修士論文

圧力ポアソン方程式と ε-ストークス方程式の境界値問題

Boundary value problems for a pressure-Poisson equation and an ε-Stokes equation

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Abstract

We generalize pressure boundary conditions of an ε-Stokes problem. Our ε-Stokes problem connects the classical Stokes problem and the corresponding pressure-Poisson equation using one parameter $\varepsilon > 0$. For the Dirichlet boundary condition, it is proven in K. Matsui and A. Muntean (2018) that the solution for the ε-Stokes problem converges to the one for the Stokes problem as ε tends to 0, and to the one for the pressure-Poisson problem as ε tends to ∞. Here, we extend these results to the Neumann and mixed boundary conditions. We also establish error estimates in suitable norms between the solutions to the ε-Stokes problem, the pressure-Poisson problem and the Stokes problem, respectively. Several numerical examples are provided to show that several such error estimates are optimal in ε. Our error estimates are improved if one uses the Neumann boundary conditions. In addition, we show that the solution to the ε-Stokes problem has a nice asymptotic structure.

1 Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n (n \geq 2, n \in \mathbb{N})$ and let $F : \Omega \to \mathbb{R}^n$ be a given applied force field and $u_b : \Gamma := \partial\Omega \to \mathbb{R}^n$ be a given Dirichlet boundary data satisfying $\int_\Gamma u_b \cdot \nu = 0$, where ν is the unit outward normal vector on Γ. A strong form of the Stokes problem is given as follows. Find $u_S : \Omega \to \mathbb{R}^n$ and $p_S : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u_S + \nabla p_S = F & \text{in } \Omega, \\ \operatorname{div} u_S = 0 & \text{in } \Omega, \\ u_S = u_b & \text{on } \Gamma, \end{cases} \tag{S}$$

where u_S and p_S are the velocity and the pressure of the flow governed by (S), respectively. We refer to [26] for the details on the Stokes problem (i.e., physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u_S + \nabla p_S) = -\Delta(\operatorname{div} u_S) + \Delta p_S = \Delta p_S. \tag{1.1}$$

This equation is often called the pressure-Poisson equation and is used in numerical schemes such as MAC (marker and cell), SMAC (simplified MAC) and the projection methods (see, e.g., [3, 7, 10, 16, 17, 20, 23, 25]).

We need an additional boundary condition for solving the equation (1.1). In the real-would applications, the additional boundary condition is usually given by using experimental or plausible values. We consider the following problem: Find $u_{PP} : \Omega \to \mathbb{R}^n$ and $p_{PP} : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u_{PP} + \nabla p_{PP} = F & \text{in } \Omega, \\ -\Delta p_{PP} = -\operatorname{div} F & \text{in } \Omega, \\ u_{PP} = u_b & \text{on } \Gamma, \\ +\text{boundary condition for } p_{PP}. \end{cases} \tag{PP}$$

We call this problem the pressure-Poisson problem. The idea of using (1.1) instead of $\operatorname{div} u_S = 0$ is useful for calculating the pressure numerically in the Navier–Stokes equation. For example, this idea is used in MAC, SMAC and projection methods. The Dirichlet boundary condition for the pressure is used in an outflow boundary [6, 27]. See also [8, 9, 21].

We introduce an "interpolation" between problems (S) and (PP). For $\varepsilon > 0$, find $u_\varepsilon : \Omega \to \mathbb{R}^n$ and $p_\varepsilon : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = F & \text{in } \Omega, \\ -\varepsilon\Delta p_\varepsilon + \operatorname{div} u_\varepsilon = -\varepsilon \operatorname{div} F & \text{in } \Omega, \\ u_\varepsilon = u_b & \text{on } \Gamma, \\ +\text{boundary condition for } p_\varepsilon. \end{cases} \tag{ES}$$

This problem is called the ε-Stokes problem (ES) in [22]. In [11, 15, 19], the authors treat this problem as an approximation of the Stokes problem to avoid numerical instabilities. The ε-Stokes problem approximates the Stokes problem (S) as $\varepsilon \to 0$ and the pressure-Poisson problem (PP) as $\varepsilon \to \infty$ (Fig. 1). It is shown in [22] that if $p_S \in H^1(\Omega)$ then there exists a constant $c > 0$ independent of ε such that

$$\begin{aligned} \|u_S - u_{PP}\|_{H^1(\Omega)^n} &\leq c\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}, \\ \|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\leq c\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}, \end{aligned}$$

where $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ is the standard trace operator [14]. From the first inequality, if we have a good prediction value for pressure on Γ, then u_{PP} is a good approximation of u_S. Moreover, u_ε is also a good approximation of u_S from the second inequality.

$$\begin{tikzcd}
(PP) \arrow[rr, dotted, no head] & & (S) \\
& (ES) \arrow[ul, "\varepsilon\to\infty"] \arrow[ur, "\varepsilon\to 0"'] &
\end{tikzcd}$$

Figure 1: Sketch of the connections between the problems (S), (PP) and (ES).

Next we specify the boundary conditions for p_{PP} and p_ε. We assume that the boundary Γ is a union of two open subsets Γ_D and Γ_N such that

$$|\Gamma \setminus (\Gamma_D \cup \Gamma_N)| = 0, \quad |\Gamma_D| > 0, \quad \Gamma_D \cap \Gamma_N = \emptyset,$$

and number of connected components of Γ_D and Γ_N with respect to the relative topology of Γ are finite. We consider a Neumann boundary condition (1.2) and a mixed boundary condition (1.3),

$$\frac{\partial p_{PP}}{\partial \nu} = g_b \text{ on } \Gamma, \quad \frac{\partial p_\varepsilon}{\partial \nu} = g_b \text{ on } \Gamma, \tag{1.2}$$

$$\begin{cases} p_{PP} = p_b & \text{on } \Gamma_D, \\ \dfrac{\partial p_{PP}}{\partial \nu} = g_b & \text{on } \Gamma_N, \end{cases} \quad \begin{cases} p_\varepsilon = p_b & \text{on } \Gamma_D, \\ \dfrac{\partial p_\varepsilon}{\partial \nu} = g_b & \text{on } \Gamma_N, \end{cases} \tag{1.3}$$

where $p_b : \Gamma_D \to \mathbb{R}$ and $g_b = \Gamma \to \mathbb{R}$ satisfying $\int_\Gamma g_b = \int_\Gamma \operatorname{div} F$ are given boundary data.

In [22], the authors impose Dirichlet boundary conditions for p_{PP} and p_ε (i.e., (1.3) with $\Gamma_D = \Gamma$ and $\Gamma_N = \emptyset$.) For such boundary conditions, they introduce a weak solution $(u_\varepsilon, p_\varepsilon)$ to the ε-Stokes problem (ES) and prove that $(u_\varepsilon, p_\varepsilon)$ strongly converges in $H^1(\Omega)^n \times H^1(\Omega)$ to a weak solution to the pressure-Poisson problem (PP) as $\varepsilon \to \infty$ and weakly converges in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to a weak solution (u_S, p_S) to the Stokes problem (S) as $\varepsilon \to 0$. Moreover, if $p_S \in H^1(\Omega)$, then strong convergence of $(u_\varepsilon, p_\varepsilon)$ to (u_S, p_S) as $\varepsilon \to 0$ holds.

In this paper, we generalize the Dirichlet boundary condition of p_{PP} and p_ε to both the Neumann boundary condition (1.2) and the mixed boundary condition (1.3), and prove the corresponding convergence result (see Theorem 3.1, 4.2 and 4.3). Since the mixed boundary condition for pressure often appears in engineering problems, this generalization of the boundary conditions for pressure is important. In addition, for the Neumann boundary condition, we estimate the error between the weak solutions to (ES) and (S) provided $p_S \in H^1(\Omega)$. We also give an asymptotic expansion for the weak solution to (ES). We furthermore check this convergence result using numerical computations.

The organization of this paper is as follows. In Section 2 we introduce the notation used in this work and the weak form of these problems. We also prove the well-posedness of the problems (PP) and (ES). In Section 3 we establish error estimates between solutions to the problems (PP), (ES) and (S) in terms of the additional boundary conditions. In Section 4 we study that the solution to (ES) converges to the solution to (PP) in the strong topology as $\varepsilon \to \infty$. We also explore here the structure of the regular perturbation asymptotics. Section 5 is devoted to proving that the solution to (ES) converges to the solution to (S) in the weak and strong topology as $\varepsilon \to 0$. In Section 6, we show several numerical examples of these problems. The numerical errors between the problems (ES) and (PP), and between the problems (ES) and (S) using the P2/P1 finite element method. We conclude this paper with several comments on future works in Section 7. In Appendix A we prove the Nečas inequality on a bounded Lipschitz domain. Proofs for several theorems which are similar to ones in [22] are described in Appendix B.

