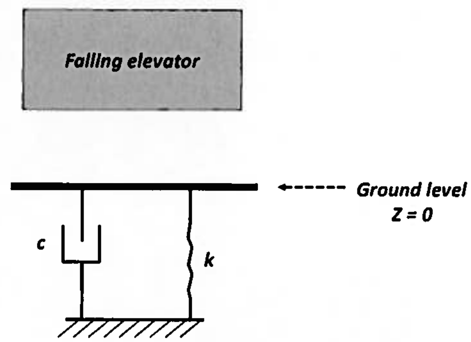
MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 2
Due Thursday, February 12th at the beginning of class

NOTE: Please make sure to show and explain all the steps followed to arrive at the solution. Otherwise we are unable to assess your understanding of the material and give you credit for your answers.

1. Laplace transforms: Solve the following differential equation using the Laplace transform method:

$$a\dot{x} + bx = A\sin(\omega t), \; x(0) = 1$$



2. Falling elevator – safety design application: An engineer is studying the effect of external accelerations on human health with particular focus on industrial accidents. As part of his research, he is evaluating different types of elevator safety systems. Specifically, consider a simple elevator of mass 1000 Kg, loaded with eight persons of total mass 500 Kg, that is free falling from a height of 5 meters (a little less than two floors) due to a broken cable. A spring-damper combination is located on the ground, which has a platform *of negligible mass* located at the ground level as indicated in the figure (the spring and damper are located inside a pit and allow the elevator to travel below the ground level). The spring force constant is k = 7500 N/m and the damper constant is 3000 N s/m. **For simplicity assume that the acceleration of gravity is equal to 10 m/s^2.**

(a) Calculate the velocity of the elevator for the instant when it first contacts the platform.
(b) Draw the free body diagram and write the equation of motion of the elevator for the time it is in contact with the platform (assume that the elevator attaches to the platform as soon as it comes into contact with it, and assume that this happens at time = 0).
(c) Solve for the position of the elevator, $x(t)$, for time > 0.
(d) Using the above solution, calculate the minimum depth required for the pit. That is, what is the maximum displacement of the platform in the negative direction? **[For simplicity you can solve this numerically by plotting $x(t)$ and finding the minimum]**
(e) Calculate the absolute value of the maximum acceleration that the persons inside the elevator are subjected to and express it in terms of 'g' (the acceleration of gravity). That is, how many g's will the persons in the elevator be subjected to? **[For simplicity you can calculate the acceleration as a function of time numerically by differentiating the position twice, and then you can look for its largest absolute value]**

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SPRING 2015

HOMEWORK #2
SOLUTION

PROBLEM 1

$$a\dot{x} + bx = A\sin(\omega t) \; ; \quad x(0) = 1$$

Taking the Laplace transform

$$a\left[sX(s) - X(0)\right] + bX(s) = \frac{A\omega}{s^2+\omega^2}$$

$$\Rightarrow X(s)\left[as+b\right] - a = \frac{A\omega}{s^2+\omega^2}$$

$$\Rightarrow X(s) = \frac{a}{as+b} + \frac{A\omega}{(as+b)(s^2+\omega^2)}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{a}{as+b}\right] + \mathcal{L}^{-1}\left[\frac{A\omega}{(as+b)(s^2+\omega^2)}\right]$$

Working out the terms separately

(i) $$\frac{a}{as+b} = \frac{1}{s+\frac{b}{a}}$$

from the table, $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+\frac{b}{a}}\right] = e^{-\frac{b}{a}t}$$

(ii) $$\frac{A\omega}{(as+b)(s^2+\omega^2)}$$

Using partial fractions

$$\frac{A\omega}{(as+b)(s^2+\omega^2)} = \frac{M}{as+b} + \frac{Ns+Q}{s^2+\omega^2}$$

$$\Rightarrow A\omega = M(s^2+\omega^2) + (Ns+Q)(as+b)$$

$$\Rightarrow A\omega = \cancel{Ms^2} + M\omega^2 + \cancel{Nas^2} + \cancel{Nsb} + \cancel{Qas} + Qb$$

$$= s^2(M+Na) + s(Nb+Qa) + (M\omega^2+Qb)$$

This gives us three equations:

(I) $M + Na = 0 \longrightarrow M = -Na$

(II) $Nb + Qa = 0 \longrightarrow Q = -\frac{Nb}{a}$

(III) $M\omega^2 + Qb = A\omega$

Inserting (I) and (II) into (III)

$$M\omega^2 + Qb = A\omega$$

$$\Rightarrow (-Na)\omega^2 + \left(-\frac{Nb}{a}\right)b = A\omega$$

$$\Rightarrow -Na\omega^2 - \frac{Nb^2}{a} = A\omega$$

$$\Rightarrow N\left(-a\omega^2 - \frac{b^2}{a}\right) = A\omega$$

$$\Rightarrow N = \frac{-A\omega}{a\omega^2 + \frac{b^2}{a}}$$

from (II) we have that

$$Q = -\frac{Nb}{a} = -\left(\frac{-A\omega}{a\omega^2+\frac{b^2}{a}}\right)\frac{b}{a} = \frac{A\omega b}{a^2\omega^2+b^2}$$

from (I) we have that

$$M = -Na = -\left(\frac{-A\omega}{a\omega^2+\frac{b^2}{a}}\right)a = \frac{A\omega a}{a\omega^2+\frac{b^2}{a}}$$

Therefore, the term we seek to invert can be written as:

$$\left(\frac{Aa\omega}{a\omega^2+\frac{b^2}{a}}\right)\cdot\frac{1}{as+b} - \left(\frac{A\omega}{a\omega^2+\frac{b^2}{a}}\right)\cdot\frac{s}{s^2+\omega^2}$$

$$+\left(\frac{A\omega b}{a^2\omega^2+b^2}\right)\cdot\frac{1}{s^2+\omega^2}$$

This term will give an exponential

This term will give a cosine

$$= \frac{A\omega}{a\omega^2+\frac{b^2}{a}}\cdot\left[\frac{1}{s+\frac{b}{a}}\right] - \frac{A\omega}{a\omega^2+\frac{b^2}{a}}\cdot\left[\frac{s}{s^2+\omega^2}\right]$$

$$+\frac{Ab}{a^2\omega^2+b^2}\cdot\left[\frac{\omega}{s^2+\omega^2}\right]$$

this term will give a sine.

The inverse Laplace transform of the above can be directly found using the table:

$$\frac{A\omega}{a\omega^2+\frac{b^2}{a}}e^{-\frac{b}{a}t} - \frac{A\omega}{a\omega^2+\frac{b^2}{a}}\cos(\omega t) + \frac{Ab}{a^2\omega^2+b^2}\sin\omega t$$

The final step is to combine all terms in the solution

$$x(t) = e^{-\frac{b}{a}t} + \frac{A\omega}{a\omega^2+\frac{b^2}{a}} e^{-\frac{b}{a}t} - \frac{A\omega}{a\omega^2+\frac{b^2}{a}} \cos\omega t + \frac{Ab}{a^2\omega^2+b^2} \sin\omega t$$

$$\Rightarrow \boxed{x(t) = \left[\frac{A\omega}{a\omega^2+\frac{b^2}{a}} + 1\right] e^{-\frac{b}{a}t} - \frac{A\omega}{a\omega^2+\frac{b^2}{a}} \cos\omega t + \frac{Ab}{a^2\omega^2+b^2} \sin\omega t}$$

