exp\{-(ay+bz+1)x\}, \quad x>0\ y>0,\ z>0$$

$$\Rightarrow X|Y,Z \sim Exp(ay+bz+1)$$

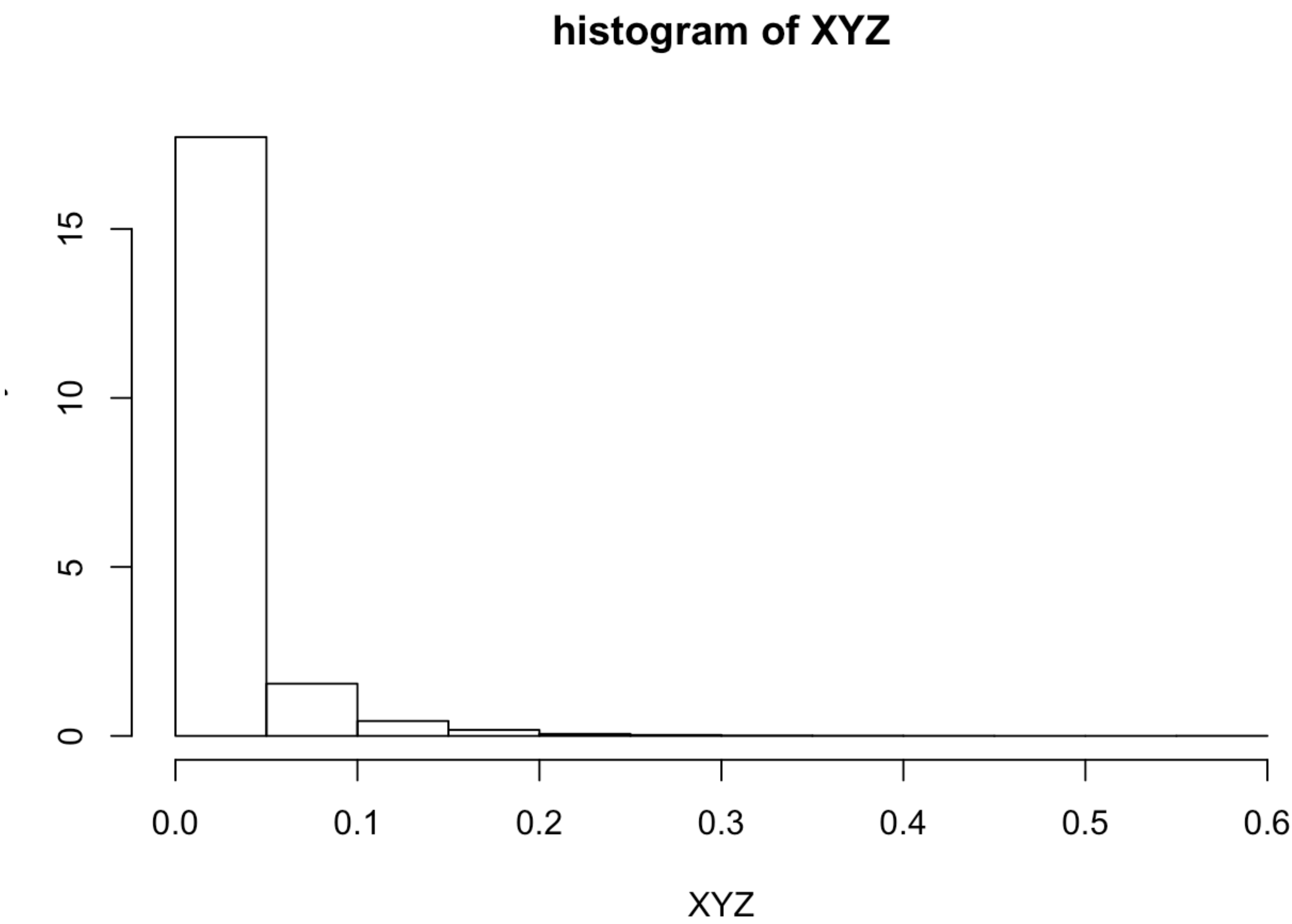
Similarly, $Y|X,Z \sim Exp(ax+cz+1)$

$Z|X,Y \sim Exp(bx+cy+1)$

```
gibbs2=function(n,a,b,c){
  par=matrix(rep(NA,freq*3),nrow=freq,ncol = 3)
  par[1,]=c(1,1,1)
  for(i in 2:freq){
    par[i,1]=rexp(1,a*par[i-1,2]+b*par[i-1,3]+1)
    par[i,2]=rexp(1,a*par[i,1]+c*par[i-1,3]+1)
    par[i,3]=rexp(1,b*par[i,1]+c*par[i,2]+1)
  }
  return(par)
}

a=2
b=3
c=6
freq=30000
res=gibbs2(freq,a,b,c)



library("scatterplot3d") # load
scatterplot3d(res[seq(0.1*freq,freq),], angle = 245,xlab="X",ylab="Y",zlab="Z")
```




```
hist(res[seq(0.1*freq,freq),1]*res[seq(0.1*freq,freq),2]*res[seq(0.1*freq,freq),3]
,probability = T,xlab="XYZ",main="histogram of XYZ")
```




```
#E(XYZ)
