THE GEORGE WASHINGTON UNIVERSITY
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Optimal Low Thrust Transfers in the Circular Restricted Three Body Problem

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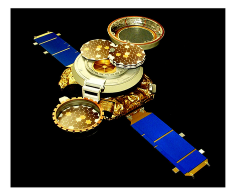
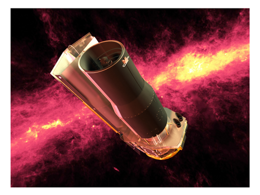
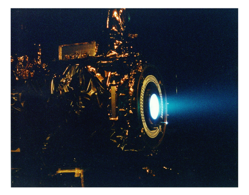
I. Background and Motivation

- Spacecraft serve a critical role in enabling and understanding modern life
  - The Global Positioning System (GPS) is vital to navigation, commerce, and emergency services
  - Scientific instruments such as the Hubble Space Telescope, Voyager missions, and the recent Rosetta mission have provided great insight into our solar system
- Designing spacecraft trajectories is a classic and ongoing topic of research
  - Predicting the motion of celesial bodies has motivated the likes of Newton, Euler, and Lagrange
  - The launch of Sputnik in 1957 shifted the focus to the motion of man-made objects
  - Incorporating additional perturbing forces increases the model accuracy at the expense of complexity
- The addition of propulsion devices enables the spacecraft to depart from the free motion trajectory
  - Previous work assume impulsive or instantaneous changes in spacecraft velocity (ΔV) to ease complexity
  - Continuous low thrust propulsion devices, such as electric propulsion, increase analysis complexity but improve capability
  - Electric Propulsion has higher exhaust speeds and offers higher specific impulse than chemical propulsion methods
  - Can operate for extended periods which is ideal for deep space missions
- Miniaturization of electric propulsion enables their use on small satellites ($1-100$ kg)
- Several recent missions have exploited the system dynamics to enable very low energy orbital transfers


Genesis

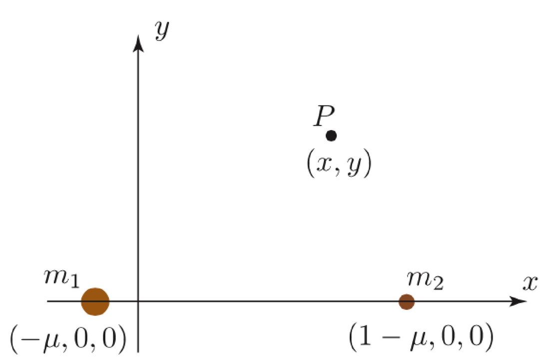

Spitzer


Ion Propulsion

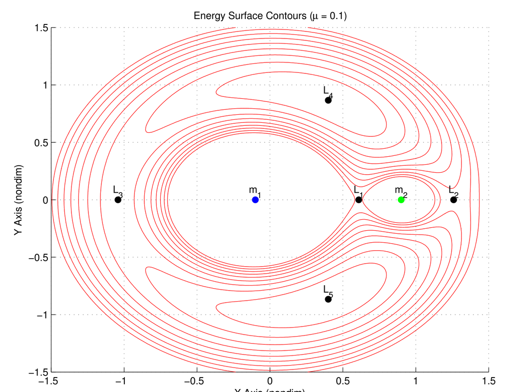
II. Dynamic Model

- Circular Restricted Three Body Problem describes the motion of a spacecraft under the gravitational force of two massive primaries
  - No closed form analytic solutions available
  - Use of a rotating reference frame allows for additional insight that produces an effective system for trajectory design in a multi-body environment
- The Jacobi Integral, is a scalar energy like term that divides realms of possible motion

$$E = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \bar{U}(x, y, z)$$



Rotating Reference Frame


Energy Contours

III. Dynamic Structure

- Equilibrium points classified by local stability properties
  - Collinear lagrange points (L1, L2, L3) are dynamically unstable while equilateral (L4, L5) are stable
  - Numerical technique of Differential correction allows for infinite number of periodic solutions

$$\delta x(t) = \frac{\partial x}{\partial x_0}\delta x_0 = \Phi(t, t_0)\delta x_0$$

