HW 2 SOLUTION 9/25/17

CONSIDER JUNO ENTERING JOVIAN SYSTEM

$\odot$ ———— s/c ———— CALLISTO ———— $\text{♃} \longrightarrow \hat{x}$

ASSUME ALL BODIES ON CIRCULAR ORBITS W/ RADIUS EQUAL TO SEMI-MAJOR AXIS

DIST BTWN S/C + CALLISTO IS 4000 km

POSITION VECTORS RELATIVE TO JUPITER

$\bar{r}_{\text{♃}c} = -1.883 \times 10^6 \,\hat{x}$ km (CALLISTO)

$\bar{r}_{\text{♃}\odot} = -7.78412 \times 10^8 \,\hat{x}$ km (SUN)

$\bar{r}_{\text{♃}s/c} = -\bar{r}_{\text{♃}c} - 4000\,\hat{x} = -1.887 \times 10^6$ km $\hat{x}$ (S/C)

$\mu_J = GM_J = 1.26687 \times 10^8 \text{ km}^3/\text{sec}^2$

$\mu_c = GM_c = 7179.29 \text{ km}^3/\text{sec}^2$

$\mu_s = GM_s = 1.32712 \times 10^{11} \text{ km}^3/\text{sec}^2$

EQUATIONS OF MOTION OF JUNO WRT JUPITER

RELATIVE N-BODY PROBLEM

$$\ddot{\bar{r}}_{\text{♃}s/c} + \frac{G(m_{s/c} + m_{\text{♃}})}{r_{\text{♃}s/c}^3}\bar{r}_{\text{♃}s/c} = G\sum_{\substack{k=1\\ k\neq s/c,\text{♃}}}^{4}\left(\frac{\bar{r}_{s/ck}}{r_{s/ck}^3} - \frac{\bar{r}_{\text{♃}k}}{r_{\text{♃}k}^3}\right)$$

EXPANDING SUMMATION

$$\underbrace{\ddot{\bar{r}}_{\text{♃}s/c} + \frac{G(m_{s/c} + m_{\text{♃}})}{r_{\text{♃}s/c}^3}\bar{r}_{\text{♃}s/c}}_{\text{DOMINANT}} = GM_\odot\left(\underbrace{\frac{\bar{r}_{s/c\odot}}{r_{s/c\odot}^3}}_{\text{DIRECT SUN ON S/C}} - \underbrace{\frac{\bar{r}_{\text{♃}\odot}}{r_{\text{♃}\odot}^3}}_{\text{INDIRECT SUN ON JUPITER}}\right) + GM_c\left(\underbrace{\frac{\bar{r}_{s/cc}}{r_{s/cc}^3}}_{\text{DIRECT CALLISTO ON S/C}} - \underbrace{\frac{\bar{r}_{\text{♃}c}}{r_{\text{♃}c}^3}}_{\text{INDIRECT CALL. ON S/C}}\right)$$

SINCE MASS JUPITER >> MASS JUNO

WE ASSUME $G(m_{s/c} + m_{4}) \approx G m_{4} \approx \mu_{4}$

COMPUTE THE ACCELERATIONS:

DOMINANT $\bar{a}_{DOM} = -\frac{G(m_{s/c} + m_{4})}{r_{4s/c}^2} \hat{r}_{4s/c}$

$= \frac{-G m_{4}}{r_{4s/c}^2}(-\hat{x}) = \boxed{3.557855 \times 10^{-5} \hat{x} \frac{km}{sec^2}}$

SUN DIRECT: $\bar{a}_{s,D} = \frac{G m_{\odot}}{r_{s/c\,s}^2}(-\hat{x}) = \boxed{-2.200895 \times 10^{-7} \hat{x} \; km/sec^2}$

INDIRECT: $\bar{a}_{s,i} = -\frac{G m_{\odot}}{r_{4\odot}^2}(-\hat{x}) = \boxed{2.190237 \times 10^{-7} \hat{x} \; km/sec^2}$

CALLISTO DIRECT $\bar{a}_{c,d} = \frac{G m_c}{r_{s/c\,c}^2}(\hat{x}) = \boxed{4.487056 \times 10^{-4} \hat{x} \frac{km}{sec^2}}$