mean=mean(res[seq(0.1*freq,freq),1]*res[seq(0.1*freq,freq),2]*res[seq(0.1*freq,fre
q),3])
mean
```

```
## [1] 0.02069958
```

E(XYZ) =0.0206996

3

$$f(x,y) = \frac{n!}{x!(n-x)!} y^{(x+\alpha)} (1-y)^{n-x+\beta} \quad x=0,1,2,3 \cdots n \quad 0 \le y \le 1$$

$n=10 \quad \alpha=3 \quad \beta=5$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\frac{n!}{x!(n-x)!} y^{x+\alpha} (1-y)^{n-x+\beta}}{\sum_{x=0}^{n} \frac{n!}{x!(n-x)!} y^{(x+\alpha)} (1-y)^{n-x+\beta}}$$

$$= \frac{\frac{n!}{x!(n-x)!} y^{x+\alpha} (1-y)^{n-x+\beta}}{y^{\alpha}(1-y)^{\beta} \underbrace{\sum_{x=0}^{n} \frac{n!}{x!(n-x)!} y^{x} (1-y)^{n-x}}_{1}} = \frac{\frac{n!}{x!(n-x)!} y^{x+\alpha} (1-y)^{n-x+\beta}}{y^{\alpha}(1-y)^{\beta}}$$

$$= \frac{n!}{x!(n-x)!} y^{x}(1-y)^{n-x} \quad x=0,1,2 \cdots n \quad 0 \le y \le 1 \quad \Rightarrow X|Y \sim \text{Binomial}(n, Y)$$