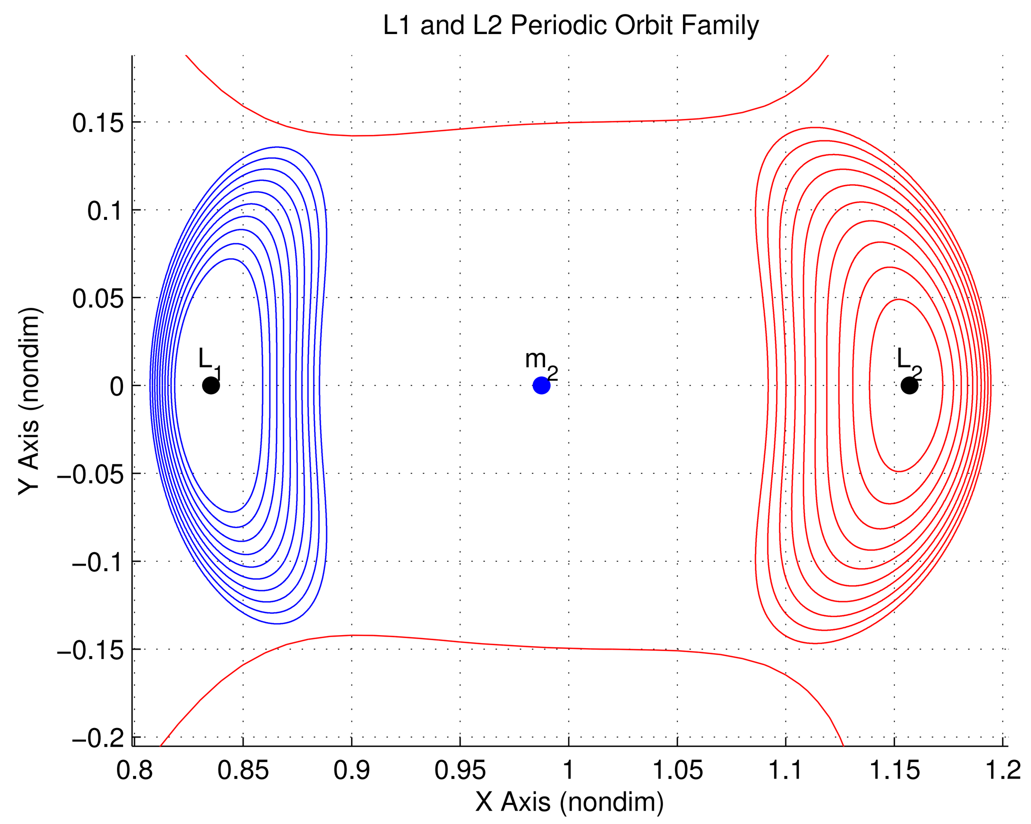
  - The State Transition Matrix gives the linear relationship between initial and final displacements

$$\dot{\Phi(t, t_0)} = \frac{\partial f(x, u, t)}{\partial x}\Phi(t, t_0)$$

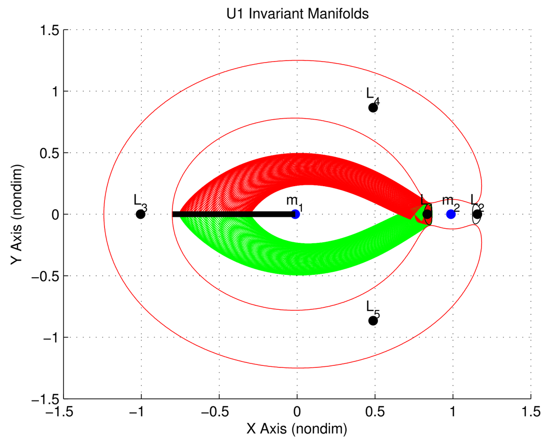
- The Monodomy Matrix of the periodic solutions allows insight into the local stability

