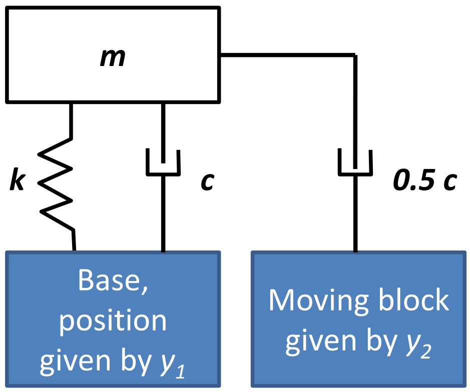
MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 6
Due Thursday, April 2nd at the beginning of class



Consider the system shown in the figure, with m = 5 Kg, k = 2 N/m and c = 2 Ns/m.

1. Derive the equation of motion of the mass around the static equilibrium position, taking the upward direction as the positive y-direction.

2. Derive the transfer function for the case when the position of the mass is the output and the <u>*position*</u> of the base, y_1 is the only input (y_2 = 0 in this case).

3. Derive the transfer function for the case when the position of the mass is the output and the <u>*velocity*</u> of the block, $\dot{y}_2$ is the only input (y_1 = 0 in this case).

4. Solve for the time-dependent position of the mass when the initial conditions with respect to the static equilibrium position are $y(0) = 1\ m$ and $\dot{y}(0) = -3\frac{m}{s}$, and the position of the base and the block are both kept fixed at $y_1(t) = y_2(t) = 0$.

5. Solve for the time-dependent position of the mass when the initial conditions with respect to the static equilibrium position are zero and the position of the base is prescribed as $y_1(t) = 2\ t$. Assume that the block is kept fixed at $y_2(t) = 0$. Solve this part of the problem <u>using the transfer function you found in part 2</u>.

6. Solve for the time-dependent position of the mass when the initial conditions with respect to the static equilibrium position are zero, the base is kept fixed at $y_1 = 0$, and the <u>*velocity*</u> of the block is prescribed as $\dot{y}_2(t) = \exp(-2t)$. Solve this part of the problem <u>using the transfer function you found in part 3</u>.

7. Solve for the time-dependent position of the mass when,
   a. The initial conditions with respect to the static equilibrium position are $y(0) = -5\ m$ and $\dot{y}_2(0) = +15\frac{m}{s}$,
   b. The position of the base is prescribed as $y_1(t) = 0.5(t-1)u(t-1)$, and
   c. The velocity of the block is prescribed as $\dot{y}_2(t) = 8\exp(-2t)$.

8. Assuming that the initial conditions are zero and the base is moving according to $y_1 = A\cos(\omega t)$. Provide an equation that describes the motion of the block, $y_2(t)$, such that the mass does not move at all. That is, what should be the motion of the block such that its effect completely cancels out the effect of the motion of the base?

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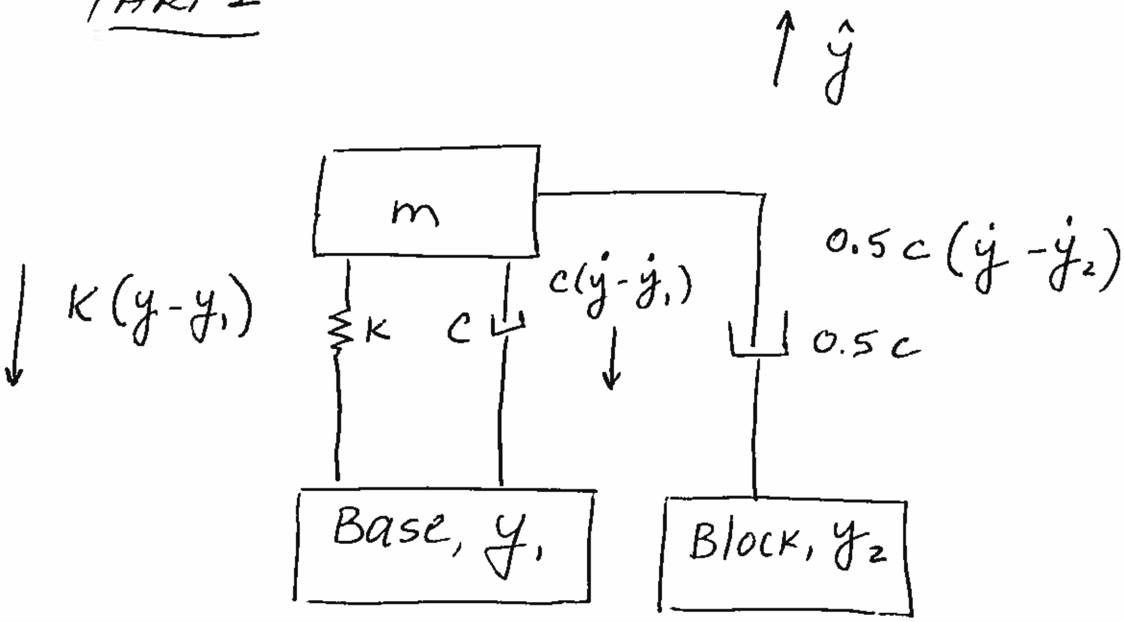
HOMEWORK 6 SOLUTION

