Constants

$$\omega_\oplus = 15\,^\circ\,\text{sidereal hr}^{-1}$$
$$\omega_\oplus = 0.000\,072\,921\,151\,467\,\text{rad solar sec}^{-1}$$
$$\mu_\oplus = 398\,600.5\,\text{km}^3\,\text{s}^{-2}$$
$$R_\oplus = 6378.137\,\text{km}$$
$$e_\oplus = 0.081\,819\,190\,842\,6$$
$$J_2 = 1.082\,63 \times 10^{-3}$$
$$\rho_o = 1.225\,\text{kg}\,\text{m}^{-3}$$

Satellites - Basics

$$0 = \ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} \qquad r = \frac{a\left(1-e^2\right)}{1+e\cos\nu}$$
$$h = rv\cos\gamma = \sqrt{\mu a\left(1-e^2\right)} \qquad \bar{h} = \bar{r}\times\bar{v}$$
$$a = \frac{r_a + r_p}{2} \qquad e = \frac{2c}{2a} = \frac{r_a - r_p}{r_a + r_p}$$
$$r_p = a\left(1-e\right) \qquad r_a = a\left(1+e\right)$$
$$p = a\left(1-e^2\right) = \frac{h^2}{\mu} \qquad \bar{e} = \frac{\bar{v}\times\bar{h}}{\mu} - \frac{\bar{r}}{r}$$
$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad \varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Time

$$\theta_g = \theta_{g_0} + 1.00273790935 \times 2\pi * D$$
$$D = \text{Day Num} - 1 + \frac{\text{HR}}{24} + \frac{\text{MIN}}{1440} + \frac{\text{SEC}}{86400} \quad \text{where D is elapsed days}$$
$$\text{LST} = \theta_g + \lambda_E$$
$$\text{EST} = \text{UT} - 5 \qquad \text{EDT} = UT - 4$$

Preliminary Orbit Determination

$$x = \left|\frac{a_e}{\sqrt{1-e^2\sin^2 L}} + H\right|\cos L$$
$$z = \left|\frac{a_e\left(1-e^2\right)}{\sqrt{1-e^2\sin^2 L}} + H\right|\sin L$$
$$\bar{r}_s = x\cos\text{LST}\hat{\imath} + x\sin\text{LST}\hat{\jmath} + z\hat{k}$$
$$\rho_s = -\rho\cos\alpha\cos\beta$$
$$\rho_e = \rho\sin\alpha\cos\beta$$
$$\rho_z = \rho\sin\beta$$
$$\dot{\rho}_s = -\dot{\rho}\cos\alpha\cos\beta + \rho\dot{\alpha}\sin\alpha\cos\beta + \rho\dot{\beta}\cos\alpha\sin\beta$$
$$\dot{\rho}_e = \dot{\rho}\sin\alpha\cos\beta + \rho\dot{\alpha}\cos\alpha\cos\beta - \rho\dot{\beta}\sin\alpha\sin\beta$$
$$\dot{\rho}_z = \dot{\rho}\sin\beta + \rho\dot{\beta}\cos\beta$$
$$[\text{IJK}] = ROT_3(\text{LST})ROT_2(\text{COLAT})\,[\text{SEZ}]$$

$$\bar{r} = \bar{\rho} + \bar{r}_s$$
$$\bar{v} = \dot{\bar{\rho}} + \bar{\omega}_\oplus \times \bar{r}$$
$$\bar{v}_2 = -\Delta t_{32}\left(\frac{1}{\Delta t_{21}\Delta t_{31}} + \frac{\mu}{12r_1^3}\right)\bar{r}_1 + \left(\Delta t_{32} - \Delta t_{21}\right)\left(\frac{1}{\Delta t_{21}\Delta t_{32}} + \frac{\mu}{12r_2^3}\right)\bar{r}_2 + \Delta t_{21}\left(\frac{1}{\Delta t_{32}\Delta t_{31}} + \frac{\mu}{12r_3^3}\right)\bar{r}_3$$

