Low-Thrust Trajectory Design Using Reachability Sets near Asteroid 4769 Castalia

Shankar Kulumani

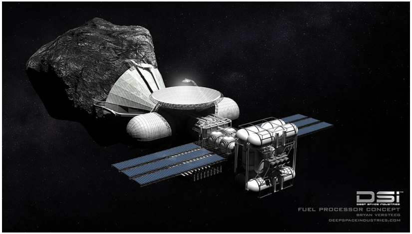
Flight Dynamics & Control Lab

THE GEORGE WASHINGTON UNIVERSITY

WASHINGTON, DC

Asteroid Missions

- Science - insight into the early formation of the solar system
- Mining - vast quantities of useful materials
- Impact - high risk from hazardous near-Earth asteroids



Asteroid Mining

- Useful materials can be extracted from asteroids to support:
  - Propulsion, construction, life support, agriculture, and precious/strategic metals
- Commercialization of near-Earth asteroids is feasible[1]

Element	Price ($/kg)	Sales ($M/yr)
Phosphorous (P)	0.08	2167
Gallium (Ga)	300.00	1544
Germanium (Ge)	745.00	6145
Platinum (Pt)	12 394.00	1705
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- Potential is a function of only the shape model
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$$U(\boldsymbol{r}) = \frac{1}{2} G\sigma \sum_{e \in \text{edges}} \boldsymbol{r}_e \cdot \boldsymbol{E}_e \cdot \boldsymbol{r}_e \cdot L_e - \frac{1}{2} G\sigma \sum_{f \in \text{faces}} \boldsymbol{r}_f \cdot \boldsymbol{F}_f \cdot \boldsymbol{r}_f \cdot \omega_f$$

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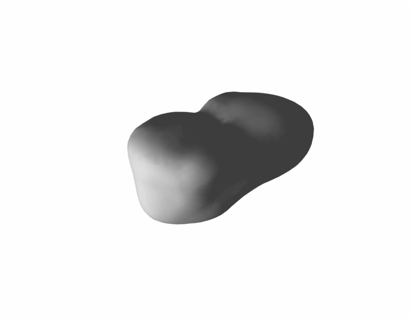

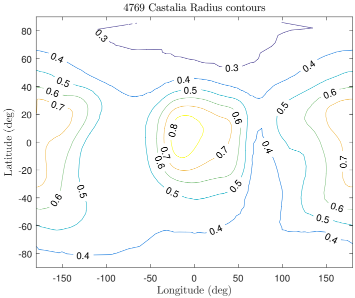

Equations of Motion

- Many similarities to the three-body problem
- Equations are also defined in rotating frame

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{g}\left(\boldsymbol{r}\right) + \boldsymbol{h}\left(\boldsymbol{v}\right) + \boldsymbol{u} \end{bmatrix}$$

- Dynamics allow for a single integral of motion
- Differential correction used to find periodic orbits

$$J\left(\boldsymbol{r}, \boldsymbol{v}\right) = \frac{1}{2}\omega^2\left(x^2 + y^2\right) + U(\boldsymbol{r}) - \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

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Proposed Approach

- Reachability set on Poincaré section allows for systematic transfer design
  - Transfer design on lower dimensional subspace
  - Simple method to incorporate effects of low-thrust
  - Avoids the issue of determining initial conditions
- Extension of previous work in planar three-body problem

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Reachability Sets and the Poincaré map

- Intersections of trajectories with a lower dimensional surface
  - The Poincaré section, Σ, reduces the computational complexity
  - Useful for investigating the stability and structure of the system

$$\Sigma = \{(x, \dot{x}, z, \dot{z})\,|y(t_f) = 0\}$$

- Poincaré section serves as subspace for the reachability set
  - Set of states achievable from a given initial condition over fixed t_f s.t. maximum control constraint

$$R(\boldsymbol{x}_0, \mathcal{U}, t_f) = \{\boldsymbol{x}_f \subseteq \mathcal{X} | \exists \boldsymbol{u} \in \mathcal{U}, \boldsymbol{x}(t_f) = \boldsymbol{x}_f\}$$

  - Directly derivable from optimal control
  - Frequently used for safety planning, e.g. air traffic avoidance
  - We extend its use to the design of spacecraft transfers

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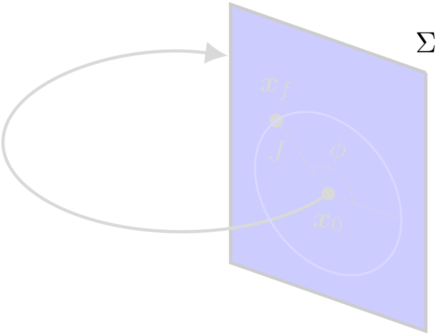

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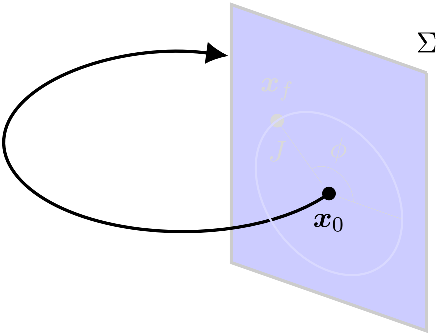