2 Preliminaries

In this section, we introduce the notation and the weak form of the problems (S), (PP) and (ES), and prove their well-posedness. We give estimates between these solutions by using a pressure error on the boundary Γ.

2.1 Notation

We set

$$\begin{aligned}
C^\infty(\overline{\Omega}) &:= \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is infinitely differentiable on } \Omega \text{ which can be} \\ \text{continuously extended with all their derivatives} \\ \text{to the closure } \overline{\Omega} \text{ of } \Omega. \end{array} \right\}, \\
C_0^\infty(\Omega) &:= \{ f \in C^\infty(\overline{\Omega}) \mid \mathrm{supp}(f) \text{ is compact subset in } \Omega \}, \\
L^2(\Omega) &:= \left\{ f : \Omega \to \mathbb{R} : \text{measurable} \;\middle|\; \int_\Omega |f|^2 < \infty \right\}, \\
H^1(\Omega) &:= \left\{ f \in L^2(\Omega) \;\middle|\; \frac{\partial f}{\partial x_i} \in L^2(\Omega) \text{ for all } i = 1, \cdots, n \right\}, \\
H^2(\Omega) &:= \left\{ f \in H^1(\Omega) \;\middle|\; \frac{\partial^2 f}{\partial x_i \partial x_j} \in L^2(\Omega) \text{ for all } i, j = 1, \cdots, n \right\}, \\
H_0^1(\Omega) &:= \overline{C_0^\infty(\Omega)}^{H^1(\Omega)}, \\
L^2(\Omega)/\mathbb{R} &:= \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega f = 0 \right\}, \\
H^1(\Omega)/\mathbb{R} &:= H^1(\Omega) \cap (L^2(\Omega)/\mathbb{R}), \\
H_{0,D}^1(\Omega) &:= \{ \psi \in H^1(\Omega) \mid \psi|_{\Gamma_D} = 0 \},
\end{aligned}$$

where $\mathrm{supp}(f)$ means the support of the function f. $\mathscr{D}'(\Omega)$ denotes the space of distributions on Ω.

The spaces $L^2(\Omega)^m (m \in \mathbb{N}), H^1(\Omega)$ and $H^1(\Omega)^n$ are Hilbert spaces with the scalar products

$$\begin{aligned}
(f, g)_{L^2(\Omega)^m} &:= \int_\Omega f \cdot g && \text{for } f = (f_1, \cdots, f_m), \\
& && \quad g = (g_1, \cdots, g_m) \in L^2(\Omega)^m, \\
(p, q)_{H^1(\Omega)} &:= \int_\Omega pq + \int_\Omega \nabla p \cdot \nabla q && \text{for } p, q \in H^1(\Omega), \\
(u, v)_{H^1(\Omega)^n} &:= \int_\Omega u \cdot v + \int_\Omega \nabla u : \nabla v && \text{for } u, v \in H^1(\Omega)^n,
\end{aligned}$$

where

$$\begin{aligned}
f \cdot g &:= \sum_{i=1}^m f_i g_i, \\
\nabla p &:= \left(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_n} \right)^T, \\
\nabla u : \nabla v &:= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j},
\end{aligned}$$

respectively. Here, A^T denotes the transpose of matrix A. We define the following objects:

$$
\boxed{
\begin{array}{rcl}
[f] &:=& f - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f,\\
\|f\|_{L^2(\Omega)} &:=& \sqrt{(f,f)_{L^2(\Omega)}}\\
\|F\|_{L^2(\Omega)^n} &:=& \sqrt{(F,F)_{L^2(\Omega)^n}}\\
\|p\|_{H^1(\Omega)} &:=& \sqrt{(p,p)_{H^1(\Omega)}}\\
\|u\|_{H^1(\Omega)^n} &:=& \sqrt{(u,u)_{H^1(\Omega)^n}}\\
\|\nabla u\|_{L^2(\Omega)^{n\times n}} &:=& \sqrt{(u,u)_{H^1(\Omega)^n} - (u,u)_{L^2(\Omega)^n}}\\
\|p\|_{H^1(\Omega)/H^1_0(\Omega)} &:=& \inf\limits_{\psi\in H^1_0(\Omega)} \|p+\psi\|_{H^1(\Omega)},\\
\|f\|_{L^2(\Omega)/\mathbb{R}} &:=& \inf\limits_{a\in\mathbb{R}} \|f+a\|_{L^2(\Omega)} = \|[f]\|_{L^2(\Omega)},\\
\langle \nabla f, \varphi\rangle &:=& -\displaystyle\int_\Omega f \operatorname{div}\varphi,\\
\Delta v &:=& \displaystyle\sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}
\end{array}
}
$$

for all $f \in L^2(\Omega), F \in L^2(\Omega)^n, p \in H^1(\Omega), u \in H^1(\Omega)^n, \varphi \in H^1_0(\Omega)$ and $v : \Omega \to \mathbb{R}$, where $|\Omega|$ is the volume of Ω.

We also use the Lebesgue space $L^2(\Gamma)$ and Sobolev space $H^{1/2}(\Gamma)$ defined on Γ. The norm $\|\eta\|_{H^{1/2}(\Gamma)}$ is defined by

$$\|\eta\|_{H^{1/2}(\Gamma)} := \left(\|\eta\|^2_{L^2(\Gamma)} + \int_\Gamma\int_\Gamma \frac{|\eta(x)-\eta(y)|^2}{|x-y|^3} ds(x)ds(y)\right)^{1/2} \qquad \text{for } \eta \in H^{1/2}(\Gamma).$$

For $m = 1$ or $m = n$, $H^{-1}(\Omega)^m = (H^1_0(\Omega)^m)^*$ is equipped with the norm

$$\|f\|_{H^{-1}(\Omega)^m} := \sup_{\varphi\in S_m} \langle f, \varphi\rangle$$

for $f \in H^{-1}(\Omega)^m$, where

$$S_m := \{\varphi \in H^1_0(\Omega)^m \mid \|\nabla\varphi\|_{L^2(\Omega)^{n\times m}} = 1\}.$$

Let $Q \subset H^1(\Omega)$ be a closed subspace such that there exists a constant $c > 0$ for which $\|q\|_{L^2(\Omega)} \le c\|\nabla q\|_{L^2(\Omega)^n}$ for all $q \in Q$. The dual space Q^* is equipped with the norm

$$\|f\|_{Q^*} := \sup_{\psi\in S_Q} \langle f, \psi\rangle$$

for $f \in Q^*$, where

$$S_Q := \{\psi \in Q \mid \|\nabla\psi\|_{L^2(\Omega)^n} = 1\}.$$

2.2 Preliminary results

Let $\gamma_0 \in \mathscr{L}(H^1(\Omega), H^{1/2}(\Gamma))$ be the standard trace operator. The trace operator γ_0 is surjective and satisfies $\mathrm{Ker}(\gamma_0) = H_0^1(\Omega)$ [14, Theorem 1.5]. Let ν be the unit outward normal for Γ. Since ν is a unit vector, $H^1(\Omega)^n \ni u \mapsto u \cdot \nu := (\gamma_0 u) \cdot \nu \in H^{1/2}(\Gamma)$ is a linear continuous map. For all $u \in H^1(\Omega)^n$ and $p \in H^1(\Omega)$, the following Gauss divergence formula holds:

$$\int_\Omega u \cdot \nabla p + \int_\Omega (\operatorname{div} u) p = \int_\Gamma (u \cdot \nu) p.$$

