

Low-Thrust Trajectory Design Using Reachability Sets near Asteroid 4769 Castalia

Shankar Kulumani and Taeyoung Lee

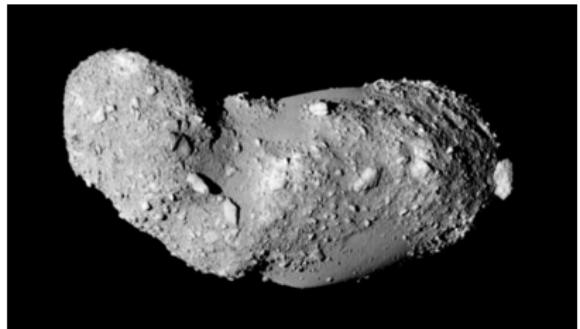
Flight Dynamics & Control Lab

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WASHINGTON, DC

Asteroid Missions

- Science - insight into the early formation of the solar system
- Mining - vast quantities of useful materials
- Impact - high risk from hazardous near-Earth asteroids



Asteroid Mining

- Useful materials can be extracted from asteroids to support:
 - Propulsion, construction, life support, agriculture, and precious/strategic metals
- Commercialization of near-Earth asteroids is feasible¹

| Element | Price (\$/kg) | Sales (\$M/yr) |
|-----------------|---------------|----------------|
| Phosphorous (P) | 0.08 | 2167 |
| Gallium (Ga) | 300.00 | 1544 |
| Germanium (Ge) | 745.00 | 6145 |
| Platinum (Pt) | 12 394.00 | 1705 |
| Gold (Au) | 12 346.00 | 49 |
| Osmium (Os) | 12 860.00 | 307 |

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Low-thrust vehicles

- Low-thrust orbital transfers offer increased mission opportunities
 - Electric propulsion is increasing in capability
 - Offers much higher specific impulse than chemical engines
 - Requires much longer operating periods for maneuvers
 - Enables long duration missions with frequent thrusting



Challenges for Optimal Transfer Design

- Optimal Trajectory Design
 - Orbital dynamics are nonlinear and chaotic
 - Very sensitive to initial conditions
 - Intuition required by designer to enable convergence
- Transfers using low-thrust propulsion
 - Requires long periods of thrusting/coasting
 - Small perturbations require accurate numerical integration
 - Difficult to capture the long-term effects accurately
- Direct Optimal Control
 - Reformulate problem as parameter optimization
 - Allows for use of nonlinear programming methods
 - High dimensional problem and computationally intensive
 - Results in suboptimal solutions due to discretization

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Gravitational Modeling

- Asteroids are extended bodies not point masses
- Spherical Harmonic - only valid outside of circumscribing sphere

$$U(r) = \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{R}{r}\right)^n P_{n,m}(\sin \phi) \{C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda)\}$$

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 - Model switching at circumscribing sphere
 - Coefficient matching used to ensure continuity

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Polyhedron Gravitation Model

- Potential is a function of only the shape model
- Globally valid, closed-form expression of potential
- Exact potential assumes a constant density
- Accuracy solely dependent on shape model

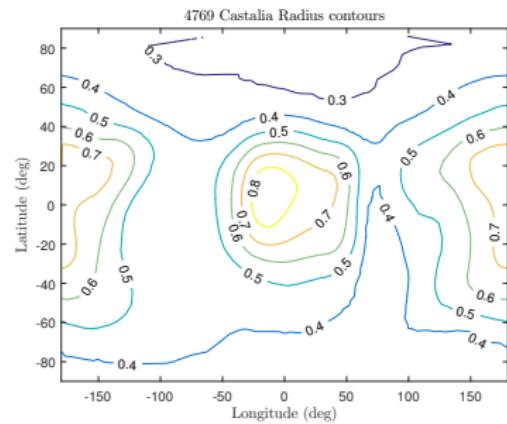
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$$U(\mathbf{r}) = \frac{1}{2}G\sigma \sum_{e \in \text{edges}} \mathbf{r}_e \cdot \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e - \frac{1}{2}G\sigma \sum_{f \in \text{faces}} \mathbf{r}_f \cdot \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f$$

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Equations of Motion

- Many similarities to the three-body problem
- Equations are also defined in body fixed frame

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g}(\mathbf{r}) + \mathbf{h}(\mathbf{v}) + \mathbf{u} \end{bmatrix}$$

- Dynamics allow for a single integral of motion
- Differential correction used to find periodic orbits

$$J(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\omega^2(x^2 + y^2) + U(\mathbf{r}) - \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

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Proposed Approach

- Reachability set on Poincaré section allows for systematic transfer design
 - Transfer design on lower dimensional subspace
 - Simple method to incorporate effects of low-thrust
 - Avoids the issue of determining initial conditions
- Extension of previous work in planar three-body problem

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Poincaré map

- Intersection of a periodic orbit with a lower dimensional subspace, called the Poincaré section
 - Can be considered a discrete map
- Useful for investigating the stability and structure
- Define a Poincaré section Σ
 - Used for initial and target periodic orbits
 - Subspace for the reachability set

