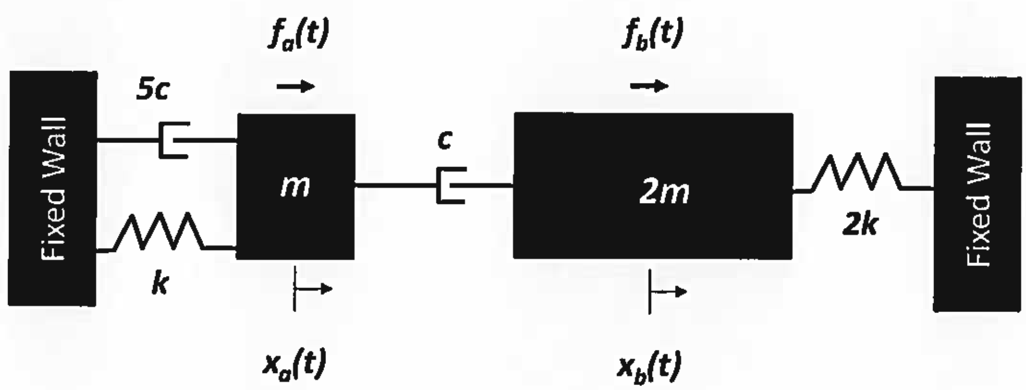
MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 8
Due Thursday, April 16th at the beginning of class

1. Consider an LRC circuit with one inductor, one resistor, one capacitor and one voltage source. Assume that the initial conditions (capacitor charge and current) are zero. Assume also that the above components are arranged clockwise, that the current direction is clockwise and that the voltage is positive for that current direction. Furthermore, $V(t) = 3\cos(0.75\,t) + 6\cosh(1.25\,t)$.

   a. Construct the space state and output equations *in matrix form* for the system, for L = 20 H, R = 10 Ω and C = 0.05 F. Consider the charge and current ($q(t)$ and $i(t)$, respectively) as the outputs and the voltage ($V(t)$) as the input.
   b. Identify the matrices **A**, **B**, **C** and **D**
   c. Calculate the transfer matrix of the system.
   d. Use the transfer matrix to find the charge and current *in Laplace space* (you do <u>not</u> need to invert the Laplace expressions).

2. Consider the system of masses shown in the figure below. If $k = 1$ N/m, $c = 0.5$ Ns/m, $m = 2$ Kg, $f_a(t) = t\,u(t)$ and $f_b(t) = \cos(t)\,u(t)$,



   a. Construct the space state and output equations in matrix form for the system, taking the positions of the two masses and the velocities of the two masses as the outputs, and the two forces as the inputs.
   b. Provide an expression to calculate the transfer matrix of the system. Here you do <u>not</u> need to calculate the inverse of the matrix $(s\mathbf{I} - \mathbf{A})$. You just need to plug in all the values of the matrix elements and simplify $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$ as far as you can.
   c. Use the above expression for the transfer matrix to derive an expression for the output matrix *in Laplace space*. You can leave the matrix operations indicated, simplifying as much as possible, as in the previous step.

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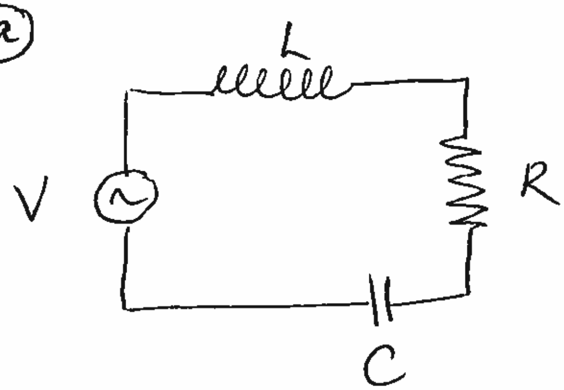
HOMEWORK 8 SOLUTION

PROBLEM 1

(a)



$i(0) = 0$
$q(0) = 0$

The governing equation is:

$$V - L\frac{di}{dt} - Ri - \frac{q}{C} = 0$$

$\Rightarrow L\dfrac{di}{dt} + Ri + \dfrac{1}{C}q = V$ but $i = \dfrac{dq}{dt}$, so we have

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V \Rightarrow \ddot{q} = -\frac{1}{LC}q - \frac{R}{L}\dot{q} + \frac{V}{L}$$

We select q and $\dot{q}$ as the state variables:

$$q_1 = q$$
$$q_2 = \dot{q}$$

We now construct the space equations:

$$\dot{q}_1 = \dot{q} = q_2 \Rightarrow \boxed{\dot{q}_1 = q_2}$$

$$\dot{q}_2 = \ddot{q} = -\frac{1}{LC}q - \frac{R}{L}\dot{q} + \frac{V}{L}$$

$$\Rightarrow \boxed{\dot{q}_2 = -\frac{1}{LC}q_1 - \frac{R}{L}q_2 + \frac{V}{L}}$$

