MAE 3134: Homework 6

Due date: Thursday, 6 April 2017, 0935

Consider an LRC circuit with one inductor, one resistor, one capacitor and one voltage source. Assume that the initial conditions (capacitor charge and current) are zero. Also assume that the above components are arranged clockwise, that the current direction is clockwise, and that the voltage is positive for the defined current direction.

1. Draw the circuit described above.

2. For each of the three cases below, find $q(t)$ and $i(t)$ using Kirchoff's Voltage Law. Ensure that you show all of the required steps for each solution

   (a) Case 1: $L = 10\,\mathrm{H}, R = 20\,\Omega, C = 0.1\,\mathrm{F}, V = 2\,\mathrm{V}$

   (b) Case 2: $L = 10\,\mathrm{H}, R = 40\,\Omega, C = 0.1\,\mathrm{F}, V = 2\,\mathrm{V}$

   (c) Case 3: $L = 10\,\mathrm{H}, R = 5\,\Omega, C = 0.1\,\mathrm{F}, V = 2\,\mathrm{V}$

3. Plot $q(t)$ for all three cases on a single graph. On a seperate graph, plot $i(t)$ for all three cases. Using your plots, answer the following questions:

   (a) For each case above, indicate which would be:
   - underdamped,
   - critically damped,
   - overdamped.

   Also, explain how you reached your conclusions.

   (b) Convert each electrical system above into the equivalent mechanical system. Give the effective mass, damping constant, and spring constant for each case. In addition, compute the damping ratio ζ and the natural frequency ω_n for each case.

For the following questions, use the system defined in Case 3 above.

4. Compute the transfer function $G(s)$.

5. Compute the frequency response function $G(j\omega)$

6. Using $G(j\omega)$ find analytical expressions for the magnitude and phase responses:

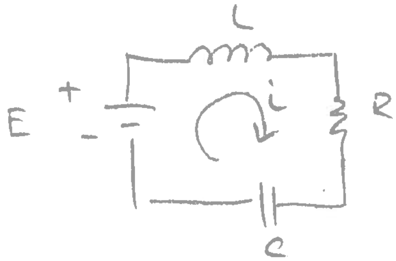
$$M(\omega) = \|G(j\omega)\|,$$
$$\phi(\omega) = < G(j\omega)$$

7. By hand, generate two plots which show the magnitude and phase response of the system for $0.1 \leq \omega \leq 10\,\mathrm{rad\,s^{-1}}$

   (a) On your plots, identify the magnitude/phase at $\omega = 0.05\,\mathrm{rad\,s^{-1}}$?

HOMEWORK 6 2017 SOLUTION

1 LRC CIRCUIT



$q(0) = 0$

$i(0) = \dot{q}(0) = 0$

2 USE KIRCHOFF'S VOLTAGE LAW TO SUM UP THE VOLTAGES

$$E(t) - L\frac{di}{dt} - Ri - \frac{q}{C} = 0 \qquad i = \frac{dq}{dt}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t) \rightarrow \text{EOMS}$$