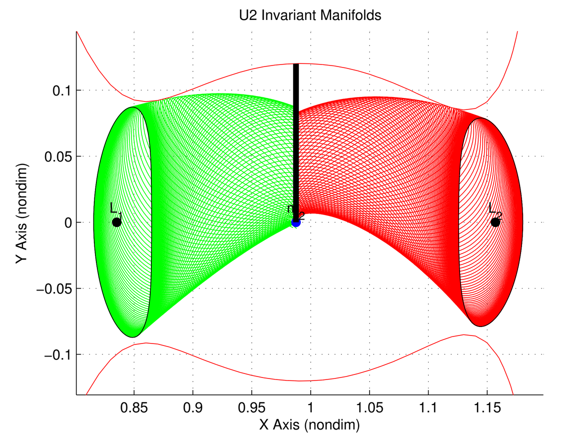
$$M = \Phi(T)$$

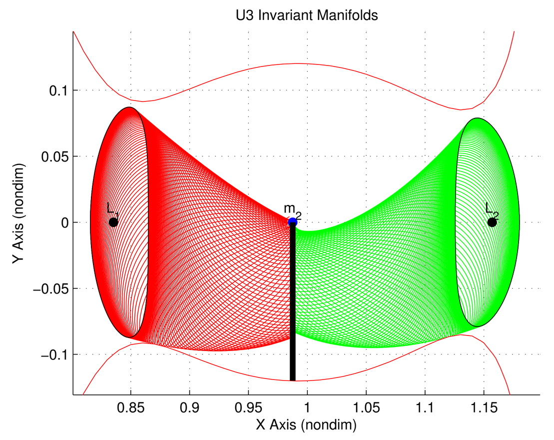
- Globalization of the Invariant Manifolds define regions of the state space that asymptotically arrive or depart the periodic orbit

