MAE 3134 – Linear System Dynamics
Spring 2015

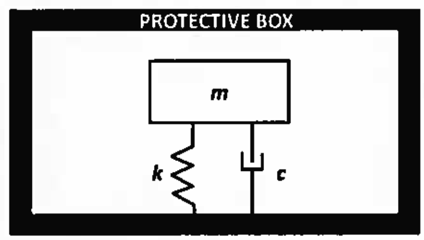
Homework # 3
Due Thursday, February 19th at the beginning of class

NOTE: Please make sure to show and explain all the steps followed to arrive at the solution. Otherwise we are unable to assess your understanding of the material and give you credit for your answers.

1. A kitchen door with a mass moment of inertia J_{door} = 20 Kgm^2 is equipped with (i) a damper having a rotational damper constant of 48 Nms/rad and (ii) a rotational spring with stiffness 28.8 Nm/rad. Assuming that the initial position of the door is $\theta = 0$ (closed) and its initial angular speed is 4 rad/s,

(a) Calculate the analytical solution for the door displacement as a function of time.
(b) Plot the analytical solution for the door displacement as a function of time.
(c) Determine *analytically* the maximum displacement of the door with respect to the equilibrium (closed) position.
(d) Calculate *numerically or analytically* the amount of time it takes for the door to return to a displacement equal to 0.01 times the maximum displacement.

2. Consider an electronic component of mass m = 1 Kg packed inside a box that is dropped from a shelf. The box impacts the floor upright with an initial downward velocity of 10 m/s. The electronic component is attached to the box through a spring of force constant k = 9 N/m and a damper with damping constant = 10 Ns/m. Assume that (i) the box does *not* bounce off the floor (i.e., the box sticks to the floor upon impact), (ii) the action of the spring and damper is perfectly aligned with the vertical direction, and (iii) the electronic component is located at $x = -mg/k$ just before the impact, such that we do not need to consider the weight in solving the problem (for simplicity).



(a) Derive an analytical expression for the time-dependent motion of the electronic component.
(b) What is the force experienced by the electronic component upon impact?
(c) What can parameters of the system can be changed to reduce that force, and how do those changes relate to the size of the box that is required to protect the electronic component? Explain your answer mathematically.

3. For the door of problem 1,

(a) Derive the transfer function when the input is an arbitrary torque, $\tau(t)$?
(b) What is the response of the door in the Laplace domain for each of the following three imputs?
    (a) $\tau(t)$ = A δ(t) Nm
    (b) $\tau(t)$ = A cos (ωt)
    (c) $\tau(t)$ = A cosh (ωt)

HOMEWORK 3 SOLUTION

PROBLEM 1

(a) Equation of motion:

$$J\ddot{\theta} + C\dot{\theta} + K\theta = 0$$

$$20\ddot{\theta} + 48\dot{\theta} + 28.8\theta = 0$$

$\theta(0) = 0$
$\dot{\theta}(0) = 4$ rad/s
$J = 20$ Kg m^2
$C = 48$ Nms/rad
$K = 28.8$ Nm/rad

We apply the Laplace transform:

$$20\left(s^2 T(s) - s\,\cancel{\theta(0)}^{\,0} - \dot{\theta}(0)\right) + 48\left(s\,T(s) - \cancel{\theta(0)}^{\,0}\right) + 28.8\,T(s) = 0$$

$$\Rightarrow 20s^2 T(s) - 20(4) + 48s\,T(s) + 28.8\,T(s) = 0$$

$$\Rightarrow T(s) = \frac{80}{20s^2 + 48s + 28.8} = \frac{4}{s^2 + 2.4s + 1.44}$$

We must now invert this expression.

We note that the denominator is a perfect square:

$$s^2 + 2.4s + 1.44 = (s+1.2)^2$$

$$\Rightarrow T(s) = \frac{4}{(s+1.2)^2}$$

Using the Laplace transform table, $\theta(t) = 4te^{-1.2t}$ rad.

(b) See attached plot

(c) To find the maximum displacement we look for the critical point of $\dot{\theta}(t)$

$$\dot{\theta}(t) = \frac{d}{dt}\theta(t) = 4\left[e^{-1.2t} + t(-1.2)e^{-1.2t}\right] = 0$$

$$\Rightarrow e^{-1.2t} = 1.2t\, e^{-1.2t}$$

$$\Rightarrow 1 = 1.2t \Rightarrow t = \frac{1}{1.2} \approx 0.83s$$

Therefore, $\theta_{max} = \theta(0.83) \approx 1.23$ rad.

