LINEARIZATION CH 7-4

- STATE SPACE LETS US MODEL NONLINEAR SYSTEMS $\dot{x} = f(x) \longrightarrow x(t)$

- EVEN IF WE CAN REPRESENT IN STATE SPACE DOES NOT MEAN WE CAN SOLVE IT!

- FIND THE SYSTEM REPRESENTATION IN STATE SPACE → THEN LINEARIZE ABOUT AN EQUILIBRIUM

SIMPLE PENDULUM

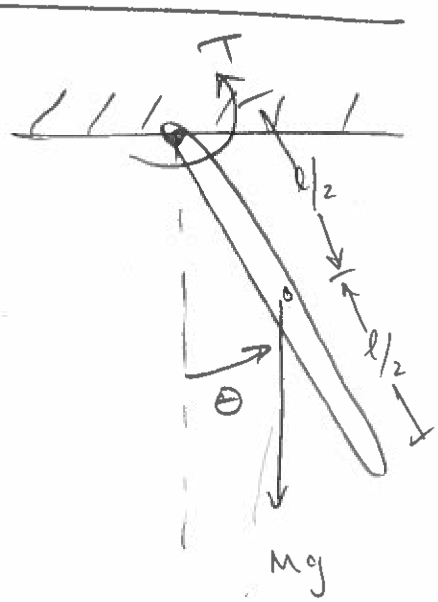
- CLASSIC PROBLEM IN CONTROLS

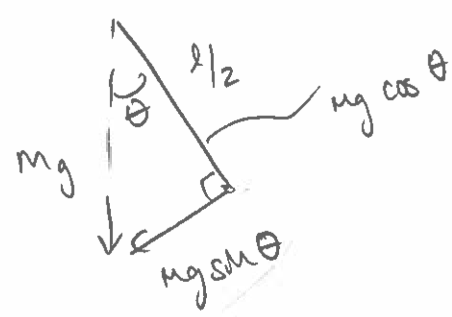
WEIGHT IS EVENLY DISTRIBUTED

Mg - WEIGHT

T - APPLIED TORQUE

l - LENGTH OF PENDULUM





$$\boxed{J\ddot{\theta} + \frac{Mgl}{2}\sin\theta = T}$$

$$\alpha f(x) + \beta f(y) = f(\alpha x + \beta y)$$

LINEAR FCN ⟶ PRINCIPLE OF SUPERPOSITION APPLIES.

NONLINEAR TERMS

- TRIG $\quad e^{x}$
- x^2, x^3, x^4

STATE SPACE

$x_1 = \theta$

$x_2 = \dot{\theta}$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgl}{2J}\sin x_1 + \frac{T}{J} \end{aligned}}$$

CAN'T DO THIS. $Ax + Bu$

WE NEED A LINEAR SYSTEM TO APPLY OUR TOOLS.

- FIND EQUILIBRIUM POINTS

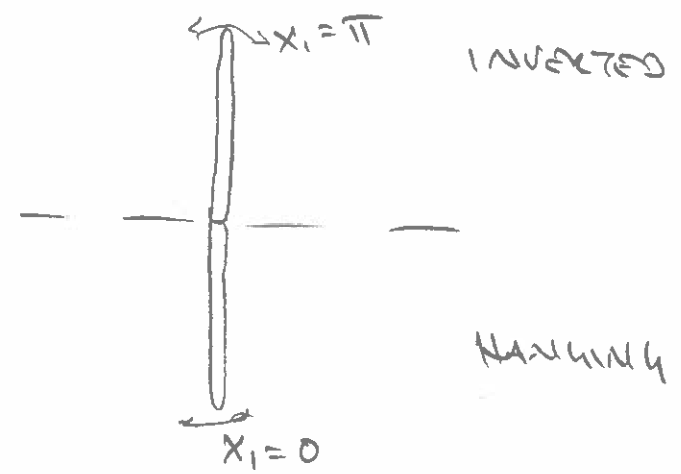
$$\dot{x} = f(x) = 0 \quad \leftarrow \text{FIND } x^* \text{ S.T } f(x) = 0$$

$$0 = x_2$$

$$0 = -\frac{mgl}{2}\sin x_1 + \frac{0}{J} \quad \leftarrow \text{NO INPUT}$$

EQ. POINTS

$$\left.\begin{aligned} x_1^* &= \pm 0, k\pi \\ x_2^* &= 0 \end{aligned}\right\}$$