Transfers

$$\Delta V_{OTB}^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\Delta\gamma$$
$$\tan\gamma = \frac{e\sin\nu}{1+e\cos\nu} \qquad \Delta V_s = 2V_i\sin\frac{\theta}{2}$$
$$\Delta V_{COMB}^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\Delta i$$

Rendezvous

$$\text{TOF} = \pi\sqrt{\frac{a^3}{\mu}} \qquad \omega = \sqrt{\frac{\mu}{r_{circ}^3}}$$
$$\alpha_{lead} = \omega_t \times \text{TOF} \qquad \phi_f = \pi - \alpha_{lead}$$
$$\text{Wait time}_{coplanar} = \frac{\phi_f - \phi_i \pm 2\pi n}{\omega_t - \omega_i}$$
$$\text{Wait time}_{noncoplanar} = \frac{\phi_f - \phi_i \pm 2\pi n}{\omega_t} = \frac{\alpha_i - \alpha_f + 2\pi n}{\omega_t}$$

Keplers Problem

$$n = \sqrt{\frac{\mu}{a^3}} \qquad M_f = M_i + n \times \text{TOF} - 2k\pi$$
$$M = E - e\sin E$$
$$\cos E = \frac{e+\cos\nu}{1+e\cos\nu} \qquad \cos\nu = \frac{\cos E - e}{1 - e\cos E}$$
$$E_{n+1} = E_n + \frac{M - E_n + e\sin E_n}{1 - e\cos E_n}$$

COEs to RV

$$\bar{r} = r\left[\cos\nu\hat{p} + \sin nu\hat{q}\right]$$
$$\bar{v} = \sqrt{\frac{\mu}{p}}\left[-\sin\nu\hat{p} + (e+\cos\nu)\hat{q}\right]$$
$$[\text{IJK}] = ROT_3(\Omega)ROT_1(i)ROT_3(\omega)\,[PQW]$$

Perturbations

$$\bar{n} = n_0 \left[1 + \frac{3}{2} J_2 \left(\frac{R_\oplus}{p_0}\right)^2 \sqrt{1-e_0^2}\left(1 - \frac{3}{2}\sin^2 i_0\right)\right]$$

$$n = n_0 + \dot{n}_0 \Delta t$$

$$\dot{e}_{drag} = \frac{-2(1-e_0)\dot{n}_0}{3\bar{n}}$$

$$e = e_0 + \dot{e}_{drag}\Delta t$$

$$\dot{\Omega}_{J_2} = \left[-\frac{3}{2} J_2 \left(\frac{R_\oplus}{p_0}\right)^2 \cos i_0\right]\bar{n}$$

$$\Omega = \Omega_0 + \dot{\Omega}_{J_2}\Delta t$$

$$\dot{\omega}_{J_2} = \left[\frac{3}{2} J_2 \left(\frac{R_\oplus}{p_0}\right)^2 \left(2 - \frac{2}{5}\sin^2 i_0\right)\right]\bar{n}$$

$$\omega = \omega_0 + \dot{\omega}_{J_2}\Delta t$$

$$M = M_0 + n_0 \Delta t + \frac{\dot{n}_0}{2}\Delta t^2$$

$$\bar{a}_{drag} = -\frac{1}{2}\rho \frac{C_D A}{m} v\bar{v}$$

Entry - $p_0 = 1.225\,\mathrm{kg\,m^{-3}}$

$$\dot{r} = v \sin\phi_E$$

$$\dot{v} = \frac{\rho_0}{2\Delta} v^2 \exp -\beta h \qquad \beta = \frac{1}{7.315\,\mathrm{km}}$$

$$\Delta = \frac{m}{C_D A}$$

$$v = v_e \exp\left(\frac{\rho_0}{2\Delta\beta\sin\phi_E}\exp(-\beta h)\right)$$

$$h_{max_g} = \frac{1}{\beta}\ln\left(\frac{-\rho_0}{\Delta\beta\sin\phi_E}\right)$$

