RELATING POSITION WITH TIME

CURTIS CH 3, BMW CH 4.

- PLANETS DON'T MOVE WITH UNIFORM SPEED
- BEGINNING OF CH 4 BMW IS VERY INTERESTING
- WE CAN ALREADY DEFINE POSITION IN ORBIT

$$r = \frac{p}{1+e\cos\nu} \qquad \nu - \text{TRUE ANOMALY}$$

HOW DO WE RELATE THIS TO TIME?

WHERE IS SPACECRAFT IN 2 HOURS?

HOW LONG (TIME) UNTIL PERIAPSIS?

. ALREADY KNOW ABOUT PERIOD OF ELLIPTICAL ORBIT

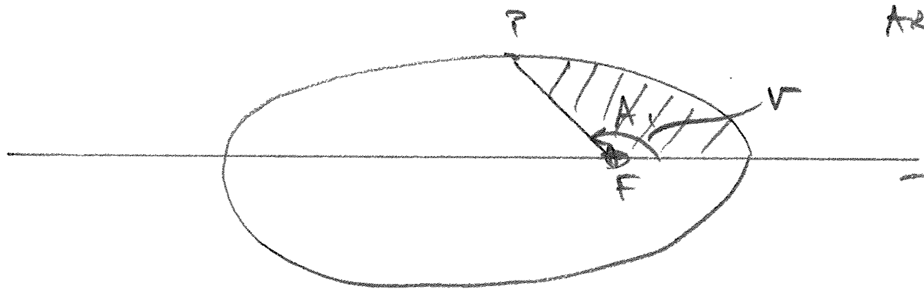
$$\frac{P}{\pi ab} = \frac{t-T}{A_1} \tag{1}$$