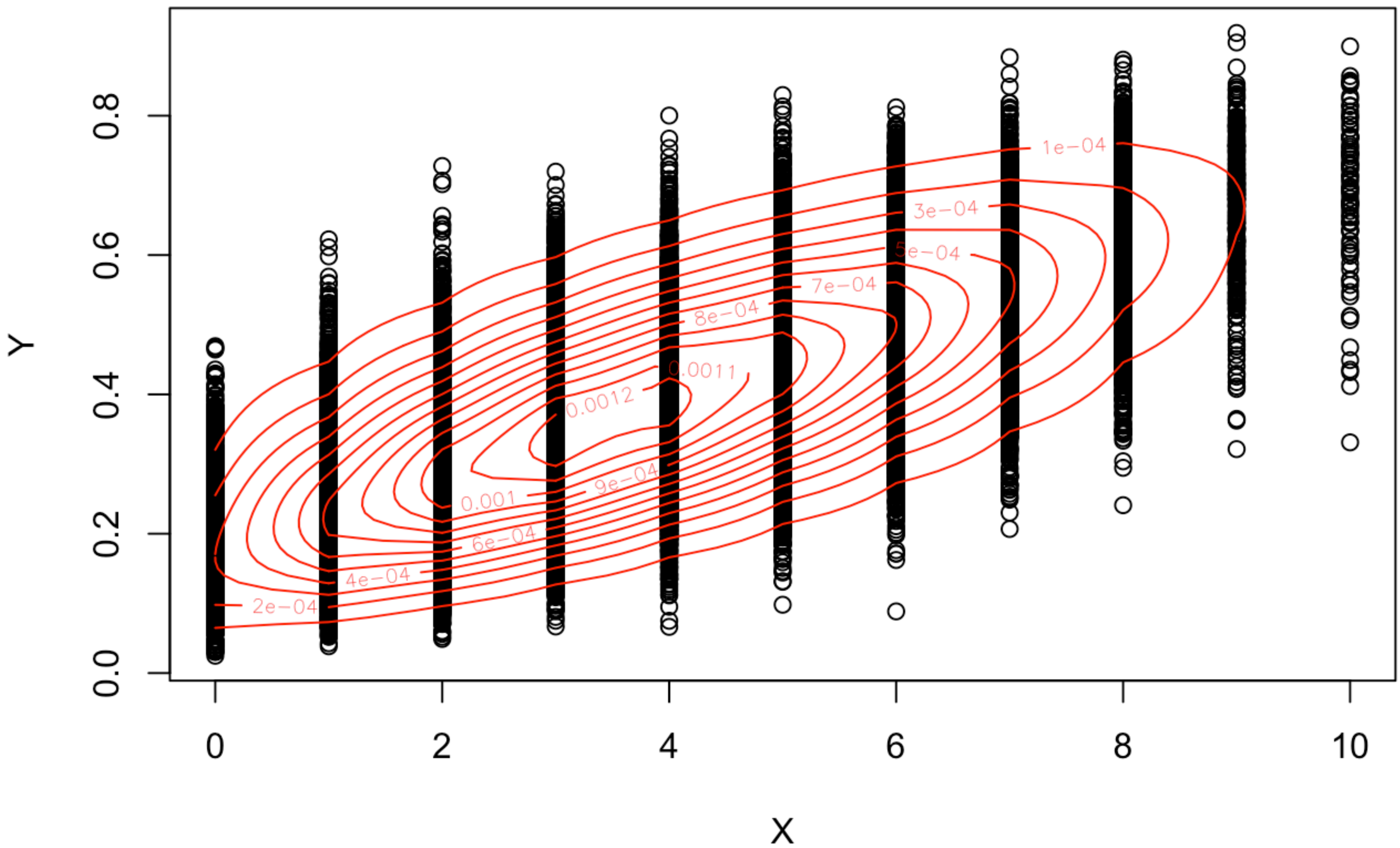
$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{n!}{x!(n-x)!} y^{x+\alpha}(1-y)^{n-x+\beta}}{\int_0^1 \frac{n!}{x!(n-x)!} y^{x+\alpha} (1-y)^{n-x+\beta} dy}$$

$$= \frac{\frac{n!}{x!(n-x)!} y^{x+\alpha} (1-y)^{n-x+\beta}}{\frac{n!}{x!(n-x)!} \frac{(x+\alpha)!(n-x+\beta)!}{(n+\alpha+\beta+1)!} \underbrace{\int_0^1 \frac{(n+\alpha+\beta+1)!}{(x+\alpha)!(n-x+\beta)!} y^{x+\alpha+1-1} (1-y)^{n-x+\beta+1-1} dy}_{1}}$$

$$= \frac{(n+\alpha+\beta+1)!}{(x+\alpha)!(n-x+\beta)!} y^{x+\alpha+1-1} (1-y)^{n-x+\beta+1-1} \quad x=0,1,\cdots n \quad 0 \le y \le 1$$