Composition of the trace operator γ_0 and the restriction $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_D)$ is denoted by $\psi \mapsto \psi|_{\Gamma_D}$, which is a continuous map from $H^1(\Omega)$ to $H^{1/2}(\Gamma_D)$. Since the kernel of this map is $H_{0,D}^1(\Omega)$, there exists a constant $c > 0$ such that

$$\|\psi\|_{H^1(\Omega)/H_{0,D}^1(\Omega)} \leq c \|\psi|_{\Gamma_i}\|_{H^{1/2}(\Gamma_D)},$$

where $\|\psi\|_{H^1(\Omega)/H_{0,D}^1(\Omega)} := \inf_{q \in H_{0,D}^1(\Omega)} \|\psi + q\|_{H^1(\Omega)}$. We simply write ψ instead of $\psi|_{\Gamma_D}$ when there is no ambiguity. For $\tilde{\Gamma} = \Gamma$ or Γ_N, we denote by $\langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ the duality pairing between $H^{-1/2}(\tilde{\Gamma}) := H^{1/2}(\tilde{\Gamma})^*$ and $H^{1/2}(\tilde{\Gamma})$. We remark that $\eta^* \in L^2(\tilde{\Gamma})$ can be identified with an element of $H^{-1/2}(\tilde{\Gamma})$ by

$$\langle \eta^*, \eta \rangle_{\tilde{\Gamma}} := \int_{\tilde{\Gamma}} \eta^* \eta \qquad \text{for all } \eta \in H^{1/2}(\tilde{\Gamma}).$$

For $p \in H^1(\Omega)$ satisfying $\Delta p \in L^2(\Omega)$, we set

$$\begin{aligned} \left\langle \frac{\partial p}{\partial \nu}, \psi_1 \right\rangle_{\Gamma_N} &:= \int_\Omega (\nabla p \cdot \nabla \psi_1 + (\Delta p) \psi_1) \quad \text{for all } \psi_1 \in H_{0,D}^1(\Omega), \\ \left\langle \frac{\partial p}{\partial \nu}, \psi_2 \right\rangle_{\Gamma} &:= \int_\Omega (\nabla p \cdot \nabla \psi_2 + (\Delta p) \psi_2) \quad \text{for all } \psi_2 \in H^1(\Omega). \end{aligned}$$

We remark that $p \in H^2(\Omega)$ satisfies

$$\left\langle \frac{\partial p}{\partial \nu}, \psi_1 \right\rangle_{\Gamma_N} = \int_{\Gamma_N} \frac{\partial p}{\partial \nu} \psi_1, \qquad \left\langle \frac{\partial p}{\partial \nu}, \psi_2 \right\rangle_{\Gamma} = \int_\Gamma \frac{\partial p}{\partial \nu} \psi_2$$

for all $\psi_1 \in H_{0,D}^1(\Omega)$ and $\psi_2 \in H^1(\Omega)$.

We use the following lemmas and theorems. For their proofs, we refer the reader for instance to [5, 14, 26].

Theorem 2.1 (Cauchy–Schwarz inequality). *For $f, g \in L^2(\Omega)$, we have the following inequality:*

$$\left| \int_\Omega fg \right| \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Theorem 2.2 (Poincaré inequality). *There exists a constant $c > 0$ such that*

$$\|f\|_{L^2(\Omega)} \leq c \|\nabla f\|_{L^2(\Omega)^n}$$

for all $f \in H_0^1(\Omega)$.

Theorem 2.3. *Assume that E is a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Then there exist $x \in E$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$x_{n_k} \rightharpoonup x \text{ weakly in } E \text{ as } k \to \infty.$$

Theorem 2.4. *Assume that E is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of E. Then M is reflexive.*

Theorem 2.5 (Lax–Milgram)**.** *Assume that $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a continuous coercive bilinear form on a Hilbert space H. Then, given any $f \in H^*$, there exists a unique element $u \in H$ such that*

$$a(u, v) = \langle f, v\rangle$$

for all $v \in H$.

Theorem 2.6 (Rellich–Kondrachov)**.** *The space $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

The following Theorem 2.7 is necessary for the existence and the uniqueness of a solution to the Stokes problem.

Theorem 2.7. [14, Corollary 4.1] *Let $(X, \|\cdot\|_X)$ and $(Q, \|\cdot\|_Q)$ be two real Hilbert spaces. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bilinear and continuous maps and let $f \in X^*$. If there exist two constants $\alpha > 0$ and $\beta > 0$ such that*

$$\begin{aligned} a(v, v) &\geq \alpha\|v\|_X^2 \quad \text{for all } v \in V, \\ \sup_{0\neq v\in X} \frac{b(v, q)}{\|v\|_X} &\geq \beta\|q\|_Q \quad \text{for all } q \in Q, \end{aligned}$$

where $V = \{v \in X \mid b(v, q) = 0 \text{ for all } q \in Q\}$, then there exists a unique solution $(u, p) \in X \times Q$ to the following problem:

$$\begin{cases} a(u, v) + b(v, p) = f(v) & \text{for all } v \in X, \\ b(u, q) = 0 & \text{for all } q \in Q. \end{cases}$$

We recall the following Theorem 2.8 that plays an important role in the proof of the existence of pressure solution of Stokes problem; see Appendix A for the proof.

Theorem 2.8. [24, Lemma 7.1] *There exists a constant $c > 0$ such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)})$$

for all $p \in L^2(\Omega)$.

The following two results follow from Theorem 2.8.

Theorem 2.9. [14, Corollary 2.1, 2°] *There exists a constant $c > 0$ such that*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla p\|_{H^{-1}(\Omega)^n}$$

for all $p \in L^2(\Omega)$.

Theorem 2.10. [14, Corollary 2.4, 2°] *The operator* $\operatorname{div} : H_0^1(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ *is surjective.*

The following two embedding theorems are often called the Poincaré inequality.

Theorem 2.11. [24, Theorem 7.8] *There exists a constant* $c > 0$ *such that*

$$\|\psi\|_{L^2(\Omega)} \leq c\|\nabla\psi\|_{L^2(\Omega)^n}$$

for all $\psi \in H^1(\Omega)/\mathbb{R}$.

Theorem 2.12. [14, Lemma 3.1] *There exists a constant* $c > 0$ *such that*

$$\|\psi\|_{L^2(\Omega)} \leq c\|\nabla\psi\|_{L^2(\Omega)^n}$$

for all $\psi \in H_{0,D}^1(\Omega)$.

2.3 Weak formulations of the problems (S), (PP) and (ES)

We assume the following conditions for F, u_b, g_b and p_b:

$$F \in L^2(\Omega)^n, \qquad u_b \in H^{1/2}(\Gamma), \qquad \int_\Gamma u_b \cdot \nu = 0, \tag{2.4}$$

$$g_b \in L^2(\Gamma), \qquad \operatorname{div} F \in L^2(\Omega), \tag{2.5}$$

$$\int_\Gamma g_b = \int_\Omega \operatorname{div} F, \tag{2.6}$$

$$p_b \in H^1(\Omega). \tag{2.7}$$

We start by defining the weak solution to (S). For all $\varphi \in H_0^1(\Omega)^n$, we obtain from the first equation of (S) that

$$\begin{aligned} \int_\Omega F \cdot \varphi &= -\int_\Gamma \frac{\partial u_S}{\partial \nu} \cdot \varphi + \int_\Omega \nabla u_S : \nabla\varphi + \int_\Gamma p_S \varphi \cdot \nu - \int_\Omega p_S \operatorname{div}\varphi \\ &= \int_\Omega \nabla u_S : \nabla\varphi + \langle \nabla p_S, \varphi\rangle. \end{aligned}$$

Using this expression, the weak form of the Stokes problem becomes as follows: Find $u_S \in H^1(\Omega)^n$ and $p_S \in L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_S : \nabla\varphi + \langle \nabla p_S, \varphi\rangle = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \operatorname{div} u_S = 0 & \text{in } L^2(\Omega), \\ u_S = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases} \tag{S'}$$

Remark 2.13. *If* $(u_S, p_S) \in H^1(\Omega)^n \times L^2(\Omega)$ *satisfies* $u_S \in H^2(\Omega)^n, p_S \in H^1(\Omega)$ *and (S'), then we have*

$$\begin{cases} \displaystyle\int_\Omega (-\Delta u_S + \nabla p_S - F) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \operatorname{div} u_S = 0 & \text{in } L^2(\Omega), \\ u_S = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases}$$

Therefore, (u_S, p_S) *satisfies (S).*

Next, we define the weak formulations of (PP) and (ES) first for the Neumann boundary condition (1.2) and them for the mixed boundary condition (1.3). After that, we define generalized weak formulations for (PP) and (ES) which cover both cases.

