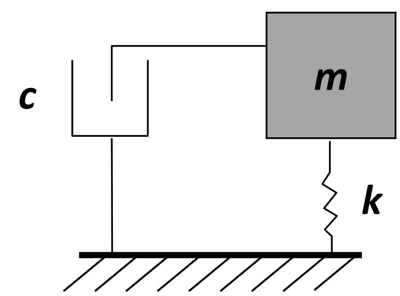
MAE 3134 – Linear System Dynamics
Spring 2015

Homework # 5
Due Thursday, March 19th at the beginning of class

Problem 1

Consider a vibratory system as shown in the figure, with k = 2 N/m, natural frequency, f_o = 1/π Hz and damping factor ζ = 0.25.



i) If the mass is displaced to an initial position located 1.5 m *above the static equilibrium* position and then released at time t = 0 without any time-dependent forces acting on it, what will be its height after one complete oscillation?

ii) If one wishes to completely stop the oscillation of the mass sometime *before* it completes one full oscillation, and this is to be done by hitting the mass with a hammer, provide a mathematical expression for an impact that will accomplish the objective.

Problem 2

In class we studied the transient and steady state responses of a vibratory system to a force of the form $f(t) = F_o \sin(\omega t)$. Specifically, we saw that the *steady state* response was given by:

$$x(t) = \frac{F_o}{k} A(\omega) \sin[\omega t - \theta(\omega)]$$

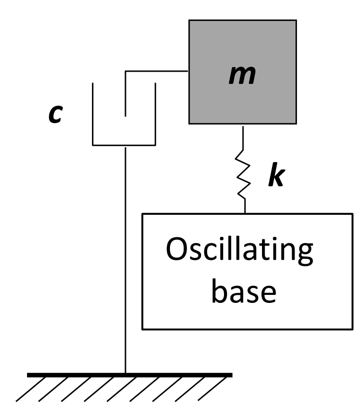
With $A(\omega) = \frac{1}{\sqrt{(1-\left(\frac{\omega}{\omega_o}\right)^2)^2 + (2\zeta\left(\frac{\omega}{\omega_o}\right))^2}}$; $\theta(\omega) = tan^{-1}[2\zeta(\frac{\omega}{\omega_o})/(1 - \left(\frac{\omega}{\omega_o}\right)^2)]$; $\omega_o = \sqrt{\frac{k}{m}}$;

and $\zeta = c/(2\sqrt{k\,m})$

Based on simple mathematical arguments, derive the *steady state* response for the case where the excitation force is of the form $f(t) = F_o$ **cos**(ωt).

Problem 3

Consider the vibratory system shown in the figure, with k = 2 N/m, m = 1 Kg and c = 0.5 N s/m. If a vertical oscillatory force, $f(t)$ = (0.75 N) cos (ω_o t) is applied to the mass and the base position oscillates according to $y(t)$ = (5 m) cos [(2/3) ω_o t],



i) Calculate the steady-state response of the mass, $x(t)$.

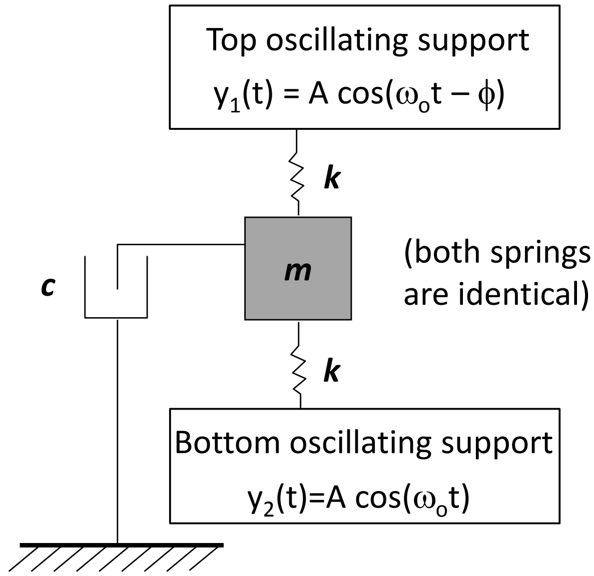
ii) Calculate the period of oscillation.

