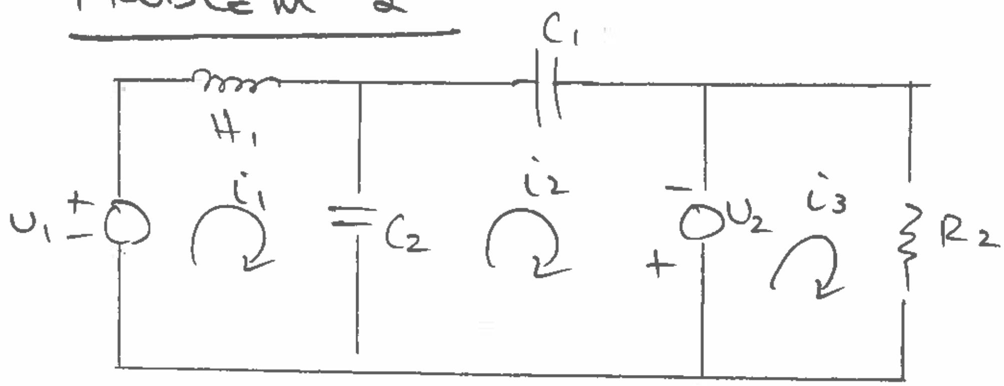
PROBLEM 2



(1) $+u_1 - H_1 \frac{di_1}{dt} - \frac{1}{C_2}(q_1 - q_2) = 0$

(2) $-\frac{1}{C_2}(q_2 - q_1) - \frac{1}{C_1} q_2 + U_2 = 0$

(3) $-U_2 - i_3 R_2 = 0$

STANDARD FORM $\quad i = \frac{dq}{dt} = \dot{q}$

$$u_1 - H_1 \ddot{q}_1 - \frac{1}{C_2}(q_2 - q_1) = 0$$

$$-\frac{1}{C_2}(q_2 - q_1) - \frac{1}{C_1} q_2 + U_2 = 0$$

$$-U_2 - \dot{q}_3 R_2 = 0$$

$$\ddot{q}_1 = \frac{1}{H_1 C_2}(q_2 - q_1) - \frac{1}{H_1} u_1$$

$$\left(\frac{1}{C_2} + \frac{1}{C_1}\right) q_2 = \frac{1}{C_2} q_1 + U_2 \longrightarrow \text{SOLVE FOR } q_2$$

$$\dot{q}_3 = -\frac{1}{R_2} U_2$$

$$q_2 = \left(\frac{C_1+C_2}{C_1C_2}\right)^{-1}\left(\frac{1}{C_2}q_1 + u_2\right)$$

$$q_2 = \frac{C_1C_2}{C_1+C_2}\left(\frac{1}{C_2}q_1 + u_2\right)$$

$$q_2 = \frac{C_1}{C_1+C_2}q_1 + \frac{C_1C_2}{C_1+C_2}u_2 \longrightarrow \text{REMOVE } q_2 \text{ AS STATE}$$

$$\ddot{q}_1 = \frac{1}{H_1C_2}\left(\frac{C_1}{C_1+C_2}q_1 + \frac{C_1C_2}{C_1+C_2}u_2\right) - \frac{1}{H_1C_2}q_1 - \frac{1}{H_1}u_1$$

$$= \frac{C_1}{H_1C_2(C_1+C_2)}q_1 + \frac{C_1}{H_1(C_1+C_2)}u_2 - \frac{1}{H_1C_2}q_1 - \frac{1}{H_1}u_1$$

$$\ddot{q}_1 = \left(\frac{C_1}{H_1C_2(C_1+C_2)} - \frac{1}{H_1C_2}\right)q_1 + \frac{C_1}{H_1(C_1+C_2)}u_2 - \frac{1}{H_1}u_1$$

$$\dot{q}_3 = -\frac{1}{R_2}u_2$$

STATE

$x_1 = q_1$ $\qquad \dot{x}_1 = x_2$

$x_2 = \dot{q}_1$ $\qquad \dot{x}_2 = \ddot{q}_1 = a x_1 - \frac{1}{H_1}u_1 + \frac{C_1}{H_1(C_1+C_2)}u_2$

