MAE3145: Homework 0 - Skills Review

Due date: September 8, 2017

Problem 1. The magnitude of a vector is defined as

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

for the vector

$$\vec{a} = \begin{bmatrix} a_1 & a_2 & a_3. \end{bmatrix}$$

(a) Find the magnitude of $\|\vec{a}\|$ where $\vec{a} = \begin{bmatrix} \sqrt{5} & \sqrt{3} & 1 \end{bmatrix}$.

(b) Find the magnitude of $\left\|\vec{b}\right\|$ where $\vec{b} = \begin{bmatrix} -2 & 4 & -4 \end{bmatrix}$.

(c) Find the magnitude of $\|\vec{c}\|$ where $\vec{c} = \begin{bmatrix} 0 & 0 & -9 \end{bmatrix}$.

Problem 2. Consider two vectors defined as

$$\vec{a} = \begin{bmatrix} 0 & 3000 & 0 \end{bmatrix},$$
$$\vec{b} = \begin{bmatrix} 4000 & 0 & 0 \end{bmatrix}.$$

(a) Find $\vec{c} = \vec{a} + \vec{b}$.

(b) Find $\|\vec{c}\|$.

(c) Find $\|\vec{a}\| + \left\|\vec{b}\right\|$.

(d) True or False. $\|\vec{a}\| + \left\|\vec{b}\right\| > \|\vec{c}\|$.

Problem 3. Two vectors $\vec{a}, \vec{b} \in \Re^3$ are given as follows:

$$\vec{a} = [a_1,\, a_2,\, a_3]^T, \quad \vec{b} = [b_1,\, b_2,\, b_3]^T.$$

The dot product of two vectors is a scalar defined as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{1}$$

Let $a = \sqrt{\vec{a} \cdot \vec{a}}$, $b = \sqrt{\vec{b} \cdot \vec{b}}$ be the length of the vectors $\vec{a}$, and $\vec{b}$, respectively. Let θ be the angle between $\vec{a}$ and $\vec{b}$. The dot product can be re-written in these variables as

$$\vec{a} \cdot \vec{b} = ab\cos\theta \tag{2}$$

(a) Suppose that $\vec{a} \cdot \vec{b} = 0$. What is the corresponding value of the angle θ. (Assume that $\vec{a} \neq 0$, $\vec{b} \neq 0$, and $0 \leq \theta < 2\pi$.)

(b) Let $\vec{a} = [1,\, 3,\, -2]^T$, $\vec{b} = [-4,\, -1,\, -2]^T$. Find $\vec{a} \cdot \vec{b}$ using Equation (1).

(c) For $\vec{a}$ and $\vec{b}$ defined at (b), find the angle between those two vectors.

(d) Check that your answer of (c) is consistent with your answer to (b).

Problem 4. The cross product of two vectors is another vector defined as

$$\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]^T. \tag{3}$$

Alternatively, it is written as the determinant of the following matrix:

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{4}$$

We can show that its magnitude is given by

$$\|\vec{a} \times \vec{b}\| = ab\sin\theta, \tag{5}$$

and the direction of $\vec{a} \times \vec{b}$ is determined by the right-handed rule. Note that $\vec{a} \times \vec{b}$ is always perpendicular to either $\vec{a}$, $\vec{b}$, or the plane spanned by $\vec{a}$ and $\vec{b}$.

(a) Suppose that $\vec{a} \times \vec{b} = \vec{0}$. What is the corresponding value of the angle θ. (Assume $0 \leq \theta < 2\pi$.)

For the remaining parts (b)-(e), let $\vec{a} = [1,\ 3,\ -2]^T$, $\vec{b} = [-4,\ -1, -2]^T$.

(b) Show that $\vec{c} = \vec{a} \times \vec{b} = [-8,\ 10,\ 11]^T$ using Equation (3) or Equation (4). Also compute its length using $c = \sqrt{\vec{c} \cdot \vec{c}}$.

