MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 7

Due Thursday, April 9th at the beginning of class

Consider an LRC circuit with one inductor, one resistor, one capacitor and one voltage source. Assume that the initial conditions (capacitor charge and current) are zero. Assume also that the above components are arranged clockwise, that the current direction is clockwise and that the voltage is positive for that current direction.

1. Draw the circuit describe above.

2. For each of the three cases below, find $q(t)$ and $i(t)$ using the Laplace transform method (make sure to show all the algebra steps required to reach the solution):

   a. **Case 1:** L = 10 H; R = 20 Ω; C = 0.1 F; V = 2 V
   b. **Case 2:** L = 10 H; R = 40 Ω; C = 0.1 F; V = 2 V
   c. **Case 3:** L = 10 H; R = 5 Ω; C = 0.1 F; V = 2 V

3. Plot $q(t)$ for all three cases of part 2 together on the same graph, plot $i(t)$ for all three cases together on a separate graph (there will be a total of two graphs: one for $q(t)$ for all cases together, and one for $i(t)$ for all cases together) and answer the following questions:

   a. If the responses corresponded to mechanical systems instead of electrical systems, indicate which response would be underdamped, which response would be critically damped and which response would be overdamped, explaining what features of the solutions lead to the conclusions you reached.

   b. If the above responses corresponded to mechanical systems instead of electrical systems, what would be the damping factor ζ in each of the three cases of part 2? Give the equivalent mass, damper constant and force constant in each case.

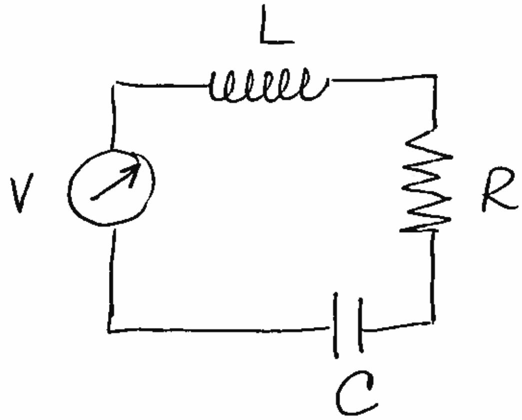
4. Calculate the steady state solution for $q(t)$ and $i(t)$ for case 3 of part 2 for V = 10 sin(0.05 t). To solve this part, use the equations that were discussed in class for the response of a vibratory mechanical system to a sinusoidal input force.

MAE 3134

SPRING 2015

HOMEWORK 7
SOLUTION

PART #1



$i(0) = \dot{q}(0) = 0$

$q(0) = 0$

PART #2

Case 1:

$L = 10\ H$

$R = 20\ \Omega$

$C = 0.1\ F$

$V = 2\ V$

Equation $\quad -L\dfrac{di}{dt} - Ri - \dfrac{q}{C} + V = 0$

$$L\frac{di}{dt} + Ri + \frac{q}{C} = V$$

but $i = \dfrac{dq}{dt} \Rightarrow \quad L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = V$

Introducing values, $\quad 10\ddot{q} + 20\dot{q} + 10q = 2$

Laplace transform: $\quad 10s^2Q(s) + 20sQ(s) + 10Q(s) = \dfrac{2}{s}$

$$\Rightarrow Q(s) = \frac{2}{s(10s^2+20s+10)} = \frac{0.2}{s(s^2+2s+1)} = \frac{0.2}{s(s+1)^2}$$

$$\Phi(s) = \frac{0.2}{s(s+1)^2}$$

PARTIAL FRACTIONS: $\dfrac{0.2}{s(s+1)^2} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$

$$\Rightarrow 0.2 = A(s+1)^2 + Bs(s+1) + Cs$$
$$\Rightarrow 0.2 = A(s^2+2s+1) + Bs^2+Bs+Cs$$
$$\Rightarrow 0.2 = As^2+2As+A+Bs^2+Bs+Cs$$
$$\Rightarrow 0.2 = s^2(A+B) + s(2A+B+C) + A$$

