ADMIN

3/27

- HOMEWORK DUE - TUESDAY START OF CLASS

- GRADES - LOTS OF CONCERN
  - WILL DROP LOWEST HOMEWORK GRADE → ONLY 1
  - THOSE WITH "HIGH ENOUGH" FINAL GRADE WILL BE EXEMPT FROM FINAL
  - PERFORMING "WELL ENOUGH" ON FINAL WILL ALLOW ONE TO GAIN AN EXTRA "GRADE BOOST"

DEFINITIONS

HIGH ENOUGH - > 95% OR SIMILAR.

WELL ENOUGH - > 95% OR SIMILAR

GRADE BOOST - HALF | FULL LETTER GRADE BOOST

$C+ \longrightarrow B \xrightarrow{\text{BOOST}} A$

EXACT DETAILS MAY VARY ←

SOLUTION OF STATE EQUATION

CH 8-5

FINDING TIME RESPONSE FOR LINEAR STATE EQUATIONS. $\dot{x} = f \longrightarrow x(t)$, $x(0) = x_0$

REVIEW SOLUTION OF FIRST ORDER SCALAR DIFF. EQ.

$$\dot{x} = ax \quad \longleftarrow \text{WHAT IS THE SOLUTION ?}$$

$$x(t) = e^{-at} x(0)$$

WE CAN ASSUME A SOLUTION OF THE FORM

$$x(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_k t^k + \ldots$$

PLUG INTO $\dot{x}$

LEFT SIDE:

$$b_1 + 2b_2 t + 3b_3 t^2 + \ldots + k b_k t^{k-1} + \ldots =$$

RIGHT SIDE

$$a(b_0 + b_1 t + b_2 t^2 + \ldots + b_k t^k + \ldots)$$

SOLUTION MUST HOLD FOR ALL t

$\longrightarrow$ EQUATE THE COEFFICIENTS.

$$b_1 = a b_0$$

$$2b_2 = a b_1$$

$$3b_3 = a b_2$$

$$\vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

EVERY COEFFICIENT IS A FUNCTION OF b_0!

WE FIND b_0 BY FINDING THE INITIAL CONDITION $t=0 \Rightarrow \underline{x(t=0) = b_0}$

SOLUTION BECOMES

$$x(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_k t^k + \dots$$

$$= b_0 + a b_0 t + \frac{1}{2} a^2 b_0 t^2 + \dots + \frac{1}{k!} a^k b_0 t^k$$

$$= b_0 \left(1 + at + \frac{1}{2} a^2 t^2 + \dots + \frac{1}{k!} a^k t^k \right)$$ POWER SERIES.

$$= x(0) \left[\sum_{k=0}^{\infty} \frac{(at)^k}{k!} \right] = \boxed{x(0)\, e^{at} = x(t)}$$

NOW LET'S EXTEND THIS TO A VECTOR CASE.

$$\dot{\bar{x}} = A\,\bar{x}$$

$\bar{x}$ – $n \times 1$ VECTOR

A – $n \times n$ STATE MATRIX (CONSTANT)

EACH ELEMENT OF $\dot{\bar{x}}$ IS A SCALAR AND WILL HAVE A SOLUTION JUST LIKE PREVIOUSLY $\Rightarrow$ PUT THEM ALL INTO A MATRIX EQ.

$$\bar{x}(t) = \bar{b}_0 + \bar{b}_1 t + \bar{b}_2 t^2 + \dots + \bar{b}_k t^k$$

AGAIN WE PLUG THIS INTO OUR DIFF. EQ AND EQUATE COEFFICIENTS.

<u>LEFT SIDE</u>

$$\bar{b}_1 + 2\bar{b}_2 t + \dots + k\,\bar{b}_k t^{k-1} \quad =$$

<u>RIGHT SIDE</u>

$$A(\bar{b}_0 + \bar{b}_1 t + \bar{b}_2 t^2 + \dots + \bar{b}_k t^k)$$

EQUATE COEFFICIENTS. AND FIND $\bar{b}_0 = \bar{x}(0)$

$$\bar{x}(t) = \bar{x}(0)\left(I + At + \frac{1}{2!}A^2t^2 + \cdots \frac{1}{k!}A^kt^k\right)$$

$$\boxed{\bar{x}(t) = \bar{x}(0)\, e^{At}}$$ ← MATRIX EXPONENTIAL

PROPERTIES OF MATRIX EXPONENTIAL

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

A IS $n \times n$ MATRIX.
STATE MATRIX

$$\frac{d}{dt} e^{At} = e^{At} A$$ ← JUST LIKE SCALAR CASE

$$e^{A(t+s)} = e^{At} e^{As}$$

$$e^{At} e^{-At} = e^{-At} e^{At} = e^{A(t-t)} = I$$

$$A \; A^{-1} = A^{-1} \; A = I$$

INVERSE OF e^{At} IS e^{-At} ← e^{At} IS ALWAYS

INVERTIBLE ⟹ NON-SINGULAR

IN GENERAL DO NOT DO THE FOLLOWING !!

$$e^{(A+B)t} \neq e^{At} e^{Bt}$$

ONLY WORKS IF $AB = BA$ (IN GENERAL $AB \neq BA$)

$$2 \cdot 3 = 3 \cdot 2$$

HOMOGENEOUS STATE SOLUTION

$\dot{x}(t) = A\,x(t)$ A is $n \times n$ $\bar{x}$ is $n \times 1$

LAPLACE $\Longrightarrow$ $s\,X(s) - x(0) = A\,X(s)$

$$(SI - A)\,X(s) = X(0)$$

$$X(s) = (SI - A)^{-1} x(0)$$

$$\boxed{x(t) = \mathcal{L}^{-1}\{(SI-A)^{-1}\}\,x(0)}$$

SOLUTION.