INDIRECT $\bar{a}_{c,i} = -\frac{G m_c}{r_{4c}^2}(-\hat{x}) = \boxed{2.024794 \times 10^{-9} \hat{x} \frac{km}{sec^2}}$

NET ACCEL DUE TO SUN

$\bar{a}_s = \bar{a}_{s,D} + \bar{a}_{s,i} = -1.065774 \times 10^{-9} \hat{x} \frac{km}{sec^2}$

NET ACCEL DUE TO CALLISTO

$\bar{a}_c = \bar{a}_{c,d} + \bar{a}_{c,i} = 4.487076 \times 10^{-4} \hat{x} \; km/sec^2$

CALLISTO HAS A LARGER EFFECT THAN THE SUN. THE ACCEL MAGNITUDE IS 5 ORDERS OF MAGNITUDE LARGER. DESPITE THE SMALL MASS, THE MUCH CLOSER CALLISTO IS MUCH MORE SIGNIFICANT.

THE SMALLEST ACCEL IS THE INDIRECT ACCEL OF CALLISTO WHILE THE LARGEST IS THE DIRECT ACCEL OF CALLISTO ON THE S/C. THIS A DIFF. OF 5 ORDERS OF MAGNITUDE!

THE DOMINANT ACCEL (JUPITER) IS 1 ORDER OF MAGNITUDE SMALLER THAN THE DIRECT ACCEL OF CALLISTO $\Longrightarrow$ JUPITER IS NOT THE LARGEST ACCEL.

TWO BODY APPROXIMATION (JUPITER + JUNO) IS NOT APPROPRIATE

B THE PERTURBING ACCELERATIONS FROM ABOVE ARE:

$$\boxed{\begin{aligned} \bar{a}_S &= -1.065774 \times 10^{-9} \ \frac{km}{sec^2} \ \hat{x} \\ \bar{a}_C &= 4.487076 \times 10^{-4} \ \frac{km}{sec^2} \ \hat{x} \end{aligned}}$$

$\leftarrow$ LARGEST

CALLISTO PULLS THE S/C TOWARDS JUPITER WHILE THE SUN PULLS S/C AWAY FROM JUPITER

EVEN WITH A SMALL MASS, CALLISTO HAS A MUCH LARGER ACCEL., EVEN COMPARED TO JUPITER.

WE HAVE AN INVERSE SQUARE GRAVITY LAW

$$a \propto \frac{GM}{r^2} \hat{r}$$

THE ROLE OF DISTANCE (r) PLAYS A MUCH MORE SIGNIFICANT PART, AS COMPARED TO MASS (PARABOLIC VS. LINEAR)

MUCH SHORTER DISTANCE $>$ MUCH LARGER MASS (IN THIS CASE)

C GIVEN THE FOUR BODIES + ACCEL. CALCULATED

A TWO-BODY MODEL IS NOT SUFFICIENT.

THE PERTURBING ACCEL. ARE ACTUALLY LARGER THAN THE DOMINANT ACCEL - AND ARE SIGNIFICANT!

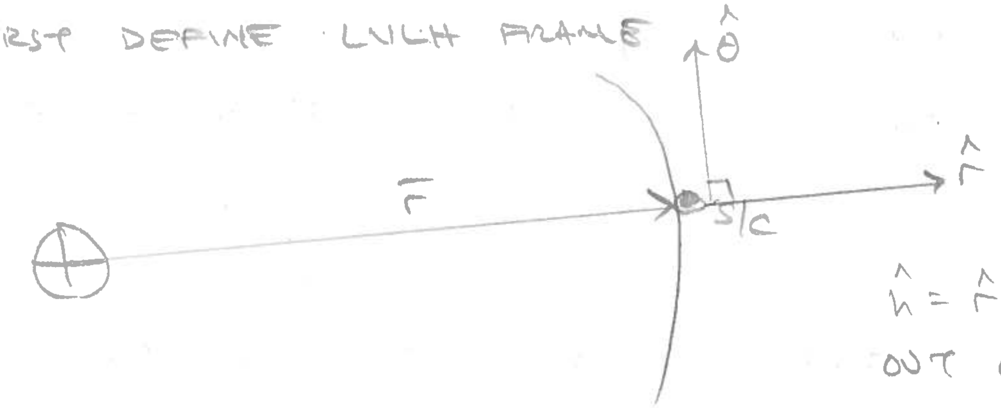
WITH ONLY 3 BODIES I WOULD CHOOSE

JUPITER, CALLISTO, S/C TO KEEP

AT THIS INSTANCE.

