Systematic Design of Optimal Low-Thrust Transfers for the Three-Body Problem

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Motivation - Small Satellites

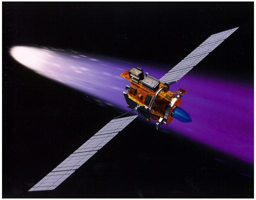
- Spacecraft design/launch and development are costly endeavors
  - Long development timelines with extensive component testing
  - Prohibitive launch costs and manifest scheduling

Vehicle	LEO Capacity (kg)	LEO Cost per kg
Space Shuttle	28 803	$ 10 416
Atlas 2A	8618	$ 11 314
Falcon 9	13 150	$ 4654
Falcon 9 Heavy	53 000	$ 1698

- Small spacecraft enable cost effective and rapid development
  - Reduced size/mass allows for 'piggyback' on larger vehicles
  - Cheaper designs allow for mass production/standardization

Motivation - Low Thrust Transfers

- Low-thrust orbital transfers
  - Electric propulsion has increased in popularity

  - Offers much higher specific impulse than chemical engines
  - Requires much longer operating periods for maneuvers
  - Small satellites with electric propulsion allows for new mission types
    - Formation flight (distributed aperture sensing)
    - On-orbit servicing
    - Interplanetary swarms

Previous Work
Challenges

- Optimal Trajectory Design
  - Orbital dynamics are nonlinear and chaotic
  - Very sensitive to initial conditions
  - Intuition required by designer
- Direct Optimal Control
  - Reformulate problem as parameter optimization
  - Allows for use of nonlinear programming methods
  - High dimensional problem and computationally intensive
- Numerical Integration
  - Typical methods use Runge-Kutta schemes
  - Numerical Instability and Energy drift
  - Difficult to capture long-term effects of low-thrust propulsion accurately
- Invariant Manifolds
  - Associated with periodic orbits
  - Control-free and transfers require manifolds to intersect

Proposed Approach

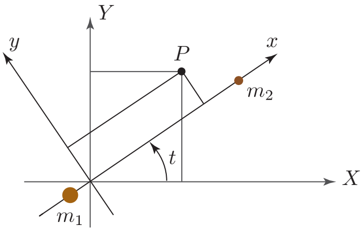
- Extend invariant manifold concept with low-thrust control
- Variational integrator allows for accurate and stable computation
- *Computational geometric optimal control* used to generate reachable set
- *Reachability set* on Poincaré section allows for systematic transfer design

Planar Circular Restricted Three-Body Problem

- Motion of spacecraft P under the mutual gravitational attraction of two primaries m_1 and m_2
- Nondimensionalization of the system
  - $G = 1$
  - Orbital period $= 2\pi$
  - Single parameter to describe dynamics

$$\mu = \frac{m_2}{m_1 + m_2}$$

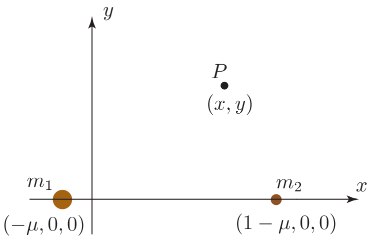
- x axis aligned along primaries
- Allows for greater structure in the dynamics



Planar Circular Restricted Three-Body Problem

- Distances to each primary

$$r_1 = \sqrt{(x+\mu)^2 + y^2}$$
$$r_2 = \sqrt{(x-1+\mu)^2 + y^2}$$



- Effective gravitational potential

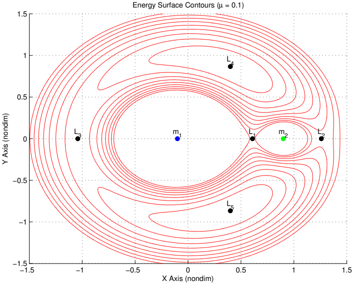
$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

- Planar Spacecraft Dynamics

$$\begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} \bar{v} \\ A\bar{v} + \nabla U + \bar{u}(t) \end{bmatrix}$$
$$= f\left(t, x, u\right)$$

Lagrange Points and Jacobi Integral

- 5 equilibrium points
  - Collinear - L_1, L_2, L_3
  - Equlilateral - L_4, L_5



- Jacobi energy integral defines a 3D surface of constant energy

$$E\left(\bar{r}, \bar{v}\right) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - U\left(x, y\right)$$

- Contours of constant E define zero velocity curves and regions of allowable motion

Variational Principle

- Variational Integrators
  - Structure-preserving integrators for Hamiltonian systems
  - Obtained by discretizing variational principle

