Circular Restricted Three-Body Problem

Problem Definition Consider three point masses m_1, m_2 and m, acting under their mutual gravity. Assume that $m \ll m_1, m_2$, and m_2 is on a circular orbit around m_1 with orbital radius r_{12}. Define a reference frame $G - xyz$ such that the origin G is on the mass center, and the x-axis points toward m_2. The z axis is parallel to the angular momentum vector of the circular orbit, and the y-axis is chosen according to the rigid-handed rule. This frame is non-inertial, as it is rotating about the z axis with the angular velocity $\Omega = \sqrt{\frac{\mu}{r_{12}^3}}$, where $\mu = G(m_1 + m_2)$.

With respect to this frame, the masses m_1 and m_2 are fixed. Their location on the x axis is given by

$$x_1 = -\frac{m_2}{m_1 + m_2} r_{12} = -\frac{\mu_2}{\mu} r_{12}, \quad x_2 = \frac{m_1}{m_1 + m_2} r_{12} = \frac{\mu_1}{\mu} r_{12},$$

where $\mu_1 = Gm_1$, and $\mu_2 = Gm_2$.

Equations of Motion The equations of motion for the mass m are given by

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3}(x - x_1) - \frac{\mu_2}{r_2^3}(x - x_2), \tag{1}$$

$$\ddot{y} + 2\Omega\dot{y} - \Omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y, \tag{2}$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z, \tag{3}$$

where

$$\Omega = \sqrt{\frac{\mu}{r_{12}^3}}, \quad r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}.$$

Lagrange Points The above equations of motion has five fixed solution, or relative equilibria, referred to as Lagrange points. The first three Lagrange points are on the x-axis, and the remaining two Lagrange points are at the vertex of the equilateral triangle composed of m_1 and m_2.

Jacobi Constant Suppose that $z(0) = \dot{z}(0) = 0$ such that $z(t) = 0$ for all $t \geq 0$, i.e., we consider planar motions. Along the solution of the equations of motion, the following constant, referred to as Jacobi Constant, is fixed:

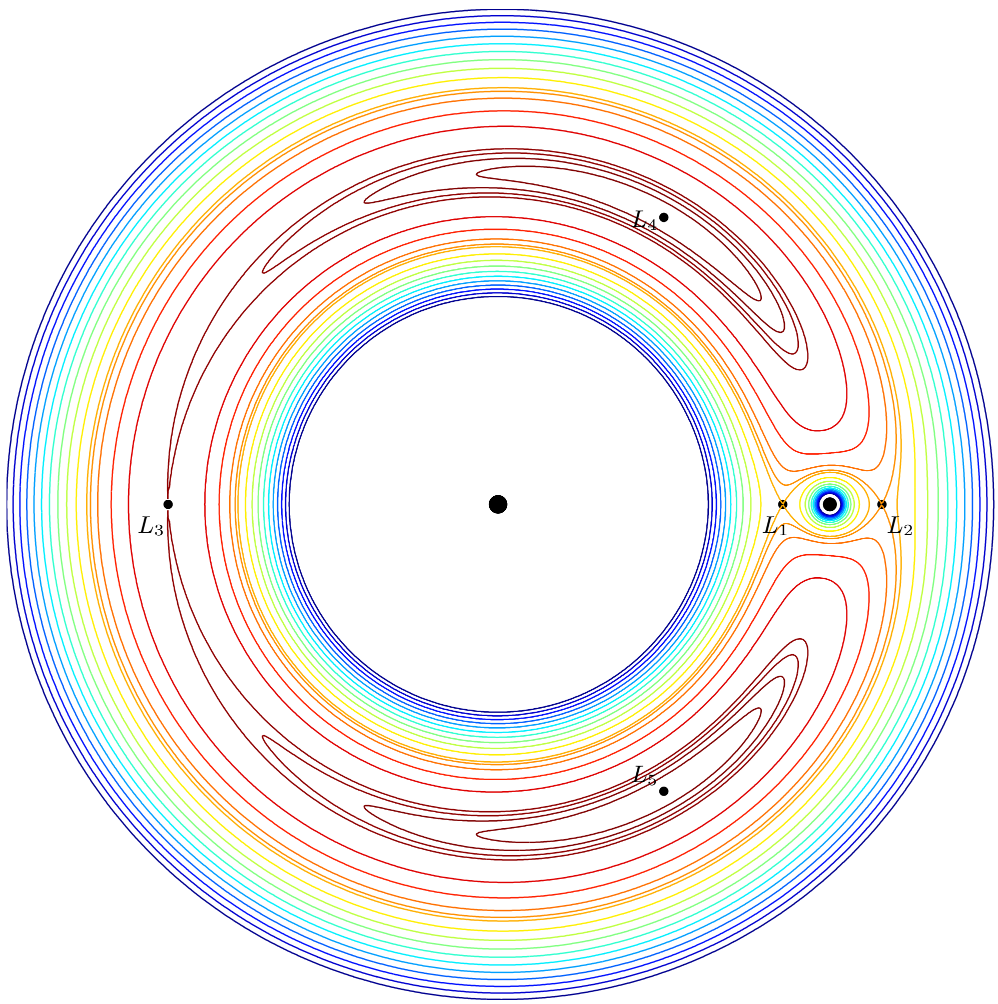
$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\Omega^2(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}. \tag{4}$$