$$\Rightarrow Y|X \sim \text{Beta}(x+\alpha+1, n-x+\beta+1)$$

```r
#3
gibbs3=function(freq,n,alpha,beta){
  par=matrix(rep(NA,freq*2),nrow=2,ncol = freq)
  par[,1]=c(1,1)
  for(i in 2:freq){
    par[1,i]=rbinom(1,n,par[2,i-1])
    par[2,i]=rbeta(1,par[1,i]+alpha+1,n-par[1,i]+beta+1)
  }
  return(par)
}

freq=30000
n=10
alpha=3
beta=5
res=gibbs3(freq,n,alpha,beta)
plot(res[1,seq(0.1*freq,freq)],res[2,seq(0.1*freq,freq)],xlab="X",ylab="Y")

pdf <- function(x, y,n,alpha,beta){
  fxy <- matrix(NA, length(x), length(y))
  for(i in 1:length(x)){
    for(j in 1:length(y)){
      fxy[i, j] =choose(n,x[i])*y[j]^(x[i]+alpha)*(1-y[j])^(n-x[i]+beta)
    }
  }
  return(fxy)
}
xx <- seq(0, 10, 1)
yy <- seq(0, 1, 0.01)
contour(xx, yy, pdf(xx, yy, n,alpha,beta),nlevels = 15, col = 2, add = TRUE)
```



Y
0.0
0.2
0.4
0.6
0.8
X
0
2
4
6
8
10
1e−04
2e−04
3e−04
4e−04
5e−04
6e−04
7e−04
8e−04
9e−04
0.001
0.0011
0.0012

4.

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad w_1 = 0.7,\quad w_2 = 0.3$$

$$\mu_1 = \begin{bmatrix} -5 \\ -7 \end{bmatrix} = \begin{bmatrix} \mu_{11} \\ \mu_{12} \end{bmatrix} \qquad \mu_{11} = -5,\quad \mu_{12} = -7$$

$$\mu_2 = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \mu_{21} \\ \mu_{22} \end{bmatrix} \qquad \mu_{21} = 5,\quad \mu_{22} = 7$$

$$\Sigma_1 = \begin{bmatrix} 1 & -0.7 \\ -0.7 & 1 \end{bmatrix} = \begin{bmatrix} \sigma_{11}^2 & \rho_1\sigma_{11}\sigma_{12} \\ \rho_1\sigma_{11}\sigma_{12} & \sigma_{12}^2 \end{bmatrix} \qquad \sigma_{11} = 1,\quad \sigma_{12} = 1,\quad \rho_1 = -0.7$$

$$\Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \sigma_{21}^2 & \rho_2\sigma_{21}\sigma_{22} \\ \rho_2\sigma_{21}\sigma_{22} & \sigma_{22}^2 \end{bmatrix} \qquad \sigma_{21} = \sqrt{2},\quad \sigma_{22} = \sqrt{3},\quad \rho_2 = 0$$

$$f(x) \propto 0.7 \times \underbrace{\frac{1}{\sqrt{|\Sigma_1|}} \exp\left\{ \frac{-(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)}{2} \right\}}_{\text{Bivariate normal distribution}} + 0.3 \times \underbrace{\frac{1}{\sqrt{|\Sigma_2|}} \exp\left\{ \frac{-(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)}{2} \right\}}_{\text{Bivariate normal distribution}}$$

composite method
$$\begin{cases} 0.7 & BN(\mu_{11} = -5,\ \mu_{12} = -7\ \ \rho_1 = -0.7) \\ 0.3 & BN(\mu_{21} = 5,\ \mu_{22} = 7\ \ \rho_2 = 0) \end{cases}$$

$$x_{11} | x_{12} \sim N\left(\mu_{11} + \rho_1 \frac{\sigma_{11}}{\sigma_{12}}(x_{12} - \mu_{12}), (1-\rho_1)^2 \sigma_{11}^2\right),\ x_{12} | x_{11} \sim N\left(\mu_{12} + \rho_1 \frac{\sigma_{12}}{\sigma_{11}}(x_{11} - \mu_{11}), (1-\rho_1^2)\sigma_{12}^2\right)$$

$$x_{21} | x_{22} \sim N\left(\mu_{21} + \rho_2 \frac{\sigma_{21}}{\sigma_{22}}(x_{22} - \mu_{22}), (1-\rho_2)^2 \sigma_{21}^2\right),\ x_{22} | x_{21} \sim N\left(\mu_{22} + \rho_2 \frac{\sigma_{22}}{\sigma_{21}}(x_{21} - \mu_{21}), (1-\rho_2^2)\sigma_{22}^2\right)$$