This gives: I $\boxed{A = 0.2}$

II $2A+B+C = 0$

III $A+B = 0 \Rightarrow B = -A \Rightarrow \boxed{B = -0.2}$

From II we have $C = -B-2A$

$$\Rightarrow \boxed{C = +0.2 - 2(0.2) = -0.2}$$

$$\Rightarrow \Phi(s) = \frac{0.2}{s} - \frac{0.2}{(s+1)} - \frac{0.2}{(s+1)^2}$$

Inverting using the table,

$$\boxed{q(t) = 0.2u(t) - 0.2e^{-t} - 0.2te^{-t}}$$

we find $i(t)$ by differentiation:

$$\boxed{i(t) = \frac{dq(t)}{dt} = 0.2\delta(t) + 0.2e^{-t} - 0.2e^{-t} + 0.2te^{-t}}$$

please see note about this delta function term, which is an artifact showing up in all three cases (the note is at the top of page 8)

Case 2: $L = 10\ H$
$R = 40\Omega$
$C = 0.1\ F$
$V = 2\ V$

The equation is symbolically the same:

$$L\ddot{q} + R\dot{q} + \frac{1}{C} q = V$$

$$\Rightarrow 10\ddot{q} + 40\dot{q} + 10q = 2$$

$$\Rightarrow \ddot{q} + 4\dot{q} + q = 0.2$$

Laplace transform:

$$s^2 Q(s) + 4s\,Q(s) + Q(s) = \frac{0.2}{s}$$

$$Q(s)\left(s^2 + 4s + 1\right) = \frac{0.2}{s} \Rightarrow Q(s) = \frac{0.2}{s(s^2+4s+1)}$$

Partial fractions

$$\frac{0.2}{s(s^2+4s+1)} = \frac{a}{s} + \frac{bs+c}{s^2+4s+1} = \frac{a(s^2+4s+1) + s(bs+c)}{s(s^2+4s+1)}$$

$$\Rightarrow 0.2 = as^2 + 4as + a + bs^2 + Cs$$

$$\Rightarrow 0.2 = s^2(a+b) + s(4a+c) + a$$

$\Rightarrow$ I $a = 0.2$

II $4a + C = 0 \Rightarrow C = -4a \Rightarrow \boxed{C = -0.8}$

III $a + b = 0 \Rightarrow \boxed{b = -0.2}$

$$\Rightarrow Q(s) = \frac{0.2}{s} + \frac{-0.2s - 0.8}{s^2 + 4s + 1}$$

We can factor the denominator of the second term using the quadratic formula:

for $s^2+4s+1=0$ we have zeros at

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$\Rightarrow s = -2 \pm \sqrt{3} \Rightarrow s_1 = -2+\sqrt{3} \ ; \ s_2 = -2-\sqrt{3}$$

$$\Rightarrow s^2+4s+1 = \left(s-(-2+\sqrt{3})\right)\left(s-(-2-\sqrt{3})\right)$$
$$= \left(s+(2-\sqrt{3})\right)\left(s+(2+\sqrt{3})\right)$$

Therefore,

$$\Phi(s) = \frac{0.2}{s} + \frac{-0.2s-0.8}{[s+(2-\sqrt{3})][s+(2+\sqrt{3})]}$$

$$= \frac{0.2}{s} - \frac{0.2s}{[s+(2-\sqrt{3})][s+(2+\sqrt{3})]} - \frac{0.8}{[s+(2-\sqrt{3})][s+(2+\sqrt{3})]}$$

Using entries #15 and 16 of the Laplace transform table in the book we have:

$$\varphi(t) = 0.2u(t) - \frac{0.2}{(2+\sqrt{3})-(2-\sqrt{3})}\left[(2+\sqrt{3})e^{-(2+\sqrt{3})t} - (2-\sqrt{3})e^{-(2-\sqrt{3})t}\right] - \frac{0.8}{(2+\sqrt{3})-(2-\sqrt{3})}\left[e^{-(2-\sqrt{3})t} - e^{-(2+\sqrt{3})t}\right]$$

$$\Rightarrow \boxed{\varphi(t) = 0.2u(t) - \frac{0.2}{2\sqrt{3}}\left[(2+\sqrt{3})e^{-(2+\sqrt{3})t} - (2-\sqrt{3})e^{-(2-\sqrt{3})t}\right] - \frac{0.8}{2\sqrt{3}}\left[e^{-(2-\sqrt{3})t} - e^{-(2+\sqrt{3})t}\right]}$$