PART 1



Equation of motion:

$$\Sigma F_y = m\ddot{y} = -K(y-y_1) - c(\dot{y}-\dot{y}_1) - 0.5c(\dot{y}-\dot{y}_2)$$

$$\Rightarrow m\ddot{y} + 1.5c\dot{y} + Ky = Ky_1 + c\dot{y}_1 + 0.5c\dot{y}_2$$

$$\Rightarrow \boxed{5\ddot{y} + 3\dot{y} + 2y = 2y_1 + 2\dot{y}_1 + \dot{y}_2}$$

PART 2

Output $= y(t) \longrightarrow Y(s)$ in Laplace space

Input $= y_1(t) \longrightarrow Y_1(s)$ in Laplace space

We are told that $y_2 = 0$, so the equation of motion becomes:

$$5\ddot{y} + 3\dot{y} + 2y = 2y_1 + 2\dot{y}_1$$

Applying the Laplace transform,

$$5s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) = 2\,Y_1(s) + 2s\,Y_1(s)$$

$$\Rightarrow Y(s)\left[5s^2+3s+2\right] = Y_1(s)\left[2+2s\right]$$

$$\Rightarrow \boxed{G_1(s) = \frac{Y(s)}{Y_1(s)} = \frac{2+2s}{5s^2+3s+2}}$$

PART 3

Now we set $y_1 = 0$, so the equation of motion becomes,

$$5\ddot{y} + 3\dot{y} + 2y = \dot{y}_2$$

Applying the Laplace transform,

$$5s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) = s\,Y_2(s)$$

$$\Rightarrow Y(s)\left[5s^2+3s+2\right] = s\,Y_2(s)$$

The output is $Y(s)$, as before. The input is $\dot{y}_2(t)$, the <u>velocity</u> of the block, which is $sY_2(s)$ in Laplace space. Therefore,

$$G_2(s) = \frac{Y(s)}{s\,Y_2(s)} = \frac{1}{5s^2+3s+2}$$

PART 4

We now have initial conditions, with $y_1 = y_2 = 0$.

The equation of motion simplifies to:

$$5\ddot{y} + 3\dot{y} + 2y = 0$$

Applying the Laplace transform

$$5\left(s^2 Y(s) - s y(0) - \dot{y}(0)\right) + 3\left(sY(s) - y(0)\right) + 2Y(s) = 0$$

$$\Rightarrow 5s^2 Y(s) - 5s(1) - 5(-3) + 3sY(s) - 3(1) + 2Y(s) = 0$$

$$\Rightarrow Y(s) * \left(5s^2 + 3s + 2\right) = 5s - 15 + 3$$

$$\Rightarrow Y(s) = \frac{5s - 12}{5s^2 + 3s + 2} = \frac{s - 12/5}{s^2 + \frac{3}{5}s + \frac{2}{5}}$$

