## PROBLEM 1

SYSTEM OF 5 PARTICLES

$\hat{x}$ → SUN —— EARTH —— MOON —— S/C —— JUPITER

REF. FRAME $\odot$ $(\oplus)$ $(☾)$ $(♃)$

DISTANCE BTWN EACH BODY IS DEFINED AS THE SEMI-MAJOR AXIS GIVEN ON PLANETARY CONSTANTS TABLE

SO :

$$|\bar{r}_{\odot\oplus}| = a_{\oplus} = 1.495 \, 89800 \, E8 \text{ km}$$

$$|\bar{r}_{\oplus ☾}| = a_{☾} = 3.844 \, E5 \text{ km}$$

$$|\bar{r}_{\odot ♃}| = a_{♃} = 7.78412 \, E8 \text{ km}$$

UNITS!!

$$|\bar{r}_{☾ s/c}| = 150{,}000 \text{ km} \qquad m_{s/c} = 130 \text{ kg}$$

PART A – LOCATE THE CENTER OF MASS

$$G\left(M_T \bar{r}_{cm}\right) = \left(M_{\odot}\bar{r}_{\odot} + M_{\oplus}\bar{r}_{\oplus} + M_{☾}\bar{r}_{☾} + M_{s/c}\bar{r}_{s/c} + M_{♃}\bar{r}_{♃}\right) G$$

$Gm_i$ IS AVAILABLE ON THE CONSTANTS TABLE

$$Gm_T = Gm_{\odot} + Gm_{\oplus} + Gm_{☾} + Gm_{s/c} + Gm_{♃} \quad \left(\text{km}^3/\text{sec}^2\right)$$

IS THE SUM OF $Gm_i$ VALUES

$$Gm_{s/c} = 6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg sec}^2} \cdot 130 \text{ kg}$$

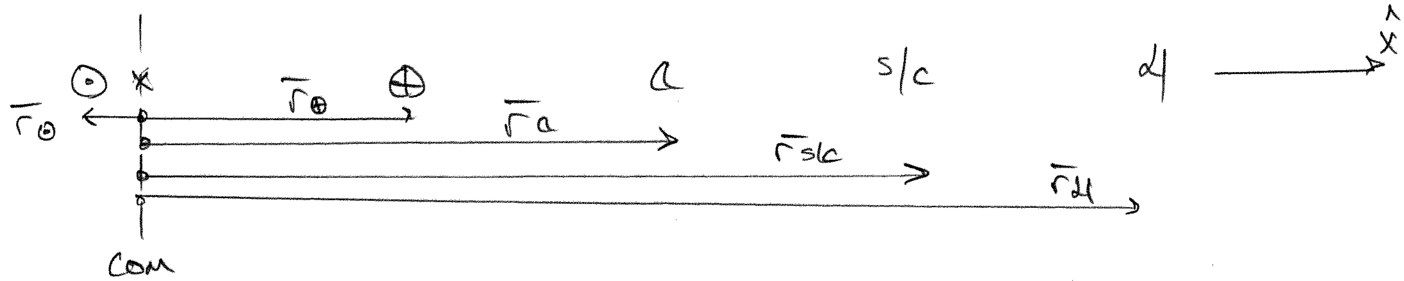
$$= 8.6749 \times 10^{-9} \frac{\text{m}^3}{\text{sec}^2} = 8.6749 \times 10^{-18} \frac{\text{km}^3}{\text{sec}^2}$$

VERY SMALL #

THE ORIGIN OF OUR REF. SYSTEM IS THE SUN.

$\bar{r}_i = \bar{r}_{\odot i} = r_{\odot i}\,\hat{x}$ ← ALL BODIES (i) COLLINEAR

$$\boxed{\bar{r}_{cm} = 742816.309\,\hat{x}\ \text{km} = 1.06728\, r_{\odot}\,\hat{x}}$$ OUTSIDE OF SUN!!




