NCCRS OF TORIC SINGULARITIES WITH DIVISOR CLASS RANK ONE

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The notion of non-commutative crepant resolutions (NCCRs) is introduced in [17] as a virtual space of crepant resolutions of a given Gorenstein normal singularity. After that, NCCRs turn out to have deep connections with Cohen-Macaulay representation theory [9, 23], higher Auslander-Reiten theory [5, 10, 14], Calabi-Yau algebras [11] and additive categorification of cluster algebras [1, 3, 6]. The existence of NCCRs is proven for quotient singularities [10, 16], compound Du Val singularities having crepant resolutions [18], Du Val del Pezzo cones [15, 17], some toric singularities and so on [4, 7, 19]. In these cases, NCCRs give rich information to their geometry, derived categories and Cohen-Macaulay representations.

For toric singularities, it is convenient to restrict ourselves to toric NCCRs which are NCCRs given by a direct sum of divisorial modules. For a Gorenstein toric singularity R, the existence of toric NCCRs is proven if $\dim R \leq 3$ [2], $\mathrm{Cl}(R)$ is torsion, $\mathrm{Cl}(R) \cong \mathbb{Z}$ [17] or some other cases [8, 13, 20, 21]. We remark here that there exists a Gorenstein toric singularity R with $\dim R = 4$ and $\mathrm{Cl}(R) \cong \mathbb{Z}^2$ such that R has an NCCR, but has *no* toric NCCRs [19, 20]. Based on these facts, it is conjectured that all Gorenstein toric singularities have NCCRs.

In this talk, we prove the existence of toric NCCRs of Gorenstein toric singularities R with $\mathrm{rk}\,\mathrm{Cl}\,R = 1$. Remark that we include the case when $\mathrm{Cl}(R)$ has a torsion. Moreover, we give a classification of toric NCCRs which is new even when $\mathrm{Cl}(R) \cong \mathbb{Z}$. To state our main result, we prepare some notations. Let $\vec{x_1}, \cdots, \vec{x_{d+2}} \in \mathrm{Cl}(R)$ be weights. If we put $\pi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(R)/\mathrm{Cl}(R)_{\mathrm{tors}}$, then we may assume that $\pi(\vec{x_i}) \begin{cases} > 0 & (1 \leq i \leq l) \\ < 0 & (l+1 \leq i \leq l+l') \\ = 0 & (l+l'+1 \leq i \leq d+2) \end{cases}$ holds. Here, we can prove $l, l' \geq 2$. We put $H := G/(\sum_{i=l+l'+1}^{d+2} \mathbb{Z}\vec{x_i})$ and let $q\colon G \to H$ be the natural surjection. Then we can define a partial order on H as

$$h_1 \geq h_2 :\Leftrightarrow h_1 - h_2 \in \sum_{i=1}^{l} \mathbb{Z}_{\geq 0} q(\vec{x_i}) + \sum_{j=1}^{l'} \mathbb{Z}_{\geq 0} q(-\vec{x_{l+j}}) \subseteq H \text{ for } h_1, h_2 \in H.$$

Put $p := \sum_{i=1}^{l} q(\vec{x_i}) = \sum_{j=1}^{l'} q(-\vec{x_{l+j}}) \in H$. For a finite subset $J \subseteq \mathrm{Cl}(R)$, let $M_J \in \mathsf{ref}\,R$ denote the direct sum of divisorial modules corresponding to elements in J.

Theorem 0.1. Let R be a Gorenstein toric singularity with $\mathrm{rk}\,\mathrm{Cl}(R) = 1$. In the above notations, we have a bijection between the following sets.

(1) The set of non-trivial upper sets in H.
(2) $\{J \subseteq \mathrm{Cl}(R) \mid M_J \text{ gives an NCCR.}\}$

A bijection from (1) to (2) is given by $I \mapsto q^{-1}(J(I))$ where $J(I) := I \cap (I^c + p) \subseteq H$.

According to our classification, we can find R with $\mathrm{Cl}(R) \cong \mathbb{Z}$ having a toric NCCR other than one constructed in [17].

Finally, we investigate Iyama-Wemyss mutations of our toric NCCRs. The operation of mutations of NCCRs is introduced in [12] and they proved that two NCCRs connected by mutations are derived equivalent. Moreover, for compound Du Val singularities, the mutations of NCCRs correspond to the flips of crepant resolutions [22]. For these reasons, it is a fundamental task to compute mutations of a given NCCR. We show that in our setting, Iyama-Wemyss mutations are compatible with mutations of upper sets.

Theorem 0.2. In the notation of Theorem 0.1, let $I \subseteq H$ be a non-trivial upper set. Take a minimal element $m \in I$ and write $\mu_m^-(I) := I \setminus \{m\}$. Put $M := M_{q^{-1}(J(I) \setminus \{m\})}$. Then we have

$$(\mu_M^+)^{l'-1}(M_{q^{-1}(J(I))}) = M_{q^{-1}(J(\mu_m^-(I)))} = (\mu_M^-)^{l-1}(M_{q^{-1}(J(I))}).$$

Combining those our results with combinatorics of upper sets, we obtain the following corollary.

Corollary 0.3. Let R be a Gorenstein toric singularity with $\operatorname{rk} \operatorname{Cl}(R) = 1$. Then all toric NCCRs of R are connected by iterated Iyama-Wemyss mutations. In particular, they are all derived equivalent to each other.

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