GIVEN ANY LINEAR SYSTEM

1. DIFF. EQ.
2. TRANSFER FCN (FREQ. RESPONSE FCN)
3. STATE SPACE
4. ELEC | MECH. MODEL

YOU SHOULD BE ABLE TO

- CONVERT BTWN EACH OF THE FORMS
- FIND THE ANALYTICAL RESPONSE TO INPUTS
- DETERMINE POLES + (ζ, ω_n) + PLOT POLES
- CALCULATE PERFORMANCE SPECS.
- PLOT THOSE SPECS ON COMPLEX PLANE + UNDERSTAND RELATIONSHIP TO POLES | ζ, ω_n
- FIND FREQ. RESPONSE FCN $G(s)\Big|_{s=j\omega}$
- DETERMINE M, ϕ OF SYSTEM ANALYTICALLY
- FIND STEADY STATE OUTPUT TO A SINUSOIDAL INPUT.
- UNDERSTAND FREQ. RESP. PLOT (APPROXIMATE BODE PLOT) + READ IT.
- IMPACT OF ζ, ω_n ON BODE PLOT.
- UNDERSTAND BODE APPROXIMATIONS ⟶ APPLY THEM
- RECOGNIZE BENEFITS + STATE SPACE OVER CLASSICAL CONTROL
- LINEARIZE N.L. EQ.
- SOLVE S.S. EQ. FOR SOLUTION Φ
- HANDLE INPUT DERIVATIVES $u, \dot{u}, \ddot{u}$

$$\ddot{q}_1 + \frac{q_1}{H_1 C_2} = \frac{u_1}{H_1} + \frac{q_2}{H_1 C_2}$$

$$q_2 = \frac{C_1 C_2}{C_1 + C_2} u_2 + \frac{C_1}{C_1 + C_2} q_1 \qquad \dot{x} = Ax + Bu$$

$$\dot{q}_3 = -\frac{u_2}{R_2}$$

$$\ddot{q}_1 + \frac{q_1}{H_1 C_2} = \frac{u_1}{H_1} + \frac{1}{H_1 \cancel{C_2}} \frac{C_1 \cancel{C_2}}{C_1 + C_2} u_2$$

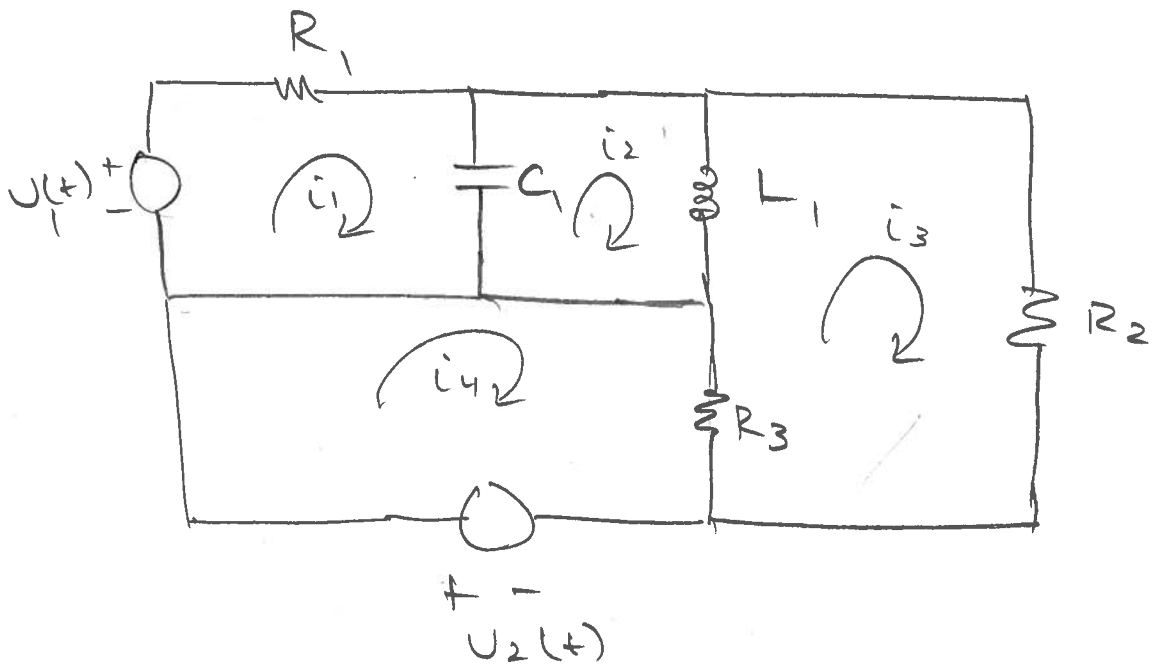
$$+ \frac{1}{H_1 C_2} \frac{C_1}{(C_1 + C_2)} q_1$$