$x_3 = q_3$ $\qquad \dot{x}_3 = -\frac{1}{R_2}u_2$

$y = Cx + Du$

$$a = \frac{C_1}{H_1C_2(C_1+C_2)} - \frac{1}{H_1C_2}$$

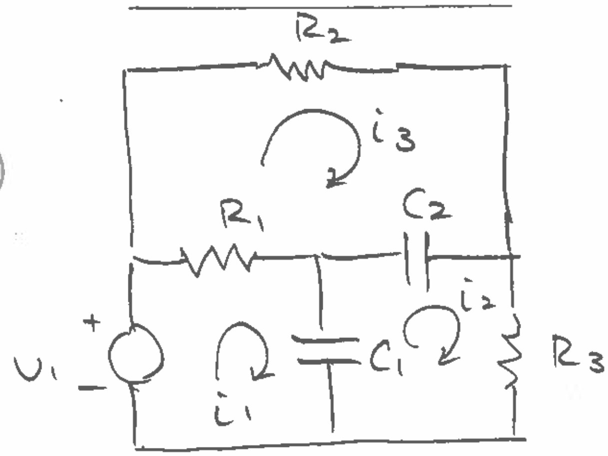
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ \frac{-1}{H_1} & \frac{C_1}{H_1(C_1+C_2)} \\ 0 & \frac{-1}{R_2} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & \frac{-1}{R_2} \end{bmatrix}$$

PROBLEM 3



(1) $+v_1 - R_1(i_1 - i_3) - \dfrac{q_1 - q_2}{C_1} = 0$

(2) $-\dfrac{q_2 - q_1}{C_1} - \dfrac{q_2 - q_3}{C_2} - R_3 i_2 = 0$

(3) $-R_2 i_3 - \dfrac{q_3 - q_2}{C_2} - R_1(i_3 - i_1) = 0$

STANDARD FORM $\quad i = \dfrac{dq}{dt}.$

$$v_1 - R_1(\dot{q}_1 - \dot{q}_3) - \frac{q_1 - q_2}{C_1} = 0$$

$$-\frac{(q_2 - q_1)}{C_1} - \frac{(q_2 - q_3)}{C_2} - R_3 \dot{q}_2 = 0$$

$$-R_2 \dot{q}_3 - \frac{(q_3 - q_2)}{C_2} - R_1(\dot{q}_3 - \dot{q}_1) = 0$$

SOLVE FOR $[\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3]^T$

$$-R_1 \dot{q}_1 + R_1 \dot{q}_3 = -U_1 + \frac{q_1 - q_2}{C_1}$$

$$-R_3 \dot{q}_2 = \frac{q_2 - q_1}{C_1} + \frac{q_2 - q_3}{C_2}$$

$$R_1 \dot{q}_1 - (R_2 - R_1)\dot{q}_3 = \frac{q_3 - q_2}{C_2}$$

$$\begin{bmatrix} -R_1 & 0 & R_1 \\ 0 & -R_3 & 0 \\ R_1 & 0 & R_1 - R_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \frac{q_1 - q_2}{C_1} - U_1 \\ \frac{q_2 - q_1}{C_1} + \frac{q_2 - q_3}{C_2} \\ \frac{q_3 - q_2}{C_2} \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{\text{COFACTOR}(A)^T}{\det(A)} = \frac{\text{MINOR}(M) \cdot (-1)^{i+j}}{\det(A)}$$

$$\text{MINOR}(A) = \begin{bmatrix} -R_3R_1 + R_3R_2 & 0 & +R_1R_3 \\ 0 & -2R_1^2 + R_1R_2 & 0 \\ R_1R_3 & 0 & R_1R_3 \end{bmatrix}$$