## PART 3

VECTOR ODE FOR s/c

$$\boxed{m_{s/c}\,\ddot{\bar{r}}_{s/c} = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_{s/c}\, m_j}{r_{j\,s/c}^{3}}\, \bar{r}_{j\,s/c}}$$

$j = \odot, \oplus, ☾, ♃$

THE BASE POINT OF $\bar{r}_{s/c}$ IS THE COM SO THE EQN APPLIES AS WRITTEN SINCE THE COM IS INERTIALLY FIXED DUE TO THE CONSERVATION OF LINEAR MOM.

ACCELERATION ON s/c

$$\ddot{\bar{r}}_{s/c} = \frac{-G m_{\odot} \bar{r}_{\odot s/c}}{r_{\odot s/c}^{3}} - \frac{G m_{\oplus} \bar{r}_{\oplus s/c}}{r_{\oplus s/c}^{3}} - \frac{G m_{☾} \bar{r}_{☾ s/c}}{r_{☾ s/c}^{3}} - \frac{G m_{♃} \bar{r}_{♃ s/c}}{r_{♃ s/c}^{3}}$$

NOW IN ORDER OF DESCENDING MAGNITUDE (km/sec²)

SUN $-5.88856 \times 10^{-6}\ \hat{x}$

EARTH $-1.39574 \times 10^{-6}\ \hat{x}$

MOON $-2.17902 \times 10^{-7}\ \hat{x}$

JUPITER $3.20933 \times 10^{-10}\ \hat{x}$

NET ACCELERATION

$$\boxed{\ddot{\bar{r}}_{s/c} = -7.50187 \times 10^{-6}\ \frac{\text{km}}{\text{sec}^2}\ \hat{x}}$$

WHILE THE RESULTS ONLY SHOW 5 DECIMAL PLACES, YOU <u>MUST</u> USE A PROGRAM TO ACHIEVE ACCURACY.

QUANTITIES OF THE ORDER OF $10^{-5}/10^{-6}$ <u>ARE SIGNIFICANT</u>.

BE VERY CAREFUL WITH NUMERICAL ACCURACY !!

<u>PART C</u>

THE SUN DOMINATES THE MOTION OF THE S/C WHILE BOTH THE EARTH/MOON ARE MUCH CLOSER.

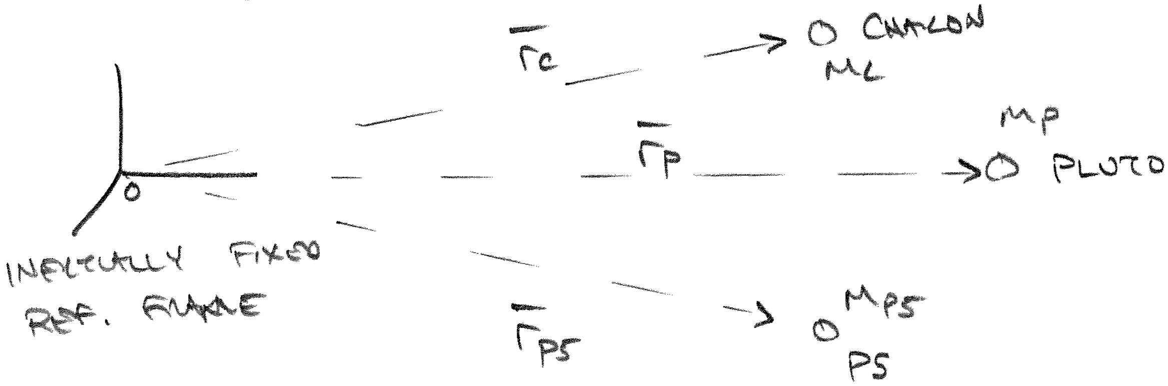
IN THE 5-BODY SYSTEM THE S/C WOULD BE SEEN ORBITING THE SYSTEM C.M IN A LARGE ORBIT.

THUS IF THE S/C IS ORBITING THE COM THEN THE ACCELERATION (NET) WOULD BE DIRECTED TOWARDS THE COM.