As before, we find $i(t)$ by differentiation:

$$i(t) = \frac{dq(t)}{dt} = 0.2\,\delta(t) - \frac{0.2}{2\sqrt{3}}\Big[-(2+\sqrt{3})^2 e^{-(2+\sqrt{3})t}$$

$$+ (2-\sqrt{3})^2 e^{-(2-\sqrt{3})t}\Big] - \frac{0.8}{2\sqrt{3}}\Big[-(2-\sqrt{3})e^{-(2-\sqrt{3})t}$$

$$+ (2+\sqrt{3})e^{-(2+\sqrt{3})t}\Big]$$

$$\Rightarrow \boxed{\begin{aligned} i(t) = 0.2\,\delta(t) &- \frac{0.2}{2\sqrt{3}}\Big[-(5+2\sqrt{3})e^{-(2+\sqrt{3})t} + (5-2\sqrt{3})e^{-(2-\sqrt{3})t}\Big] \\ &- \frac{0.8}{2\sqrt{3}}\Big[-(2-\sqrt{3})e^{-(2-\sqrt{3})t} + (2+\sqrt{3})e^{-(2+\sqrt{3})t}\Big] \end{aligned}}$$

<u>Case 3:</u> $L = 10\ H$
$R = 5\ \Omega$
$C = 0.1\ F$
$V = 2\ V$

Our equation is again $L\ddot{q} + R\dot{q} + \frac{1}{C}q = V$

$$\Rightarrow 10\ddot{q} + 5\dot{q} + 10q = 2$$

$$\Rightarrow \ddot{q} + 0.5\dot{q} + q = 0.2$$

Laplace transform:

$$\phi(s)\left[s^2 + 0.5s + 1\right] = \frac{0.2}{s}$$

$$\Rightarrow \phi(s) = \frac{0.2}{s\left(s^2 + \frac{1}{2}s + 1\right)}$$

Partial fractions

$$\frac{0.2}{s(s^2+\frac{1}{2}s+1)} = \frac{A}{s} + \frac{Bs+C}{(s^2+\frac{1}{2}s+1)}$$

$$\Rightarrow 0.2 = A(s^2+\tfrac{1}{2}s+1) + s(Bs+c)$$

$$\Rightarrow 0.2 = As^2 + \tfrac{1}{2}As + A + Bs^2 + Cs$$

$$\Rightarrow 0.2 = s^2(A+B) + s(\tfrac{1}{2}A+c) + A$$

$$\Rightarrow \text{I} \quad \boxed{A = 0.2}$$

$$\text{II} \quad \tfrac{1}{2}A + C = 0 \Rightarrow C = -\tfrac{1}{2}A \Rightarrow \boxed{C = -0.1}$$

$$\text{III} \quad A + B = 0 \Rightarrow \boxed{B = -0.2}$$

$$\Rightarrow Q(s) = \frac{0.2}{s} + \frac{-0.2s - 0.1}{s^2 + \frac{1}{2}s + 1}$$

Completion of the square in the denominator of the second term goes as follows

$$s^2 + \tfrac{1}{2}s + 1 = s^2 + \tfrac{1}{2}s + \left(\tfrac{1}{4}\right)^2 - \left(\tfrac{1}{4}\right)^2 + 1$$

$$= \left(s+\tfrac{1}{4}\right)^2 + 1 - \tfrac{1}{16} = \left(s+\tfrac{1}{4}\right)^2 + \tfrac{15}{16}$$

$$\Rightarrow Q(s) = \frac{0.2}{s} + \frac{-0.2s - 0.1}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}$$