$$\text{COFACTOR}(A) = \begin{bmatrix} -R_1R_3 + R_2R_3 & 0 & R_1R_3 \\ 0 & R_1R_2 - 2R_1^2 & 0 \\ R_1R_3 & 0 & R_1R_3 \end{bmatrix}$$

$$\det(A) = -R_1(-R_1R_3 + R_2R_3) + R_1(R_1R_3)$$

$$= +R_1^2R_3 - R_1R_2R_3 + R_1^2R_3$$

$$\det(A) = 2R_1^2R_3 - R_1R_2R_3$$

$$A^{-1} = \frac{1}{2R_1^2R_3 - R_1R_2R_3}\begin{bmatrix} -R_1R_3 + R_2R_3 & 0 & R_1R_3 \\ 0 & R_1R_2 - 2R_1^2 & 0 \\ R_1R_3 & 0 & R_1R_3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{R_2 - R_1}{2R_1^2 - R_1R_2} & 0 & \frac{1}{2R_1 - R_2} \\ 0 & \frac{-1}{R_3} & 0 \\ \frac{1}{2R_1 - R_2} & 0 & \frac{1}{2R_1 - R_2} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{21} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

$$\dot{q}_1 = a_{11}\left(\frac{q_1 - q_2}{C_1} - U_1\right) + a_{13}\left(\frac{q_3 - q_2}{C_2}\right)$$

$$\dot{q}_2 = a_{21}\left(\frac{q_2 - q_1}{C_1} + \frac{q_2 - q_3}{C_2}\right)$$

$$\dot{q}_3 = a_{31}\left(\frac{q_1 - q_2}{C_1} - U_1\right) + a_{33}\left(\frac{q_3 - q_2}{C_2}\right)$$

STATE

$x_1 = q_1$

$x_2 = q_2$

$x_3 = q_3$

$x_4 = \dot{q}_3$

$$\dot{x}_1 = \frac{a_{11}}{C_1} x_1 + x_2\left(-\frac{a_{11}}{C_1} - \frac{a_{13}}{C_2}\right) - a_{11} U_1$$

$$\dot{x}_2 = -\frac{a_{21}}{C_1} x_1 + x_2\left(\frac{a_{21}}{C_1} + \frac{a_{21}}{C_2}\right) - \frac{a_{21}}{C_2} x_3$$

$$\dot{x}_3 = \frac{a_{31}}{C_1} x_1 + x_2\left(-\frac{a_{31}}{C_1} - \frac{a_{33}}{C_2}\right) + \frac{a_{33}}{C_2} x_3 - a_{31} U_1$$

$$\dot{x}_4 = 0$$

$$\dot{x} = Ax + BU$$

$$y = Cx + DU$$

$$A = \begin{bmatrix} \frac{a_{11}}{C_1} & \frac{-C_2 a_{11} - C_1 a_{13}}{C_1 C_2} & 0 & 0 \\ \frac{-a_{21}}{C_1} & \frac{C_2 a_{21} + C_1 a_{21}}{C_1 C_2} & \frac{-a_{21}}{C_2} & 0 \\ \frac{a_{31}}{C_1} & \frac{-C_2 a_{31} - C_1 a_{33}}{C_1 C_2} & \frac{a_{33}}{C_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -a_{11} \\ 0 \\ -a_{31} \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & R_3 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

PROBLEM 2

$$\frac{\Phi(s)}{\delta(s)} = \frac{K_a s + K_b}{K_3 s^3 + K_2 s^2 + K_1 s + K_0}$$

INPUT DERIVATIVES

$$\xrightarrow{\delta(s)} \boxed{\frac{Z(s)}{\delta(s)}} \xrightarrow{Z(s)} \boxed{\frac{\Phi(s)}{Z(s)}} \xrightarrow{\Phi(s)}$$

$$\frac{1}{K_3 s^3 + K_2 s^2 + K_1 s + K_0} \qquad\qquad \frac{K_a s + K_b}{1}$$

$$\downarrow$$

$$K_3 \dddot{z} + K_2 \ddot{z} + K_1 \dot{z} + K_0 z = \delta \qquad\qquad \phi = K_a \dot{z} + K_b z$$

