INVERSE SQUARE LAW

NEWTON'S LAW OF GRAVITY

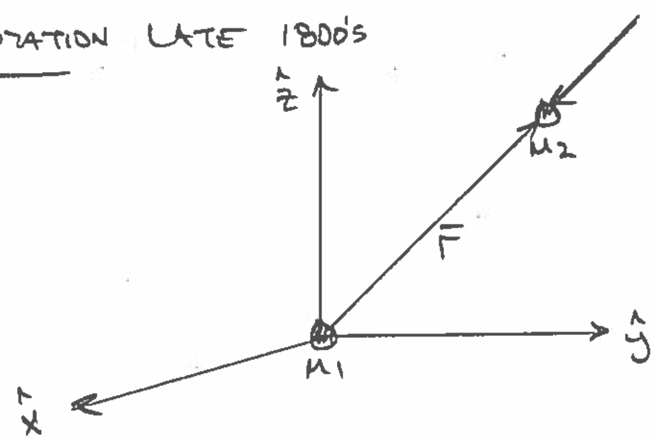
ANY TWO BODIES ATTRACT ONE ANOTHER WITH A FORCE PROPORTIONAL TO THE PRODUCT OF THEIR MASSES AND INVERSELY PROPORTIONAL TO THE SQUARE OF THE DISTANCE BETWEEN THEM

$$F = \frac{G m_1 m_2}{d^2}$$

NEWTON DID NOT USE VECTORS

G IS UNIVERSAL GRAVITATIONAL CONSTANT

VECTOR NOTATION LATE 1800's



$\bar{F}$ - FORCE ON m_2 DUE TO m_1

FORCE OPPOSITE DIRECTION OF $\bar{r}$

$$\bar{F} = -\frac{G m_1 m_2}{r^2}\frac{\bar{r}}{r}$$

ATTRACTIVE

$$G = 6.6742 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

ASSUMPTION: POINT MASSES $\Longrightarrow$ LAW ONLY VALID WHEN BODIES CAN BE MODELED AS POINT MASSES

WHY DOES THIS WORK FOR PLANETS?

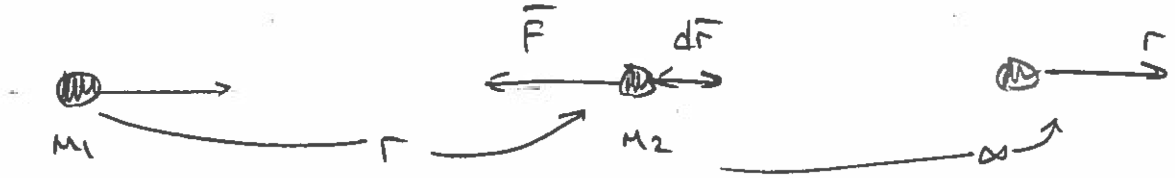
CENTROBARIC $\rightarrow$ CLASS OF BODIES FOR WHICH THE TRUE GRAVITATIONAL FORCE IS EQUIV. TO A POINT MASS

EASY TO MODEL PLANETS AS SPHERICALLY SYMMETRIC

- DENSITY ONLY VARIES WITH RADIUS
- GRAVITY OUTSIDE $\Longrightarrow$ POINT MASS.

LON 2

GRAVITATIONAL POTENTIAL ENERGY



$F = \dfrac{G M_1 M_2}{r^2}$ – DEPENDS ON POSITION

GRAVITY IS A CONSERVATIVE FORCE $\iff$ $F = -\nabla U(r)$

WORK DONE BY GRAVITY $= -$ CHANGE IN POTENTIAL ENERGY

$$dW = \bar{F} \cdot d\bar{r} = -\frac{G M_1 M_2}{r^2}\, dr$$

$$W = \int_{r_1}^{r_2} dW = -\int_{r_1}^{r_2} \frac{G M_1 M_2}{r^2}\, dr = +\frac{G M_1 M_2}{r}\Bigg|_{r_1}^{r_2}$$ CONVENTION

$$= \frac{G M_1 M_2}{r_2} - \frac{G M_1 M_2}{r_1} \Longrightarrow \boxed{U(r) = -\frac{G M_1 M_2}{r}}$$ GRAVITATIONAL POTENTIAL ENERGY SCALAR

