Towards a structure theorem for quasi-Buchsbaum bundles on $\mathbb{P}^3$

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This talk describes a study on the structure of on quasi-Buchsbaum bundle on $\mathbb{P}^n$ with simple cohomologies. Horrocks' theorem says that a vector bundle on the projective space without intermediate cohomologies is isomorphic to a direct sum of line bundles, that is, a Cohen-Macaulay graded module over the polynomial ring is graded free. There are several widely-known proofs, while I prefer the proof by using m-regular, which also gives the Evans-Griffith theorem. The original proof, more importantly, is based on the categorical equivalence that a stable equivalence class of vector bundles corresponding to $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\mathcal{E})$. Moreover, a Buchsbaum vector bundle on $\mathbb{P}^n$ is isomorphic to a direct sum of differential p-forms, which is proved by Chang and Goto. On the other hand, there are lots of attempts towards Hartshorne's conjecture, that is, a vector bundle of rank 2 is isomorphic to a direct sum of line bundles on $\mathbb{P}^n$, $n \geq 7$, which gives construction methods for vector bundles of lower rank.

Given this background, I will talk on a study of the structure of quasi-Buchsbaum bundle, and the first simplest example, a null-correlation bundle are chracterized in Theorem 1. Then our interest goes on the classification of quasi-Buchsbaum bundles on $\mathbb{P}^3$ which have different ring-theoretic properties.

Let $S = k[x_0, \cdots, x_n]$ be the polynomial ring over an algebraically closed field k with $\mathfrak{m} = (x_0, \cdots, x_n)$, $\operatorname{char} k \neq 2$. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n = \operatorname{Proj} S$. Then $M = \Gamma_*(\mathcal{E}) = \oplus_\ell \Gamma(\mathcal{E}(\ell)))$ is a finitely generated graded S-module with $M_\mathfrak{p}$ free for $\mathfrak{p} \neq \mathfrak{m}$. Note that $\mathrm{H}^i_*(\mathcal{E}) = \oplus_\ell \mathrm{H}^i(\mathcal{E}(\ell)) \cong \mathrm{H}^{i+1}_\mathfrak{m}(M)$ for $i \geq 1$ and $\mathrm{H}^0_\mathfrak{m}(M) = \mathrm{H}^1_\mathfrak{m}(M) = 0$. A vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ is called quasi-Buchsbaum if $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}) = 0$ for $1 \leq i \leq n-1$.

A null-correlation bundle, n odd, is defined as the following short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\psi} \Omega^1_{\mathbb{P}^n}(1) \to \mathcal{E}^\vee \to 0,$$

where $\psi : \mathcal{O}_{\mathbb{P}^n}(-1) \to \Omega^1_{\mathbb{P}^n}(1) \subset \mathcal{O}^{n+1}_{\mathbb{P}^n}$ is given by a skew-symmetric matrix of rank $n+1$.

Theorem 1. Let $\mathcal{E}$ be an indecomposable vector bundle on $\mathbb{P}^n$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^{n-1}_*(\mathcal{E}) = 1$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $2 \leq i \leq n-2$. If $\operatorname{rank} \mathcal{E} \leq n-1$, then n is odd and $\mathcal{E}$ is isomorphic to a null-correlation bundle on $\mathbb{P}^n$.

From now on we consider the case $n = 3$.

Theorem 2. Let $\mathcal{E}$ be an indecomposable vector bundle on $\mathbb{P}^3$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^2_*(\mathcal{E}) = 1$. Then $\mathcal{E}$ is isomorphic to either
(1) a null-correlation bundle or
(2) a vector bundle $\mathcal{E}$ with an short exact sequence: $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\psi} \Omega^1_{\mathbb{P}^3}(1) \oplus \mathcal{O}^2_{\mathbb{P}^3} \to \mathcal{E}^\vee \to 0$, where (i) $\psi = (x_3, x_3, x_3, -x_0 - x_1 + x_2) \times (x_0, x_1)$, (ii) $\psi = (0, x_3, x_3, -x_1 + x_2) \times (x_0, x_1)$, or (iii) $\psi = (0, 0, x_3, -x_2) \times (x_0, x_1)$.

Remark 3. For a vector bundle $\mathcal{E}$ in (1) and (2-i), there are no hyperplanes H such that $\mathcal{E}|_H$ is Buchsbaum. However, in (2-ii) and (2-iii) cases, there exists a hyperplane H such that $\mathcal{E}|_H$ is Buchsbaum.

Definition 4. A part of system of parameters $y_1, \cdots, y_j \in S_1$ for M is called standard if $I \cdot \mathrm{H}^2_\mathfrak{m}(M) = I \cdot \mathrm{H}^3_\mathfrak{m}(M) = 0$ and $I \cdot \mathrm{H}^2_\mathfrak{m}(M/x_iM) = 0$ for all i, where $I = (y_1, \cdots, y_j)$

Remark 5. In terms of the maximal number of elements which gives a part of standard system of parameters, I will explain the difference of a Buchsbaum bundle, a null-correlation bundle and a vector bundle of (2) in Theorem 2.

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