1

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} \sin 3t \qquad y = \underbrace{[1 \ \ 2]}_{C} x \qquad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Y(s) = ?$$

$$sX(s) - X(0) = AX(s) + BU(s) \rightarrow sX(s) - AX(s) = BU(s) + X(0)$$

$$\rightarrow X(s) = [sI - A]^{-1}[BU(s) + X(0)]$$

$$\rightarrow Y(s) = C\{[sI - A]^{-1}[BU(s) + X(0)]\}$$

$$U(s) = \mathcal{L}[\sin 3t] = \frac{3}{s^2+9}$$

$$= [1 \ \ 2]\left(\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}\right\}^{-1} \cdot \left\{\begin{bmatrix} 3/s^2+9 \\ 3/s^2+9 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right)$$

$$= [1 \ \ 2]\left\{\begin{bmatrix} s-1 & -2 \\ 3 & s+1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{2s^2+21}{s^2+9} \\ \frac{s^2+12}{s^2+9} \end{bmatrix}\right\} = [1 \ \ 2]\left\{\begin{bmatrix} \frac{s+1}{s^2+5} & \frac{2}{s^2+5} \\ \frac{-3}{s^2+5} & \frac{s-1}{s^2+5} \end{bmatrix} \cdot \begin{bmatrix} (2s^2+21)/(s^2+9) \\ (s^2+12)/(s^2+9) \end{bmatrix}\right\}$$

$$= \frac{2\cdot(3/(s^2+9)+1)}{s^2+5} - \frac{6(3/(s^2+9)+2)}{s^2+5} + \frac{2[(3/(s^2+9)+1)\cdot(s-1)]}{s^2+5} + \frac{(3/(s^2+9)+2)(s+1)}{s^2+5}$$

$$= \boxed{\frac{4s^3 - 10s^2 + 45s - 105}{s^4 + 14s^2 + 45}}$$

2

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1} \qquad Y(s) = \frac{1}{(s+1)(s^3+10s^2+26s+12)} = \frac{1}{s^4+11s^3+36s^2+38s+12}$$

3

$$\mathcal{L}[u(t)] = 1/s \qquad Y(s) = \frac{1}{s(s^2+3s+2)}$$

$$y(t) = \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2}$$

4

state transition matrix: $\Phi(t) = e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$

$$\Phi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

state history: $x(t) = \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$

output history: $y(t) = 2\sin(t) - 3\cos(t) + 3$

5

$$\Phi = \begin{bmatrix} e^{-2t} & \dfrac{e^{-t/2}\left(\cos(t\sqrt{23}/2) + (13\cdot\sqrt{23}\cdot\sin(t\sqrt{23}/2)/23)\right)}{8} - e^{-2t} & \dfrac{e^{-2t} - e^{-t/2}\left(\cos\left(\frac{t\sqrt{23}}{2}\right) - (3\sqrt{23}\sin\left(\frac{t\sqrt{23}}{2}\right)/23)\right)}{8} \\ 0 & \dfrac{e^{-t/2}\left(\cos\left(\frac{t\sqrt{23}}{2}\right) + \sqrt{23}\sin\left(\frac{t\sqrt{23}}{2}\right)\right)}{23} & \dfrac{2\sqrt{23}\,e^{-t/2}\sin\left(\frac{t\sqrt{23}}{2}\right)}{23} \\ 0 & \dfrac{-12\cdot\sqrt{23}\,e^{-t/2}\sin\left(\frac{t\sqrt{23}}{2}\right)}{23} & \dfrac{e^{-t/2}\left(\cos\left(\frac{t\sqrt{23}}{2}\right) - \sqrt{23}\cdot\sin\left(\frac{t\sqrt{23}}{2}\right)\right)}{23} \end{bmatrix}$$

state history: $x(t) = \begin{bmatrix} 1/2 - \frac{e^{-2t}}{2} \\ 0 \\ 0 \end{bmatrix}$

output history: $y(t) = \frac{1}{2} - \frac{e^{-2t}}{2}$

6

a) $A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$ $\quad e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$

$$= \begin{bmatrix} \dfrac{e^{-2t}+e^{8t}}{2} & \dfrac{e^{-2t}-e^{8t}}{2} \\ \dfrac{e^{-2t}-e^{8t}}{2} & \dfrac{e^{-2t}+e^{8t}}{2} \end{bmatrix}$$