STATE

$$x_1 = z \qquad\qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{z} \qquad\qquad \dot{x}_2 = x_3$$

$$x_3 = \ddot{z} \qquad\qquad \dot{x}_3 = -\frac{K_0}{K_3} x_1 - \frac{K_1}{K_3} x_2 - \frac{K_2}{K_3} x_3 + \frac{\delta}{K_3}$$

$$y = \phi = K_a x_2 + K_b x_1$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & \frac{1}{K_3} \end{bmatrix}^T$$

$$C = \begin{bmatrix} K_b & K_a & 0 \end{bmatrix}$$

$$D = [0]$$

PROBLEM 3

$$\begin{bmatrix} \dot{a}_n \\ \dot{q} \\ \dot{\delta}_e \end{bmatrix} = \begin{bmatrix} -1.702 & 50.72 & 263.38 \\ 0.22 & -1.418 & -31.99 \\ 0 & 0 & -14 \end{bmatrix} \begin{bmatrix} a_n \\ q \\ \delta_e \end{bmatrix} + \begin{bmatrix} -272.06 \\ 0 \\ 14 \end{bmatrix} \delta_{com}$$

$$\dot{x} = Ax + Bu \longrightarrow sX(s) = AX(s) + BU(s)$$

$$X(s) = \left[(sI - A)^{-1} B\right] U(s)$$

$$(sI - A) = \begin{bmatrix} s + 1.702 & -50.72 & -263.38 \\ -0.22 & s + 1.418 & 31.99 \\ 0 & 0 & s + 14 \end{bmatrix}$$

$$(sI - A)^{-1} B = \begin{bmatrix} -\dfrac{(10112.19s + (s+14)(929.9s + 1582.7) + 17210.97)}{(s+1.702)(s+14)(3.418s - 5.34)} \\[2ex] -\dfrac{(507.71s + 788.9)}{(s+14)(3.418s + 5.34)} \\[2ex] \dfrac{14}{s+14} \end{bmatrix}$$

$$G_1 = \frac{A_n(s)}{\delta_{com}(s)}$$

$$G_2(s) = \frac{Q(s)}{\delta_{com}(s)}$$

PROBLEM 4

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$x = [w \quad q \quad z \quad \theta]^T$$

$$y = [z \quad \theta]^T$$

$$u = [\delta_B \quad \delta_S]^T$$

$$A = \begin{bmatrix} -0.038 & 0.896 & 0 & 0.0015 \\ 0.0017 & -0.092 & 0 & -0.0056 \\ 1 & 0 & 0 & -3.086 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.0075 & -0.023 \\ 0.0017 & -0.0022 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$s\,X(s) = A\,X(s) + B\,U(s)$$

$$Y(s) = C\,X(s)$$

$$Y(s) = \left[C(sI - A)^{-1}B\right]U(s)$$

$$G(s) = \begin{bmatrix} \dfrac{Z(s)}{\delta_B(s)} & \dfrac{Z(s)}{\delta_S(s)} \\[2ex] \dfrac{\theta(s)}{\delta_B(s)} & \dfrac{\theta(s)}{\delta_S(s)} \end{bmatrix}$$

USING PYTHON $\longrightarrow$

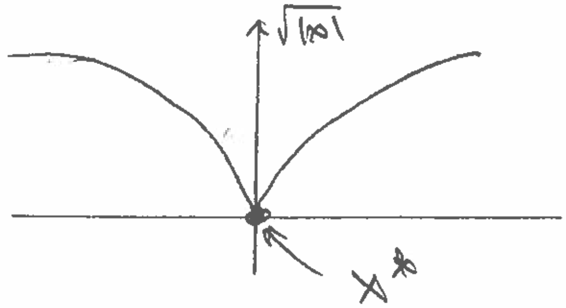
PROBLEM 5

$\underline{A}$ $\dot{x} = x^3 = F(x)$ EQ PT $F(x^*) = 0 \to x^* = 0$

$$\frac{\partial}{\partial x} F(x) = 3x^2 \implies \frac{\partial}{\partial x} F(x^*) = 0$$

$$\boxed{\delta \dot{x} = 0 \implies \dot{x} = Ax \quad A = [0]}$$

$\underline{B}$ $\dot{x} = \sqrt{|x|} = F(x)$ $0 = F(x^*)$ $0 = \sqrt{|x^*|}$



$x^* = 0$ IS ONLY EQ. PT.

