CH-2 LAPLACE TRANSFORM

1/29/18

RECALL: GOAL IS TO TRANSFORM FROM

TIME DOMAIN $\xrightarrow{\mathcal{L}[\cdot]}$ S-(LAPLACE) DOMAIN

$$\ddot{x} + b\dot{x} + cx = g(t) \longrightarrow s^2X(s) + bsX(s) + cX(s) = G(s)$$

DEFINITION: FOR $f(t)$ S.T $f(t) = 0 \ \forall \ t < 0$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt\,[f(t)] = \int_0^\infty f(t)\,e^{-st}\,dt$$

SOME BASIC EXAMPLES:

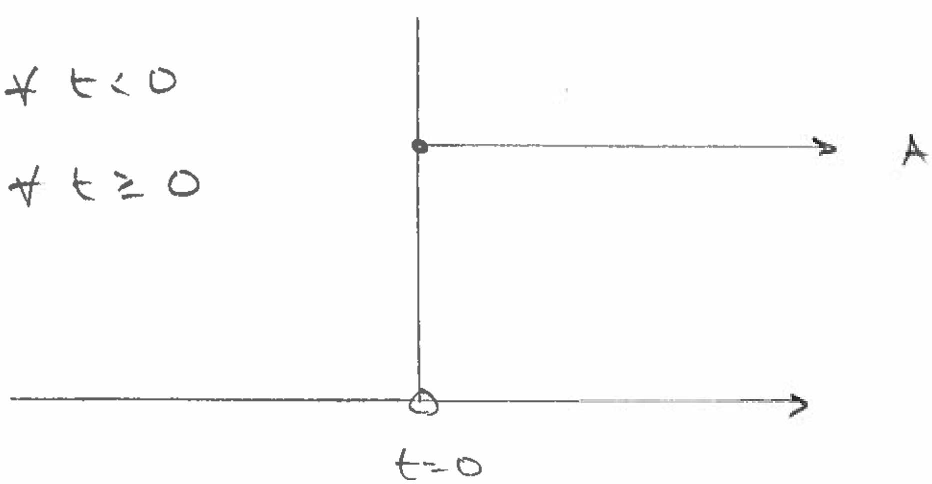
$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = \int_0^\infty A\,e^{-(\alpha+s)t}\,dt$$

$$= A \cdot \frac{1}{-\alpha - s}\,e^{-(\alpha+s)t}\Big|_0^\infty = \frac{A}{s+\alpha}$$

STEP:

$f(t) = 0 \ \forall \ t < 0$

$f(t) = A \ \forall \ t \geq 0$



$$\mathcal{L}[A] = \int_0^\infty A e^{-st}\, dt = \frac{A}{s}$$

SINUSOIDAL FCN: $f(t) = A\sin(\omega t) \quad \forall\ t \geq 0$

$$A\sin\omega t = A\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

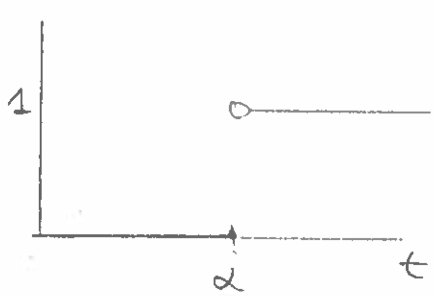
$$\mathcal{L}[f(t)] = \frac{A}{2j}\int_0^\infty \left[e^{j\omega t} - e^{-j\omega t}\right] e^{-st}\, dt$$

$$= \frac{A}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{A\omega}{s^2+\omega^2} = \mathcal{L}[A\sin\omega t]$$

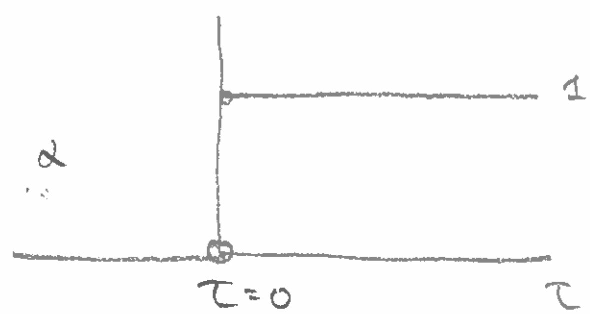
→ AFTER DOING THIS ONCE YOU CAN BUILD A TABLE.

