Estimation of Information-Theoretic Quantities for Particle Clouds

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Motivation

- Particle systems represent uncertainty with higher fidelity, especially for long term propagation
- Information theoretic quantities such as entropy and divergence require integration of analytical functions
- Given a particle cloud, how do we estimate information theoretic properties of the underlying unknown probability density function (PDF)?
    - Need to estimate the underlying PDF

Applications

- Entropy: Computing entropy of nonlinear systems for non-linearity detection (DeMars et al., 2013)
- Divergence: Solving track-to-track association for UCT correlation problems (Hussein et al., 2015 and 2016)
- Mutual information: Solving observation-to-observation data association/IOD (Hussein et al., 2014, 2015)

PDF Estimation Approaches

- Histogram: Construct a histogram from the particle system
  - Drawbacks: non-smooth, PDF estimate is highly sensitive to the bin parameters, initractable for higher dimensional systems
- Kernel Methods: Use positive basis functions that integrate to 1 to fit the data
  - Use k-NN algorithm to estimate the PDF
  - Use the EM algorithm to estimate the PDF
  - Implemented the k-NN algorithm in this paper
  - Compare k-NN performance against EM in future work

Information Quantities

- Rényi entropy:

$$
\begin{aligned}
H_\alpha(p) &= \frac{1}{1-\alpha}\log\int p^\alpha(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{1-\alpha}\log\left(E[p^{\alpha-1}(\boldsymbol{x})]\right) \\
&\simeq \frac{1}{1-\alpha}\log\left(\frac{1}{N}\sum_{i=1}^{N} p^{\alpha-1}(\boldsymbol{x}_i)\right)
\end{aligned}
$$

- Shannon information entropy

$$
\begin{aligned}
H(p) &= -\int p(\boldsymbol{x})\log(p(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \\
&= -E[\log(p(\boldsymbol{x}))] \\
&\simeq -\frac{1}{N}\sum_{i=1}^{N}\log(p(\boldsymbol{x}_i))
\end{aligned}
$$

Information Quantities

- Bhattacharyya divergence:

$$\begin{aligned}
D_{\mathrm{B}}(p_1||p_2) &= -\log \mathrm{B}_C(p_1, p_2) = -\log\left[\int \sqrt{p_1(\boldsymbol{x})p_2(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\right] \\
&= -\log\left[\int p_1(\boldsymbol{x})\sqrt{\frac{p_2(\boldsymbol{x})}{p_1(\boldsymbol{x})}}\mathrm{d}\boldsymbol{x}\right] \\
&= -\log\left[E_{p_1}\left[\sqrt{\frac{p_2(\boldsymbol{x})}{p_1(\boldsymbol{x})}}\right]\right] \\
&\simeq -\log\left[\frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{p_2(\boldsymbol{x}_i)}{p_1(\boldsymbol{x}_i)}}\right], \ \boldsymbol{x}_i \sim p_1 \\
&\simeq -\log\left[\frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{p_1(\boldsymbol{x}_i)}{p_2(\boldsymbol{x}_i)}}\right], \ \boldsymbol{x}_i \sim p_2
\end{aligned}$$

The k-NN Algorithm

- A non-parametric method that relies on locally approximating the PDF based on particle concentration in a neighborhood

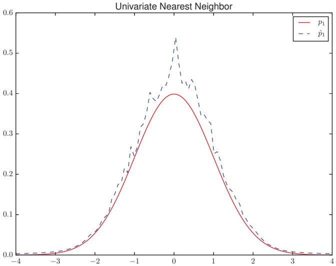
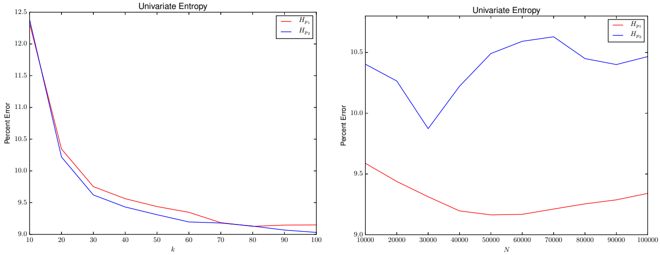


Figure: Univariate nearest neighbor estimate, $N = 20,000$, $k = 200$

The k-NN Algorithm

- Sensitivity to the parameters k and N (1-d)

$$p_1(x) = \mathcal{N}\{1, 1\}$$
$$p_2(y) = \mathcal{N}\{-1, 1\}$$

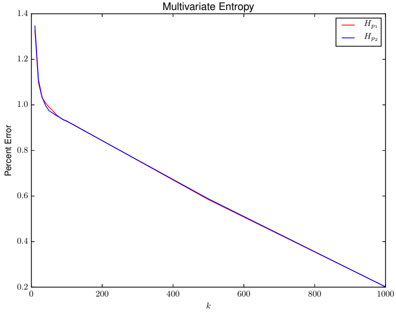


(a) Sensitivity to kth nearest neighbor. (b) Sensitivity to sample size N.

