Spring 2018 MAE3134: Midterm Exam

SOLUTION

8 March 2018

Resources allowed: One sided note sheet, calculator, ruler. No computers or mobile devices.

Name: ______________________ GWID: ______________________

	Prob. 1	Prob. 2	Prob. 3	Prob. 4	Prob. 5	Total
MEAN	16	15	12	10	18	71
	20	20	20	20	20	100

MEAN 71

MEDIAN 71

MAX 99

MIN LOW

Problem 1 Elon Musk, CEO of SpaceX and Tesla Motors, is developing his newest spacecraft. The output response of a critical subsystem can be defined by the following function, $X(s)$.

$$X(s) = \frac{30}{s\,(s^2 + 2s + 10)}$$

Find the output response in the time domain, i.e. find $x(t)$. Ensure you show all of your work, as Elon believes in the maxim "trust but verify".

COMPLEX POLES $s=0$ $s=-1 \pm 3j$ +2

COMPLETING THE SQUARE

$$s^2+2s+10=(s+1)^2+3^2$$ +2

EXPANSION

$$X(s)=\frac{A}{s}+\frac{Bs+C}{(s+1)^2+3^2}$$ +2

$$30=A(s^2+2s+10)+(Bs+C)\cdot s$$

$$30=(A+B)s^2+(2A+C)s+10A$$

$A+B=0$ $2A+C=0$ $10A=30$ +6 ALGEBRA

$B=-3$ $C=-6$ $A=3$

+4

$$X(s)=\frac{3}{s}+\frac{-3s-6}{(s+1)^2+3^2}=\frac{3}{s}+\frac{-3(s+1)}{(s+1)^2+3^2}+\frac{-3}{(s+1)^2+3^2}$$