LINEARIZED STATE SPACE

$$A = \frac{\partial}{\partial x} f(x) \Big|_{x=x^*} \qquad B = \frac{\partial}{\partial u} f(x) \Big|_{x=x^*,\, u=u^*}$$

$$\boxed{\delta \dot{x} = A\, \delta x + B\, \delta u}$$

$$f_1 = x_2$$

$$f_2 = -\frac{mgl}{2J} \sin x_1 + \frac{T}{J}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{x=x^*} = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{2J} \cos x_1^* & 0 \end{bmatrix}_{x=x^*}$$

EQ. PT.

$$B = \begin{bmatrix} \partial f_1 / \partial u \\ \partial f_2 / \partial u \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}$$

$$\boxed{\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{2J} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \delta u}$$

ABOUT HANGING

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

VALID NEAR EQ. PT.

WHERE ARE THE POLES WHEN $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

HOW ABOUT $x^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

A CONTINUOUS TIME SYSTEM

$$\dot{x} = F(x, u)$$

$$y = H(x, u)$$

← NONLINEAR IN GENERAL.

x^* IS AN EQUILIBRIUM STATE CORRESPONDING TO CONSTANT INPUT u^*

$$\rightarrow \quad F(x^*, u^*) = 0 = \dot{x}$$

$$H(x^*, u^*) = y^*$$

← CONSTANT OUTPUT.

INTRODUCE THE PERTURBED STATE INPUT, OUTPUT

$$\delta x = x - x^* \qquad \delta u = u - u^* \qquad \delta y = y - y^*$$

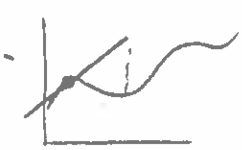
$$\delta \dot{x} = F(x^* + \delta x, u^* + \delta u)$$

$$\delta y = H(x^* + \delta x, u^* + \delta u) - H(x^*, u^*)$$

FROM THE TAYLOR SERIES

HIGHER ORDER TERMS

$$\rightarrow \quad f(x) \approx f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*} (x - x^*) + \text{HOT} \ldots$$



$$F(x^* + \delta x, u^* + \delta u) \approx F(x^*, u^*) + \left.\frac{\partial F}{\partial x}\right|_{x^*, u^*} \delta x + \left.\frac{\partial F}{\partial u}\right|_{(*)} \delta u$$

$$= \left.\frac{\partial F}{\partial x}\right|_{(*)} \delta x + \left.\frac{\partial F}{\partial u}\right|_{(*)} \delta u$$

JACOBIAN

EVALUATED AT EQ. POINT.

$$H(x^* + \delta x, u^* + \delta u) \approx H(x^*, u^*) + \left.\frac{\partial H}{\partial x}\right|_{(*)} \delta x + \left.\frac{\partial H}{\partial u}\right|_{(*)} \delta u \quad (-H$$

$$\delta \dot{x} \approx \left.\frac{\partial F}{\partial x}\right|_{(*)} \delta x + \left.\frac{\partial F}{\partial u}\right|_{(*)} \delta u$$

$$\delta y \approx \left.\frac{\partial H}{\partial x}\right|_{(*)} \delta x + \left.\frac{\partial H}{\partial u}\right|_{(*)} \delta u$$

$$A = \left.\frac{\partial F}{\partial x}\right|_{(*)} \qquad B = \left.\frac{\partial F}{\partial u}\right|_{(*)} \qquad C = \left.\frac{\partial H}{\partial x}\right|_{(*)} \qquad D = \left.\frac{\partial H}{\partial u}\right|_{(*)}$$

SOME STATE SPACE EXAMPLES

$$2\ddot{q}_1 + \ddot{q}_2 + \sin q_1 = 0$$

$$\ddot{q}_1 + 2\ddot{q}_2 + \sin q_2 = 0.$$

FIND: S-S. REPRESENTATION

LET'S DE COUPLE THE EQUATIONS

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} -\sin q_1 \\ -\sin q_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -\sin q_1 \\ -\sin q_2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -\sin q_1 \\ -\sin q_2 \end{bmatrix}$$

$$\ddot{q}_1 = -2/3 \sin q_1 + \frac{1}{3} \sin q_2$$

$$\ddot{q}_2 = 1/3 \sin q_1 - \frac{2}{3} \sin q_2$$

STATE VARIABLES

$x_1 = q_1$

$x_2 = \dot{q}_1$

$x_3 = q_2$

$x_4 = \dot{q}_2$

NONLINEAR STATE EQ.