We have then the following:

$$\dot{q}_1 = 0 q_1 + 1 q_2 + 0 V$$
$$\dot{q}_2 = -\frac{1}{LC} q_1 - \frac{R}{L} q_2 + \frac{1}{L} V$$

Or in matrix form:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V$$

Plugging in values we have:

STATE SPACE EQUATION

$$\boxed{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.05 \end{bmatrix} (3 \cos(0.75t) + 6 \cosh(1.25t))}$$

Now we construct the output equation:

$$y_1 = q(t) = q_1$$
$$y_2 = \dot{q}(t) = q_2$$

So we have

$$y_1 = 1 q_1 + 0 q_2 + 0 V$$
$$y_2 = 0 q_1 + 1 q_2 + 0 V$$

So in matrix form we have:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} (3 \cos(0.75t) + 6 \cosh(1.25t))$$

Or simply

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}$$

OUTPUT EQUATION

(b) Therefore,

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix} \qquad \underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(c) Calculation of the transfer matrix:

$$\underline{G}(s) = \underline{C} \left(s\underline{I} - \underline{A} \right)^{-1} \underline{B} + \underline{D}$$

$$\Rightarrow \underline{G}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{G}(s) = \begin{bmatrix} s & -1 \\ 1 & s+0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$$

Calculating the above inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ (I) $as + b = 1$

(II) $-a + b(s+\frac{1}{2}) = 0 \Rightarrow a = b(s+\frac{1}{2})$

(III) $cs + d = 0$

(IV) $-c + d(s+\frac{1}{2}) = 1$

substituting II into I

$$b\left(s+\tfrac{1}{2}\right)s + b = 1 \Rightarrow b\left[\left(s+\tfrac{1}{2}\right)+1\right] = 1$$

$$\Rightarrow \boxed{b = \frac{1}{s^2+\frac{1}{2}s+1}}$$

Using II $\boxed{a = \left(s+\tfrac{1}{2}\right)b = \frac{s+\frac{1}{2}}{s^2+\frac{1}{2}s+1}}$

From III $d = -cs$

Substituting III into IV

$$-c - cs\left(s+\tfrac{1}{2}\right) = 1 \Rightarrow -c\left[1+s\left(s+\tfrac{1}{2}\right)\right] = 1$$

$$\Rightarrow \boxed{c = \frac{-1}{s^2+\frac{1}{2}s+1}}$$

Using III $d = -cs \Rightarrow \boxed{d = \frac{s}{s^2+\frac{1}{2}s+1}}$

Therefore,

$$\underline{G}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0.05 \end{bmatrix} = \frac{1}{s^2+\frac{1}{2}s+1}\begin{bmatrix} s+\frac{1}{2} & 1 \\ -1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$$

$$\Rightarrow \boxed{G(s) = \begin{bmatrix} \dfrac{0.05}{s^2+\frac{1}{2}s+1} \\[2ex] \dfrac{0.05s}{s^2+\frac{1}{2}s+1} \end{bmatrix}}$$

(d) We know that $\underline{Y}(s) = \underline{G}(s)\,\underline{U}(s)$.

In our case there is only one input in $\underline{U}$, which is the voltage.

$$\underline{U}(s) = \mathcal{L}[V(t)] = \mathcal{L}[3\cos(0.75t) + 6\cosh(1.25t)]$$

$$= \frac{3s}{s^2+0.75^2} + \frac{6s}{s^2-1.25^2}$$

(since there is only one element, U is actually a scalar).

$$\Rightarrow Y(s) = \begin{bmatrix} \dfrac{0.05}{s^2+\frac{1}{2}s+1} \\ \dfrac{0.05s}{s^2+\frac{1}{2}s+1} \end{bmatrix} \left(\frac{3s}{s^2+0.75^2} + \frac{6s}{s^2-1.25^2} \right) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}$$

Therefore, we have

Charge in Laplace space:

$$\boxed{Y_1(s) = \left(\frac{0.05}{s^2+\frac{1}{2}s+1} \right) \left(\frac{3s}{s^2+0.75^2} + \frac{6s}{s^2-1.25^2} \right)}$$

Current in Laplace space:

$$\boxed{Y_2(s) = \left(\frac{0.05s}{s^2+\frac{1}{2}s+1} \right) \left(\frac{3s}{s^2+0.75^2} + \frac{6s}{s^2-1.25^2} \right)}$$

Note that $Y_2(s) = s\,Y_1(s)$ as it should be, since $y_2(t) = \frac{d}{dt} y_1(t)$ $\left(\text{or } i(t) = \frac{dq(t)}{dt} \right)$