PROBLEM 2 :

FIRST DEFINE LVLH FRAME



$\hat{h} = \hat{r} \times \hat{\theta}$

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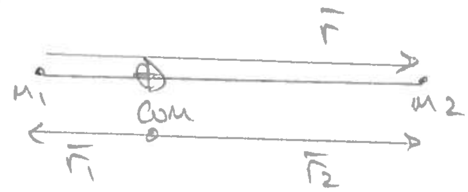
FIRST CONSIDER THE RELATIVE POSITION VECTOR OF S/C WRT EARTH

THE CENTER OF MASS



$$\bar{r}_{COM} = \frac{m_2}{m_1 + m_2} \bar{r}$$

WHICH GIVES

$$\bar{r}_1 = -\frac{m_2}{m_1+m_2} \bar{r} \qquad \bar{r}_2 = \frac{m_1}{m_1+m_2} \bar{r}$$

ALSO $\bar{r} = r\hat{r} = 2000\,km + R_{\oplus} = 8378.14\ km\ \hat{r}$

VELOCITY $\mathring{\bar{r}} = \mathring{r}\hat{r} + r\mathring{\theta}\hat{\theta} = -1.2\hat{r} + 6.7\hat{\theta}\ \frac{km}{sec^2}$

$\bar{C}_3$ SYSTEM ANGULAR MOMENTUM

$$\bar{C}_3 = m_1(\bar{r}_1 \times \mathring{\bar{r}}_1) + m_2(\bar{r}_2 \times \mathring{\bar{r}}_2) \leftarrow \text{PLUG IN } \bar{r}_1, \bar{r}_2 \text{ FROM ABOVE}$$

$$= \frac{m_1 m_2}{m_1+m_2}(\bar{r} \times \mathring{\bar{r}}) = \frac{m_1 m_2}{m_1+m_2} r^2 \mathring{\theta}\,\hat{h}$$

$r = 8378.14\ km$

$r\mathring{\theta} = 6.7\ \frac{km}{sec}$

$\Bigg\}$ $m_1 = \frac{GM_{\oplus}}{G} = 5.974 \times 10^{24}\ kg$

$m_2 = 500\ kg$

VECTOR !!

$$\Longrightarrow \boxed{\bar{C}_3 = 2.80668 \times 10^7\ \hat{h}\ \frac{km^2 kg}{sec}}$$

SPECIFIC ANGULAR MOMENTUM

$$|\bar{h}| = |\bar{r} \times \dot{\bar{r}}| = \frac{m_1 + m_2}{m_1 m_2} |\bar{C}_3|$$

$$= r^2 \dot{\theta} = \boxed{5.61335 \times 10^4 \ \frac{km^2}{sec} = |\bar{h}|}$$

TOTAL KINETIC ENERGY

$$T = \frac{1}{2} m_1 (\dot{\bar{r}}_1 \cdot \dot{\bar{r}}_1) + \frac{1}{2} m_2 (\dot{\bar{r}}_2 \cdot \dot{\bar{r}}_2) \quad \leftarrow \text{PLUG IN } \bar{r}_1, \bar{r}_2$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\dot{\bar{r}} \cdot \dot{\bar{r}}) = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$\boxed{T = 1.15825 \times 10^4 \ \frac{km^2 \, kg}{sec^2}}$$

$$C_4 = T - U = 1.15825 \times 10^4 - \frac{G m_1 m_2}{r}$$

$$\boxed{C_4 = -1.22056 \times 10^4 \ \frac{km^2 \, kg}{sec^2}} \qquad \text{SCALAR!!}$$

SPECIFIC ENERGY

$$\mathcal{E} = C_4 \frac{m_1 + m_2}{m_1 m_2} \Rightarrow \boxed{\mathcal{E} = -24.4112 \ \frac{km^2}{sec^2}} \qquad \boxed{\frac{m_1 + m_2}{m_1 m_2} = 2 \times 10^{-3}}$$