We now invert this transform:

$$\Rightarrow Y(s) = \frac{s - 12/5}{s^2 + \frac{3}{5}s + \frac{2}{5}} \xrightarrow{\text{completing the square}} \frac{s - 12/5}{s^2 + \frac{3}{5}s + \left(\frac{3}{10}\right)^2 + \frac{2}{5} - \left(\frac{3}{10}\right)^2}$$

$$= \frac{s - 12/5}{\left(s + \frac{3}{10}\right)^2 + \frac{31}{100}} = \frac{s - 12/5}{\left(s + \frac{3}{10}\right)^2 + \left(\sqrt{\frac{31}{100}}\right)^2}$$

$$= \frac{s + \frac{3}{10}}{\left(s + \frac{3}{10}\right)^2 + \left(\sqrt{\frac{31}{100}}\right)^2} + \frac{\frac{-12}{5} - \frac{3}{10}}{\left(s + \frac{3}{10}\right)^2 + \left(\sqrt{\frac{31}{100}}\right)^2}$$

$$= \frac{s + \frac{3}{10}}{\left(s + \frac{3}{10}\right)^2 + \left(\sqrt{\frac{31}{100}}\right)^2} + \frac{\left(\frac{-27}{10}\right)\sqrt{\frac{31}{100}}}{\sqrt{\frac{31}{100}}\left(\left(s + \frac{3}{10}\right)^2 + \left(\sqrt{\frac{31}{100}}\right)^2\right)}$$

And this inverts to:

$$y(t) = e^{-\frac{3}{10}t}\cos\left(\frac{\sqrt{31}}{10}t\right) - \frac{\frac{27}{10}}{\frac{\sqrt{31}}{10}} e^{-\frac{3}{10}t}\sin\left(\frac{\sqrt{31}}{10}t\right)$$

$$\Rightarrow y(t) = e^{-\frac{3}{10}t}\left[\cos\left(\frac{\sqrt{31}}{10}t\right) - \frac{27}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right]$$

PART 5

We already know from part 2 that for this scenario,

$$G_1(s) = \frac{Y(s)}{Y_1(s)} = \frac{2+2s}{5s^2+3s+2}$$

Therefore, $Y(s) = \frac{2+2s}{5s^2+3s+2} Y_1(s)$

We are told that $y_1(t) = 2t \Rightarrow Y_1(s) = \frac{2}{s^2}$ (from the table), so,

$$Y(s) = \left(\frac{2+2s}{5s^2+3s+2}\right)\frac{2}{s^2} = \frac{4+4s}{5s^2+3s+2}$$

Applying the partial fractions method,

$$\frac{4+4s}{5s^2+3s+2} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{5s^2+3s+2}$$

$$\Rightarrow 4+4s = As\left(5s^2+3s+2\right) + B\left(5s^2+3s+2\right) + (Cs+D)s^2$$

$$\Rightarrow 4+4s = 5As^3 + 3As^2 + 2As + 5Bs^2 + 3Bs + 2B + Cs^3 + Ds^2$$

$$\Rightarrow (4) + s(4) = (2B) + s(2A+3B) + s^2(3A+5B+D) + s^3(5A+C)$$

This gives four equations

I $\quad 2B = 4 \Rightarrow \boxed{B = 2}$

II $\quad 2A + 3B = 4 \Rightarrow A = \frac{4 - 3B}{2} = \frac{4 - 3(2)}{2} \Rightarrow \boxed{A = -1}$

III $\quad 3A + 5B + D = 0$

IV $\quad 5A + C = 0 \Rightarrow C = -5A \Rightarrow \boxed{C = 5}$

From III we have $D = -3A - 5B = -3(-1) - 5(2)$

$$\Rightarrow \boxed{D = -7}$$

$$\Rightarrow X(s) = \underbrace{\frac{-1}{s} + \frac{2}{s^2}}_{} + \frac{5s - 7}{5s^2 + 3s + 2}$$

These terms invert to $-u(t) + 2t$.

The last term needs a bit of work.

$$\frac{5s - 7}{5s^2 + 3s + 2} = \frac{5 - \frac{7}{5}}{5s^2 + 3s + 2} = \frac{s - \frac{7}{5}}{\underbrace{\left(s + \frac{3}{10}\right)^2 + \left(\frac{\sqrt{31}}{10}\right)^2}_{}}$$

we had already completed the square for this denominator

$$= \frac{s + \frac{3}{10}}{\left(s + \frac{3}{10}\right)^2 + \left(\frac{\sqrt{31}}{10}\right)^2} + \frac{\left(-\frac{7}{5} - \frac{3}{10}\right)}{\frac{\sqrt{31}}{10}} \frac{\frac{\sqrt{31}}{10}}{\left(s + \frac{3}{10}\right)^2 + \left(\frac{\sqrt{31}}{10}\right)^2}$$