(c) Compute the length $\|\vec{a} \times \vec{b}\|$ using Equation (5), and show that it is equal to your second answer to (b).

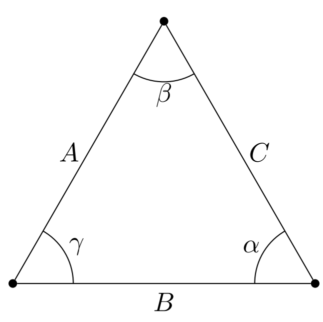
(d) Find the angle between $\vec{a}$ and $\vec{c}$.

Problem 5.

(a) Plot a cosine curve by hand and label both axes as well as the minimum and maximum values.

(b) Plot a sine curve by hand and label both axes as well as the minimum and maximum values.

Problem 6. Write the law of cosines relating each of the three sides and one interior angle of a triangle. You should show three equations defining the relationships between each angle and the other sides of the triangle.



(a) Show that the law of cosines reduces to the well-known Ptyhagorean formula for a right triangle.

Consider three vectors defined as

$$\vec{a} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix},$$
$$\vec{b} = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix},$$
$$\vec{c} = \begin{bmatrix} -3 & 0 & 4 \end{bmatrix}.$$

(a) Find the angle between $\vec{a}$ and $\vec{b}$.

(b) Find the angle between $\vec{b}$ and $\vec{c}$.

(c) Find the angle between $\vec{a}$ and $\vec{c}$.

Problem 7. In the matrix formula, $Ax = b$, where $b \in \mathbb{R}^{n\times 1}$ is a column vector, what are the dimensions of A and x. You can assume that A is a square matrix.

Problem 8. Consider the matrix

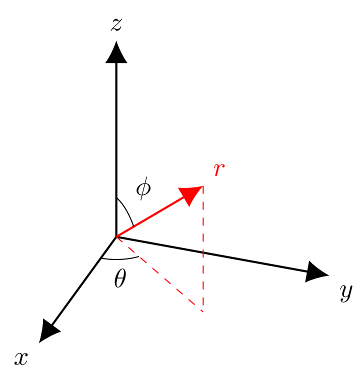
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6}$$

(a) What is A^{-1}?

(b) What is $A^{-1}A$?

(c) What is AA^{-1}?

Problem 9. Convert the vector $x = \begin{bmatrix} 6 & \frac{\pi}{2} & \pi \end{bmatrix}$ from spherical to rectangular coordinates. The spherical coordinates are defined as $x = \begin{bmatrix} r & \phi & \theta \end{bmatrix}$ with the angles given in radians.



(a) Sketch the vector.

(b) What is the magnitude of the vector?

Problem 10. Consider the function

$$y = \exp x. \tag{7}$$

(a) What is the Taylor Series approximation of $y(x)$ about the point $x = a$?

(b) Using the approximation, find $y(0.1)$ using a third order Taylor series about the point $a = 0.0$.

(c) Estimate the error of your answer and compare it to the true value of $y(0.1)$.

Problem 11. The following vector identities will be used later in this class.

$$\vec{x} \cdot (\vec{y} \times \vec{z}) = \vec{y} \cdot (\vec{z} \times \vec{x}) = \vec{z} \cdot (\vec{x} \times \vec{y}), \tag{8}$$
$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}. \tag{9}$$

(a) Using Equation (8), show that $\vec{x}$ is perpendicular to $\vec{x} \times \vec{y}$, i.e., show $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0$.

(b) Suppose that $\vec{q}$ is a unit vector, i.e., $\|\vec{q}\| = 1$. Show that $-\vec{q} \times (\vec{q} \times \vec{x})$ is the orthogonal projection of $\vec{x}$ to the plane normal to $\hat{q}$, i.e., show that $-\vec{q} \times (\vec{q} \times \vec{x}) = \vec{x} - (\vec{q} \cdot \vec{x})\vec{q}$.