CASE 1 $\quad L = 10\,H \quad R = 20\,\Omega \quad C = 0.1\,F \quad E = 2\,V$

$$10\ddot{q} + 20\dot{q} + 10q = 2$$

LAPLACE (ZERO I.C) $\rightarrow$ $10s^2Q(s) + 20sQ(s) + 10Q(s) = 2/s$

$$Q(s)\left[10s^2 + 20s + 10\right] = 2/s$$

$$Q(s) = \frac{0.2}{s(s^2+2s+1)} = \frac{0.2}{s(s+1)^2}$$ OUTPUT RESPONSE

PARTIAL FRACTION EXPANSION

$$Q(s) = \frac{A}{s} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)}$$

$$A = s\,\frac{0.2}{s(s+1)^2}\Bigg|_{s=0} = 0.2$$

$$C = \frac{d}{ds}\,\frac{0.2}{s}\Bigg|_{s=-1} = -0.2s^{-2}\Bigg|_{s=-1} = 0.2$$

$$B = (s+1)^2\,\frac{0.2}{s(s+1)^2}\Bigg|_{s=-1} = -0.2$$

$$\boxed{Q(s) = \frac{0.2}{s} - \frac{0.2}{(s+1)^2} - \frac{0.2}{(s+1)}}$$

INVERT USING LAPLACE TRANSFORM TABLE

$q(t) = 0.2\,u(t) - 0.2\,te^{-t} - 0.2\,e^{-t}$

$i(t) = \frac{dq}{dt} = 0.2\,\delta(t) - 0.2e^{-t} + 0.2te^{-t} + 0.2e^{-t}$

CASE 2 $\quad L = 10\,H \quad R = 40\,\Omega \quad C = 0.1\,F \quad V = 2V$

$10\ddot{q} + 40\dot{q} + 10q = 2$

$\ddot{q} + 4\dot{q} + q = 0.2$

$Q(s)\left[s^2 + 4s + 1\right] = \frac{0.2}{s}$

$Q(s) = \frac{0.2}{s(s^2+4s+1)} = \frac{0.2}{s(s+a)(s+b)}$

QUADRATIC FORMULA

$s^2 + 4s + 1 = 0 \implies s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16-4}}{2}$

$s = -2 \pm \sqrt{3}$ $\quad a = 2 + \sqrt{3} \approx 3.732051$

$-3.73 \quad -0.267$ $\quad b = 2 - \sqrt{3} \approx 0.267949$

PARTIAL FRACTION EXPANSION

$Q(s) = \frac{0.2}{s(s+a)(s+b)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 1}$

$= \frac{A(s^2+4s+1) + s(Bs+C)}{s(s^2+4s+1)} = \frac{(A+B)s^2 + (4A+C)s + A}{s(s^2+4s+1)}$

$0.2 = A \rightarrow A = 0.2$

$A + B = 0 \quad B = -0.2$

$4A + C = 0 \quad C = -0.8$

$$\boxed{Q(s) = \frac{0.2}{s} + \frac{-0.2s - 0.8}{(s+2-\sqrt{3})(s+2+\sqrt{3})}}$$

INVERSE LAPLACE USING OGATA TABLE

$$q(t) = 0.2u(t) - \frac{0.2}{a-b}\left[a e^{-at} - b e^{-bt}\right]$$

$$- \frac{0.8}{a-b}\left[e^{-bt} - e^{-at}\right] \qquad a = 2+\sqrt{3}$$

$$b = 2-\sqrt{3}$$

$$i(t) = 0.2\,\delta(t) - \frac{0.2}{a-b}\left[-a^2 e^{-at} + b^2 e^{-bt}\right]$$

$$- \frac{0.8}{a-b}\left[-be^{-bt} + a e^{-at}\right]$$

CASE 3 $\quad L = 10\,H \quad R = 5\,\Omega \quad C = 0.1\,F \quad V = 2\,V$

$$10\ddot{q} + 5\dot{q} + 10q = 2$$

$$\ddot{q} + 0.5\dot{q} + q = 0.2$$

$$Q(s)\left[s^2 + 0.5s + 1\right] = \frac{0.2}{s}$$

$$Q(s) = \frac{0.2}{s(s^2+0.5s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+\frac{1}{2}s+1}$$

$$A(s^2+\tfrac{1}{2}s+1) + (Bs+C)s = 0.2$$

$$(A+B)s^2 + (\tfrac{1}{2}A+C)s + A = 0.2$$

$$A = 0.2 \rightarrow A = 0.2$$

$$A+B = 0.2 \rightarrow B = -0.2$$

$$\tfrac{1}{2}A + C = 0 \rightarrow C = -0.1$$

COMPLETE SQUARE

$$s^2 + \tfrac{1}{2}s + 1 = s^2 + \tfrac{1}{2}s + \left(\tfrac{1}{4}\right)^2 - \left(\tfrac{1}{4}\right)^2 + 1$$

$$= \left(s+\tfrac{1}{4}\right)^2 + \frac{15}{16}$$

$$Q(s) = \frac{0.2}{s} - \frac{0.2s + 0.2(\frac{1}{4})}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{0.1 - 0.2(\frac{1}{4})}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

$$Q(s) = \frac{0.2}{s} - 0.2\,\frac{s+1/4}{\left(s+\frac{1}{4}\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2} - \frac{0.05}{\left(s+1/4\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2}$$

$$= \frac{0.2}{s} - 0.2\,\frac{s+1/4}{\left(s+1/4\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2} - \frac{0.05}{\sqrt{15/16}}\,\frac{\sqrt{15/16}}{\left(s+1/4\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2}$$

$$q(t) = 0.2\,u(t) - 0.2\,e^{-1/4\,t}\cos\sqrt{\frac{15}{16}}\,t - \frac{0.2}{\sqrt{15}}\,e^{-1/4\,t}\sin\sqrt{\frac{15}{16}}\,t$$

$$i(t) = 0.2\,\delta(t) + \frac{0.8}{\sqrt{15}}\,e^{-1/4\,t}\sin\left(\sqrt{\frac{15}{16}}\,t\right)$$