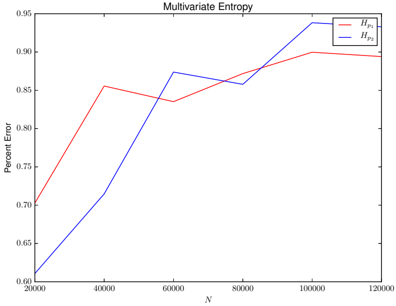
Figure: Univariate Entropy Estimation.

The k-NN Algorithm

- Sensitivity to the parameters k and N (6-d)



(a) Sensitivity to kth nearest neighbor ($N = 50,000$).

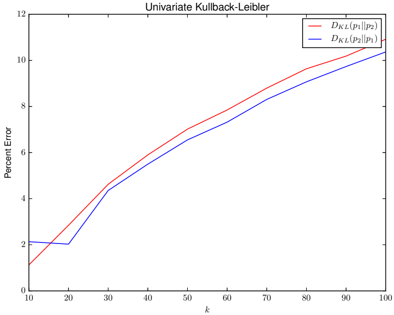


(b) Sensitivity to sample size N ($k = 200$).

Figure: Multivariate Entropy Estimation.

The k-NN Algorithm

- Sensitivity to the parameters k and N (1-d)



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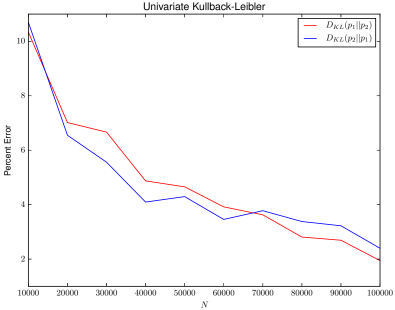
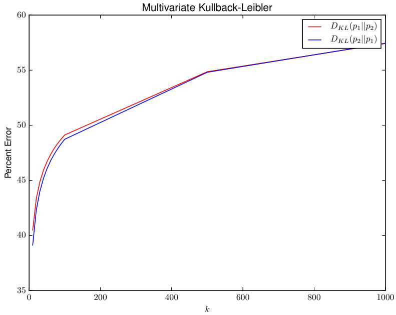


(b) Sensitivity to sample size N ($k = 50$).

Figure: Univariate Kullback-Leibler Estimation.

The k-NN Algorithm

- Sensitivity to the parameters k and N (6-d)



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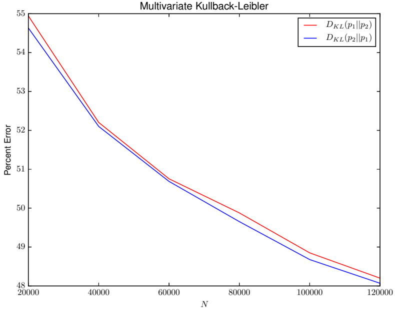

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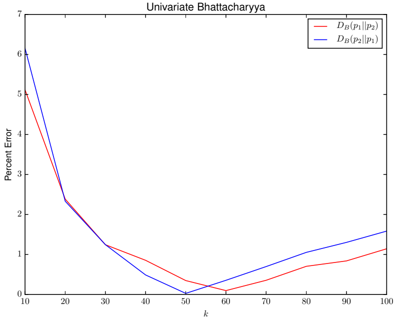
The k-NN Algorithm

- Sensitivity to the parameters k and N (1-d)



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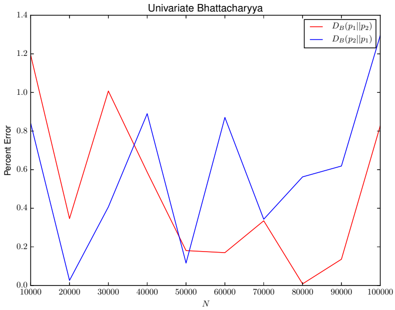
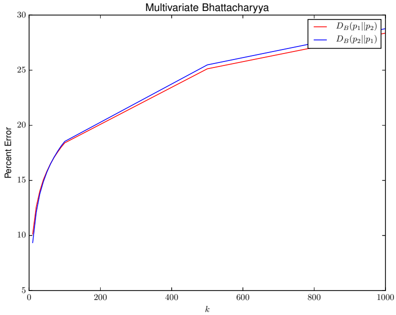


(b) Sensitivity to sample size N ($k = 50$).