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{2}{3} \sin x_1 + \frac{1}{3} \sin x_3$

$\dot{x}_3 = x_4$

$\dot{x}_4 = \frac{1}{3} \sin x_1 - \frac{2}{3} \sin x_3$

ANOTHER EXAMPLE

$\ddot{q}_1 \longrightarrow \ddot{q}_1 + \dot{q}_2 + q_1^3 = 0$

$\dot{q}_2 \longrightarrow \dot{q}_1 + \dot{q}_2 + q_2^3 = 0$

DECOUPLE

$\longrightarrow$ SOLVE FOR $\ddot{q}_1, \dot{q}_2$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$

$\ddot{q}_1 = -q_1^3 + \dot{q}_1 + q_2^3$

$\dot{q}_2 = -\dot{q}_1 - q_2^3 \longrightarrow$ STATE SPACE

$x_1 = q_1$

$x_2 = \dot{q}_1$

$x_3 = q_2$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + x_2 + x_3^3 \\ \dot{x}_3 &= -x_2 - x_3^3 \end{aligned}} \not\Longrightarrow \dot{x} = Ax + Bu$$

NUMBERS

DECOUPLE

STATE SPACE

$\longrightarrow$ LINEARIZE

EXAMPLE - LINEARIZATION

$$\dot{x}_1 = 2x_2(1-x_1) - x_1 = F_1(x_1, x_2)$$

$$\dot{x}_2 = 3x_1(1-x_2) - x_2 = F_2(x_1, x_2)$$

FIND EQUILIBRIUM POINTS $x_1^*, x_1^e, \ldots$

$$\left.\begin{array}{l} F_1(x_1^*, x_2^*) = 0 \\ F_2(x_1^*, x_2^*) = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} 0 = 2x_2^*(1-x_1^*) - x_1^* & (1) \\ 0 = 3x_1^*(1-x_2^*) - x_2^* & (2) \end{array}$$

ONE SOLUTION $\bar{x}^* = [0 \quad 0]^T$ (TRIVIAL)

SOLVE FOR x_1^*, x_2^* FOR OTHER EQ. PT.

(1) $2x_2^*(1-x_1^*) = x_1^* \Rightarrow x_2^* = \dfrac{x_1^*}{2-2x_1^*} \rightarrow (2)$

(2) $3x_1^* - 3x_1^*x_2^* - x_2^* = 0 \Rightarrow 3x_1^* - 3x_1^*\left(\dfrac{x_1^*}{2-2x_1^*}\right) - \dfrac{x_1^*}{(2-2x_1^*)} =$

$\Rightarrow (6x_1^* - 6(x_1^*)^2) - 3(x_1^*)^2 - x_1^* = 0$

$\rightarrow 5 - 9x_1^* = 0 \rightarrow x_1^* = 5/9$

QUADRATIC EQ.

$$x_2^* = \frac{5/9}{2-2(5/9)} = \frac{5}{8}$$

EQ. POINTS $\rightarrow \bar{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{x}^* = \begin{bmatrix} 5/9 \\ 5/8 \end{bmatrix}$

LINEARIZE ABOUT ZERO EQ. STATE $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$F_1(x_1, x_2,) = 2x_2 - 2x_2 x_1 - x_1 = \dot{x}_1$ $\quad \dfrac{\partial F_1}{\partial x_1}$

$F_2(x_1, x_2,) = 3x_1 - 3x_1 x_2 - x_2 = \dot{x}_2$

$\left.\dfrac{\partial f_1}{\partial x_1}\right|_* = -2x_2^* - 1 \qquad \left.\dfrac{\partial F_1}{\partial x_2}\right|_* = 2 - 2x_1^*$