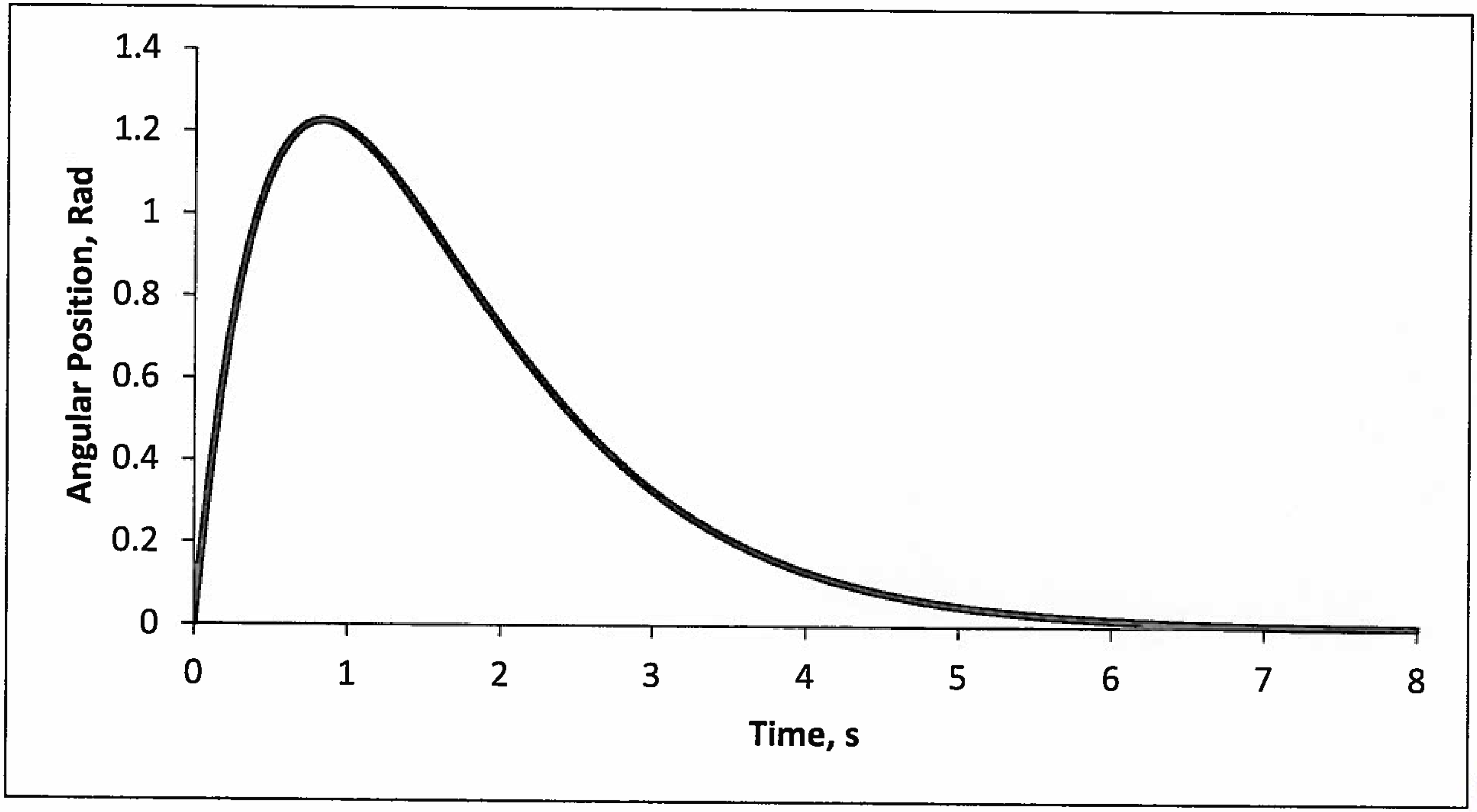
(d) The maximum displacement is 1.23 rad, so 0.01 times that is 0.0123 rad.

We are therefore looking for the solution to

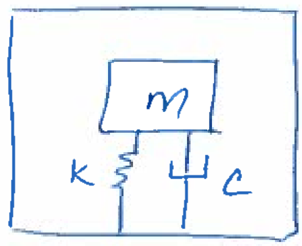
$$\theta(t) = 4t\, e^{-1.2t} = 0.0123$$

This can be easily found numerically and is $t \approx 6.36$ s.

Answer to (b)



PROBLEM 2



At the time of impact, $t=0$, we make $x(t)=0$ and $\dot{x}(t) = -10\ m/s$

We know that,

$m = 1\ Kg$

$c = 10\ Ns/m$

$k = 9\ N/m$

(a) Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{(not considering the weight; Solving about the position } -\frac{mg}{k}\text{)}$$

$$\ddot{x} + 10\dot{x} + 9x = 0$$

Taking the Laplace transform,

$$1\left[s^2 X(s) - s\,\cancel{x(0)}^{\,0} - \dot{x}(0)\right] + 10\left(sX(s) - \cancel{x(0)}^{\,0}\right) + 9X(s) = 0$$

$$\Rightarrow s^2 X(s) - (-10) + 10s\,X(s) + 9X(s) = 0$$

$$\Rightarrow X(s) = \frac{-10}{s^2 + 10s + 9}$$

We now invert the above expression,

$$X(s) = \frac{-10}{s^2+10s+9} = \frac{-10}{(s^2+10s+25)+9-25}$$

$$\Rightarrow X(s) = \frac{-10}{(s+5)^2 - 16}$$

The above looks similar to

$$\mathcal{L}\left[e^{-\alpha t} \sin \omega t\right] = \frac{\omega}{(s+\alpha) + \omega^2}$$ with imaginary ω