$$max_g = \frac{-\beta v_E^2 \sin\phi_E}{2 g_0 \exp(1)}$$

$$v_{max_g} = v_E \exp(-0.5) \approx 0.61 v_E$$

Proximity Operations

$$x(t) = \frac{\dot{x}_0}{\omega}\sin\omega t - \left(3x_0 + \frac{2\dot{y}_0}{\omega}\right)\cos\omega t + \left(4x_0 + \frac{2\dot{y}_0}{\omega}\right)$$

$$\dot{x}(t) = \dot{x}_0 \cos\omega t + (3\omega x_0 + 2\dot{y}_0)\sin\omega t$$

$$y(t) = \left(6x_0 + \frac{4\dot{y}_0}{\omega}\right)\sin\omega t + \frac{2\dot{x}_0}{\omega}\cos\omega t - (6\omega x_0 + 3\dot{y}_0)\,t + \left(y_0 - \frac{2\dot{x}_0}{\omega}\right)$$

$$\dot{y}(t) = (6\omega x_0 + 4\dot{y}_0)\cos\omega t - 2\dot{x}_0 \sin\omega t - (6\omega x_0 + 3\dot{y}_0)$$

$$z(t) = z_0 \cos\omega t + \frac{\dot{z}_0}{\omega}\sin\omega t$$

$$\dot{z}(t) = -z_0\omega\sin\omega t + \dot{z}_0\cos\omega t$$

Attitude Kinematics

$$ROT_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$ROT_2 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$ROT_3 = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: Assuming column vectors.

Summary of Orbital Properties

Any Type of Orbit

$$\vec{h} = \vec{r} \times \vec{v}, \qquad \vec{r} = r\hat{u}_r \qquad \vec{v} = v_r\hat{u}_r + v_\theta\hat{u}_\theta,$$

$$h = rv\cos\gamma = rv_\theta = r^2\dot{\theta}, \qquad r = \frac{h^2/\mu}{1 + e\cos\theta}, \qquad v_r = \frac{\mu}{h}e\sin\theta = \dot{r},$$

$$\vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}, \qquad r_p = \frac{h^2/\mu}{1+e}, \qquad v_\theta = \frac{\mu}{h}(1 + e\cos\theta) = r\dot{\theta},$$

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2), \; \tan\gamma = \frac{v_r}{v_\theta} = \frac{e\sin\theta}{1 + e\cos\theta}.$$

Circular Orbits: ($e = 0$)

$$v = \sqrt{\frac{\mu}{r}}, \qquad \mathcal{E} = -\frac{\mu}{2r}, \qquad T = \frac{2\pi}{\sqrt{\mu}}r^{3/2}.$$

Elliptic Orbits: ($0 < e < 1$)

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}, \qquad a = \frac{h^2/\mu}{1-e^2} = \frac{1}{2}(r_a + r_p), \qquad T = \frac{2\pi}{\sqrt{\mu}}a^{3/2},$$

$$r_p = \frac{h^2/\mu}{1+e} = a(1-e), \qquad b = a\sqrt{1-e^2}, \qquad e = \frac{r_a - r_p}{r_a + r_p},$$

$$r_a = \frac{h^2/\mu}{1-e} = a(1+e), \qquad \mathcal{E} = -\frac{\mu}{2a}, \qquad h = \sqrt{\mu a(1-e^2)}.$$

