Introduction to Data Science - 1MS041

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Recall from last time

- Concentration of measure is a statement of the form, for every $0 < \delta < 1$ there is an $\epsilon > 0$ such that

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- Hoeffding, we only know boundedness $a \leq X \leq b$

$$\mathbb{P}(|\overline{X}_n - \mathbb{E}[\overline{X}_n]| \geq \epsilon) \leq 2e^{-\frac{2n\epsilon^2}{(b-a)^2}}.$$

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$$\mathbb{P}(|\overline{X}_n - \mathbb{E}[\overline{X}_n]| \geq \epsilon) \leq 2\exp\left(-\frac{n\sigma^2}{b^2} h\left(\frac{b\epsilon}{\sigma^2}\right)\right)$$

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- The confidence $1-\delta$ tells us that **before** we compute the interval, the probability that our interval I covers $\mathbb{E}[X]$ is at least $1-\delta$.

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- A **parametric model** is a model where the indexing parameter θ is a vector in k-dimensional Euclidean space. That is, θ is finite dimensional.
- A **non-parametric model** is a model where Θ is infinite dimensional.

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Example

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The truth is out there

Under a statistical model $\mathcal{F}$, there is a hidden $f^* \in \mathcal{F}$ that generates the data, we would like to infer something about f^* using observations.

Here are some examples of inference problems:

Estimation of the distribution function

This is the fundamental problem. Once we have the distribution function we have everything!
Often we can only get it up to some error, i.e. we get something like

$$\mathbb{P}(\hat{F} - \epsilon \leq F \leq \hat{F} + \epsilon) \geq 1 - \delta$$

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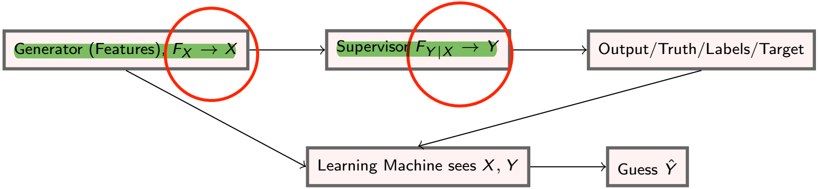
Estimating the density directly

If we can estimate the density up to some error, then we can also estimate the distribution function up to some error.
This is the **holy grail** of estimation.

Supervised learning

Setup

1. The generator of the data G
2. The supervisor S
3. The learning machine LM.



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- LM, The Learning Machine observing pairs of data of house and price.

Learning Machine

Examples

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SMS spam and not spam

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- G, the recieving of a sms
- S, You saying if the text is spam or not
- LM, The Learning Machine observing pairs of text and spam/ham. We "learned" that free appearing in the text is a good predictor, so we could use that as our guess: free $\rightarrow$ guess spam.

What is not supervised

If there is no supervisor there is no supervision

Lets say that all we have access to is the output of the generator. For instance house data. This is not a supervised problem as there is no supervisor, **no labels**.

??

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- You have a bunch of photos of 6 people but without information about who is on which one and you want to divide this dataset into 6 piles, each with the photos of one individual.

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- You have molecules, part of them are drugs and part are not but you do not know which are which and you want the algorithm to discover the drugs.

Mathematical formulation

- Generator: represented by F_X.
- Supervisor: represented by $F_{Y|X}$
- Learning Machine: Trying to learn the relation between X, Y and use that to guess Y given X.

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Risk

The risk is expected loss

$$R(g) = \mathbb{E}[L((X,Y),g)]$$

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Goal of the LM

The goal of the Learning machine is to minimize its risk!
In the quadratic case this is called least squares. Using what?

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Pattern recognition

The supervisor uses a discrete distribution for Y, i.e. $F_{Y|X}$ is a discrete distribution. Here the learning machine uses $0 - 1$ loss.

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Regression function

積分y

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- We should specify a so called "model space", where are we searching? Linear functions? We denote this by $\mathcal{M}$.

The risk minimization problem

The learning machine wants to solve the following problem

$$g^* = \arg\min_{g \in \mathcal{M}} R(g)$$

Goal

The learning machine tries to minimize the risk among the possible models in $\mathcal{M}$. In linear regression we where searching among functions of the type $g_{(k,m)}(x) = kx + m$.