$t-T$ ← TIME SINCE PERIAPSIS PASSAGE

A_1 ← AREA SWEPT OUT IN TIME $(t-T)$

P: ORBITAL PERIOD

πab: AREA OF ELLIPSE

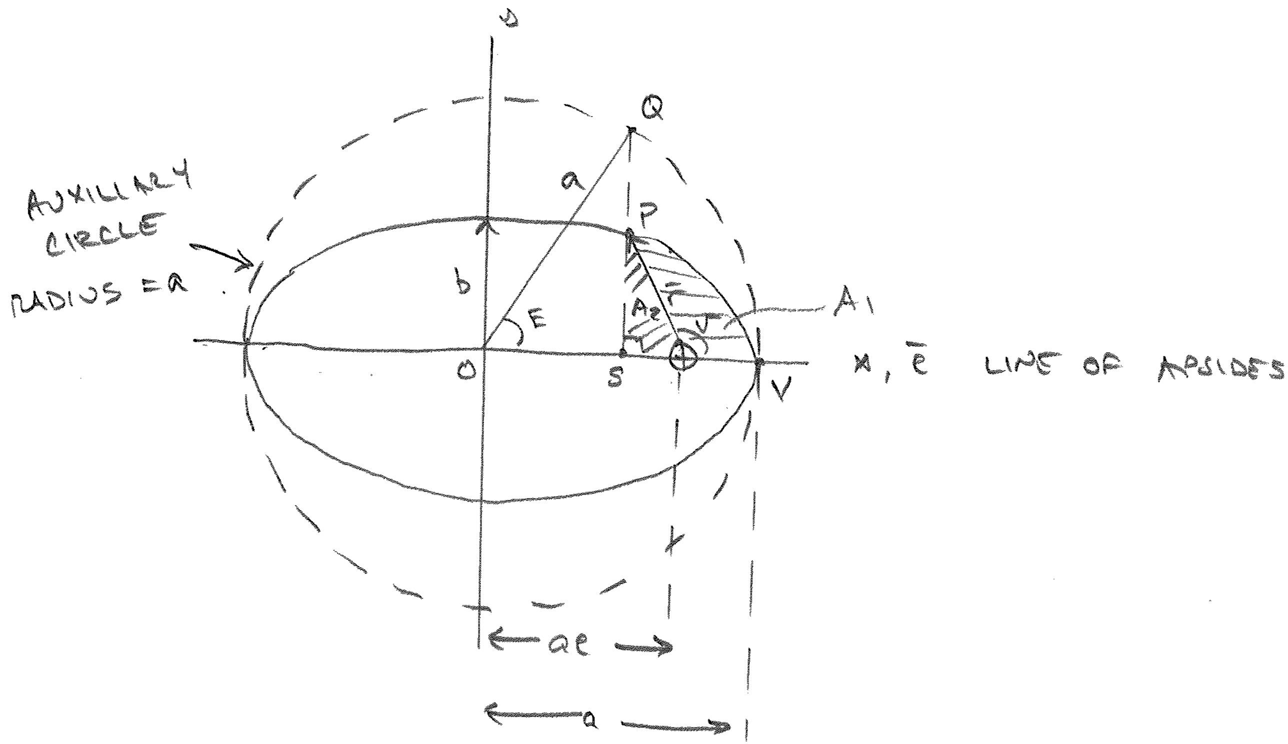


NEED ONLY FIND AREA OF A_1

↓

- AREA SWEPT OUT IN TIME $(t-T)$
- ANGLE ν

- WE'LL USE GEOMETRY - JUST LIKE KEPLER
- BETTER METHODS EXIST NOW f, g FCNS - UNIVERSAL VARIABLE FORMULATION



E - ECCENTRIC ANOMALY - ANGLE ON AUXILIARY CIRCLE AT Q.

* FROM GEOMETRY

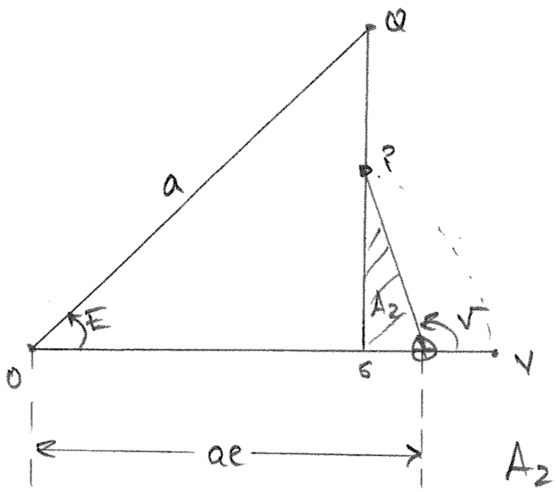
ELLIPSE : $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ CIRCLE : $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$

DEFINE y POSITION AS FCN OF x

ELLIPSE : $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$ CIRCLE : $y = \sqrt{a^2 - x^2}$

$\Rightarrow \boxed{\dfrac{y_{\text{ELLIPSE}}}{y_{\text{CIRCLE}}} = \dfrac{b}{a}}$ WILL USE LATER

FROM DRAWING — AREA A_1 = AREA PSV − AREA A_2 (2)



$\cos E = \dfrac{OS}{a} \rightarrow OS = a\cos E$

$\sin E = \dfrac{QS}{a} \rightarrow QS = a\sin E$

AREA OF TRIANGLE A_2

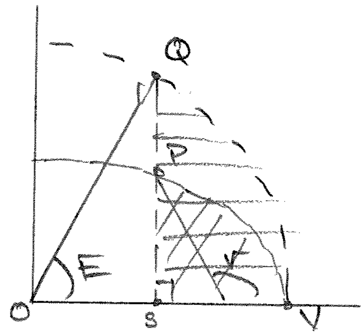
$\frac{1}{2}$(BASE)(HEIGHT)

$$A_2 = \frac{1}{2}(ae - a\cos E)\left(\frac{b}{a} a\sin E\right)$$

MORE ALGEBRA

$$A_2 = \frac{1}{2}\left(\cancel{a} e \frac{b}{\cancel{a}} a \sin E - \cancel{a} \frac{b}{\cancel{a}} a \sin E \cos E\right)$$

$$A_2 = \frac{ab}{2}\left(e \sin E - \sin E \cos E\right) \qquad (3)$$



(4) AREA PSV = $\frac{b}{a}$ (AREA QSV)

