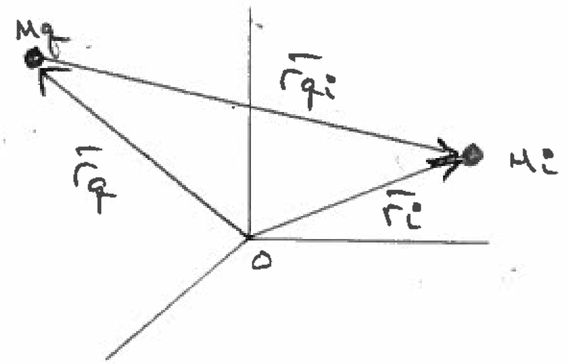
RELATIVE n-BODY

ONLY 10 INTEGRALS OF MOTION, 12 ARE REQUIRED TO SOLVE 2BP

WE ONLY REALLY CARE ABOUT RELATIVE MOTION



$$\ddot{\bar{r}}_{q} = \ddot{\bar{r}}_i - \ddot{\bar{r}}_q$$

$\downarrow$

$\bar{r}_{qi}$ IS MORE USEFUL

REFORMULATE OUR PROBLEM

REDO PROBLEM IN TERMS $\bar{r}_{qi}$ FROM q TO i

HOW TO GET EOMS $(\ddot{\bar{r}}_{qi})$

FOR ANY ACCELERATION $\left\{ \begin{array}{l} \text{POSITION VECTOR} \\ \text{FRAME OF DIFFERENTIATION (INERTIAL OBSERVER} \end{array} \right.$

TO APPLY NEWTON'S LAW OF MOTION MUST DIFFERENTIATE IN INERTIAL FRAME AND BASE POINT OF POSITION VECTOR MUST BE FIXED IN THAT FRAME

CANNOT APPLY $\bar{F} = m\bar{a}$

BUT $\bar{r}_q$ CAN BE WRITTEN IN CORRECT TERMS

$$\bar{r}_i = \bar{r}_{qi} + \bar{r}_q$$

$$\ddot{\bar{r}}_i = \ddot{\bar{r}}_{qi} + \ddot{\bar{r}}_q$$

APPLY NEWTON'S LAW TO $\bar{r}_i$ AND $\bar{r}_q$

$$\ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3} \bar{r}_{ji} \quad \text{EOM APPLIED TO } m_i \text{ (SAT)}$$

$$\ddot{\bar{r}}_q = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3} \bar{r}_{jq} \quad \text{EOM APPLIED TO } m_q \text{ (EARTH)}$$

SUBSTITUTE INTO $\bar{r}_{qi} + \bar{r}_q = \bar{r}_i$

$$\ddot{\bar{r}}_{qi} - G\sum_{\substack{j=1\\ j\neq q}}^{n} \frac{m_j}{r_{jq}^3}\bar{r}_{jq} = -G\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ji}^3}\bar{r}_{ji}$$

← ONLY RELATIVE POSITIONS APPEAR

REMOVE i TERM — REMOVE q TERM

$$\ddot{\bar{r}}_{qi} - G\frac{m_i}{r_{iq}^3}\bar{r}_{iq} - \underbrace{G\sum_{\substack{j=1\\ j\neq i,q}}^{n} \frac{m_j}{r_{jq}^3}\bar{r}_{jq}}_{\text{MOVE RIGHT}} = \underbrace{-G\frac{m_q}{r_{qi}^3}\bar{r}_{qi}}_{\text{MOVE LEFT}} - G\sum_{\substack{j=1\\ j\neq i,q}}^{n} \frac{m_j}{r_{ji}^3}\bar{r}_{ji}$$

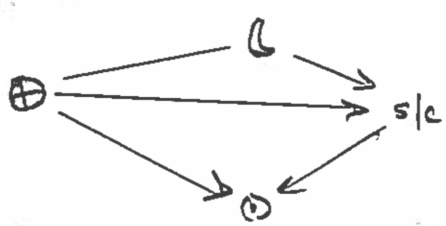
COMBINE THE SUMMATIONS : $j \to n,\ j\neq i,q$

