

Low-Thrust Trajectory Design Using Reachability Sets near Asteroid 4769 Castalia

Shankar Kulumani

Flight Dynamics & Control Lab

THE GEORGE WASHINGTON UNIVERSITY

WASHINGTON, DC

Asteroid Missions

- Science - insight into the early formation of the solar system
- Mining - vast quantities of useful materials
- Impact - high risk from hazardous near-Earth asteroids



Asteroid Mining

- Useful materials can be extracted from asteroids to support:
 - Propulsion, construction, life support, agriculture, and precious/strategic metals
- Commercialization of near-Earth asteroids is feasible¹

Element	Price (\$/kg)	Sales (\$M/yr)
Phosphorous (P)	0.08	2167
Gallium (Ga)	300.00	1544
Germanium (Ge)	745.00	6145
Platinum (Pt)	12 394.00	1705
Gold (Au)	12 346.00	49
Osmium (Os)	12 860.00	307

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Gravitational Modeling

- Potential is a function of only the shape model
- Globally valid, closed-form expression of potential
- Exact potential assumes a constant density
- Accuracy solely dependent on shape model

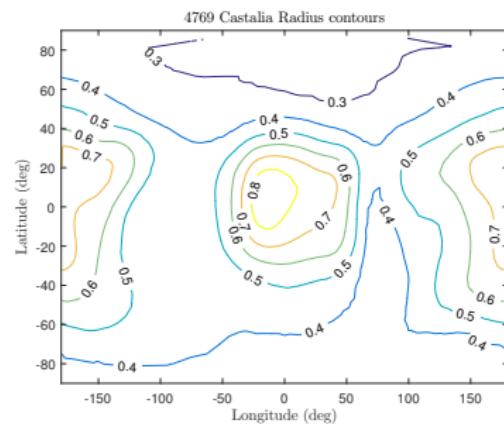
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$$U(\mathbf{r}) = \frac{1}{2}G\sigma \sum_{e \in \text{edges}} \mathbf{r}_e \cdot \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e - \frac{1}{2}G\sigma \sum_{f \in \text{faces}} \mathbf{r}_f \cdot \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f$$

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Equations of Motion

- Many similarities to the three-body problem
- Equations are also defined in rotating frame

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g}(\mathbf{r}) + \mathbf{h}(\mathbf{v}) + \mathbf{u} \end{bmatrix}$$

- Dynamics allow for a single integral of motion
- Differential correction used to find periodic orbits

$$J(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\omega^2(x^2 + y^2) + U(\mathbf{r}) - \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

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Proposed Approach

- Reachability set on Poincaré section allows for systematic transfer design
 - Transfer design on lower dimensional subspace
 - Simple method to incorporate effects of low-thrust
 - Avoids the issue of determining initial conditions
- Extension of previous work in planar three-body problem

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Reachability Sets and the Poincaré map

- Intersections of trajectories with a lower dimensional surface
 - The Poincaré section, Σ , reduces the computational complexity
 - Useful for investigating the stability and structure of the system

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \mid y(t_f) = 0\}$$

- Poincaré section serves as subspace for the reachability set
 - Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(x_0, \mathcal{U}, t_f) = \{x_f \subseteq \mathcal{X} \mid \exists \mathbf{u} \in \mathcal{U}, x(t_f) = x_f\}$$

- Directly derivable from optimal control
- Frequently used for safety planning, e.g. air traffic avoidance
- We extend its use to the design of spacecraft transfers