ω_o is the natural frequency.

BONUS PROBLEM (20 POINTS)

Problem 4

A mass is suspended between two blocks which are sinusoidally oscillating as indicated in the figure.



a) Set up the equation of motion of the system
b) For what value of ϕ will the oscillation amplitude of the mass be greatest? ϕ is a constant phase angle in the expression describing the oscillation of the top block, $y_1(t)$.
c) For what value of ϕ will the oscillation amplitude of the mass be smallest? What is the smallest possible value of the oscillation amplitude at steady state?

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HOMEWORK 5
SOLUTION

PROBLEM 1

(i) We find the parameters of the system from the information given:

$K = 2\ N/m$

$f_0 = \frac{1}{\pi}\ Hz \Rightarrow \omega_0 = 2\pi f_0 = 2\pi \frac{1}{\pi} \frac{Rad}{s} = 2 \frac{Rad}{s}$

Furthermore, $\omega_0 = \sqrt{\frac{K}{m}} \Rightarrow m = \frac{K}{\omega_0^2}$

$\Rightarrow m = \frac{2\ N/m}{(2\ rad/s)^2} = \frac{1}{2}\ Kg.$

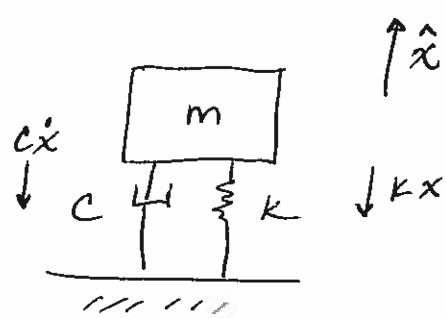
Now, $\zeta = \frac{c}{2\sqrt{Km}} = 0.25$

$\Rightarrow C = (0.25)\, 2\sqrt{2\ N/m \cdot 1/2\ Kg} = \frac{1}{2}\ Ns/m$

Therefore, we have

$$\boxed{\begin{aligned} K &= 2\ N/m \\ m &= 1/2\ Kg \\ C &= 1/2\ Ns/m \end{aligned}}$$

Our equation of motion is:



$m\ddot{x} + c\dot{x} + Kx = 0$

we don't have to include the weight if we solve about the static equilibrium position.

We apply the Laplace transform to the equation:

$$m\left[s^2 X(s) - s\,x(0) - \underbrace{\cancel{\dot{x}(0)}}_{\text{ZERO}}\right] + c\left[s X(s) - x(0)\right] + k X(s) = 0$$

$$\Rightarrow X(s)\left[ms^2 + cs + k\right] = ms\,x(0) + c\,x(0)$$

$x(0) = \frac{3}{2}$

$$\Rightarrow X(s) = \frac{ms\,x(0) + c\,x(0)}{ms^2 + cs + k} = \frac{\frac{1}{2}\left(\frac{3}{2}s + \frac{3}{2}\right)}{\frac{1}{2}s^2 + \frac{1}{2}s + 2}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2}\right)(s+1)}{\frac{1}{2}\left(s^2 + s + 4\right)} = \frac{3}{2}\,\frac{(s+1)}{s^2+s+4}$$