ARTEMIS LEVERAGED THE GRAVITY OF THE SUN, EARTH, AND MOON TO BE SUCCESSFUL. ALL WHILE USING VERY LITTLE PROPELLANT.

## PROBLEM 2

SYSTEM COMPOSED OF THREE PARTICLES




PART A VECTOR ODE'S FOR EACH PARTICLE

RECALL THE N-BODY PROBLEM FOR $m_i$

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}$$

FOR PLUTO ($m_p$)

$$\boxed{\begin{aligned} m_p \ddot{\bar{r}}_p &= \frac{-G m_p m_c}{r_{cp}^3} \bar{r}_{cp} - \frac{G m_p m_{ps}}{r_{psp}^3} \bar{r}_{psp} \\ \bar{r}_{cp} &= \bar{r}_p - \bar{r}_c \qquad \bar{r}_{psp} = \bar{r}_p - \bar{r}_{ps} \end{aligned}} \qquad (1)$$

FOR CHARON ($m_c$)

$$\boxed{\begin{aligned} m_c \ddot{\bar{r}}_c &= -\frac{G m_c m_p}{r_{pc}^3} - \frac{G m_{ps} m_c}{r_{psc}^3} \bar{r}_{psc} \\ \bar{r}_{pc} &= \bar{r}_c - \bar{r}_p \qquad \bar{r}_{psc} = \bar{r}_c - \bar{r}_{ps} \end{aligned}} \qquad (2)$$

FOR PS ($m_{ps}$)

$$\boxed{\begin{aligned} m_{ps} \ddot{\bar{r}}_{ps} &= -\frac{G m_c m_{ps}}{r_{cps}^3} \bar{r}_{cps} - \frac{G m_p m_{ps}}{r_{pps}^3} \bar{r}_{pps} \\ \bar{r}_{cps} &= \bar{r}_{ps} - \bar{r}_c \qquad \bar{r}_{pps} = \bar{r}_{ps} - \bar{r}_p \end{aligned}} \qquad (3)$$

ALL VECTORS DEFINED WRT INERTIAL POINT O.

PART B SUM ALL OF THE EQUATIONS

$$M_p\ddot{\bar{r}}_p + M_c\ddot{\bar{r}}_c + M_{ps}\ddot{\bar{r}}_{ps} = -\frac{G m_p m_c}{r_{cp}^3}\bar{r}_{cp} - \frac{G M_{ps} M_p}{r_{psp}^3}\bar{r}_{psp}$$

(*)

$$- \frac{G M_c M_p}{r_{pc}^3}\bar{r}_{pc} - \frac{G M_{ps} M_c}{r_{psc}^3}\bar{r}_{psc}$$

$$- \frac{G m_{ps} M_p}{r_{pps}^3}\bar{r}_{pps} - \frac{G M_c M_{ps}}{r_{cps}^3}\bar{r}_{cps}$$

FROM NEWTON'S THIRD LAW

$$-\frac{G m_p m_c}{r_{cp}^3}\bar{r}_{cp} = +\frac{G m_c m_p}{r_{pc}^3}\bar{r}_{pc}$$

$$-\frac{G m_{ps} M_p}{r_{psp}^3}\bar{r}_{psp} = +\frac{G m_{ps} M_p}{r_{pps}^3}\bar{r}_{pps}$$

$$-\frac{G M_c M_{ps}}{r_{psc}^3}\bar{r}_{psc} = +\frac{G m_c M_{ps}}{r_{cps}^3}\bar{r}_{cps}$$

PLUG BACK INTO (*)

THE ENTIRE RHS CANCELS !

$$M_p\ddot{\bar{r}}_p + m_c\ddot{\bar{r}}_c + M_{ps}\ddot{\bar{r}}_{ps} = \bar{0}$$

INTEGRATE THIS TWICE TO GET

$$\boxed{M_p\dot{\bar{r}}_p + M_c\dot{\bar{r}}_c + M_{ps}\dot{\bar{r}}_{ps} = \bar{C}_1 t}$$