$$x(t)=3\,u(t)-3e^{-t}\cos 3t-e^{-t}\sin 3t$$

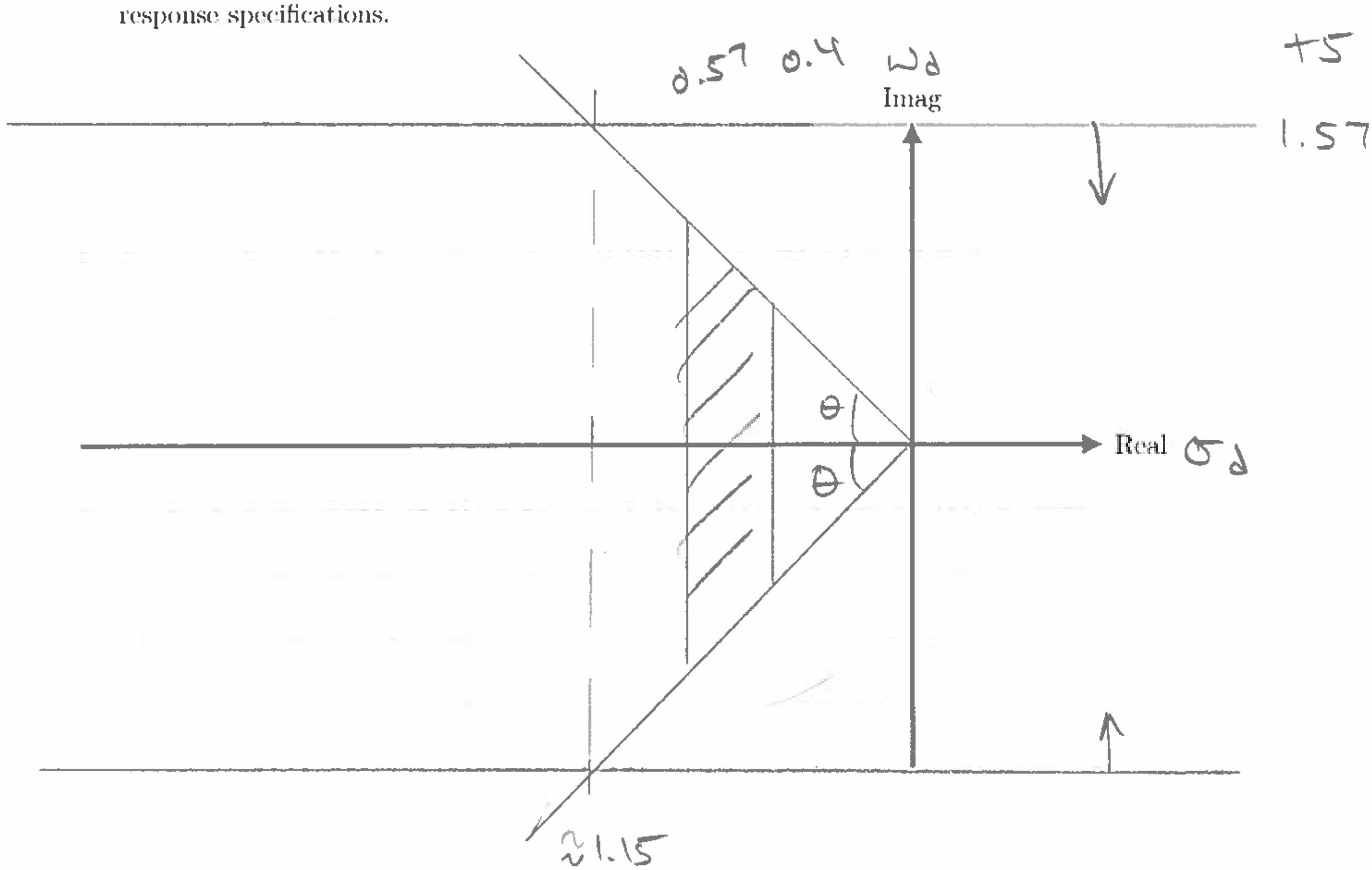
+4

Problem 2 Elon Musk, CEO of SpaceX and Tesla Motors, has a background in physics but unfortunately has never passed a Linear Dynamics course. His newest space vehicle must satisfy the following second order time response specifications for a unit step input:

- Percent Overshoot must be less than 9%.
- Peak time greater than 2 s.
- Settling Time greater than 7 s but less than 10 s.

Elon needs your help to choose a set of poles which will satisfy the specifications and save humanity from impending disaster.

1. On the s-plane, or complex plane, map out the acceptable regions where you could locate poles and meet the requirements.
2. Label the specifications lines and show your work.
3. Choose a set of poles that will meet the requirements.
4. Determine the transfer function representation for this sytem.
5. Use the initial and final value theorems to determine the initial and final values of the output response assuming a step input.
6. Describe the effect of moving the poles to the LEFT, i.e. more negative, on the system response specifications.



NOT TO SCALE

$$\zeta = \frac{\left(\ln \frac{OS}{100}\right)^2}{\sqrt{\pi^2 + \left(\ln \frac{OS}{100}\right)^2}}$$

$\zeta > 0.59$ +5

$\cos\theta = \zeta$

$\theta < 53.8^\circ$

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d}$$

$0.4 < \sigma_d < 0.57$

$0.67 < \omega_n < 0.96$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$\omega_d < 1.57$

POLES $s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$

$= -\sigma_d \pm j\omega_d$ +2

CANDIDATE POLES

$$\left.\begin{array}{l}\zeta = 0.65 \\ \omega_n = 0.7\end{array}\right\} \boxed{s = -0.45 \pm 0.53j}$$

TRANSFER FUNCTION

$$G = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.49}{s^2 + 0.9s + 0.49}$$ +2

INITIAL VALUE THEOREM

OUTPUT $F(s) = \frac{1}{s} G(s) = \frac{0.49}{s(s^2+0.9s+0.49)}$ +2

$\lim_{t\to 0} f(t) =$

$\lim_{s\to\infty} sF(s) = \frac{1}{s^2+0.9s+0.49} \to 0$ ZERO INITIAL VALUE

FINAL VALUE

+2

$\lim_{s\to 0} sF(s) = \frac{0.49}{0.49} = 1$ STEADY STATE OF 1

$\lim_{t\to\infty} f(t)$

MOVE POLES TO THE LEFT.

+2

T_s WILL DECREASE

ζ WILL INCREASE $\to$ %OS WILL DECREASE

Problem 3 Elon has read the Wikipedia page on "State Space Control" and is intrigued. However, he is having difficulty with converting the state space representation to the equivalent transfer function.

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

Starting with the standard state space form, **DERIVE** the expression for the transfer function $\frac{Y(s)}{U(s)}$. Remember to show all of your work.

$$s\,X(s) = A\,X(s) + B\,U(s) \qquad +5$$

$$Y(s) = C\,X(s) + D\,U(s) \qquad +5$$

$$Y(s) = \left[C(sI-A)^{-1}B + D\right]U(s) \qquad +5$$

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

Problem 3.1 "Modern! Sch-Modern!, transfer functions are fine..." exlaims Elon during a particular heated engineering review meeting. List at least two advantages of state-space or "modern control" techniques as compared to "classical control" approaches to convince Elon of your superior knowledge. +5

- MULTIPLE INPUT | OUTPUT
- FIRST ORDER DIFF. EQ
- NON-ZERO + NONLINEAR SYSTEMS.

