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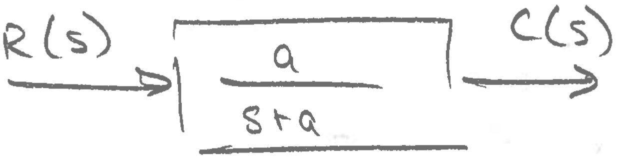
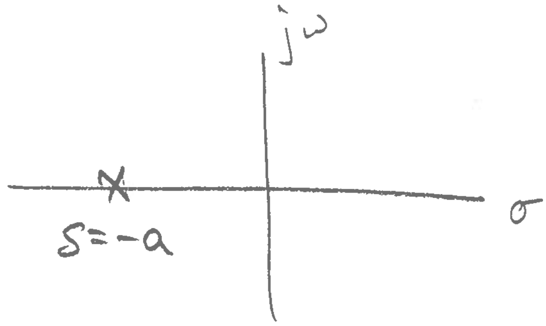
FIRST ORDER RESPONSE

CONSIDER THE FIRST ORDER SYSTEM W/O ANY ZEROS





STEP INPUT $R(s) = \frac{1}{s} \rightarrow C(s) = \frac{a}{s(s+a)} \rightarrow c(t)$

TAKE THE INVERSE LAPLACE TRANSFORM

$C(s) = \frac{A}{s} + \frac{B}{s+a}$ $\quad A = 1 \quad B = -1$

$$\boxed{c(t) = 1 - e^{-at}}$$

FORCED NATURAL

WHAT IS THE EFFECT OF a ?

$$e^{-at}\Big|_{t=\frac{1}{a}} = e^{-1} = 0.37$$

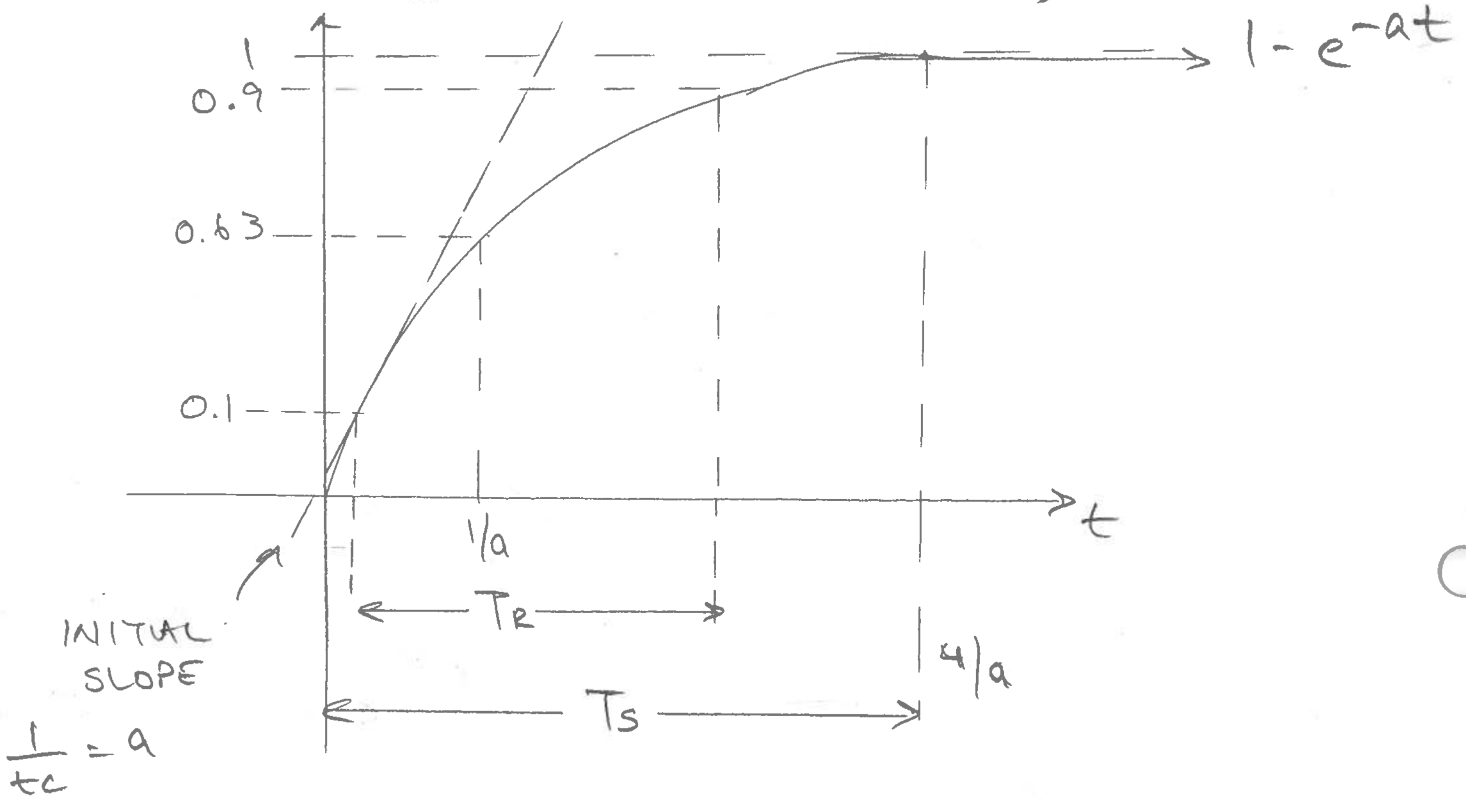
$$c(t)\Big|_{t=\frac{1}{a}} = 1 - e^{-at}\Big|_{t=\frac{1}{a}} = 1 - 0.37 = 0.63$$