Problem 2

Equations of motion:

mass a :
$$m_a \ddot{x}_a = -Kx_a - 5c\dot{x}_a - c(\dot{x}_a - \dot{x}_b) + f_a(t)$$
$$m\ddot{x}_a = -Kx_a - 6c\dot{x}_a + C\dot{x}_b + f_a(t)$$
$$2\ddot{x}_a = -1\,x_a - 3\dot{x}_a + \tfrac{1}{2}\dot{x}_b + t\,u(t)$$

mass b:
$$m_b\ddot{x}_b = -2Kx_b - c(\dot{x}_b - \dot{x}_a) + f_b(t)$$
$$2m\ddot{x}_b = -2Kx_b - C\dot{x}_b + C\dot{x}_a + f_b(t)$$
$$4\ddot{x}_b = -2x_b - \tfrac{1}{2}\dot{x}_b + \tfrac{1}{2}\dot{x}_a + \cos(t)u(t)$$

So we have:
$$2\ddot{x}_a = -1\,x_a + 0\,x_b - 3\dot{x}_a + \tfrac{1}{2}\dot{x}_b + t\,u(t)$$
$$4\ddot{x}_b = 0\,x_a - 2\,x_b + \tfrac{1}{2}\dot{x}_a - \tfrac{1}{2}\dot{x}_b + \cos(t)u(t)$$

We choose the following state variables:
$$x_1 = x_a \Rightarrow \dot{x}_1 = \dot{x}_a = x_2$$
$$x_2 = \dot{x}_a \Rightarrow \dot{x}_2 = \ddot{x}_a$$
$$x_3 = x_b \Rightarrow \dot{x}_3 = \dot{x}_b = x_4$$
$$x_4 = \dot{x}_b \Rightarrow \dot{x}_4 = \ddot{x}_b$$

Therefore, we have
$$\dot{x}_1 = 0x_1 + 1x_2 + 0x_3 + 0x_4 + 0\,t\,u(t) + 0\cos(t)u(t)$$
$$\dot{x}_2 = -\tfrac{1}{2}x_1 + 0x_2 - \tfrac{3}{2}x_3 + \tfrac{1}{4}x_4 + 1\,t\,u(t) + 0\cos(t)u(t)$$
$$\dot{x}_3 = 0x_1 + 0x_2 + 0x_3 + 1\,x_4 + 0\,t(u(t)) + 0\cos(t)u(t)$$
$$\dot{x}_4 = 0x_1 - \tfrac{1}{2}x_2 + \tfrac{1}{8}x_3 - \tfrac{1}{8}x_4 + 0\,t\,u(t) + 1\cos(t)u(t)$$

Or in matrix form.

$$\boxed{\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{bmatrix}0&1&0&0\\-\frac{1}{2}&0&-\frac{3}{2}&\frac{1}{4}\\0&0&0&1\\0&-\frac{1}{2}&\frac{1}{8}&-\frac{1}{8}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}t\,u(t)\\\cos(t)\,u(t)\end{bmatrix}}$$

The above is the state space equation.

Now, we have four outputs as per the problem statement,

$$y_1 = x_1$$
$$y_2 = x_2$$
$$y_3 = x_3$$
$$y_4 = x_4$$

So the output equation is:

$$\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}0&0\\0&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}t\,u(t)\\\cos(t)\,u(t)\end{bmatrix}$$

or simply,

$$\boxed{\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}$$

The above completes the answer to part a.

(b) Now we are looking for the transfer matrix:

$$\underline{G}(s) = \underline{C}\,(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D}$$

In our case,

$$\underline{G}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{3}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix} \right)^{-1}$$

$$\times \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\Rightarrow \underline{G}(s) = \begin{bmatrix} s & -1 & 0 & 0 \\ \frac{1}{2} & s & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & s & -1 \\ 0 & \frac{1}{2} & -\frac{1}{8} & s+\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}$$

(c) The two inputs are $tu(t)$ and $\cos(t)u(t)$

$$\Rightarrow \underline{U}(t) = \begin{bmatrix} tu(t) \\ \cos t\, U(t) \end{bmatrix} \Rightarrow \underline{U}(s) = \begin{bmatrix} \frac{1}{s^2} \\ \frac{s}{s^2+1} \end{bmatrix}$$

$$\Rightarrow \underline{Y}(s) = \underline{G}(s)\,\underline{U}(s)$$

$$\Rightarrow \underline{Y}(s) = \begin{bmatrix} s & -1 & 0 & 0 \\ \frac{1}{2} & s & \frac{3}{2} & \frac{-1}{4} \\ 0 & 0 & s & -1 \\ 0 & \frac{1}{2} & -\frac{1}{8} & s+\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2} \\ \frac{s}{s^2+1} \end{bmatrix}$$

which can be simplified to

$$\underline{Y}(s) = \begin{bmatrix} s & -1 & 0 & 0 \\ \frac{1}{2} & s & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & s & -1 \\ 0 & \frac{1}{2} & -\frac{1}{8} & s+\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s^2} \\ 0 \\ \frac{s}{s^2+1} \end{bmatrix}$$