$\bar{C}_1, \bar{C}_2$ ARE VECTOR INTEGRATION CONSTANTS

$$\boxed{M_p\bar{r}_p + M_c\bar{r}_c + M_{ps}\bar{r}_{ps} = \bar{C}_1 t + \bar{C}_2}$$

THE VELOCITY OF THE CENTER OF MASS IS BY DEFINITION

$$\dot{\bar{r}}_{com} = \frac{m_p \dot{\bar{r}}_p + m_{ps} \dot{\bar{r}}_{ps} + m_c \dot{\bar{r}}_{pc}}{m_p + m_{ps} + m_c} = \frac{\bar{C}_1}{m_p + m_{ps} + m_c}$$

$\bar{C}_1$ IS CONSTANT AND ALL $m_i$ ARE CONSTANT $\Rightarrow$

$$\boxed{\dot{\bar{r}}_{com} = \text{CONSTANT}}$$

PART C

CROSS (1) WITH $\bar{r}_p$

$$m_p \ddot{\bar{r}}_p \times \bar{r}_p = -\frac{G m_p m_c}{r_{cp}^3}(\bar{r}_p - \bar{r}_c) \times \bar{r}_p - \frac{G m_p m_{ps}}{r_{psp}^3}(\bar{r}_p - \bar{r}_{ps}) \times \bar{r}_p$$

$$= \frac{G m_p m_c}{r_{cp}^3}(\bar{r}_c - \cancel{\bar{r}_p}) \times \bar{r}_p + \frac{G m_p m_{ps}}{r_{psp}^3}(\bar{r}_{ps} - \cancel{\bar{r}_p}) \times \bar{r}_p$$

NOTE: $m_p \ddot{\bar{r}}_p \times \bar{r}_p = m_p \frac{{}^I d}{dt}(\dot{\bar{r}}_p \times \bar{r}_p)$

$$= m_p(\ddot{\bar{r}}_p \times \bar{r}_p) + m_p\cancel{(\dot{\bar{r}}_p \times \dot{\bar{r}}_p)}$$

$$(4) \Rightarrow m_p \frac{{}^I d}{dt}(\dot{\bar{r}}_p \times \bar{r}_p) = \frac{G m_p m_c}{r_{cp}^3} \bar{r}_c \times \bar{r}_p + \frac{G m_p m_{ps}}{r_{psp}^3} \bar{r}_{ps} \times \bar{r}_p$$

SIMILARLY FOR $m_{ps}, m_c$,

$$(5) \quad m_{ps} \frac{{}^I d}{dt}(\dot{\bar{r}}_{ps} \times \bar{r}_{ps}) = \frac{G m_p m_{ps}}{r_{pps}^3} \bar{r}_p \times \bar{r}_{ps} + \frac{G m_c m_{ps}}{r_{cps}^3} \bar{r}_c \times \bar{r}_{ps}$$

$$(6) \quad m_c \frac{{}^I d}{dt}(\dot{\bar{r}}_c \times \bar{r}_c) = \frac{G m_c m_p}{r_{pc}^3} \bar{r}_p \times \bar{r}_c + \frac{G m_c m_{ps}}{r_{psc}^3} \bar{r}_{ps} \times \bar{r}_c$$

SUM (4) - (6) AND USE $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$

$$\frac{{}^I d}{dt}(m_p \dot{\bar{r}}_p \times \bar{r}_p + m_{ps} \dot{\bar{r}}_{ps} \times \bar{r}_{ps} + m_c \dot{\bar{r}}_c \times \bar{r}_c) = \bar{0}$$