First, we apply the Neumann boundary condition (1.2) for (PP) and (ES). We take a test function $\psi \in H^1(\Omega)$. From the second equation of (PP), we obtain

$$\begin{aligned} -\int_\Omega (\operatorname{div} F)\psi &= -\int_\Omega (\Delta p_{PP})\psi = -\int_\Gamma \frac{\partial p_{PP}}{\partial \nu}\psi + \int_\Omega \nabla p_{PP} \cdot \nabla \psi \\ &= -\int_\Gamma g_b \psi + \int_\Omega \nabla p_{PP} \cdot \nabla \psi. \end{aligned}$$

Hence,

$$\int_\Omega \nabla p_{PP} \cdot \nabla \psi = \int_\Gamma g_b \psi - \int_\Omega (\operatorname{div} F)\psi.$$

We note that $\int_\Gamma g_b \psi - \int_\Omega (\operatorname{div} F)\psi = \int_\Gamma g_b[\psi] - \int_\Omega (\operatorname{div} F)[\psi]$ for all $\psi \in H^1(\Omega)$ by (2.6). Therefore, the weak form of the pressure-Poisson problem with the Neumann boundary condition (1.2) becomes as follows. Find $u_{PP} \in H^1(\Omega)^n$ and $p_{PP} \in H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_{PP} : \nabla \varphi + \int_\Omega \nabla p_{PP} \cdot \varphi = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla p_{PP} \cdot \nabla \psi = \langle G_1, \psi \rangle & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \end{cases} \tag{PP$_1$}$$

where $G_1 \in H^1(\Omega)^*$ such that for $\psi \in H^1(\Omega)$

$$\langle G_1, \psi \rangle = \int_\Gamma g_b \psi - \int_\Omega (\operatorname{div} F)\psi. \tag{2.8}$$

The weak form of (ES) with the Neumann boundary condition can be defined similarly to that of (PP). Find $u_\varepsilon \in H^1(\Omega)^n$ and $p_\varepsilon \in H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_\varepsilon : \nabla \varphi + \int_\Omega \nabla p_\varepsilon \cdot \varphi = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon \int_\Omega \nabla p_\varepsilon \cdot \nabla \psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = \varepsilon \langle G_1, \psi \rangle & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u_\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases} \tag{ES$_1$}$$

Remark 2.14. *If $(u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)$ satisfies $u_{PP} \in H^2(\Omega)^n, p_{PP} \in H^1(\Omega)$ and (PP$_1$), then we have*

$$\begin{cases} \displaystyle\int_\Omega (-\Delta u_{PP} + \nabla p_{PP} - F) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega (-\Delta p_{PP} + \operatorname{div} F) \cdot \psi = \left\langle -\frac{\partial p_{PP}}{\partial \nu} + g_b, \psi \right\rangle_\Gamma & \text{for all } \psi \in H^1(\Omega), \\ u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases}$$

Therefore, (u_{PP}, p_{PP}) satisfies (PP) and the Neumann boundary condition (1.2).

In the same way, if $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)$ satisfies $u_\varepsilon \in H^2(\Omega)^n, p_\varepsilon \in H^1(\Omega)$ and (ES$_1$), then we have

$$\begin{cases} \displaystyle\int_\Omega (-\Delta u_\varepsilon + \nabla p_\varepsilon - F) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega (-\varepsilon\Delta p_\varepsilon + \operatorname{div} u_\varepsilon + \varepsilon \operatorname{div} F) \cdot \psi = \varepsilon \left\langle -\frac{\partial p_\varepsilon}{\partial \nu} + g_b, \psi \right\rangle_\Gamma & \text{for all } \psi \in H^1(\Omega), \\ u_\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases}$$

Therefore, $(u_\varepsilon, p_\varepsilon)$ satisfies (ES) and the Neumann boundary condition (1.2).

Secondly, we apply the mixed boundary condition (1.3) for (PP) and (ES). We take a test function $\psi \in H^1_{0,D}(\Omega)$. From the second equation of (PP), we obtain

$$\begin{aligned} -\int_\Omega (\operatorname{div} F)\psi &= -\int_\Omega (\Delta p_{PP})\psi = -\int_\Gamma \frac{\partial p_{PP}}{\partial \nu}\psi + \int_\Omega \nabla p_{PP} \cdot \nabla\psi \\ &= -\int_{\Gamma_N} g_b \psi + \int_\Omega \nabla p_{PP} \cdot \nabla \psi. \end{aligned}$$

Hence,

$$\int_\Omega \nabla p_{PP} \cdot \nabla \psi = \int_{\Gamma_N} g_b \psi - \int_\Omega (\operatorname{div} F)\psi.$$

The weak form of the pressure-Poisson problem with the mixed boundary condition (1.3) becomes as follows. Find $u_{PP} \in H^1(\Omega)^n$ and $p_{PP} \in H^1(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_{PP} : \nabla\varphi + \int_\Omega \nabla p_{PP} \cdot \varphi = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla p_{PP} \cdot \nabla \psi = \langle G_2, \psi \rangle & \text{for all } \psi \in H^1_{0,D}(\Omega), \\ u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p_{PP} = p_b & \text{in } H^{1/2}(\Gamma_D), \end{cases} \tag{PP$_2$}$$

where $G_2 \in H^1_{0,D}(\Omega)^*$ such that

$$\langle G_2, \psi \rangle = \int_{\Gamma_N} g_b \psi - \int_\Omega (\operatorname{div} F)\psi \tag{2.9}$$

for $\psi \in H^1_{0,D}(\Omega)$. The weak form of (ES) with the mixed boundary condition (1.3) can be defined similarly to that of (PP). It reads as follows. Find $u_\varepsilon \in H^1(\Omega)^n$ and $p_\varepsilon \in H^1(\Omega)$ such that

$$\begin{cases} \int_\Omega \nabla u_\varepsilon : \nabla\varphi + \int_\Omega \nabla p_\varepsilon \cdot \varphi = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\ \varepsilon \int_\Omega \nabla p_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = \varepsilon\langle G_2, \psi\rangle & \text{for all } \psi \in H^1_{0,D}(\Omega), \\ u_\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p_\varepsilon = p_b & \text{in } H^{1/2}(\Gamma_D). \end{cases} \tag{ES$_2$}$$

Remark 2.15. *If $(u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)$ satisfies $u_{PP} \in H^2(\Omega)^n, p_{PP} \in H^1(\Omega)$ and (PP$_2$), then we have*

$$\begin{cases} \int_\Omega (-\Delta u_{PP} + \nabla p_{PP} - F) \cdot \varphi = 0 & \textit{for all } \varphi \in H^1_0(\Omega)^n, \\ \int_\Omega (-\Delta p_{PP} + \operatorname{div} F) \cdot \psi = \left\langle -\frac{\partial p_{PP}}{\partial \nu} + g_b, \psi \right\rangle_{\Gamma_N} & \textit{for all } \psi \in H^1_{0,D}(\Omega), \\ u_{PP} = u_b & \textit{in } H^{1/2}(\Gamma)^n, \\ p_{PP} = p_b & \textit{in } H^{1/2}(\Gamma_D). \end{cases}$$

Therefore, (u_{PP}, p_{PP}) satisfies (PP) and the mixed boundary condition (1.3).

In the same way, if $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)$ satisfies $u_\varepsilon \in H^2(\Omega)^n, p_\varepsilon \in H^1(\Omega)$ and (ES$_2$), then we have

$$\begin{cases} \int_\Omega (-\Delta u_\varepsilon + \nabla p_\varepsilon - F) \cdot \varphi = 0 & \textit{for all } \varphi \in H^1_0(\Omega)^n, \\ \int_\Omega (-\varepsilon\Delta p_\varepsilon + \operatorname{div} u_\varepsilon + \varepsilon \operatorname{div} F) \cdot \psi = \varepsilon \left\langle -\frac{\partial p_\varepsilon}{\partial \nu} + g_b, \psi \right\rangle_{\Gamma} & \textit{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u_\varepsilon = u_b & \textit{in } H^{1/2}(\Gamma)^n, \\ p_\varepsilon = p_b & \textit{in } H^{1/2}(\Gamma_D). \end{cases}$$

Therefore, $(u_\varepsilon, p_\varepsilon)$ satisfies (ES) and the mixed boundary condition (1.3).