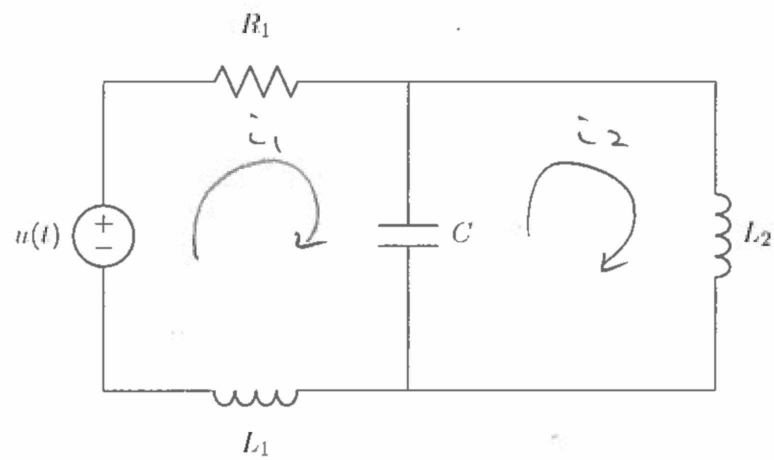
Problem 4 For the electrical system in Fig. 1:

1. Find the differential equations of motion for the system.
2. Find the state space representation of the system with your state vector defined as

$$x = \begin{bmatrix} q_1 & i_1 & q_2 & i_2 \end{bmatrix}^T,$$

where q_1, i_1 represent the charge and current in the left loop while q_2, i_2 represent the charge and current in the right loop, respectively. The output is defined as

$$y = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T.$$



$i = \frac{d}{dt} q$

Figure 1: Electrical Circuit

LEFT

$$u - i_1 R_1 - \frac{q_1 - q_2}{C} - L\frac{di_1}{dt} = 0$$

+5

$$u = L\ddot{q}_1 + R_1\dot{q}_1 + \frac{1}{C}(q_1 - q_2) = 0$$

RIGHT

$$\frac{-1}{C}(q_2 - q_1) - L\frac{d}{dt}i_2 = 0$$

$$L\ddot{q}_2 + \frac{1}{C}(q_2 - q_1) = 0$$

+5

STATE

$$x_1 = q_1 \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{q}_1 \qquad \dot{x}_2 = -\frac{R_1}{L} x_2 - \frac{1}{LC} x_1 + \frac{1}{LC} x_3 + \frac{u}{L}$$

$$x_3 = q_2 \qquad \dot{x}_3 = x_4$$

$$x_4 = \dot{q}_2 \qquad \dot{x}_4 = -\frac{1}{LC} x_3 + \frac{1}{LC} x_1$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du \longrightarrow 2\times1 = (2\times4)\ 4\times1$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{LC} & -\frac{R_1}{L} & \frac{1}{LC} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{LC} & 0 & -\frac{1}{LC} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/L \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad +5$$

Problem 5 Given the following differential equation

$$\ddot{\theta} + 3\dot{\theta} + 2\theta = 12\dot{u}(t) + 24u(t).$$

1. Find the transfer function $G(s) = \frac{\theta(s)}{U(s)}$.
2. Find the state space representation assuming the state is defined as $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$.
3. Find the matrix $\Phi(s) = (sI - A)^{-1}$.

$$s^2\Theta(s) + 3s\,\Theta(s) + 2\,\Theta(s) = 12s\,U(s) + 24\,U(s)$$

$$\frac{\Theta(s)}{U(s)} = \frac{12s + 24}{s^2 + 3s + 2}$$

+5

$$U(s) \longrightarrow \boxed{\frac{Z(s)}{U(s)}} \longrightarrow \boxed{\frac{\Theta(s)}{Z(s)}} \longrightarrow \Theta(s)$$

+5

$$\frac{1}{s^2 + 3s + 2} \qquad \frac{12s + 24}{1}$$

$$\ddot{z} + 3\dot{z} + 2z = 0 \qquad 12\dot{z} + 24z = \theta$$

$x_1 = z$

$x_2 = \dot{z}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 24 & 12 \end{bmatrix} x$$

+5

$$\Phi = (sI - A)^{-1}$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \cdot \frac{1}{s(s+3)+2}$$

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{+1}{s^2+3s+2} \\ \dfrac{-2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix} \qquad +5$$