WE CAN SHOW (LINEAR ALGEBRA CLASS)

$$(SI - A)^{-1} = \frac{I}{S} + \frac{A}{S^{2}} + \frac{A^2}{S^2} + \frac{A^3}{S^3} + \cdots$$

$$\Rightarrow \mathcal{L}\{(SI-A)^{-1}\} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$= e^{At}$$

$$\Longrightarrow \boxed{x(t) = e^{At}\,x(0)}$$

MATRIX EXPONENTIAL.

$\dot{x}(t) = A\,x(t)$ WITH $x(0) = x_0$

$\downarrow$

$$x(t) = \mathcal{L}^{-1}\{(SI-A)^{-1}\}\,x(0)$$

$$= e^{At}\,x(0)$$

SOLUTION

$$e^{At} = \mathcal{L}^{-1}\{(SI-A)^{-1}\}$$

THE SOLUTION OF $\dot{x} = Ax$ CAN BE WRITTEN AS:

Φ – CAPITAL PHI

$$x(t) = \Phi(t)\, x(0) = e^{At}\, x(0)$$

WHERE $\Phi(t)$ IS A $n \times n$ MATRIX AND IS THE SOLUTION OF

$$\dot{\Phi}(t) = A\,\Phi(t) \quad ; \quad \Phi(0) = I$$

<u>STATE TRANSITION MATRIX</u> $\Phi(t)$

MATRIX WHICH TRANSFORMS INITIAL CONDITION $x(0)$ TO $x(t)$.

$$x(0) = \Phi(0)\, x(0) = x(0) \quad \text{DEF.}$$

$$\dot{x}(t) = \dot{\Phi}(t)\, x(0) = A\,\underbrace{\Phi(t)\, x(0)} = A\,x(t)$$

$$\boxed{\Phi(t) = e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}}$$

$$\boxed{\Phi^{-1}(t) = e^{-At} = \Phi(-t)}$$

← USEFUL TO GO BACKWARDS IN TIME.

AVOID INVERSE

EXAMPLE

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t)$$

FIND: $\Phi = e^{At} = \mathcal{L}^{-1}\{(SI-A)^{-1}\}$ ← EASIEST.

$$(SI-A) = \begin{bmatrix} S & -1 \\ 8 & S+6 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ IDENTITY MATRIX}$$

$$(SI-A)^{-1} = \begin{bmatrix} S+6 & 1 \\ -8 & S \end{bmatrix} \cdot \frac{1}{S^2+6S+8}$$

$S^2+6S+8 = (S+2)(S+4)$

$$= \begin{bmatrix} \frac{S+6}{S^2+6S+8} & \frac{1}{S^2+6S+8} \\ \frac{-8}{S^2+6S+8} & \frac{S}{S^2+6S+8} \end{bmatrix}$$

PARTIAL FRACTION ON EACH TERM.

$$\frac{S+6}{S^2+6S+8} = \frac{A}{S+2} + \frac{B}{S+4} = \frac{2}{S+2} - \frac{1}{S+4}$$

$$A = \left.\frac{\cancel{(S+2)}(S+6)}{(S+4)\cancel{(S+2)}}\right|_{S=-2} = 2$$

$$B = \left.\frac{\cancel{(S+4)}(S+6)}{(S+2)\cancel{(S+4)}}\right|_{S=-4} = \frac{2}{-2} = -1$$

REPEAT FOR EACH TERM!

$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{2}{s+2} - \dfrac{1}{s+4} & \dfrac{1/2}{s+2} - \dfrac{1/2}{s+4} \\ \dfrac{-4}{s+2} + \dfrac{4}{s+4} & \dfrac{-1}{s+2} + \dfrac{2}{s+4} \end{bmatrix}$$

INVERSE LAPLACE $\Phi(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

$$\Phi = e^{At} = \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2e^{-4t} \end{bmatrix}$$

WHAT IS $x_1(t)$ GIVEN $x(0) = [1 \quad 2]^T$

$\longrightarrow x(t) = \Phi(t)\, x(0)$

$x_1(t) = \Phi_{11}(t)\, x_1(0) + \Phi_{12}(t)\, x_2(0)$

$a\ddot{x} + b\dot{x} + cx = 0 \longleftarrow$ x TO FIND OUTPUT SOLUTION $x(t)$

$x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

$X(s) = \dfrac{(\qquad)}{as^2 + bs + c}$

AVOID REDOING THIS.