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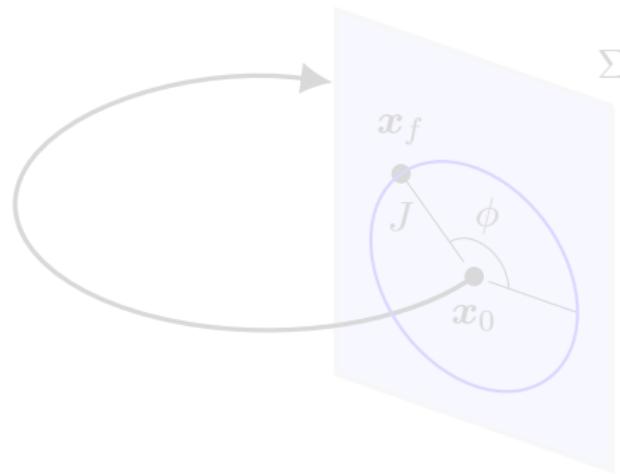
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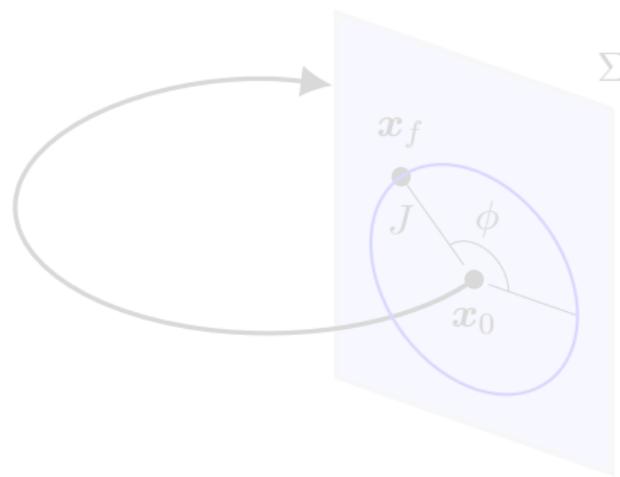


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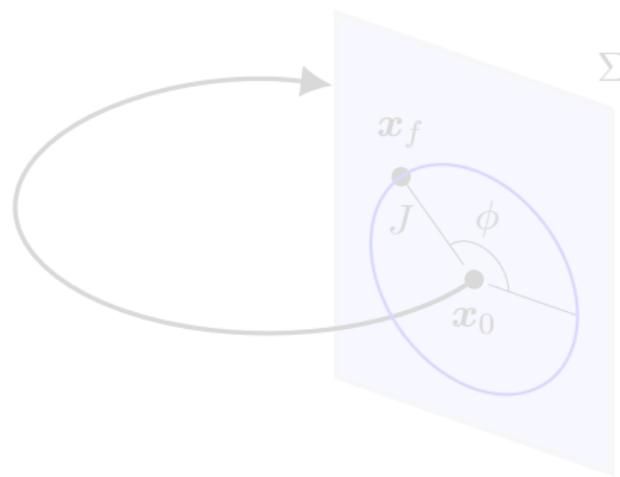


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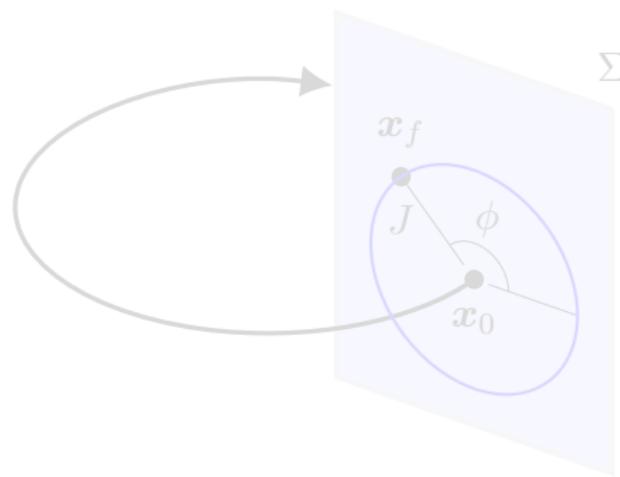


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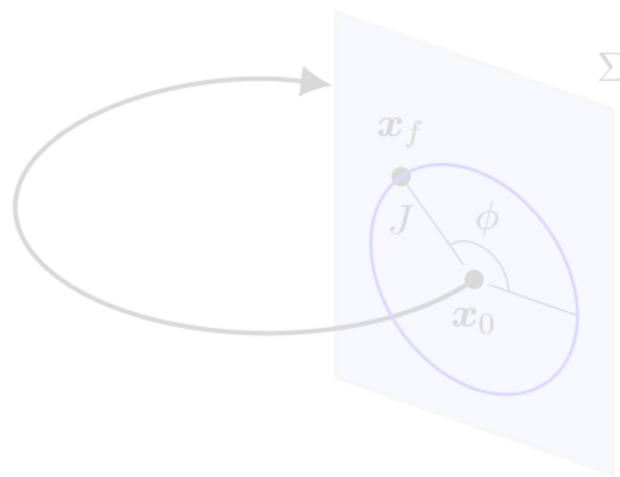


