SERRE DEPTH AND SYMBOLIC POWERS ON STANLEY–REISNER RINGS

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This talk is based on our recent preprint [1]. In commutative ring theory, Serre's condition (S_r) provides a natural and essential generalization of the Cohen–Macaulay property, which has been a foundational concept for many years. Let us recall the definition of Serre's condition (S_r). Let $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$ and I be a homogeneous ideal of S and let $r \geq 2$. Then S/I satisfies *Serre's condition* (S_r), if the inequality $\operatorname{depth}((S/I)_{\mathfrak{p}}) \geq \min\{r, \dim(S/I)_{\mathfrak{p}}\}$ holds for every $\mathfrak{p} \in \operatorname{Spec}(S)$. Motivated by [4, Lemma 3.2.1], the Serre depth for (S_r) of S/I is defined in [3, Definition 3.8] for an unmixed homogeneous ideal I:

Definition 1.1. The *Serre depth* for (S_r) is defined as

$$S_r\text{-depth}(S/I) = \min\{j \ : \dim K^j_{S/I} \geq j - r + 1\},$$

where $K^j_{S/I} = \operatorname{Hom}_{\Bbbk}(H^j_{\mathfrak{m}}(S/I), \Bbbk)$ and $\dim K^j_{S/I}$ denotes the Krull dimension of $K^j_{S/I}$ as an S-module. Here, if $K^j_{S/I} = 0$, then we set $\dim K^j_{S/I} = -\infty$.

By the definition and [4, Lemma 3.2.1], one can see that $S_r\text{-depth}(S/I) = \dim S/I$ if and only if S/I satisfies Serre's condition (S_r). Also, one can see that

$$\dim S/I \geq S_2\text{-depth}(S/I) \geq \cdots \geq S_d\text{-depth}(S/I) = \operatorname{depth}(S/I),$$

where $d = \dim S/I$. Therefore, the Serre depth is defined as an analogue of the depth: just as the depth measures how far a module is from being Cohen–Macaulay, the Serre depth does so with respect to Serre's condition (S_r).

We will focus on Stanley–Reisner ideals. Concretely, we relate the Serre depth both to the minimal free resolution of a Stanley–Reisner ring and to that of its Alexander dual. Also, we obtain a generalization of a known result that describes the depth of Stanley–Reisner rings in terms of skeletons [5, Theorem 3.7]:

Theorem 1.2. *Let Δ be a simplicial complex on $[n]$. Then we have*

(1) $\max\{i \ : \beta_{i,i+j}(\Bbbk[\Delta]) \neq 0$ *for all* $j < r\} \leq n - S_r\text{-depth}(\Bbbk[\Delta])$.

(2) $\max\{j \mid \beta_{i,i+j}(\Bbbk[\Delta^{\vee}]) \neq 0$ *for some* $i \leq r\} = n - S_r\text{-depth}(\Bbbk[\Delta]) - 1$.

(3) $\operatorname{reg}_{\leq r-1} I_{\Delta^{\vee}} - \operatorname{indeg} I_{\Delta^{\vee}} = \dim \Bbbk[\Delta] - S_r\text{-depth}(\Bbbk[\Delta])$.

(4) $S_r\text{-depth}(\Bbbk[\Delta]) = 1 + \max\{i \mid \Bbbk[\Delta^i]$ *satisfies Serre's condition* $(S_r)\}$.

Here, $\operatorname{reg}_{\leq r-1} I_{\Delta^{\vee}} = \max\{j \ : \beta_{i,i+j}(I_{\Delta^{\vee}}) \neq 0$ *for some* $i \leq r-1\}$ *and* $\Delta^i = \{F \mid \dim F \leq i\}$ *for* $-1 \leq i \leq \dim \Delta$.

Moreover, we discuss the Serre depth for (S_2) and the depth on symbolic powers of Stanley–Reisner ideals. It had been an open question whether the depth on the symbolic powers of Stanley–Reisner ideals satisfies the non-increasing property, but Nguyen and Trung provided a negative answer, where the ideal they construct is not unmixed [2, Theorem 2.8]. Accordingly, we construct a Stanley–Reisner ideal that does not satisfy the non-increasing property for both depth and Serre depth for (S_2), whereas it is Cohen–Macaulay and its second symbolic power is also Cohen–Macaulay:

Theorem 1.3. *There exists a non-cone simplicial complex* Δ *with* $\dim \Delta = d-1 \geq 2$ *such that the following conditions are satisfied:*

(1) Δ *is pure and shellable,*
(2) $S/I_\Delta^{(2)}$ *is Cohen–Macaulay,*
(3) $\operatorname{depth}(S/I_\Delta^{(d+1)}) = 1$ *and* $\operatorname{depth}(S/I_\Delta^{(d+2)}) \geq 2$*,*
(4) $S_2\text{-depth}(S/I_\Delta^{(d+1)}) = 1$ *and* $S_2\text{-depth}(S/I_\Delta^{(d+2)}) \geq 2$*.*

Moreover, we obtain that the Serre depth for (S_2) and the depth satisfy the non-increasing property when $\dim \Delta = 1$:

Theorem 1.4. *Let* Δ *be a simplicial complex with* $\dim \Delta = 1$ *and* $\ell \geq 1$*. Then we have*

$$\operatorname{depth}(S/I_\Delta^{(\ell)}) \geq \operatorname{depth}(S/I_\Delta^{(\ell+1)}).$$

Moreover, if Δ *is pure, then we have* $S_2\text{-depth}(S/I_\Delta^{(\ell)}) \geq S_2\text{-depth}(S/I_\Delta^{(\ell+1)})$*.*

As a corollary, we completely classify the Serre depth for (S_2) and the depth of the symbolic powers of Stanley–Reisner ideals when $\dim \Delta = 1$.

Finally, we study the Serre depth on edge and cover ideals. It has been an open question whether the depth on the symbolic powers of edge ideals satisfies the non-increasing property. As a natural analogue of this important problem, while it remains open for the depth, we obtain that the non-increasing property of the Serre depth for (S_2) holds for the edge ideals of any well-covered graph. Note that well-coveredness means the edge ideal is unmixed.

Theorem 1.5. *For a well-covered graph* G*, we have*

$$S_2\text{-depth}(S/I(G)^{(\ell)}) \geq S_2\text{-depth}(S/I(G)^{(\ell+1)}) \text{ for all } \ell.$$

In addition, we obtain that the Serre depth for (S_2) on the cover ideals of any graph also satisfies the non-increasing property:

Theorem 1.6. *For a graph* G*, we have*

$$S_2\text{-depth}(S/J(G)^{(\ell)}) \geq S_2\text{-depth}(S/J(G)^{(\ell+1)}) \text{ for all } \ell.$$

References

[1] Y. Muta and N. Terai, The Serre depth of Stanley-Reisner rings and the depth of their symbolic powers, Preprint (2025), arXiv: 2509. 14838

[2] H.D. Nguyen and N.V. Trung, Depth functions of symbolic powers of homogeneous ideals, *Invent. math.* **218** (2019), 779–827.

[3] M.R. Pournaki, M. Poursoltani, N. Terai, and S. Yassemi, Simplicial complexes satisfying Serre's condition versus the ones which are Cohen–Macaulay in a fixed codimension, *SIAM J. Discrete Math.* **36** (2022), no. 4, 2506–2522.

[4] P. Schenzel, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, Lecture Notes in Math., vol. 907, Springer, Berlin, 1982.

[5] D.E. Smith, On the Cohen–Macaulay property in commutative algebra and simplicial topology, *Pacific J. Math.* **141** (1990), no. 1, 165–196.