MAE3145: Solution for Homework 3

Problem 1 Consider Asteroid 5 discussed at Question 3 of HW#2. Its specific energy and specific angular momentum are given by $\mathcal{E} = 10\,\mathrm{km}^2/\mathrm{s}^2$, and $h = 8\times 10^4\,\mathrm{km}^2/\mathrm{s}$. We want to determine the time after periapsis passage t when the true anomaly is $\theta = 100°$.

(a) Compute the semi-major axis a, and the eccentricity e.

Sol: Since $\mathcal{E} > 0$, it is a hyperbolic orbit. We have

$$a = \frac{\mu}{2\mathcal{E}} = 19930\,\mathrm{km}, \quad e = \sqrt{2\mathcal{E}\frac{h^2}{\mu^2} + 1} = 1.3427.$$

(b) Compute the maximum true anomaly θ_∞. Is $\theta < \theta_\infty$?

Sol: The maximum true anomaly is given by $\theta_\infty = \cos^{-1} 1/e = 2.4101\,\mathrm{rad} = 138.09°$. We have $\theta = 100° < \theta_\infty$.

(c) Compute the hyperbolic eccentric anomaly F, and the hyperbolic mean anomaly M_h.

Sol: The hyperbolic eccentric anomaly and the hyperbolic mean anomaly are given by

$$F = 2\tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}\right) = 0.9855\,\mathrm{rad}, \quad M_h = e\sinh F - F = 0.5638\,\mathrm{rad}.$$

(d) Show that the time after the periapsis passage is given by $t = 0.6979\,\mathrm{hrs}$.

$$t = M_h \Big/ \left(\frac{\mu^2}{h^3}(e^2-1)^{3/2}\right) = 2.5124\times 10^3\,\mathrm{sec} = 0.6979\,\mathrm{hrs}.$$

Problem 2 An Earth-orbiting satellite has a period of $T = 15.743$ hours and a periapsis radius $r_p = 12756\,\mathrm{km}$. We want to determine the location of this satellite at time $t = 1$ hour after periapsis passage.

(a) Compute the semi-major axis a, and the eccentricity e.

Sol: Since $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$ and $r_p = a(1-e)$, we have

$$a = \left(T\frac{\sqrt{\mu}}{2\pi}\right)^{2/3} = 31890\,\mathrm{km}, \quad e = 1 - \frac{r_p}{a} = 0.6.$$

(b) Compute the mean anomaly M_e.

Sol: We have $M_e = 2\pi t/T = 0.3991\,\mathrm{rad}$.

(c) Write a Matlab program to compute the eccentric anomaly E.

```
clear all;
close all;

T=15.743*3600;
rp=12756;

mu=398600;
```

```
a=(T*sqrt(mu)/2/pi)^(2/3);
e=1-rp/a;
t=1*3600;
Me=2*pi/T*t;

E=Me;
errE=1;
while errE > 1e-15
    f=(E-e*sin(E)-Me);
    fp=1-e*cos(E);
    Enew=E-f/fp;
    errE=norm(Enew-E);
    E=Enew;
end

theta=2*atan(sqrt((1+e)/(1-e))*tan(E/2));
disp(theta*180/pi);
```

(d) Show that the true anomaly is given by $\theta = 84.2850°$. **Sol:** The above code returns $E = 0.8498\,\mathrm{rad}$. The true anomaly is given by

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) = 1.4711\,\mathrm{rad} = 84.2850°.$$

Problem 3 We observed the position and the velocity of a spacecraft orbiting the Earth as follows:

$$\vec{r}_0 = [6000, 6000, 6000]\,\mathrm{km}, \quad \vec{v}_0 = [-5, -5, 0]\,\mathrm{km/s}.$$

Assume that $\mu = 398600\,\mathrm{km}^3/\mathrm{s}^2$.

(a) Using the Matlab code shown in the class, find the orbital elements $(h, e, \theta, \Omega, \omega, i)$.

Sol: We use the Matlab function `rv2oe.m` as follows:

```
r_vec=[6000 6000 6000]';
v_vec=[-5 -5 0]';
[h,e,theta,Omega,omega,i]=rv2oe(r_vec,v_vec)
```

These commands yield:

```
h =   4.2426e+04 (km^2/s)
e =    0.8351
theta =   -2.3146 (rad)
Omega =    0.7854 (rad)
omega =    2.9301 (rad)
i =    1.5708 (rad)
```

(b) Write a Matlab function `oe2rv.m` that computes the position and the velocity vector for given orbital elements.

Sol:

```
function [r_vec, v_vec]=oe2rv(h,e,theta,Omega,omega,i)
xhat=[1;0;0];
yhat=[0;1;0];
zhat=[0;0;1];
mu=398600;

Nhat=cos(Omega)*xhat+sin(Omega)*yhat;
hhat=sin(i)*sin(Omega)*xhat-sin(i)*cos(Omega)*yhat+cos(i)*zhat;
Nthat=-sin(Omega)*cos(i)*xhat+cos(Omega)*cos(i)*yhat+sin(i)*zhat;

urhat=cos(theta+omega)*Nhat+sin(theta+omega)*Nthat;
uthat=-sin(theta+omega)*Nhat+cos(theta+omega)*Nthat;

r=h^2/mu*1/(1+e*cos(theta));
E=-1/2*mu^2/h^2*(1-e^2);
v=sqrt(2*(E+mu/r));
gam=atan2(e*sin(theta),1+e*cos(theta));

r_vec=r*urhat;
v_vec=v*cos(gam)*uthat+v*sin(gam)*urhat;
```

(c) Evaluate the function `oe2rv.m` for varying `theta=linspace(0,2*pi,200)`. The other five orbital elements $(h, e, \Omega, \omega, i)$ are fixed at your solution of (a). Plot the position and the velocity vector in a three-dimensional space.

