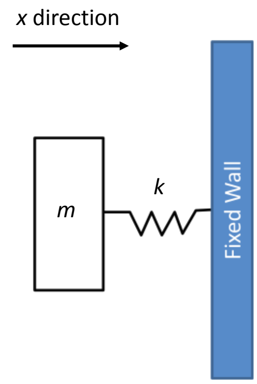
**The George Washington University**
MAE 3134 – Linear System Dynamics
Spring 2015

**Midterm Exam**

**Problem # 1 [25 points]**

An engineer is working with a vibratory system and has determined that its transfer function is $G(s) = \frac{X(s)}{F(s)} = \frac{1}{(s+1)^2}$ . Calculate the time-dependent response of this system to an input consisting of a double hammer impact. The first impact is given at $t$ = 1 s and the second at $t$ = 1.25 s. Both impacts have identical magnitude $\gamma$ Kg m/s. The initial velocity and position (before the impacts are given) are zero.

**Problem # 2 [25 points]**



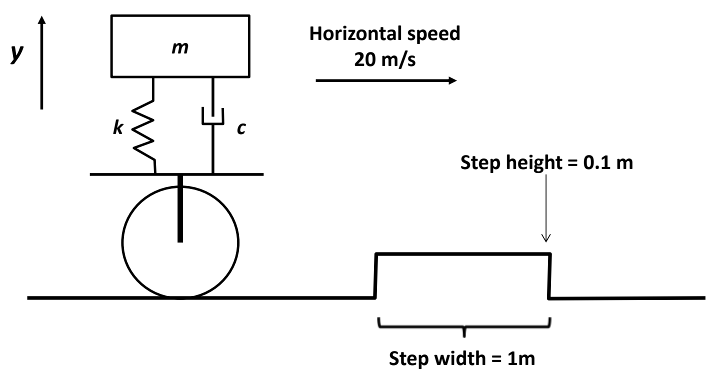

A scientist is conducting experiments with the undamped oscillator shown in the figure. The oscillator is set into motion with the initial conditions, $x(0) = 2\,m$, and $\dot{x}(0) = 0$. $m$ = 4 Kg and $k$ = 9 N/m.

(i) Derive an expression for the time-dependent motion of the mass, $x$(t). ***[10 points]***

**(ii)** The scientist wishes to stop the motion of the mass through a single impact, which is to be given when the mass crosses the equilibrium position for the *second* time. Provide an expression for the required $f$(t). ***[15 points]***

**Problem # 3 [50 points]**

Consider the model for a car-tire system in the figure, in which the vertical position of the mass exhibits a time-dependent response due to changes in the road topography. The parameters are $m$ = 1000 Kg, $k$ = 4000 N/m and $c$ = 2000 Ns/m.




(i) Write the equation of motion of the system. ***[5 points]***

(ii) Derive the transfer function of the system for the vertical motion of the mass, taking the road height as the input. ***[10 points]***

(iii) Provide an expression for the response of the mass in the Laplace domain, $Y(s)$ when the input is a unit step ($y_{tire}$ = u(t)) and the initial conditions are zero. ***[5 points]***

(iv) Calculate the time-dependent response for the Laplace-domain expression you derived for $Y(s)$ in step (iii). ***[15 points]***

(v) Calculate the time-dependent vertical response of the mass in the car-tire system to the road feature shown in the figure. Take $t$ = 0 as the instant when the tire encounters the up-step. The initial conditions in the vertical direction $y(0)\,and\;\dot{y}(0)$ are both zero. The car is traveling at a horizontal speed of 20 m/s. ***[15 points]***