ELLIPSE / CIRCLE RATIO

AREA QSV IS PART OF A SECTOR OF A CIRCLE

AREA QSV = AREA QOV − TRIANGLE QOS

$$= \frac{1}{2}a^2 E - \frac{1}{2}(\text{BASE})(\text{HEIGHT})$$

$$= \frac{1}{2}a^2 E - \frac{1}{2}(a \cos E)(a \sin E)$$

AREA QSV $= \frac{1}{2}a^2(E - \cos E \sin E) \Rightarrow (4)$

AREA PSV $= \frac{b}{a}\frac{1}{2}a^2(E - \cos E \sin E)$

PSV $= \frac{ab}{2}(E - \cos E \sin E) \rightarrow (2)$ WITH (3)

$$A_1 = \frac{ab}{2}(E - \cancel{\cos E \sin E}) - \frac{ab}{2}(e \sin E - \cancel{\sin E \cos E})$$

$$A_1 = \frac{ab}{2}(E - e \sin E) \qquad (5)$$

PLUG (5) $\Rightarrow$ (1) WITH $P = 2\pi\sqrt{\frac{a^3}{\mu}}$

$$\cancel{2}\cancel{\pi}\sqrt{\frac{a^3}{\mu}}\,\frac{\cancel{ab}}{\cancel{2}}(E - e\sin E) = (t - T)(\cancel{\pi ab})$$

$$\boxed{t - T = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)}$$

TIME OF FLIGHT TO GO FROM PERIAPSIS TO E

DEFINE MEAN ANOMALY $M = E - e\sin E$

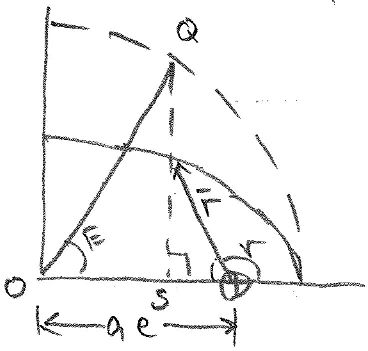
MEAN MOTION $n = \sqrt{\dfrac{\mu}{a^3}}$

$$\boxed{M = n(t-T) = E - e\sin E}$$

KEPLER'S EQN.
TIME SINCE PERIAPSIS

HOW DO WE RELATE E TO ν ?

FROM PREVIOUS DRAWING



$$a\cos E = ae - r\cos(180° - \nu)$$

$$a\cos E = ae + r\cos\nu \qquad (6)$$

KNOW $r = \dfrac{a(1-e^2)}{1+e\cos\nu} \implies \cos\nu = \dfrac{a(1-e^2)}{re} - \dfrac{1}{e}$

PLUG INTO $\cos\nu \rightarrow (6)$

$$\cos E = \frac{ae + r\cos\nu}{a} = e + \frac{r}{a}\cos\nu = e + \frac{\cancel{r}}{\cancel{a}}\left(\frac{\cancel{a}(1-e^2)}{\cancel{r}e} - \frac{1}{e}\right)$$

$$\cos E = e + \left(\frac{1-e^2}{e} - \frac{r}{ae}\right) = \frac{ae^2}{ae} + \frac{a(1-e^2)}{ae} - \frac{r}{ae}$$

$$= \frac{\cancel{ae^2} + a - \cancel{ae^2} - r}{ae} = \boxed{\frac{a-r}{ae} = \cos E}$$

OR $\boxed{r = a(1 - e\cos E)}$ RADIUS IN TERMS OF ECCENTRIC ANOMALY

HOW DO WE RELATE E, ν ?