TIME CONSTANT : $\frac{1}{a} = t_c \rightarrow$ ~~UNITS OF $\frac{1}{SEC}$~~ (Hz)

TIME CONSTANT

1. TIME REQUIRED FOR e^{-at} TO DECAY TO 37% OF ORIGINAL

2. TIME FOR RESPONSE $c(t)$ TO REACH 63% OF FINAL VALUE. (STEADY STATE)



$\underline{a}$: EXPONENTIAL FREQUENCY ← GOVERNS RESP.

GOVERNS THE TRANSIENT RESPONSE (INITIAL)

$a \uparrow \rightarrow$ FASTER RESPONSE.

RISE TIME

$$\boxed{T_R = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}}$$

TIME TO GO

10% → 90% OF $c(t)_{ss}$

$c(t) = 1 - e^{-at}$

SETTLING TIME

$$\boxed{T_S = \frac{4}{a}}$$

TIME FOR $c(t)$ TO REACH 98% (OR 90% / 95%)

2%

FIRST ORDER RESPONSE FROM TESTING

- MIGHT NOT BE POSSIBLE TO ANALYTICALLY FIND TRANSFER FCN
- CLOSED / DIFFICULT / BLACK BOX

GOAL: FIND TRANSFER FCN OF UNKNOWN SYS.

APPROACH: APPLY STEP INPUT → MEASURE RESPONS

BLACK BOX

$$R(s) \longrightarrow \boxed{G(s) = \frac{K}{s+a}} \longrightarrow C(s) = \frac{K/a}{s} - \frac{K/a}{s+a}$$

FIND K, a FROM $c(t) = K/a - K/a\, e^{-at}$

STEADY STATE VALUE $\longrightarrow$ $SS = \frac{K}{a}$ TWO UNK + TWO EQ.

TIME CONSTANT $\longrightarrow$ $t_c = 0.63\,(SS) = \frac{1}{a}$

EXAMPLE – $G(s) = \frac{5}{s+7}$

SECOND ORDER SYSTEM RESPONSE

- FIRST ORDER DEFINED BY SINGLE PARAMETER

  $a \rightarrow t_c \rightarrow$ SPEED OF RESPONSE

- SECOND ORDER HAS MORE PARAMETERS (2)

  SPEED + FORM OF RESPONSE CAN CHANGE

TWO DEFINING PARAMETERS

NATURAL FREQUENCY - ω_n (OMEGA)

FREQ OF OSCILLATIONS WITHOUT DAMPING

DAMPING RATIO ζ (ZETA)

OSCILLATIONS REGARDLESS OF TIME SCALE

$$\left.\begin{array}{l}\text{3 CYCLES IN 1 MILLISECOND} \\ \text{3 CYCLES IN A CENTURY}\end{array}\right\} \text{SAME } \zeta$$

$$\zeta = \frac{\text{EXP DECAY}}{\text{NAT FREQ}} = \frac{1}{2\pi} \frac{\text{NATURAL PERIOD}}{\text{EXP. TIME CONSTANT}}$$

NO UNITS

GENERAL SECOND ORDER $G(s) = \dfrac{b}{s^2 + as + b}$

ROOT LOCUS

NO DAMPING $\Rightarrow$ PURE OSCILLATIONS $\Rightarrow$ IMAGINARY

$$G(s) = \frac{b}{s^2 + b} \qquad a = 0$$

POLES ARE AT $s = \pm j\sqrt{b} \leftarrow = \pm j\omega$

$$\omega_n = \sqrt{b} \rightarrow b = \omega_n^2$$

UNDERDAMPED RESPONSE POLES AT $s = -\frac{a}{2} \pm j\omega_n$

$$\zeta = \frac{a/2}{\omega_n} \Rightarrow a = 2\zeta\omega_n$$

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

ζ, ω_n COMPLETELY DEFINE RESP.