$$= \frac{0.2}{s} - \frac{0.2s + 0.2\left(\frac{1}{4}\right)}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}} - \frac{0.1 - 0.2\left(\frac{1}{4}\right)}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}$$

$$\Phi(s) = \frac{0.2}{s} - 0.2\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2} - \frac{0.05}{\left(s+\frac{1}{4}\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2}$$

$$= \frac{0.2}{s} - 0.2\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2} - \frac{0.05}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\left(\sqrt{\frac{15}{16}}\right)^2}$$

which is easily inverted using the table to give:

$$\boxed{\begin{aligned} q(t) = {} & 0.2u(t) - 0.2e^{-\frac{1}{4}t}\cos\left(\sqrt{\tfrac{15}{16}}t\right) \\ & - \frac{0.2}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\sqrt{\tfrac{15}{16}}t\right) \end{aligned}}$$

Additionally, $i(t) = \frac{dq(t)}{dt}$

$$\Rightarrow i(t) = 0.2\delta(t) + \frac{1}{4}(0.2)e^{-\frac{1}{4}t}\cos\left(\sqrt{\tfrac{15}{16}}t\right)$$
$$+0.2e^{-\frac{1}{4}t}\sqrt{\tfrac{15}{16}}\sin\left(\sqrt{\tfrac{15}{16}}t\right) + \frac{1}{4}\frac{0.2}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\sqrt{\tfrac{15}{16}}t\right)$$
$$-\frac{0.2}{\sqrt{15}}\sqrt{\tfrac{15}{16}}e^{-\frac{1}{4}t}\cos\left(\sqrt{\tfrac{15}{16}}t\right)$$

$$\Rightarrow i(t) = 0.2\delta(t) + \left[0.05e^{-\frac{1}{4}t}\cos\left(\sqrt{\tfrac{15}{16}}t\right)\right.$$
$$\left. - 0.05e^{-\frac{1}{4}t}\cos\left(\sqrt{\tfrac{15}{16}}t\right)\right] + \left[0.05\sqrt{15}e^{-\frac{1}{4}t}\sin\left(\sqrt{\tfrac{15}{16}}t\right)\right.$$
$$\left. + \frac{0.05}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\sqrt{\tfrac{15}{16}}t\right)\right]$$

$$\Rightarrow \boxed{i(t) = 0.2\delta(t) + \frac{0.8}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\sqrt{\tfrac{15}{16}}t\right)}$$

Note: In all expressions for $i(t)$ we obtained a term $0.2\delta(t)$ which is the derivative of $0.2u(t)$ and gives a spike in the current at $t=0$. This spike is an unphysical result that comes from the fact that our solution contains step functions since it is defined only for $t>0$.

PART #3

The graphs for $q(t)$ and $i(t)$ are shown on the next page.

a) We note that cases 1 and 2 do not show oscillations in $q(t)$, and we also see that case 2 takes longer for the system to settle. Therefore, case 2 would correspond to an overdamped system. From the graph we don't know for sure that case 1 corresponds to the fastest possible settling, so we can't tell if it is critically damped or overdamped. However, if we know that one of the systems is for sure critically damped, then it would be the one for case 1, which settles the fastest. Case 3 is clearly underdamped since it exhibits oscillations.

b) To find the equivalent mass, damping constant and force constant, we compare the equation of our system with the

