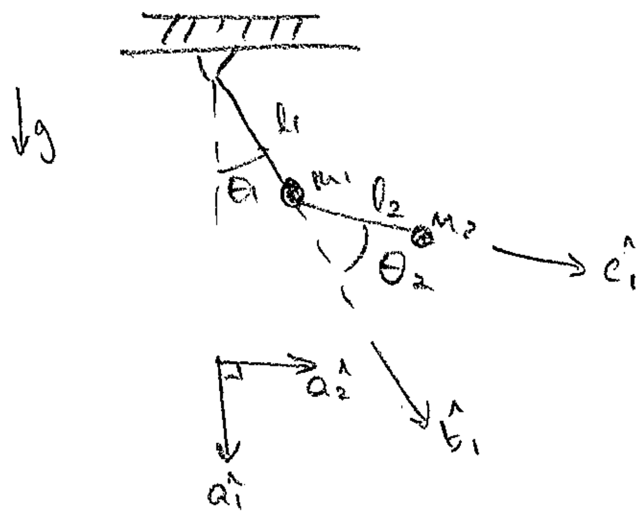
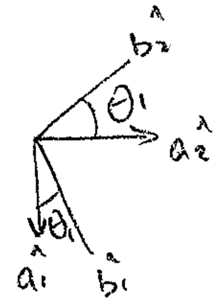
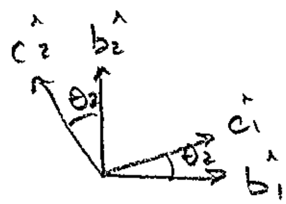
DOUBLE PENDULUM





	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1$	$\cos\theta_1$	$-\sin\theta_1$
$\hat{a}_2$	$\sin\theta_1$	$\cos\theta_1$



	$\hat{b}_1$	$\hat{b}_2$
$\hat{c}_1$	$\cos\theta_2$	$\sin\theta_2$
$\hat{c}_2$	$-\sin\theta_2$	$\cos\theta_2$

POSITION OF MASSES

$$\bar{r}_1 = l_1 \hat{b}_1 \qquad \hat{r}_2 = l_1 \hat{b}_1 + l_2 \hat{c}_1$$

$$\hat{r}_2 = l_1 \hat{b}_1 + l_2 (\cos\theta_2 \hat{b}_1 + \sin\theta_2 \hat{b}_2$$

VELOCITY (INERTIAL)

$$\bar{v}_A = \frac{dr}{dt}_B + {}^A\omega^B \times \bar{r} \qquad {}^A\bar{\omega}^B = \dot{\theta}_1 \hat{b}_3$$

$$\bar{v}_1 = \dot{\theta}_1 \hat{b}_3 \times l_1 \hat{b}_1$$

$$= l_1 \dot{\theta}_1 \hat{b}_2$$

$$\bar{v}_2 = \frac{dr_2}{dt} + {}^A\bar{\omega}^B \times \bar{r}_2$$

$$\frac{d\bar{r}_2}{dt} = -l_2 \dot{\theta}_2 \sin\theta_2 \hat{b}_1 + l_2 \dot{\theta}_2 \cos\theta_2 \hat{b}_2$$

$${}^A\bar{\omega}^B \times \bar{r}_2 = \dot{\theta}_1 \hat{b}_3 \times \left((l_1 + l_2 \cos\theta_2) \hat{b}_1 + l_2 \sin\theta_2 \hat{b}_2 \right)$$

$$= (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_1 \cos\theta_2) \hat{b}_2 - l_2 \dot{\theta}_1 \sin\theta_2 \hat{b}_1$$

$$\bar{v}_2 = (-l_2 \dot{\theta}_2 \sin\theta_2 - l_2 \dot{\theta}_1 \sin\theta_2) \hat{b}_1 + (l_2 \dot{\theta}_2 \cos\theta_2 + l_1 \dot{\theta}_1 + l_2 \dot{\theta}_1 \cos\theta_2) \hat{b}_2$$

$$\bar{V}_1 = l_1 \dot{\theta}_1 \hat{b}_2$$

$$\bar{V}_2 = -l_2 (\dot{\theta}_2 + \dot{\theta}_1) \sin\theta_2 \, \hat{b}_1 + \left(l_1 \dot{\theta}_1 + l_2 (\dot{\theta}_2 + \dot{\theta}_1) \cos\theta_2\right) \hat{b}_2$$