$$F(x) = \begin{cases} \sqrt{x} & \text{FOR } x > 0 \\ 0 & x = 0 \\ \sqrt{-x} & \text{FOR } x < 0 \end{cases} \implies \frac{\partial}{\partial x} F(x) = \begin{cases} 1/2\sqrt{x} & x > 0 \\ \text{NONE} & x = 0 \\ \frac{-1}{2\sqrt{x}} & x < 0 \end{cases}$$

$\frac{\partial}{\partial x} F(x^*) = \frac{\partial F}{\partial x}(0)$ DOES NOT EXIST AT $x^* = 0$

CANNOT BE LINEARIZED ABOUT $x^* = 0$

D ATTITUDE DYNAMICS $(I_1 \neq I_2 \neq I_3)$

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2$$

$$\bar{x} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

- RIGID BODY DYNAMIC
- EULER'S EQNS
-

REDEFINE IN TERMS OF STATE

$$\dot{x}_1 = \frac{I_2 - I_3}{I_1} x_2 x_3 = F_1(\bar{x}, u)$$

$$\dot{x}_2 = \frac{I_3 - I_1}{I_2} x_3 x_1 = F_2(\bar{x}, u)$$

$$\dot{x}_3 = \frac{I_1 - I_2}{I_3} x_1 x_2 = F_3(\bar{x}, u)$$

EQ. PT

$$0 = \dot{x}_1 = F_1(\bar{x}^*, u^*)$$

$$0 = \dot{x}_2 = F_2(\bar{x}^*, u^*)$$

$$0 = \dot{x}_3 = F_3(\bar{x}^*, u^*)$$

$$\Longrightarrow \quad x_2^* x_3^* = 0, \quad x_3^* x_1^* = 0, \quad x_1^* x_2^* = 0$$

SPIRALING ROCKET SATELLITE?

4 EQ. PT $\longrightarrow$ CONSTANT ANG. VEL ABOUT EACH AXIS.

(1) $\bar{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (2) $x^* = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$ (3) $x^* = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}$ (4) $x^* = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$

WHERE c IS A CONSTANT. $\neq 0$

PARTIAL DERIVATIVES

$$\frac{\partial f_1}{\partial x_1}\bigg|_* = 0 \qquad \frac{\partial f_1}{\partial x_2}\bigg|_* = \frac{I_2 - I_3}{I_1} x_3^* \qquad \frac{\partial f_1}{\partial x_3}\bigg|_* = \frac{I_2 - I_3}{I_1} x_2^*$$

$$\frac{\partial f_2}{\partial x_1}\bigg|_* = \frac{I_3 - I_1}{I_2} x_3^* \qquad \frac{\partial f_2}{\partial x_2}\bigg|_* = 0 \qquad \frac{\partial f_2}{\partial x_3}\bigg|_* = \frac{I_3 - I_1}{I_2} x_1^*$$

$$\frac{\partial f_3}{\partial x_1}\bigg|_* = \frac{I_1 - I_2}{I_3} x_2^* \qquad \frac{\partial f_3}{\partial x_2}\bigg|_* = \frac{I_1 - I_2}{I_3} x_1^* \qquad \frac{\partial f_3}{\partial x_3}\bigg|_* = 0$$

LINEARIZED SYSTEM

$$\delta \dot{x}_1 = \frac{I_2 - I_3}{I_1} x_3^* \delta x_2 + \frac{I_2 - I_3}{I_1} x_2^* \delta x_3$$

$$\delta \dot{x}_2 = \frac{I_3 - I_1}{I_2} x_3^* \delta x_1 + \frac{I_3 - I_1}{I_2} x_1^* \delta x_3$$

$$\delta \dot{x}_3 = \frac{I_1 - I_2}{I_3} x_2^* \delta x_1 + \frac{I_1 - I_2}{I_3} x_1^* \delta x_2$$

- AT EACH EQUILIBRIUM POINT.