## Table of Laplace Transforms

| | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ | | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ |
|---|---|---|---|---|---|
| 1. | $1$ | $\frac{1}{s}$ | 2. | $\mathbf{e}^{at}$ | $\frac{1}{s-a}$ |
| 3. | $t^n, \quad n=1,2,3,\ldots$ | $\frac{n!}{s^{n+1}}$ | 4. | $t^p, p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$ |
| 5. | $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$ | 6. | $t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$ | $\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$ |
| 7. | $\sin(at)$ | $\frac{a}{s^2+a^2}$ | 8. | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| 9. | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ | 10. | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| 11. | $\sin(at)-at\cos(at)$ | $\frac{2a^3}{(s^2+a^2)^2}$ | 12. | $\sin(at)+at\cos(at)$ | $\frac{2as^2}{(s^2+a^2)^2}$ |
| 13. | $\cos(at)-at\sin(at)$ | $\frac{s(s^2-a^2)}{(s^2+a^2)^2}$ | 14. | $\cos(at)+at\sin(at)$ | $\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$ |
| 15. | $\sin(at+b)$ | $\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$ | 16. | $\cos(at+b)$ | $\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$ |
| 17. | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ | 18. | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| 19. | $\mathbf{e}^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ | 20. | $\mathbf{e}^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| 21. | $\mathbf{e}^{at}\sinh(bt)$ | $\frac{b}{(s-a)^2-b^2}$ | 22. | $\mathbf{e}^{at}\cosh(bt)$ | $\frac{s-a}{(s-a)^2-b^2}$ |
| 23. | $t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ | 24. | $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ |
| 25. | $u_c(t)=u(t-c)$ Heaviside Function | $\frac{\mathbf{e}^{-cs}}{s}$ | 26. | $\delta(t-c)$ Dirac Delta Function | $\mathbf{e}^{-cs}$ |
| 27. | $u_c(t)f(t-c)$ | $\mathbf{e}^{-cs}F(s)$ | 28. | $u_c(t)g(t)$ | $\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$ |
| 29. | $\mathbf{e}^{ct}f(t)$ | $F(s-c)$ | 30. | $t^n f(t), \quad n=1,2,3,\ldots$ | $(-1)^n F^{(n)}(s)$ |
| 31. | $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)\,du$ | 32. | $\int_0^t f(v)\,dv$ | $\frac{F(s)}{s}$ |
| 33. | $\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ | 34. | $f(t+T)=f(t)$ | $\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$ |
| 35. | $f'(t)$ | $sF(s)-f(0)$ | 36. | $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| 37. | $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ | | | |

# MIDTERM EXAM SOLUTION

## PROBLEM #1

$$G(s) = \frac{1}{(s+1)^2} \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

Consider a single input at $t=0$, $f(t) = \gamma\delta(t)$. The response to this input can be calculated using

$$X(s) = G(s)F(s)$$

$$\mathcal{L}\left[\gamma\delta(t)\right] = \gamma \Rightarrow X(s) = G(s)\gamma$$

$$= \frac{1}{(s+1)^2}\gamma$$

To find the time dependent response we have to invert $X(s)$. Using the table (item #23) we find that

$$x(t) = \gamma t e^{-t}$$

Using time invariance we find that for an input $f(t) = \gamma\delta(t - t_0)$ the output is

$$x(t) = \gamma(t - t_0)e^{-(t-t_0)}u(t-t_0)$$

In our case, $f(t) = \gamma\delta(t-1) + \gamma(t-1.25)$ so, using linearity, the desired output to the double impact is:

$$\boxed{x(t) = \gamma\left[(t-1)e^{-(t-1)}u(t-1) + (t-1.25)e^{-(t-1.25)}u(t-1.25)\right]}$$

## PROBLEM #2

(i) We know that $x(0) = 2m$ and $\dot{x}(0) = 0$. We also know that the response of an undamped oscillator to initial conditions is a pure sinusoidal. A sinusoidal with a maximum, $x = 2$ at $t = 0$ and zero slope $(\dot{x}(0) = 0)$ is:

$$x(t) = 2\cos(\omega_n t)$$

We know that the undamped oscillation frequency is $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$ rad/s.

$$\Rightarrow x(t) = 2\cos\left(\tfrac{3}{2}t\right)$$

Alternatively, we could have found this solution using the Laplace transform method:

(i) Equation of motion: $m\ddot{x} + kx = 0$

Laplace transform:

$$m[s^2 X(s) - s\underbrace{(2)}_{x(0)}] + kX(s) = 0$$

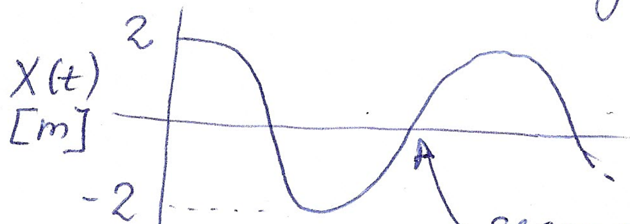
$$\Rightarrow X(s)\left[4s^2 + 9\right] = 8s$$

$$\Rightarrow X(s) = \frac{8}{4s^2+9} = \frac{2s}{s^2 + \frac{9}{4}} = \frac{2s}{s^2 + \left(\frac{3}{2}\right)^2}$$

which can be easily inverted using the table (item #8) to give

$$x(t) = 2\cos\left(\tfrac{3}{2}t\right), \text{ as above}$$

(ii) We now look for the time of the second axis crossing of $x(t)$.