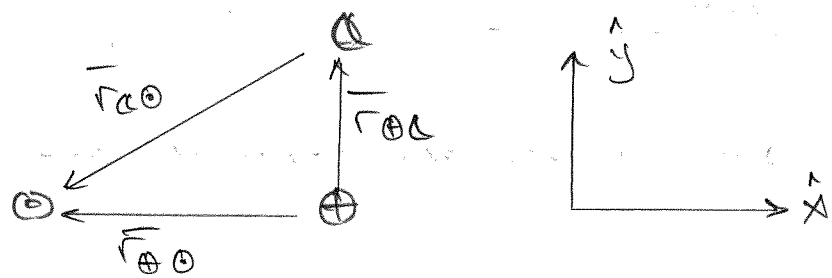
INTEGRATE ONCE TO GET

$$\boxed{\bar{r}_p \times M_p \dot{\bar{r}}_p + \bar{r}_c \times M_c \dot{\bar{r}}_c + \bar{r}_{ps} \times M_{ps} \dot{\bar{r}}_{ps} = \bar{C}_3}$$

SYSTEM ANGULAR MOMENTUM VECTOR IS CONSERVED.

PROBLEM 3

SYSTEM OF THREE PARTICLES




EOM OF MOON RELATIVE TO THE EARTH

$$\boxed{\bar r_{\oplus\leftmoon} + \frac{G(M_\oplus + M_\leftmoon)}{r_{\oplus\leftmoon}^3}\bar r_{\oplus\leftmoon} = GM_\odot\left(\frac{\bar r_{\leftmoon\odot}}{r_{\leftmoon\odot}^3} - \frac{\bar r_{\oplus\odot}}{r_{\oplus\odot}^3}\right)}$$

DOMINANT DIRECT INDIRECT

PART B

DOMINANT ACCELERATION DUE TO EARTH ON THE MOON
THIS WILL BE IN THE $-\hat y$ DIRECTION

$$-\frac{G(M_\oplus + M_\leftmoon)}{r_{\oplus\leftmoon}^3}\bar r_{\oplus\leftmoon} \Rightarrow \boxed{-2.7307\times10^{-6}\ \frac{km}{sec^2}\ \hat y \quad \text{DOMINANT}}$$

DIRECT PERTURBATION IS DUE TO THE SUN ON THE
MOON THIS WILL BE IN BOTH THE $-\hat x, -\hat y$ DIRECTIONS

$$GM_\odot\frac{\bar r_{\leftmoon\odot}}{r_{\leftmoon\odot}^3} = \frac{GM_\odot(-\bar r_{\oplus\leftmoon} + \bar r_{\oplus\odot})}{\|-\bar r_{\oplus\leftmoon} + \bar r_{\oplus\odot}\|^3} =$$

$$\boxed{-5.9306\times10^{-6}\ \hat x - 1.5239\times10^{-8}\ \hat y\ \ km/sec^2} \quad \text{DIRECT}$$

INDIRECT PERTURBATION IS DUE TO THE SUN ON THE EARTH.
THIS WILL BE IN THE $-\hat{x}$ DIRECTION

$$\frac{G m_{\odot} \bar{r}_{\oplus\odot}}{r_{\oplus\odot}^3} = \boxed{-5.9307 \times 10^{-6} \hat{x} \ \frac{km}{sec^2} \quad \text{INDIRECT}}$$

COMPARING THE MAGNITUDES IN DESCENDING ORDER

| | | |
|---|---|---|
| INDIRECT | $5.9307 \times 10^{-6}$ | |
| DIRECT | $5.930665 \times 10^{-6}$ | $\frac{km}{sec^2}$ |
| DOMINANT | $2.730736 \times 10^{-6}$ | |