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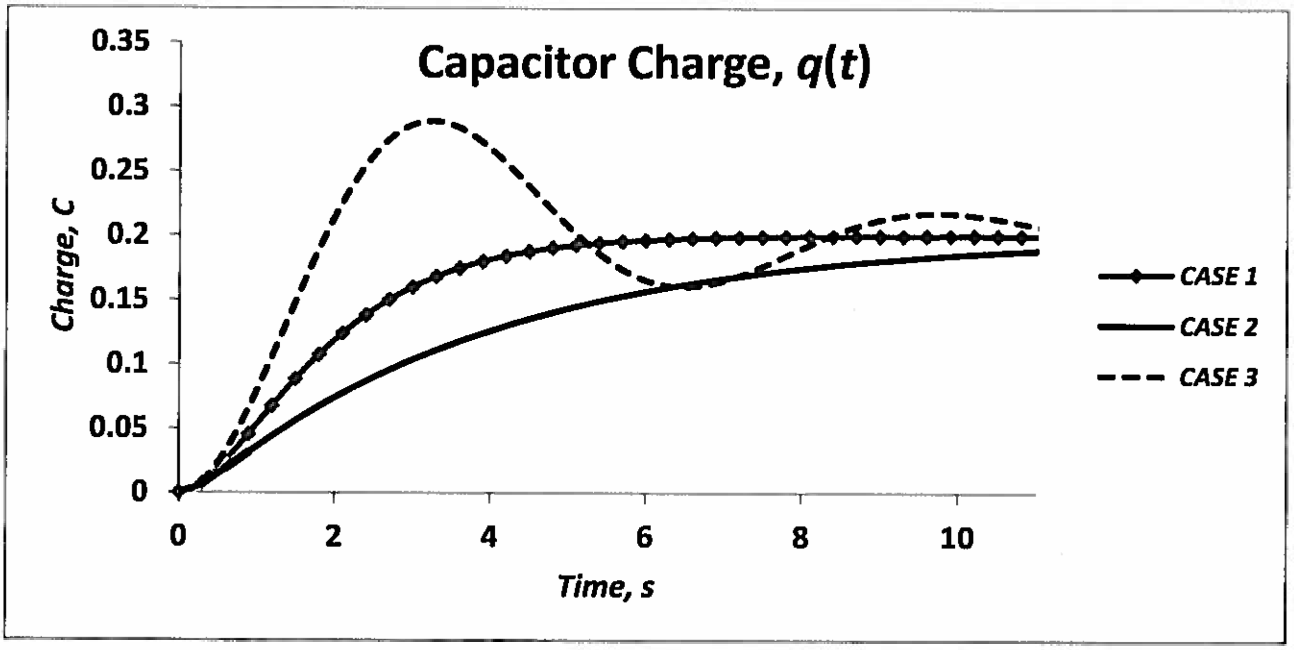
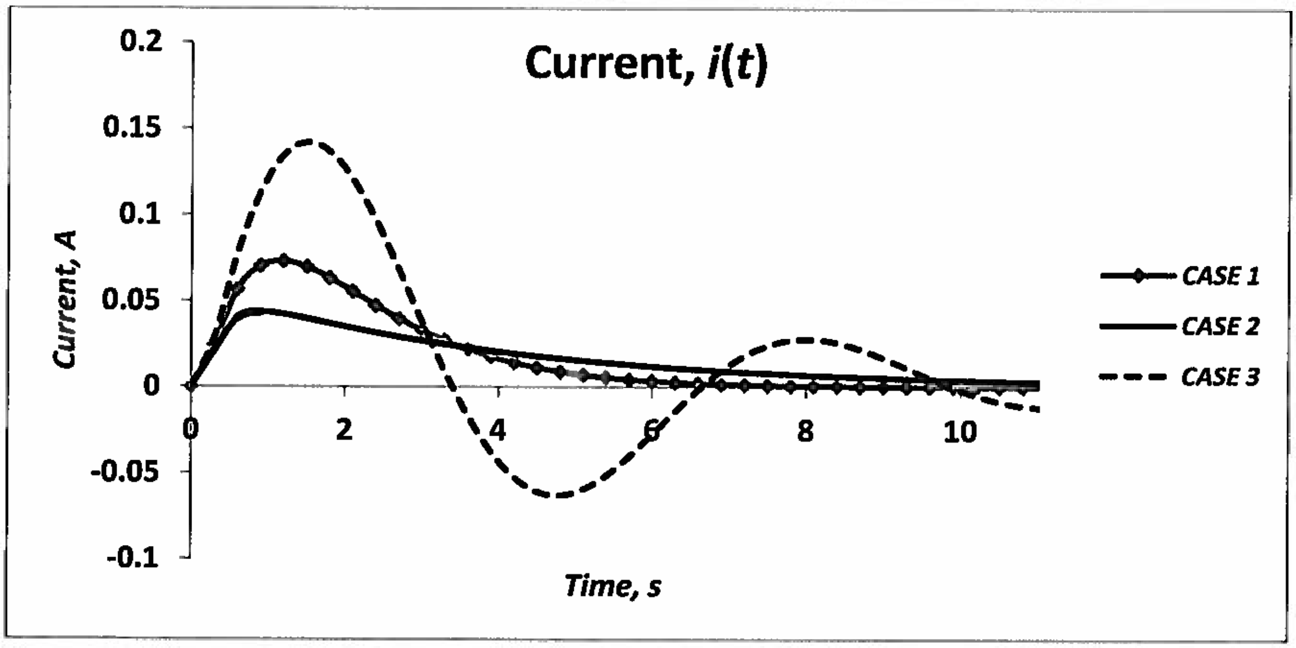