PREVIOUSLY $r\cos\nu = a\cos E - ae$

TRIG IDENTITY $\cos 2\alpha = 2\cos^2\alpha - 1$

$$r\left(2\cos^2\frac{\nu}{2} - 1\right) = a\cos E - ae$$

$$2r\cos^2\frac{\nu}{2} = a\cos E - ae + r \longleftarrow r = a(1 - e\cos E)$$

$$2r\cos^2\frac{\nu}{2} = a\cos E - ae + a(1 - e\cos E)$$

$$= (a - ae)\cos E + (a - ae)$$

$$2r\cos^2\frac{\nu}{2} = a(1 - e)(1 + \cos E) \qquad \text{(A)}$$

TRIG IDENTITY $\cos 2\alpha = 1 - 2\sin^2\alpha \rightarrow r\cos\nu = a\cos E - ae$

$$r\left(1 - 2\sin^2\frac{\nu}{2}\right) = a\cos E - ae$$

$$-2\sin^2\frac{\nu^2}{2} = a\cos E - ae - a(1 - e\cos E)$$

$$= (a + ae)\cos E - (a + ae)$$

$$-2r\sin^2\frac{\nu}{2} = a(1 + e)(\cos E - 1) \qquad \text{(B)}$$

DIVIDE (B)/(A)

$$\frac{-2r\sin^2 \nu/2}{2r\cos^2 \nu/2} = \frac{a(1+e)(\cos E - 1)}{a(1-e)(1+\cos E)} \swarrow$$

IDENTITY

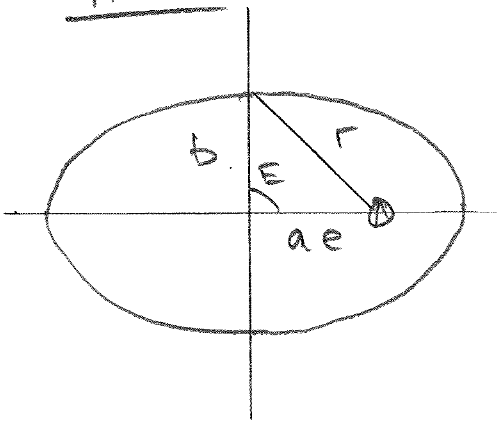
$$\tan^2\frac{E}{2} = \frac{1-\cos E}{1+\cos E}$$

$$\boxed{\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2}}$$

CONVERT $\nu \Leftrightarrow E$

ALWAYS IN THE SAME HALF PLANE

NOTE



AT $E = 90°$

$r = a(1-\cos E) = a$

$b^2 = r^2 - a^2e^2$

$b^2 = a^2(1-e^2)$

$$\boxed{b = a\sqrt{(1-e^2)}}$$

$$r = \frac{p}{1+e\cos\theta} \rightarrow \cos\theta = \frac{p}{re} - \frac{1}{e} = \frac{a(1-e^2)}{ae} - \frac{1}{e} = -e$$

$$\Rightarrow \boxed{\theta = \cos^{-1}(-e)} \quad \text{WHEN } E = 90°$$

TIME OF FLIGHT SINCE PERIAPSIS $\quad t - T = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)$

KEPLER'S EQN $\quad M = n(t-T) = (E - e\sin E)$

$n = \sqrt{\frac{\mu}{a^3}}$ MEAN MOTION

CONVERT $\nu \leftrightarrow E$ $\quad \tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2}$

- SIMILIAR RELATIONSHIPS EXIST FOR OTHER CONIC SECTIONS
- WE'RE SKIPPING THEM

ORBITAL POSITION AS FCN OF TIME

ELLIPSE

$$\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2}$$

$$M = n(t-T) = E - e\sin E$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

HYPERBOLIC

$$\tan\frac{\nu}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\frac{H}{2}$$

$$M = \sqrt{\frac{\mu}{|a|^3}}\,(t-T) = (e\sinh H) - H$$

PARABOLIC

BARKER'S EQ.