EQS OF MOTION OF m_i RELATIVE TO m_q

$$\boxed{\ddot{\bar{r}}_{qi} + \frac{G(m_i+m_q)}{r_{qi}^3}\bar{r}_{qi} = G\sum_{\substack{j=1\\ j\neq i,q}}^{n} m_j\left(\frac{\bar{r}_{ij}}{r_{ij}^3} - \frac{\bar{r}_{qj}}{r_{qj}^3}\right)}$$

DOMINANT — PERTURBING — DIRECT — INDIRECT

EXAMPLE ⊕ ☉ ☾ s/c HOW DOES s/c MOVE RELATIVE TO ⊕



MOTION OF MASS 2 RELATIVE TO MASS 1 ; PERTURBED BY MASSES 3, 4

$\therefore q=1 \quad i=2 \quad j=3,4$

$$\ddot{\bar{r}}_{12} + G\frac{m_1+m_2}{r_{12}^3}\bar{r}_{12} = Gm_3\left(\frac{\bar{r}_{23}}{r_{23}^3} - \frac{\bar{r}_{13}}{r_{13}^3}\right) + Gm_4\left(\frac{\bar{r}_{24}}{r_{24}^3} - \frac{\bar{r}_{14}}{r_{14}^3}\right)$$

$$\ddot{\bar{r}}_{\oplus s/c} + G\frac{m_\oplus + m_{s/c}}{r_{\oplus s/c}^3}\bar{r}_{\oplus s/c} = Gm_\odot\left(\frac{\bar{r}_{s/c\odot}}{r_{s/c\odot}^3} - \frac{\bar{r}_{\oplus\odot}}{r_{\oplus\odot}^3}\right) + Gm_☾\left(\frac{\bar{r}_{s/c☾}}{r_{s/c☾}^3} - \frac{\bar{r}_{\oplus☾}}{r_{\oplus☾}^3}\right)$$

ACCEL — DOMINANT — DIRECT ☉ ON s/c — INDIRECT ☉ ON ⊕ — DIRECT ☾ ON s/c — INDIRECT ☾ ON ⊕

GIVEN RELATIVE EOMS — CAN WE SOLVE

STILL CAN'T SOLVE FOR $n \geq 3$

$n = 3$: REQUIRES POS OF ☉ REL TO ⊕ OR ☾

→ REQUIRES ANOTHER VECTOR DE.

⟹ 12 SCALAR 1st ODE → NOT SOLVABLE

$n = 2$: NO M_j PERTURBATIONS

2nd ODE IN ONLY ONE UNKNOWN

6 SCALAR EQS, 6 DEP. VARIABLES → REQUIRES

6 CONSTANTS (10 KNOWN)

RELATIVE MOTION OF TWO B.P. IS SOLVABLE

$$\boxed{\ddot{\bar{r}} + \frac{G(m_1+m_2)}{r^3}\bar{r} = 0}$$

$\mu = G(m_1+m_2)$

SOLUTION : RELATIVE MOTION OF TWO BODIES

SOLVE : $\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0$ RELATIVE 2BP

$\mu = G(M_1 + M_2)$

$\bar{r}_{\oplus s/c}$ ← DROP SUBSCRIPTS

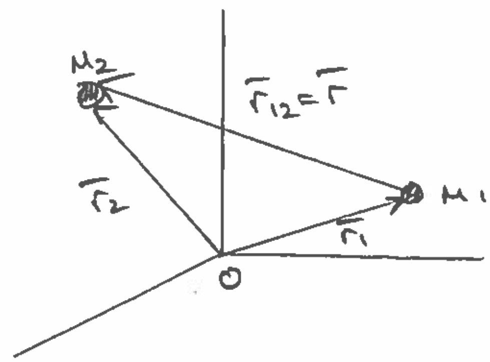
DERIVATION

1. USE ANGULAR MOMENTUM - SYSTEM ANGULAR MOMENTUM

$$\sum_{i=1}^{n} m_i (\bar{r}_i \times \dot{\bar{r}}_i) = \bar{C}_3$$

CONSTANT VECTOR ← ONLY FOR TWO BODIES

LET $n = 2$



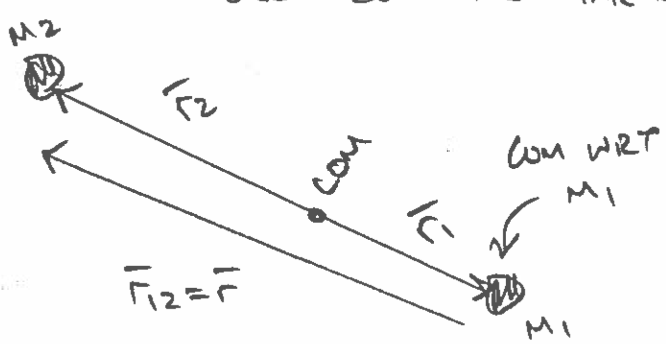
$$\bar{C}_3 = m_1(\bar{r}_1 \times \dot{\bar{r}}_1) + m_2(\bar{r}_2 \times \dot{\bar{r}}_2) \quad (*)$$