SOLVING FOR THE POLES

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \leftarrow$$

WE'LL SPECIFICALLY DEFINE THE TYPES OF RESPONSE

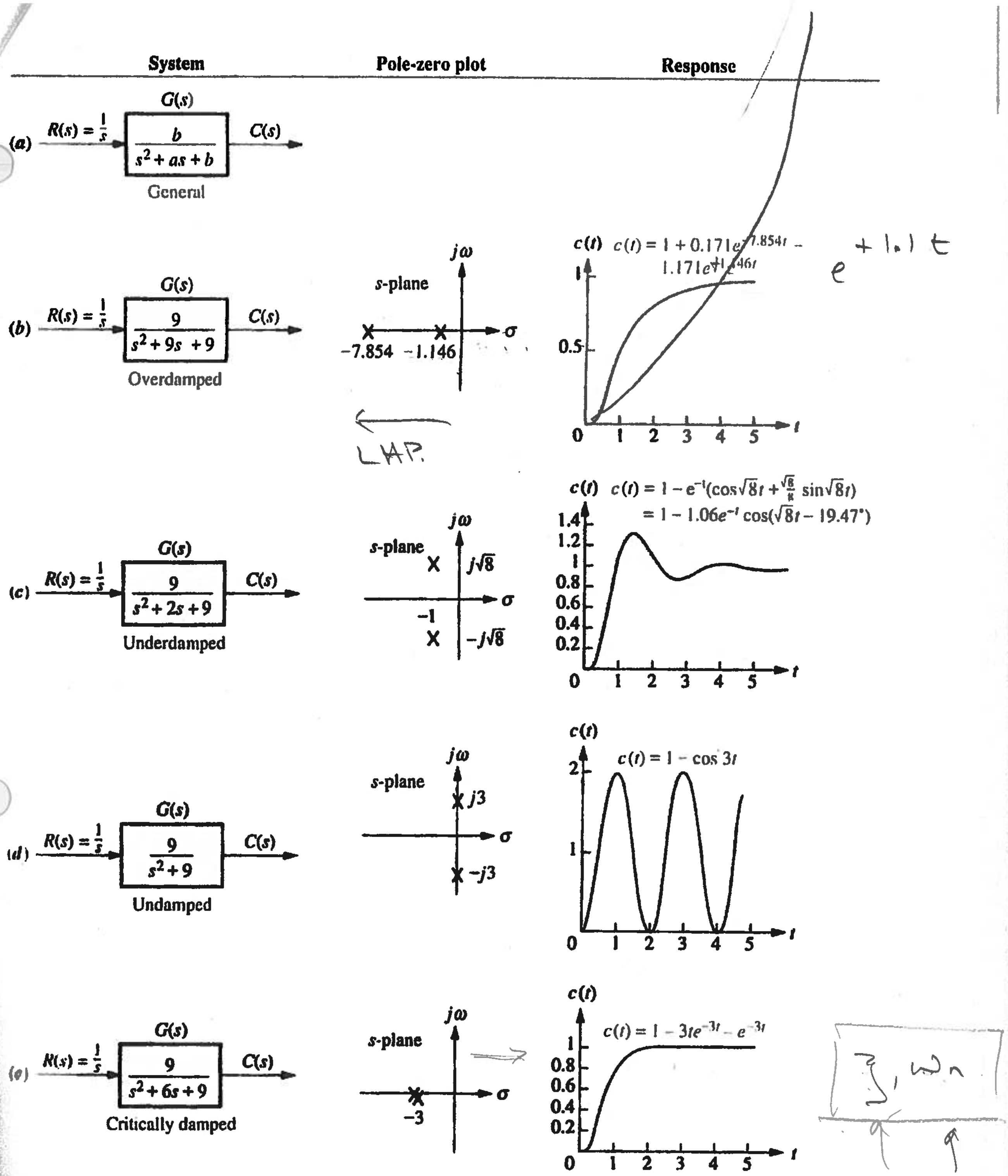


Figure 4.7
Second-order systems, pole plots, and step responses

OVERDAMPED RESPONSE

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{9}{s^2+9s+9}} \xrightarrow{C(s)}$$

$$\boxed{\begin{aligned} s &= -7.854 \\ &= -1.146 \end{aligned}}$$

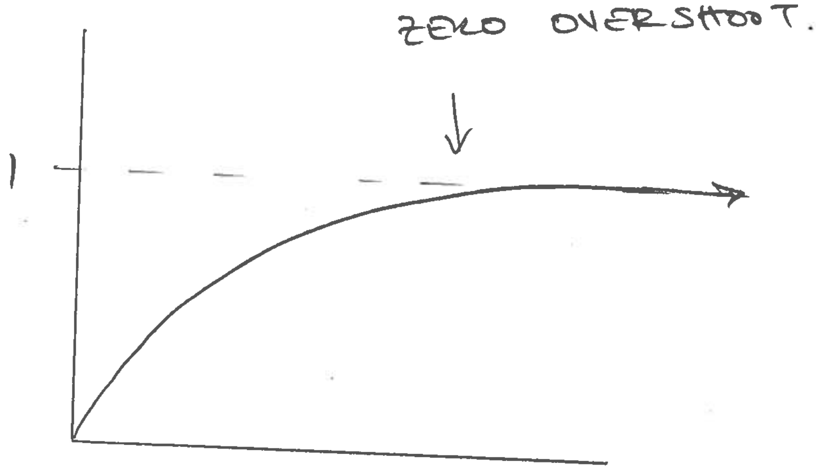
OUTPUT $c(t)$ TO A STEP INPUT

$$c(t) = 1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

FORCED RESPONSE $(s=0)$

POLES

- TWO REAL POLES
- RESPONSE HAS NO OSCILLATIONS – NO OVERSHOOT FOR STEP INPUT.