$$B = 3\sqrt{\frac{\mu}{p^3}}\,(t-T)$$

$$\tan\frac{\nu}{2} = \left(B + \sqrt{1+B^2}\right)^{1/3} - \left(B + \sqrt{1+B^2}\right)^{-1/3}$$

$$M = 6\sqrt{\frac{\mu}{p^3}}\,(t-T) = \tan^3\frac{\nu}{2} + 3\tan\frac{\nu}{2}$$

RELATING ORBITAL POSITION $\Leftrightarrow$ TIME

- LAST TIME WE USED GEOMETRY (JUST LIKE KEPLER) TO RELATE POSITION WITH TIME IN ELLIPTICAL ORBITS.

- WE CAN DERIVE THE SAME RELATIONSHIPS IN A DIFFERENT APPROACH!

SAME ANSWER $\Leftrightarrow$ TWO DIFF. METHODS.

BEGIN WITH SOME KNOWN RELATIONSHIPS

$$p = \frac{h^2}{\mu} \qquad r = \frac{p}{1+e\cos\theta} \qquad h = r^2\dot{\theta} = r^2\frac{d\theta}{dt}$$

COMBINE ALL THREE TO ELIMINATE h, r

$$h = r^2 \frac{d\theta}{dt}$$

$$\sqrt{\mu p} = \frac{p^2}{(1+e\cos\theta)^2}\frac{d\theta}{dt}$$

OR EQUIVALENTLY BY ALGEBRA

$$\sqrt{\frac{\mu}{p^3}}\, dt = \frac{d\theta}{(1+e\cos\theta)^2}$$

NEED TO INTEGRATE TO GET A RELATIONSHIP BTWN t, θ $\rightarrow$ DIRECTLY INTEGRATING IS DIFFICULT.

HOWEVER WE CAN USE E AND

$r = a(1-e\cos E)$ TO SIMPLIFY THINGS.

USE E

1. $r = a(1 - e\cos E) \implies \cos E = \dfrac{a-r}{ae}$ (1)

2. DIFFERENTIATE (1) + REARRANGE $\dot{r} = ae\dot{E}\sin E$ (2)

3. GIVEN $\mathcal{E} = \dfrac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \dfrac{\mu}{r} = -\dfrac{\mu}{2a}$

MULTIPLY BY

$\dfrac{2r^2a}{\mu}$ $\qquad \dfrac{ar^2\dot{r}^2}{\mu} = \dfrac{-a(r^4\dot{\theta}^2)}{\mu} + 2ra - r^2$

BUT $r^4\dot{\theta}^2 = h^2 = \mu p = \mu a(1-e^2)$

$\implies \dfrac{ar^2\dot{r}^2}{\mu} = a^2e^2 - (a-r)^2$

4. SQUARE (1) $\implies a^2e^2\cos^2 E = (a-r)^2$

COMBINE WITH (2), (3) $\dfrac{ar^2}{\mu}\left[a^2e^2\dot{E}^2\sin^2 E\right] = a^2e^2 - a^2e^2\cos^2 E$

$$\frac{ar^2}{\mu}\left[a^2e^2\dot{E}^2\sin^2 E\right] = a^2e^2 - a^2e^2\cos^2 E \implies r\dot{E} = \sqrt{\frac{\mu}{a}}$$

5. RE-WRITE $r\,dE = \sqrt{\dfrac{\mu}{a}}\,dt$

$r = a(1 - e\cos E)$

$$\boxed{\sqrt{\frac{\mu}{a}}\,dt = a(1 - e\cos E)\,dE}$$

EASY TO INTEGRATE.

6. INTEGRATE (SHOULD BE EASY FOR YOU!)

$$\boxed{\sqrt{\frac{\mu}{a^3}}\,(t - T) = E - e\sin E}$$

INTEGRATION CONSTANT.