Continuous Time	Discrete Time
Configuration Space $(q, \dot{q}) \in TQ$	Configuration Space $(q_k, q_{k+1}) \in Q \times Q$
Lagrangian $L(q, \dot{q})$	Lagrangian $L_d(q_k, q_{k+1})$
Action Integral $S = \int_0^T L(q, \dot{q})\, dt$	Action Sum $S_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$
Stationary Action $\delta S = 0$	Stationary Action $\delta S_d = 0$
Equation of Motion $\ddot{q} = f(q, \dot{q})$	Equation of Motion $q_{k+2} = f_d(q_k, q_{k+1})$

Discrete Lagrangian

Continuous time Lagrangian

$$L = \frac{1}{2}\left((\dot{x} - y)^2 + (\dot{y} + x)^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Choice of Quadrature rule affects accuracy

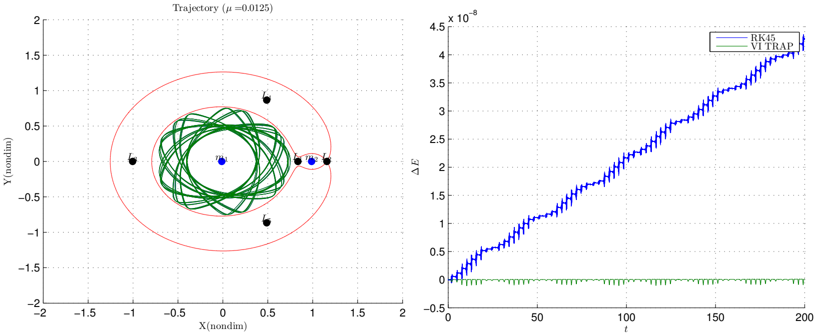
Rectangle	$L_d(q_0, q_1) = L(q_0, \frac{q_1-q_0}{h})h$
Midpoint	$L_d(q_0, q_1) = L(\frac{q_0+q_1}{2}, \frac{q_1-q_0}{h})h$
Trapezoidal	$L_d(q_0, q_1) = \frac{1}{2}\left[L(q_0, \frac{q_1-q_0}{h}) + L(q_1, \frac{q_1-q_0}{h})\right]h$

Discrete time Lagrangian using Trapezoidal Rule

$$\begin{aligned} L_d = \frac{h}{2}\Bigg(&\frac{1}{2}\left[\left(\frac{x_{k+1}-x_k}{h} - y_k\right)^2 + \left(\frac{y_{k+1}-y_k}{h} + x_k\right)^2\right] + \frac{1-\mu}{r_{1_k}} + \frac{\mu}{r_{2_k}} \\ &+ \frac{1}{2}\left[\left(\frac{x_{k+1}-x_k}{h} - y_{k+1}\right)^2 + \left(\frac{y_{k+1}-y_k}{h} + x_{k+1}\right)^2\right] + \frac{1-\mu}{r_{1_{k+1}}} + \frac{\mu}{r_{2_{k+1}}}\Bigg) \end{aligned}$$

Integrator Comparison

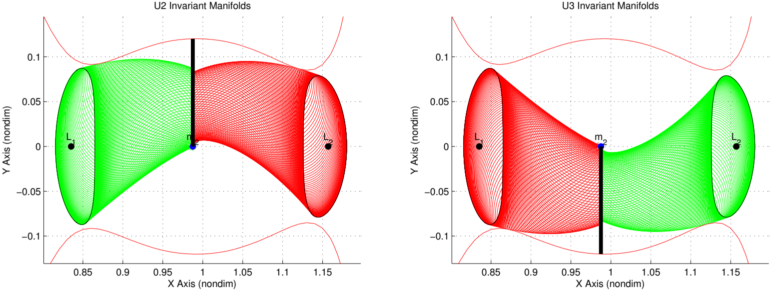
- Numerical Simulation of PCRTBP ($t_f = 200 \approx 15$ years)
  - Variable step Runge-Kutta method (`ode45.m`)
  - Variational integrator $\bar{x}_k \to \bar{x}_{k+1}$



- Typical integration methods do not preserve structure

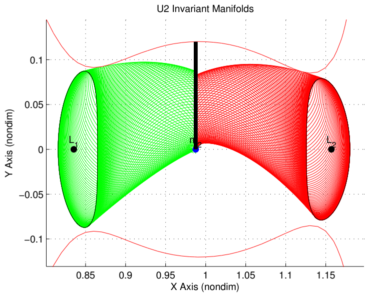
Invariant Manifolds

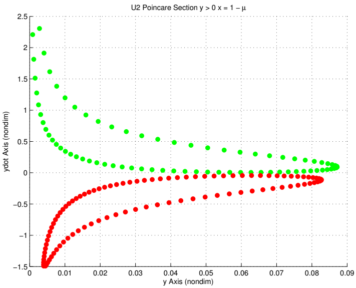
- Tube structure governs orbital motion
- Stable/Unstable 'tubes' divide orbits
- Poincaré section visualizes intersection