$$\ddot{q}_1 + q_1 \underbrace{\left(\frac{1}{H_1 C_2} - \frac{C_1}{H_1 C_2 (C_1 + C_2)} \right)}_{a} = \frac{u_1}{H_1} + \underbrace{\frac{C_1}{H_1 (C_1 + C_2)}}_{b} u_2$$

$$\dot{q}_3 = -\frac{u_2}{R_2}$$

$X_1 = q_1$	$\dot{x}_1 = x_2$
$X_2 = \dot{q}_1$	$\dot{x}_2 = -a x_1 + \frac{u_1}{H_1} + b u_2$
$X_3 = q_3$	$\dot{x}_3 = -\frac{u_2}{R_2} \leftarrow y$

ELECTRICAL SYSTEM EXAMPLE



FINAL REVIEW

- TIME RESPONSE SPECIFICATIONS FOR FIRST + SECOND ORDER
- FREQUENCY RESPONSE
  - STEADY STATE RESPONSE TO SINUSOIDAL INPUT
  - MAG + PHASE OF FREQ. RESP. FCN
  - BODE PLOTS +
  - APPROXIMATIONS FOR BODE PLOTS
  -

$$M_i < \phi_i \longrightarrow \boxed{M_G < \phi_G} \longrightarrow M_o < \phi_o$$

$$M_G = \frac{M_o}{M_i} \qquad \phi = \phi_w - \phi_i$$

8 PROBLEMS

≈ 4 HAVE NO MATH INVOLVED

- WILL HAVE TO DRAW BODE APPROXIMITION NEATNESS + ACC. WILL COUNT!
- NO CURVE

- ELECTRICAL SYSTEMS
  - MODEL USING KIRCHOFF VOLTAGE LAW
  - CAN FIND A MECHANICAL ANALOG.

- STATE SPACE APPROACH
  - ADVANTAGES / DISADVANTAGES
    MIMO, NONLINEAR, ~~EASIER~~ IN COMPUTER
  - CONVERT TF $\longleftrightarrow$ S.S.
  - INPUT DERIVATIVES $u, \dot{u}, \ddot{u}$ ETC.
  - SOLVING $\dot{x} = Ax + Bu$ WITH Φ

$$X(s) = (sI - A)^{-1} B\, U(s)$$

$$\frac{Y(s)}{U(s)} = \left[C\,(sI-A)^{-1} B\, U(s) + D\,U(s)\right]$$

$$\dot{x} = Ax$$

$$x(t) = e^{At} x(0)$$

$$= \mathcal{L}^{-1}\left((sI-A)^{-1}\right) x(0)$$

EXAMPLE

$\dot{x} = Ax + Bu \qquad x(0) = 0$

GIVEN ANY LINEAR SYSTEM

$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y = [1 \quad 0]\, x + 0\, u$

