MAE3145: Homework 0

Due date: September 9, 2015

Problem 1 Two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ are given as follows:

$$\vec{a} = [a_1,\, a_2,\, a_3]^T, \quad \vec{b} = [b_1,\, b_2,\, b_3]^T.$$

The dot product of two vectors is a scalar defined as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{1}$$

Let $a = \sqrt{\vec{a} \cdot \vec{a}},\ b = \sqrt{\vec{b} \cdot \vec{b}}$ be the length of the vectors $\vec{a}$, and $\vec{b}$, respectively. Let θ be the angle between $\vec{a}$ and $\vec{b}$. The dot product can be re-written in these variables as

$$\vec{a} \cdot \vec{b} = ab \cos\theta \tag{2}$$

(a) Suppose that $\vec{a} \cdot \vec{b} = 0$. What is the corresponding value of the angle θ. (Assume that $\vec{a} \neq 0$, $\vec{b} \neq 0$, and $0 \leq \theta < 2\pi$.)

(b) Let $\vec{a} = [1,\, 3,\, -2]^T$, $\vec{b} = [-4,\, -1, -2]^T$. Find $\vec{a} \cdot \vec{b}$ using (1).

(c) For $\vec{a}$ and $\vec{b}$ defined at (c), find the angle between those two vectors.

(d) Check that your answer of (d) is consistent with your answer to (b).

Problem 2 The cross product of two vectors is another vector defined as

$$\vec{a} \times \vec{b} = [a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1]^T. \tag{3}$$

Alternatively, it is written as the determinant of the following matrix:

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{4}$$

We can show that its magnitude is given by

$$\|\vec{a} \times \vec{b}\| = ab \sin\theta, \tag{5}$$

and the direction of $\vec{a} \times \vec{b}$ is determined by the right-handed rule. Note that $\vec{a} \times \vec{b}$ is always perpendicular to either $\vec{a}$, $\vec{b}$, or the plane spanned by $\vec{a}$ and $\vec{b}$.

(a) Suppose that $\vec{a} \times \vec{b} = \vec{0}$. What is the corresponding value of the angle θ. (Assume $0 \leq \theta < 2\pi$.)

For the remaining parts (b)-(e), let $\vec{a} = [1,\, 3,\, -2]^T$, $\vec{b} = [-4,\, -1, -2]^T$.

(b) Show that $\vec{c} = \vec{a} \times \vec{b} = [-8,\, 10,\, 11]^T$ using (3) or (4). Also compute its length using $c = \sqrt{\vec{c} \cdot \vec{c}}$.

(c) Compute the length $\|\vec{a} \times \vec{b}\|$ using (5), and show that it is equal to your second answer to (b).

(d) Find the angle between $\vec{a}$ and $\vec{c}$.

Problem 3 The following vector identities will be used in this class later.

$$\vec{x} \cdot (\vec{y} \times \vec{z}) = \vec{y} \cdot (\vec{z} \times \vec{x}) = \vec{z} \cdot (\vec{x} \times \vec{y}), \tag{6}$$

$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}. \tag{7}$$

(a) Using (6), show that $\vec{x}$ is perpendicular to $\vec{x} \times \vec{y}$, i.e., show $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0$.

(b) Suppose that $\vec{q}$ is a unit vector, i.e., $\|\vec{q}\| = 1$. Show that $-\vec{q} \times (\vec{q} \times \vec{x})$ is the orthogonal projection of $\vec{x}$ to the plane normal to $\hat{q}$, i.e., show that $-\vec{q} \times (\vec{q} \times \vec{x}) = \vec{x} - (\vec{q} \cdot \vec{x})\vec{q}$.

Problem 4 A satellite is on a circular orbit around the Earth, i.e. the trajectory of the satellite is a circle centered at the center of the Earth. We observe that the satellite is located at the following point from the center of the Earth:

$$\vec{r}_A = 3741.7\,\hat{\imath} + 5612.5\,\hat{\jmath} - 1870.8\,\hat{k}\,(\mathrm{km}).$$

Sometime later, we observe that the satellite moved to the following position:

$$\vec{r}_B = -441.8\,\hat{\imath} + 6627.5\,\hat{\jmath} + 2209.2\,\hat{k}\,(\mathrm{km}).$$

(a) Find the radius of the circular orbit. (Specify the unit!)

(b) What is the rotation angle of the satellite on its circular orbit, i.e. find the angle between $\vec{r}_A$ and $\vec{r}_B$.

(c) Find a **unit** vector, namely $\hat{h}$ that is perpendicular to the orbital plane. (Hint: find any vector, say $\vec{h}$, that is perpendicular to both of $\vec{r}_A$ and $\vec{r}_B$, and normalize it with its length, i.e. $\hat{h} = \vec{h}/h$, where $h = \sqrt{\vec{h} \cdot \vec{h}}$).

Problem 5 Email a casual photo of yourself to `tylee@gwu.edu` with your name. Please, make the email subject start with `MAE3145`. This is NOT mandatory, but it would be very helpful to me memorize the name of the students.