Figure: Univariate Bhattacharyya Estimation.

The k-NN Algorithm

- Sensitivity to the parameters k and N (6-d)



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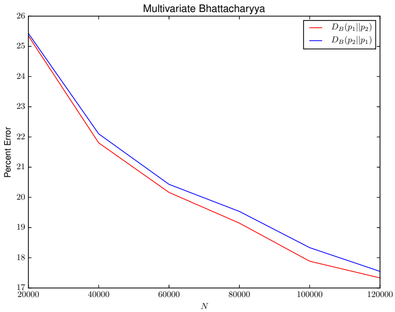
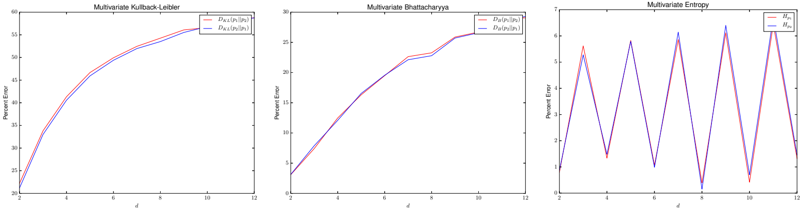


(b) Sensitivity to sample size N ($k = 200$).

Figure: Multivariate Bhattacharyya Estimation.

The k-NN Algorithm

- Sensitivity to the parameters k and N ($k = 50$, $N = 50,000$)



(a) Kullback-Leibler divergence estimation.

(b) Bhattacharyya divergence estimation.

(c) Entropy estimation.

Figure: Sensitivity of estimation to varying dimensions.

Entropy Conservation in Hamiltonian Systems

- Entropy is conserved for a Hamiltonian system when canonical coordinates are used to parametrize the system
- Object in a zero eccentricity, inclined (97.9 degrees) LEO orbit
- Initial uncertainty is Gaussian with 10 m position and 0.1 m/s velocity standard deviation (uniform in all directions)
- 10,000 particles were generated
- Transform the particles' to Delaunay orbital elements, and use to approximate the underlying PDF
- Initial entropy was found to be 44.90855570500575
- After a 30 day orbit propagation, the final entropy was estimated to be 44.90800388940583 (5.5×10^{-2}% error)

Bhattacharyya Divergence for UCT Correlation

- Two objects with same mean orbital elements:
  - Position standard deviation: 5 km to 200 km
  - Velocity standard deviation: 0.5 m/s
- Duration between t_* and t^*: 30 sidereal days
- 1000 MC runs for each position separation (1000 particles)

Table: Parameters of the True Orbit

Parameter	**Value**
Semimajor Axis, km	26 600.0
Eccentricity	0.2
Inclination, deg	55.0
Argument of Perigee, deg	-120.0
Right Ascension of the Ascending Node, deg	207.0
True Anomaly, deg	20.0

Bhattacharyya Divergence for UCT Correlation

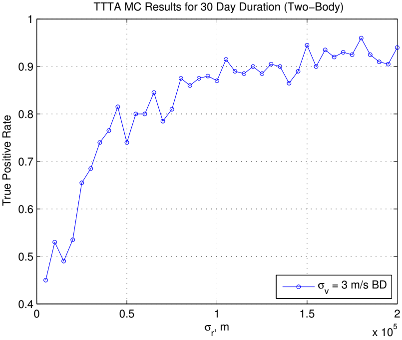


Figure: UCT Correlation results over a 30 day period.

Conclusion

- Introduced the k-NN algorithm for density estimation
- Used k-NN algorithm for information theoretic quantity estimation for particle systems
  - Entropy
  - Divergence
- Applied k-NN to demonstrate conservation of entropy for Hamiltonian systems and for solving UCT correlation problems
- Currently:
  - Implementing EM algorithm for PDF estimation
  - Comparing EM algorithm against k-NN algorithm performance
  - Applying approaches to estimating mutual information (for observation-to-observation association/IOD)