1. FIND $x(t)$ ANALYTICALLY TO A STEP INPUT

2. FIND FREQ. RESPONSE FCN

3. FIND TF. $Y(s)/U(s)$

4. FIND $|G(j\omega)|$, $\angle G(j\omega)$

5. PLOT THE BODE PLOT.

6. WHAT IS PEAK TIME, %OS, T_s + ζ, ω_n

FINAL REVIEW

4/23/18

- STATE SPACE REPRESENTATION
  - ADVANTAGES / DISADVANTAGES
  - TF $\Leftrightarrow$ ODE $\longleftrightarrow$ S.S.
  - HOW TO HANDLE $u, \dot{u}, \ddot{u} \ldots$
  - SOLVE $\dot{x} = Ax + Bu$ USING $\Phi = e^{At}$
  - LINEARIZATION
- FREQUENCY RESPONSE
  - STEADY STATE RESPONSE TO A SINUSOIDAL INPUT
  - MAG + PHASE OF FREQ. RESP. FCN.
  - DRAW BODE APPROXIMATION

$$M_i \angle \phi_i \longrightarrow \boxed{M_q \angle \phi_q} \longrightarrow M_o \angle \phi_o$$

$$M_q = \frac{M_o}{M_i} \qquad \phi_q = \phi_o - \phi_i$$

PLUS EVERYTHING FROM BEFORE MIDTERM IS STILL VALID.

- MODELLING
- TIME RESPONSE SPECIFICATIONS.
- LAPLACE TRANSFORM

EXAMPLE — GIVEN ANY LINEAR (N.L) SYSTEM.

$$\dot{x} = Ax + Bu \qquad x(0) = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0u$$