UNDERDAMPED RESPONSE

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$$R(s) \longrightarrow \boxed{\frac{9}{s^2+2s+9}} \longrightarrow C(s) \qquad \boxed{s = -1 \pm j\sqrt{8}}$$

$$\sigma \pm j\omega$$

OUTPUT $c(t)$ FOR A STEP INPUT IS

(EXERCISE FOR THE READER)

$$c(t) = 1 - e^{-t}\left(\cos(\sqrt{8}\,t) + \frac{\sqrt{8}}{8}\sin(\sqrt{8}\,t)\right)$$

DECAY FREQUENCY $= -1 = \sigma$

FORCED $(s=0)$

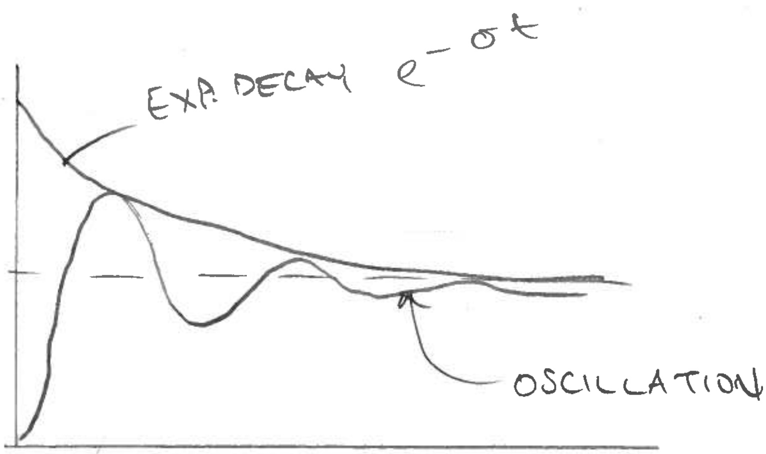
IMAGINARY PART OF POLE

- RESPONSE IS AN EXPONENTIALLY DECAYING SINUSOIDAL RESPONSE

- COMPLEX POLES $\longrightarrow$ DAMPED SINUSOIDAL OSCILLATIONS

- TIME CONSTANT OF DECAY $\Rightarrow \dfrac{1}{\text{REAL PART OF POLE}}$

- FREQUENCY OF OSCILLATIONS $\longrightarrow$ IMAGINARY PART OF POLE



CRITICALLY DAMPED

$$R(s) \longrightarrow \boxed{\frac{9}{(s+3)^2}} \longrightarrow C(s)$$

$\boxed{s = -3 \quad \text{TWO POLES}}$

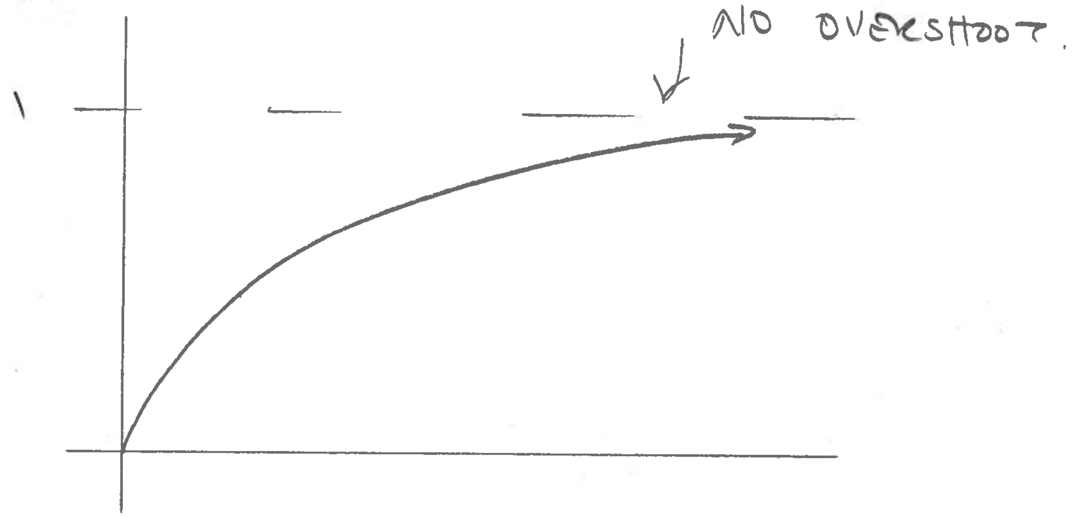
OUTPUT RESPONSE

$$c(t) = \underset{1}{K_1} - 3te^{-3t} - e^{-3t}$$

REPEATED

EXP. DECAY → POLES

- FASTEST RESPONSE <u>w/o</u> OVERSHOOT



SEPERATION BETWEEN UNDERDAMPED + OVERDAMPED

UNDAMPED RESPONSE

$$R(s) \longrightarrow \boxed{\frac{9}{s^2+9}} \longrightarrow C(s)$$

$$\boxed{s = \pm j3}$$

OUTPUT WILL BE

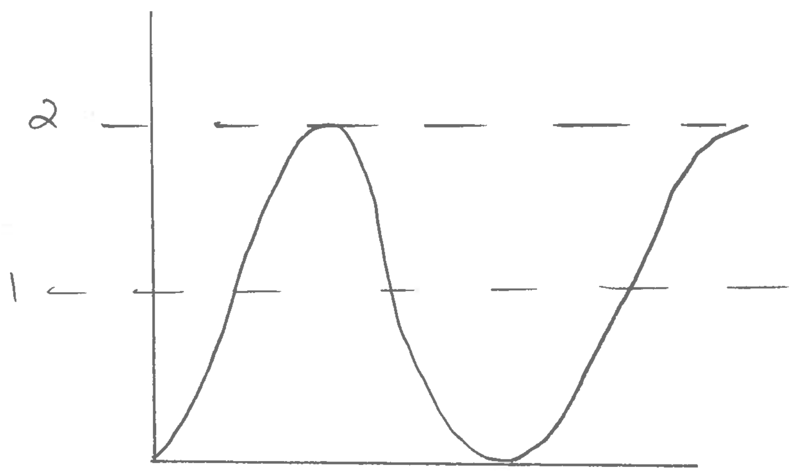
$$c(t) = K_1 + K_4 \cos(3t)$$

EXPONENTIAL DECAY

$e^{0} = 1$ ← REAL PART

IMAGINARY PART OF POLE

- PURE COMPLEX POLES → PURE OSCILLATION
- NO DECAY OF RESPONSE