AREAL VELOCITY COMPUTED FROM $\hat{h}$

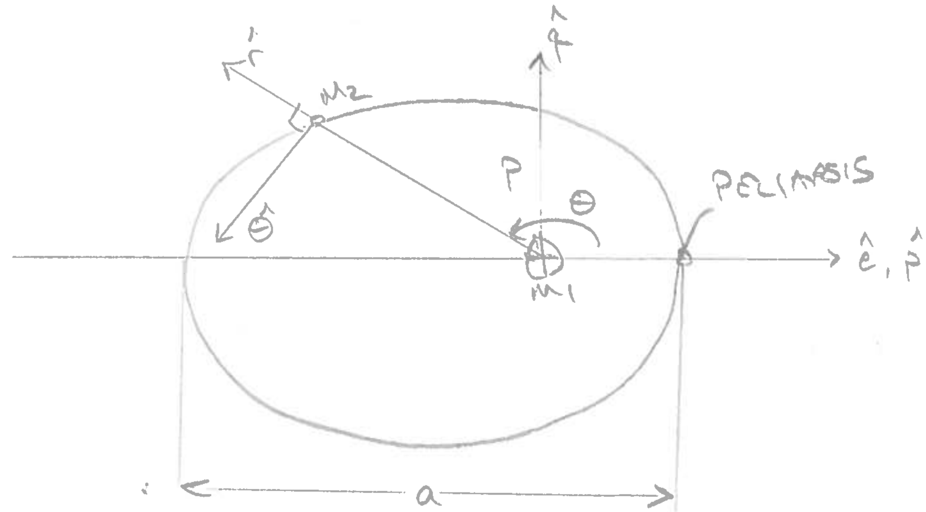
$$\frac{dA}{dt} = \frac{h}{2} \rightarrow \frac{dA}{dt} = \boxed{2.80668 \times 10^4 \ \frac{km^2}{sec}}$$

B FIND THE FOLLOWING CHARACTERISTICS IN RELATIVE 2BP

$p, e, a, \mathbb{P}, \theta$

POSITION IN PERIFOCAL FRAME

RECALL OUR ELLIPTICAL CONIC SECTION



$$p = \frac{h^2}{\mu} \Rightarrow \boxed{7905.10 \text{ km}}$$

$$e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}} = \boxed{0.17817}$$

$$a = \frac{p}{(1-e^2)} = \boxed{8164.29 \text{ km}}$$

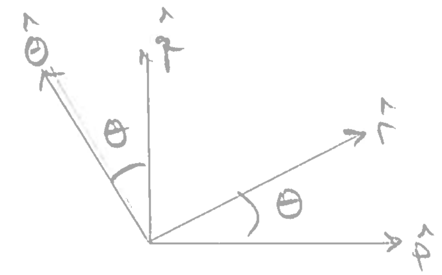
$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = \boxed{7341.57 \text{ sec} = 2.0393 \text{ hrs}}$$

$$r = \frac{p}{1+e\cos\theta} \Longrightarrow \theta = \cos^{-1}\left(\frac{1}{e}\left(\frac{p}{r} - 1\right)\right) = 108.47° \text{ OR } 251.52°$$

INVERSE TRIG FCNS ARE ALWAYS DOUBLE VALUED !!

VELOCITY IS IN $-\hat{r}$ DIRECTION → DESCENDING → MOVING TOWARD PERIAPSIS → $\boxed{\theta = 251.52°}$

RECALL THE RELATIONSHIP OF LVLH AND PERIFOCAL FRAMES



$$\hat{h} = \hat{p} \times \hat{q} = \hat{r} \times \hat{\theta} = \hat{w}$$

$$\underset{\text{LVLH TO PQW}}{[R]} = \begin{array}{c|ccc} & \hat{r} & \hat{\theta} & \hat{h} \\ \hline \hat{p} & \cos\theta & -\sin\theta & 0 \\ \hat{q} & \sin\theta & \cos\theta & 0 \\ \hat{w} & 0 & 0 & 1 \end{array}$$

$$\boxed{\begin{aligned} \bar{r}_{LVLH} &= 8378.14\,\hat{r}\ \text{km} \\ &= -2654.91\,\hat{p} - 7946.36\,\hat{q}\ \text{km} \end{aligned}}$$

c CIRCULAR RELATIVE VELOCITY AT THIS ALTITUDE

$$\bar{V}_c = \sqrt{\frac{\mu}{r}}\,\hat{\theta} = 6.8975\ \frac{\text{km}}{\text{sec}}\ \hat{\theta}$$