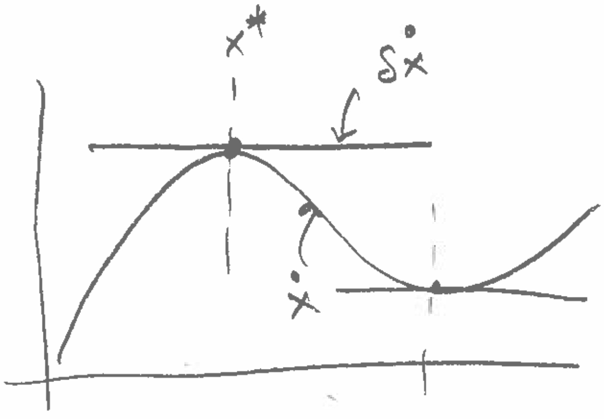
$\left.\dfrac{\partial F_2}{\partial x_1}\right|_* = 3 - 3x_2^* \qquad \left.\dfrac{\partial F_2}{\partial x_2}\right|_* = -3x_1^* - 1$

$\delta\dot{x}_1 = (-2x_2^* - 1)\,\delta x_1 + (2 - 2x_1^*)\,\delta x_2$

$\delta\dot{x}_2 = (3 - 3x_2^*)\,\delta x_1 + (-3x_1^* - 1)\,\delta x_2$

AT $\bar{x}^* = [0 \quad 0]^T$

$$\left.\begin{aligned} \delta\dot{x}_1 &= -\delta x_1 + 2\,\delta x_2 \\ \delta\dot{x}_2 &= 3\,\delta x_1 - \delta x_2 \end{aligned}\right\} \quad \begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$



THE LINEARIZATION ABOUT $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\downarrow$

$x^* = \begin{bmatrix} 5/9 \\ 5/8 \end{bmatrix} \leftarrow$

LETS LINEARIZE THESE SYSTEMS

USUALLY EASIEST TO LINEARIZE THEN

* FIND S.S. REPRESENTATION * ←

EX 1

$$\ddot{q}_1 = -q_1^3 + \dot{q}_1 + q_2^3$$

$$\dot{q}_2 = -\dot{q}_1 - q_2^3$$

FIND $\boxed{\text{EQ. POINT}}$ $q_1^*, q_2^* \longrightarrow \dot{q}_1^* = \ddot{q}_1^* = 0 = \dot{q}_2^*$

$$0 = -q_1^{*3} + 0 + q_2^{*3} \rightarrow q_1^* = 0$$

$$0 = 0 - q_2^{*3} \longrightarrow q_2^* = 0$$

$$q^e = (q_1^*, q_2^*, \cancel{\dot{q}_1^*}, \cancel{\dot{q}_2^*}, \cancel{\ddot{q}_1}, \cancel{\ddot{q}_2})$$

$$F_1: \left.\frac{\partial F_1}{\partial q_1}\right|_* \delta q_1 + \left.\frac{\partial F_1}{\partial \ddot{q}_1}\right|_{(*)} \delta \ddot{q}_1 + \left.\frac{\partial F_1}{\partial \dot{q}_1}\right|_{(*)} \delta \dot{q}_1 + \left.\frac{\partial F_1}{\partial q_2}\right|_{(*)} \delta q_2 = 0$$

$$-3q_1^2\Big|_{(*)} \delta q_1 - 1\Big|_{(*)} \delta \ddot{q}_1 + 1\,\delta \dot{q}_1 + 3q_2^2\Big|_{(*)} \delta q_2 = 0$$

$$F_2: \left.\frac{\partial F_2}{\partial \dot{q}_2}\right|_* \delta \dot{q}_2 + \left.\frac{\partial F_2}{\partial q_2}\right|_* \delta q_2 + \left.\frac{\partial F_2}{\partial \dot{q}_1}\right|_{(*)} \delta \dot{q}_1 = 0$$

$$-1\,\delta \dot{q}_2 - 3q_2^2\Big|_* \delta q_2 - 1\,\delta \dot{q}_1 = 0$$

EVALUATE AT EQ. PT. (PLUG IN) $\bar{q}^* = [0 \quad 0]^T$

DIFF. FROM $\bar{q}^*$

$$\left.\begin{aligned} \delta\ddot{q}_1 &= \delta\dot{q}_1 \\ \delta\dot{q}_2 &= -\delta\dot{q}_1 \end{aligned}\right\}$$

$x_1 = \delta q_1$ $\qquad$ $\dot{x}_1 = x_2$

$x_2 = \delta\dot{q}_1$ $\qquad$ $\dot{x}_2 = x_2$

$x_3 = \delta q_2$ $\qquad$ $\dot{x}_3 = -x_2$

$$\boxed{\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \delta x}$$

$$\boxed{\dot{x} = A\,x}$$

$x_1 = \delta q_1$ $\qquad$ $\dot{x}_1 = \delta\dot{q}_1 = x_2$

$x_2 = \delta\dot{q}_1$ $\qquad$ $\dot{x}_2 = \delta\ddot{q}_1 = \delta\dot{q}_1 = x_2$

$x_3 = \delta q_2$ $\qquad$ $\dot{x}_3 = \delta\dot{q}_2 = -\delta\dot{q}_1 = -x_2$

STATE EQ.