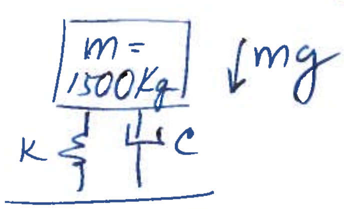
Problem 2

(a) The velocity of the elevator after falling five meters and taking $g = 10\ m/s^2$ will be:

$$v = -\sqrt{12gh} = -\sqrt{12\times 10\times 5} = -10\ m/s$$

height

(b)



with $\begin{cases} \dot{x}(0) = -10\ m/s \\ x(0) = 0 \end{cases}$

Equation of motion

$$m\ddot{x} + c\dot{x} + kx = -mg$$

$$\Rightarrow 1500\ddot{x} + 3000\dot{x} + 7500x = -1500 * 10$$

$$\Rightarrow 1500\ddot{x} + 3000\dot{x} + 7500x = -15000$$

(c) We Laplace transform the equation:

$$1500\left[s^2X(s) - sx(0) - \dot{x}(0)\right]$$

$V(0) = -10$ m/s

zero

$$+ 3000\left[sX(s) - x(0)\right] + 7500X(s) = \frac{-15000}{s}$$

zero

$$\Rightarrow 1500\left[s^2X(s) + 10\right] + 3000\left[sX(s)\right] + 7500X(s) = -\frac{15000}{s}$$

Dividing through by 1500,

$$s^2X(s) + 10 + 2sX(s) + 5X(s) = \frac{-10}{s}$$

$$\Rightarrow X(s)\left(s^2 + 2s + 5\right) = \frac{-10}{s} - 10$$

$$\Rightarrow X(s) = \frac{-10}{s(s^2+2s+5)} - \frac{10}{s^2+2s+5}$$

We expand the first term using partial fractions:

$$\frac{-10}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

$$\Rightarrow s(Bs+C) + A(s^2+2s+5) = -10$$

$$\Rightarrow Bs^2 + Cs + As^2 + 2As + 5A = -10$$

$$\Rightarrow s^2(B+A) + s(C+2A) + 5A = -10$$

This leads to three equations:

I $5A = -10 \Rightarrow \boxed{A = -2}$

II $C + 2A = 0 \Rightarrow C = -2A \Rightarrow \boxed{C = 4}$

III $B + A = 0 \Rightarrow B = -A \Rightarrow \boxed{B = 2}$

$$\Rightarrow \frac{-10}{s(s^2+2s+5)} = \frac{-2}{s} + \frac{2s}{s^2+2s+5} + \frac{4}{s^2+2s+5}$$

with this we have that

$$X(s) = \frac{-2}{s} + \frac{2s}{s^2+2s+5} + \frac{4}{s^2+2s+5} - \frac{10}{s^2+2s+5}$$

$$= \frac{-2}{s} + \frac{2s}{s^2+2s+5} - \frac{6}{s^2+2s+5}$$

We now complete the square in the denominator of the last two terms to get

$$X(s) = \frac{-2}{s} + \frac{2s}{(s+1)^2+2^2} - \frac{6}{(s+1)^2+2^2}$$

from the Laplace table we have

$$\mathcal{L}\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\mathcal{L}\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

So the first and last terms of $X(s)$ give us

$$-2u(t) \quad - 3e^{-t}\sin 2t$$

step function

This leaves the middle term

$$\frac{2s}{(s+1)^2+2^2} = \frac{2s+2}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

Which using the above information from the Laplace table gives in real space:

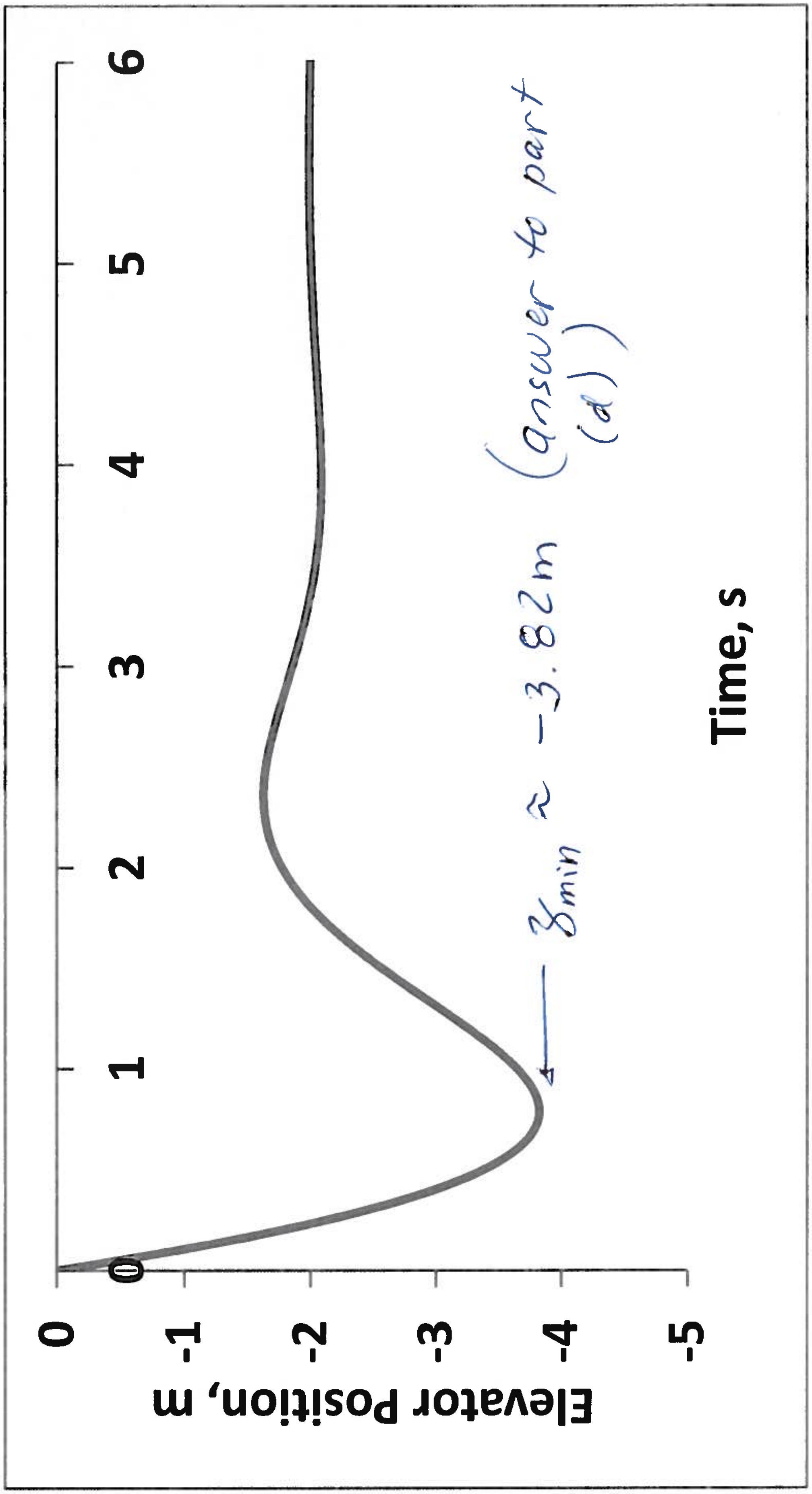
$$2e^{-t}\cos 2t \quad - e^{-t}\sin 2t$$

Adding all parts of the solution we get:

$$x(t) = -2u(t) - 3e^{-t}\sin 2t + 2e^{-t}\cos 2t$$
$$- e^{-t}\sin 2t$$

$$\Rightarrow x(t) = -2u(t) - 4e^{-t}\sin 2t + 2e^{-t}\cos 2t$$

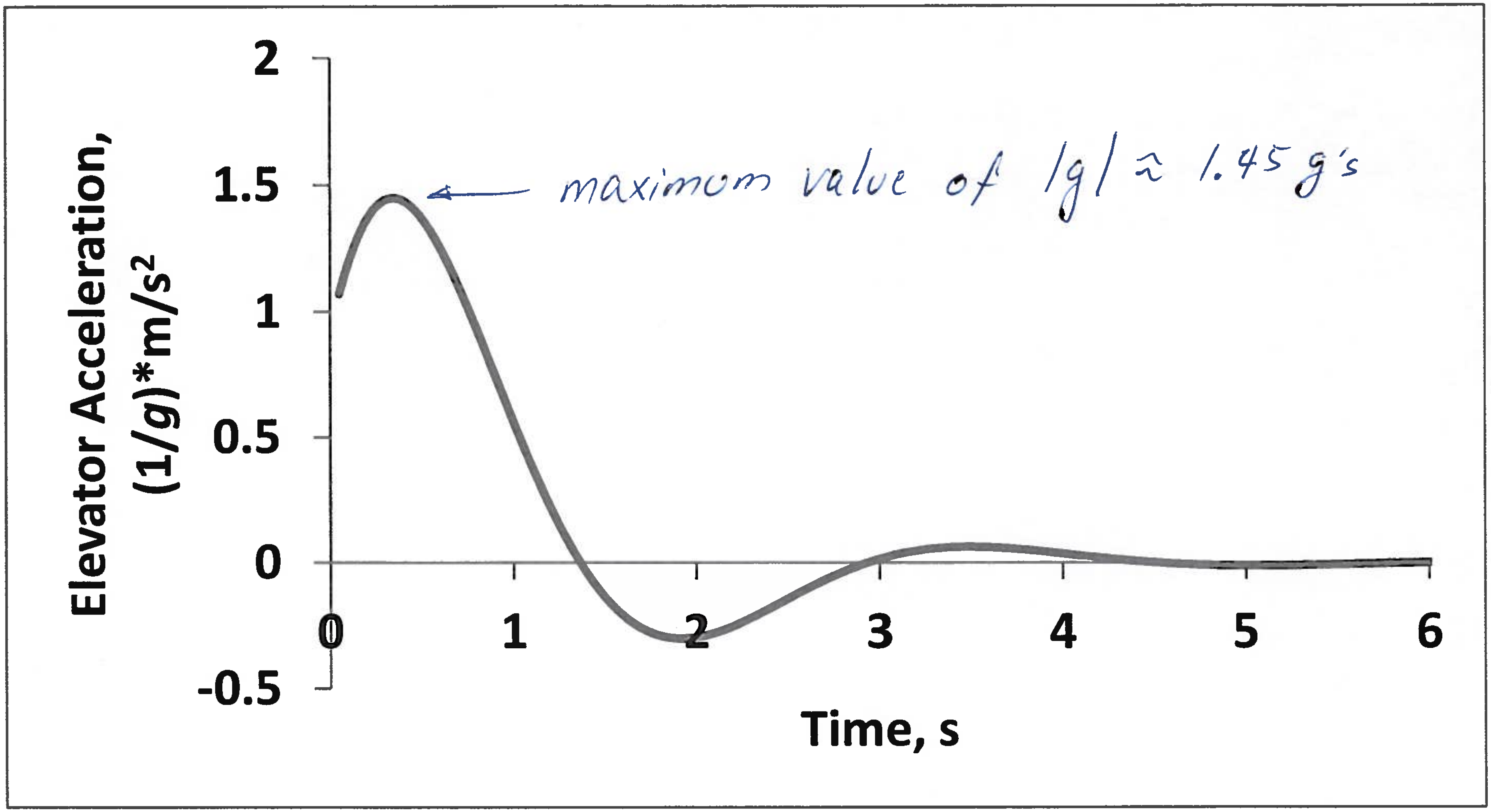
which is plotted on the next page.


Elevator Position, m
Time, s
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6
$z_{min} \approx -3.82$m
(answer to part (d))

(d) See answer on the graph on the previous page (the value of z_{min} was obtained from the graph, but can of course also be obtained analytically)

(e) The next page shows a plot of the acceleration of the elevator divided by g. This chart was obtained by numerically differentiating $x(t)$ and dividing it by $g = 10 \, m/s^2$.

From the graph we see that the maximum absolute value of the acceleration is approximately 1.45 g's.


Elevator Acceleration, (1/g)*m/s²
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
5
6
Time, s
maximum value of |g| ≈ 1.45 g's