$$\frac{3}{2}t = \frac{3}{2}\pi \Rightarrow t = \pi$$

We can find the velocity at $t=\pi$ by differentiation of $x(t)$:

$$\dot{x}(t) = 2\left(-\frac{3}{2}\sin\left(\frac{3}{2}t\right)\right)$$

$$\Rightarrow \dot{x}(\pi) = -3\sin\left(\frac{3}{2}\pi\right) = 3 \text{ m/s}$$

The momentum is $mv = 4\,\text{Kg} \cdot 3\,\text{m/s}$
$= 12\,\text{Kg}\cdot\frac{m}{s}$. Therefore, we need an impact that removes this amount of momentum, at $t=\pi$ (note the negative coefficient)

$$\Rightarrow \boxed{f(t) = -12\,\text{Kg}\frac{m}{s}\,\delta(t-\pi)}$$

## PROBLEM 3

(i) $m\ddot{y} = -c(\dot{y} - \dot{y}_{tire}) - K(y - y_{tire})$

$$m\ddot{y} = -c\dot{y} + c\dot{y}_{tire} - Ky + Ky_{tire}$$

$$m\ddot{y} + c\dot{y} + Ky = c\dot{y}_{tire} + Ky_{tire}$$

Introducing the values of $m$, $c$ and $k$:

$$1000\ddot{y} + 2000\dot{y} + 4000y = 2000\dot{y}_{tire} + 4000y_{tire}$$

$$\Rightarrow \boxed{\ddot{y} + 2\dot{y} + 4y = 2\dot{y}_{tire} + 4y_{tire}}$$

(ii) To derive the transfer function we take the Laplace transform of the equation with zero initial conditions:

$$s^2Y(s) + 2sY(s) + 4Y(s) = 2sY_{tire}(s) + 4Y_{tire}(s)$$

$$Y(s)\left[s^2 + 2s + 4\right] = Y_{tire}(s)\left[2s + 4\right]$$

$$\Rightarrow \boxed{G(s) = \frac{Y(s)}{Y_{tire}(s)} = \frac{2s + 4}{s^2 + 2s + 4}}$$

(iii) We are now told that $y_{tire}(t) = u(t)$.

We know that $\mathcal{L}\left[u(t)\right] = \frac{1}{s}$

$$\Rightarrow Y_{tire}(s) = \frac{1}{s} \quad \text{(item \# 25 in the table)}$$

Now, $Y(s) = G(s)\,Y_{tire}(s)$

$$\Rightarrow \boxed{Y(s) = \left[\frac{2s + 4}{s^2 + 2s + 4}\right]\frac{1}{s}}$$

This is the response to a unit input in the Laplace domain.

(iv) We now need to invert

$$Y(s) = \left[\frac{2s+4}{s^2+2s+4}\right]\left[\frac{1}{s}\right]$$

Using partial fractions

$$\frac{2s+4}{s^2+2s+4} \cdot \frac{1}{s} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+4}$$

$$\Rightarrow 2s+4 = A(s^2+2s+4) + (Bs+C)s$$
$$\Rightarrow 2s+4 = As^2 + 2As + 4A + Bs^2 + Cs$$
$$\Rightarrow 2s+4 = s^2(A+B) + s(2A+C) + 4A$$

We get three equations:

I $\quad A+B=0$

II $\quad 2A+C=2$

III $\quad 4A=4 \quad \Rightarrow \boxed{A=1} \Rightarrow \boxed{B=-1}$ from I

Using II $\quad 2(1)+C=2 \quad \Rightarrow \boxed{C=0}$

Our expression becomes

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+2s+4}$$

$$= \frac{1}{s} - \frac{s}{(s+1)^2+4} = \frac{1}{s} - \frac{(s+1)}{(s+1)^2+3} + \frac{1}{(s+1)^2+3}$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1)^2+(\sqrt{3})^2} + \frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2} \cdot \frac{1}{\sqrt{3}}$$

This is easily inverted using items # 25, 20 and 19 of the table, respectively, to give:

$$y(t) = u(t) - e^{-t}\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}} e^{-t}\sin(\sqrt{3}t)$$

(v) To solve this part we note that the feature on the road can be written as the sum of two unit steps. The first one is positive and happens at $t=0$:

$$0.1\ u(t)$$

step height from the figure

The second one is negative and happens at

$$t = \frac{1\ m}{20\ m/s}$$

1 m ← step width (from the figure)

20 m/s ← car speed

So the second step is $-0.1\,u(t-0.05)$

The total input is then:

$$y_{tire}(t) = 0.1\,u(t) - 0.1\,u(t-0.05)$$

Using linearity and time invariance and the solution of part (iv), the desired solution here is:

$$\boxed{\begin{aligned} y(t) = {} & 0.1\left[u(t) - e^{-t}\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}} e^{-t}\sin(\sqrt{3}t)\right] \\ & -0.1\left[u(t-0.05) - e^{-(t-0.05)}\cos(\sqrt{3}(t-0.05))\,u(t-0.05) \right. \\ & \left. + \frac{1}{\sqrt{3}} e^{-(t-0.05)}\sin(\sqrt{3}(t-0.05))\,u(t-0.05)\right] \end{aligned}}$$