Finally, we generalize (PP$_1$) and (PP$_2$) to an abstract pressure-Poisson problem. Let $Q \subset H^1(\Omega)$ be a closed subspace as defined in Section 2.1. Find $u_{PP} \in H^1(\Omega)^n$ and $p_{PP} \in Q$ such that

$$\begin{cases} \int_\Omega \nabla u_{PP} : \nabla\varphi + \int_\Omega \nabla p_{PP} \cdot \varphi = \int_\Omega F \cdot \varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\ \int_\Omega \nabla p_{PP} \cdot \nabla\psi = \langle G, \psi\rangle & \text{for all } \psi \in Q, \\ u_{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p_{PP} - p_b \in Q, \end{cases} \tag{PP'}$$

with $G \in Q^*$. Indeed, by Theorem 2.11 and 2.12, we obtain (PP$_1$) from (PP') by putting $Q := H^1(\Omega)/\mathbb{R}$ and $G := G_1$. Similarly, we obtain (PP$_2$) from (PP') by putting $Q := H^1_{0,D}(\Omega)$ and $G := G_2$.

We generalize (ES_1) and (ES_2) to an abstract ε-Stokes problem. Find $u_\varepsilon \in H^1(\Omega)^n$ and $p_\varepsilon \in Q$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_\varepsilon : \nabla\varphi + \int_\Omega \nabla p_\varepsilon \cdot \varphi = \int_\Omega F\cdot\varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla p_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = \varepsilon\langle G, \psi\rangle & \text{for all } \psi \in Q, \\ u_\varepsilon - u_b \in H_0^1(\Omega)^n, \\ p_\varepsilon - p_b \in Q. \end{cases} \tag{ES'}$$

Indeed, by Theorem 2.11, 2.12, we obtain (ES_1) from (ES') by putting $Q := H^1(\Omega)/\mathbb{R}$ and $G := G_1$. Similarly, we also obtain (ES_2) from (ES') by putting $Q := H^1_{0,D}(\Omega)$ and $G := G_2$.

2.4 Well-posedness of (S'), (PP') and (ES')

We show the well-posedness of problems (S'), (PP') and (ES') in Theorem 2.16, 2.17 and 2.18.

Theorem 2.16. *Under the condition (2.4), there exists a unique solution $(u_S, p_S) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ satisfying (S').*

Proof. We take arbitrary $u_1 \in H^1(\Omega)^n$ with $\gamma_0 u_1 = u_b$. By Theorem 2.10, there exists $u_2 \in H_0^1(\Omega)^n$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$. We put $u_0 := u_1 - u_2$, and note that $\gamma_0 u_0 = u_b$ and $\operatorname{div} u_0 = 0$. The problem (S') is equivalent to

$$\begin{cases} \displaystyle\int_\Omega \nabla(u_S - u_0) : \nabla\varphi - \int_\Omega p_S \operatorname{div}\varphi = \int_\Omega F\cdot\varphi - \int_\Omega \nabla u_0 : \nabla\varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \psi \operatorname{div}(u_S - u_0) = 0 & \text{for all } \psi \in L^2(\Omega)/\mathbb{R} \\ u_S - u_0 \in H_0^1(\Omega)^n. \end{cases} \tag{2.10}$$

By Theorems 2.12, the continuous bilinear form $H_0^1(\Omega)^n \times H_0^1(\Omega)^n \ni (u, \varphi) \mapsto \int_\Omega \nabla u : \nabla\varphi \in \mathbb{R}$ is coercive. By Theorems 2.7 and 2.9, there exists a unique solution $(u_S, p_S) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ satisfying (2.10). $\square$

Theorem 2.17. *Under the conditions (2.4) and (2.7), for $G \in Q^*$, there exists a unique solution $(u_{PP}, p_{PP}) \in H^1(\Omega)^n \times Q$ satisfying (PP').*

Proof. Using the Lax–Milgram theorem, since $Q \times Q \ni (p, \psi) \mapsto \int_\Omega \nabla p \cdot \nabla\psi \in \mathbb{R}$ is a continuous and coercive bilinear form, $p_{PP} \in H^1(\Omega)$ is uniquely determined from the second and fourth equations of (PP'). Then $u_{PP} \in H^1(\Omega)^n$ is also uniquely determined from the first and third equations, again using the Lax–Milgram theorem. $\square$

Theorem 2.18. *Under the conditions (2.4) and (2.7), for $\varepsilon > 0$ and $G \in Q^*$, there exists a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)$ satisfying (ES').*

This is a generalization of Theorem 2.6 in [22]. See Appendix B for the proof.

3 Error estimates in terms of the additional boundary condition

In this section, we give estimates of the difference between the solutions to the pressure-Poisson problem, the ε-Stokes problem and the Stokes problem, respectively. Roughly speaking, from (1.1) and the second equation of (PP), $\Delta(p_S - p_{PP}) = 0$ holds. Hence, we get

$$\|p_S - p_{PP}\|_{H^1(\Omega)} \lesssim (\text{ difference between } p_S \text{ and } p_{PP} \text{ on } \Gamma),$$

where $A \lesssim B$ means that there exists a constant $c > 0$ independent of A and B such that $A \leq cB$. From (S) and the second equation of (PP), we have

$$-\Delta(u_S - u_{PP}) = -\nabla(p_S - p_{PP}).$$

Since $u_S - u_{PP} = 0$ on Γ, we have

$$\|u_S - u_{PP}\|_{H^1(\Omega)^n} \lesssim \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}.$$

Therefore, it follows that

$$\|u_S - u_{PP}\|_{H^1(\Omega)^n} \lesssim (\text{ difference between } p_S \text{ and } p_{PP} \text{ on } \Gamma).$$

In other words, if we have a good prediction for the pressure boundary data, then (PP) is good approximation for (S).

We prove the following lemma about estimates of the difference between the solutions to the ε-Stokes problem and the Stokes problem.

Lemma 3.1. *If $p_S \in H^1(\Omega)$, then there exists a constant $c > 0$ independent of ε such that*

$$\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} \leq c\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}.$$

Proof. Let $w_\varepsilon := u_S - u_\varepsilon \in H^1_0(\Omega)^n$ and $r_\varepsilon := p_{PP} - p_\varepsilon \in Q$. By (S'), (PP') and (ES'), we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla w_\varepsilon : \nabla\varphi + \int_\Omega (\nabla r_\varepsilon)\cdot\varphi = -\int_\Omega (\nabla(p_S - p_{PP}))\cdot\varphi & \text{for all } \varphi \in H^1_0(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla r_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} w_\varepsilon)\psi = 0 & \text{for all } \psi \in Q. \end{cases} \tag{3.11}$$

Putting $\varphi := w_\varepsilon$ and $\psi := r_\varepsilon$ and adding the two equations of (3.11), we get

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla r_\varepsilon\|^2_{L^2(\Omega)^n} \leq \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}\|w_\varepsilon\|_{L^2(\Omega)^n} \tag{3.12}$$

from $\int_\Omega(\nabla r_\varepsilon)\cdot w_\varepsilon = -\int_\Omega(\operatorname{div} w_\varepsilon)r_\varepsilon$. Thus we find

$$\|w_\varepsilon\|_{H^1(\Omega)^n} \leq c\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}$$

for a constant $c > 0$ independent of ε. □

By Lemma 3.1, if we have a good prediction for the pressure boundary data, then (ES) is also good approximation for (S). In this section, we prove these types of estimates for the weak solutions.

Theorem 3.2. *Suppose that $p_S \in H^1(\Omega)$, $H_0^1(\Omega) \subset Q$ and $\langle G, \psi\rangle = -\int_\Omega (\operatorname{div} F)\psi$ for all $\psi \in H_0^1(\Omega)$. Then there exists a constant $c > 0$ independent of ε such that*

$$\begin{aligned} \|u_S - u_{PP}\|_{H^1(\Omega)^n} &\le c\|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}, \\ \|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\le c\|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}. \end{aligned} \tag{3.13}$$

In particular, if $p_S = p_{PP}$, then $(u_S, p_S) = (u_{PP}, p_{PP}) = (u_\varepsilon, p_\varepsilon)$ holds for all $\varepsilon > 0$.

This is a generalization of Proposition 2.7 in [22]. See Appendix B for the proof.

Since $H_0^1(\Omega) \not\subset H^1(\Omega)/\mathbb{R}$, Theorem 3.2 does not apply directly for the case of the Neumann boundary condition (1.2). However, we add natural assumptions, then it leads to (3.13).

Corollary 3.3. *Suppose that $p_S \in H^1(\Omega)$ and $Q = H^1(\Omega)/\mathbb{R}$. If $G = G_1$ defined by (2.8), then we have (3.13).*

Proof. By (2.8), it holds that

$$\int_\Omega \nabla p_{PP} \cdot \nabla \psi = -\int_\Omega (\operatorname{div} F)\psi$$

for all $\psi \in H_0^1(\Omega)$ from the second equation of (PP'). Hence, it leads the second equation of (B.22). Using the proof of Theorem 3.2, we obtain (3.13). $\square$

We focus on the mixed boundary conditions (1.3), i.e. (PP_2) and (ES_2). We establish the following lemma.

