MATLAB: Initial Value Problem for Ordinary Differential Equations

Consider an ordinary differential equation

$$\dot{x} = f(t, x), \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. An initial value problem is to find the solution $x(t)$ satisfying the ordinary differential equation and an initial condition $x(0) = x_0$. Matlab provides several solvers for initial value problems. Here, we illustrate an example for a pendulum.

Pendulum Model A pendulum is a point mass connected to a frictionless pivot point by a massless link acting under a gravity. The motion of a pendulum is described by the following differential equation.

$$\ddot{\theta} = -\frac{g}{l}\sin\theta, \tag{2}$$

where θ is the angle of the link from the hanging position, l is the length of the link, and g is the gravitational acceleration.

We will solve an initial value problem of this pendulum model using the Matlab `ode45` function. The initial conditions and the properties of the pendulum is given by

$$\theta(0) = \frac{\pi}{4}, \quad \dot{\theta}(0) = 0, \quad l = 9.81\,\mathrm{m}, \quad g = 9.81\,\mathrm{m/s^2}.$$

Step 1. Standard First-Order Form The first step is to rewrite the differential equation (2) into the standard first-order form (1). Define

$$x_1 = \theta, \quad x_2 = \dot{\theta}. \tag{3}$$

Then, (2) can be written as two first-order differential equations of $x = [x_1, x_2] \in \mathbb{R}^2$.

$$\dot{x}_1 = \dot{\theta} = x_2, \tag{4}$$

$$\dot{x}_2 = \ddot{\theta} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\sin x_1. \tag{5}$$

These equations can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}, \tag{6}$$

which has the same form as (1).

Step 2. Matlab Function for f The next step is writing a Matlab function for $f(t, x)$: the input of this function is t and x, and the output is $\dot{x}$. For example, the following function is saved as `eomPend.m`.

```
function dotX=eomPend(t,X)
g=9.81;
l=9.81;

theta=X(1);
dottheta=X(2);

ddottheta=-g/l*theta;

dotX=[dottheta; ddottheta];
```

Step 3. Use `ode45` Function The Matlab function `eomPend.m` is integrated by the Matlab initial value problem solver `ode45`. The syntax is as follows

```
[t,X] = ode45(@odefun,tspan,X0);
```

where `@odefun` is the handle of the differential equation, `tspan=[t0 tf]` specifies the simulation time for the initial time `t0` and the terminal time `tf`. The initial condition is specified by `X0`. Then, it returns the column vector `t` of time points, and the solution array `X`, where each row in `X` corresponds to the solution at a time returned in the corresponding row of `t`.

For our initial value problem for the pendulum, use the following commands.

```
clear all;
close all;

theta0=pi/4;
dottheta0=0;
X0=[theta0; dottheta0];

[t,X]=ode45(@eomPend,[0 10],X0);

theta=X(:,1);
dottheta=X(:,2);

plot(theta,dottheta);
```