Low-Thrust Trajectory Design Using Reachability Sets near Asteroid 4769 Castalia

Shankar Kulumani and Taeyoung Lee

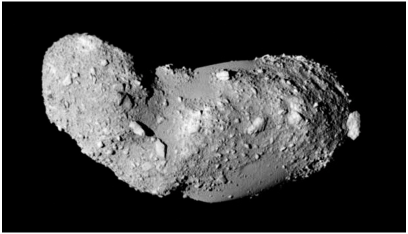
Flight Dynamics & Control Lab

THE GEORGE WASHINGTON UNIVERSITY

WASHINGTON, DC

Asteroid Missions

- Science - insight into the early formation of the solar system
- Mining - vast quantities of useful materials
- Impact - high risk from hazardous near-Earth asteroids



Asteroid Mining

- Useful materials can be extracted from asteroids to support:
  - Propulsion, construction, life support, agriculture, and precious/strategic metals
- Commercialization of near-Earth asteroids is feasible[1]

Element	Price ($/kg)	Sales ($M/yr)
Phosphorous (P)	0.08	2167
Gallium (Ga)	300.00	1544
Germanium (Ge)	745.00	6145
Platinum (Pt)	12 394.00	1705
Gold (Au)	12 346.00	49
Osmium (Os)	12 860.00	307

[1] Shane D Ross. "Near-Earth Asteroid Mining". In: *Space* (2001).

Asteroid Mining

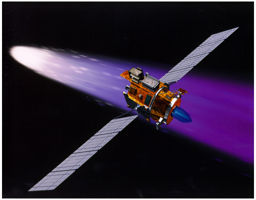
- Useful materials can be extracted from asteroids to support:
  - Propulsion, construction, life support, agriculture, and precious/strategic metals
- Commercialization of near-Earth asteroids is feasible[1]

Element	Price ($/kg)	Sales ($M/yr)
Phosphorous (P)	0.08	2167
Gallium (Ga)	300.00	1544
Germanium (Ge)	745.00	6145
Platinum (Pt)	12 394.00	1705
Gold (Au)	12 346.00	49
Osmium (Os)	12 860.00	307

[1] Shane D Ross. "Near-Earth Asteroid Mining". In: *Space* (2001).

Low-thrust vehicles

- Low-thrust orbital transfers offer increased mission oportunities
    - Electric propulsion is increasing in capability
    - Offers much higher specific impulse than chemical engines
    - Requires much longer operating periods for maneuvers
    - Enables long duration missions with frequent thrusting

Challenges for Optimal Transfer Design

- Optimal Trajectory Design
  - Orbital dynamics are nonlinear and chaotic
  - Very sensitive to initial conditions
  - Intuition required by designer to enable convergence
- Transfers using low-thrust propulsion
  - Requires long periods of thrusting/coasting
  - Small perturbations require accurate numerical integration
  - Difficult to capture the long-term effects accurately
- Direct Optimal Control
  - Reformulate problem as parameter optimization
  - Allows for use of nonlinear programming methods
  - High dimensional problem and computationally intensive
  - Results in suboptimal solutions due to discretization

Challenges for Optimal Transfer Design

- Optimal Trajectory Design
  - Orbital dynamics are nonlinear and chaotic
  - Very sensitive to initial conditions
  - Intuition required by designer to enable convergence
- Transfers using low-thrust propulsion
  - Requires long periods of thrusting/coasting
  - Small perturbations require accurate numerical integration
  - Difficult to capture the long-term effects accurately
- Direct Optimal Control
  - Reformulate problem as parameter optimization
  - Allows for use of nonlinear programming methods
  - High dimensional problem and computationally intensive
  - Results in suboptimal solutions due to discretization

Challenges for Optimal Transfer Design

- Optimal Trajectory Design
  - Orbital dynamics are nonlinear and chaotic
  - Very sensitive to initial conditions
  - Intuition required by designer to enable convergence
- Transfers using low-thrust propulsion
  - Requires long periods of thrusting/coasting
  - Small perturbations require accurate numerical integration
  - Difficult to capture the long-term effects accurately
- Direct Optimal Control
  - Reformulate problem as parameter optimization
  - Allows for use of nonlinear programming methods
  - High dimensional problem and computationally intensive
  - Results in suboptimal solutions due to discretization