UNITS : $J = Nm - kg \dfrac{m^2}{s^2}$

CHOOSE REF VALUE
$r = \infty \Rightarrow U = 0$

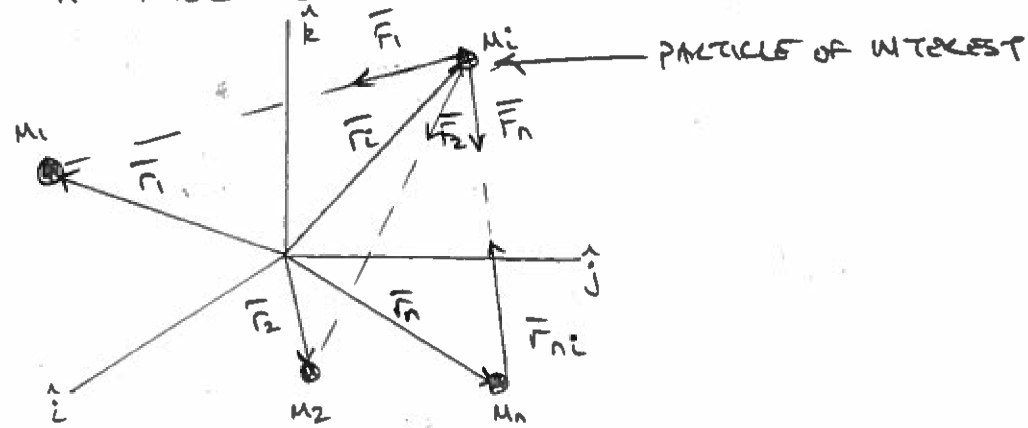
$r \to \infty$ $U \to 0$ (NEGATIVE) $\Longrightarrow$ FUNCTION OF POSITION

NOTE

- AS BODIES APPROACH $\rightarrow$ SPEED (KE) INCREASES BUT TOTAL ENERGY MUST BE CONSERVED, YET AT THE SAME TIME PE INCREASES $\Rightarrow$ PE IS NEGATIVE
- SINGULARITY AT $r=0$ BUT NOT TYPICALLY ENCOUNTERED ALSO AVOIDS LARGE NUMERICAL VALUES OF U

N-BODY PROBLEM

SYSTEM OF n-SPHERICALLY SYMMETRIC BODIES



FORCE ON m_i DUE TO m_n

$$\bar{F}_n = \frac{-G m_i m_n}{r_{ni}^2}\hat{r}_{ni} = \frac{-G m_i m_n}{r_{ni}^3}\bar{r}_{ni}$$

← $\bar{r}_{ni}$ – VECTOR FROM n TO i

$\bar{r}_i = \bar{r}_n + \bar{r}_{ni}$

SUM ALL THE FORCES ON m_i ← WHAT WE CARE ABOUT

$$\bar{F}_T = \frac{-G m_i m_1}{r_{1i}^3}\bar{r}_{1i} - \frac{G m_i m_2}{r_{2i}^3}\bar{r}_{2i} + \dots - \frac{G m_i m_n}{r_{ni}^3}\bar{r}_{ni}$$

WHAT ABOUT $\frac{-G m_i m_i}{r_{ii}^3}\bar{r}_{ii}$?

