Perfectoid towers and lim Cohen–Macaulay sequences

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This talk is based on the joint work [3] with Kazuma Shimomoto.

Fix a prime number p. Perfectoid towers are a certain class of direct systems of rings in mixed characteristic $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$, which I introduced with Nakazato and Shimomoto in [2]. This class of towers provides one of the connections between perfectoid theory and Noetherian ring theory. Indeed, the tilting operation for perfectoid towers can be defined, and the tilted tower $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ of a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ consisting of Noetherian rings also consists of Noetherian rings.

Recently, B. Bhatt, M. Hochster, and L. Ma introduced the notion of lim Cohen–Macaulay sequences of modules. These are sequences of modules that serve as substitutes for small Cohen–Macaulay modules. In fact, if every complete Noetherian local domain of mixed characteristic admits a lim Cohen–Macaulay sequence, then the positivity conjecture of Serre's intersection multiplicity holds. For an F-finite Noetherian local ring of characteristic $p > 0$, they proved in [1] that the sequence $\{F_*^{(n)}(R)\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence. However, it remains unknown whether any Noetherian local ring of mixed characteristic has a lim Cohen–Macaulay sequence.

To explore a new approach to this problem, we establish the following theorem, which connects perfectoid towers and lim Cohen–Macaulay sequences:

Theorem 1 (I.–Shimomoto [3])**.** *Let R be a Noetherian local domain in mixed characteristic. Suppose that R has a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from some pair (R, I). Then $\{R_n\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence. In particular, if every complete Noetherian local domain of mixed characteristic admits a perfectoid tower, then the positivity conjecture of Serre's intersection multiplicity holds.*

In this talk, I will provide an introduction to some basic properties of perfectoid towers and lim Cohen–Macaulay sequences, as well as a sketch of the proof of Theorem 1.

References

[1] B. Bhatt, M. Hochster, and L. Ma, *Lim Cohen–Macaulay sequences of modules*, arXiv:2410.18372.

[2] S. Ishiro, K. Nakazato, and K. Shimomoto, *Perfectoid towers and their tilts: with an application to the étale cohomology groups of local log-regular rings*, Algebra & Number Theory 19.12 (2025): 2307-2358.

[3] S. Ishiro and K. Shimomoto, *δ-rings, perfectoid towers, and lim Cohen–Macaulay sequences*, arXiv:2509.06527.