Lemma 3.4. *If $p \in H^1(\Omega)$, $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$ satisfy*

$$\int_\Omega \nabla p \cdot \nabla \psi = \int_\Omega f\psi + \langle g, \psi\rangle_{\Gamma_N} \qquad \text{for all } \psi \in H_{0,D}^1(\Omega), \tag{3.14}$$

then there exists a constant $c > 0$ such that

$$\|p\|_{H^1(\Omega)} \le c\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|p\|_{H^{1/2}(\Gamma_D)}\right).$$

Proof. Let $p_0 \in H^1(\Omega)$ such that $p_0 - p \in H_{0,D}^1(\Omega)$. Putting $\psi := p - p_0$ in (3.14), we obtain

$$\begin{aligned} & \|\nabla(p - p_0)\|_{L^2(\Omega)^n}^2 \\ = & \int_\Omega \nabla(p - p_0) \cdot \nabla(p - p_0) \\ = & \int_\Omega f(p - p_0) + \langle g, p - p_0\rangle_{\Gamma_D} - \int_\Omega \nabla p_0 \cdot \nabla(p - p_0) \\ \le & \|f\|_{L^2(\Omega)}\|p - p_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)}\|p - p_0\|_{H^{1/2}(\Gamma_N)} + \|\nabla p_0\|_{L^2(\Omega)^n}\|\nabla(p - p_0)\|_{L^2(\Omega)^n} \\ \le & (\|f\|_{L^2(\Omega)} + c_1\|g\|_{H^{-1/2}(\Gamma_N)} + \|p_0\|_{H^1(\Omega)})\|p - p_0\|_{H^1(\Omega)} \end{aligned}$$

for a constant $c_1 > 0$. By Theorem 2.12, there exists a constant $c_2 > 0$ such that

$$c_2\|p - p_0\|_{H^1(\Omega)}^2 \le (\|f\|_{L^2(\Omega)} + c_1\|g\|_{H^{-1/2}(\Gamma_N)} + \|p_0\|_{H^1(\Omega)})\|p - p_0\|_{H^1(\Omega)}.$$

Hence,

$$\|p - p_0\|_{H^1(\Omega)} \le c_3(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|p_0\|_{H^1(\Omega)}),$$

where $c_3 := \frac{1}{c_2}\max\{1, c_1\}$. Since $\|p\|_{H^1(\Omega)} - \|p_0\|_{H^1(\Omega)} \le \|p - p_0\|_{H^1(\Omega)}$, it follows that

$$\|p\|_{H^1(\Omega)} \le c_4(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|p_0\|_{H^1(\Omega)}), \tag{3.15}$$

where $c_4 := 1 + c_3$. For all $p_0 \in H^1(\Omega)$ satisfying $p_0 - p \in H^1_{0,D}(\Omega)$, (3.15) holds. Therefore,

$$\begin{aligned}
\|p\|_{H^1(\Omega)} &\le c_4\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \inf_{q \in H^1_{0,D}(\Omega)} \|p + q\|_{H^1(\Omega)}\right) \\
&= c_4(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|p\|_{H^1(\Omega)/H^1_{0,D}(\Omega)}) \\
&\le c_5(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|p\|_{H^{1/2}(\Gamma_D)})
\end{aligned}$$

for a constant $c_5 > 0$. □

We show a property of the solution to (S').

Proposition 3.5. *If (u_S, p_S) satisfies $p_S \in H^1(\Omega)$ and $\Delta p_S \in L^2(\Omega)$, then we have*

$$\int_\Omega \nabla p_S \cdot \nabla \psi = -\int_\Omega (\operatorname{div} F)\psi + \left\langle \frac{\partial p_S}{\partial \nu}, \psi \right\rangle_{\Gamma_N}$$

for all $\psi \in H^1_{0,D}(\Omega)$.

Proof. From the first equation of (S'), we obtain

$$-\Delta u_S + \nabla p_S = F \qquad \text{in } \mathscr{D}'(\Omega).$$

Taking the divergence, we get

$$\operatorname{div} F = \operatorname{div}(-\Delta u_S + \nabla p_S) = -\Delta(\operatorname{div} u_S) + \Delta p_S = \Delta p_S \qquad \text{in } \mathscr{D}'(\Omega).$$

By the assumptions $\Delta p_S \in L^2(\Omega)$ and $\operatorname{div} F \in L^2(\Omega)$, $\Delta p_S = \operatorname{div} F$ holds in $L^2(\Omega)$. Multiplying $\psi \in H^1_{0,D}(\Omega)$ and integrating over Ω, we get

$$-\int_\Omega (\operatorname{div} F)\psi = -\int_\Omega (\Delta p_S)\psi = \int_\Omega \nabla p_S \cdot \nabla \psi - \left\langle \frac{\partial p_S}{\partial \nu}, \psi \right\rangle_{\Gamma_N},$$

which is the desired result. □

Using Proposition 3.5, we prove the following theorem.

Theorem 3.6. *Let $Q = H^1_{0,D}(\Omega)$ and $G = G_2$ defined by (2.9). If $p_S \in H^1(\Omega)$ and $\Delta p_S \in L^2(\Omega)$, there exists a constant $c > 0$ such that*

$$\begin{aligned}
\|u_S - u_{PP}\|_{H^1(\Omega)^n} &\le c\left(\left\|\frac{\partial p_S}{\partial \nu} - g_b\right\|_{H^{-1/2}(\Gamma_N)} + \|p_S - p_b\|_{H^{1/2}(\Gamma_D)}\right), \\
\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\le c\left(\left\|\frac{\partial p_S}{\partial \nu} - g_b\right\|_{H^{-1/2}(\Gamma_N)} + \|p_S - p_b\|_{H^{1/2}(\Gamma_D)}\right).
\end{aligned} \tag{3.16}$$

Proof. Using Proposition 3.5, we obtain from (S') and (PP'),

$$\left\{\begin{array}{lll}\displaystyle\int_\Omega \nabla(u_S - u_{PP}) : \nabla\varphi & = & \displaystyle\int_\Omega (p_S - p_{PP})\,\mathrm{div}\,\varphi \\ & & \qquad\qquad \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla(p_S - p_{PP}) \cdot \nabla\psi & = & \displaystyle\left\langle \frac{\partial p_S}{\partial\nu} - g_b, \psi \right\rangle_{\Gamma_N} \\ & & \qquad\qquad \text{for all } \psi \in H_{0,D}^1(\Omega).\end{array}\right. \tag{3.17}$$

Putting $\varphi := u_S - u_{PP} \in H_0^1(\Omega)^n$ in (3.17), we get

$$\begin{aligned}\|\nabla(u_S - u_{PP})\|^2_{L^2(\Omega)^{n\times n}} &= \int_\Omega (p_S - p_{PP})\,\mathrm{div}(u_S - u_{PP}) \\ &\le \|p_S - p_{PP}\|_{L^2(\Omega)} \|\,\mathrm{div}(u_S - u_{PP})\|_{L^2(\Omega)} \\ &\le \sqrt{n}\|p_S - p_{PP}\|_{H^1(\Omega)} \|u_S - u_{PP}\|_{H^1(\Omega)^n}.\end{aligned}$$

From Theorem 2.12, it follows that

$$\|u_S - u_{PP}\|_{H^1(\Omega)^n} \le c_1 \|p_S - p_{PP}\|_{H^1(\Omega)}$$

for a constant $c_1 > 0$. By the second equation of (3.17) and Lemma 3.4, there exists a constant $c_2 > 0$ such that

$$\begin{aligned}\|p_S - p_{PP}\|_{H^1(\Omega)} &\le c_2 \left(\left\| \frac{\partial p_S}{\partial\nu} - g_b \right\|_{H^{-1/2}(\Gamma_N)} + \|p_S - p_{PP}\|_{H^{1/2}(\Gamma_D)} \right) \\ &= c_2 \left(\left\| \frac{\partial p_S}{\partial\nu} - g_b \right\|_{H^{-1/2}(\Gamma_N)} + \|p_S - p_b\|_{H^{1/2}(\Gamma_D)} \right).\end{aligned}$$

Hence, we obtain the first inequality of (3.16) with $c = c_1 c_2$. By Lemma 3.1, it holds that

$$\begin{aligned}\|u_S - u_{PP}\|_{H^1(\Omega)^n} &\le c_3 \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} \\ &\le c_2 c_3 \left(\left\| \frac{\partial p_S}{\partial\nu} - g_b \right\|_{H^{-1/2}(\Gamma_N)} + \|p_S - p_b\|_{H^{1/2}(\Gamma_D)} \right).\end{aligned}$$

□

In the same way as above, we also obtain estimates of the difference between the solutions to (S'), (PP_1) and (ES_1), respectively.