FORCE MODEL

$$\boxed{\bar{F}_T = -G m_i \sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{m_j}{r_{ji}^3}\bar{r}_{ji}}$$

i – PARTICLE OF INTEREST

j – OTHER PARTICLE

USE THIS FORCE MODEL TO WRITE EQNS USING NEWTON'S SECOND LAW

$$\frac{{}^{I}d}{dt}(m_i \bar{v}_i) = \bar{F}_T$$

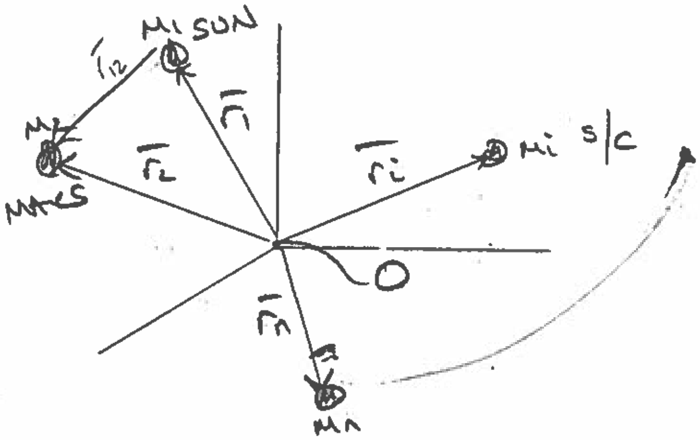
ONLY IF DERIVATIVE WRT INERTIAL FRAME

$$m_i \frac{{}^{I}d\bar{v}_i}{dt} + \bar{v}_i \cancel{\frac{dm_i}{dt}} = \bar{F}_T \implies \boxed{m_i \frac{d^2\bar{r}_i}{dt^2} = -G \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3}\bar{r}_{ji}}$$

CONSTANT MASS

VECTOR EQN FOR m_i

WHERE IS THE ORIGIN?



INERTIALLY FIXED REF FRAME

WHERE IS POINT O?

- CENTER OF SUN ?
- CENTER OF SOLAR SYS. ?
- CENTER OF UNIVERSE !

EXAMPLE - THREE PARTICLES

$$\bar{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$ ← PARTICLE OF INTEREST

$$\bar{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$$

$$\bar{r}_3 = x_3 \hat{i} + y_3 \hat{j} + z_3 \hat{k}$$

$$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1$$ ← 1 → 2

$$\bar{r}_{13} = \bar{r}_3 - \bar{r}_1$$

$$\bar{F}_T = -G \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}$$

$$\bar{F}_1 = G\left(\frac{m_1 m_2}{r_{12}^3} \bar{r}_{12} + \frac{m_1 m_3}{r_{13}^3} \bar{r}_{13} \right) = m_1 \frac{{}^{I}d^2 \bar{r}_1}{dt^2}$$

ALTERNATIVELY - USE POTENTIAL FCN

$$\bar{F}_1 = \bar{\nabla}_1 U \qquad \bar{\nabla}_1 = \hat{i} \frac{\partial}{\partial x_1} + \hat{j} \frac{\partial}{\partial y_1} + \hat{k} \frac{\partial}{\partial z_1}$$

$$U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ij}}$$ POTENTIAL FCN - SCALAR

$$U = \frac{1}{2} G \left(\frac{m_1 m_2}{r_{12}} + \boxed{\frac{m_1 m_3}{r_{13}}} + \frac{m_2 m_1}{r_{21}} + \frac{m_2 m_3}{r_{23}} + \boxed{\frac{m_3 m_1}{r_{31}}} + \frac{m_3 m_2}{r_{32}} \right)$$

$$= G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right)$$

NOW WRITE THE RELATIVE POSITION VECTORS

$$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\bar{r}_{13} = \bar{r}_3 - \bar{r}_1 =$$

$$\bar{r}_{23} = \bar{r}_3 - \bar{r}_2 =$$

SCALAR MAGNITUDES $\rightarrow$ $|\bar{r}_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}$

$$|\bar{r}_{13}| =$$

$|\bar{r}_{23}|$ NOT NEEDED

TAKE THE PARTIALS $\bar{\nabla}_1 U = \frac{\partial U}{\partial x_1}\hat{i} + \frac{\partial U}{\partial y_1}\hat{j} + \frac{\partial U}{\partial z_1}\hat{k}$