T — TIME OF PERIAPSIS PASSAGE, WHEN $E = \theta = \nu = 0°$

DEFINE SOME VARIABLES. $n = \sqrt{\frac{\mu}{a^3}}$ MEAN MOTION AVERAGE ANG. VELOCITY.

$\Rightarrow$ NOT AN ANGLE!

$M = n(t - T)$ MEAN ANOMALY $\approx$ TIME

$\Rightarrow$ NOT AN ANGLE!!

$$\boxed{n(t - T) = M = E - e\sin E}$$

KEPLER'S EQ. FOR ELLIPTICAL ORBITS.

THIS IS A SIMPLE BUT TRANSCENDENTAL EQ

GIVEN $M \rightarrow$ CANNOT SOLVE FOR E IN CLOSED FORM.

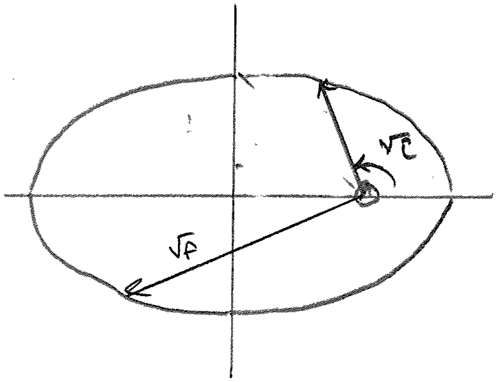
$\rightarrow$ MUST RESORT TO NUMERICAL METHODS.

PROBLEM

HOW LONG TO GO FROM $\nu_i \rightarrow \nu_f$ IN YOUR ELLIPTICAL ORBIT

GIVEN: ν_i, ν_f

FIND: Δt OR TIME OF FLIGHT



IDEA FIND $t - T$ @ ν_i, ν_f THEN SUBTRACT.

$$\left.\begin{array}{l} \nu_i \rightarrow E_i \rightarrow M_i \\ \nu_f \rightarrow E_f \rightarrow M_f \end{array}\right\} \Rightarrow \Delta t$$

1. FIND E_i, E_f, M_i, M_f

$$M_f - M_i = n(t_f - T) - n(t_i - T) = n(t_f - t_i) = n\,\Delta t$$

2. GET $\Delta t = \dfrac{M_f - M_i}{n}$

IF S/C PASSES THROUGH PERIAPSIS $\Rightarrow \nu_i > \nu_f$

$$n\,\Delta t = 2k\pi + M_f - M_i$$

k = # OF PERIAPSIS PASSAGES.

PROBLEM

GIVEN: $\nu_i, \Delta t$

FIND: ν_f

} PROPOGATE! FIND FUTURE POSITION AS FCN OF TIME

BIG ISSUE WITH KEPLERS EQN

$$\nu_i \rightarrow E_i \rightarrow M_i \xrightarrow{\Delta t} M_f \overset{?}{\rightarrow} E_f \rightarrow \nu_f$$

HOW TO SOLVE FOR E_f GIVEN $M_f = E_f - e \sin E_f$?

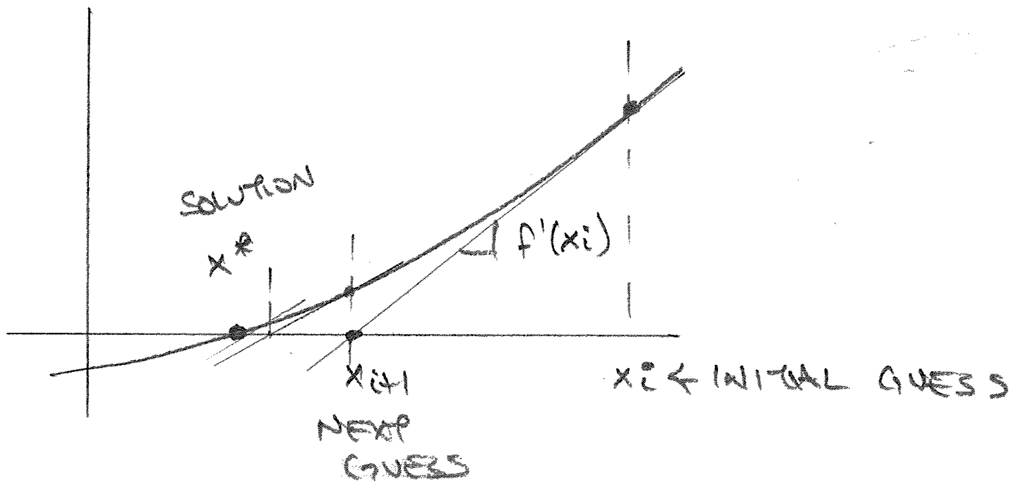
GIVEN M NEED TO SOLVE TRANSCENDENTAL EQN.