We complete the square in the denominator and look for solutions of the form $e^{-at}\cos\omega t$ and $e^{-at}\sin\omega t$:

$$X(s) = \frac{3}{2}\,\frac{s+1}{\left(s^2+s+\left(\frac{1}{2}\right)^2\right)+4-\left(\frac{1}{2}\right)^2} = \frac{3}{2}\,\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{15}{4}}$$

$$= \frac{3}{2}\left[\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{15}{4}}\right]$$

$$= \frac{3}{2}\,\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{15}{4}}\right)^2} + \frac{3}{2}\,\frac{1}{2}\,\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{15}{4}}\right)^2}$$

$$X(s) = \frac{3}{2}\,\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{15}{4}}\right)^2} + \frac{3}{4}\,\frac{\sqrt{\frac{15}{4}}}{\sqrt{\frac{15}{4}}}\,\frac{1}{(s+1)^2+\left(\sqrt{\frac{15}{4}}\right)^2}$$

$$\Rightarrow x(t) = \frac{3}{2}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{3}{4}\frac{1}{\sqrt{\frac{15}{4}}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

$$\Rightarrow x(t) = \frac{3}{2}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{3}{2\sqrt{15}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

Now, the question is about the height after one full oscillation. At the end of one full oscillation, the argument of the sinusoidals will be equal to 2π, because $\omega = \frac{\sqrt{15}}{2} \Rightarrow f = \frac{1}{2\pi}\frac{\sqrt{15}}{2}$

$$\Rightarrow T\ (\text{period}) = \frac{1}{f} = \frac{2\,(2\pi)}{\sqrt{15}} = 3.24\,s$$

$$\Rightarrow \omega T = \frac{\sqrt{15}}{2} \times \frac{2\,(2\pi)}{\sqrt{15}} = 2\pi$$

$$\Rightarrow x\,(t=T) = \frac{3}{2}e^{-\frac{1}{2}T}\underbrace{\cos(2\pi)}_{\text{UNITY}} + \frac{3}{2\sqrt{15}}e^{-\frac{1}{2}T}\underbrace{\cancel{\sin(2\pi)}}_{\text{ZERO}}$$

$$= \frac{3}{2}e^{-\frac{1}{2}\left(\frac{2\,(2\pi)}{\sqrt{15}}\right)}$$

$$\Rightarrow \boxed{x(T) = 0.296\ m}$$

(ii) Now we must find the times at which $x(t)$ crosses the equilibrium position:

$$x(t) = \frac{3}{2} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{15}}{2} t\right) + \frac{3}{2\sqrt{15}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{15}}{2} t\right) = 0$$

$$\Rightarrow \cos\left(\frac{\sqrt{15}}{2} t\right) + \frac{1}{\sqrt{15}} \sin\left(\frac{\sqrt{15}}{2} t\right) = 0$$

$$\Rightarrow \cos\left(\frac{\sqrt{15}}{2} t\right) = -\frac{1}{\sqrt{15}} \sin\left(\frac{\sqrt{15}}{2} t\right)$$

$$\Rightarrow -\sqrt{15} = \frac{\sin\left(\frac{\sqrt{15}}{2} t\right)}{\cos\left(\frac{\sqrt{15}}{2} t\right)} = \tan\left(\frac{\sqrt{15}}{2} t\right)$$

$$\Rightarrow \tan^{-1}(-\sqrt{15}) = \frac{\sqrt{15}}{2} t$$

$$\Rightarrow t = \frac{2}{\sqrt{15}} \tan^{-1}(-\sqrt{15})$$

any integer

$$\Rightarrow t = \frac{2}{\sqrt{15}} \left[-1.318 \text{ Rad} + \underbrace{n \pi \text{ Rad}}_{} \right]$$

This is added because tangent is a periodic function

Using the above formula we find that the possible times are:

$t_1 = -0.681 \text{ s}$

$t_2 = 0.942 \text{ s}$

$t_3 = 2.564 \text{ s}$

$t_4 = 4.186 \text{ s} \quad \ldots$

We discard t_1 because t must be positive. We also discard t_4 and any subsequent values of t because they exceed the period, 3.24 s. We are left with two possibilities:

$$t = 0.942\,s \quad \text{or} \quad 2.564\,s$$

Our next step is to find the velocity at one of these times. We can use any of them, let's say the first one ($t = 0.942\,s$). The velocity is $\dot{x} = \frac{dx}{dt}$:

$$\frac{d}{dt}x(t) = \frac{3}{2}\left(-\frac{1}{2}\right)e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right)$$
$$+ \frac{3}{2}e^{-\frac{1}{2}t}\left(\frac{\sqrt{15}}{2}\right)\left(-\sin\frac{\sqrt{15}}{2}t\right)$$
$$+ \frac{3}{2\sqrt{15}}\left(-\frac{1}{2}\right)e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$
$$+ \frac{3}{2\sqrt{15}}e^{-\frac{1}{2}t}\left(\frac{\sqrt{15}}{2}\right)\cos\left(\frac{\sqrt{15}}{2}t\right)$$

We evaluate the above expression at $t = 0.942\,s$ and find

$$\dot{x}(0.942) = \frac{-3}{4}e^{-\frac{0.942}{2}}\cos(1.824)$$
$$- \frac{3\sqrt{15}}{4}e^{-\frac{0.942}{2}}\sin(1.824)$$
$$+ \frac{-3}{4\sqrt{15}}e^{-\frac{0.942}{2}}\sin(1.824)$$
$$+ \frac{3}{4}e^{-\frac{0.942}{2}}\cos(1.824) \approx -1.9\ m/s$$

The momentum is $mv = \left(-1.9\,\frac{m}{s}\right)\left(\frac{1}{2}\,Kg\right)$

$\approx -0.95\,\frac{Kg\,m}{s}$ and is also the coefficient of the impact required multiplied by -1.

Therefore, the impact required is

$$f(t) = +0.95\,\frac{Kg\,m}{s}\,\delta(t-0.942)$$

A similar analysis can be carried out for the other time, $t = 2.564$, in which case the required impact is

$$f(t) = -0.42\,\frac{Kg\,m}{s}\,\delta(t-2.564)$$

Problem 2

This problem is very simple because

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

or $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

We also know that all transients have died out at steady state, and that steady state occurs at $t \to \infty$ (by definition).

Additionally, we can redefine our time axis in any way we want.

We have that

$$X(t) = \frac{F_0}{K}\,A(\omega)\sin[\omega t - \theta(\omega)]$$

for $f(t) = F_0 \sin \omega t$.

Let's now introduce a new variable,

$$\hat{t} = t - \frac{\pi}{2\omega} \Rightarrow t = \hat{t} + \pi/2\omega$$

we then have that for an input

$$f(t) = F_0 \sin(\omega t) = F_0 \sin(\omega(\hat{t} + \pi/2\omega))$$

$$= F_0 \sin\left(\omega\hat{t} + \frac{\pi}{2}\right) = F_0 \cos(\omega\hat{t})$$

the output will be (at steady state)

$$x(t) = x(\hat{t} + \pi/2\omega) = \frac{F_0}{K} A(\omega) \sin(\omega(\hat{t} + \pi/2\omega) - \theta(\omega))$$

$$= \frac{F_0}{K} A(\omega) \sin\left(\omega\hat{t} + \frac{\pi}{2} - \theta(\omega)\right)$$

$$= \frac{F_0}{K} A(\omega) \cos(\omega\hat{t} - \theta(\omega))$$

So after the transients have died out, the solution for a cosine excitation force will be

$$x(t) = \frac{F_0}{K} A(\omega) \cos(\omega t - \theta(\omega)),$$

Since we can arbitrarily shift the time axis by any constant we wish, as long as we focus in the region where transients no longer exist.