which inverts to:

$$e^{-\frac{3}{10}t} \cos\left(\frac{\sqrt{31}}{10} t\right) - \frac{\frac{17}{10}}{\frac{\sqrt{31}}{10}} e^{-\frac{3}{10}t} \sin\left(\frac{\sqrt{31}}{10} t\right)$$

which can be written as

$$e^{-\frac{3}{10}t}\left[\cos\left(\frac{\sqrt{31}}{10}t\right) - \frac{17}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right]$$

Therefore the full solution is

$$y(t) = -u(t) + 2t + e^{-\frac{3}{10}t}\left[\cos\left(\frac{\sqrt{31}}{10}t\right) - \frac{17}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right]$$

PART 6

From part 3 we know that for this scenario,

$$G_2(s) = \frac{Y(s)}{s\,Y_2(s)} = \frac{1}{5s^2+3s+2}$$

$$\Rightarrow Y(s) = \left(\frac{1}{5s^2+3s+2}\right) s\,Y_2(s).$$

We are told that $\dot{y}_2(t) = e^{-2t}$. Applying the Laplace transform on both sides we have that: $sY_2(s) - y_2(0) = \mathcal{L}\left[e^{-2t}\right]$

$$\Rightarrow sY_2(s) - e^{-2(0)} = \frac{1}{s+2}$$

$$\Rightarrow sY_2(s) = \frac{1}{s+2} + 1 = \frac{s+3}{s+2}$$

$$\Rightarrow Y(s) = \left(\frac{1}{5s^2+3s+2}\right) s\,Y_2(s) = \left(\frac{1}{5s^2+3s+2}\right)\frac{s+3}{s+2}.$$

We now apply the partial fractions method:

$$\frac{s+3}{(s+2)(5s^2+3s+2)} = \frac{A}{s+2} + \frac{Bs+C}{5s^2+3s+2}$$

$$\Rightarrow A(5s^2+3s+2) + (Bs+C)(s+2) = s+3$$

$$\Rightarrow 5As^2 + 3As + 2A + Bs^2 + 2Bs + Cs + 2C = s+3$$

$$\Rightarrow s^2(5A+B)+s(3A+2B+C)+(2A+2C)=s+3$$

This gives the following equations:

(I) $2A+2C=3 \Rightarrow C=\dfrac{3-2A}{2}$

(II) $3A+2B+C=1$

(III) $5A+B=0 \Rightarrow B=-5A$

Substituting (I) and (III) into (II):

$$3A+2(-5A)+\frac{3-2A}{2}=1$$

$$\Rightarrow 3A-10A-A=1-\frac{3}{2}=-\frac{1}{2}$$

$$\Rightarrow -8A=-\frac{1}{2} \Rightarrow \boxed{A=\frac{1}{16}}$$

from (III) $B=-5A \Rightarrow \boxed{B=\dfrac{-5}{16}}$

from (I) $C=\dfrac{3-2A}{2}=\dfrac{3-2\left(\frac{1}{16}\right)}{2}=\dfrac{3}{2}-\dfrac{1}{16}$

$$\Rightarrow \boxed{C=\frac{23}{16}}$$

$$\Rightarrow Y(s)=\frac{1/16}{s+2}+\frac{\frac{-5}{16}s+\frac{23}{16}}{5s^2+3s+2}$$

The first term inverts to $\frac{1}{16}e^{-2t}$. The second term needs a little work:

$$\frac{-\frac{5}{16}s+\frac{23}{16}}{5s^2+3s+2}=\frac{-\frac{1}{16}s+\frac{23}{80}}{\left(s+\frac{3}{10}\right)^2+\left(\frac{\sqrt{31}}{10}\right)^2}$$

$$= \frac{-\frac{1}{16}s - \frac{1}{16}\left(\frac{3}{10}\right)}{\left(s+\frac{3}{10}\right)^2 + \left(\frac{\sqrt{31}}{10}\right)^2} + \frac{\left(\frac{23}{80}+\frac{3}{16}\right)}{\frac{\sqrt{31}}{10}} \frac{\frac{\sqrt{31}}{10}}{\left(s+\frac{3}{10}\right)^2 + \left(\frac{\sqrt{31}}{10}\right)^2}$$