SYSTEM LINEAR MOMENTUM IS CONSERVED

- $\bar{V}_{COM}$ = CONSTANT
- A COORD FRAME MOVING WITH COM IS INERTIAL

SO WE ASSUME COM IS "FIXED" IN NEW INERTIAL FRAME

USE COM AS INERTIALLY FIXED BASE POINT



$$(M_1 + M_2)\bar{r}_{COM} = M_2 \bar{r} + M_1(0)$$

$$\bar{r}_{COM} = \frac{M_2}{M_1 + M_2}\bar{r} = -\bar{r}_1$$

$$\bar{r}_1 = \frac{-M_2}{M_1 + M_2}\bar{r}$$

$$\bar{r}_2 = \frac{M_1}{M_1 + M_2}\bar{r}$$

SUBSTITUTE INTO (*) $\bar{C}_3 = m_1(\bar{r}_1 \times \dot{\bar{r}}_1) + m_2(\bar{r}_2 \times \dot{\bar{r}}_2)$

$$\bar{C}_3 = m_1\left(\frac{-m_2}{m_1+m_2}\,\bar{r} \times \frac{-m_2}{m_1+m_2}\,\dot{\bar{r}}\right) + m_2\left(\frac{m_1}{m_1+m_2}\,\bar{r} \times \frac{m_1}{m_1+m_2}\,\dot{\bar{r}}\right)$$

$$\bar{C}_3 = \frac{m_1 m_2}{m_1+m_2}\left(\bar{r} \times \dot{\bar{r}}\right) \Rightarrow \frac{m_1+m_2}{m_1 m_2}\,\bar{C}_3 = \boxed{\bar{r} \times \dot{\bar{r}} = \bar{h}}$$

SPECIFIC ANGULAR MOMENTUM

CONSTANT VECTOR

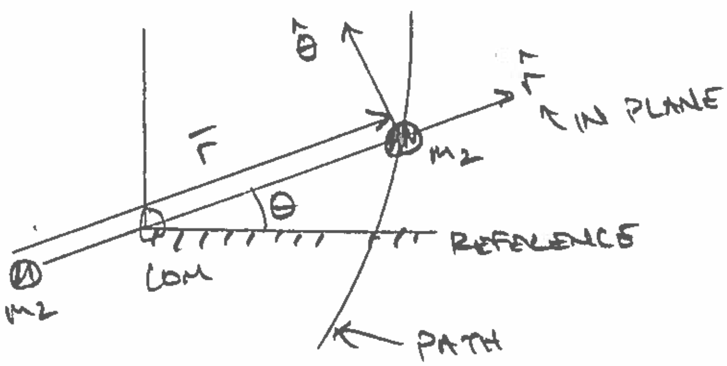
* $\dot{\bar{r}} = \frac{{}^{I}d\bar{r}}{dt}$ RELATIVE VELOCITY

$\bar{h} \perp \bar{r}, \dot{\bar{r}}$ → PLANE OF MOTION CONSTANT

2-D MOTION

<u>INVARIABLE PLANE</u> — PLANE CONTAINS COM WHOSE NORMAL COINCIDES WITH $\bar{h}$

CAN REPRESENT $\bar{h}$ IN SCALAR FORM

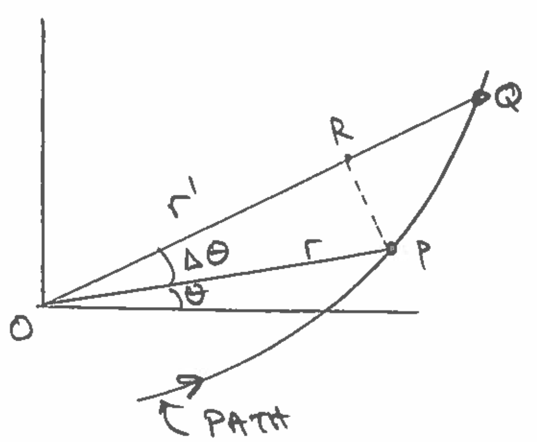


$$\bar{r} = r\,\hat{r}$$
$$\dot{\bar{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$
$$|\bar{h}| = |\bar{r} \times \dot{\bar{r}}| = r^2\dot{\theta}$$
$$h = r^2\dot{\theta}$$ ← MAGNITUDE