Gravitational Modeling

- Asteroids are extended bodies not point masses
- Spherical Harmonic - only valid outside of circumscribing sphere

$$U(\boldsymbol{r}) = \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{R}{r}\right)^{n} P_{n,m}(\sin\phi) \left\{C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right\}$$

- Infinite series is an approximation and adds complexity
  - Model switching at circumscribing sphere
  - Coefficient matching used to ensure continuity

Gravitational Modeling

- Asteroids are extended bodies not point masses
- Spherical Harmonic - only valid outside of circumscribing sphere

$$U(\boldsymbol{r}) = \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{R}{r} \right)^n P_{n,m}(\sin\phi) \left\{ C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda) \right\}$$

- Infinite series is an approximation and adds complexity
  - Model switching at circumscribing sphere
  - Coefficient matching used to ensure continuity

Gravitational Modeling

- Asteroids are extended bodies not point masses
- Spherical Harmonic - only valid outside of circumscribing sphere

$$U(\boldsymbol{r}) = \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{R}{r} \right)^n P_{n,m}(\sin\phi) \left\{ C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda) \right\}$$

- Infinite series is an approximation and adds complexity
  - Model switching at circumscribing sphere
  - Coefficient matching used to ensure continuity

Polyhedron Gravitation Model

- Potential is a function of only the shape model
- Globally valid, closed-form expression of potential
- Exact potential assumes a constant density
- Accuracy solely dependent on shape model

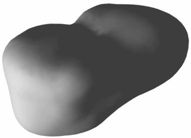
Polyhedron Gravitation Model

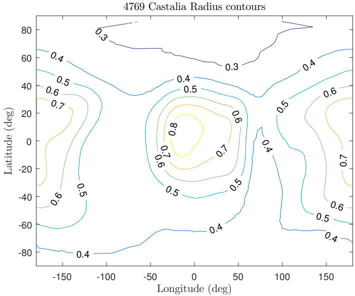
- Potential is a function of only the shape model
- Globally valid, closed-form expression of potential
- Exact potential assumes a constant density
- Accuracy solely dependent on shape model

$$U(\boldsymbol{r}) = \frac{1}{2} G\sigma \sum_{e \in \text{edges}} \boldsymbol{r}_e \cdot \boldsymbol{E}_e \cdot \boldsymbol{r}_e \cdot L_e - \frac{1}{2} G\sigma \sum_{f \in \text{faces}} \boldsymbol{r}_f \cdot \boldsymbol{F}_f \cdot \boldsymbol{r}_f \cdot \omega_f$$

Polyhedron Gravitation Model

- Potential is a function of only the shape model
- Globally valid, closed-form expression of potential
- Exact potential assumes a constant density
- Accuracy solely dependent on shape model





Equations of Motion

- Many similarities to the three-body problem
- Equations are also defined in body fixed frame

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{g}\left(\boldsymbol{r}\right) + \boldsymbol{h}\left(\boldsymbol{v}\right) + \boldsymbol{u} \end{bmatrix}$$

- Dynamics allow for a single integral of motion
- Differential correction used to find periodic orbits

$$J\left(\boldsymbol{r},\boldsymbol{v}\right) = \frac{1}{2}\omega^2\left(x^2+y^2\right) + U(\boldsymbol{r}) - \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

Equations of Motion

- Many similarities to the three-body problem
- Equations are also defined in body fixed frame

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{g}\left(\boldsymbol{r}\right) + \boldsymbol{h}\left(\boldsymbol{v}\right) + \boldsymbol{u} \end{bmatrix}$$

- Dynamics allow for a single integral of motion
- Differential correction used to find periodic orbits

$$J\left(\boldsymbol{r}, \boldsymbol{v}\right) = \frac{1}{2}\omega^2\left(x^2 + y^2\right) + U(\boldsymbol{r}) - \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Proposed Approach