CURRENT VELOCITY IN ORBIT

$$\bar{v} = -1.2\,\hat{r} + 6.7\,\hat{\theta}\ \frac{\text{km}}{\text{sec}} \qquad |\bar{v}| = 6.8061\ \text{km/sec}$$

MAG AT t_0 THE VELOCITY MAG. IS LESS THAN IF IT WERE IN A CIRCULAR ORBIT

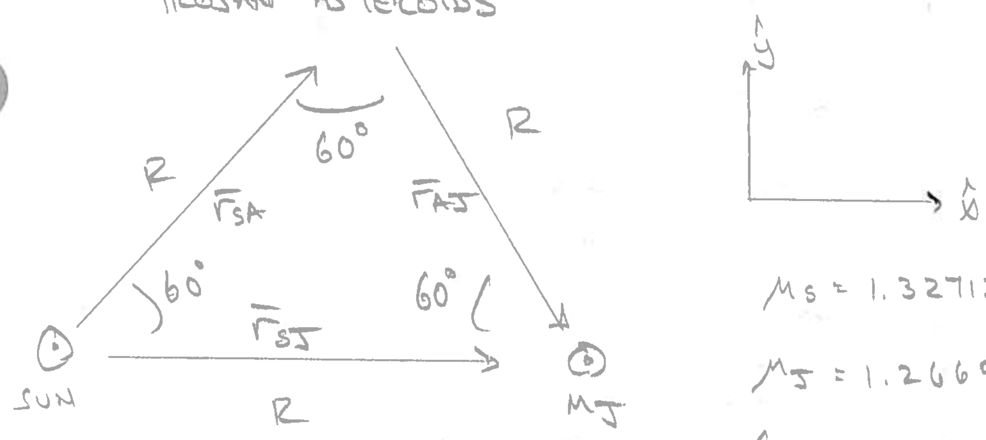
DIRECTION WE'RE DECREASING IN THE RADIAL DIRECTION

→ GETTING CLOSER TO BODY

IN A CIRCULAR ORBIT THERE IS NO VELOCITY IN THE $\hat{r}$ DIRECTION.

PROBLEM 3

TROJAN ASTEROIDS



$\mu_S = 1.32712 \times 10^{11}\ km^3/sec^2$

$\mu_J = 1.26687 \times 10^{8}\ km^3/sec^2$

GM_A = VERY SMALL

TOTAL MASS OF ALL TROJAN ASTEROIDS IS $10^{-4} M_\oplus$

ASSUME ONLY SUN + JUPITER

A MOTION OF ASTEROID RELATIVE TO SUN

$$\ddot{\bar{r}}_{SA} + \frac{G(M_A + M_S)}{r_{SA}^3}\bar{r}_{SA} = GM_J\left(\frac{\bar{r}_{AJ}}{r_{AJ}^3} - \frac{\bar{r}_{SJ}}{r_{SJ}^3}\right)$$

FROM DIAGRAM $r_{SA} = r_{AJ} = r_{SJ} = R = 7.78412 \times 10^8$ km

JUPITER SEMI-MAJOR AXIS

$\bar{r}_{SJ} = R\hat{x}$

$\bar{r}_{AJ} = R\cos 60°\,\hat{x} - R\sin 60\,\hat{y} = \frac{R}{2}\hat{x} - \frac{\sqrt{3}}{2}R\hat{y}$

$\bar{r}_{SA} = R\cos 60°\,\hat{x} + R\sin 60°\,\hat{y} = \frac{R}{2}\hat{x} + \frac{\sqrt{3}}{2}R\hat{y}$