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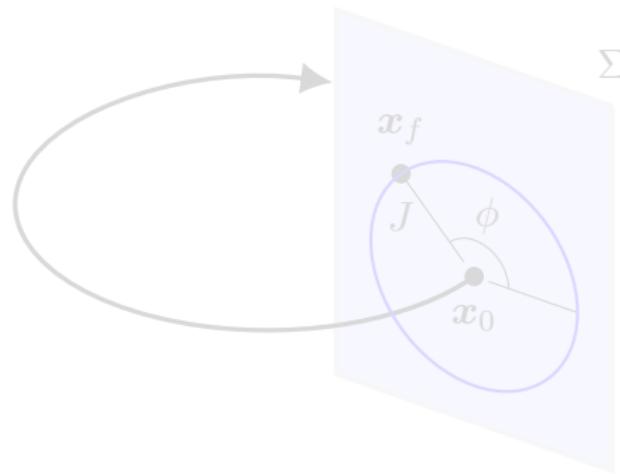


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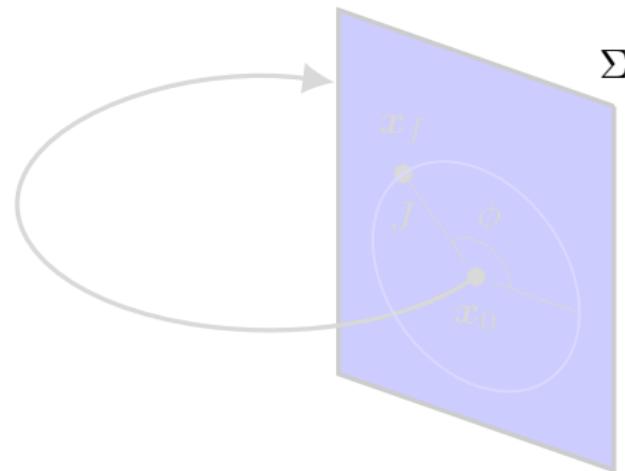


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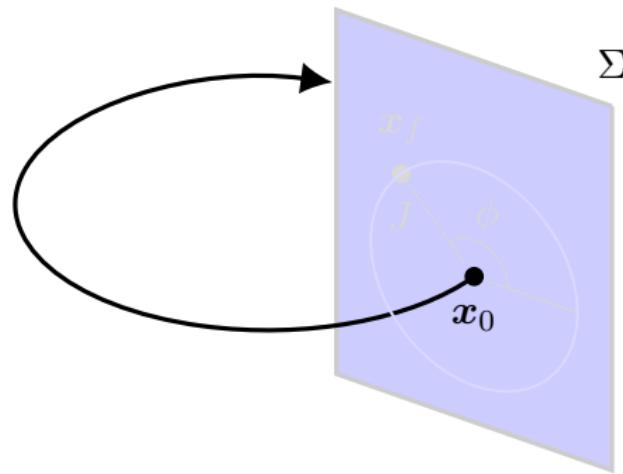


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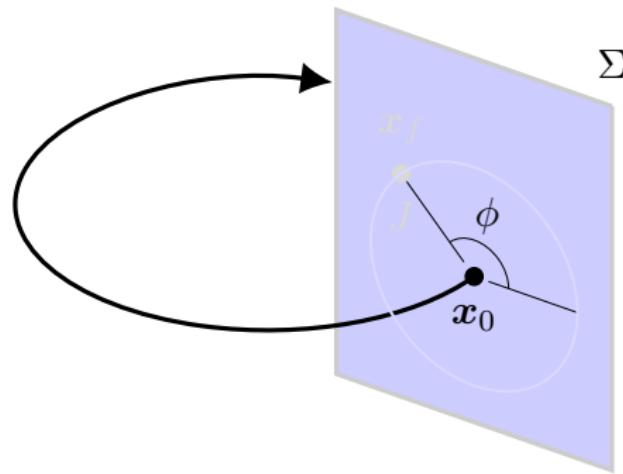