TRANSLATED FUNCTION

$f(t-\alpha)\, 1(t-\alpha)$ ← UNIT STEP FCN. ⇒ $u(t)$ ← STEP.



SHIFTED BY α



$$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = \int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}\, dt$$

CHANGE OF VARIABLES $\tau = t - \alpha$, $d\tau = dt$

$$\int_{-\alpha}^{\infty} f(\tau)\, 1(\tau)\, e^{-s(\tau+\alpha)}\, d\tau$$

STEP FCN $1(\tau) = 0 \;\; \forall \; \tau < 0$ — CHANGE LOWER LIMIT.

$$= \int_{0}^{\infty} f(\tau)\, 1(\tau)\, e^{-s(\tau+\alpha)}\, d\tau$$

$$= \int_{0}^{\infty} f(\tau)\, e^{-s\tau}\, e^{-\alpha s}\, d\tau$$

$\tau = t - \alpha$

$$= e^{-\alpha s} \int_{0}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau = e^{-\alpha s}\, \mathcal{L}[f(\cancel{t})]$$

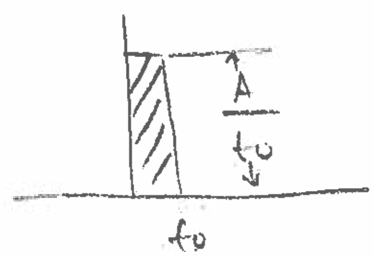
$$\boxed{\mathcal{L}[f(t-\alpha)\,1(t-\alpha)] = e^{-\alpha s} F(s) \qquad \alpha \geq 0}$$

TIME TRANSLATION $\Longrightarrow e^{-\alpha s} F(s) \longleftarrow$ S-DOMAIN.

PULSE FCN

$$f(t) = \frac{A}{t_0} \quad \text{FOR } 0 < t < t_0$$

$$= 0 \quad \text{FOR } t < 0,\; t > t_0$$



TREAT AS $f(t) = \frac{A}{t_0}\, 1(t-0) - \frac{A}{t_0}\, 1(t-t_0)$

$$\mathcal{L}[f(t)] = \frac{A}{t_0 s}\left(1 - e^{-s t_0}\right)$$

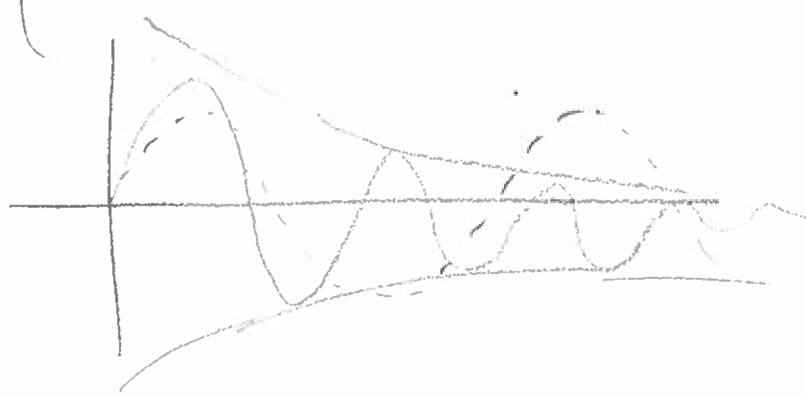
MULTIPLICATION BY $e^{-\alpha t}$ - SCALING BY EXPONENT.

$$\mathcal{L}\left[e^{-\alpha t} f(t)\right] = \int_0^\infty e^{-\alpha t} f(t)\, e^{-st}\, dt = F(s+\alpha)$$

EXAMPLE

$$\mathcal{L}\left[\sin \omega t\right] = \frac{\omega}{s^2+\omega^2} = F(s) \longleftarrow$$

$$\mathcal{L}\left[e^{-\alpha t} \sin \omega t\right] = \frac{\omega}{(s+\alpha)^2+\omega^2}$$