$x^* = [0 \quad 0 \quad 0]^T$

(1) $$\delta \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x \quad \longleftarrow \text{OKAY}$$

(2) $x^* = [0 \quad 0 \quad c]^T$

$$\delta \dot{x} = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} c & 0 \\ \frac{I_3 - I_1}{I_2} c & 0 & 0 \\ \end{bmatrix} \delta x$$

(3) $x^* = [0 \quad c \quad 0]^T$

$$\delta \dot{x} = \begin{bmatrix} 0 & 0 & \frac{I_2 - I_3}{I_1} c \\ 0 & 0 & 0 \\ \frac{I_1 - I_2}{I_3} c & 0 & 0 \end{bmatrix} \delta x$$

(4) $x^* = [c \quad 0 \quad 0]^T$

$$\delta \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{I_3 - I_1}{I_2} c \\ 0 & \frac{I_1 - I_2}{I_3} c & 0 \end{bmatrix} \delta x$$

PROBLEM 6

A $F(y,\dot{y},\ddot{y}) = \ddot{y} + (y^2-1)\dot{y} + y = 0$ EQ PT $\dot{y} = \ddot{y} = 0$

$y^* = 0$ ONLY EQ. PT. $* = (y^*, 0, 0)$

$F(y,\dot{y},\ddot{y}) = \ddot{y} + \dot{y}y^2 - \dot{y} + y = 0$

$\frac{\partial F}{\partial y}\Big|_* = 2y^*\dot{y}^* + 1 = 1$ $\frac{\partial F}{\partial \dot{y}}\Big|_* = 2y^{*2} - 1 = -1$

$\frac{\partial F}{\partial \ddot{y}}\Big|_* = 1$

$\delta\ddot{y} + \delta y - \delta\dot{y} = 0$ $\boxed{\delta\ddot{y} = -\delta y + \delta\dot{y}}$

$\boxed{\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\delta x}$

$x_1 = \delta y$

$x_2 = \delta\dot{y}$

B

$\ddot{y} + \dot{y} + y - y^3 = 0 = F(y,\dot{y},\ddot{y})$

AT EQ. PT $\dot{y} = \ddot{y} = 0$

$y^* - (y^*)^3 = 0 \longrightarrow y^*(1-(y^*)^2) = 0$

EQ. PT. $y^* = 0,\ y^* = 1,\ y^* = -1$

$(*) = (y^*, 0, 0)$

$$\frac{\partial F}{\partial y}\Big|_* = 1 - 3(y^*)^2 \qquad \frac{\partial F}{\partial \dot{y}}\Big|_* = 1 \qquad \frac{\partial F}{\partial \ddot{y}}\Big|_* = 1$$

LINEARIZED EQ. $\delta\ddot{y} = [3(y^*)^2 - 1]\,\delta y - \delta\dot{y}$

$x_1 = \delta y \qquad x_2 = \delta\dot{y}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (3(y^*)^2 - 1)\,x_1 - x_2 \end{aligned} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3(y^{*2}) - 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

AT $y^* = (0)$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \delta x$$

AT $y^* = 1$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \delta x$$

AT $y^* = -1$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \delta x$$

PROBLEM 6 C

$$F_1(y, \dot{y}, \ddot{y}, \theta, \dot{\theta}, \ddot{\theta}) = (M+m)\ddot{y} + m l \ddot{\theta}\cos\theta - m l \dot{\theta}^2 \sin\theta + ky = 0$$

$$F_2(y, \dot{y}, \ddot{y}, \theta, \dot{\theta}, \ddot{\theta}) = m l \ddot{y}\cos\theta + m l^2 \ddot{\theta} + m g l \sin\theta = 0$$

AT EQ. $\dot{y} = \ddot{y} = \dot{\theta} = \ddot{\theta} = 0$

$k(y^*) = 0$ $\quad$ $\boxed{y^* = 0}$ $\quad$ EQ. PT.