1. FIND $x(t)$ ANALYTICALLY FOR VARIOUS INPUTS (STEP | SINUSOID | RAMP)

2. FIND FREQ. RESP. FCN.

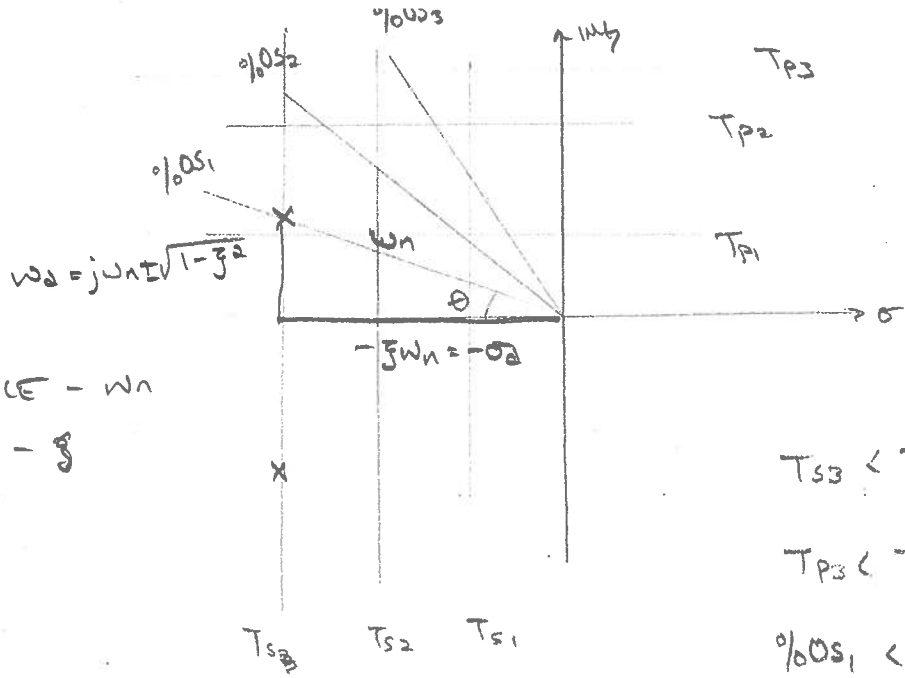
3. FIND TF. $G(s) = \dfrac{Y(s)}{U(s)}$

4. FIND $|G(j\omega)|$, $\angle G(j\omega)$

5. PLOT BODE APPROXIMATION

6. COMPUTE T_p, %OS, T_s, T_R (ζ, ω_n)

7. DETERMINE POLES TO MEET SPECS.



RADIAL – CONSTANT ζ / %OS

HORIZONTAL – CONSTANT T_p

VERTICAL – CONSTANT T_s

DISTANCE – ω_n

$\cos\theta$ – ζ

$T_{s3} < T_{s2} < T_{s1}$

$T_{p3} < T_{p2} < T_{p1}$

$\%OS_1 < \%OS_2 < \%OS_3$

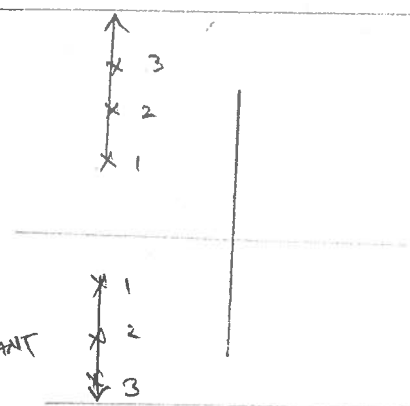
$\omega_d \uparrow \quad T_p \downarrow$

$\%OS \uparrow \quad T_r \downarrow$

$\sigma_d \uparrow \quad T_s \downarrow$

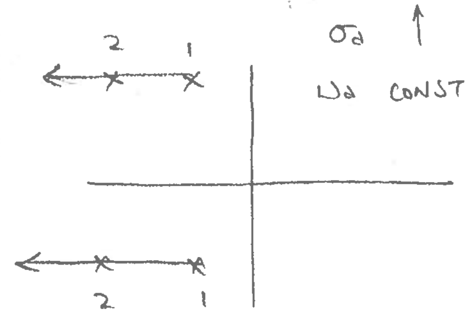
$\zeta = \cos\theta \quad \theta \uparrow \quad \zeta \downarrow \quad \%OS \uparrow$

S-PLANE



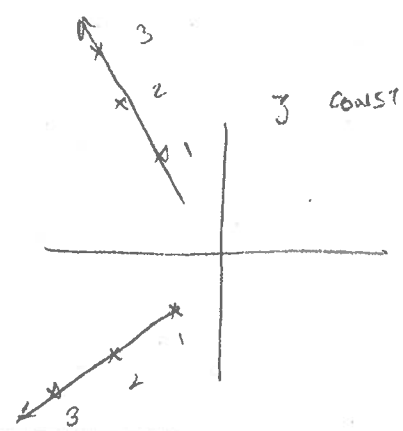
$\omega_d \uparrow$

σ_d CONSTANT



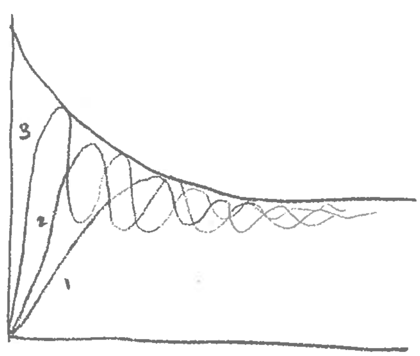
$\sigma_d \uparrow$

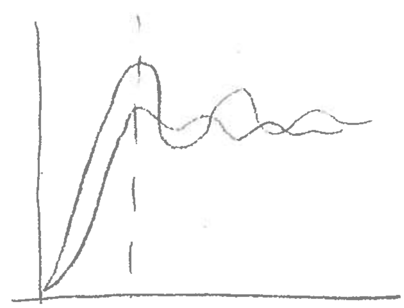
ω_d CONST.

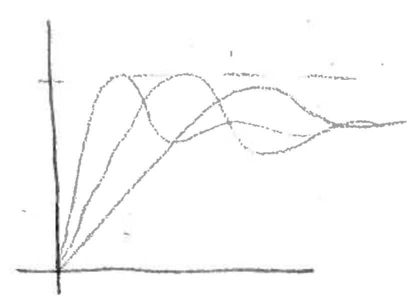


ζ CONST.

RESPONSE







SAME T_s – DIFFERENT FREQ.

T_p SAME T_s DIFF.

SPEED OF DAMPING / SETTLING

%OS CONST.