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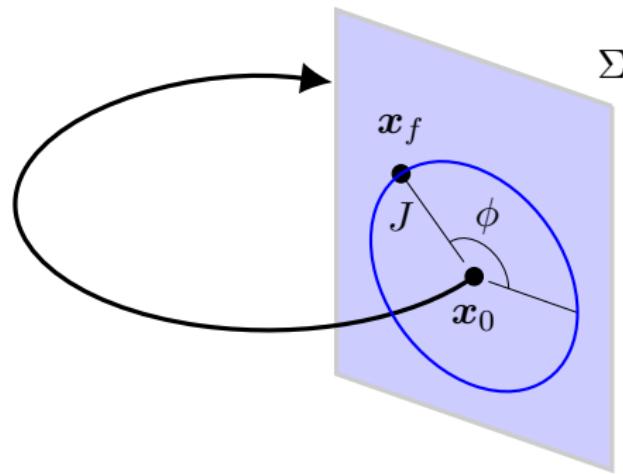


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Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2} (\mathbf{x}(t_f) - \mathbf{x}_n(t_f))^T Q (\mathbf{x}(t_f) - \mathbf{x}_n(t_f))$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$m_1 = y = 0$$

$$m_2 = (\sin \phi_{1_d}) (x_1^2 + x_2^2 + x_3^2 + x_4^2) - x_1^2 = 0$$

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- Control constraint used to emulate realistic system

$$c(\mathbf{u}) = \mathbf{u}^T \mathbf{u} - u_m^2 \leq 0$$

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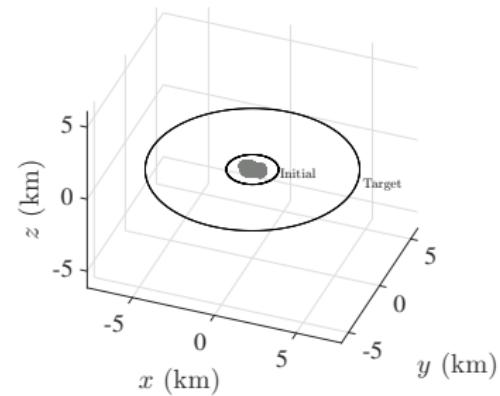
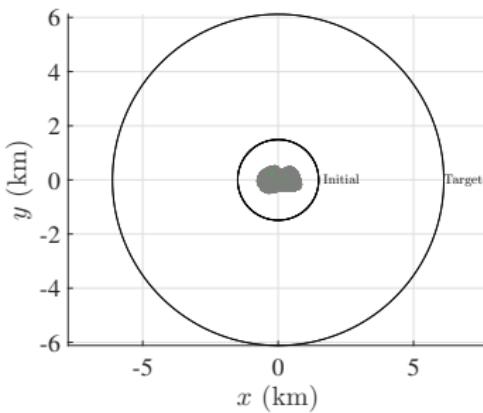
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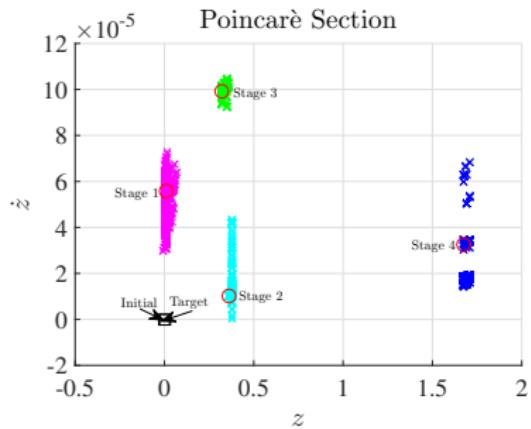
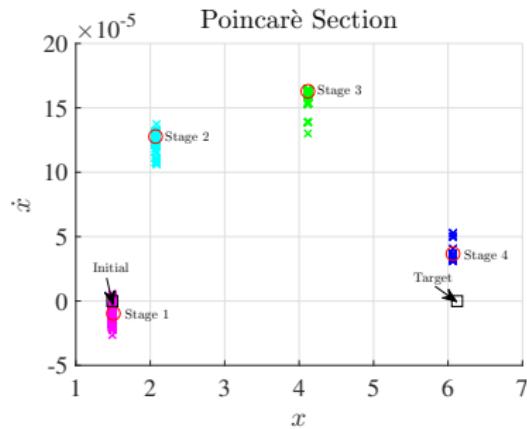
Transfer Objective