<u>EXTRA</u>

$\bar{h}$ RELATED TO AREAL VELOCITY

(KEPLER'S THIRD LAW : LINE JOINING PLANET + SUN SWEEPS OUT EQUAL AREAS IN EQUAL TIMES)



AREAL VELOCITY : RATE AT WHICH RADIUS VECTOR DESCRIBES A CURVE

ASSUME MOTION IN PLANE $\longrightarrow$ KNOWN FROM h CONSTANT

ΔA REPRESENTS AREA OF TRIANGLE OPQ SWEPT OVER BY RADIUS VECTOR IN INTERVAL Δt

AREA OF TRIANGLE $= \frac{1}{2}$ (BASE) (HEIGHT)

$$\Delta A = \frac{1}{2}(r')(r \sin \Delta\theta) = \frac{r' r \sin \Delta\theta}{2}$$

$$\frac{\Delta A}{\Delta t} = \frac{r' r}{2} \; \frac{\sin \Delta\theta}{\Delta\theta} \, \frac{\Delta\theta}{\Delta t}$$

AS $\Delta\theta$ DIMINISHES, RATIO OF AREA OF TRIANGLE TO THAT OF A SECTOR APPROACHES UNITY AS A LIMIT

LIMIT OF $\frac{\sin \Delta\theta}{\Delta\theta}$ IS UNITY

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}$$

AREAL VELOCITY CONSTANT BECAUSE h IS CONSTANT

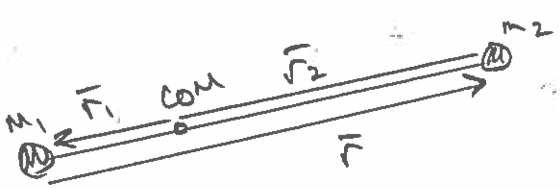
EXTRA

USE ENERGY $\longleftarrow$ SCALAR

GRAVITY FIELD IS CONSERVATIVE

$$T - U = C_4$$

REWRITE IN RELATIVE VARIABLES



$$\bar{r}_1 = \frac{-m_2}{m_1+m_2}\bar{r} \qquad \dot{\bar{r}}_1 = \frac{-m_2}{m_1+m_2}\dot{\bar{r}}$$

$$\bar{r}_2 = \frac{m_1}{m_1+m_2}\bar{r} \qquad \dot{\bar{r}}_2 = \frac{m_1}{m_1+m_2}\dot{\bar{r}}$$

$n = 2$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \, \bar{V}_i \cdot \bar{V}_i$$

$$= \frac{1}{2} m_1 (\dot{\bar{r}}_1 \cdot \dot{\bar{r}}_1) + \frac{1}{2} m_2 (\dot{\bar{r}}_2 \cdot \dot{\bar{r}}_2)$$

$$T = \frac{1}{2} m_1 \left(\frac{-m_2}{m_1+m_2} \dot{\bar{r}} \cdot \frac{-m_2}{m_1+m_2} \dot{\bar{r}} \right) + \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2} \dot{\bar{r}} \cdot \frac{m_1}{m_1+m_2} \dot{\bar{r}} \right)$$

$$\boxed{T = \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \left(\dot{\bar{r}} \cdot \dot{\bar{r}} \right)}$$

$$U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}}$$ ← RELATIVE DISTANCE

$$U = \frac{1}{2} G \left(\frac{m_1 m_2}{r} + \frac{m_2 m_1}{r} \right) = \frac{G m_1 m_2}{r}$$

$$T - U = \frac{1}{2} \left(\dot{\bar{r}} \cdot \dot{\bar{r}} \right) \frac{m_1 m_2}{m_1+m_2} - \frac{G m_1 m_2}{r} = C_4$$