DIFFERENTIATION THEOREM

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

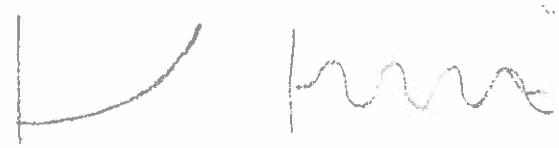
LAPLACE AUTOMATICALLY TAKES INITIAL CONDITIONS ☆

INTO ACCOUNT - UNLIKE DIRECT INTEGRATION

FINAL VALUE THEOREM

- PROVIDES DETAIL ABOUT SYSTEM BEHAVIOR AT STEADY STATE $(t \Rightarrow \infty)$

- ONLY APPLICABLE IF $\lim_{t \to \infty} f(t)$ i.e. $f(t)$ CONVERGES TO A VALUE



- ALL THE POLES OF $sF(s)$ MUST BE IN LEFT HALF PLANE POLES

THEOREM

IF $f(t)$ AND $\frac{dF(t)}{dt}$ ARE LAPLACE TRANSFORMABLE AND $\lim_{t \to \infty} f(t)$ EXISTS THEN

FINAL $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

INITIAL $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$

BOTH FINAL VALUE + INITIAL VALUE THEOREMS GIVE INSIGHT INTO TIME RESPONSE WITHOUT HAVING TO PERFORM THE INVERSE TRANSFORM

INTEGRATION THEOREM

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

IF $f(t)$ IS OF EXPONENTIAL ORDER

AND $f^{-1}(0) = \int f(t)\,dt$ AT $t = 0$

INVERSE LAPLACE USING TABLES

$$\boxed{x(t) = 2e^{-t} - 2e^{-2t}}$$ ← MOTION OF SYSTEM.

EXAMPLE - FIRST ORDER SYSTEM

$\dot{c} + 2c = 0$ WITH $c(0) = 1$

LAPLACE

$$s\,C(s) - c(0) + 2\,C(s) = 0$$

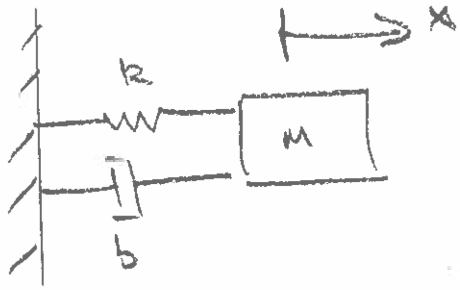
$$C(s)\left[s + 2\right] = 1$$

$$C(s) = \frac{1}{s+2}$$

INVERSE LAPLACE

$$\boxed{c(t) = e^{-2t}}$$

EXAMPLE - FREE MASS SPRING + DAMPER



$$m\ddot{x} = -kx - b\dot{x}$$

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0}$$

FIND: $X(t)$

FIND: THE MOTION $x(t)$ FOR $x(0) = 0$ $\dot{x}(0) = 2$

WITH $b = 3$ $k = 2$ $m = 1$

$$\ddot{x} + 3\dot{x} + 2x = 0$$ - HOMOGENEOUS ODE

LAPLACE TRANSFORM

$$s^2 X(s) - s\,\cancel{x(0)} - \dot{x}(0) + 3s\,X(s) - \cancel{x(0)} + 2X(s) = 0$$

COMBINE TERMS + SIMPLIFY

$$X(s)\left[s^2 + 3s + 2\right] - 2 = 0 \implies \boxed{X(s) = \frac{2}{s^2 + 3s + 2}}$$

NOW INVERSE LAPLACE TRANSFORM

$$X(s) = \frac{2}{(s+2)(s+1)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

$$a_1 = \left.\frac{2}{s+2}\right|_{s=-1} = 2$$

$$a_2 = \left.\frac{2}{s+1}\right|_{s=-2} = -2$$

$$\boxed{X(s) = \frac{2}{s+1} + \frac{-2}{s+2}}$$

$$\boxed{x(t) = 2e^{-t} - 2e^{-2t}}$$