which inverts to:

$$-\frac{1}{16} e^{-\frac{3}{10}t} \cos\left(\frac{\sqrt{31}}{10}t\right) + \frac{\frac{38}{80}}{\frac{\sqrt{31}}{10}} e^{-\frac{3}{10}t} \sin\left(\frac{\sqrt{31}}{10}t\right)$$

$$= -\frac{1}{16} e^{-\frac{3}{10}t} \cos\left(\frac{\sqrt{31}}{10}t\right) + \frac{19}{4\sqrt{31}} e^{-\frac{3}{10}t} \sin\left(\frac{\sqrt{31}}{10}t\right)$$

So the full solution is,

$$y(t) = \frac{+1}{16} e^{-2t} + e^{-\frac{3}{10}t}\left[-\frac{1}{16}\cos\left(\frac{\sqrt{31}}{10}t\right) + \frac{19}{4\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right]$$

PART 7

In this problem, the initial conditios are those of part 4 multiplied by -5, so we have to add the solution of part 4 (call it $y_4(t)$) multiplied by -5. Next we have an input that is 0.25 times the input of part 5, but shifted by 1 s to the right. So if we call the solution of part 5 $y_5(t)$, we have to add $0.25\, y_5(t-1)\, u(t-1)$. Finally, we have an input that is eight times the input of part 6, so we have to add eight times the solution of part 6 ($8\, y_6(t)$). This is because of linearity and time invariance. The desired solution is then:

$$y(t) = -5\,y_4(t) + 0.25\,y_5(t-1)\,u(t-1) + 8\,y_6(t)$$

$$\Rightarrow y(t) = -5\,e^{-\frac{3}{10}t}\left[\cos\left(\frac{\sqrt{31}}{10}t\right) - \frac{27}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right]$$
$$+ 0.25\,u(t-1)\left[-1 + 2(t-1) + e^{-\frac{3}{10}(t-1)}\left\{\cos\left(\frac{\sqrt{31}}{10}(t-1)\right) - \frac{17}{\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}(t-1)\right)\right\}\right] + 8\left[\frac{1}{16}e^{-2t}\right.$$
$$\left. + e^{-\frac{3}{10}t}\left\{-\frac{1}{16}\cos\left(\frac{\sqrt{31}}{10}t\right) + \frac{19}{4\sqrt{31}}\sin\left(\frac{\sqrt{31}}{10}t\right)\right\}\right].$$

PART 8

We know from parts 2 and 3 that

$$G_1(s) = \frac{Y(s)}{Y_1(s)} \quad \text{and} \quad G_2(s) = \frac{Y(s)}{sY_2(s)}$$

$\Rightarrow$ For zero initial conditions,

$$Y(s) = G_1(s)\,Y_1(s) + G_2(s)\,sY_2(s)$$

$$= \frac{2+2s}{5s^2+3s+2}\,Y_1(s) + \frac{1}{5s^2+3s+2}\,sY_2(s).$$

We are told that $y_1(t) = A\cos(\omega t)$

$$\Rightarrow Y_1(s) = \frac{As}{s^2+\omega^2}$$

$$\Rightarrow Y(s) = \left(\frac{2+2s}{5s^2+3s+2}\right)\left(\frac{As}{s^2+\omega^2}\right) + \frac{sY_2(s)}{5s^2+3s+2}$$

But we want the mass to remain static, so $y(t)$ must be zero, which implies that $Y(s)=0$. Therefore,

$$\left(\frac{2+2s}{5s^2+3s+2}\right)\left(\frac{As}{s^2+\omega^2}\right) + \frac{sY_2(s)}{5s^2+3s+2} = 0$$

$$\Rightarrow -\frac{(2A+2As)}{s^2+\omega^2} = Y_2(s)$$

$$\Rightarrow Y_2(s) = -\frac{2A}{\omega}\frac{\omega}{s^2+\omega^2} - \frac{2As}{s^2+\omega^2}.$$

We easily invert this to get,

$$y_2(t) = -\frac{2A}{\omega}\sin(\omega t) - 2A\cos(\omega t).$$