b) $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ $\quad e^{At} = \begin{bmatrix} \dfrac{e^{2t}+e^{4t}}{2} & \dfrac{e^{2t}-e^{4t}}{2} \\ \dfrac{e^{2t}-e^{4t}}{2} & \dfrac{e^{2t}+e^{4t}}{2} \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad e^{At} = \begin{bmatrix} \dfrac{e^{-t}+e^{t}}{2} & \dfrac{e^{t}-e^{-t}}{2} \\ \dfrac{e^{t}-e^{-t}}{2} & \dfrac{e^{-t}+e^{t}}{2} \end{bmatrix}$

d) $A = \begin{bmatrix} -11 & 20 \\ -6 & 11 \end{bmatrix}$ $\quad e^{At} = \begin{bmatrix} 6e^{-t} - 5e^{t} & 10e^{t} - 10e^{-t} \\ 3e^{-t} - 3e^{t} & 6e^{t} - 5e^{-t} \end{bmatrix}$

```
%----- Problem 1 -----
clear

syms s
a = [1 2; -3 -1];
b = [3/(s^2+9); 3/(s^2+9)];
c = [1 2];
d = 0;
x0 = [2; 1];

a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ab = at*(b+x0);
Y = c*ab;
Ys = simplify(Y)
```

```
Ys =

(4*s^3 - 10*s^2 + 45*s - 105)/(s^4 + 14*s^2 + 45)
```

```
%----- Problem 2 -----
clear

syms s
a = [0 1 0; -2 -4 1; 0 0 -6];
b = [0; 0; 1/(s+1)];
c = [1 0 0];
d = 0;
x0 = [0; 0; 0];

a2 = ((s*(eye(3,3)))-a);
at = a2^-1;
ab = at*(b+x0);
Y = c*ab;
Ys = simplify(Y)
```

```
Ys =

1/((s + 1)*(s^3 + 10*s^2 + 26*s + 12))
```

```
%----- Problem 3 -----
clear

syms s
a = [-2 0; -1 -1];
b = [1/s; 1/s];
c = [0 1];
d = 0;
x0 = [1; 0];

a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ab = at*(b+x0);
Y = c*ab;
yt = ilaplace(Y)
```

```
yt =

exp(-2*t)/2 - exp(-t) + 1/2
```

```
%----- Problem 4 -----
clear

syms s
a = [0 1; -1 0];
b = [0; 1];
us = 1/s;
c = [3 2];
d = 0;
x0 = [0; 0];

a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
p = ilaplace(at)

Xs = at*((b*us)+x0);
xt = ilaplace(Xs)

Ys = (c*Xs)+(d*us);
yt = ilaplace(Ys)
```

```
p =

[  cos(t), sin(t)]
[ -sin(t), cos(t)]


xt =

 1 - cos(t)
     sin(t)


yt =

2*sin(t) - 3*cos(t) + 3
```

```
%----- Problem 5 -----
clear

syms s
a = [-2 1 0; 0 0 1; 0 -6 -1];
b = [1; 0; 0];
us = 1/s;
c = [1 0 0];
d = 0;
x0 = [0; 0; 0];

a2 = ((s*(eye(3,3)))-a);
at = a2^-1;
p = ilaplace(at)

Xs = at*((b*us)+x0);
xt = ilaplace(Xs)

Ys = (c*Xs)+(d*us);
yt = ilaplace(Ys)
```

```
%----- Problem 6 Part a -----
clear

syms s
a = [3 -5; -5 3];
a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ilaplace(at)
```

```
ans =

[ exp(-2*t)/2 + exp(8*t)/2, exp(-2*t)/2 - exp(8*t)/2]
[ exp(-2*t)/2 - exp(8*t)/2, exp(-2*t)/2 + exp(8*t)/2]
```

```
%----- Problem 6 Part b -----
clear

syms s
a = [3 -1; -1 3];
a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ilaplace(at)
```

```
ans =

[ exp(2*t)/2 + exp(4*t)/2, exp(2*t)/2 - exp(4*t)/2]
[ exp(2*t)/2 - exp(4*t)/2, exp(2*t)/2 + exp(4*t)/2]
```

```
%----- Problem 6 Part c -----
clear

syms s
a = [0 1; 1 0];
a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ilaplace(at)
```

```
ans =

[ exp(-t)/2 + exp(t)/2, exp(t)/2 - exp(-t)/2]
[ exp(t)/2 - exp(-t)/2, exp(-t)/2 + exp(t)/2]
```

```
%----- Problem 6 Part d -----
clear

syms s
a = [-11 20; -6 11];
a2 = ((s*(eye(2,2)))-a);
at = a2^-1;
ilaplace(at)
```

```
ans =

[ 6*exp(-t) - 5*exp(t), 10*exp(t) - 10*exp(-t)]
[ 3*exp(-t) - 3*exp(t),    6*exp(t) - 5*exp(-t)]
```