SUMMARY : NATURAL RESPONSE

OVERDAMPED : REAL POLES $\quad s = -\sigma_1, -\sigma_2$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

EXPONENTIAL

UNDERDAMPED : COMPLEX PAIR $\quad s = -\sigma_d \pm j\omega_d$

$$c(t) = A e^{-\sigma_d t} \cos(\omega_d t - \phi)$$

REAL

IMG

DAMPED SINUSOID

UNDAMPED : IMAGINARY $\quad s = \pm j\omega_1$

$$c(t) = A \cos(\omega_1 t - \phi)$$

IMAG.

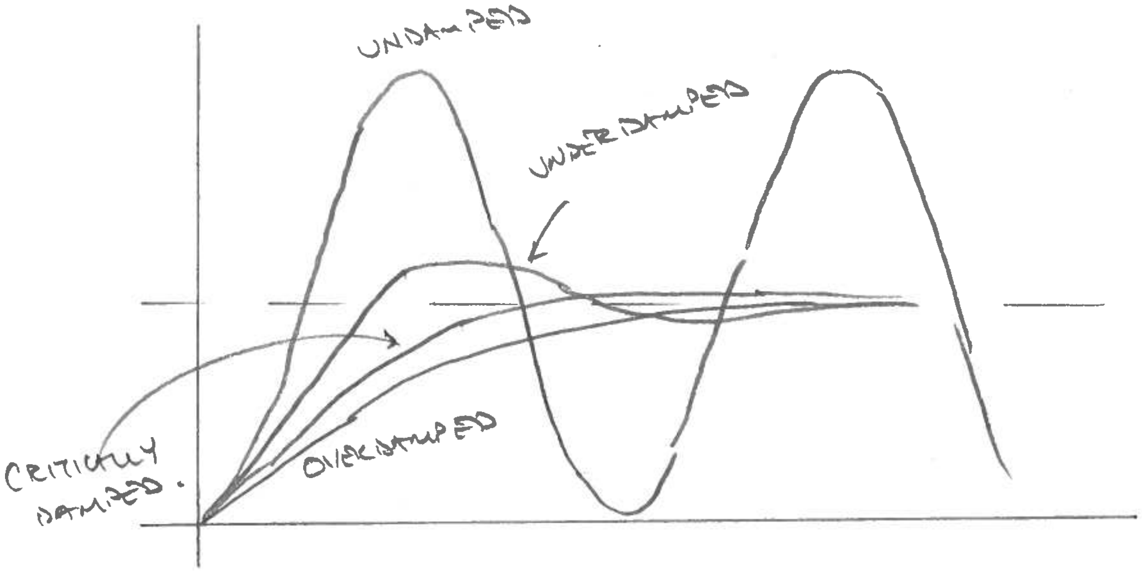
UNDAMPED SINSOID – OSCILLATES FOREVER

CRITICALLY DAMPED : REPEATED REAL $\quad s = -\sigma_1, -\sigma_1$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

POLE

EXPONENTIAL + $t \times$ EXPONENTIAL



- LOCATION OF POLES DEFINE THE RESPONSE!
- RELATE POLES TO $\zeta, \omega_n \rightarrow$ THEN DEFINE RESPONSE SPECIFICATIONS

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

POLES LOCATED AT

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

FIND THE STEP RESPONSE OF $G(s)$

$$C(s) = \frac{1}{s} G(s)$$

THE OUTPUT $c(t)$ IS:

$$c(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ASSUME $(\zeta < 1)$

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi\right)$$

WHERE $\phi = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}$

SHOW PLOT IN PYTHON

$\zeta \downarrow \Rightarrow$ MORE OSCILLATIONS

$\zeta \uparrow \Rightarrow$ MORE DAMPING - LESS OSCILLATION.

NOW WE CAN DEFINE SOME SPECIFICATIONS.

1. RISE TIME - T_R - TIME FOR 10% → 90%

$T_R \approx$ NO EXP.