We can rewrite $X(s)$ as:

$$X(s) = -10\left(\frac{i}{i}\right) \frac{1}{(s+5)^2 + (4i)^2}$$ with $i = \sqrt{-1}$

$$= \frac{-2.5}{i} \frac{4i}{(s+5)^2 + (4i)^2}$$

So $x(t) = \frac{-2.5}{i} e^{-5t} \sin(4i\, t)$

But we know that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$,

so $$x(t) = \frac{-2.5}{i} e^{-5t} \left[\frac{e^{+i(4it)} - e^{-i(4it)}}{2i}\right]$$

$$= 1.25\, e^{-5t} \left[e^{-4t} - e^{4t}\right]$$

$$\Rightarrow x(t) = 1.25\left(e^{-9t} - e^{-t}\right)$$

(b) To find the force at $t=0$, we find the acceleration, since $F = m\ddot{x}$.

$$x(t) = 1.25\left[e^{-9t} - e^{-t}\right]$$

$$\Rightarrow \dot{x}(t) = 1.25\left[-9e^{-9t} + e^{-t}\right]$$

$$\Rightarrow \ddot{x}(t) = 1.25\left[81\, e^{-9t} - e^{-t}\right]$$

at $t=0$, $\ddot{x}(0) = 1.25(80) = 100\ m/s^2$

$$\Rightarrow F = m\ddot{x} = (1\ Kg)(100\ m/s^2) = 100\ N$$

(c) We notice from the non-oscillatory nature of the solution that the system is overdamped, which we can verify by calculating the damping factor,

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{10\ N\ s/m}{2\sqrt{9\ N/m \cdot 1\ Kg}} = \frac{10}{6}$$

$$= 1.67 > 1$$

We can also see that right upon impact the forces experienced by the system come primarily (exclusively) from the damper.

Looking at the equation of motion,

$$\Sigma F_x = m\ddot{x} = -c\dot{x} - \underbrace{\cancel{kx}}_{\text{zero because } x(0)=0}$$

$$\Rightarrow \Sigma F_x\ (t=0) = -c\dot{x}$$

Therefore, we can reduce the initial force by reducing the value of c.

Tradeoffs: as we decrease the value of c, the system gradually goes from overdamped to underdamped. The latter case exhibits oscillations and larger displacements, which call for a larger box in order to keep the mass m from impacting the box. The stiffness of the spring also plays a role (stiffer spring = smaller displacements = smaller box required). In practice the engineer needs to evaluate the most likely forces and impacts applied to the box and design the spring and damper accordingly.

Problem 3

(a) We recall that the initial conditions are zero in the derivation of the transfer function, so we start with the equation of motion and $\theta(0) = 0$ and $\dot{\theta}(0) = 0$.

From problem #1, $20\ddot{\theta} + 48\dot{\theta} + 28.8\theta = 0$

Applying the Laplace transform and adding the input, we have,

$$\mathcal{L}\left[20\ddot{\theta} + 48\dot{\theta} + 28.8\theta = \underbrace{J(t)}_{\text{input}}\right]$$

$$20s^2\hat{\theta}(s) + 48s\hat{\theta}(s) + 28.8\hat{\theta}(s) = \hat{J}(s)$$

$$\hat{\Theta}(s)\left[20s^2 + 48s + 28.8\right] = \hat{J}(s)$$

$$\Rightarrow G(s) = \frac{\text{Output}}{\text{Input}} = \frac{1}{20s^2 + 48s + 28.8}$$

Now we analyze the various inputs and write an expression for the output.

$$\text{Output} = G(s) * \text{Input}$$

(a) $J(t) = A\,\delta(t) \Rightarrow \hat{J}(s) = A$

$$\Rightarrow \hat{\Theta}(s) = A \cdot G(s) = \frac{A}{20s^2 + 48s + 28.8}$$

(b) $J(t) = A\cos(\omega t) \Rightarrow \hat{J}(s) = \dfrac{As}{s^2 + \omega^2}$

$$\Rightarrow \hat{\Theta}(s) = \left(\frac{As}{s^2+\omega^2}\right) G(s)$$

$$= \left(\frac{As}{s^2+\omega^2}\right)\left(\frac{1}{20s^2 + 48s + 28.8}\right)$$

(c) $J(t) = A\cosh(\omega t) = \hat{J}(s) = \dfrac{As}{s^2 - \omega^2}$

$$\Rightarrow \hat{\Theta}(s) = \left(\frac{As}{s^2-\omega^2}\right) G(s)$$

$$= \left(\frac{As}{s^2-\omega^2}\right)\left(\frac{1}{20s^2 + 48s + 28.8}\right)$$

In each case above, $\Theta(t)$ can be found by inverting $\hat{\Theta}(s)$. That is, $\Theta(t) = \mathcal{L}^{-1}\left[\hat{\Theta}(s)\right]$.