Problem 12. A satellite is on a circular orbit around the Earth, i.e. the trajectory of the satellite is a circle centered at the center of the Earth. We observe that the satellite is located at the following point from the center of the Earth:

$$\vec{r}_A = 3741.7\,\hat{\imath} + 5612.5\,\hat{\jmath} - 1870.8\,\hat{k}\,(\mathrm{km}).$$

Sometime later, we observe that the satellite moved to the following position:

$$\vec{r}_B = -441.8\,\hat{\imath} + 6627.5\,\hat{\jmath} + 2209.2\,\hat{k}\,(\mathrm{km}).$$

(a) Find the radius of the circular orbit. (Specify the units!)

(b) What is the rotation angle of the satellite on its circular orbit, i.e. find the angle between $\vec{r}_A$ and $\vec{r}_B$.

(c) Find a **unit** vector, namely $\hat{h}$ that is perpendicular to the orbital plane. (Hint: find any vector, say $\vec{h}$, that is perpendicular to both of $\vec{r}_A$ and $\vec{r}_B$, and normalize it with its length, i.e. $\hat{h} = \vec{h}/h$, where $h = \sqrt{\vec{h} \cdot \vec{h}}$).

Problem 13. The specific orbital energy is defined as

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}. \tag{10}$$

(a) Using Equation (10), find an equation for v.

(b) Substitute $\varepsilon = \frac{-\mu}{2r}$ into the above expression for v and simplify.

(c) Substitute $\varepsilon = 0$ into your original expression for v and simplify.

(d) If $r = \infty$ is substituted into Equation (10), what is the resulting expression for ϵ.

(e) Suppose $\varepsilon_1 = \varepsilon_2$, this implies that

$$\frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{v_2^2}{2} - \frac{\mu}{r_2}.$$

Given the following parameters, solve for v_2.

$$\begin{aligned} \mu &= 398\,600.5\,\mathrm{km}^3\,\mathrm{s}^{-2}, \\ v_1 &= 6.5\,\mathrm{km\,s}, \\ r_1 &= 20\,000\,\mathrm{km}, \\ r_2 &= \infty. \end{aligned}$$

Problem 14. You will be required to develop your own software tools to peform astrodynamic tasks, including but not limited to:

- Converting between coordinate systems,
- Simulating orbits,
- Computing visibility conditions,

- Predicting satellite passes.

Regardless of your previous/future computing courses here at GWU, we will be using Python to develop our software in this course.

- Python is free and open - it costs nothing to use and you are free to commerically develop using Python
- Python is cross-platform - the skills you learn will apply to any operating system or platform.
- Python is a general purpose language - everyone uses Python, from Google, Instagram, Dropbox.
  - Google Tensorflow - https://www.tensorflow.org/
  - Instagram - https://engineering.instagram.com/web-service-efficiency-at-instagram-with-python-4976d078e366
  - Dropbox - https://github.com/dropbox/dropbox-sdk-python
- Python can do science - NASA uses SciPy to easily enable mathematics, science, and engineering.
  - NASA - https://www.python.org/about/success/usa/
  - Scipy - https://www.scipy.org/

You will need to complete the following steps:

- Install the latest Anaconda onto your system. This includes Python as well as a collection of related packages for science.
  - https://www.continuum.io/downloads
- Use some/all of the following to learn a little about Python and SciPy. You do not need to complete all of them but you should familarize yourself with some of the concepts. The internet is full of useful learning material.
  - Scipy Lecture Notes - http://www.scipy-lectures.org/
  - Numpy for Matlab Users - https://docs.scipy.org/doc/numpy-dev/user/numpy-for-matlab-users.html
  - Learn Python the Hard Way - https://learnpythonthehardway.org/book/
  - MAE3145 Python Introduction - https://github.com/fdcl-gwu/scientific_python
- Install a real text editor, some options include Atom, Sublime, Vim.

Problem 15. Email a casual photo of yourself to `skulumani@gwu.edu` with your name. Please, make the email subject starts with `MAE3145`. This is NOT mandatory, but it would be much appreciated to help memorize all of your names.