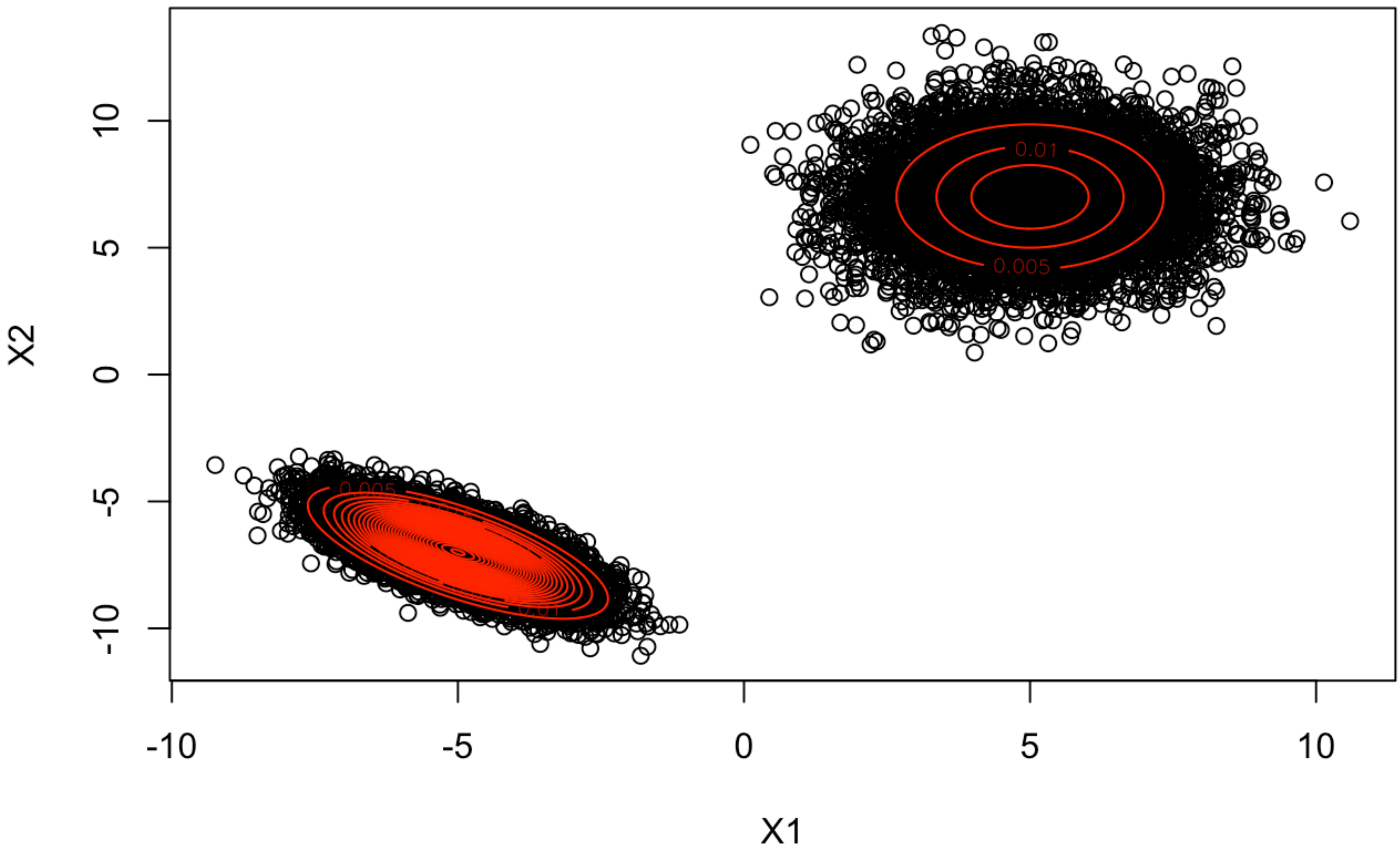
(a)

```r
#(a)
c.normal=function(freq,mu1,mu2,sigma1,sigma2,rho){
  par=matrix(rep(NA,freq*2),nrow=2,ncol = freq)
  par[,1]=c(0,0)
  c.sigma1=sqrt((1-rho^2)*sigma1^2)
  c.sigma2=sqrt((1-rho^2)*sigma2^2)
  for(i in 2:freq){
    c.mean1=mu1+rho*sigma1/sigma2*(par[2,i-1]-mu2)
    par[1,i]=rnorm(1,c.mean1,c.sigma1)
    c.mean2=mu2+rho*sigma2/sigma1*(par[1,i]-mu1)
    par[2,i]=rnorm(1,c.mean2,c.sigma2)
  }
  return(par)
}


gibbs4=function(freq,w1,w2,mu11,mu12,mu21,mu22,sigma11,sigma12,sigma21,sigma22,rho
1,rho2){
  x=matrix(rep(NA,freq*2),nrow=2,ncol = freq)
  res1=c.normal(freq,mu11,mu12,sigma11,sigma12,rho1)
  res2=c.normal(freq,mu21,mu22,sigma21,sigma22,rho2)
  for(i in 1:freq){
    U=runif(1)
    if(U<w1){
      x[,i]=res1[,i]
    }else{
      x[,i]=res2[,i]
    }
  }
  return(x)
}
```