Corollary 3.7. *Let $Q = H^1(\Omega)/\mathbb{R}$ and $G = G_1$ defined by (2.8). If $p_S \in H^1(\Omega)$ and $\Delta p_S \in L^2(\Omega)$, there exists a constant $c > 0$ such that*

$$\begin{aligned}\|u_S - u_{PP}\|_{H^1(\Omega)^n} &\le c \left\| \frac{\partial p_S}{\partial\nu} - g_b \right\|_{H^{-1/2}(\Gamma)}, \\ \|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\le c \left\| \frac{\partial p_S}{\partial\nu} - g_b \right\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

4 Links between (ES) and (PP)

In this section, we show that $(u_\varepsilon, p_\varepsilon)$ converges to (u_{PP}, p_{PP}) strongly in $H^1(\Omega)^n \times H^1(\Omega)$ as $\varepsilon \to \infty$. We also treat the case of the regular perturbation asymptotics by exploring the structure of the lower order terms and their effect on the convergence rate.

4.1 Convergence as $\varepsilon \to \infty$

We use the following Lemma 4.1 for the proofs of the theorems in this section.

Lemma 4.1. *Let $h \in Q^*$ and $(v_\varepsilon, q_\varepsilon) \in H_0^1(\Omega)^n \times Q$ satisfy*

$$\begin{cases} \displaystyle\int_\Omega \nabla v_\varepsilon : \nabla\varphi + \int_\Omega (\nabla q_\varepsilon)\cdot\varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla q_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} v_\varepsilon)\psi = \langle h, \psi\rangle & \text{for all } \psi \in Q \end{cases} \tag{4.18}$$

for an arbitrarily fixed $\varepsilon > 0$. Then there exists a constant $c > 0$ such that

$$\|v_\varepsilon\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon}\|h\|_{Q^*}, \qquad \|q_\varepsilon\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|h\|_{Q^*}.$$

Proof. Putting $\varphi := v_\varepsilon$ and $\psi := q_\varepsilon$ and adding two equations of (4.18), we obtain

$$\|\nabla v_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla q_\varepsilon\|^2_{L^2(\Omega)^n} \le \|h\|_{Q^*}\|\nabla q_\varepsilon\|_{L^2(\Omega)^n}.$$

where we have used $\int_\Omega \nabla q_\varepsilon \cdot v_\varepsilon = -\int_\Omega (\operatorname{div} v_\varepsilon) q_\varepsilon$. Thus

$$\|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \le \frac{1}{\varepsilon}\|h\|_{Q^*}.$$

In addition, from the first equation of (4.18) by putting $\varphi := v_\varepsilon$, we have

$$\begin{aligned} \|\nabla v_\varepsilon\|^2_{L^2(\Omega)^n} &= \int_\Omega \nabla v_\varepsilon : \nabla v_\varepsilon = -\int_\Omega (\nabla q_\varepsilon)\cdot v_\varepsilon \le \|\nabla q_\varepsilon\|_{L^2(\Omega)^n}\|v_\varepsilon\|_{L^2(\Omega)^n} \\ &\le c\|\nabla q_\varepsilon\|_{L^2(\Omega)^n}\|\nabla v_\varepsilon\|_{L^2(\Omega)^{n\times n}} \end{aligned}$$

for a constant $c > 0$, and then

$$\|\nabla v_\varepsilon\|_{L^2(\Omega)^n} \le c\|\nabla q_\varepsilon\|_{L^2(\Omega)^n} \le \frac{c}{\varepsilon}\|h\|_{Q^*}.$$

□

Using Lemma 4.1, we obtain the following theorem.

Theorem 4.2. *There exists a constant $c > 0$ independent of $\varepsilon > 0$ such that*

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon}\|\operatorname{div} u_{PP}\|_{Q^*}, \quad \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|\operatorname{div} u_{PP}\|_{Q^*}.$$

for all $\varepsilon > 0$. In particular, we have

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \to 0, \ \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to \infty.$$

Proof. Combining (PP') and (ES'), we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla v_\varepsilon : \nabla\varphi + \int_\Omega \nabla q_\varepsilon \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla q_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} v_\varepsilon)\psi = -\int_\Omega (\operatorname{div} u_{PP})\psi & \text{for all } \psi \in Q, \end{cases} \tag{4.19}$$

where $v_\varepsilon := u_\varepsilon - u_{PP}$ and $q_\varepsilon := p_\varepsilon - p_{PP}$. By Lemma 4.1, we conclude the proof. □

Corollary 4.3. *If u_{PP} satisfies $\operatorname{div} u_{PP} = 0$, then $u_\varepsilon = u_{PP}$ and $p_\varepsilon = p_{PP}$ hold for all $\varepsilon > 0$. Furthermore, $u_S = u_\varepsilon = u_{PP}$ and $p_S = [p_\varepsilon] = [p_{PP}]$ hold for all $\varepsilon > 0$.*

4.2 Regular Perturbation Asymptotics

By Theorem 4.2, there exists a constant $c > 0$ such that $\|\varepsilon(u_\varepsilon - u_{PP})\|_{H^1(\Omega)^n} \leq c$ and $\|\varepsilon(p_\varepsilon - p_{PP})\|_{H^1(\Omega)} \leq c$ for all $\varepsilon > 0$. It implies that there exists a subsequence of $(\varepsilon(u_\varepsilon - u_{PP}), \varepsilon(p_\varepsilon - p_{PP}))$ which converges weakly to $(v^{(1)}, q^{(1)}) \in H_0^1(\Omega)^n \times Q$ if $\varepsilon \to \infty$. The next theorem states properties of the limit functions $v^{(1)}$ and $q^{(1)}$.

Theorem 4.4. *Let $v_\varepsilon^{(1)} := \varepsilon(u_\varepsilon - u_{PP}) \in H_0^1(\Omega)^n, q_\varepsilon^{(1)} := \varepsilon(p_\varepsilon - p_{PP}) \in Q$ and let $(v^{(1)}, q^{(1)}) \in H_0^1(\Omega)^n \times Q$ satisfy*

$$\begin{cases} \displaystyle\int_\Omega \nabla v^{(1)} : \nabla\varphi + \int_\Omega (\nabla q^{(1)}) \cdot \varphi = 0 & \textit{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla q^{(1)} \cdot \nabla\psi = -\int_\Omega (\operatorname{div} u_{PP})\psi & \textit{for all } \psi \in Q. \end{cases} \tag{4.20}$$

Then there exists a constant $c > 0$ independent of ε such that

$$\|v_\varepsilon^{(1)} - v^{(1)}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon}\|\operatorname{div} v^{(1)}\|_{Q^*}, \qquad \|q_\varepsilon^{(1)} - q^{(1)}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\|\operatorname{div} v^{(1)}\|_{Q^*}.$$

Proof. The existence and the uniqueness of the pair $(v^{(1)}, q^{(1)}) \in H_0^1(\Omega)^n \times Q$ as a solution to (4.20) follows from Theorem 2.17. As in (4.19), we have

$$\begin{cases} \displaystyle\int_\Omega \nabla v_\varepsilon^{(1)} : \nabla\varphi + \int_\Omega (\nabla q_\varepsilon^{(1)}) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla q_\varepsilon^{(1)} \cdot \nabla\psi + \frac{1}{\varepsilon}\int_\Omega (\operatorname{div} v_\varepsilon^{(1)})\psi = -\int_\Omega (\operatorname{div} u_{PP})\psi & \text{for all } \psi \in Q. \end{cases} \tag{4.21}$$

Subtracting (4.20) from (4.21), it holds that

$$\begin{cases} \displaystyle\int_\Omega \nabla(v_\varepsilon^{(1)} - v^{(1)}) : \nabla\varphi + \int_\Omega (\nabla(q_\varepsilon^{(1)} - q^{(1)})) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla(q_\varepsilon^{(1)} - q^{(1)}) \cdot \nabla\psi + \frac{1}{\varepsilon}\int_\Omega (\operatorname{div} v_\varepsilon^{(1)})\psi = 0 & \text{for all } \psi \in Q. \end{cases}$$

