MAE3145: Homework 2

Due date: October 5, 2016

Problem 1 The relative motion of the two-body problem is described by

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}. \tag{1}$$

The specific angular momentum $\vec{h}$ and the eccentricity $\vec{e}$ are defined as follows:

$$\vec{h} = \vec{r} \times \vec{v}, \quad \vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}.$$

In class, we found that $\vec{h}$ is fixed, i.e. $\dot{\vec{h}} = 0$. Here, we wish to show $\vec{e}$ is fixed according to the following steps:

(a) Using (1), show that $\frac{d}{dt}(\vec{v} \times \vec{h}) = -\frac{\mu}{r^3}\vec{r} \times \vec{h}$.

(b) Using the definition of $\vec{h}$, show that $\frac{1}{r^3}\vec{r} \times \vec{h} = \frac{\vec{r}\dot{r} - \dot{\vec{r}}r}{r^2}$.
(Hint: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$, $\vec{r} \cdot \vec{r} = r^2$, and $\vec{r} \cdot \dot{\vec{r}} = r\dot{r}$).

(c) Show that $\frac{d}{dt}\frac{\vec{r}}{r} = -\frac{\vec{r}\dot{r} - \dot{\vec{r}}r}{r^2}$.

(d) By combining the results of parts (a), (b), and (c), show that $\frac{d}{dt}\vec{e} = 0$, i.e, the eccentricity vector is fixed.

Problem 2 A satellite is on an elliptic orbit around the Earth with a perigee radius of $r_p = 7000\,\mathrm{km}$ and an apogee radius of $r_a = 70000\,\mathrm{km}$. Assume that the gravitational parameter and the radius of the Earth are $\mu = 398600\,\mathrm{km}^3/\mathrm{s}^2$, and $R_E = 6378\,\mathrm{km}$, respectively. Determine the following parameters (specify units in km, sec, degree).

(a) eccentricity e

(b) period T

(c) specific energy $\mathcal{E}$

(d) true anomaly θ at which the altitude is $1000\,\mathrm{km}$.

(e) velocity v_r, v_θ at the point found in part (d).

Problem 3 The specific energy and angular momentum of several asteroids heading toward the Earth have been measured as follows:

Asteroid	$\mathcal{E}$ ($\mathrm{km^2/s^2}$)	h ($\mathrm{km^2/s}$)
1	1	1×10^5
2	100	1×10^5
3	0	7×10^4
4	0	8×10^4
5	10	8×10^4

We wish to determine whether any asteroid is likely to hit the Earth. The trajectory of an asteroid is assumed to be the solution of the two-body problem of the asteroid and the Earth, where $\mu_E = 398600\,\mathrm{km^3/s^2}$.

(a) Using the fact that $\mathcal{E}$ and h are conserved, show that the distance at the periapsis r_p satisfies the following quadratic equation:

$$2\mathcal{E}\, r_p^2 + 2\mu_E\, r_p - h^2 = 0. \tag{2}$$

(Hint: at the periapsis, $h = rv$ since $\vec{r}$ is perpendicular to $\vec{v}$.)

(b) Calculate r_p for all asteroids, and determine which asteroid will hit the surface of the Earth: an asteroid will hit the Earth if $r_p < R_E = 6378\,\mathrm{km}$.
(Hint: In Matlab, the quadratic equation $ax^2 + bx + c = 0$ can be solved by the command `roots([a b c])`.)

(c) For each asteroid that hits the surface of the Earth, calculate its impact velocity at the surface of the Earth.
(Note: the impact velocity is not same as the velocity at the periapsis.)

(d) For each asteroid that does not hit the surface of the Earth, calculate its velocity when it is closest to the Earth.