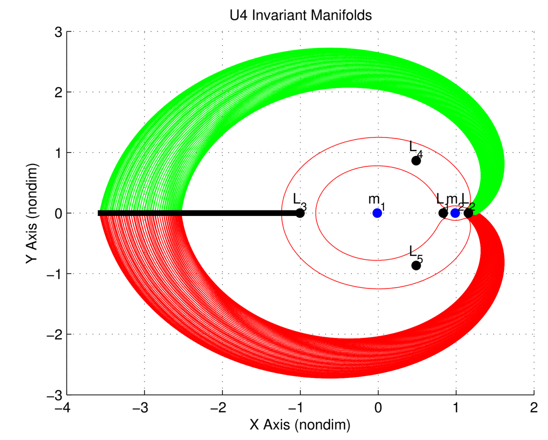


Family of Periodic Orbits


U1 Manifolds

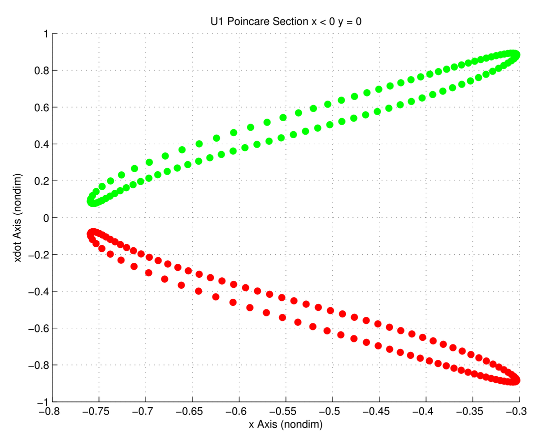

U2 Manifolds

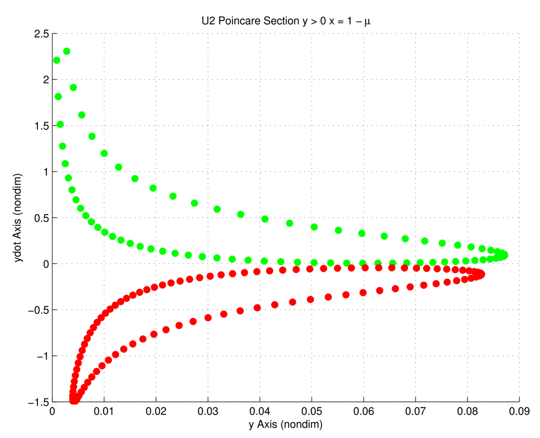

U3 Manifolds

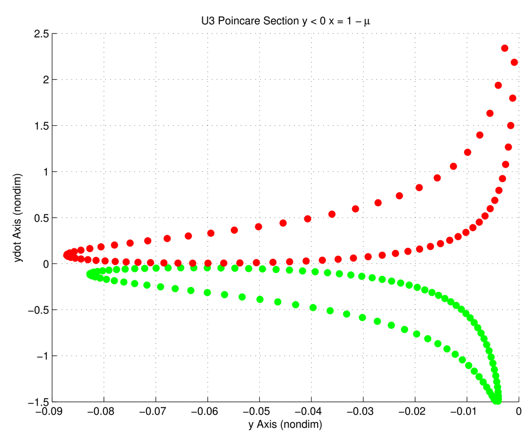

U4 Manifolds

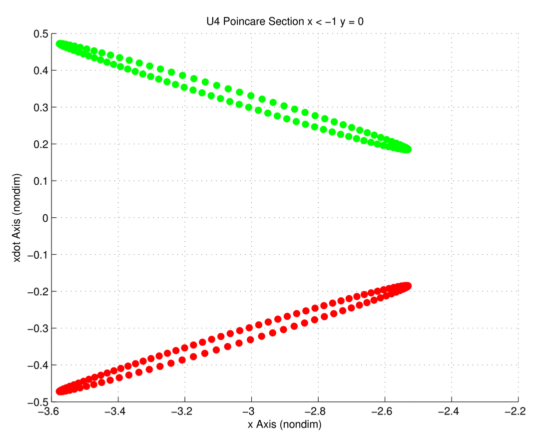
IV. Optimal Control Formulation

- Poincaré Section, or return map, is an essential tool in examining the behavior of periodic orbits
  - Use of the Jacobi integral and intersection hyperplane reduces the problem dimensionality and allows for improved visualization


U1 Poincaré


U2 Poincaré


U3 Poincaré


U4 Poincaré

- Optimal Control techniques enables the design of control histories that allow transfers between regions of space
- Incorporating control magnitude constraints lets us approximate the reachable set of the spacecraft
  - Calculus of variations leads to a two point boundary value problem with split boundary conditions

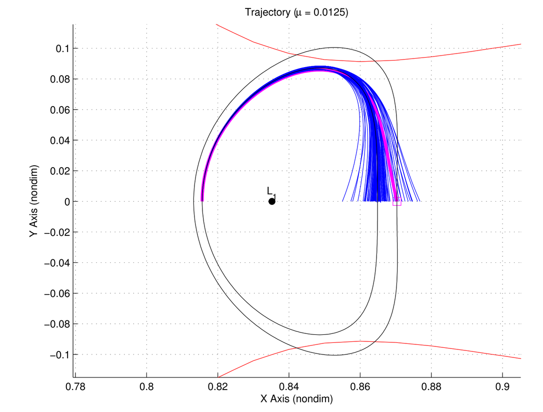
$$\min J = -\frac{1}{2}\left(\bar{x}(t_f) - \bar{x}_n(t_f)\right)^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\bar{x}(t_f) - \bar{x}_n(t_f)\right)$$