MULTIPLY BY $\frac{m_1+m_2}{m_1 m_2}$

$$\frac{1}{2} \left(\dot{\bar{r}} \cdot \dot{\bar{r}} \right) - \frac{G(m_1+m_2)}{r} = C_4 \left(\frac{m_1+m_2}{m_1 m_2} \right) \longrightarrow \mathcal{E}$$ SPECIFIC ENERGY

DEFINE $\bar{v} = \dot{\bar{r}} = \frac{{}^{I} d\bar{r}}{dt}$

$$\bar{v} = \dot{\bar{r}} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$v^2 = |\dot{\bar{r}}|^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

$$\frac{v^2}{2} - \frac{G(m_1+m_2)}{r} = C_4 \left(\frac{m_1+m_2}{m_1 m_2} \right) = \mathcal{E}$$ "ENERGY"

LET $\mu = G(m_1+m_2)$

$$\boxed{\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}}$$

SPECIFIC MECHANICAL ENERGY

U' ← GENERAL POTENTIAL

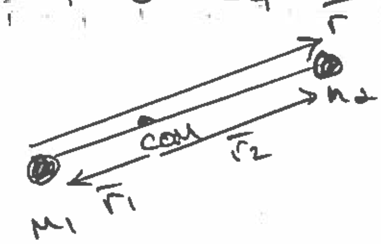
SOLUTION OF RELATIVE TWO BODY PROBLEM

$$(*)\quad \ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0 \qquad \mu = G(m_1 + m_2)$$

APPLY OUR CONSTANTS OF MOTION - TO 2BP

CONSERVATION OF ENERGY

$$T - U = C_4$$



CENTER OF MASS WRT m_1

$$(m_1 + m_2)\bar{r}_{COM} = m_1(0) + m_2\bar{r}_{12}$$

$$\bar{r}_{COM} = \frac{m_2}{m_1 + m_2}\bar{r} \leftarrow \text{COM WRT } m_1$$

SO

$$\bar{r}_1 = \frac{-m_2}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_1 = \frac{-m_2}{m_1 + m_2}\dot{\bar{r}}$$

$$\bar{r}_2 = \frac{m_1}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_2 = \frac{m_1}{m_1 + m_2}\dot{\bar{r}}$$

$$T = \frac{1}{2}\sum_{i=1}^{2} m_i(\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i) = \frac{1}{2}m_1(\dot{\bar{r}}_1 \cdot \dot{\bar{r}}_1) + \frac{1}{2}m_2(\dot{\bar{r}}_2 \cdot \dot{\bar{r}}_2)$$

$$T = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\dot{\bar{r}} \cdot \dot{\bar{r}})$$

$$U = \frac{1}{2}G\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j \neq i}}^{n}\frac{m_i m_j}{r_{ji}} \qquad \text{SCALAR}$$

$$= \frac{1}{2}G\left(\frac{m_1 m_2}{r} + \frac{m_2 m_1}{r}\right) = \frac{G m_1 m_2}{r}$$

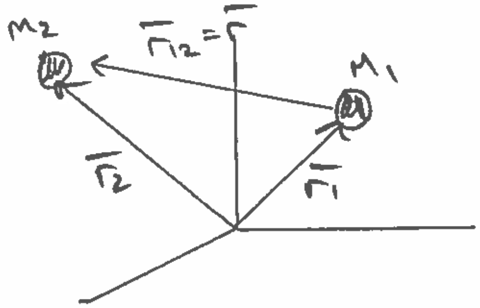
$$T - U = \frac{1}{2}(\dot{\bar{r}} \cdot \dot{\bar{r}})\frac{m_1 m_2}{m_1 + m_2} - G\frac{m_1 m_2}{r} = C_4 \qquad \text{MULTIPLY BY } \frac{m_1 + m_2}{m_1 m_2}$$

$$\frac{1}{2}(\dot{\bar{r}} \cdot \dot{\bar{r}}) - \frac{G(m_1 + m_2)}{r} = C_4\frac{m_1 + m_2}{m_1 m_2} = \mathcal{E} \qquad \text{SPECIFIC MECHANICAL ENERGY}$$

$$\frac{v^2}{2} - \frac{G(m_1 + m_2)}{r} = \boxed{\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}}$$