PART 3

PLOT ALL OF THEM

SYSTEM $\quad L\ddot{q} + R\dot{q} + \frac{1}{C}q = v(t) \qquad s^2 + 2\zeta\omega_n s + \omega_n^2$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

	M	c	k	ζ	ω_n	
CASE 1	10	20	10	1	1	CRITICALLY DAMPED
CASE 2	10	40	10	2	1	OVERDAMPED
CASE 3	10	5	10	0.25	1	UNDERDAMPED

$$\omega_n^2 = \frac{k}{M}$$

$$2\zeta\omega_n = c/m$$

$$\zeta = \frac{c/m}{2\omega_n}$$

PART 4

CASE 3 : $\ddot{q} + 0.5\dot{q} + q = e(t)$

$$\frac{Q(s)}{E(s)} = \frac{1}{s^2+0.5s+1} = G(s)$$

$$G(j\omega) = \frac{1}{(j\omega)^2+0.5(j\omega)+1} = \frac{1}{j^2\omega^2+0.5j\omega+1} = \frac{1}{-\omega^2+1+0.5j\omega}$$

$$= \frac{1}{(1-\omega^2)+0.5\omega j} \cdot \frac{(1-\omega^2)-0.5\omega j}{(1-\omega^2)-0.5\omega j}$$

$$G(j\omega) = \frac{(1-\omega^2)-0.5\omega j}{(1-\omega^2)^2-0.5^2\omega^2 j^2} = \frac{(1-\omega^2)-0.5\omega j}{(1-\omega^2)^2+0.5^2\omega^2} = \frac{(1-\omega^2)-0.5\omega j}{(1-\omega^2)^2+0.25\omega^2}$$

$$(1-\omega^2)^2 = (1-\omega^2)(1-\omega^2) = 1-2\omega^2+\omega^4$$

$$\boxed{G(j\omega) = \frac{(1-\omega^2)-0.5\omega j}{\omega^4-1.75\omega^2+1}} = \frac{(1-\omega^2)-0.5\omega j}{(1-\omega^2)^2+0.25\omega^2} = G(j\omega)$$

$$M_G(\omega) = |G(j\omega)| = \left[\left(\frac{1-\omega^2}{(1-\omega^2)^2+0.25\omega^2}\right)^2 + \left(\frac{-0.5\omega}{(1-\omega^2)^2+0.25\omega^2}\right)^2\right]^{1/2}$$

$$= \left[\frac{(1-\omega^2)^2}{((1-\omega^2)^2+0.25\omega^2)^2} + \frac{0.25\omega^2}{((1-\omega^2)^2+0.25\omega^2)^2}\right]^{1/2}$$

$$= \left[\frac{(1-\omega^2)^2+0.25\omega^2}{((1-\omega^2)^2+0.25\omega^2)^2}\right]^{1/2} = \boxed{\frac{1}{\sqrt{(1-\omega^2)^2+0.25\omega^2}} = M_G(\omega)}$$

$$\phi(\omega) = \tan^{-1} \frac{-0.5\omega \big/ \cancel{(1-\omega^2)+0.25\omega^2}}{1-\omega^2 \big/ \cancel{(1-\omega^2)+0.25\omega^2}} = \tan^{-1} \frac{-0.5\omega}{1-\omega^2}$$

$$\boxed{\phi(\omega) = -\tan^{-1} \frac{0.5\omega}{1-\omega^2}} \qquad \boxed{M(\omega) = \frac{1}{\sqrt{(1-\omega^2)^2+0.25\omega^2}}}$$