$$\frac{\partial U}{\partial x_1} = G m_1 m_2 \frac{\partial r_{12}^{-1}}{\partial x_1} + G m_1 m_3 \frac{\partial r_{13}^{-1}}{\partial x_1} + G m_2 m_3 \frac{\partial r_{23}^{-1}}{\partial x_1}$$

$$= \frac{G m_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{G m_1 m_3}{r_{13}^2}(x_3 - x_1) + 0$$

$$\frac{\partial U}{\partial y_1} = \dots$$

$$\frac{\partial U}{\partial z_1} = \dots$$

$$\bar{\nabla}_1 U = \left\{ \frac{G m_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{G m_1 m_3 (x_3 - x_1)}{r_{13}^3} \right\}\hat{i}$$

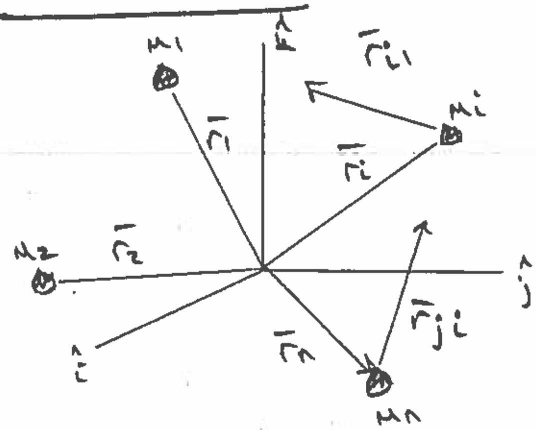
$$+ \left\{ \frac{G m_1 m_2}{r_{12}^3}(y_2 - y_1) + \frac{G m_1 m_3}{r_{13}^3}(y_3 - y_1) \right\}\hat{j}$$

$$+ \left\{ \frac{G m_1 m_2}{r_{12}^3}(z_2 - z_1) + \frac{G m_1 m_3}{r_{13}^3}(z_3 - z_1) \right\}\hat{k}$$

$$\bar{\nabla}_1 U = \bar{F}_1 = \frac{G m_1 m_2}{r_{12}^3}\bar{r}_{12} + \frac{G m_1 m_3}{r_{13}^3}\bar{r}_{13}$$

LSN 2

N-BODY PROBLEM



VECTOR EOMS FOR m_i

$$\boxed{m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}}$$

$\bar{r}_{ji} = \bar{r}_i - \bar{r}_j$

POI OTHER BODIES

3 2nd ORDER DE $\Rightarrow$ 6 1st ORDER DE CAN WE SOLVE ?

1. INDEPENDENT VARIABLE $\rightarrow$ TIME

   DEPENDENT VARIABLES $\rightarrow$ SCALAR COMPONENTS OF POSITION/VELOCITY (r_i AND $\dot{r}_i$)

   IF $\bar{r}_j(t)$ KNOWN $\Rightarrow$ WE CAN SOLVE IN THEORY

2. HOWEVER MOTION OF m_i WILL AFFECT m_j

   $\Rightarrow$ NO OF DEP. VARIABLES > NO OF EQNS.

   N-BODY PROBLEM IS NOT SOLVABLE ! !

3. WE CAN TRY TO ADD MORE EQNS $\Rightarrow$ ADD DE FOR MOTION OF m_j

4. A COMPLETE ANALYTICAL SOLUTION REQUIRES A CONSTANTS/INTEGRAL OF MOTION

   - VERY DIFFICULT TO INTEGRATE COUPLED DE

   $\Downarrow$

   $6n$ EQs WITH $6n$ DEP. VARIABLES $\Rightarrow$ $6n$ CONSTANTS

WE ONLY KNOW OF 10 INTEGRALS $\therefore$ EVEN 2-BODY PROB IS NOT SOLVABLE!

TEN KNOWN INTEGRALS - KNOWN SINCE EULER (1707-1783)
NOTHING NEW SINCE!

1. LINEAR MOMENTUM

CONSERVED FOR SYSTEM WITH NO EXTERNAL FORCES

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

TO GET TOTAL $\bar{p}$, ADD ALL OF THE EQNS.