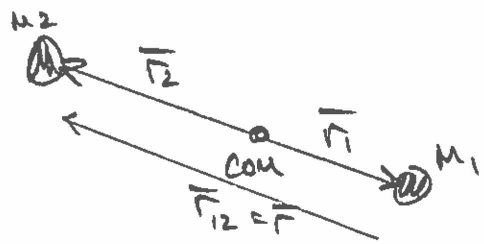
CONSERVATION OF ANGULAR MOMENTUM

$$\sum_{i=1}^{n} m_i(\bar r_i \times \dot{\bar r}_i) = \bar C_3 \quad \text{CONSTANT VECTOR}$$



$$\bar C_3 = m_1(\bar r_1 \times \dot{\bar r}_1) + m_2(\bar r_2 \times \dot{\bar r}_2)$$

- WE KNOW LINEAR MOMENTUM IS CONSERVED $\bar V_{com}$ = CONSTANT
- REF. FRAME FIXED AT COM ⇒ INERTIAL



$$(m_1+m_2)\bar r_{com} = m_2 \bar r + m_1(0)$$

$$\bar r_{com} = \frac{m_2}{m_1+m_2}\bar r = -\bar r_1$$

$$\bar r_1 = \frac{-m_2}{m_1+m_2}\bar r \qquad \bar r_2 = \frac{m_1}{m_1+m_2}\bar r$$

INTO $\bar C_3$

$$\bar C_3 = \frac{m_1 m_2}{m_1+m_2}(\bar r \times \dot{\bar r})$$

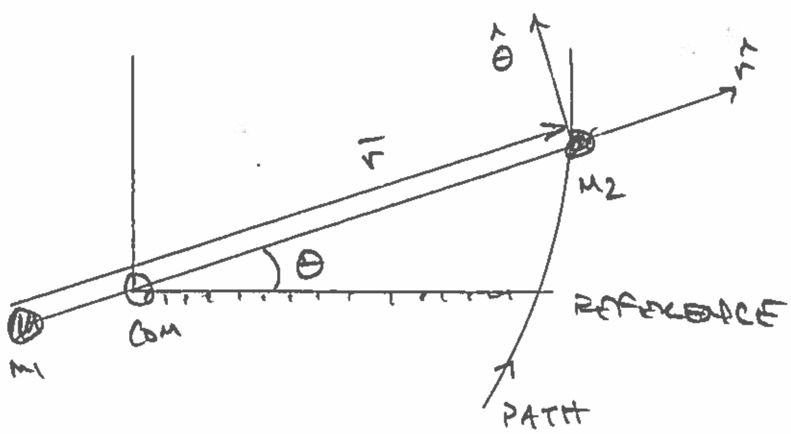
$$\frac{m_1+m_2}{m_1 m_2}\bar C_3 = \boxed{\bar r \times \dot{\bar r} = \bar h}$$

SPECIFIC ANGULAR MOMENTUM VECTOR } CONSTANT

$\bar h \perp \bar r_1 \; \dot{\bar r}_1$ – PLANE OF MOTION IS CONSTANT

INVARIABLE PLANE

REPRESENT $\bar h$ IN SCALAR FORM



$$\hat r \times \hat \theta = \hat h$$

$$\bar r = r \hat r$$

$$\dot{\bar r} = \dot r \hat r + r \dot\theta \hat\theta$$

$$|\bar h| = |\bar r \times \dot{\bar r}| = r^2 \dot\theta$$

$$\boxed{h = r^2 \dot\theta}$$

USING KNOWN CONSTANTS $(h, \mathcal{E})$

REPLACE $\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0$ 2nd ODE – VECTOR

WITH TWO 1st ODE WITH DEP VAR. (r, θ)

→ PLANAR MOTION ∴ ONLY TWO DEP VARIABLES

$$h = r^2\dot{\theta}$$

$$\mathcal{E} = \frac{1}{2}v^2 - U' = \frac{1}{2}\underbrace{(\dot{r}^2 + r^2\dot{\theta}^2)} - U'$$

$$\bar{v} = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

FIND SOLUTION FOR TIME HISTORY OF DEP. VARIABLE

CLASSICAL DERIVATION

$$h = r^2 \frac{d\theta}{dt} \Rightarrow dt = \frac{r^2}{h} d\theta$$ REMOVE TIME

$$\mathcal{E} = \frac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} - U'$$

$$2\mathcal{E} = \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{d\theta}\frac{h}{r^2}\right)^2 - 2U'$$

$$= \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^2} - 2U'$$

$$\frac{2}{h^2}[\mathcal{E} + U'] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$ ← DIFF. EQ. CAN INTEGRATE