- Reachability set on Poincaré section allows for systematic transfer design
  - Transfer design on lower dimensional subspace
  - Simple method to incorporate effects of low-thrust
  - Avoids the issue of determining initial conditions
- Extension of previous work in planar three-body problem

Proposed Approach

- Reachability set on Poincaré section allows for systematic transfer design
  - Transfer design on lower dimensional subspace
  - Simple method to incorporate effects of low-thrust
  - Avoids the issue of determining initial conditions
- Extension of previous work in planar three-body problem

Poincaré map

- Intersection of a periodic orbit with a lower dimensional subspace, called the Poincaré section
  - Can be considered a discrete map
- Useful for investigating the stability and structure
- Define a Poincaré section Σ
  - Used for initial and target periodic orbits
  - Subspace for the reachability set

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \,|y(t_f) = 0\}$$

Poincaré map

- Intersection of a periodic orbit with a lower dimensional subspace, called the Poincaré section
  - Can be considered a discrete map
- Useful for investigating the stability and structure
- Define a Poincaré section Σ
  - Used for initial and target periodic orbits
  - Subspace for the reachability set

$$\Sigma = \{(x, \dot{x}, z, \dot{z})\,|y(t_f) = 0\}$$

Poincaré map

- Intersection of a periodic orbit with a lower dimensional subspace, called the Poincaré section
    - Can be considered a discrete map
- Useful for investigating the stability and structure
- Define a Poincaré section Σ
    - Used for initial and target periodic orbits
    - Subspace for the reachability set

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \, | y(t_f) = 0\}$$

Poincaré map

- Intersection of a periodic orbit with a lower dimensional subspace, called the Poincaré section
  - Can be considered a discrete map
- Useful for investigating the stability and structure
- Define a Poincaré section Σ
  - Used for initial and target periodic orbits
  - Subspace for the reachability set

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \,|y(t_f) = 0\}$$

Reachability Set

- Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(\boldsymbol{x}_0, \mathcal{U}, t_f) = \{\boldsymbol{x}_f \subseteq \mathcal{X} | \exists \boldsymbol{u} \in \mathcal{U}, \boldsymbol{x}(t_f) = \boldsymbol{x}_f\}$$

- Directly derivable from optimal control
- Frequently used for safety planning, e.g. air traffic collision avoidance
- We extend its use to the design of spacecraft transfers

Reachability Set

- Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(\boldsymbol{x}_0, \mathcal{U}, t_f) = \{\boldsymbol{x}_f \subseteq \mathcal{X} | \exists \boldsymbol{u} \in \mathcal{U}, \boldsymbol{x}(t_f) = \boldsymbol{x}_f\}$$

- Directly derivable from optimal control
- Frequently used for safety planning, e.g. air traffic collision avoidance
- We extend its use to the design of spacecraft transfers

Reachability Set

- Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(\boldsymbol{x}_0, \mathcal{U}, t_f) = \{\boldsymbol{x}_f \subseteq \mathcal{X} | \exists \boldsymbol{u} \in \mathcal{U}, \boldsymbol{x}(t_f) = \boldsymbol{x}_f\}$$

- Directly derivable from optimal control
- Frequently used for safety planning, e.g. air traffic collision avoidance
- We extend its use to the design of spacecraft transfers

Reachability Set on Poincaré section

- Generate the reachability set on a Poincaré section

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \,|y(t_f) = 0\}$$

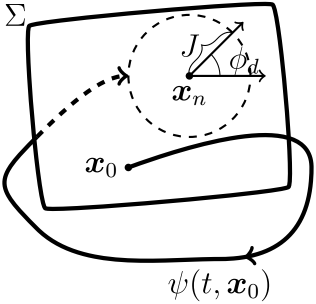
- Control input is chosen to enlarge the reachable set

