ON THE UBIQUITY OF DOMINANT LOCAL RINGS

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Generation of a module from another by applying basic operations such as taking syzygies, direct summands and extensions has been playing a key role in the homological and categorical studies of modules over rings. Takahashi [28, 29] has established a classification of the thick subcategories of the singularity category of a hypersurface using generation of modules. This does motivate us to seek for wider classes of rings than that of hypersurfaces to which similar module generation techniques apply.

In light of a result of Burch [7], Dao, Kobayashi and Takahashi [11] have introduced the notion of a *Burch ideal* of a local ring, and have shown that over the quotient ring of a regular local ring by a Burch ideal, every nonfree module generates the residue field by taking syzygies, direct summands and just one extension. As a consequence, such a ring is Tor/Ext-friendly in the sense of Avramov, Iyengar, Nasseh and Sather-Wagstaff [2]. In particular, such a ring is G-regular in the sense of Takahashi [27] if it is not a hypersurface, and the Auslander–Reiten conjecture holds for such a ring. A lot of ideals are known to be Burch ideals, including $\mathfrak{m}$-primary integrally closed ideals and nonzero ideals of the form $\mathfrak{m}I$ for some ideal I, where $\mathfrak{m}$ is the maximal ideal. However, a local ring defined by a Burch ideal must have depth zero. As an extension to the positive depth case, the notion of a *Burch ring* has been introduced also in [11]. This is a local ring whose completion is a deformation of the quotient ring of a regular local ring by a Burch ideal. A hypersurface is a typical example of a Burch ring. It is straightforward that a Burch ring is Tor/Ext-friendly. Dao and Eisenbud [10] have recently introduced the notion of a *Burch index*, and the Burch rings are characterized as the local rings with positive Burch index. A local ring with Burch index greater than one exhibits a certain extremal property on syzygies of modules; see [10, 12] for details. Many other works on Burch ideals and Burch rings have been done so far; see [8, 9, 14, 16] for instance.

The notion of a *quasi-fiber product ring* in the sense of Freitas, Jorge Pérez, R. Wiegand and S. Wiegand [15] has been investigated by several authors in recent years. This notion is the same as that of a local ring with quasi-decomposable maximal ideal in the sense of Nasseh and Takahashi [21]. Over a quasi-fiber product ring, some syzygy of the residue field can be constructed from every module of infinite projective dimension by taking a direct summand of an extension of syzygies. In particular, quasi-fiber product rings share many properties with Burch rings including Tor/Ext-friendliness. One of the next expectations is thus to unify the theories of Burch rings and quasi-fiber product rings.

The notions of a *dominant local ring* and a *uniformly dominant local ring* have been introduced by Takahashi [30, 31]. A dominant local ring is a local ring whose residue field can be classically generated in the sense of Bondal and Van den Bergh [5] by any nonzero object in the singularity category. The *dominant index* $\mathrm{dx}(R)$ of a local ring R is the least number n such that the residue field is generated in the singularity category by any nonzero object by taking finite direct sums, direct summands, shifts and at most n extensions. A local ring is called uniformly dominant if it has finite dominant index. Thus a uniformly dominant local ring is dominant, while a dominant local ring is Tor/Ext-friendly. Both a Burch ring and a quasi-fiber product ring are uniformly dominant. A remarkable fact is that the thick subcategories of the singularity category are classified completely if the localization at each prime ideal is dominant; see [30, 31] for details.

A *Golod local ring* is a local ring defined by an extremal behavior of the Poincaré series of the residue field. Basic facts on Golod rings are found in [1]. It is known that a Golod ring is Tor/Ext-friendly [2, 17]. Typical examples of Golod local rings include local hypersurfaces and Cohen–Macaulay local rings with minimal multiplicity, which are also Burch rings. In general, however, a Golod local ring is neither a Burch ring nor a quasi-fiber product ring; see [13, 19]. These observations lead us to investigate the relationship between Golodness and dominance further. It is thus quite natural to ask the following question.

Question. (1) When is a given local ring dominant, or more strongly, uniformly dominant?
(2) If a given local ring is uniformly dominant, then how big is its dominant index?
(3) Is every Golod local ring dominant, or more strongly, uniformly dominant?

In this talk, the speaker will report some answers to the above question which he has obtained in ongoing joint work with Toshinori Kobayashi [18].

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