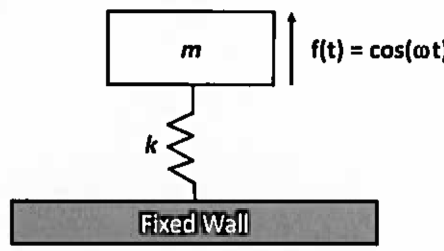
MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 4
Due Thursday, February 26th at the beginning of class

NOTE: Please make sure to show and explain (using sentences) the various followed to arrive at the solution. Otherwise we are unable to assess your understanding of the material and give you credit for your answers.

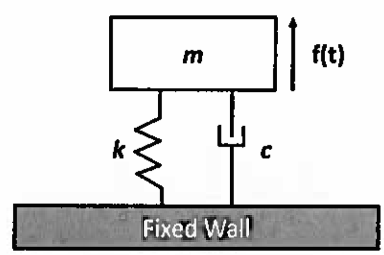
1. (i) Set up the equation of motion for the spring-mass system shown in the diagram, which is under the influence of a sinusoidal force (assume that ω is an arbitrary frequency, *different* from the natural frequency ω_n). (ii) Solve the differential equation for $t > 0$ using the Laplace transform method, for the case when the initial position and velocity are both zero.



2. Derive the *steady state* solution for a system similar to that of problem 1, for which the excitation force consists of three sinusoidal forces: $f(t) = F_1 \cos(0.5\ \omega_n t) + F_2 \cos(0.6\ \omega_n t) + F_3 \cos(0.7\ \omega_n t)$. $F_1 = 2$ N, $F_2 = 1$ N and $F_3 = 0.5$ N. ω_n is the natural *angular* frequency given by $\omega_n = \sqrt{\frac{k}{m}}$.

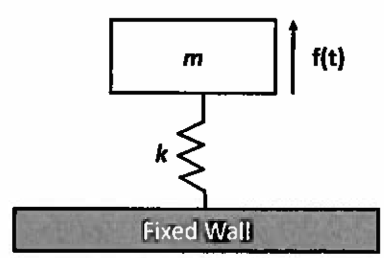
3. Consider the system shown in the figure, which has parameters $m = 1$ Kg, $c = 2$ Ns/m and $k = 1$ N/m, and zero initial position and velocity. Derive the time-dependent position of the system $x(t)$ for the case when $f(t)$ consists of an infinite number of impacts according to the following:

   $f(t) = \sum_{k=0}^{\infty} f_o \delta(t-k)$, with $f_o = 1$ Kgm/s. In other words, one impact is given to the system every k units of time, and all impacts are identical.



BONUS PROBLEM (30 points)

4. An engineer excites the system in the figure with an impact $f(t) = f_o \delta(t)$ with $f_o > 0$, and observes that the system oscillates with an amplitude of 0.25 m. (i) Provide an expression for $x(t)$. (ii) Calculate the value of f_o. For this system $m = 6$ Kg and $k = 12$ N/m.



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Homework #4 Solution

PROBLEM 1

(i) Equation of motion about the equilibrium position:

$+x$ is positive in the upward direction.

$$m\ddot{x} = -Kx + f(t)$$

$$\Rightarrow m\ddot{x} + Kx = \cos(\omega t)$$

(ii) Applying the Laplace transform to the equation:

$$ms^2 X(s) + K X(s) = \frac{s}{s^2+\omega^2}$$

$$\Rightarrow X(s)(ms^2+K) = \frac{s}{s^2+\omega^2}$$

$$\Rightarrow X(s) = \frac{s}{(s^2+\omega^2)(ms^2+K)}$$

To invert $X(s)$ we use the partial fractions method:

$$\frac{s}{(s^2+\omega^2)(ms^2+K)} = \frac{Bs+C}{s^2+\omega^2} + \frac{Ds+E}{ms^2+K}$$

$$\Rightarrow \frac{s}{(s^2+\omega^2)(ms^2+K)} = \frac{(Bs+C)(ms^2+K)+(Ds+E)(s^2+\omega^2)}{(s^2+\omega^2)(ms^2+K)}$$

Equating the numerators

$$s = (Bs+C)(ms^2+K) + (Ds+E)(s^2+\omega^2)$$

$$s = Bms^3 + Bsk + Cms^2 + Ck + Ds^3 + Dsw^2 + Es^2 + Ew^2$$

$$s = s^3[Bm+D] + s^2[Cm+E] + s[Bk + Dw^2] + [Ck + Ew^2]$$

This gives four equations

I $\quad Bm + D = 0 \quad \Rightarrow \quad D = -Bm \quad$ Ia

II $\quad Cm + E = 0 \quad \Rightarrow \quad E = -Cm \quad$ IIa

III $\quad Bk + Dw^2 = 1$

IV $\quad Ck + Ew^2 = 0$

Substituting Ia into III

$$Bk + Dw^2 = 1$$

$$\Rightarrow Bk + (-Bm)w^2 = 1$$

$$\Rightarrow Bk - Bmw^2 = 1 \quad \Rightarrow \quad B(k - mw^2) = 1$$

$$\Rightarrow \boxed{B = \frac{1}{k - mw^2}}$$

Returning to Ia we have

$$D = -Bm \quad \Rightarrow \quad \boxed{D = -\frac{m}{k - mw^2}}$$

Substituting IIa into IV:

$$Ck + Ew^2 = 0$$

$$\Rightarrow Ck + (-Cm)w^2 = 0$$

$$\Rightarrow C(k - mw^2) = 0 \quad \Rightarrow \quad \boxed{C = 0}$$

Returning to IIa: $\quad E = -Cm \quad \Rightarrow \quad \boxed{E = 0}$

With the above constants,

$$X(s) = \frac{Bs + \cancelto{\text{ZERO}}{C}}{s^2+\omega^2} + \frac{Ds + \cancelto{\text{ZERO}}{E}}{ms^2+K}$$

$$= \left(\frac{1}{K-m\omega^2}\right)\left(\frac{s}{s^2+\omega^2}\right) + \left(\frac{-m}{K-m\omega^2}\right)\left(\frac{s}{ms^2+K}\right)$$

$$= \left(\frac{1}{K-m\omega^2}\right)\left(\frac{s}{s^2+\omega^2}\right) + \left(\frac{1}{K-m\omega^2}\right)\left(\frac{-ms}{ms^2+K}\right)$$

$$= \left(\frac{1}{K-m\omega^2}\right)\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\underbrace{\tfrac{K}{m}}_{\omega_n^2}}\right]$$

Inverting using the table,

$$x(t) = \frac{1}{K-m\omega^2}\left(\cos(\omega t) - \cos(\omega_n t)\right)$$

PROBLEM 2

We are now given a problem similar to the previous one, but in which the force contains three terms

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

with

$$f_1(t) = 2\cos(0.5\,\omega_n t)$$
$$f_2(t) = 1\cos(0.6\,\omega_n t)$$
$$f_3(t) = 0.5\cos(0.7\,\omega_n t)$$

Since the problem is linear, we can write three separate (similar) problems, one for each term in $f(t)$, and then add their

Solutions. The $f(t)$ terms have different coefficients but, due to linearity, this is not a problem. If the input is multiplied by a constant A, then the output will also be multiplied by A. The three problems we write are:

$$(i)\quad m\ddot{x} + Kx = f_1(t) = 2\cos(0.5\,\omega_n t)$$
$$(ii)\quad m\ddot{x} + Kx = f_2(t) = \cos(0.6\,\omega_n t)$$
$$(iii)\quad m\ddot{x} + Kx = f_3(t) = 0.5\cos(0.7\,\omega_n t).$$

Each one will have a solution similar to that of problem 1:

$$(i)\quad x_1(t) = 2\left[\frac{1}{K - m\omega^2}\left[\cos(\omega t) - \cos(\omega_n t)\right]\right]$$

This is the coefficient of the input, which is also applied to the output

$$\Rightarrow x_1(t) = 2\left[\frac{1}{K - m(0.5\,\omega_n)^2}\left[\cos(0.5\,\omega_n t) - \cos(\omega_n t)\right]\right]$$

$$\Rightarrow x_1(t) = \frac{2}{K - 0.25\,m\omega_n^2}\left[\cos(0.5\,\omega_n t) - \cos(\omega_n t)\right]$$

(ii) We proceed similarly,

$$x_2(t) = 1\left[\frac{1}{K - m(0.6\,\omega_n)^2}\left[\cos(0.6\,\omega_n t) - \cos(\omega_n t)\right]\right]$$

$$\Rightarrow x_2(t) = \frac{1}{K - 0.36\,m\omega_n^2}\left[\cos(0.6\,\omega_n t) - \cos(\omega_n t)\right]$$

(iii) $$x_3(t) = \frac{0.5}{k - m(0.7\omega_n)^2}\left[\cos(0.7\omega_n t) - \cos(\omega_n t)\right]$$

$$\Rightarrow x_3(t) = \frac{0.5}{k - 0.49 m\omega_n^2}\left[\cos(0.7\omega_n t) - \cos(\omega_n t)\right]$$

Finally, the answer to our problem is the sum of all three solutions:

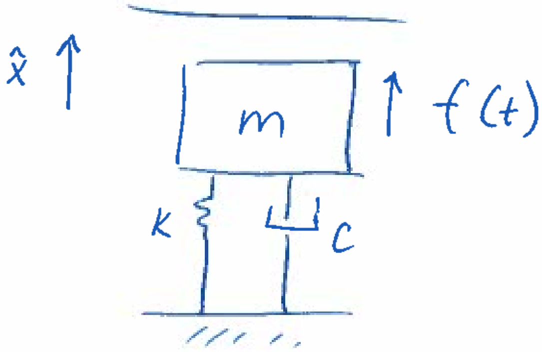
$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

$$\Rightarrow x(t) = \frac{2}{k - 0.25 m\omega_n^2}\left[\cos(0.5\omega_n t) - \cos(\omega_n t)\right]$$

$$+ \frac{1}{k - 0.36 m\omega_n^2}\left[\cos(0.6\omega_n t) - \cos(\omega_n t)\right]$$

$$+ \frac{0.5}{k - 0.49 m\omega_n^2}\left[\cos(0.7\omega_n t) - \cos(\omega_n t)\right]$$

PROBLEM 3



Equation of motion

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\left.\begin{array}{l} m = 1 \text{ Kg} \\ c = 2 \text{ Ns/m} \\ k = 1 \text{ N/m} \end{array}\right\} \Rightarrow \ddot{x} + 2\dot{x} + x = f(t)$$

We are told that an impact is given every k units of time. In other words,

$$f(t) = \delta(t) + \delta(t-k) + \delta(t-2k) + \delta(t-3k) + \dots$$

Since the system is linear and time invariant, We only need to solve for one impact. Once we have that solution, we can repeat it for all other impacts every κ units of time.

