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CS 575 HW #1

1. Prove the following using the original definitions of O, Ω, θ, and ω.

1.1. $15n^2 + 5n + 8 = O(n^2)$
g(n) is O(f(n)) if for positive constants c and N, 0 <= g(n) <= cf(n) for all n >= N
$g(n) = 15n^2 + 5n + 8$
$f(n) = n^2$
$0 <= 15n^2 + 5n + 8 <= cn^2$
$0 <= 15 + 5/n + 8/n^2 <= c$ (Divide both sides by n^2)
For N = 1, c = 28, 0 <= 28 <= 28
Expression is satisfied.

1.2. $n^3 + 2n + 5 = \Omega(n^3)$
g(n) is Ω(f(n)) if for positive constants c and N, 0 <= cf(n) <= g(n) for all n >= N
$g(n) = n^3 + 2n + 5$
$f(n) = n^3$
$0 <= cn^3 <= n^3 + 2n + 5$
$0 <= c <= 1 + 2/n^2 + 5/n^3$ (Divide both sides by n^3)
For N = 1, c = 8, 0 <= 8 <= 8
Expression is satisfied.

1.3. $\sum_{i=1}^{k} a_i n^i = \Omega(n^k)$ for k > 1. Assume that $a_i > 0$ for all i.
g(n) is Ω(f(n)) if for positive constants c and N, 0 <= cf(n) <= g(n) for all n >= N
$g(n) = \sum_{i=1}^{k} a_i n^i$
$f(n) = n^k$
$0 <= cn^k <= \sum_{i=1}^{k} a_i n^i$
$0 <= cn^k <= a(1-n^k)/(1-n)$ (Geometric Series)
$0 <= cn^k <= a/(1-n) - an^k/(1-n)$ (Separate terms)
$0 <= c <= a/n^k(1-n) - a(1-n)$ (Divide both sides by n^k)
For N = 2, k = 1, a = 1, c = ½, 0 <= ½ <= ½

1.4. $n^7 + 3n^2 = \theta(n^7)$
g(n) is θ(f(n)) if for positive constants c, d, and N, 0 <= cf(n) <= g(n) <= df(n) for all n >= N
$0 <= cn^7 <= n^7 + 3n^2 <= dn^7$
$0 <= cn^7 <= n^7 + 3n^2$
$0 <= c <= 1 + 3/n^5$
For N = 1, c = 4, 0 <= 4 <= 4
Expression is satisfied.

$n^7 + 3n^2 <= dn^7$
For N = 1, d = 4, 4 <= 4
Expression is satisfied.

2. Prove the following using limits.

$$\text{Master Theorem} = \lim_{n\to\infty} f(n)/g(n) = \begin{cases} c \text{ then } f(n) = \theta(g(n)) \text{ if } c > 0 \\ 0 \text{ then } f(n) = o(g(n)) \\ \infty \text{ then } f(n) = \omega(g(n)) \end{cases}$$

2.1. $n^k = o(3^n)$ where $k >= 0$
$f(n) = n^k$
$g(n) = 3^n$
$\lim_{n\to\infty} n^k/3^n$
$\lim_{n\to\infty} kn^{k-1}/\ln(3) * 3^n$ (L'Hopital's)
$\lim_{n\to\infty} kn^{k-1}/\ln(3) * 3^n = \lim_{n\to\infty} k(k-1)n^{k-2}/ln^2(3) * 3^n = \ldots$ (Continue taking derivative)
$\lim_{n\to\infty} k!/\ln^k(3) * 3^n = 0$

2.2. $n \ln n^7 = o(n^3)$
$f(n) = n \ln n^7$
$g(n) = n^3$
$\lim_{n\to\infty} n \ln n^7 \;/\; n^3$
$\lim_{n\to\infty} \ln n^7 \;/\; n^2$
$\lim_{n\to\infty} 7 \;/\; 2n^2 = 0$ (L'Hopital's)

3. Prove that $7^n - 1$ is divisible by 6 for n = 1, 2, 3, … using induction
m mod n = m - n floor(m / n)
<u>Base</u>: For n = 1
$7^n - 1 = 7^1 - 1 = 6$. 6 - 6 floor(1) = 0. Satisfied.
<u>Hypothesis</u>: Assume true for (n+1).
<u>Induction</u>: $7^{n+1} - 1$ is divisible by 6.
$7*7^n - 1 - (7^n - 1)$. If we subtract a multiple of 6 from a value, we should get another multiple of 6.
$6*7^n$ is a multiple of 6 as it is multiplied by 6.

4. Analyze the worst-case time complexity in terms of Big Oh notation. Briefly, yet clearly explain your answer.

4.1. O(lg(n)). By instruction count, we see that lg(n) + 1 instructions are performed in the loop due to the statement i=2*i.

4.2. O(n lg(n)). By instruction count, the outer for loop executes n times. The three function calls have time complexities of O(1), O(lg(n)), and O(1), respectively. Since each of these are called n times, the highest order term is then n lg(n).

4.3. O(2^s), where s is the number of bits for a value n. When the value computed for a factorial gets large enough, it will no longer fit in a word and multiplication will no longer be a constant time operation.