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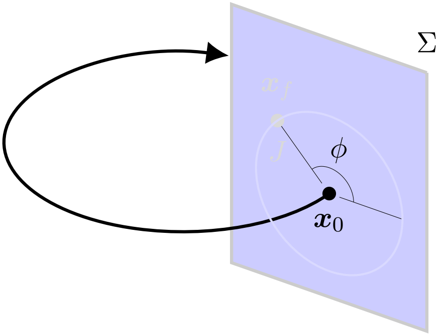

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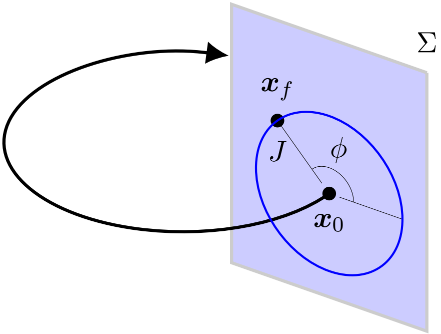

Optimal Control Problem

- Reachability defined as distance between controlled and uncontrolled states

$$J = -\frac{1}{2}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)^T Q \left(\boldsymbol{x}(t_f) - \boldsymbol{x}_n(t_f)\right)$$

- Terminal constraints used to ensure correct section and specific direction on $\Sigma \in \mathbb{R}^4$

$$\begin{aligned}
m_1 &= y = 0 \\
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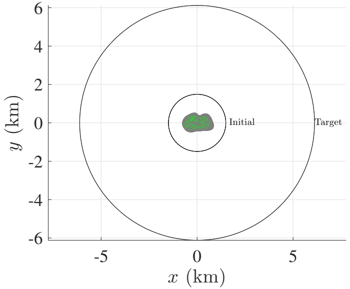
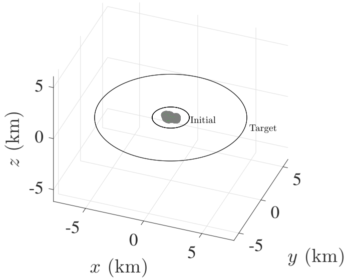
Transfer Objective

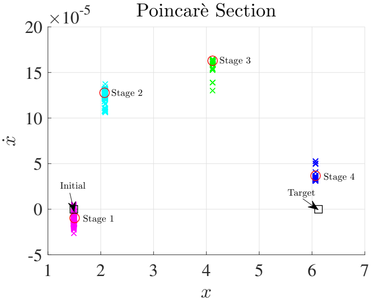
- Goal is to transfer between two equatorial periodic orbits
- Typical scenario during study of an asteroid





Simulation

- Generate the reachability set through discretization of ϕ_i
- Visualize $\Sigma \in \mathbb{R}^4$ through the use of two 2-D sections
- Control input allows for large deviation in velocity components



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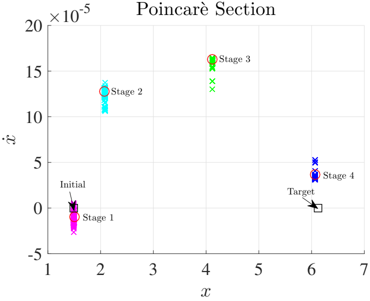
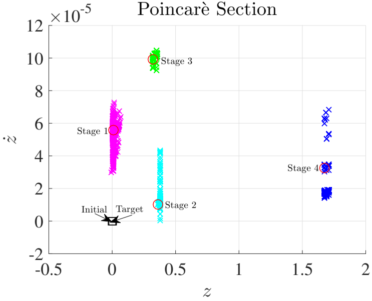
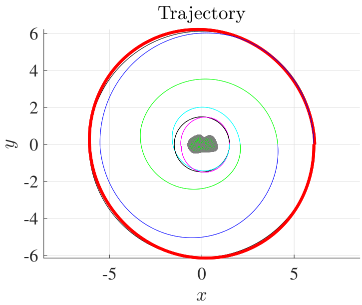
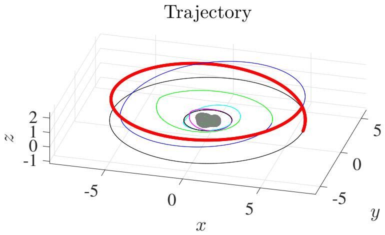




Simulation

- Four iterations of the reachable state to meet the target set
- Final transfer is computed with a fixed terminal state constraint





Complete transfer

- We can visualize the complete trajectory in both the body and inertial frames

Conclusions

- Transfer using multiple iterations of the reachability set
- Alleviates the need for selecting accurate initial guesses
- Gives insight into the possible motion of the spacecraft
- Extension of work in the planar three-body problem
- Future research goals
    - Apply Computational Geometric Optimal Control
    - Lyapunov based feedback control for orbital transfers

Thank you

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