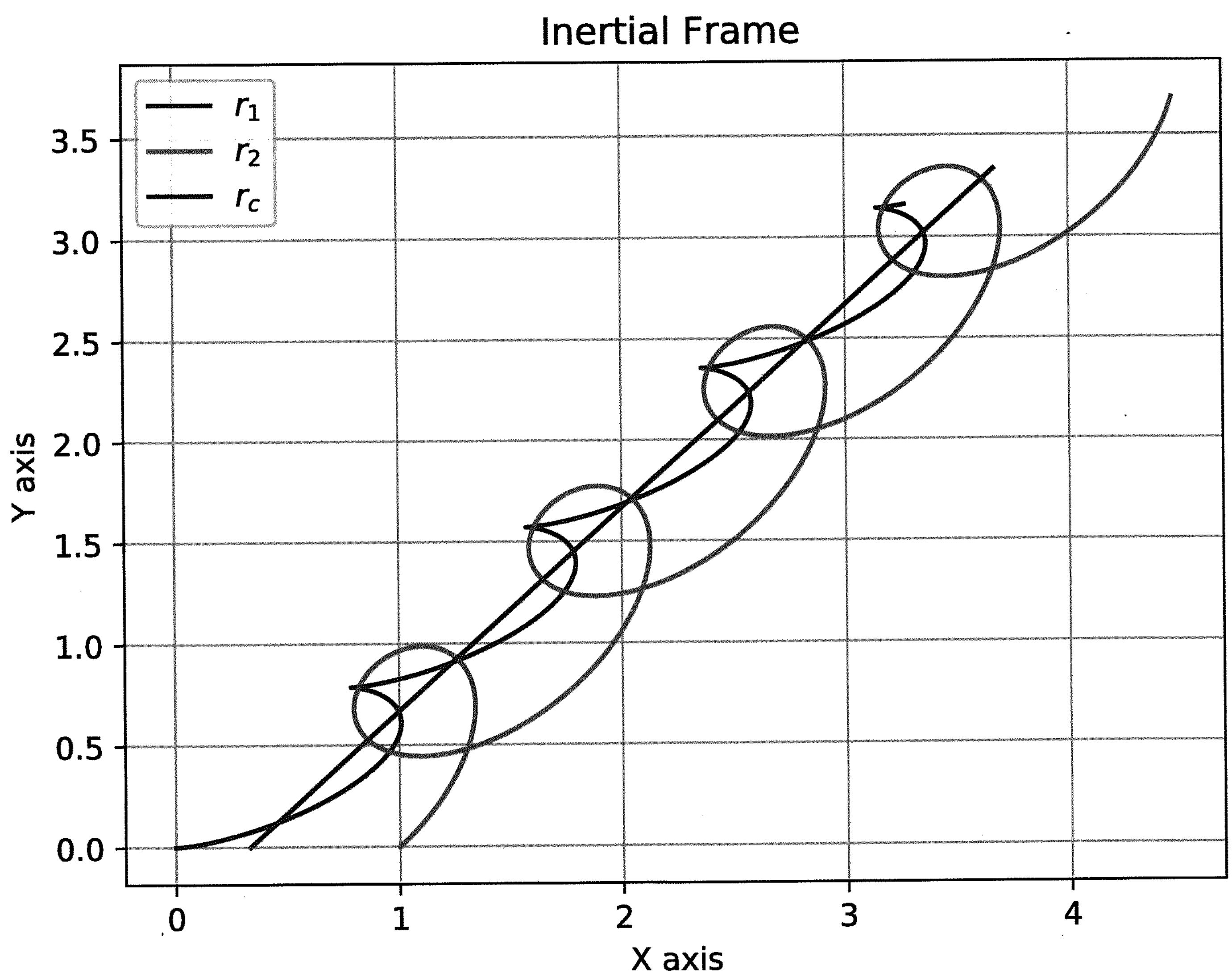
NET PERTURBING ACCEL ON SPACECRAFT / MOON

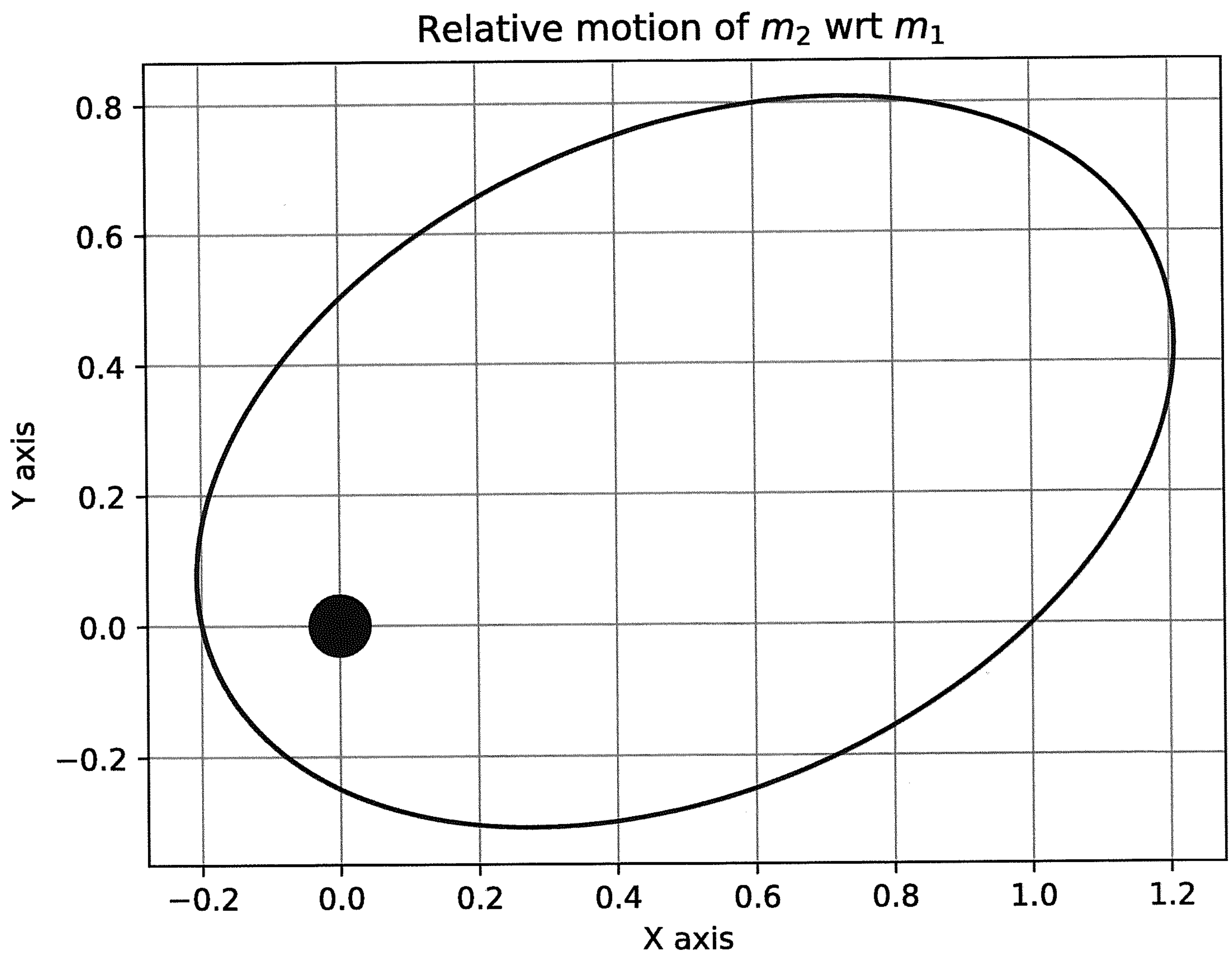
$$\bar{a}_{pert} = 5.8743 \times 10^{-11} \hat{x} - 1.523994 \times 10^{-8} \hat{y}$$

FROM THIS ANALYSIS THE ACCELERATIONS DUE TO THE SUN IS LARGER THAN THAT OF THE EARTH.

IF THIS WERE MODELLED AS A 2BP, INCLUDING ONLY THE EARTH, THIS WOULD IGNORE THE LARGEST CONTRIBUTOR OF ACCELERATION (SUN).

WITHOUT CONSIDERING THE SUN, THE MOTION OF THE MOON CANNOT BE ACCURATELY PREDICTED!


Inertial Frame
$r_1$
$r_2$
$r_c$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Y axis
0
1
2
3
4
X axis



Relative motion of $m_2$ wrt $m_1$
Y axis
0.8
0.6
0.4
0.2
0.0
−0.2
X axis
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2