(b)

```
freq=30000
w1=0.7
w2=0.3
mu11=-5
mu12=-7
mu21=5
mu22=7
rho1=-0.7
rho2=0
sigma11=1
sigma12=1
sigma21=sqrt(2)
sigma22=sqrt(3)
x=gibbs4(freq,w1,w2,mu11,mu12,mu21,mu22,sigma11,sigma12,sigma21,sigma22,rho1,rho2)
plot(x[1,seq(0.1*freq,freq)],x[2,seq(0.1*freq,freq)],xlab="X1",ylab="X2")
```

```
pdf <- function(x1,x2,w1,w2,mu11,mu12,mu21,mu22,sigma11,sigma12,sigma21,sigma22,rh
o1,rho2){
  fx1x2 <- matrix(NA, length(x1), length(x2))
  for(i in 1:length(x1)){
    for(j in 1:length(x2)){
     z1 <- (x1[i]-mu11)^2/sigma11^2 + (x2[j]-mu12)^2/sigma12^2 -
2*rho1*(x1[i]-mu11)*(x2[j]-mu12)/(sigma11*sigma12)
      z2<- (x1[i]-mu21)^2/sigma21^2 + (x2[j]-mu22)^2/sigma22^2 -
2*rho2*(x1[i]-mu21)*(x2[j]-mu22)/(sigma21*sigma22)
      fx1x2[i, j] =0.7*exp(-z1/(2*(1-rho1^2)))/(2*3.14*sigma11*sigma12*sqrt(1-rho1
^2))+0.3*exp(-z2/(2*(1-rho2^2)))/(2*3.14*sigma21*sigma22*sqrt(1-rho2^2))
    }
  }
  return(fx1x2)
}

x1x1 <- seq(-10, 10, 0.01)
x2x2 <- seq(-10, 10, 0.01)

contour(x1x1, x2x2, pdf(x1x1,x2x2,w1,w2,mu11,mu12,mu21,mu22,sigma11,sigma12,sigma2
1,sigma22,rho1,rho2),nlevels = 30, col = 2, add = TRUE)
```


X2
X1
-10
-5
0
5
10
0.01
0.005
0.005