USE NEWTON'S METHOD

- ITERATIVE / NUMERICAL METHOD TO SOLVE

$$f(x) = 0$$

- APPROXIMATE $f(x)$ BY A TANGENT LINE (SLOPE)



$$\text{SLOPE} = f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}} \implies x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

ONLY NEED $f(x)$ + $f'(x)$

IF WE CAN FIND DERIVATIVE WE CAN USE NEWTON'S METHOD

REPEAT UNTIL $|x_{i+1} - x_i| < \varepsilon$ ← SOME SMALL NUMBER

FOR KEPLER'S EQN

$$f(E) = E - e\sin E - M$$

$$f'(E) = 1 - e\cos E$$

$$E_{NEW} = E_{OLD} - \frac{f(E)}{f'(E)} = E_{OLD} - \frac{E_{OLD} - e\sin E_{OLD} - M}{1 - e\cos E_{OLD}}$$

ALGORITHM - NEWTON'S METHODS

GIVEN: $M, e \longrightarrow E$

FIND: E

GUESS FOR $E = M \rightarrow E_{OLD}$

- NOT ACTUAL PYTHON CODE!
- YOU CAN LOOK UP LOOPS

$\varepsilon = 1 \times 10^{-10}$

$\Delta = 2\varepsilon$

WHILE $\Delta > \varepsilon$

  FIND $E_{NEW} = f(E_{OLD}, M, e)$

  $\Delta = |E_{NEW} - E_{OLD}|$

  $E_{OLD} = E_{NEW}$

LOOP

PROPOGATE

GIVEN: $\vec{r}_i, \Delta t$

FIND: $\vec{r}_f$

1. $\vec{r}_i \rightarrow E_i \rightarrow M_i$

2. $M_f - M_i = n\Delta t \Rightarrow M_f = n\Delta t + M_i$

3. NEWTON'S METHOD $M_f \rightarrow E_f$

4. $E_f \rightarrow \vec{r}_f$

EXAMPLE - SOLVING KEPLER'S EQ

GIVEN: $M = 235.4^\circ$ $\quad e = 0.4$

FIND: E

GUESS $E_n = M = 4.108505$ RAD

FIRST ITERATION

$$E_{n+1} = E_n + \frac{M - E_n + e \sin E_n}{1 - e \cos E_n}$$

$$= 3.840194 \text{ RAD}$$

$E_2 = 3.84865$ RAD

$E_3 = 3.8486617$ RAD

$E_4 = 3.8486617$ RAD $\leftarrow$ CONVERGED QUICKLY

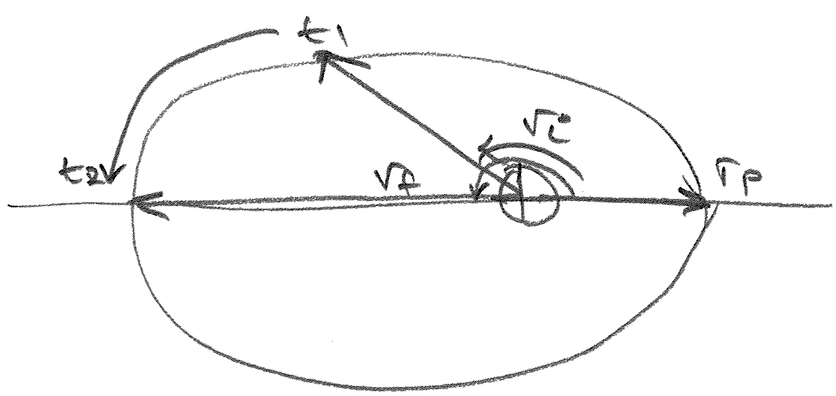
EXAMPLE FIND TIME OF FLIGHT

GIVEN: $r_p = 9600$ km $\qquad \mu_\oplus = 398600.5 \text{ km}^3/\text{sec}^2$

$r_a = 21000$ km

$\nu_i = 120^\circ$ $\qquad \nu_f = 180^\circ$

FIND: FIND TIME TO TRAVEL FROM ν_i TO ν_f



1. DEFINE OUR ORBIT