KINETIC ENERGY $\quad T = \frac{1}{2} m \bar{V} \cdot \bar{V}$

$$T_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$T_2 = \frac{1}{2} m_2 \left[l_2^2 (\dot{\theta}_2 + \dot{\theta}_1)^2 \sin^2\theta_2 + l_1^2 \dot{\theta}_1^2 + 2 l_1 \dot{\theta}_1 l_2 (\dot{\theta}_2 + \dot{\theta}_1) \cos\theta_2 + l_2^2 (\dot{\theta}_2 + \dot{\theta}_1)^2 \cos^2\theta_2 \right.$$

$$= \frac{1}{2} m_2 \left[l_2^2 (\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_1^2) \sin^2\theta_2 + l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 + 2 l_1 l_2 \dot{\theta}_1^2 \cos\theta_2 \right.$$

$$\left. + l_2^2 (\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_1^2) \cos^2\theta_2 \right]$$

$$= \frac{1}{2} m_2 \left[l_2 \dot{\theta}_2^2 + 2 l_2^2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_1^2 + l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 + 2 l_1 l_2 \dot{\theta}_1^2 \cos\theta_2 \right]$$

$$T = T_1 + T_2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_1^2 + l_2^2 m_2 \dot{\theta}_1 \dot{\theta}_2$$

$$+ l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 m_2 \dot{\theta}_1^2 \cos\theta_2$$

POTENTIAL $\quad V = mgh$

$$h_1 = \bar{r}_1 \cdot \hat{a}_1 = l_1 \hat{b}_1 \cdot \hat{a}_1 = l_1 (\cos\theta_1 \hat{a}_1 + \sin\theta_1 \hat{a}_2) \cdot \hat{a}_1 = l_1 \cos\theta_1$$

$$V_1 = -m_1 g \, l_1 \cos\theta_1$$

$$h_2 = \bar{r}_2 \cdot \hat{a}_1 = (l_1 + l_2 \cos\theta_2) \hat{b}_1 + (l_2 \sin\theta_2) \hat{b}_2 \cdot \hat{a}_1$$

$$= (l_1 + l_2 \cos\theta_2)(\cos\theta_1 \hat{a}_1 + \sin\theta_1 \hat{a}_2) + l_2 \sin\theta_2 (-\sin\theta_1 \hat{a}_1 + \cos\theta_1 \hat{a}_2) \cdot \hat{a}_1$$

$$= \left((l_1 + l_2 \cos\theta_2) \cos\theta_1 - l_2 \sin\theta_2 \sin\theta_1\right) \hat{a}_1 + \left((l_1 + l_2 \cos\theta_2) \sin\theta_1 + l_2 \sin\theta_2 \cos\theta_1\right) \hat{a}_2 \cdot \hat{a}$$

$$h_2 = (l_1 \cos\theta_1 + l_2 \cos\theta_2 \cos\theta_1 - l_2 \sin\theta_2 \sin\theta_1)$$

$$V_2 = -m_2 g \, l_1 \cos\theta_1 - m_2 g l_2 \cos(\theta_1 + \theta_2)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_1^2 + l_2^2 m_2 \dot{\theta}_1 \dot{\theta}_2$$
$$+ l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 m_2 \dot{\theta}_1^2 \cos\theta_2$$

$$V = -m_1 g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos(\theta_1 + \theta_2)$$

$$L = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_1^2 + l_2^2 m_2 \dot{\theta}_1 \dot{\theta}_2$$
$$+ l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 m_2 \dot{\theta}_1^2 \cos\theta_2 + m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1$$
$$+ m_2 g l_2 \cos(\theta_1 + \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad \delta W = \tau_1 \delta\theta_1 + \tau_2 \delta\theta_2$$

$\underline{\theta_1}$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_2^2 \dot{\theta}_2 + l_2^2 m_2 \dot{\theta}_2 + l_1 l_2 m_2 \dot{\theta}_2 \cos\theta_2$$
$$+ 2 l_1 l_2 m_2 \dot{\theta}_1 \cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + l_2^2 m_2 \ddot{\theta}_1 + l_1 l_2 m_2 \ddot{\theta}_2 \cos\theta_2$$
$$- l_1 l_2 m_2 \dot{\theta}_2^2 \sin\theta_2 + 2 l_1 l_2 m_2 \ddot{\theta}_1 \cos\theta_2 - 2 l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 - m_2 g l_2 \sin(\theta_1 - \theta_2)$$

$$\ddot{\theta}_1 \left(m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 l_1 l_2 m_2 \cos\theta_2\right) + \ddot{\theta}_2 \left(m_2 l_2^2 + l_1 l_2 m_2 \cos\theta_2\right)$$
$$+ \left(- l_1 l_2 m_2 \dot{\theta}_2^2 \sin\theta_2 - 2 l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2\right) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 + m_2 g l_2 \sin(\theta$$
$$= \tau_1$$