Sol: Matlab code is given as follows.

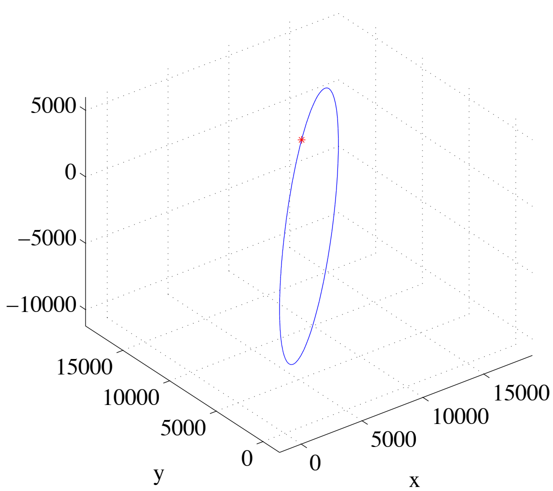
```
[h,e,theta,Omega,omega,i]=rv2oe(r_vec,v_vec);

theta=linspace(0,2*pi,200);
for k=1:200
     [r_vec_theta(:,k),v_vec_theta(:,k)]=oe2rv(h,e,theta(k),Omega,omega,i);
end
plot3(r_vec_theta(1,:),r_vec_theta(2,:),r_vec_theta(3,:));
hold on;
plot3(r_vec(1),r_vec(2),r_vec(3),'r*');
figure;
plot3(v_vec_theta(1,:),v_vec_theta(2,:),v_vec_theta(3,:));
hold on;
plot3(v_vec(1),v_vec(2),v_vec(3),'r*');
```

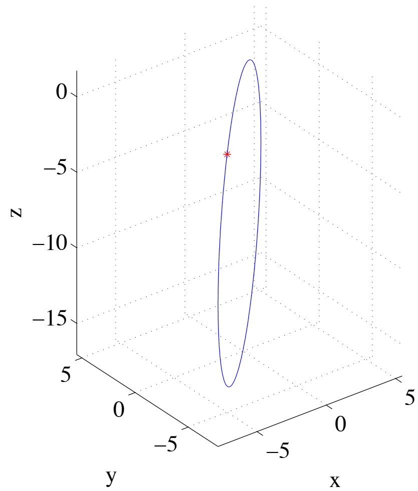
These commands generate the following figures in the next page.

(d) Check that $\vec{r}_0$ and $\vec{v}_0$ are on your curves at (c).

Sol: In the above figures, $\vec{r}_0$ and $\vec{v}_0$ are denoted by red stars, which are on the curves generated at (c).



(a) Position $\vec{r}$



(b) Velocity $\vec{v}$

Problem 4 A satellite satisfies the following condition at the current time.

- $\vec{r} = [-6634.2, -1261.8, -5230.9]\,\text{km}, \quad \vec{e} = [-0.40907, -0.48751, -0.63640]$
- It is flying toward its periapsis.

(a) What is the type of orbit.

Sol: Since $e = \|\vec{e}\| = 0.9$, it is an elliptic orbit.

(b) Find the direction of the specific angular momentum $\hat{h} = \frac{\vec{h}}{h}$.

Sol: The vectors $\vec{r}$ and $\vec{e}$ are on the orbital plane, and the vector $\hat{h}$ is normal to the orbital plane. Therefore, $\hat{h}$ can be obtained by the cross product of $\vec{r}$ and $\vec{e}$. Since it is flying toward its periapsis, we have $180^\circ < \theta < 360^\circ$. These imply

$$\hat{h} = \frac{\vec{r} \times \vec{e}}{\|\vec{r} \times \vec{e}\|} = [-0.4545\ -0.5417,\ 0.7071].$$

(c) Find the inclination i.

Sol: The inclination is given by $i = \cos^{-1}(\hat{h} \cdot \hat{z}) = 0.7854\,\text{rad} = 45^\circ$.

(d) Find the direction of the node vector $\hat{N} = \frac{\vec{N}}{N}$. **Sol:** The direction of the node vector can be written as

$$\hat{N} = \frac{\hat{z} \times \vec{h}}{\|\hat{z} \times \vec{h}\|} = \frac{\hat{z} \times \hat{h}}{\|\hat{z} \times \hat{h}\|} = [0.7661, -0.6428, 0].$$

(e)-(g) **Sol:** Similarly, we have

$$\Omega = \tan^{-1}\left(\frac{\hat{y} \cdot \hat{N}}{\hat{x} \cdot \hat{N}}\right) = -0.6981\,\text{rad} = -40^\circ,$$

$$\omega = \tan^{-1}\left(\frac{\hat{h} \cdot (\hat{N} \times \vec{e})}{(\hat{N} \cdot \vec{e})}\right) = -1.5708\,\text{rad} = -90^\circ,$$

$$\theta = \tan^{-1}\left(\frac{\hat{h} \cdot (\vec{e} \times \vec{r})}{(\vec{e} \cdot \vec{r})}\right) = -0.5236\,\text{rad} = -30^\circ.$$