NEW VARIABLES

$$\xi = \frac{1}{r} = r^{-1} \qquad \frac{d\xi}{d\theta} = -r^{-2}\frac{dr}{d\theta}$$

$$\frac{2}{h^2}[\mathcal{E} + U'] = \left(\frac{d\xi}{d\theta}\right)^2 + \xi^2$$

$$\frac{d\xi}{d\theta} = \pm\sqrt{\frac{2}{h^2}[\mathcal{E} + U'] - \xi^2}$$

$$d\theta = \frac{d\zeta}{\pm\sqrt{\frac{2}{h^2}\left(\varepsilon + U'\right) - \zeta^2}} \quad \rightarrow \quad 3 \text{ PARAMETERS } \varepsilon, h, \mu$$

$$U' = \frac{\mu}{r} = \mu\zeta$$

INTEGRATE

$$\theta = \cos^{-1} \frac{\zeta - \frac{\mu}{h^2}}{\sqrt{\frac{\mu^2}{h^4} + \frac{2\varepsilon}{h^2}}} + \omega$$

INTEGRATION CONSTANT

DIRECTION OF "PERIAPSIS"

MEASURED FROM SOME REFERENCE

$$\frac{1}{r} = \frac{\mu}{h^2} + \sqrt{\frac{\mu^2}{h^4} + \frac{2\varepsilon}{h^2}} \cos(\theta - \omega)$$

$$\boxed{r = \frac{p}{1 + e\cos(\theta - \omega)}}$$

STANDARD POLAR EQUATION OF A CONIC SECTION FOCUS AS THE ORIGIN

$p = \frac{h^2}{\mu}$ PARAMETER OR SEMI-LATUS RECTUM

$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$ ECCENTRICITY

SOLUTION: RELATIVE 2BP — VECTOR DERIVATION

ROTATION

$\bar r \cdot \dot{\hat r} = 0$

(1) $\ddot{\bar r} + \frac{\mu}{r^3}\bar r = \bar 0$

$\bar h = \bar r \times \dot{\bar r}$

NOTE: ${}^I\frac{d}{dt}\hat r = {}^I\frac{d}{dt}\left(\frac{\bar r}{r}\right) = \frac{\dot{\bar r}}{r} - \frac{\bar r}{r^2}\dot r = \frac{r^2\dot{\bar r} - r\dot r\bar r}{r^3}$

$\bar r = r\hat r$

$\dot{\bar r} = \dot r\hat r + r\dot{\hat r}$

$\bar r\cdot\dot{\bar r} = (\bar r\cdot\hat r)\dot r$

$\bar r\cdot\dot{\bar r} = r\dot r$

REWRITE ${}^I\frac{d}{dt}\hat r = \frac{(\bar r\cdot\bar r)\dot{\bar r} - (\bar r\cdot\dot{\bar r})\bar r}{r^3}$

IDENTITY $\bar A\times(\bar B\times\bar C) = (\bar A\cdot\bar C)\bar B - (\bar A\cdot\bar B)\bar C$

$\bar r \quad \dot{\bar r} \quad \bar r = (\bar r\cdot\bar r)\dot{\bar r} - (\bar r\cdot\dot{\bar r})\bar r$

${}^I\frac{d}{dt}\hat r = -\frac{\bar r\times(\bar r\times\dot{\bar r})}{r^3} = \frac{(\bar r\times\dot{\bar r})\times\bar r}{r^3} = \bar h\times\frac{\bar r}{r^3}$

FROM (1) $\frac{\bar r}{r^3} = -\frac{\ddot{\bar r}}{\mu}$

${}^I\frac{d}{dt}\hat r = \dot{\hat r} = \bar h \times -\frac{\ddot{\bar r}}{\mu}$ OR $\dot{\hat r} = \frac{\ddot{\bar r}\times\bar h}{\mu}$ CONSTANTS