Parabolic Orbits: ($e = 1$)

$$v = \sqrt{\frac{2\mu}{r}}, \qquad \mathcal{E} = 0.$$

Hyperbolic Orbits: ($e > 1$)

$$r = \frac{a(e^2-1)}{1 + e\cos\theta}, \qquad a = \frac{h^2/\mu}{e^2-1}, \qquad \theta_\infty = \cos^{-1}(-1/e),$$

$$r_p = a(e-1), \qquad b = a\sqrt{e^2-1}, \qquad \beta = \cos^{-1}(1/e),$$

$$\mathcal{E} = \frac{\mu}{2a}, \qquad h = \sqrt{\mu a(e^2-1)}.$$

Transformation between Orbital Elements and $(\vec{r}, \vec{v})$

Given $(\vec{r}, \vec{v})$, find the orbital elements $(h, e, \theta, \Omega, i, \omega)$

$$
\begin{aligned}
r &= |\vec{r}|, \\
\vec{h} &= \vec{r} \times \vec{v}, \qquad h = |\vec{h}|, \\
\vec{e} &= \frac{1}{\mu}\vec{v} \times \vec{h} - \frac{\vec{r}}{r}, \qquad e = |\vec{e}|, \\
\vec{N} &= \hat{z} \times \vec{h}, \\
i &= \cos^{-1}\left(\frac{\vec{h} \cdot \hat{z}}{h}\right) \quad (0 \leq i \leq \pi), \\
\Omega &= \tan^{-1}\left(\frac{\hat{y} \cdot \vec{N}}{\hat{x} \cdot \vec{N}}\right) = \texttt{numpy.arctan2}(\hat{y} \cdot \vec{N}, \hat{x} \cdot \vec{N}), \\
\omega &= \tan^{-1}\left(\frac{\vec{h} \cdot (\vec{N} \times \vec{e})}{h(\vec{N} \cdot \vec{e})}\right) = \texttt{numpy.arctan2}(\vec{h} \cdot (\vec{N} \times \vec{e}), h(\vec{N} \cdot \vec{e})), \\
\theta &= \tan^{-1}\left(\frac{\vec{h} \cdot (\vec{e} \times \vec{r})}{h(\vec{e} \cdot \vec{r})}\right) = \texttt{numpy.arctan2}(\vec{h} \cdot (\vec{e} \times \vec{r}), h(\vec{e} \cdot \vec{r})).
\end{aligned}
$$

(Use the Numpy `numpy.arctan2` function to compute $\tan^{-1}$, i.e. $\tan^{-1}(y/x) = \texttt{numpy.arctan2(y,x)}$).

Given the orbital elements $(h, e, \theta, \Omega, i, \omega)$, find $(\vec{r}, \vec{v})$

$$
\begin{aligned}
\hat{N} &= \cos\Omega\,\hat{x} + \sin\Omega\,\hat{y}, \\
\hat{h} &= \sin i \sin\Omega\,\hat{x} - \sin i \cos\Omega\,\hat{y} + \cos i\,\hat{z}, \\
\hat{N}_t &= -\sin\Omega\cos i\,\hat{x} + \cos\Omega\cos i\,\hat{y} + \sin i\,\hat{z}, \\
\hat{u}_r &= \cos(\theta + \omega)\,\hat{N} + \sin(\theta + \omega)\,\hat{N}_t, \\
\hat{u}_\theta &= -\sin(\theta + \omega)\,\hat{N} + \cos(\theta + \omega)\,\hat{N}_t, \\
r &= \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}, \\
\mathcal{E} &= -\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2), \\
v &= \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right)}, \\
\gamma &= \tan^{-1}\left(\frac{e\sin\theta}{1 + e\cos\theta}\right), \\
\vec{r} &= r\,\hat{u}_r, \\
\vec{v} &= v\cos\gamma\,\hat{u}_\theta + v\sin\gamma\,\hat{u}_r.
\end{aligned}
$$

Orbital Position as a Function of Time

Time since periapsis passage: t **(for any type of orbit)**

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}. \tag{1}$$

Circular orbit: $\theta \sim t$

$$t = \frac{\theta}{2\pi}T, \tag{2}$$

where $T = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$.

Elliptic orbit: $\theta \sim E \sim M_e \sim t$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}, \tag{3}$$

$$M_e = E - e\sin E, \tag{4}$$

$$t = \frac{M_e}{2\pi}T, \tag{5}$$

where $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$.