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \mid y(t_f) = 0\}$$

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Reachability Set

- Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(\mathbf{x}_0, \mathcal{U}, t_f) = \{\mathbf{x}_f \subseteq \mathcal{X} | \exists \mathbf{u} \in \mathcal{U}, \mathbf{x}(t_f) = \mathbf{x}_f\}$$

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- We extend its use to the design of spacecraft transfers

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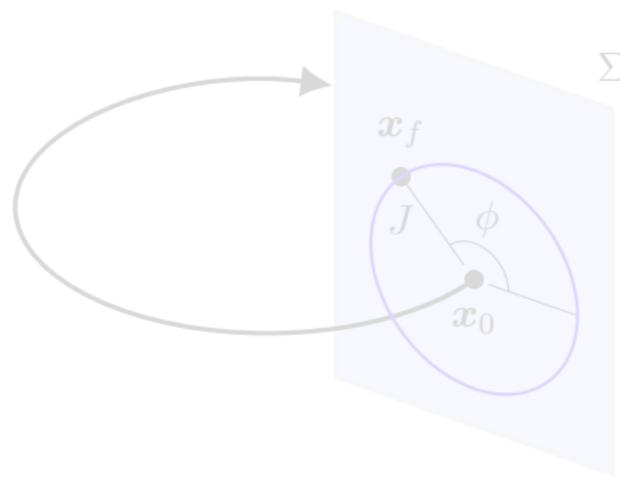
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Reachability Set on Poincaré section

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- Control input is chosen to enlarge the reachable set

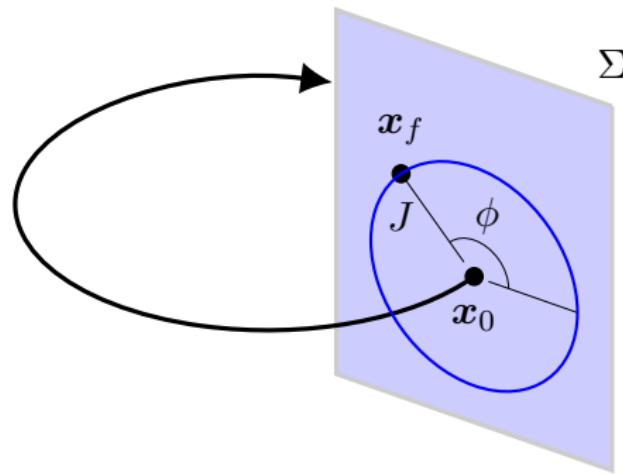


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Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2} (\mathbf{x}(t_f) - \mathbf{x}_n(t_f))^T Q (\mathbf{x}(t_f) - \mathbf{x}_n(t_f))$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$m_1 = y = 0$$

$$m_2 = (\sin \phi_{1_d}) (x_1^2 + x_2^2 + x_3^2 + x_4^2) - x_1^2 = 0$$

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- Control constraint used to emulate realistic system

$$c(\mathbf{u}) = \mathbf{u}^T \mathbf{u} - u_m^2 \leq 0$$

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Two Point Boundary Value Problem

- Multiple shooting used to solve necessary conditions
- Approximate the reachable set via ϕ_1, ϕ_2, ϕ_3
- From the reachable set we chose the state which minimizes d
- Compute another reachable set if target is not feasible

$$d = \sqrt{k_x (x_f - x_t)^2 + k_z (z_f - z_t)^2 + k_{\dot{x}} (\dot{x}_f - \dot{x}_t)^2 + k_{\dot{z}} (\dot{z}_f - \dot{z}_t)^2}$$

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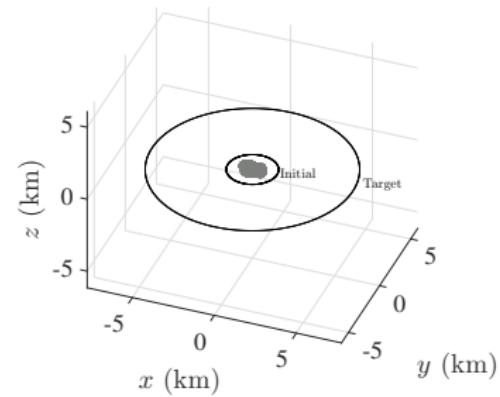
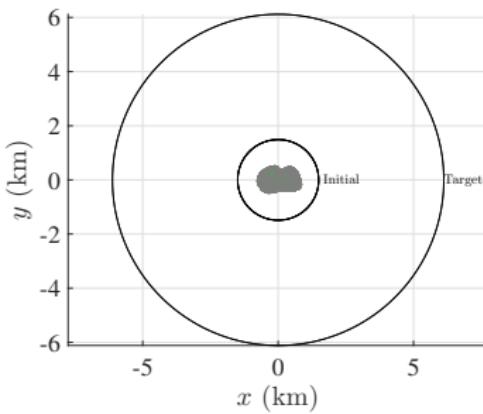
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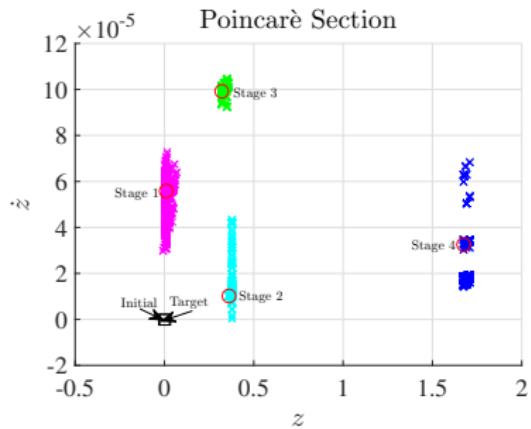
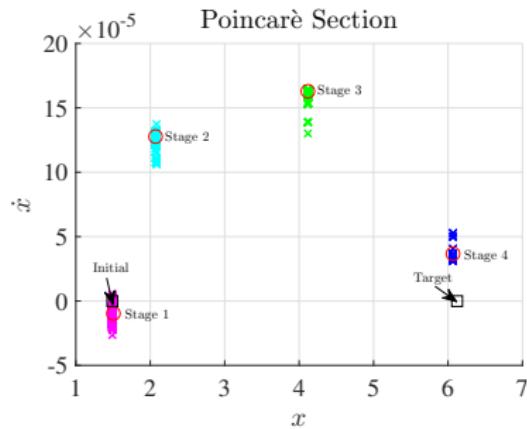
Transfer Objective