INTEGRATE ONCE $\hat r = \frac{\dot{\bar r}\times\bar h}{\mu} - \bar e$

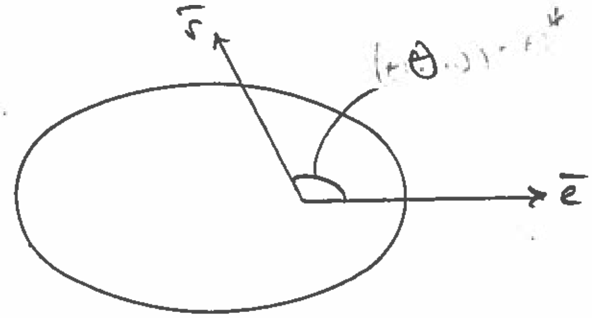
$\bar e$ – ECCENTRICITY VECTOR POINTS TOWARDS PERIAPSIS

DOT PRODUCT WITH $\bar r$

$\bar r\cdot\hat r = \bar r\cdot\frac{\dot{\bar r}\times\bar h}{\mu} - \bar r\cdot\bar e$ APPLY $\bar A\cdot\bar B\times\bar C = \bar C\cdot\bar A\times\bar B$

$r = \frac{\bar h}{\mu}\cdot\underbrace{(\bar r\times\dot{\bar r})}_{h} - \bar r\cdot\bar e$

$r = \frac{h^2}{\mu} - \underbrace{\bar r\cdot\bar e}_{re\cos(\theta)}$ $\qquad r + re\cos(\theta) = \frac{h^2}{\mu}$



$$\boxed{r = \frac{h^2/\mu}{1 + e\cos(\theta)}}$$

TRAJECTORY / POLAR EQN OF CONIC SEC

SOLUTION OF RELATIVE TWO BODY EOMS

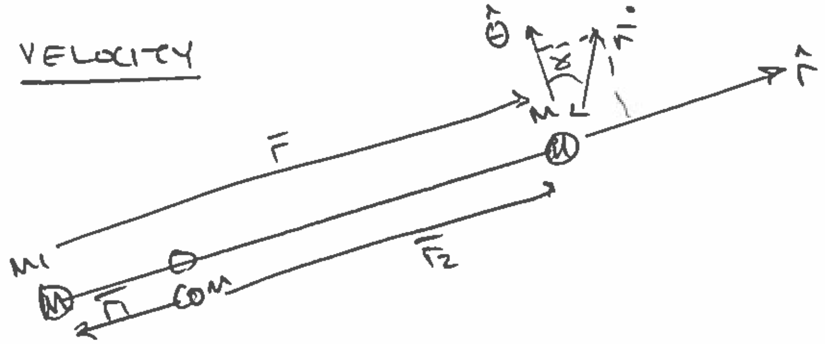
$\bar{h} = \bar{r} \times \dot{\bar{r}}$ FIXED – NORMAL TO PLANE

$\bar{e} = \dfrac{\dot{\bar{r}} \times \bar{h}}{\mu} - \dfrac{\bar{r}}{r}$ ECCENTRICITY – POINTS TO PERIAPSIS LIES IN THE PLANE

$\bar{h}, \bar{e} \Rightarrow$ DEPEND ON EACH OTHER $\Rightarrow$ DEFINE SIZE, SHAPE + ORIENTATION OF CONIC WRT FOCUS

$\theta \Rightarrow$ TIME – POSITION IN ORBIT

VELOCITY



$$\bar{v} = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = V_R\hat{r} + V_\theta\hat{\theta}$$

$$\bar{h} = \bar{r} \times \dot{\bar{r}} = r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = r^2\dot{\theta}\hat{h} \quad \text{FIXED}$$

$$V_\theta = r\dot{\theta} = \frac{h}{r} = h\frac{1+e\cos\theta}{h^2/\mu} \quad \boxed{\frac{\mu}{h}(1+e\cos\theta) = V_\theta}$$

$$V_R = \dot{r} = \frac{\partial r}{\partial\theta}\frac{\partial\theta}{\partial t} = -\frac{h^2/\mu(-e\sin\theta)}{(1+e\cos\theta)^2} \times \frac{h}{r^2}$$

$$\boxed{V_R = \frac{\mu}{h} e\sin\theta}$$

$$\boxed{\tan\gamma = \frac{V_R}{V_\theta} \quad \frac{e\sin\theta}{1+e\cos\theta}}$$ FLIGHT PATH ANGLE