# TIME RESPONSE SPECIFICATION

## EXAMPLES

EX $$T(s) = \frac{16}{s^2 + 3s + 16}$$

FIND: $\xi$, $\omega_n$, ($T_s$, $T_p$, $T_r$, % OS) STEP INPUT.

GENERAL SECOND ORDER SYSTEM

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\boxed{\omega_n = \sqrt{16} = 4} \ \frac{rad}{sec.}$$

$$2\xi\omega_n = 3 = 2\xi(4) = 3$$

$$\boxed{\xi = \frac{3}{8} \approx 0.375}$$ UNDERDAMPED!

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}$$

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$
$$= -\sigma_d \pm j\omega_d.$$

$$= \frac{4}{0.375 \cdot 4} = \frac{8}{3} \text{ SEC.}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d} = \frac{\pi}{4\sqrt{1-0.375^2}} = 0.847 \text{ sec.}$$

$T_r$ = GENERATE A PLOT AND FIND TIME TO GO FROM 10% TO 90%

$$\% \, OS = \exp\frac{-\xi\pi}{\sqrt{1-\xi^2}} \times 100 = 28\%$$

## STATE SPACE

2/19

- CLASSICAL | FREQUENCY DOMAIN APPROACH
  - CONVERT ODE → TRANSFER FUNCTION
  - \+ ALGEBRAIC RELATIONSHIP BTWN INPUTS/OUT. (SISO)
  - = ONLY APPLICABLE FOR (LTI) SYSTEMS
  - \+ EASY TO DETERMINE STABILITY + TRANSIENT RESPONSE
- MODERN | TIME DOMAIN APPROACH
  - DEVELOPED IN 50'S WITH SPACE PROGRAM
  - \+ CAN MODEL WIDE RANGE OF SYSTEMS NONLINEAR OR TIME VARYING STD
  - \+ HANDLE NON ZERO INITIAL CONDITIONS
  - \+ MULTIPLE INPUT MULTIPLE OUTPUT (MIMO)
  - INTRODUCE CONCEPT HERE → GRAD CLASS 6445 IF INTERESTED.

# DEFINITIONS

## STATE VARIABLES / VECTOR

SMALLEST SET OF LINEARLY INDEPENDENT VARIABLES SUCH THAT KNOWLEDGE OF INITIAL CONDITIONS + INPUT COMPLETELY DEFINES STATE HISTORY FOR $t \geq t_0$

## LINEAR COMBINATION

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

$K_i$ : CONSTANT SCALARS

$x_i$ : VARIABLES (STATE)

## LINEAR INDEPENDENCE

NO VARIABLES CAN BE WRITTEN AS A LINEAR COMBINATION OF OTHER VARIABLES

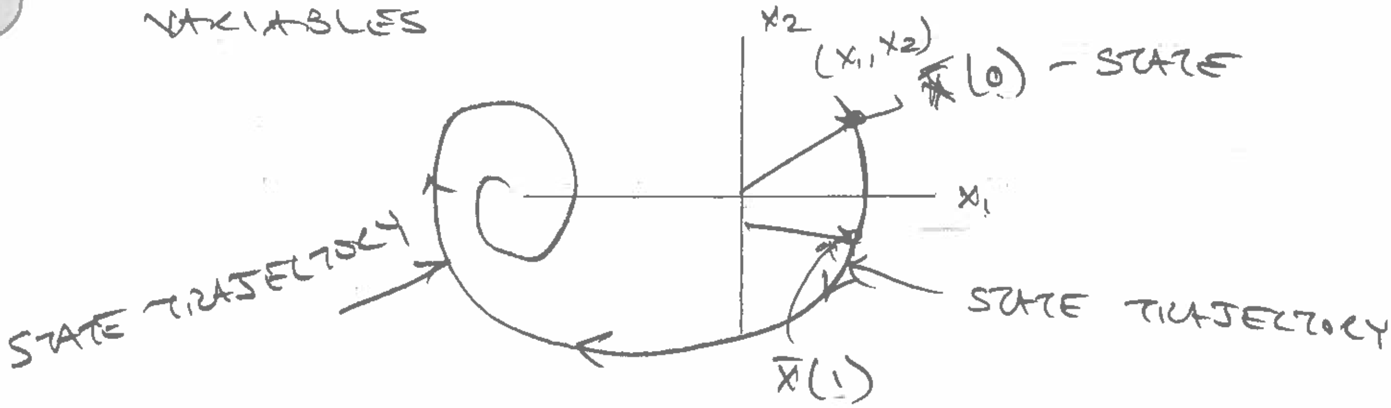
EX $x_1, x_2, x_3$

IF $x_2 = 5x_1 + 6x_3$ ← NOT LINEAR INDEPENDENT

IF $0 = \sum K_i x_i$ FOR ANY $x_i$ ONLY IF $K_i = 0$ → LINEARLY INDEPENDENT

## STATE SPACE

n- DIMENSIONAL SPACE OF ALL STATE VARIABLES




## STATE EQUATION

n - SIMULTANEOUS FIRST ODE WITH

n - VARIABLES → DEFINE EVOLUTION OF

STATE VARIABLES

## OUTPUT EQUATION

OUTPUT OF SYSTEM AS A LINEAR COMBINATION

OF STATE VARIABLES.