Poincaré Section

- Define hyperplane transverse to dynamic flow
- Analysis on a lower dimensional space





- Transfer is limited to intersecting regions

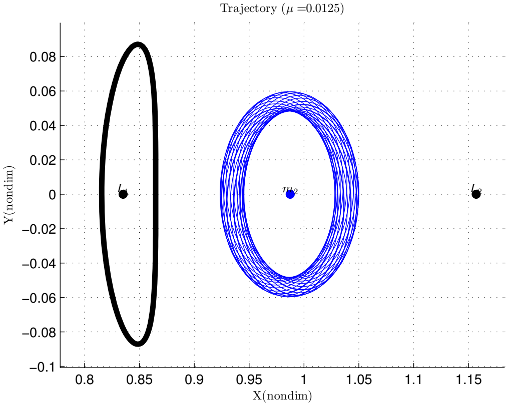
Reachability Set

- Reachable set on Poincaré section
  - The set of states that can be attained from a given initial state via admissable control input
  - Enlarge the intersection region on the Poincaré section
- *Computational Geometric Optimal Control*
  - Poincaré section defined by α_d in m_1
  - Direction on Poincaré section defined by θ_d in m_2

$$
\begin{aligned}
J &= -\frac{1}{2}\left(\bar{x}(N) - \bar{x}_n(N)\right)^T Q_f \left(\bar{x}(N) - \bar{x}_n(N)\right) \\
m_1 = 0 &= \frac{y(N) - L_{1y}}{x(N) - L_{1x}} - \tan\alpha_d \\
m_2 = 0 &= \frac{\dot{x}(N) - \dot{x_n}(N)}{x(N) - x_n(N)} - \tan\theta_d \\
0 &\geq \bar{u}^T \bar{u} - u_{max}^2
\end{aligned}
$$

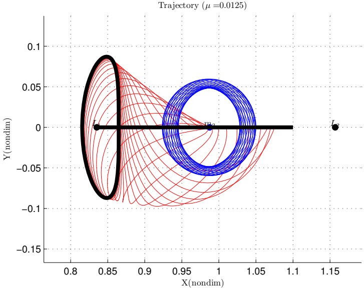
Transfer Problem

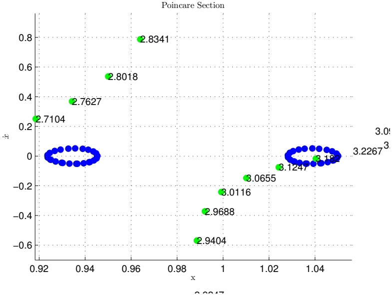
- Transfer from L_1 orbit to periodic orbits near the Moon
- Bounded control input and fixed time horizon



Invariant Manifold Transfer

- Unstable invariant manifolds generated from periodic orbit
- Poincaré section intersections generated

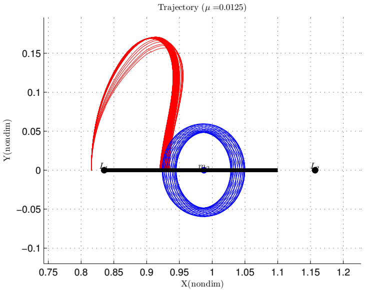


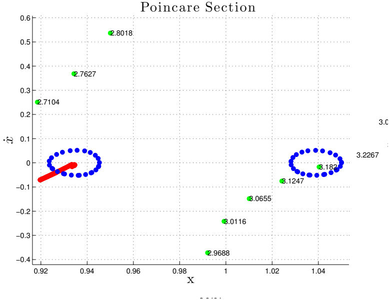


- Only portion of invariant manifold intersects the target
- Long time of flight $t_f \approx 3.1$

Reachable Set Transfer

- Reachablility set generated on Poincaré section
- Intersection point used to generate a transfer





- Reachable set intersect the target
- Shorter time of flight $t_f \approx 1.4$

Conclusions and Future Direction

1. Developed a systematic method of designing low-thrust orbital transfer
2. Extension of the previous control free work on invariant manifolds
3. *Computational geometric optimal control*: captures the long-term effects of low-thrust accurately and efficiently

Future research goals

- Extension to spatial problem
- Lyapunov feedback control rather than open loop
- Incorporate attitude control/constraints

Questions

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