(c) $\omega = 0.05$ rad/sec

$M(\omega) = 1.00$ dB

$\phi(\omega) = -1.43$ rad

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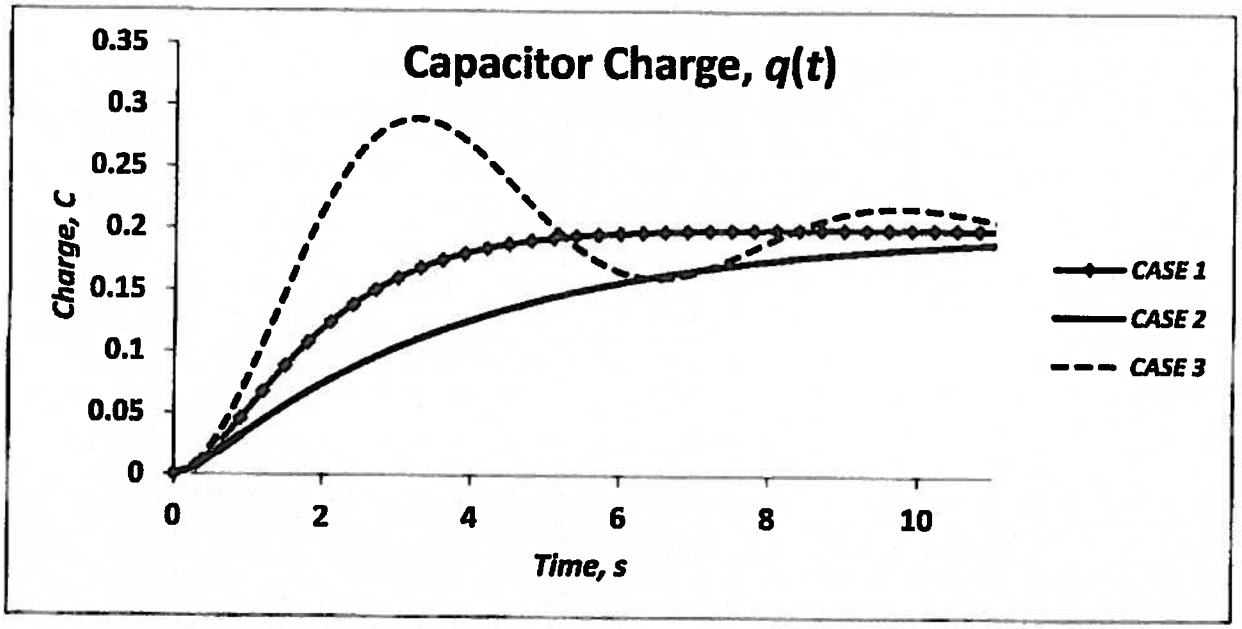
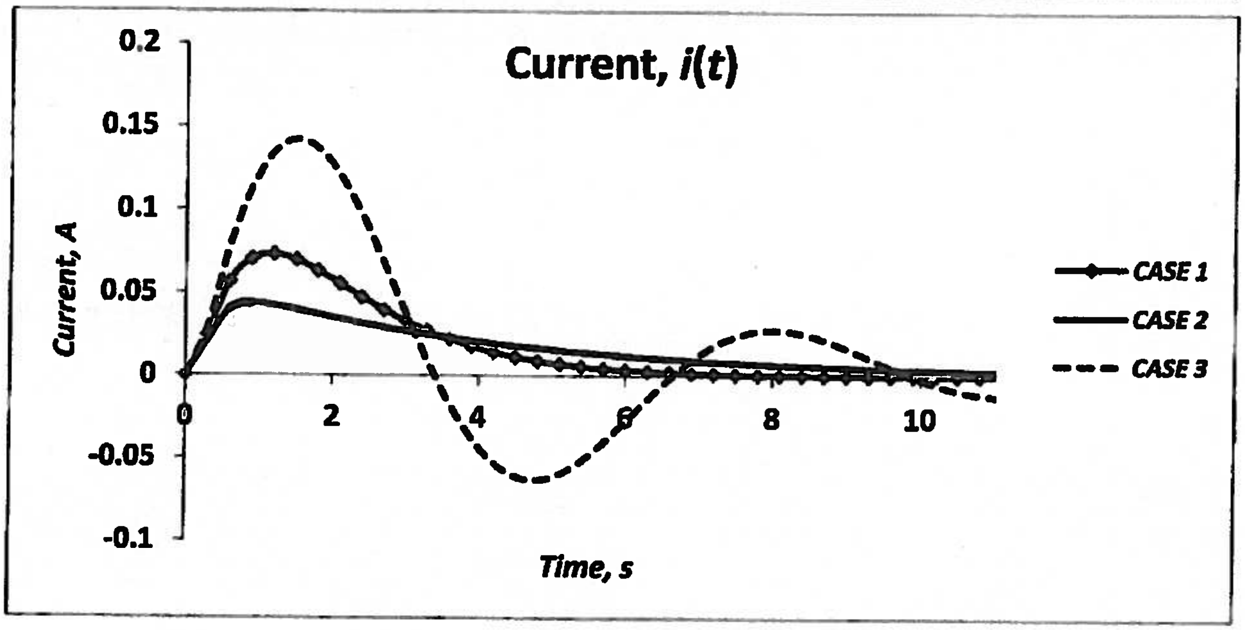