2. PEAK TIME - T_p FIRST MAXIMUM PEAK TIME

$$T_p = \pi / \omega_n\sqrt{1-\zeta^2}$$

3. SETTLING TIME - T_s TIME TO REMAIN ± 2% OF STEADY STATE VALUE

$$T_s \approx \frac{4}{\zeta\omega_n}$$

SPEED

4. % OS - PERCENT OVERSHOOT - OVERSHOOT AT T_p

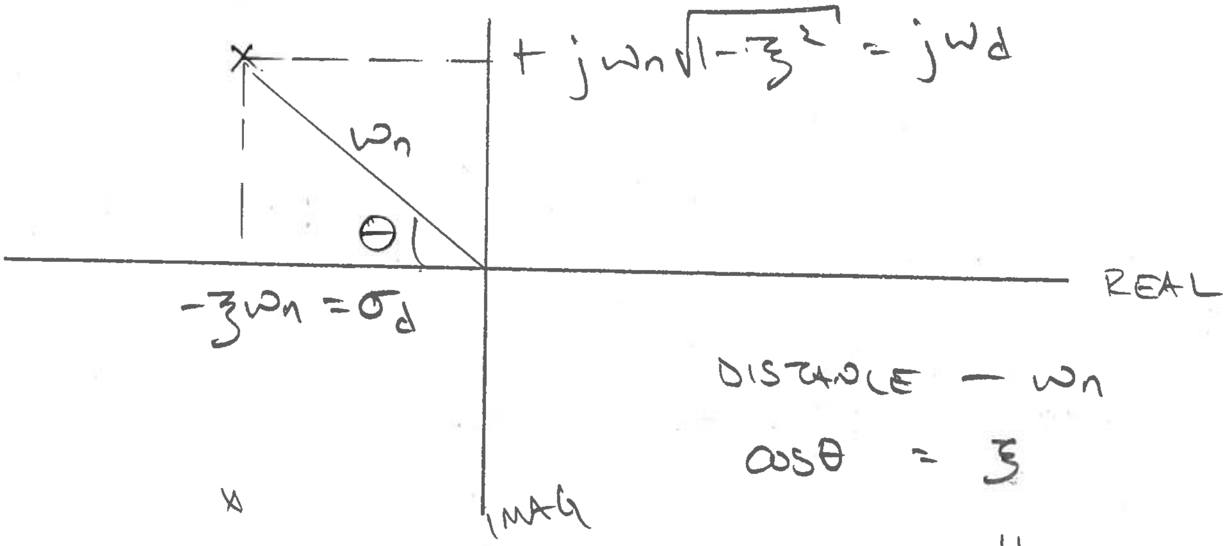
$$\%OS = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

ONLY VALID FOR SYSTEMS THAT CAN BE APPROXIMATED AS <u>SECOND ORDER</u>. (≥ 2)

RELATE BACK TO POLE LOCATIONS

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



DISTANCE — ω_n

$\cos\theta = \zeta$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \qquad T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

HORIZONTAL LINES VERTICAL LINES

ω_d - DAMPED FREQ. OF OSCILLATION

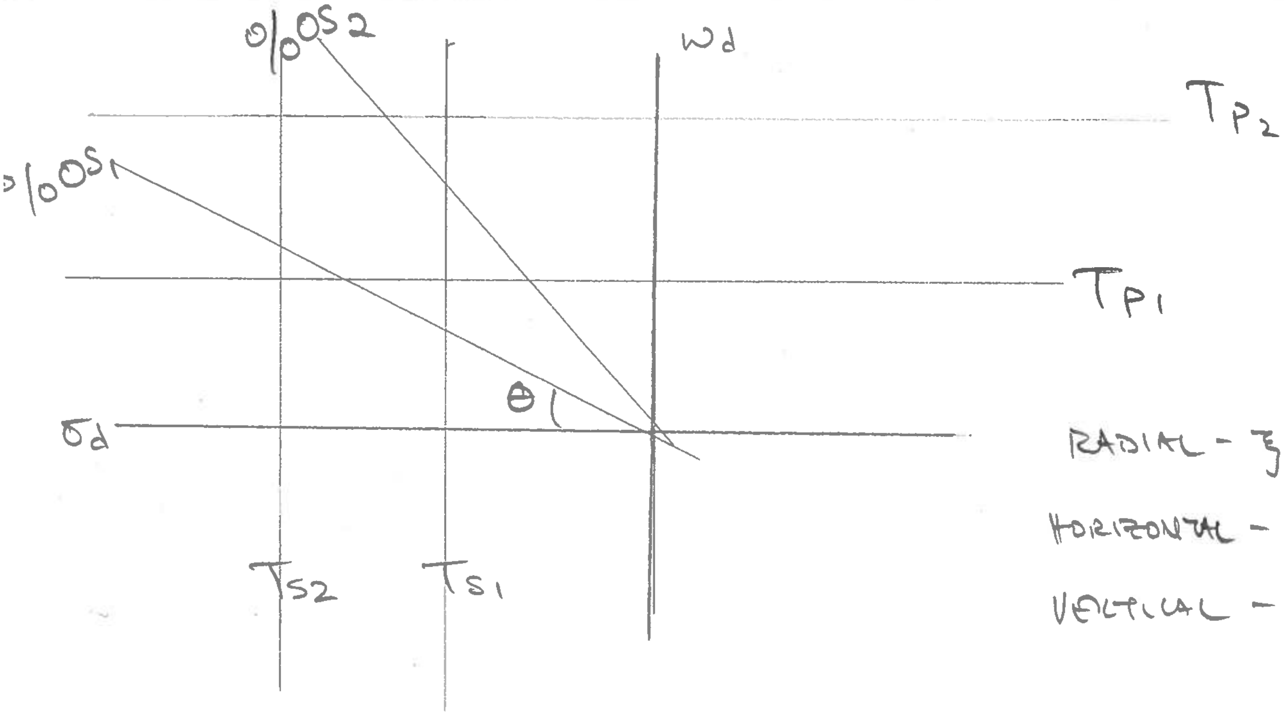
$\omega_d = \omega_n\sqrt{1-\zeta^2}$ MAG. OF IMAG. COMP.