- Goal is to transfer between two equatorial periodic orbits
- Typical scenario during study of an asteroid



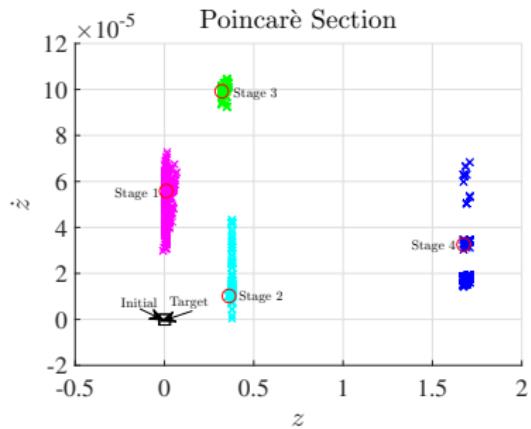
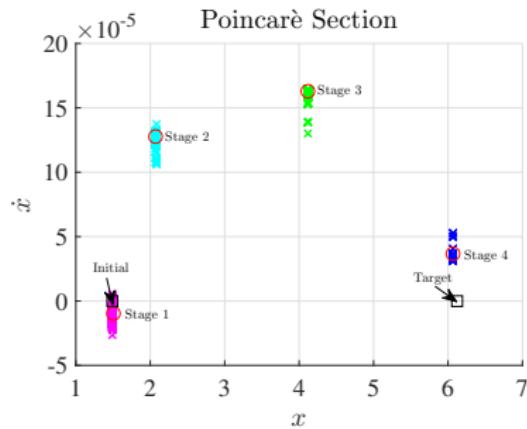
Simulation

- Generate the reachability set through discretization of ϕ_i
- Visualize $\Sigma \in \mathbb{R}^4$ through the use of two 2-D sections
- Control input allows for large deviation in velocity components



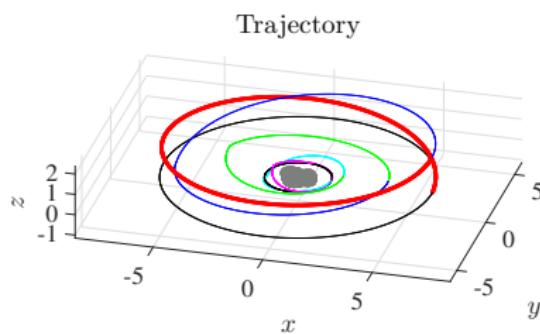
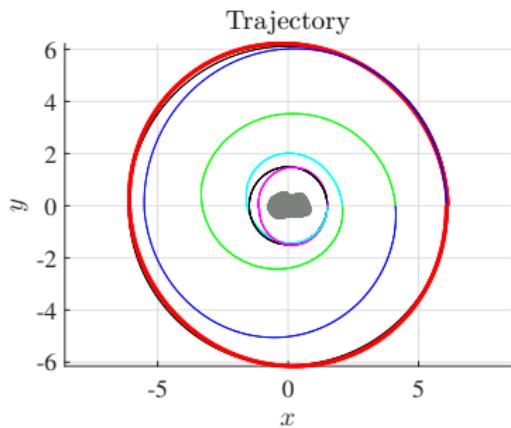
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Simulation

- Four iterations of the reachable state to meet the target set
- Final transfer is computed with a fixed terminal state constraint



Complete transfer

- We can visualize the complete trajectory in both the body and inertial frames

Conclusions

- Transfer using multiple iterations of the reachability set
- Alleviates the need for selecting accurate initial guesses
- Gives insight into the possible motion of the spacecraft
- Extension of work in the planar three-body problem
- Future research goals
 - Apply Computational Geometric Optimal Control
 - Lyapunov based feedback control for orbital transfers

Thank you

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