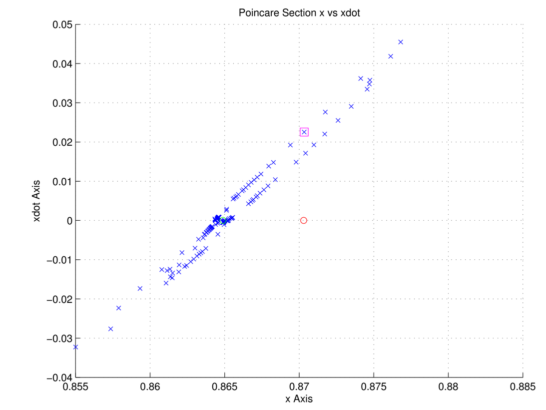
$$\frac{y(t_f) - L1_y}{x(t_f) - L1_x} - \tan\alpha_d = 0$$
$$\frac{\dot{x}(t_f) - \dot{x}_n(t_f)}{x(t_f) - x_n(t_f)} - \tan\theta_d = 0$$
$$\bar{u}^T\bar{u} - u_{max}^2 \leq 0$$

- Reachable set approximates state space that can be attained by spacecraft
- Constraints ensure trajectories intersect the return map and maximize the attainable state

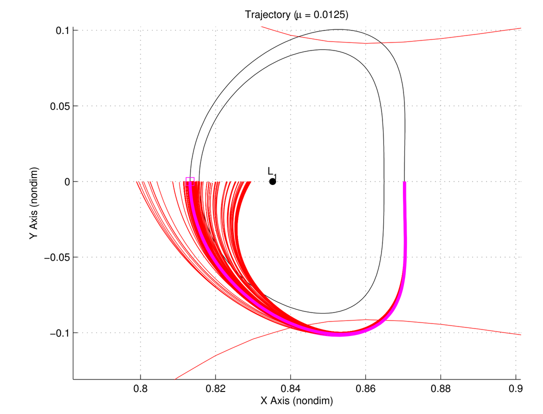
V. Experimental Results

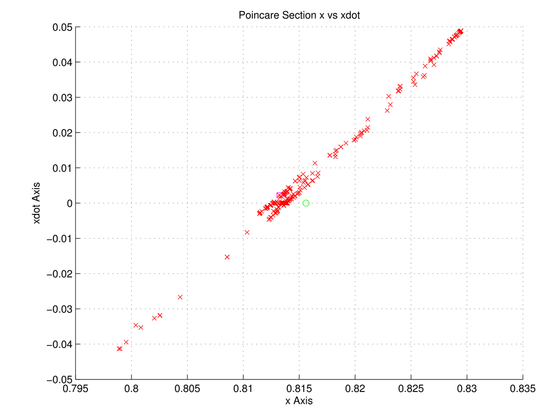
- Design control history that transfers spacecraft between periodic orbits about the Earth-Moon L1 Lagrange point
- An algorithm is developed that approximates the reachable state space on a chosen Poincaré section
  - 2D Poincaré section allows straightforward visualization and decreases the problem complexity
  - Variation of θ_d generates the reachable set which approximates the states that can be achieved subject to the bounded control input


Norther Reachable Trajectories


Northern Poncaré Reachable set

- Multiple iterations of the reachable set computation allows for transfer trajectory design


Southern Reachable Trajectories


Southern Poncaré Reachable set

- Poincaré section chosen such that periodic solutions will intersect the section twice per orbital period
- Projection of optimal trajectories onto the return map enables a decision making process in a lower dimensional space

VI. Conclusion

- Developed an optimal transfer design process which utilizes the concept of reachable sets on lower dimensional Poincaré maps
- Indirect optimal control formulation enables straightforward method of incorporating additional path and control constraints
- Current formulation is open loop and not robust to model uncertainties or disturbances.
  - Nonlinear Lyapunov theory is being pursued to develop a feedback control algorithm that offers verifiable stability properties
- Planar assumption is a valid model for popular mission scenarios such as the Earth-Moon or Sun-Earth system
  - Extension to the spatial problem is necessary for more exotic missions, such as multi-moon orbit missions of the outer planets
  - Poincaré map representation would require extension from 2D to 4D
  - Innovative visualization techniques required to display higher dimensional state spaces