(MAGNITUDE)

σ_d - EXP. DAMPING FREQ.

$\sigma_d = \zeta\omega_n$ MAG. OF REAL COMP.

$$\boxed{\begin{aligned} s &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \\ s &= -\sigma_d \pm j\omega_d \end{aligned}}$$



RADIAL - ζ / %OS

HORIZONTAL - T_P

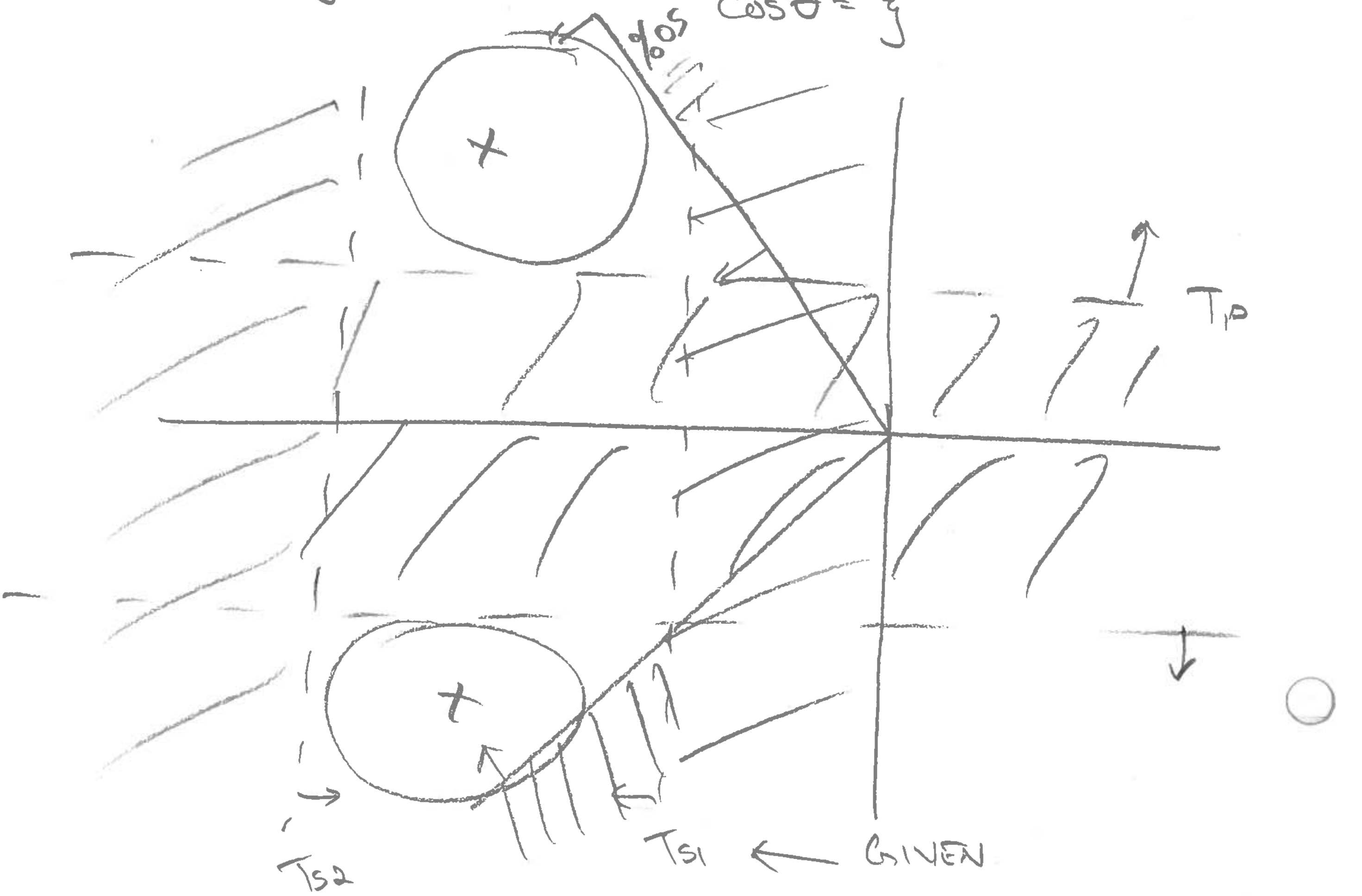
VERTICAL - T_S

$\omega_d \uparrow$ $\quad T_p \downarrow$ $\quad T_p = \dfrac{\pi}{\omega_d}$ $\quad T_{S_2} < T_{S_1}$