$m g l \sin\theta^* = 0$ $\quad$ $\boxed{\theta^* = n\pi}$ $\quad$ n IS AN INTEGER

$$\frac{\partial F_1}{\partial y}\Big|_* = k \qquad \frac{\partial F_1}{\partial \dot{y}}\Big|_* = 0 \qquad \frac{\partial F_1}{\partial \ddot{y}}\Big|_* = M+m$$

$$\frac{\partial F_1}{\partial \theta}\Big|_* = 0 \qquad \frac{\partial F_1}{\partial \dot{\theta}}\Big|_* = 0 \qquad \frac{\partial F_1}{\partial \ddot{\theta}}\Big|_* = m l \cos\theta^*$$

$$\frac{\partial F_2}{\partial y}\Big|_* = 0 \qquad \frac{\partial F_2}{\partial \dot{y}}\Big|_* = 0 \qquad \frac{\partial F_2}{\partial \ddot{y}}\Big|_* = m l \cos\theta^*$$

$$\frac{\partial F_2}{\partial \theta}\Big|_* = m g l \cos\theta^* \qquad \frac{\partial F_2}{\partial \dot{\theta}}\Big|_* = 0 \qquad \frac{\partial F_2}{\partial \ddot{\theta}}\Big|_* = m l^2$$

LINEARIZED EQ.

$$(M+m)\,\delta\ddot{y} + m l \cos\theta^*\,\delta\ddot{\theta} + k\,\delta y = 0$$

$$m l \cos\theta^*\,\delta\ddot{y} + m l^2\,\delta\ddot{\theta} + m g l \cos\theta^*\,\delta\theta = 0$$

TWO POSSIBLE EQ. STATES

$y^* = 0$ $\quad$ (1) n IS EVEN

$\theta^* = n\pi$ $\quad$ (2) n IS ODD

CASE 1 n IS EVEN

$$(M+m)\,\delta\ddot{y} + ml\,\delta\ddot{\theta} + k\delta y = 0$$

$$ml\,\delta\ddot{y} + ml^2\delta\ddot{\theta} + mgl\,\delta\theta = 0$$

$$\begin{bmatrix}\delta\ddot{y}\\ \delta\ddot{\theta}\end{bmatrix} = \begin{bmatrix}M+m & ml\\ ml & ml^2\end{bmatrix}^{-1}\begin{bmatrix}-k\,\delta y\\ -mgl\,\delta\theta\end{bmatrix} = \frac{1}{(M+m)ml^2 - m^2l^2}\begin{bmatrix}ml^2 & -ml\\ -ml & M+m\end{bmatrix}\begin{bmatrix}-k\delta y\\ -mgl\delta\theta\end{bmatrix}$$

$$= \frac{1}{Mml^2}\begin{bmatrix}-kml^2\delta y + m^2gl^2\delta\theta\\ kml\delta y - (M+m)mgl\delta\theta\end{bmatrix}$$

$$\delta\ddot{y} = -\frac{k}{M}\delta y + \frac{mg}{M}\delta\theta$$

$$\delta\ddot{\theta} = \frac{k}{ml}\delta y - \frac{(M+m)}{Ml}g\,\delta\theta$$

$x_1 = \delta y$

$x_2 = \delta\dot{y}$

$x_3 = \delta\theta$

$x_4 = \delta\dot{\theta}$

$$\delta\dot{x} = \begin{bmatrix}0 & 1 & 0 & 0\\ -k/M & 0 & \frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ \frac{k}{Ml} & 0 & -\frac{(M+m)}{Ml}g & 0\end{bmatrix}\delta x$$