Reachability Set on Poincaré section

- Generate the reachability set on a Poincaré section

$$\Sigma = \{(x, \dot{x}, z, \dot{z}) \,|y(t_f) = 0\}$$

- Control input is chosen to enlarge the reachable set



Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)^T Q \left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$\begin{aligned} m_1 &= y = 0 \\ m_2 &= (\sin\phi_{1_d})\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - x_1^2 = 0 \\ m_3 &= (\sin\phi_{2_d})\left(x_2^2 + x_3^2 + x_4^2\right) - x_2^2 = 0 \\ m_4 &= (\sin\phi_{3_d})\left(2x_3^2 + 2x_3\sqrt{x_4^2 + 2x_4^2}\right) - x_3 - \sqrt{x_4^2 + x_3^2} = 0 \end{aligned}$$

- Control constraint used to emulate realistic system

$$c(\boldsymbol{u}) = \boldsymbol{u}^T\boldsymbol{u} - u_m^2 \leq 0$$

Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)^T Q \left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$\begin{aligned}
m_1 &= y = 0 \\
m_2 &= (\sin\phi_{1_d})\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - x_1^2 = 0 \\
m_3 &= (\sin\phi_{2_d})\left(x_2^2 + x_3^2 + x_4^2\right) - x_2^2 = 0 \\
m_4 &= (\sin\phi_{3_d})\left(2x_3^2 + 2x_3\sqrt{x_4^2 + 2x_4^2}\right) - x_3 - \sqrt{x_4^2 + x_3^2} = 0
\end{aligned}$$

- Control constraint used to emulate realistic system

$$c(\boldsymbol{u}) = \boldsymbol{u}^T\boldsymbol{u} - u_m^2 \leq 0$$

Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)^T Q \left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$\begin{aligned}
m_1 &= y = 0 \\
m_2 &= (\sin\phi_{1_d})\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - x_1^2 = 0 \\
m_3 &= (\sin\phi_{2_d})\left(x_2^2 + x_3^2 + x_4^2\right) - x_2^2 = 0 \\
m_4 &= (\sin\phi_{3_d})\left(2x_3^2 + 2x_3\sqrt{x_4^2 + 2x_4^2}\right) - x_3 - \sqrt{x_4^2 + x_3^2} = 0
\end{aligned}$$

- Control constraint used to emulate realistic system

$$c(\boldsymbol{u}) = \boldsymbol{u}^T\boldsymbol{u} - u_m^2 \leq 0$$

Two Point Boundary Value Problem

- Multiple shooting used to solve necessary conditions
- Approximate the reachable set via ϕ_1, ϕ_2, ϕ_3
- From the reachable set we chose the state which minimizes d
- Compute another reachable set if target is not feasible

$$d = \sqrt{k_x\left(x_f - x_t\right)^2 + k_z\left(z_f - z_t\right)^2 + k_{\dot{x}}\left(\dot{x}_f - \dot{x}_t\right)^2 + k_{\dot{z}}\left(\dot{z}_f - \dot{z}_t\right)^2}$$

Two Point Boundary Value Problem

- Multiple shooting used to solve necessary conditions
- Approximate the reachable set via ϕ_1, ϕ_2, ϕ_3
- From the reachable set we chose the state which minimizes d
- Compute another reachable set if target is not feasible

$$d = \sqrt{k_x\left(x_f - x_t\right)^2 + k_z\left(z_f - z_t\right)^2 + k_{\dot{x}}\left(\dot{x}_f - \dot{x}_t\right)^2 + k_{\dot{z}}\left(\dot{z}_f - \dot{z}_t\right)^2}$$

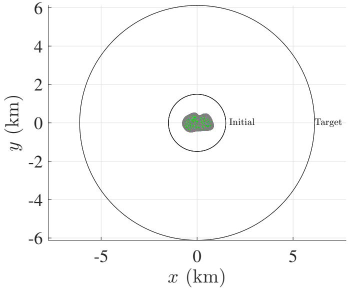
Two Point Boundary Value Problem

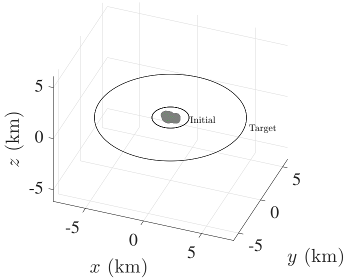
- Multiple shooting used to solve necessary conditions
- Approximate the reachable set via ϕ_1, ϕ_2, ϕ_3
- From the reachable set we chose the state which minimizes d
- Compute another reachable set if target is not feasible

$$d = \sqrt{k_x \left(x_f - x_t\right)^2 + k_z \left(z_f - z_t\right)^2 + k_{\dot{x}} \left(\dot{x}_f - \dot{x}_t\right)^2 + k_{\dot{z}} \left(\dot{z}_f - \dot{z}_t\right)^2}$$