LINEARIZATION EXAMPLE

$$\ddot{r} - r\omega^2 + \frac{\mu}{r^2} = 0 \quad = F_1 \quad \leftarrow \text{TWO BODY DYNAMICS.}$$

$$r\dot{\omega} + 2\dot{r}\omega = 0 \quad = F_2 = f(r, \omega, \dot{r}, \dot{\omega}) = 0 \; (2)$$

EQ. POINTS. AT $r^*, \omega^* \Longrightarrow \dot{r} = \ddot{r} = \dot{\omega} = 0$

$$-r^{*3}\omega^{*2} + \mu = 0$$

(1) $\boxed{\mu = r^{*3}\omega^{*2}}$ ← (1), (2) AT EQUILIBRIUM.

→ CONSTANT VALUES OF $\boxed{r^*, \omega^*}$ ← SECOND EQ.

WHAT KIND OF MOTION IS THIS ?

ALL WE HAVE TO DO IS APPLY OUR RULE. (IMPLICIT LINEARIZATION) ←

$F_1(\ddot{r}, r, \omega)$

$$(1) \quad \left.\frac{\partial F_1}{\partial \ddot{r}}\right|_* \delta\ddot{r} + \left.\frac{\partial F_1}{\partial r}\right|_* \delta r + \left.\frac{\partial F_1}{\partial \omega}\right|_* \delta\omega = 0$$

$$\frac{\partial F_1}{\partial \ddot{r}} = 1 \qquad \frac{\partial F_1}{\partial r} = -\omega^2 - \frac{2\mu}{r^3} \qquad \frac{\partial F_1}{\partial \omega} = -2\omega r$$

$$\ddot{x} \quad + \quad a x \quad + \quad b y = 0$$

$$\boxed{\delta\ddot{r} - \left(\omega^{*2} + \frac{2\mu}{r^{*3}}\right)\delta r - 2\omega^* r^* \delta\omega = 0}$$

(2) $\frac{\partial F_2}{\partial r}\Big|_* \delta r + \frac{\partial F_2}{\partial \omega}\Big|_* \delta \omega + \frac{\partial F_2}{\partial \dot{r}}\Big|_* \delta \dot{r} + \frac{\partial F_2}{\partial \dot{\omega}}\Big|_* \delta \dot{\omega} = 0$

$$\frac{\partial F_2}{\delta r} = \dot{\omega} \quad \frac{\partial F_2}{\partial \omega} = 2\dot{r} \quad \frac{\partial F_2}{\partial \dot{r}} = 2\omega \quad \frac{\partial F_2}{\partial \dot{\omega}} = r$$

(2) $\boxed{\cancel{\dot{\omega}^*}\delta r + \cancel{2\dot{r}^*}\delta\omega + 2\omega^*\delta\dot{r} + r^*\delta\dot{\omega} = 0}$

(1) $\delta\ddot{r} - \left(\omega^{*2} + 2\frac{\mu}{r^{*3}}\right)\delta r - 2\omega^* r^* \delta\omega = 0$

USE $\mu = \omega^{*2} r^{*3}$ ← CONDITIONS FOR EQUILIBRIUM

$$2\omega^*\delta\dot{r} + r^*\delta\dot{\omega} = 0$$

$$\delta\ddot{r} - 3\omega^{*2}\delta r - 2r^*\omega^*\delta\omega = 0$$

STATE

$x_1 = \delta r \qquad \dot{x}_1 = x_2$

$x_2 = \delta\dot{r} \qquad \dot{x}_2 = 3\omega^{*2}x_1 + 2r^*\omega^* x_3$

$x_3 = \delta\omega \qquad \dot{x}_3 = -\frac{2\omega^*}{r^*}x_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 3\omega^{*2} & 0 & 2r^*\omega^* \\ 0 & -\frac{2\omega^*}{r^*} & 0 \end{bmatrix} x = \underline{Ax}$$

$x(t) \leftarrow \dot{x} = Ax + Bu$