Parabolic orbit: $\theta \sim M_p \sim t$

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}, \tag{6}$$

$$\tan\frac{\theta}{2} = \left[3M_p + \sqrt{(3M_p)^2+1}\right]^{1/3} - \left[3M_p + \sqrt{(3M_p)^2+1}\right]^{-1/3}, \tag{7}$$

$$t = M_p \Big/ \left(\frac{\mu^2}{h^3}\right). \tag{8}$$

Hyperbolic orbit: $\theta \sim F \sim M_h \sim t$

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}, \tag{9}$$

$$M_h = e\sinh F - F, \tag{10}$$

$$t = M_h \Big/ \left(\frac{\mu^2}{h^3}(e^2-1)^{3/2}\right). \tag{11}$$

	Axial Rotational Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm$ (km^3/sec^2)	Semi-major Axis of Orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011 (not rigid)	695990.00	1.32712000E+11	———	———	———	———
☾ Moon*	0.0366004	1737.50	4.902800E+03	3.84400000E+05 (around Earth)	2360592 27.322 Earth days	0.05540000	5.16000
☿ Mercury	0.0170515	2.439000E+03	2.203210E+04	5.79092000E+07	7600568.601 87.97 Earth Days	0.205631	7.00487
♀ Venus	0.0041149 (retrograde)	6051.80	3.24859000E+05	1.08209000E+08	19414191.77 224.7 Earth Days	0.006773	3.39471
⊕ Earth	1.0027576	6378.14	3.98600000E+05	1.49589800E+08	31555647.16 365.23 Earth Days	0.01671020	4.98816000E-05
♂ Mars	0.9746985	3397.00	4.282840E+04	2.27937000E+08	59353583.28 686.96 Earth Days	0.09341230	1.85061
♃ Jupiter	2.4181458	71492.00	1.26687000E+08	7.7841200E+08	374396573 11.87 yr	0.04839270	1.30530
♄ Saturn	2.2522523	60330.00	3.79313000E+07	1.42673000E+09	929341659.8 29.47 yr	0.05415060	2.48446
♅ Uranus	1.3921178 (retrograde)	26200.00	5.79397000E+06	2.87097000E+09	2653128427 84.13 yr	0.04716770	0.76986
♆ Neptune	1.4897579	25225.00	6.835110E+06	4.49825000E+09	5203301252 165 yr	0.00858587	1.76917
♇ Pluto	0.1565631	1195.00	8.737670E+02	5.906638E+09	7829522968	0.24880800	17.14180

	Axial Rotational Period (Rev/Day)	Equatorial Radius (km)	Gravitational Parameter $\mu = Gm$ (km^3/sec^2)	Semi-major Axis of Orbit (km)	Orbital Period	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
Charon	0.1562500	593	108	19600 (about Pluto)	6.38725 days	0.00	96.16
Nix	?	23→68	?	48675 (about Pluto)	24.8562 d	0.0023	0.1
Hydra	?	32→84	?	64780 (about Pluto)	38.2065 d	0.0052	0.25
Ganymede	0.1397711	2631.2	9887.834000	1070000.0 (about Jupiter)	7.154553 d	0.002	0.195
Callisto	0.592059	2410.30	7179.29	1883000.0 (about Jupiter)	16.68902 d	0.007	0.281
Titan	0.6271393	2575.50	8978.190000	1221830.0 (about Saturn)	15.94542 d	0.0292	0.33
Titania	assumed synchronous	788.9	235.544000	435910.0 (about Uranus)	8.706 d	0.00220000	0.14000
⚳ Ceres	2.6448030	474.00 km	63.200000	413906175	1680.982 d	0.07990478	10.58674
Phobos	synchronous	11.10	0.000629	9377 (about Mars)	.31891 d	0.01510000	26.04000
Triton	synchronous	1350.00	1427.598000	354759 (about Neptune)	.5877 d (retrograde)	0.00001600	129.18200