$\sigma_d \uparrow$ $\quad T_s \downarrow$ $\quad T_s = \dfrac{4}{\sigma_d}$ $\quad T_{P_2} < T_{P_1}$

$\theta \uparrow$ $\quad \zeta \downarrow$ $\quad \%OS \uparrow$ $\quad \%OS_1 < \%OS_2$

$\cos\theta = \zeta$



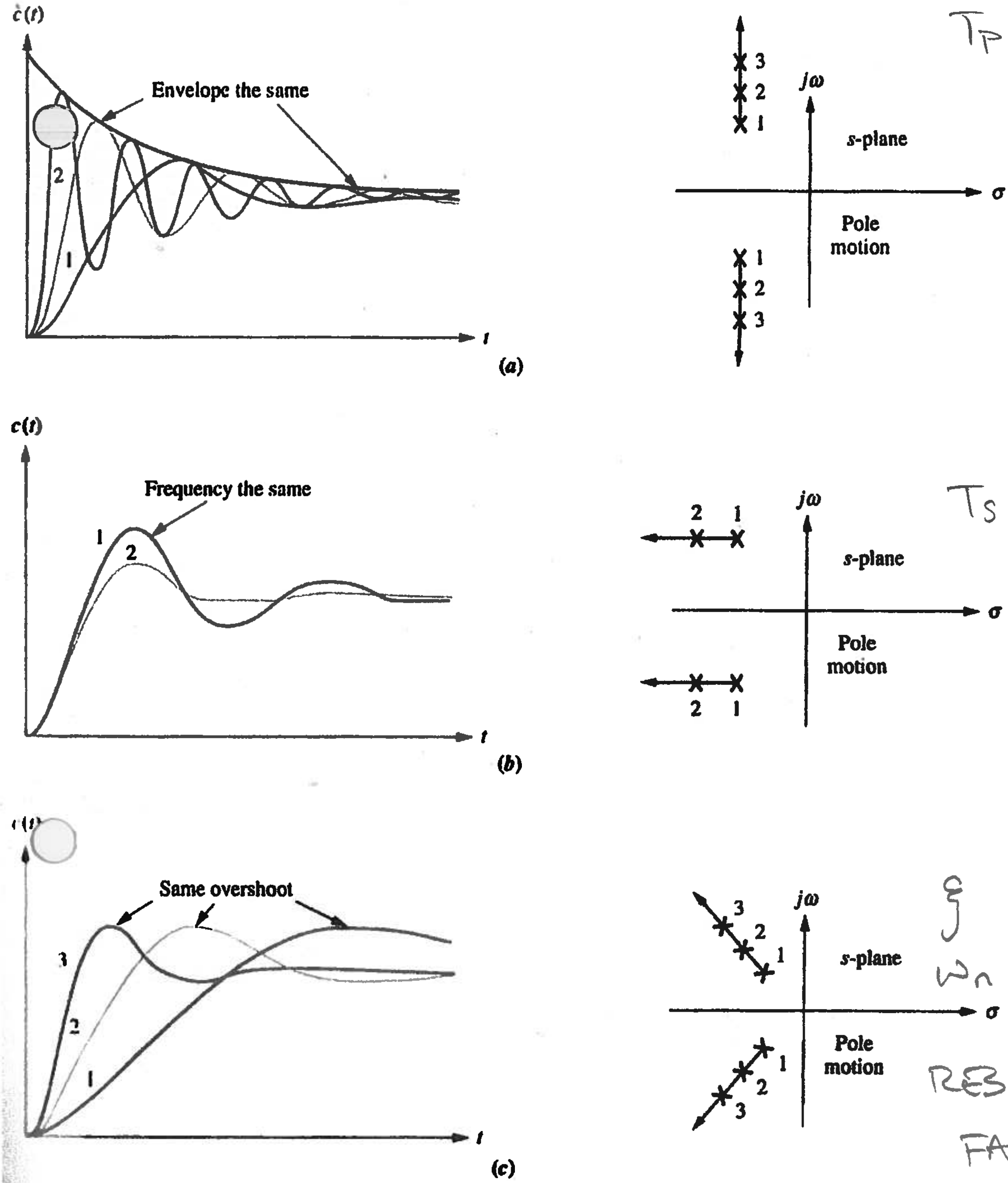


Figure 4.19
Step responses of second-order underdamped systems as poles move:
a. with constant real part;
b. with constant imaginary part;
c. with constant damping ratio

T_p IS DECREASING

T_s IS DECREASING

ξ CONSTANT
ω_n INCREASING
RESPONSE IS FASTER

SECOND ORDER SYSTEM FROM TESTING

SAME APPROACH AS FIRST ORDER CASE

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- APPLY STEP INPUT AND ANALYZE RESPONSE
- STEADY STATE VALUE GIVES K
- $\%OS \rightarrow \zeta$ $\quad T_p \rightarrow \omega_d$ $\quad T_s \rightarrow \sigma_d$