Capacitor Charge, q(t)
Charge, C
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
Time, s
CASE 1
CASE 2
CASE 3



Current, i(t)
Current, A
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
2
4
6
8
10
Time, s
CASE 1
CASE 2
CASE 3

equation of motion of a mechanical system:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$m\ddot{x} + c\dot{x} + Kx = f(t)$$

so we have the following:

	m	c	K
Case 1	10 Kg	20 Ns/m	10 N/m
Case 2	10 Kg	40 Ns/m	10 N/m
Case 3	10 Kg	5 Ns/m	10 N/m

We can now calculate ζ for each case using $\zeta = \dfrac{c}{2\sqrt{Km}}$

Case 1: $\zeta = \dfrac{20\ Ns/m}{2\sqrt{10\ N/m \cdot 10\ Kg}} = 1$ (critically damped)

Case 2: $\zeta = \dfrac{40\ Ns/m}{2\sqrt{10\ N/m \cdot 10\ Kg}} = 2$ (overdamped)

Case 3: $\zeta = \dfrac{5\ Ns/m}{2\sqrt{10\ N/m \cdot 10\ Kg}} = 0.25$ (underdamped)

The above values of ζ confirm our qualitative observations of part 3.a.

PART #4

We know that system 3 (case 3) corresponds to an underdamped case with equivalent mechanical parameters: $m = 10\ Kg$, $C = 5\ Ns/m$ and $K = 10\ N/m$. The voltage acts as the force in a mechanical system.

$$V(t) = 10 \sin(0.05t)$$
$$f(t) = F_0 \sin(\omega t)$$

Therefore, $F_0 = 10\ N$ and $\omega = 0.05\ \frac{RAD}{s}$.

Now, $\omega_n = \sqrt{\frac{K}{m}} = 1\ RAD/s$.

We know that for a mechanical system,

$$x(t) = \frac{F_0}{K} A(\omega) \sin(\omega t - \theta(\omega))$$

$$\frac{F_0}{K} = \frac{10\ N}{10\ N/m} = 1\ m$$

$$A(\omega) = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

We already know from part 3 that $\zeta = 0.25$.

$$\Rightarrow A(\omega) = \frac{1}{\sqrt{\left[1 - (0.05)^2\right]^2 + (2 * 0.25 * 0.05)^2}} = 0.9978$$

Now, $\theta(t) = \tan^{-1}\left[\dfrac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\dfrac{2 * 0.25 * 0.05}{1 - 0.05^2}\right]$

$= 0.025$ RAD (1.44°)

Therefore, if this were a mechanical system we would have:

$x(t) = \frac{F_0}{K} A(\omega)\sin(\omega t - \theta(\omega))$

$= 0.9978 \sin(0.05t - 0.025)$ meters

Therefore, $\boxed{q(t) = 0.9978 \sin(0.05t - 0.025) \text{ Coulombs}}$

$i(t)$ is found by differentiation, $i(t) = \frac{dq(t)}{dt}$

$\Rightarrow i(t) = 0.05 * 0.9978 \cos(0.05t - 0.025)$

$\Rightarrow \boxed{i(t) = 0.0499 \cos(0.05t - 0.025)}$