CHANGE IN RSP SPEED

[illegible]

STATE SPACE REVIEW

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

SOLUTION w/o INPUT

$$\dot{x} = Ax$$

$$sX(s) - x(0) = A\,X(s)$$

$$(sI - A)\,X(s) = x(0)$$

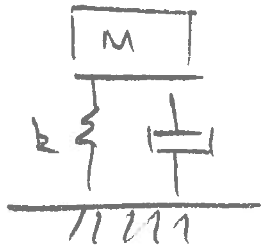
$$X(s) = (sI - A)^{-1}\,x(0)$$

$$\boxed{x(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} x(0)}$$

$$\Phi = e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

TRANSFORMATION OF INITIAL CONDITIONS.

CONSIDER



$m = 1$ kg

$c = 3$ Ns/m

$k = 2$ N/m

$x(0) = 0$

$\dot{x}(0) = 3$ m/sec

NO INPUT

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} = -2x - 3\dot{x}$$

STATE SPACE FORM

$$x_1 = x \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{x} \qquad \dot{x}_2 = \ddot{x} = -2x_1 - 3x_2$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}\, x(0)$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \Phi(t)$$

$$x(t) = \Phi(t)\, x(0)$$

$$x(t) = \begin{bmatrix} 3e^{-t} - 3e^{-2t} \\ -3e^{-t} + 6e^{-2t} \end{bmatrix}$$

FINAL EXAM   10 MAY

10:20 — 12:20

5 QUESTIONS — 1 + MULTIPLE CHOICE

- COVERS WHOLE SEMESTER
  FOCUSED ON MATERIAL SINCE MIDTERM

- 2 SIDES OF 8.5 x 11" NOTES PAPER.
  HANDWRITTEN.

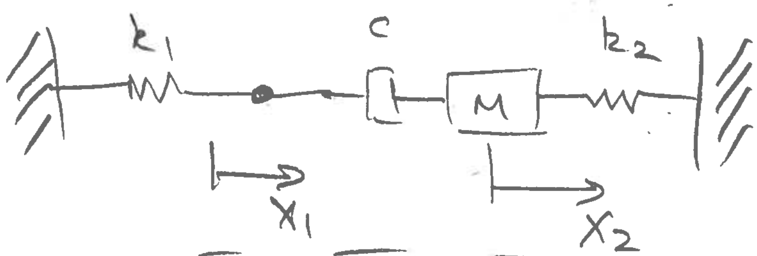
- CALCULATOR, RULER, PENCILS.

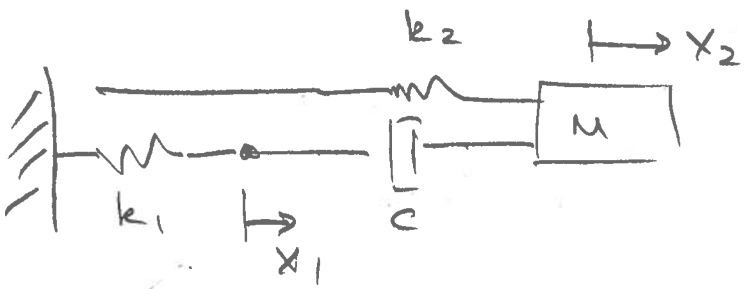
- VALIDATION > 93% → NO FINAL

- EVERYONE GETS LOWEST HW DROPPED.

- FINAL GRADE BOOST
  $\geq$ 95% → + HALF LETTER GRADE

- BLACKBOARD FINAL GRADE COLUMN.





$\underline{x_2}$

$$M\ddot{x}_2 = -k_2 x_2 - c(\dot{x}_2 - \dot{x}_1)$$

$\underline{x_1}$

$$0 = -c(\dot{x}_1 - \dot{x}_2) - k_1 x_1$$

2
SM
COS

$$0 = -k_1 q_1 - c_1 (i_1 - i_2) \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M \dot{i}_2 = -k_2 q_2 - c_1 (i_2 - i_1) \quad (sI - A)^{-1}$$

$$\begin{bmatrix} b\cdot\varepsilon & b\cdot\tau \\ b\cdot 1 & b\cdot 0 \end{bmatrix} = \begin{bmatrix} \varepsilon & \tau \\ 1 & 0 \end{bmatrix} b$$

EXAMPLES

TWO DOF SPRING MASS DAMPER

$$m\ddot{q}_1 + c(\dot{q}_1 - \dot{q}_2) + K(q_1 - q_2) + c\dot{q}_1 + k q_1 = 0$$

$$m\ddot{q}_2 + c(\dot{q}_2 - \dot{q}_1) + K(q_2 - q_1) + c\dot{q}_2 + k q_2 = u$$

FIND THE STATE SPACE DESCRIPTION

HOW MANY STATES ARE REQUIRED ?