USING THESE VECTORS

$$\ddot{\bar{r}}_{SA} + \underbrace{\frac{G(M_A+M_S)}{R^2}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)}_{\text{DOMINANT}} = \underbrace{\frac{GM_J}{R^2}\left(\underbrace{\left(\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right)}_{\text{DIRECT}} - \underbrace{\hat{x}}_{\text{INDIRECT}}\right)}_{\text{NET PERTURBATION}}$$

$$\bar{a}_{DOMINANT} = -\frac{G(\cancel{M_A}+M_S)}{R^2}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) \qquad M_A << M_S$$

$$= -1.095119 \times 10^{-7}\hat{x} - 1.896801 \times 10^{-7}\hat{y} \ \frac{km}{sec}$$

$$|\bar{a}_{DOM}| = 2.190237 \times 10^{-7} \ km/sec^2$$

NET PERTURBATION

$$\bar{a}_p = \frac{GM_J}{R^2}\left[-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]$$

$$= -1.045401 \times 10^{-10}\hat{x} - 1.810688 \times 10^{-10}\hat{y} \ km/sec^2$$

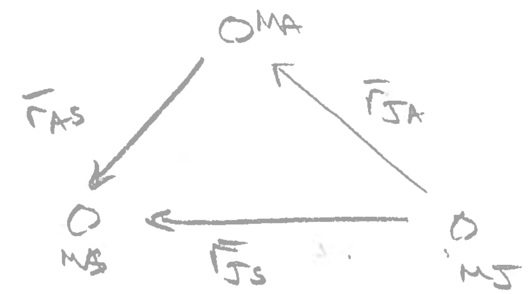
$$|\bar{a}_p| = 2.090802 \times 10^{-10} \ km/sec^2$$

DOMINANT + PERTURBATION IN THE SAME DIRECTION

BOTH PULL TROJAN + SUN CLOSER TOGETHER

THE INFLUENCE OF THE SUN IS 3 ORDERS OF MAGNITUDE LARGER THAN THAT OF JUPITER → DUE TO MUCH LOWER MASS OF JUPITER.

B REFORMULATE PROBLEM – CONSIDER MOTION OF ASTEROID WRT JUPITER



$$\bar{r}_{JS} = -R\hat{x}$$

$$\bar{r}_{JA} = R\left[-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right]$$

$$\bar{r}_{AS} = R\left[-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]$$

RELATIVE EOMS OF AST. WRT JUPITER

$$\underbrace{\ddot{\bar{r}}_{JA} + \frac{G(M_A + M_J)}{r_{JA}^3}\bar{r}_{JA}}_{\text{DOMINANT}} = \underbrace{G M_S\left(\underbrace{\frac{\bar{r}_{AS}}{r_{AS}^3}}_{\text{DIRECT}} - \underbrace{\frac{\bar{r}_{JS}}{r_{JS}^3}}_{\text{INDIRECT}}\right)}_{\text{NET PERTURBATION}}$$

PLUGGING IN OUR VECTORS + COMPUTING

$$\bar{a}_{DOM} = \frac{-G(M_A + M_J)}{R^2}\left[-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right]$$

$$= 1.045401 \times 10^{-10}\,\hat{x} - 1.810688 \times 10^{-10}\,\hat{y} \quad \frac{km}{sec^2}$$

$$|\bar{a}_{DOM}| = 2.090802 \times 10^{-10} \text{ km/sec}$$

NET PERTURBATION

$$\bar{a}_p = \frac{G M_S}{R^2}\left[\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right]$$

$$= 1.095119 \times 10^{-7}\,\hat{x} - 1.896801 \times 10^{-7}\,\hat{y} \quad \frac{km}{sec^2}$$

$$|\bar{a}_p| = 2.190237 \times 10^{-7} \text{ km/sec}^2$$

THE SUN TENDS TO DRIVE THE ASTEROID TOWARDS JUPITER

SUN PERTURBATION > JUPITER ACCEL BY 3 ORDERS OF MAGNITUDE

C EACH FORM REPRESENTS THE MOTION OF THE ASTEROID WRT TO SOME BODY. BOTH ARE EQUALLY VALID

THE DIFFERENCE IS THE BASE POINT OF THE ASTEROID POSITION. WE CAN EXPLOIT THE GEOMETRY TO EASILY DEFINE THE POSITION RELATIVE TO EITHER BODY.