PROBLEM 3

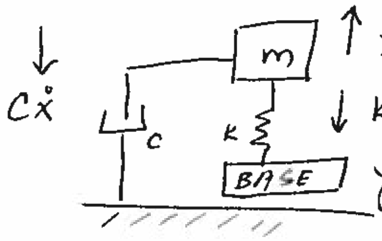
(i) $k = 2 \ N/m$

$m = 1 \ Kg$

$c = 0.5 \ Ns/m \Rightarrow \zeta = \dfrac{c}{2\sqrt{km}} = \dfrac{0.5 \ Ns/m}{2\sqrt{2 \ N/m \cdot 1 \ Kg}} = 0.177$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{2 \ N/m / 1 \ Kg} = \sqrt{2} \ rad/s$

Free body diagram:



$\uparrow f(t) = 0.75 \cos \omega_0 t$ $\qquad \uparrow \hat{y}$ axis

$\downarrow k(x - y_b)$

$y_b = 5 \cos\left(\frac{2}{3}\omega_0 t\right)$ (position of the base)

Equation of motion:

$$m\ddot{y} = -c\dot{y} - k(y - y_b) + f(t)$$

$$\Rightarrow m\ddot{y} + c\dot{y} + ky = ky_b + f(t)$$

$$m\ddot{y} + c\dot{y} + ky = \underbrace{5k \cos\left(\tfrac{2}{3}\omega_0 t\right) + 0.75 \cos \omega_0 t}$$

Each of these expressions is of the form $F_0 \cos(\omega t)$ so the response to each of them is given by the expressions derived in problem #2.

For the first input force, $\underbrace{5k}_{F_1} \cos\left(\underbrace{\tfrac{2}{3}\omega_0}_{\omega_1} t\right)$

$$A(\omega_1) = \frac{1}{\sqrt{\left(1 - \left(\frac{2}{3}\right)^2\right)^2 + \left(2 \times 0.177 \times \frac{2}{3}\right)^2}} = 1.656$$

$$\Theta(\omega_1) = \tan^{-1}\left[\frac{2 \times 0.177 \times 2/3}{1 - (2/3)^2}\right] = 0.401 \text{ rad}$$

$$\frac{F_1}{K} = \frac{5K}{K} = 5m$$

$$\Rightarrow y_1(t) = \frac{F_1}{K} A(\omega_1) \cos(\omega_1 t - \Theta(\omega_1))$$ at steady state

$$= 5 \times 1.656 \cos\left(\tfrac{2}{3}\omega_0 t - 0.401\right)$$

$$= 8.28 \cos\left(\tfrac{2}{3}\sqrt{2}\, t - 0.401\right)$$

For the second input force:

$$f_2(t) = \underbrace{0.75}_{F_2} \cos(\omega_0 t)$$

$$\frac{F_2}{K} = \frac{0.75 \text{ N}}{2 \text{ N/m}} = 0.375 \text{ m}$$

$$A(\omega_2) = \frac{1}{\sqrt{(1-1^2)^2 + (2 \times 0.177 \times 1)^2}} = 2.82$$

$$\Theta(\omega_2) = \tan^{-1}\left[\frac{2 \times 0.177 \times 1}{1 - 1^2}\right] = \frac{\pi}{2}$$

$$\Rightarrow y_2(t) = \frac{F_2}{K} A(\omega_2) \cos(\omega_2 t - \Theta(\omega_2))$$

$$= 0.375 \times 2.82 \cos\left(\sqrt{2}\, t - \frac{\pi}{2}\right)$$

$$= 1.06 \cos\left(\sqrt{2}\, t - \frac{\pi}{2}\right)$$ at steady state

The complete solution is

$$y(t) = y_1(t) + y_2(t)$$

$$\Rightarrow y(t) = 8.28 \cos\left(\tfrac{2}{3}\sqrt{2}\, t - 0.401\right) + 1.06 \cos\left(\sqrt{2}\, t - \tfrac{\pi}{2}\right)$$

(ii) To calculate the period of the combined oscillation we first calculate the individual periods for $y_1(t)$ and $y_2(t)$

$$T_1 = \frac{1}{\nu_1} = \frac{1}{\left(\frac{\omega_1}{2\pi}\right)} = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{2}{3}\omega_0} = \frac{3\pi}{\omega_0}$$