So for the first impact we have:

$$\ddot{x} + 2\dot{x} + x = \delta(t)$$

Applying the Laplace transform.

$$s^2 X(s) + 2s X(s) + X(s) = 1$$

$$\Rightarrow X(s)\left(s^2 + 2s + 1\right) = 1$$

$$\Rightarrow X(s) = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}$$

Using the table, this expression can be directly inverted to give

$$x(t) = t\,e^{-t} \quad \text{for } t > 0.$$

Now we write this solution as

$$x_{t_0}(t) = (t - t_0)\,e^{-(t-t_0)}\,u(t - t_0)$$

The above is the general solution for an impact given at $t = t_0$. The last part of the solution, the unit step $u(t-t_0)$ turns on the solution at $t = t_0$ and makes it zero before then. The solution is a function of $(t-t_0)$ instead of just t because it needs to be shifted in time, depending on when the impact occurs.

So now we need to add one of the above solutions ~~for~~ each impact.

The impact $\delta(t-t_0)$ gives rise to the response $(t-t_0)\,e^{-(t-t_0)}\,u(t-t_0)$

Since $f(t) = \sum_{k=0}^{\infty} \delta(t-k)$, the solution is then,

$$x(t) = \sum_{k=0}^{\infty} (t-k)\,e^{-(t-k)}\,u(t-k)$$

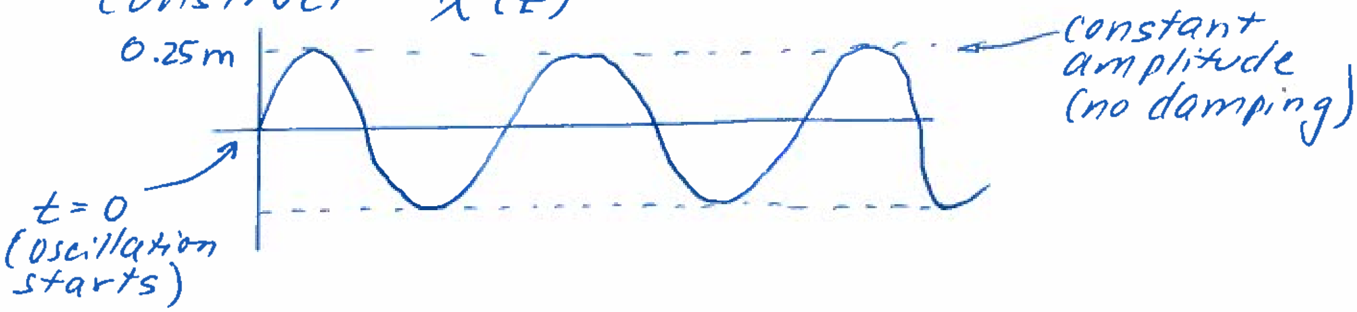
PROBLEM 4

(i) We know that the system has no damping, so it will oscillate forever with the same amplitude. This oscillation will be a pure sinusoidal wave.

Since there is no damping, the system will oscillate at its natural angular frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12\ N/m}{6\ Kg}} = \sqrt{2}$$

Finally, we know that the impact was given at $t=0$ in the positive direction. Using all of the above information we can construct $x(t)$



Since the solution must cross the $x=0$ position at $t=0$, $x(t)$ must be a sine function,

$$x(t) = 0.25 \sin(\omega_n t)$$
$$= 0.25 \sin(\sqrt{2}\, t)$$

(ii) We also know that the coefficient of the impact, f_0, corresponds to the initial momentum of the mass:

$$f_0 = m V_0 \Rightarrow V_0 = \frac{f_0}{m} = \frac{f_0}{6\ Kg}$$

So to find the answer we must first find $V_0 = \dot{x}(0)$.

$$\dot{x}(t) = \frac{d}{dt} x(t) = 0.25\sqrt{2} \cos(\sqrt{2}\, t)$$

$$\Rightarrow \dot{x}(0) = 0.25\sqrt{2} \cos(0) = 0.25\sqrt{2}\ \frac{m}{s}$$

Now, $\dot{x}(0) = V_0 = 0.25\sqrt{2}\ \frac{m}{s} = \frac{f_0}{6\ Kg}$

$$\Rightarrow f_0 = \left(0.25\sqrt{2}\ \frac{m}{s}\right)(6\ Kg) = \frac{3}{2}\sqrt{2}\ Kg\,\frac{m}{s}$$