Capacitor Charge, q(t)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Charge, C
0
2
4
6
8
10
Time, s
CASE 1
CASE 2
CASE 3


Current, i(t)
0.2
0.15
0.1
0.05
0
-0.05
-0.1
Current, A
0
2
4
6
8
10
Time, s
CASE 1
CASE 2
CASE 3

Part 5

Our Case 3 transfer function is defined as

$$G(s) = \frac{1}{s^2 + 0.5^2 + 1}$$

The frequency response function is defined as

$$G(j\omega) = \frac{(1-\omega^2) - 0.5\omega j}{(1-\omega^2)^2 + 0.25\omega^2}$$

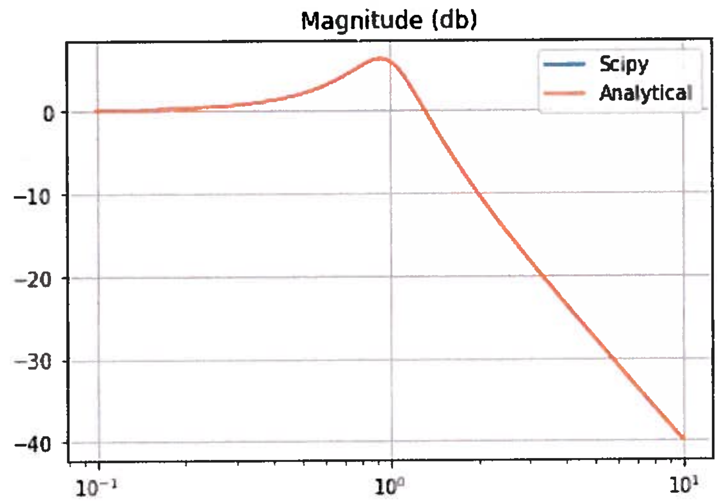
The magnitude and phase are given by

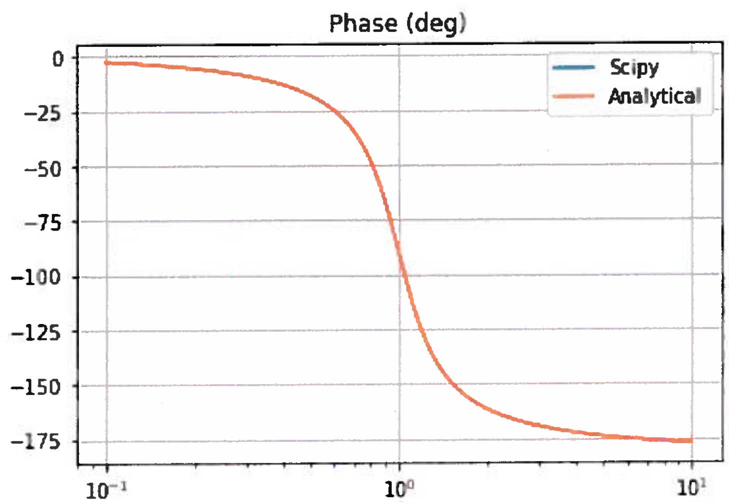
$$M = \frac{1}{\sqrt{(1-\omega^2)^2 + 0.25\omega^2}},$$
$$\phi = -\arctan\frac{0.5\omega}{1-\omega^2}.$$

In [29]:

```
import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

## define analytical functions
def magnitude(w):
    mag = 1/np.sqrt((1-w**2)**2 + 0.25 * w**2)
    return mag

def phi(w):
    phi = - np.arctan2(0.5*w, 1-w**2)
    return phi*180/np.pi

## our system
num = 1
den = [1, 0.5, 1]

sys = signal.TransferFunction(num,den)
w, mag, phase = signal.bode(sys)
```

In [25]:
```
## plot everything

plt.figure()
plt.title('Magnitude (db)')
plt.semilogx(w, mag, label='Scipy')     # Bode magnitude plot
plt.semilogx(w, 20*np.log10(magnitude(w)), label='Analytical')
plt.grid()
plt.legend()

plt.figure()
plt.title('Phase (deg)')
plt.semilogx(w, phase, label='Scipy')  # Bode phase plot
plt.semilogx(w, phi(w), label='Analytical')
plt.grid()
plt.legend()
plt.show()
```





In [30]:
```
print("M(0.05) = %f" % magnitude(0.05))
print("Phi(0.05) = %f " %  phi(0.05))
```

```
M(0.05) = 1.002192
Phi(0.05) = -1.435684
```