- Goal is to transfer between two equatorial periodic orbits
- Typical scenario during study of an asteroid



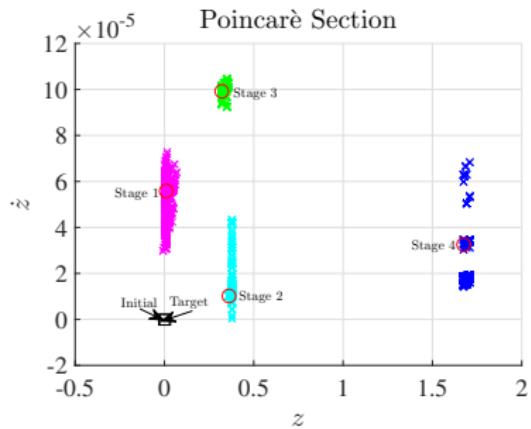
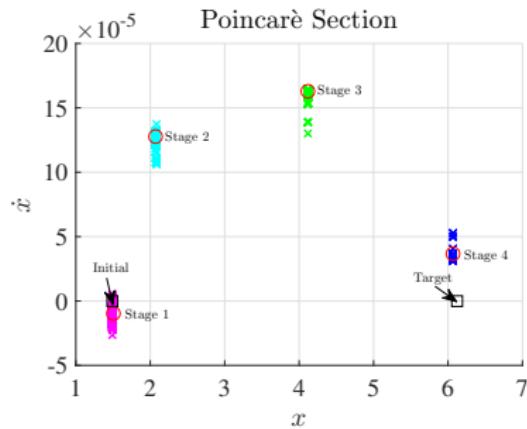
Simulation

- Generate the reachability set through discretization of ϕ_i
- Visualize $\Sigma \in \mathbb{R}^4$ through the use of two 2-D sections
- Control input allows for large deviation in velocity components



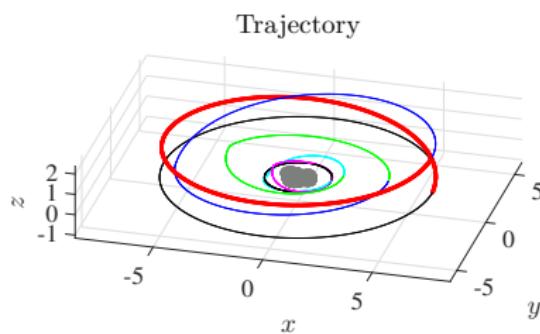
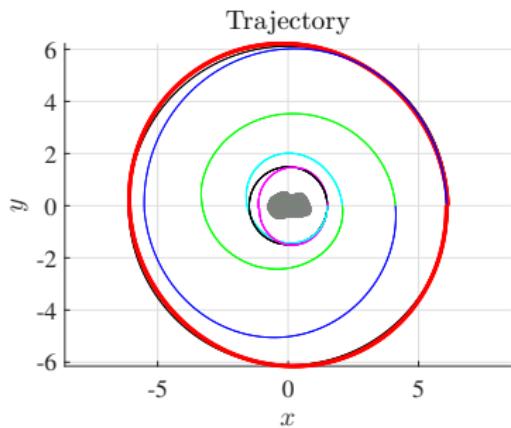
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Simulation

- Four iterations of the reachable state to meet the target set
- Final transfer is computed with a fixed terminal state constraint



Complete transfer

- We can visualize the complete trajectory in both the body and inertial frames

Conclusions

- Transfer using multiple iterations of the reachability set
- Alleviates the need for selecting accurate initial guesses
- Gives insight into the possible motion of the spacecraft
- Extension of work in the planar three-body problem
- Future research goals
 - Apply Computational Geometric Optimal Control
 - Lyapunov based feedback control for orbital transfers

Thank you

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