Transfer Objective

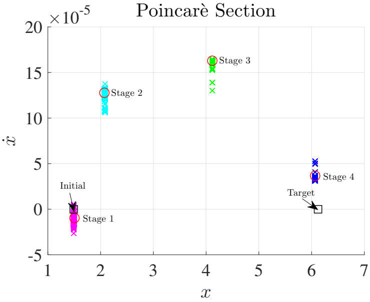
- Goal is to transfer between two equatorial periodic orbits
- Typical scenario during study of an asteroid

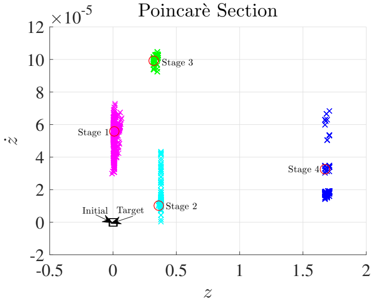




Simulation

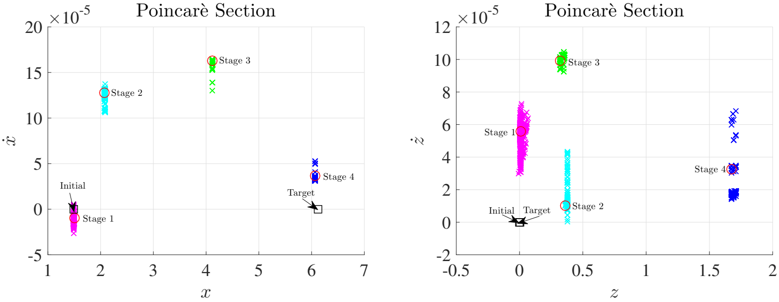
- Generate the reachability set through discretization of ϕ_i
- Visualize $\Sigma \in \mathbb{R}^4$ through the use of two 2-D sections
- Control input allows for large deviation in velocity components





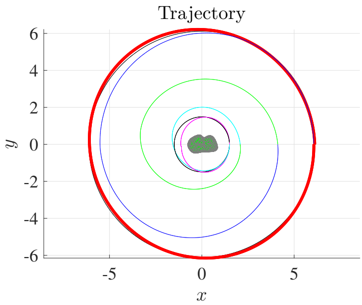
Simulation

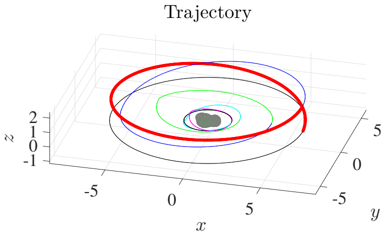
- Generate the reachability set through discretization of ϕ_i
- Visualize $\Sigma \in \mathbb{R}^4$ through the use of two 2-D sections
- Control input allows for large deviation in velocity components



Simulation

- Four iterations of the reachable state to meet the target set
- Final transfer is computed with a fixed terminal state constraint





Complete transfer

- We can visualize the complete trajectory in both the body and inertial frames

Conclusions

- Transfer using multiple iterations of the reachability set
- Alleviates the need for selecting accurate initial guesses
- Gives insight into the possible motion of the spacecraft
- Extension of work in the planar three-body problem
- Future research goals
  - Apply Computational Geometric Optimal Control
  - Lyapunov based feedback control for orbital transfers

Thank you

Flight Dynamics & Control Lab
Mechanical & Aerospace Engineering
School of Engineering & Applied Science

THE GEORGE WASHINGTON UNIVERSITY

WASHINGTON, DC

https://fdcl.seas.gwu.edu