## STATE SPACE SYSTEM

STATE EQ. → $\dot{x} = Ax + Bu$

OUTPUT EQ. → $y = Cx + Du$

C: OUTPUT MATRIX $\mathbb{R}^{m \times n}$

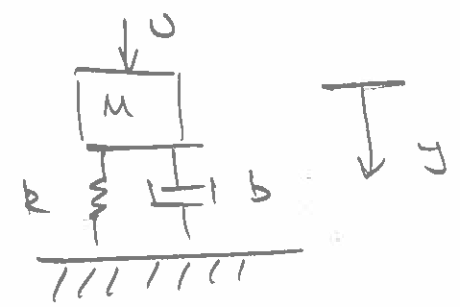
D: FEEDTHROUGH $\mathbb{R}^{m \times p}$

$x$: STATE VECTOR $\mathbb{R}^n$ $\dot{x}$: DERIVATIVE $\mathbb{R}^n$

$y$: OUTPUT $\mathbb{R}^m$ $u$: INPUT $\mathbb{R}^p$

A : STATE ... $\mathbb{R}^{n \times n}$ B: INPUT MATRIX $\mathbb{R}^{n \times p}$

EXAMPLE




INPUT : $u(t)$ FORCE

OUTPUT: $y(t)$ DISP.

EQUATIONS OF MOTION : $m\ddot{y} + b\dot{y} + ky = u$

STD FORM : $\ddot{y} + \frac{b}{M}\dot{y} + \frac{k}{M}y = \frac{u}{M}$

SECOND ORDER SYSTEM $\Rightarrow$ WE'LL NEED TWO STATE VARIABLES TO DEFINE THE MOTION.

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}\right\} \text{ STATE VARIABLES.}$$

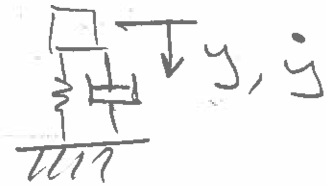
WRITE STATE EQUATIONS ($\dot{x} = Ax + Bu$) IN TERMS OF STATES

$$\left\{\begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = -\frac{b}{M}\dot{y} - \frac{k}{M}y + \frac{u}{M} \\ &= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{u}{M} \end{aligned}\right.$$

2nd ODE $\Rightarrow$ 2 1st ODE

OUTPUT EQUATION:

$$y = y(t) = x_1$$




NOW PLACE INTO MATRIX FORM

$y_1 = x_1$

$y_2 = x_2$

$y_3 = x_1 + x_2$

$y_m$

$$\dot{x}_1 = 0\,x_1 + 1\,x_2 + 0\,u$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$y = 1\,x_1 + 0\,x_2 + 0\,u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}x + \begin{bmatrix}0\end{bmatrix}u$$

DIFFERENTIAL EQUATION $\Longrightarrow$ STATE SPACE

$n^{TH}$ ORDER $\Longrightarrow$ $n$ FIRST ORDER

2nd ORDER DIFF EQUATION $\longrightarrow$ 2 FIRST ORDER DIFF. EQ

$$\ddot{y} = (\cdots)$$

$$\begin{aligned}\dot{x}_1 &= (\cdots\\ \dot{x}_2 &= \cdots)\end{aligned}$$

# CONVERT TRANSFER FUNCTION TO STATE SPACE

- STATE SPACE REPRESENTATION IS IDEAL FOR COMPUTER IMPLEMENTATION
- USUALLY EASIEST TO GO FROM DIFF. EQ TO STATE SPACE → SO WE'LL SHOW THAT.

CONSIDER: GENERAL $n$TH ORDER ODE.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

ONE CONVIENENT CHOICE OF STATE VARIABLES

$$x_1 = y \qquad x_2 = \dot{y} \qquad x_3 = \ddot{y} \qquad \cdots \qquad x_n = \frac{d^{n-1}}{dt^{n-1}}y$$

DIFFERENTIATING BOTH SIDES.

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = x_4 \qquad \cdots$$

$$\dot{x}_n = \frac{d^n}{dt^n}y = -a_0 y - a_1\frac{dy}{dt} - \cdots - a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + b_0 u$$

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + b_0 u$$

MATRIX FORM

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad x_1 \longrightarrow y(t)$$

1. CONVERT TF TO ODE
2. CONVERT ODE TO S.S.

EXAMPLE

CONVERT TRANSFER FUNCTION TO STATE SPACE

$$\frac{C(s)}{R(s)} = \frac{24}{s^3+9s^2+26s+24}$$

CROSS MULTIPLY

$$(s^3+9s^2+26s+24)\,C(s) = 24\,R(s)$$

INVERSE LAPLACE WITH ZERO I.C.

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

STATE VARIABLES

$$x_1 = c \qquad x_2 = \dot{c} \qquad x_3 = \ddot{c}$$

DERIVATIVES

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = \dddot{c} = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

MATRIX FORM

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x}$$