HOMOGENEOUS (FREE) MOTION RESPONSE

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

FIND Φ, Φ^{-1}

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

INVERSE LAPLACE (PARTIAL FRACTION)

$$(sI - A)^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & +e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$\phi^{-1}(t) = \phi(-t) \longleftarrow$ PROPERTY

$$\phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

WHAT IS THE SOLUTION ? $x(t) = \Phi(t)\, x(0)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{WITH } x(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x(t) = \phi(t)\, x(0)$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} \\ -3e^{-t} + 6e^{-2t} \end{bmatrix}$$

NONHOMOGENEOUS

LETS FIRST LOOK AT THE SCALAR CASE

$$\dot{x} = ax + bu$$

$$\dot{x} - ax = bu \qquad \text{MULTIPLY BY } e^{-at}$$

$$e^{-at}\left[\dot{x} - ax\right] = \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}\, b\, u(t)$$

INTEGRATE

$$e^{-at}x(t) = x(0) + \int_0^t e^{-a\tau} b\, u(\tau)\, d\tau$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau} b\, u(\tau)\, d\tau$$

HOMOGENEOUS FORCED (CONVOLUTION)

CAN EXTEND TO MATRIX CASE
(SKIPPING DERIVATION) LOOK IN BOOK

$$\dot{x} = Ax + Bu$$

$$x(t) = \Phi(t)\, x(0) + \int_0^t \Phi(t-\tau) B u(\tau)\, d\tau$$

$$= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

IF A^{-1} EXISTS (NONSINGULAR)

$$x(t) = e^{At}x(0) + A^{-1}\left(e^{At} - I\right)B$$

EXAMPLE

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

UNIT STEP.

SOLUTION IS $x(t) = \Phi(t)\,x(0) + \int \Phi(t-\tau)\,Bu(\tau)\,d\tau$

$$\Phi(t) = e^{At} = \mathcal{L}^{-1}\left\{ (sI-A)^{-1} \right\}$$

$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{s+6}{s^2+6s+8} & \dfrac{1}{s^2+6s+8} \\[2ex] \dfrac{-8}{s^2+6s+8} & \dfrac{s}{s^2+6s+8} \end{bmatrix}$$

EXPAND EACH TERM IN PARTIAL FRACTIONS

$$\mathcal{L}^{-1}\{\Phi(t)\} = \begin{bmatrix} \dfrac{2}{s+2} - \dfrac{1}{s+4} & \dfrac{1/2}{s+2} - \dfrac{1/2}{s+4} \\[2ex] \dfrac{-4}{s+2} + \dfrac{4}{s+4} & \dfrac{-1}{s+2} + \dfrac{2}{s+4} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} 2e^{-2t} - e^{-4t} & \tfrac{1}{2}e^{-2t} - \tfrac{1}{2}e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2e^{-4t} \end{bmatrix}$$

e^{At} PREVIOUS EXAMPLE

$$x(t) = \underbrace{\Phi(t)\, x(0)} + \int_0^t \Phi(t-\tau)\, B u(\tau)\, d\tau)$$

$$\text{(1)}\quad \Phi(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix} \qquad \text{HOMOGENEOUS}$$

$$\int_0^t \Phi(t-\tau)\, \underset{\uparrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{B}\, u(\tau)\, d\tau = \int_0^t \underset{2\times 1}{\begin{bmatrix} \tfrac{1}{2} e^{-2(t-\tau)} - \tfrac{1}{2} e^{-4(t-\tau)} \\ -e^{-2(t-\tau)} + 2e^{-4(t-\tau)} \end{bmatrix}} d\tau$$

$$= \begin{bmatrix} \tfrac{1}{2} e^{-2t} \int_0^t e^{2\tau} d\tau - \tfrac{1}{2} e^{-4t} \int_0^t e^{4\tau} d\tau \\ -e^{-2t} \int_0^t e^{2\tau} d\tau + 2e^{-4t} \int_0^t e^{4\tau} d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{2} e^{-2t} \left(\tfrac{1}{2} e^{2\tau} \Big|_0^t \right) - \tfrac{1}{2} e^{-4t} \left(\tfrac{1}{4} e^{4\tau} \Big|_0^t \right) \\ -e^{-2t} \left(\tfrac{1}{2} e^{2\tau} \Big|_0^t \right) + 2e^{-4t} \left(\tfrac{1}{4} e^{4\tau} \Big|_0^t \right) \end{bmatrix}$$

$$\text{(2)} = \begin{bmatrix} \tfrac{1}{8} - \tfrac{1}{4} e^{-2t} + \tfrac{1}{8} e^{-4t} \\ \tfrac{1}{2} e^{-2t} - \tfrac{1}{2} e^{-4t} \end{bmatrix}$$

$$\text{(1)} + \text{(2)} = \begin{bmatrix} \tfrac{1}{8} + \tfrac{7}{4} e^{-2t} - \tfrac{7}{8} e^{-4t} \\ -\tfrac{7}{2} e^{-2t} + \tfrac{7}{2} e^{-4t} \end{bmatrix} \qquad \text{STATE SOLUTION.}$$