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.37 \qquad a = \frac{r_a + r_p}{2} = 15300$$

2. DEFINE ANGLES

$$\left.\begin{matrix} \nu_i \rightarrow E_i \rightarrow M_i \\ \nu_f \rightarrow E_f \rightarrow M_f \end{matrix}\right\} \Delta t$$

$$\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2}$$

$$M = E - e\sin E = n(t_i - T) \qquad n = \sqrt{\frac{\mu}{a^3}} = 0.000333 \ \frac{\text{rad}}{\text{sec}}$$

$E_i = 1.728$ RAD $\qquad E_f = 3.1415$ RAD

$M_i = 1.3601$ RAD $\qquad M_f = 3.1415$ RAD.

3. FIND Δt

$$M_f - M_i = n\Delta t \rightarrow \Delta t = 5340.07 \text{ sec} = 1.48 \text{ hrs.}$$

TIME TO TRAVEL FROM $\nu_1 = 120° \rightarrow \nu_2 = 180°$

SEMESTER OUTLINE

HISTORY - BMW CH 1

N-BODY PROBLEM - BMW CH 1

RELATIVE | TWO-BODY PROBLEM - BMW CH 1

PROPERTIES OF CONIC SECTIONS - BMW CH 1, CURTIS CH 2.

THREE-D ORBITS / ORBITAL ELEMENTS - BMW CH 2, CURTIS CH 4.

REFERENCE FRAMES - LVLH, PERIFOCAL, ECI

KEPLER'S EQN - CURTIS CH 3, BMW CH 4.

PROJECT RV2COE → DUE 10/23

1. UPDATED ALGORITHM
2. PRINTED CODE + DOCUMENTED + TESTED
3. OUTPUT MATCHING TEST CASES
4. OUTPUT FOR FIRST THREE ADDITIONAL

MIDTERM 10/25

1. 8.5" x 11" EQ. SHEET — 1 SIDE
2. I'LL PROVIDE EQ. SHEET
3. CALCULATOR + RULER

NEXT PROJECT - PROPOCATE

ALGORITHM DUE - 10/30

PROJECT DUE 11/6 - MAY BE MODIFIED

ANOTHER H10 - POSTED BY END OF WEEK

DUE 11/1 ?

PROJECT - PROPAGATE

- GIVEN CURRENT $\bar{r}, \bar{v}, \Delta t$ FIND FUTURE $\bar{r}, \bar{v}$
- USES MANY OF THE SKILLS FROM COE2RV
- AGAIN THE MAIN GOAL IS A STRUCTURED PROGRAM WHICH CAN ACCOMPLISH OUR GOAL
- USES MANY SIMILAR SKILLS
    - READING | WRITING TEXT FILES
    - LOOPS
    - FUNCTIONS
    - TESTING
- TWO BIG REQUIRED FUNCTIONS
    - UPDATE → SOLVES KEPLER'S EQ.
    - COE2RV → CONVERT COE TO $\bar{r}, \bar{v}$