$\underline{\theta_2}$

$$\frac{\partial L}{\partial \dot{\theta}_2} = l_2^2 m_2 \dot{\theta}_1 + l_2^2 m_2 \dot{\theta}_2 + l_1 l_2 m_2 \dot{\theta}_1 \cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = l_2^2 m_2 \ddot{\theta}_1 + l_2^2 m_2 \ddot{\theta}_2 + l_1 l_2 m_2 \ddot{\theta}_1 \cos\theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -l_1 l_2 m_2 \dot{\theta}_1^2 \sin\theta_2 - l_1 l_2 m_2 \dot{\theta}_2 \dot{\theta}_1 \sin\theta_2 + l_2 m_2 g \sin(\theta_1 - \theta_2)$$

$$\ddot{\theta}_1 (l_2^2 m_2 + l_1 l_2 m_2 \cos\theta_2) + \ddot{\theta}_2 (l_2^2 m_2) - \cancel{l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2}$$
$$+ l_1 l_2 m_2 \dot{\theta}_1^2 \sin\theta_2 + \cancel{l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2} + l_2 m_2 g \sin(\theta_1 + \theta_2) = \tau_2$$

$$\bar{\tau} = M \ddot{\bar{\theta}} + V(\bar{\theta}, \dot{\bar{\theta}}) \dot{\bar{\theta}} + G$$

$$M = \begin{bmatrix} (m_1 + m_2) l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & -m_2 l_1 l_2 (2\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 \\ m_2 l_1 l_2 \dot{\theta}_1 \sin\theta_2 & 0 \end{bmatrix}$$

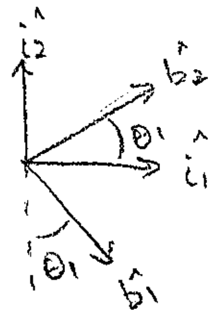
$$G = g \begin{bmatrix} (m_1 + m_2) l_1 \sin\theta_1 + m_2 l_2 \sin(\theta_1 + \theta_2) \\ m_2 l_2 \cancel{g} \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\ddot{\theta} = M^{-1} [\tau - V \dot{\theta} - G]$$

$$\ddot{\theta} = M^{-1} \tau - M^{-1} [V \dot{\theta} + G]$$

$$\tau = V \dot{\theta} + G + M [K_v (\dot{\bar{\theta}}_D - \dot{\bar{\theta}}) + K_p (\bar{\theta}_D - \bar{\theta})]$$

FEEDBACK LINEARIZATION



	$\hat{i}_1$	$\hat{i}_2$
$\hat{b}_1$	$\sin\theta_1$	$-\cos\theta_1$
$\hat{b}_2$	$\cos\theta_1$	$\sin\theta_1$

$$\boxed{\bar{r}_1 = l_1 \hat{b}_1 = l_1(\sin\theta_1 \hat{i}_1 + \cos\theta_1 \hat{i}_2)}$$

$$\bar{r}_2 = (l_1 + l_2\cos\theta_2)\hat{b}_1 + l_2 \sin\theta_2 \hat{b}_2$$

$$= (l_1 + l_2\cos\theta_2)(\sin\theta_1 \hat{i}_1 - \cos\theta_1 \hat{i}_2) + l_2\sin\theta_2(\cos\theta_1 \hat{i}_1 + \sin\theta_1 \hat{i}_2)$$

$$= \big((l_1 + l_2\cos\theta_2)\sin\theta_1 + l_2\sin\theta_2\cos\theta_1\big)\hat{i}_1$$

$$+ \big((l_1 + l_2\cos\theta_2)(-\cos\theta_1) + l_2\sin\theta_2\sin\theta_1\big)\hat{i}_2$$

$$\boxed{\bar{r}_2 = [l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2)]\hat{i}_1 + [-l_1\cos\theta_1 - l_2\cos(\theta_1+\theta_2)]\hat{i}_2}$$

FOR MATLAB PLOTTING

$$\ddot{E} + K_v\dot{E} + K_p E = 0$$

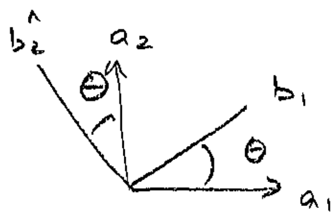
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\zeta = \frac{-\ln\frac{\%OS}{100}}{\sqrt{\pi^2 + \ln^2 \%OS/100}} \qquad T_s \approx \frac{4}{\zeta\omega_n} \qquad \omega_n = \frac{4}{T_s\zeta}$$

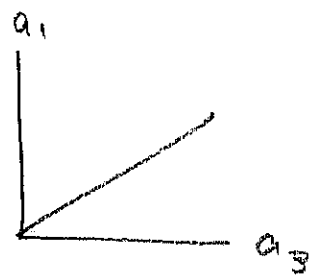
LINKS MASS 1

$I_{xx} = I_{yy} = I_{zz} = 0.004$?

$V = \frac{1}{2}\dot{\Theta}^T M \dot{\Theta} + \Theta_e^T K_p \Theta_e$ $\qquad V \geq 0 \qquad \dot{V} < 0$



	$\hat{a}_1$	$\hat{a}_2$	a_3
b_1	$c\Theta$	$s\Theta$	0
b_2	$-s\Theta$	$c\Theta$	0
b_3	0	0	1



$$\left[\begin{array}{c|c} 3\times 3 & \bar{d}_{3\times 1} \\ \hline 0_{1\times 3} & 1 \end{array}\right]$$