$x_1 = q_1$	$\dot{x}_1 = x_2$	2 SECOND ORDER
$x_2 = \dot{q}_1$	$\dot{x}_2 = \ddot{q}_1$	$\rightarrow$ 4 1st ORDER
$x_3 = q_2$	$\dot{x}_3 = x_4$	
$x_4 = \dot{q}_2$	$\dot{x}_4 = \ddot{q}_2$	

$$\dot{x}_2 = -\frac{c}{M}(x_2 - x_4) - \frac{k}{M}(x_1 - x_3) - \frac{c}{M}x_2 - \frac{k}{M}x_1$$

$$\dot{x}_4 = -\frac{c}{m}(x_4 - x_2) - \frac{k}{m}(x_3 - x_1) - \frac{c}{M}x_4 - \frac{k}{m}x_3 + u$$

$$\dot{x}_2 = -2\frac{k}{m}x_1 - 2\frac{c}{M}x_2 + \frac{k}{m}x_3 + \frac{c}{M}x_4$$

$$\dot{x}_4 = +\frac{k}{m}x_1 + \frac{c}{M}x_2 - 2\frac{k}{M}x_3 - \frac{2c}{M}x_4 + u$$

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\frac{k}{m} & -2\frac{c}{m} & \frac{k}{m} & \frac{c}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & \frac{c}{m} & \frac{-2k}{m} & \frac{-2c}{m} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

LETS ASSUME WE CAN MEASURE

POSITION OF BOTH MASSES

$$y_1 = q_1 = x_1 \qquad\qquad y = Cx + \cancel{Du}$$

$$y_2 = q_2 = x_3 \qquad\qquad 2{\times}1\ (2{\times}4)\ 4{\times}1$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

WHAT IS THE TRANSFER FCN (MATRIX)

$$Y(s) = \left[C(sI-A)^{-1}B + D\right] U(s)$$

WHAT IS $x(t)$ FOR $u = 0$ $\quad x(0) = [1\ 0\ 2\ 0]^T$

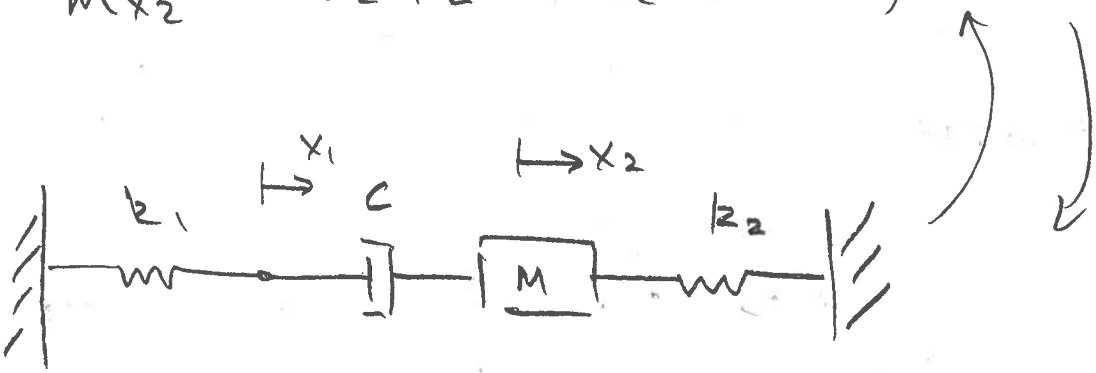
$$x(t) = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} x(0) = \Phi\, x(0)$$

EXAMPLE

DRAW MECHANICAL TRANSLATION MODEL FOR

$$0 = -k_1 x_1 + c(\dot{x}_1 - \dot{x}_2)$$

$$m\ddot{x}_2 = -k_2 x_2 - c(\dot{x}_2 - \dot{x}_1)$$



REPLACE $\dot{x}_1 = i_1$, $\dot{x}_2 = i_2$

→ DRAW ELECTRICAL CIRCUIT. ?