CASE 2 $y^* = 0$, $\theta^* = n\pi$ n IS ODD

$$(M+m)\delta\ddot{y} - ml\,\delta\ddot{\theta} + k\,\delta y = 0$$

$$-ml\,\delta\ddot{y} + ml^2\delta\ddot{\theta} - mgl\,\delta\theta = 0$$

DECOUPLE

$$\delta\ddot{y} = -\frac{k}{M}\delta y + \frac{mg}{M}\delta\theta$$

$$\delta\ddot{\theta} = -\frac{k}{Ml}\delta y + \frac{M+m}{Ml}g\,\delta\theta$$

$x_1 = \delta y$

$x_2 = \delta\dot{y}$

$x_3 = \delta\theta$

$x_4 = \delta\dot{\theta}$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{M} & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k}{Ml} & 0 & \frac{M+m}{Ml}g & 0 \end{bmatrix} \delta x$$

PROBLEM 7 FIND TRANSFER FCN. $G(s)$

$$\mathcal{L}\left\{ \begin{array}{l} \ddot{y}_1 + \ddot{y}_2 + y_1 + y_2 = u_1 + \dot{u}_2 \\ 2\ddot{y}_1 + 3\ddot{y}_2 + y_1 - y_2 = 0 \end{array} \right\}$$

$$s^2Y_1 + s^2Y_2 + Y_1 + Y_2 = U_1 + sU_2$$

$$2s^2Y_1 + 3s^2Y_2 + Y_1 - Y_2 = 0$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} s^2+1 & s^2+1 \\ 2s^2+1 & 3s^2-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 3s^2-1 & 3s^3-s \\ -2s^2-1 & -2s^3-s \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\det(\;) = s^4 - s^2 - 2 = \Delta$$

$$\boxed{G(s) = \begin{bmatrix} \dfrac{3s^2-1}{s^4-s^2-2} & \dfrac{3s^3-s}{s^4-s^2-2} \\ \dfrac{-(2s^2+1)}{s^4-s^2-2} & \dfrac{-(2s^3+s)}{s^4-s^2-s} \end{bmatrix}}$$

PROBLEM 8 FIND $G(s)$

$$\mathcal{L}\left\{ \begin{array}{l} \ddot{q}_1 + 3\dot{q}_2 + \dot{q}_1 + 2q_2 = \dot{u} + 4u \\ \ddot{q}_1 + 4\dot{q}_2 + 3q_2 = 0 \\ y = q_1 + q_2 \end{array} \right\} \Rightarrow \begin{array}{l} s^2Q_1 + 3sQ_2 + sQ + 2Q_2 = sU_1 + 4 \\ s^2Q_1 + 4sQ_2 + 3Q_2 = 0 \\ Y = Q_1 + Q_2 \end{array}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{1}{s(s^2+5s+3)} \begin{bmatrix} 4s+3 & -(3s+2) \\ -s^2 & s^2+s \end{bmatrix} \begin{bmatrix} s+4 \\ 1 \end{bmatrix} U(s)$$

$$= \frac{1}{s(s^2+5s+3)} \begin{bmatrix} 4(s+3)(s+4) - (3s+2) \\ -s^2(s+4) + s^2 + s \end{bmatrix} U(s) =$$

$$Y(s) = \frac{1}{s(s^2+5s+3)} \left[-s^3 + s^2 + 17s + 10\right] U(s) \Rightarrow$$

$$\boxed{G(s) = \frac{-s^3+s^2+17s+10}{\ldots}}$$

PROBLEM 9 TRANSFER FCN.

$$\mathcal{L}\left\{ \begin{array}{l} \dot{x} = -x + 2x(t-h) + u \\ y = [1]x \end{array} \right\} \Rightarrow \begin{array}{l} sX = -X + 2e^{-hs}X + U \\ Y = X \end{array}$$

TIME SHIFT

$$X(s) = \frac{1}{s+1-2e^{-hs}} U(s)$$

$$Y(s) = \underbrace{\frac{1}{s+1-2e^{-hs}}}_{G(s)} U(s)$$