The Golod property for Buchsbaum local rings of maximal embedding dimension by Ken-ichi Yoshida (Nihon University)

1. Introduction

Throughout this talk, let $(A, \mathfrak{m})$ be a commutative Noetherian local ring of dimension d with residue field $k = A/\mathfrak{m}$. For a finitely generated A-module M and an $\mathfrak{m}$-primary ideal $I \subset A$, $\ell_A(M)$ (resp. $\mu_A(M)$, $e_0(I, M)$) denote the *length* (resp. the *minimal number of generators* of M, the *multiplicity* of M with respect to I). In particular, we put $v(A) = \mu_A(\mathfrak{m})$ and $e_0(A) = e_0(\mathfrak{m}, A)$.

Let $\mathcal{F}$ be a minimal free resolution of M over A:

$$\mathcal{F} : \ \cdots \to F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \to M \to 0.$$

Then $\Omega_A^i(M) := \operatorname{Ker}(f_i)$ is called the i^{th} *syzygy* of M. Moreover, $\beta_i^A(M) := \dim_k \operatorname{Tor}_i^A(k, M) = \mu(\Omega_A^i(M))$ is called the i^{th} *Betti number* of M. For instance, $\beta_0^A(k) = 1$ and $\beta_1^A(k) = \mu(\mathfrak{m}) = v(A) = c + d$.

Let $A = R/\mathfrak{a}$, where $(R, \mathfrak{n})$ is a regular local ring of dimension v and $\mathfrak{a} \subset \mathfrak{n}^2$. Then Serre showed the following (coefficientwise) inequality:

$$\sum_{i=0}^{\infty} \beta_i^A(k) t^i \le \frac{(1+t)^v}{1 - t \displaystyle\sum_{i=1}^{\infty} \beta_i^R(A) t^i}. \tag{1.1}$$

Definition 1.1 (**Golod ring**). The ring $A = R/\mathfrak{a}$ is called *Golod* if Equality holds in Eq.(1.1).

Golod rings enjoy several important properties. In fact, the Poincare series $P_A(t) = \sum_{i=0}^{\infty} \beta_i^A(k) t^i$ of such a ring becomes a rational function. Many researchers have studied these properties, and so several Golod rings are known. In particular, it is known that <u>Cohen-Macaulay local rings of maximal embedding dimension</u> (... of minimal multiplicity) <u>are Golod</u>.

Now suppose that A is Buchsbaum, that is, for any parameter ideal $\mathfrak{q}$ of A, $\ell_A(A/\mathfrak{q}) - e_0(\mathfrak{q}; A)$ is constant (denoted by $I(A)$). Then

$$v(A) \le e_0(A) + d - 1 + I(A), \quad \text{and} \quad I(A) = \sum_{i=0}^{d-1} \tbinom{d-1}{i} \ell(H_{\mathfrak{m}}^i(A))$$

holds true. If equality holds, then A is said to be a *Buchsbaum local ring of maximal embedding dimension* (Goto [4]). So it is natural to ask the following question.

Question 1.2. How about Buchsbaum local ring of maximal embedding dimension?

The aim of this talk to prove the following theorem, which gives an affirmative answer to the question as above.

Theorem 1.3. *Any Buchsbaum local ring of maximal embedding dimension is Golod.*

2. Proof of Theorem

In what follows, let A be a **Buchsbaum local ring of maximal embedding dimension** with $h^j(A) = \ell(H^j_{\mathfrak{m}}(A))$ for each j. In [5, Theorem 2.4], the speaker proved that $\Omega^i_A(k)$ is a Buchsbaum A-module of dimension d with $\mu(\Omega^i_A(k)) = e_0(\Omega^i_A(k)) + I(\Omega^i_A(k))$. In the proof, we showed the following properties, which is useful for our proof of Theorem 2.3.

Lemma 2.1. *Assume* $\operatorname{depth} A > 0$. *Then there exists a non-zero divisor* $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ *such that* $\overline{A} = A/xA$ *is also a Buchsbaum local ring of maximal embedding dimension with* $\dim \overline{A} = d - 1$ *and* $e_0(\overline{A}) = e_0(A)$. *Furtheremore, for* $i \geq 2$, *we have*

$$\beta^A_i(k) = \beta^{\overline{A}}_i(k) + \beta^{\overline{A}}_{i-1}(k) \quad \text{for all } i \geq 1. \tag{2.1}$$

Lemma 2.2. *Assume* $d \geq 1$ *and* $\operatorname{depth} A = 0$. *If we put* $H = H^0_{\mathfrak{m}}(A)$, $B = A/H$, *then* B *is a Buchsbaum local ring of maximal embedding dimension with* $\operatorname{depth} B > 0$ *and* $\dim B = \dim A$, $v(B) = v(A) - h^0(A)$. *Moreover, for all* $i \geq 1$,

$$\beta^A_i(k) = \beta^B_i(k) + h^0(A) \sum_{r=0}^{i-1} \beta^B_{i-1-r}(k)\beta^A_r(k). \tag{2.2}$$

Using these lemmata, we prove the following theorem.

Theorem 2.3 (**Recursive Formula**). *Put* $c_i(A) = \sum_{j=0}^{d-i+1} \binom{d}{i+j-1} h^j(A)$ *for all* $2 \leq i \leq d+1$ *and* $c = v(A) - d$. *Then* $\{\beta^A_i(k)\}$ *satisfies the following formula*:

$$\beta^A_i(k) = c\,\beta^A_{i-1}(k) + c_2(A)\beta^A_{i-2}(k) + \cdots + c_i(A)\beta^A_0(k) + \binom{d}{i} \ (1 \leq \forall i \leq d)$$

$$\beta^A_i(k) = c\,\beta^A_{i-1}(k) + c_2(A)\beta^A_{i-2}(k) + \cdots + c_{d+1}(A)\beta^A_{i-d-1}(k) \ (\forall i \geq d+1)$$

Goto [4] showed that the associated graded ring $G(A) = \oplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is Buchsbaum and that the minimal free resolution of $G(A)$ is given obtained from one of A. Especially, one has $\beta^R_i(A) = \beta^S_i(G)$ for every $i \geq 1$, where $A = R/\mathfrak{a}$ and $S = k[X_1, \ldots, X_v]$. In order to prove **Theorem 1.3**, it is enough to show

$$\left[1 - ct - c_2t^2 - \cdots - c_{d+1}t^{d+1}\right]\left[1 + \beta^A_1(k)t + \beta^A_2(k)t^2 + \cdots\right] = (1+t)^d.$$

As another application, we can prove that any Buchsbaum local ring of maximal embedding dimension is a **Burch** ring (see [3]).

References

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