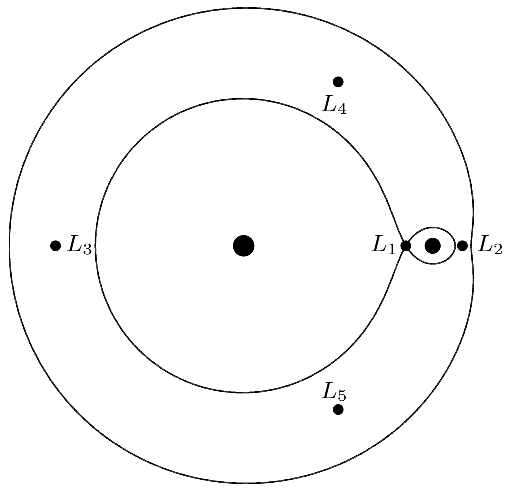
For a given initial condition $(x(0), y(0), \dot{x}(0), \dot{y}(0))$, we can compute the corresponding Jacobi constant. Then, (4) yields the following constraint on the position (x, y):

$$\Omega^2(x^2 + y^2) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \geq 0, \tag{5}$$

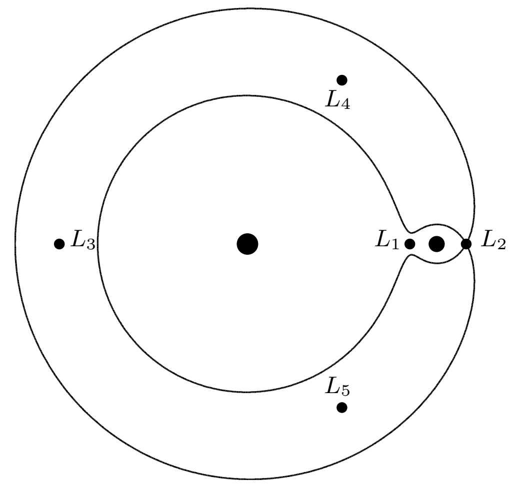
which can be used to identify the feasible region of the solution.

Earth-Moon System The following figure illustrates the Lagrange Points and the boundary of the feasible region for varying Jacobi constant, for the Earth-Moon system.

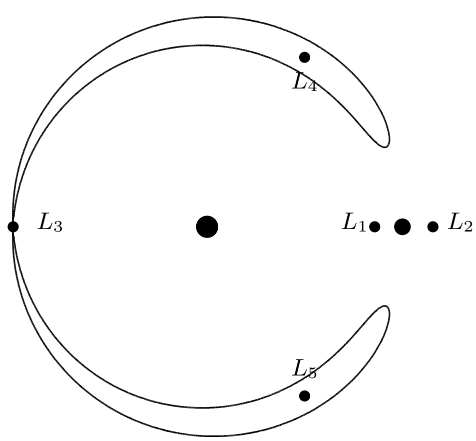




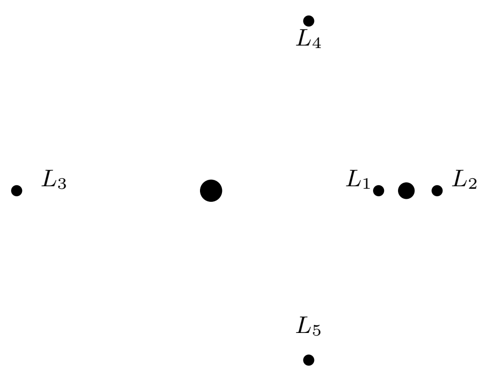
(a) $C = C_1$



(b) $C = C_2$



(c) $C = C_3$



(d) $C = C_{4,5}$