FREENESS OF RESTRICTIONS OF FREE ARRANGEMENTS AND THE HIGHT TWO COHEN-MACAULAY PROPERTY

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The theory of hyperplane arrangements is one of most attractive research areas in algebraic combinatorics. In particular, freeness of an arrangement is an important property that has a lot of algebraic and combinatorial applications. In this talk, we introduce a new commutative algebra approach to study the freeness of a restriction of a free arrangement. This is a joint work with Takuro Abe (Rikkyo University).

We quickly explain freeeness of arrangements. A hyperplane arrangement in $V = \mathbb{C}^n$ is a finite collection of hyperplanes in V. We simply call a hyperplane arrangement in V an arrangement. Let $S = \mathrm{sym}(V^*) = \mathbb{C}[x_1, \dots, x_n]$ and let $\mathrm{Der}(S) = \bigoplus S\partial_i$ be the set of all derivations of S, where $\partial_i = \frac{\partial}{\partial_{x_i}}$. The derivation module of an arrangement $\mathcal{A}$ is the S-module defined by

$$\mathrm{Der}(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) \mid \theta \cdot \ell_H \in \ell_H \cdot S \quad \text{for all } H \in \mathcal{A}\},$$

where ℓ_H is a defining linear form of H. An arrangement $\mathcal{A}$ of V is said to be free if $\mathrm{Der}(\mathcal{A})$ is a free S-module. For an arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, the deletion of H is the arrangement $\mathcal{A} \setminus H = \mathcal{A} \setminus \{H\}$ and the restriction of $\mathcal{A}$ to H is an arrangement in $H \cong \mathbb{C}^{n-1}$ defined by

$$\mathcal{A}^H = \{H \cap H' \mid H' \in \mathcal{A} \setminus H\}.$$

Although there are free arrangements $\mathcal{A}$ such that its restriction $\mathcal{A}^H$ is not free [1], people somehow experimentaly know that a restriction of a free arrangement is likely to become free. For example, a typical example of free arrangements are supersolvable arrangements, but restriction of a supersolvable arrangement is again supersolvable and hence free. Also, a well-known result of Terao [2] says that, if $\mathcal{A}$ is free, then the freeness of $\mathcal{A} \setminus H$ implies the freeness of $\mathcal{A}^H$. Finding a free arrangement $\mathcal{A}$ such that $\mathcal{A}^H$ is not free is actually not an easy problem and our motivating problem is as follows.

Problem 1. Can we explain why if $\mathcal{A}$ is free then $\mathcal{A}^H$ is likely to be free?

This is a little ambiguous problem, but in any case we do not have an good answer to this problem. Probably, to study such a problem, we need some nice way to study the freeness of restrictions of free arrangements. Motivated by this, we introduce the following ideal.

Definition 2. Let $\mathcal{A}$ a free arrangement in V and let $H \in \mathcal{A}$ be the hyperplane defined by the linear form $\ell \in S$. Let $\theta_1, \dots, \theta_n$ a free S-basis of $\mathrm{Der}(\mathcal{A})$.[1] Write $\theta_j = \sum_{i=1}^n f_{ij}\partial_i$ and let M be the $n \times n$ matrix whose i, jth entry is f_{ij}. Write $\bar{S} = S/(\ell)$ and let $\overline{M}$ be the natural image of M by the map $S \to \bar{S} = S/(\ell)$. Let $Q \in \bar{S}$ be the product of all defining linear forms of $\mathcal{A}^H$. Then it is known that any $(n-1)$-minors of $\overline{M}$ is divisible by Q. We define the ideal $I_{\mathcal{A}}^H$ of $\bar{S}$ by

$$I_{\mathcal{A}}^H = \{f/Q \mid f \in I_{n-1}(\overline{M})\}$$

where $I_{n-1}(\overline{M})$ is the ideal of $(n-1)$-minors of $\overline{M}$.

[1] it is known that if $\mathrm{Der}(A)$ is free then it must have rank n.

We remark that the definition of $I_{\mathcal{A}}^H$ does not depend on the choice of a basis of $\mathrm{Der}(S)$ so we may assume that H is the hyperplane defined by $x_1 = 0$, and in that case the first row of $\overline{M}$ is zero. Hence $I_{n-1}(\overline{M})$ is actually the ideal of $(n-1)$-minors of an $(n-1) \times n$ matrix.

Example 3. Let $\mathcal{A} = \{H = H_x, H_y, H_z, H_{x-y}, H_{x-z}\}$, where H_ℓ is the hyperplane defined by $\ell = 0$. Then $\mathrm{Der}(\mathcal{A})$ is a free $\mathbb{C}[x,y,z]$-module generated by $x\partial_x + y\partial_y + z\partial_z, x^2\partial_x + y^2\partial_y + z^2\partial_z, x^2\partial_x + xy\partial_y + z^2\partial_z$. Thus

$$M = \begin{pmatrix} x & x^2 & x^2 \\ y & y^2 & xy \\ z & z^2 & z^2 \end{pmatrix} \quad \text{and} \quad \overline{M} = \begin{pmatrix} 0 & 0 & 0 \\ y & y^2 & 0 \\ z & z^2 & z^2 \end{pmatrix}.$$

The ideal $I_{\mathcal{A}}^H \subset \overline{S} = \mathbb{C}[x,y,z]/(x)$ is given by

$$I_{\mathcal{A}}^H = \frac{1}{yz}(yz^2 - zy^2, yz^2, y^2z^2) = (y,z).$$

Our starting point is the following observation, which easily follows from known results in hyperplane arrangement theory.

Observation Assume that $\mathcal{A}$ is free and $H \in \mathcal{A}$. Then $\mathcal{A}^H$ is free if and only if $\mathrm{pd}_{\bar{S}}(\bar{S}/I_{\mathcal{A}}^H) \leq 2$.

Now Problem 1 is equivalent to the following more comuutative algebraic problem

Problem 4. Can we show that if $\mathcal{A}$ is free then $\bar{S}/I_{\mathcal{A}}^H$ is likely to have projective dimension ≤ 2?

Again, we do not have a good answer to this problem, but as a first step to study this problem we discuss basic properties of the ideal $I_{\mathcal{A}}^H$. Our first result is the conbinatorial description of the radical of this ideal. Let

$$L_{\mathcal{A}} = \{H_1 \cap \cdots \cap H_k \mid H_1, \dots, H_k \in \mathcal{A}\}$$

be the intersection lattice of an arrangement $\mathcal{A}$. For $X \in L_{\mathcal{A}}$, the arrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$ is called the localization of $\mathcal{A}$ at X.

Theorem 5. *Let $\mathcal{A}$ be a free arrangement in V and $H \in \mathcal{A}$. Let $NFT(\mathcal{A}, H)$ consist of $X \in L(\mathcal{A}^H)$ such that $\mathcal{A}_X \setminus \{H\}$ is not free. Then*

$$\sqrt{I_{\mathcal{A}}^H} = \bigcap_{X \in NFT(\mathcal{A},H)} I_X,$$

where I_X is the defining ideal of the subspace X.

If I have time, I also discuss the primary decomposition of $I_{\mathcal{A}}^H$, the radicalness of $I_{\mathcal{A}}^H$ as well as a connection to (Alexander dual of) Fröberg's theorem on edge ideals with linear resolutions.

References

[1] P.H. Edelman and V. Reiner, A counterexample to Orlik's conjecture. *Proc. Amer. Math. Soc.*, **118** (1993), 927–929.
[2] H. Terao, Arrangements of hyperplanes and their freeness I, II. *J. Fac. Sci. Univ. Tokyo* **27** (1980), 293–320.

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