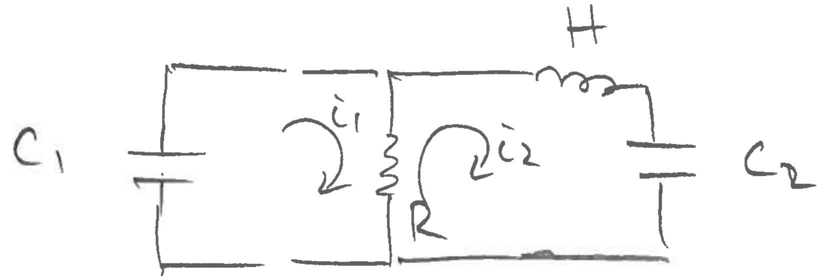
$$0 = -\underset{\text{CAPACITOR}}{k_1 q_1} - \underset{\text{RESISTOR}}{c(i_1 - i_2)}$$

$$i = \frac{dq}{dt}$$

$$\underset{\text{INDUCTOR}}{M\frac{d}{dt}i_2} = -\underset{\text{CAPACITOR}}{k_2 q_2} - \underset{\text{RESISTOR}}{c(i_2 - i_1)}$$

2 CAPACITORS
1 RESISTOR
1 INDUCTOR

TWO LOOPS.



$$-k_1 q_1 = -\frac{q_1}{C_1} \qquad C_1 = +\frac{1}{k_1}$$

$$C_2 = +\frac{1}{k_2} \qquad H = m$$

$R = C_{12} \leftarrow$ FROM BEFORE

EXAMPLE

$$G(s) = \frac{1}{s^2+3s+2}$$

(a) FIND $\dot{x} = Ax + Bu$

(b) FIND Φ

(c) FIND $x(t)$ FOR $u(t) = 1$ (STEP)

$$\frac{X(s)}{U(s)} = \frac{1}{s^2+3s+2}$$

$$s^2 X(s) + 3s\, X(s) + 2\, X(s) = U(s)$$

$$\ddot{x} + 3\dot{x} + 2x = u$$

$$x_1 = x \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{x} \qquad \dot{x}_2 = u - 3x_2 - 2x_1$$

A. $$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

B. $$\Phi = e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$(sI-A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \qquad (sI-A)^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2+3s+2}$$

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \cdot \frac{1}{(s+2)(s+1)}$$

$$= \begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} + \frac{-1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

TOTAL SOLUTION IS

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(t) = \Phi(t)\, x(0) + \underline{\int_0^t \Phi(t+\tau) B\, u(\tau)\, d\tau}$$

$$\int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \left(e^{-t}e^{\tau} - \frac{1}{2} e^{-2t} e^{2\tau}\right)\Big|_0^t \\ \left(-e^{-t}e^{\tau} + e^{-2t}e^{2\tau}\right)\Big|_0^t \end{bmatrix} \Rightarrow$$

$$= \begin{bmatrix} (1-e^{-t}) & -\frac{1}{2}(1-e^{-2t}) \\ -(1-e^{-t}) & + (1-e^{-2t}) \end{bmatrix}$$

CHECK SIGNS.

FORCED RESPONSE

$$= \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{3}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

FORCED PART.

COMPLETE SOLUTION.

$$x_1(t) = (2e^{-t} - e^{-2t})x_{10} + (e^{-t} - e^{-2t})x_{20}$$

$$+ \underline{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}$$

$$x_2(t) = (-2e^{-t} + 2e^{-2t})x_{10} + (-e^{-t} + 2e^{-2t})x_{20}$$

$$+ \underline{\frac{3}{2} - 2e^{-t} + \frac{1}{2}e^{-2t}}$$