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i = \underbrace{-G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} (\bar{r}_i - \bar{r}_j)}$$

GOES TO ZERO BECAUSE TERMS
IN THE FORM $(\bar{r}_1 - \bar{r}_2) + (\bar{r}_2 - \bar{r}_1) = 0$

$$\Rightarrow \sum_{i=1}^{n} m_i \ddot{\bar{r}}_i = 0 \leftarrow \text{WE CAN SOLVE THIS DE.}$$

INTEGRATE TWICE

$$\underbrace{\sum_{i=1}^{n} m_i \bar{r}_i}_{m \bar{r}_{CM}} = \bar{C}_1 t + \bar{C}_2 \rightarrow \bar{C}_1, \bar{C}_2$$

6 SCALAR CONSTANT (INTEGRALS)

$m \bar{r}_{CM} \leftarrow$ DEFINITION OF COM WRT TO
INERTIAL PT O (INERTIAL FRAME)

NOTE: $\bar{p} = \sum_{i=1}^{n} m_i \dot{\bar{r}}_i = \text{CONSTANT } \bar{C}_1$

$= m \bar{V}_{COM} = \bar{C}_1 \Rightarrow$ CENTER OF MASS MOVES
AT CONSTANT VELOCITY

2. ANGULAR MOMENTUM — CONSERVED FOR SYSTEM NO EXT FORCE

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

CROSS PRODUCT WITH $\bar{r}_i$ + ADD UP ALL SUMS

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = \sum_{i=1}^{n} G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} (\bar{r}_j - \bar{r}_i) \times \bar{r}_i$$

$$(\bar{r}_j \times \bar{r}_i) - (\cancel{\bar{r}_i \times \bar{r}_i}) \quad \text{ZERO}$$

$$(\bar{r}_1 \times \bar{r}_2) + (\bar{r}_2 \times \bar{r}_1) \rightarrow \text{ZERO}$$

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = 0$$ ← WE CAN INTEGRATE

INTEGRATE ONCE

$$\sum_{i=1}^{n} m_i (\bar{r}_i \times \dot{\bar{r}}_i) = \bar{C}_3$$

SYSTEM ANGULAR MOMENTUM
3 SCALAR CONSTANTS

TOTAL ANGULAR MOM OF SYSTEM OF n PARTICLES

→ CONSTANT IN MAG + DIRECTION

→ DEFINES THE INVARIABLE PLANE

PLANE CONTAINS CENTER OF MASS WHOSE NORMAL COINCIDES WITH TOTAL ANG. MOM VECTOR — $\bar{C}_3$

3. TOTAL ENERGY SCALAR CONSERVED FOR SYS

INTERNAL FORCES DERIVABLE FROM POTENTIAL FCN — CONS. SYSTEM

$$m_i \ddot{\bar{r}}_i = \bar{\nabla}_i U$$

DOT PRODUCT WITH $\dot{\bar{r}}_i$ + ADD UP EQNS.

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \cdot \dot{\bar{r}}_i = \underbrace{\sum_{i=1}^{n} \bar{\nabla}_i U \cdot \dot{\bar{r}}_i}$$

$$\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$\sum_{i=1}^{n} m_i \frac{d}{dt}\left[\frac{1}{2}\left(\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right] = \frac{dU}{dt} \quad \text{TOTAL DERIVATIVE}$$

$$\frac{d}{dt}\left[\underbrace{\sum_{i=1}^{n} m_i \left(\frac{1}{2}\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)}_{\text{TOTAL KINETIC ENERGY}}\right] = \frac{dU}{dt}$$

$$\frac{d}{dt}T = \frac{d}{dt}U \quad \text{OR} \quad \frac{d}{dt}T - \frac{d}{dt}U = 0 \quad \longleftarrow \text{CAN INTEGRATE}$$

$$T - U = C_4 \quad \text{(SCALAR)}$$

$\uparrow$

KE + PE = CONSTANT.