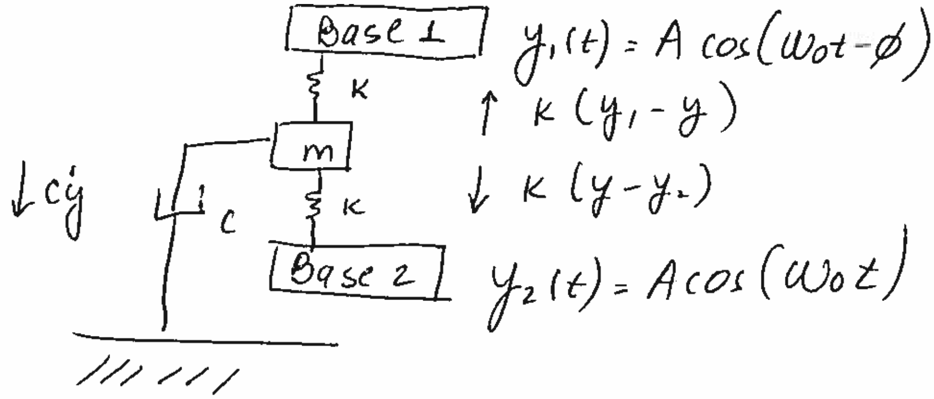
$$T_2 = \frac{1}{\nu_2} = \frac{2\pi}{\omega_0}$$

The first period is 1.5 times the second period. The minimum common multiple of the two is $\frac{6\pi}{\omega_0} = 2T_1 = 3T_2$. This is the first time where both oscillations return to the initial position together, so it is the combined period of oscillation

$$T = \frac{6\pi}{\omega_0} = \frac{6\pi \text{ rad}}{\sqrt{2}\ \frac{\text{rad}}{\text{s}}} = 13.33\text{ s}$$

Problem 4

We begin with our free body diagram:



a) Equation of motion:

$$m\ddot{y} = -c\dot{y} + K(y_1 - y) - K(y - y_2)$$

$$m\ddot{y} + c\dot{y} + 2Ky = Ky_1 + Ky_2$$

$$m\ddot{y} + c\dot{y} + 2Ky = K\left(A\cos(\omega_0 t - \phi) + A\cos(\omega_0 t)\right)$$

b) We note that this equation has two input terms on the right hand side. Both terms are of identical amplitude and frequency. They are just out of phase by the angle ϕ. The maximum oscillation amplitude will occur when both forcing terms add up with full constructive interference, which means that they will be identical:

$$A\cos(\omega_0 t - \phi) = A\cos(\omega_0 t)$$

$$\Rightarrow \phi = 0 \pm 2n\pi \qquad n = 0, \pm 1, \pm 2, \pm 3, \pm \ldots$$

(based on the periodicity and properties of the cosine function)

c) The smallest oscillation amplitude will occur when the two identical input forces cancel each other. This means they need to be completely out of phase, which means they need to be offset by exactly half an oscillation. This means that ϕ must be equal to π, since 2π is one full oscillation

The input will then be:

$$K\left(A\cos(\omega_0 t - \underset{\uparrow \atop \pi}{\phi}) + A\cos(\omega_0 t)\right)$$

$$= K\left[A\cos(\omega_0 t - \pi) + A\cos(\omega_0 t)\right)$$

$$= K\left[-A\cos(\omega_0 t) + A\cos(\omega_0 t)\right]$$

$= 0$, so at steady state the oscillation amplitude will be zero (since the input is zero). Basically our equation of motion will be reduced to:

$$m\ddot{y} + c\dot{y} + 2ky = k\left[-A\cos(\omega_0 t) + A\cos(\omega_0 t)\right]$$

$$\Rightarrow m\ddot{y} + c\dot{y} + 2ky = 0 \quad \Rightarrow$$ No oscillation takes place at steady state (minimum amplitude is zero).