Hence,

$$\begin{cases} \displaystyle\int_\Omega \nabla v_\varepsilon : \nabla\varphi + \int_\Omega \nabla q_\varepsilon \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon \int_\Omega \nabla q_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} v_\varepsilon)\psi = -\int_\Omega (\operatorname{div} v^{(1)})\psi & \text{for all } \psi \in Q, \end{cases}$$

where $v_\varepsilon := v_\varepsilon^{(1)} - v^{(1)}$ and $q_\varepsilon := q_\varepsilon^{(1)} - q^{(1)}$. By Lemma 4.1 , there exists a constant $c > 0$ independent of ε such that

$$\|v_\varepsilon^{(1)} - v^{(1)}\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(1)}\|_{Q^*}, \quad \|q_\varepsilon^{(1)} - q^{(1)}\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(1)}\|_{Q^*}$$

for all $\varepsilon > 0$. □

Next, we generalize Theorem 4.4 to the following theorem:

Theorem 4.5. *Let $k \in \mathbb{N}$ be arbitrary ($k \ge 1$) and let $v^{(0)} := u_{PP}$. If functions $v^{(1)}, v^{(2)}, \cdots, v^{(k)} \in H_0^1(\Omega)^n$ and $q^{(1)}, q^{(2)}, \cdots, q^{(k)} \in Q$ satisfy*

$$\begin{cases} \displaystyle\int_\Omega \nabla v^{(i)} : \nabla\varphi + \int_\Omega (\nabla q^{(i)}) \cdot \varphi = 0 & \textit{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla q^{(i)} \cdot \nabla\psi = -\int_\Omega (\operatorname{div} v^{(i-1)})\psi & \textit{for all } \psi \in Q, \end{cases} \tag{4.22}$$

for all $1 \le i \le k$, then there exists a constant $c > 0$ independent of ε satisfying

$$\left\| u_\varepsilon - \left(u_{PP} + \frac{1}{\varepsilon} v^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k v^{(k)} \right) \right\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon^{k+1}} \|\operatorname{div} v^{(k)}\|_{Q^*},$$
$$\left\| p_\varepsilon - \left(p_{PP} + \frac{1}{\varepsilon} q^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k q^{(k)} \right) \right\|_{H^1(\Omega)} \le \frac{c}{\varepsilon^{k+1}} \|\operatorname{div} v^{(k)}\|_{Q^*}.$$

Proof. Let $(v_\varepsilon^{(i)}, q_\varepsilon^{(i)}) \in H_0^1(\Omega)^n \times Q$ $(1 \le i \le k)$ satisfy

$$\begin{cases} \displaystyle\int_\Omega \nabla v_\varepsilon^{(i)} : \nabla\varphi + \int_\Omega (\nabla q_\varepsilon^{(i)}) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla q_\varepsilon^{(i)} \cdot \nabla\psi + \frac{1}{\varepsilon}\int_\Omega (\operatorname{div} v_\varepsilon^{(i)})\psi = -\int_\Omega (\operatorname{div} v^{(i-1)})\psi & \text{for all } \psi \in Q. \end{cases} \tag{4.23}$$

Subtracting (4.22) from (4.23), it holds that

$$\begin{cases} \displaystyle\int_\Omega \nabla(v_\varepsilon^{(i)} - v^{(i)}) : \nabla\varphi + \int_\Omega (\nabla(q_\varepsilon^{(i)} - q^{(i)})) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla(q_\varepsilon^{(i)} - q^{(i)}) \cdot \nabla\psi + \frac{1}{\varepsilon}\int_\Omega (\operatorname{div} v_\varepsilon^{(i)})\psi = 0 & \text{for all } \psi \in Q. \end{cases}$$

Setting $v_\varepsilon := v_\varepsilon^{(i)} - v^{(i)}, q_\varepsilon := q_\varepsilon^{(i)} - q^{(i)}$ and $h := -\operatorname{div} v^{(i)}$, we obtain from Lemma 4.1 that the estimates

$$\|v_\varepsilon^{(i)} - v^{(i)}\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(i)}\|_{Q^*}, \quad \|q_\varepsilon^{(i)} - q^{(i)}\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(i)}\|_{Q^*}$$

hold for all $\varepsilon > 0$. In particular, putting $i := k$, we obtain

$$\|v_\varepsilon^{(k)} - v^{(k)}\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(k)}\|_{Q^*},$$

$$\|q_\varepsilon^{(k)} - q^{(k)}\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|\operatorname{div} v^{(k)}\|_{Q^*}$$

for all $\varepsilon > 0$. By the uniqueness of the solution to (ES') in Theorem 2.18, it leads that $v_\varepsilon^{(i+1)} = \varepsilon(v_\varepsilon^{(i)} - v^{(i)}), q_\varepsilon^{(i+1)} = \varepsilon(q_\varepsilon^{(i)} - q^{(i)})$ for all $i = 1, \cdots, k-1$, and thus

$$\begin{aligned}
& v_\varepsilon^{(k)} - v^{(k)} \\
= \; & \varepsilon(v_\varepsilon^{(k-1)} - v^{(k-1)}) - v^{(k)} \\
= \; & \varepsilon\left(v_\varepsilon^{(k-1)} - \left(v^{(k-1)} + \left(\frac{1}{\varepsilon}\right)v^{(k)}\right)\right) \\
= \; & \cdots \\
= \; & \varepsilon^{k-1}\left(v_\varepsilon^{(1)} - \left(v^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-2} v^{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k-1} v^{(k)}\right)\right) \\
= \; & \varepsilon^{k}\left(u_\varepsilon - \left(u_{PP} + \frac{1}{\varepsilon}v^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-1} v^{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k} v^{(k)}\right)\right),
\end{aligned}$$

$$\begin{aligned}
& q_\varepsilon^{(k)} - q^{(k)} \\
= \; & \varepsilon(q_\varepsilon^{(k-1)} - q^{(k-1)}) - q^{(k)} \\
= \; & \varepsilon\left(q_\varepsilon^{(k-1)} - \left(q^{(k-1)} + \left(\frac{1}{\varepsilon}\right)q^{(k)}\right)\right) \\
= \; & \cdots \\
= \; & \varepsilon^{k-1}\left(q_\varepsilon^{(1)} - \left(q^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-2} q^{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k-1} q^{(k)}\right)\right) \\
= \; & \varepsilon^{k}\left(p_\varepsilon - \left(p_{PP} + \frac{1}{\varepsilon}q^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-1} q^{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k} q^{(k)}\right)\right).
\end{aligned}$$

Hence it holds that

$$\left\|u_\varepsilon - \left(u_{PP} + \frac{1}{\varepsilon}v^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k} v^{(k)}\right)\right\|_{H^1(\Omega)^n} \le \frac{c}{\varepsilon^{k+1}}\|\operatorname{div} v^{(k)}\|_{Q^*},$$

$$\left\|p_\varepsilon - \left(p_{PP} + \frac{1}{\varepsilon}q^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k} q^{(k)}\right)\right\|_{H^1(\Omega)} \le \frac{c}{\varepsilon^{k+1}}\|\operatorname{div} v^{(k)}\|_{Q^*}.$$

□

Remark 4.6. *Theorem 4.5 can be interpreted from the operator theory.*

Let $t \geq 0, X := H_0^1(\Omega)^n \times Q, Y := H^{-1}(\Omega)^n \times Q^*$ *be equipped with norms*

$$\begin{aligned} \|(u,p)\|_X^2 &:= \|u\|_{H^1(\Omega)^n}^2 + \|p\|_{H^1(\Omega)}^2, \\ \|(f,g)\|_Y^2 &:= \|f\|_{H^{-1}(\Omega)^n}^2 + \|g\|_{Q^*}^2 \end{aligned}$$

for $(u,p) \in X, (f,g) \in Y$, *and let* A *and* B *be*

$$\begin{array}{ccccccccc} A: & X & \longrightarrow & Y & \qquad & B: & X & \longrightarrow & Y \\ & \Psi & & \Psi & & & \Psi & & \Psi \\ & (u,p) & \longmapsto & (-\Delta u + \nabla p, \Delta p), & & & (u,p) & \longmapsto & (0, \operatorname{div} u). \end{array}$$

Then (u_{PP}, p_{PP}) *and* $(u_\varepsilon, p_\varepsilon)$ *satisfy*

$$A(u_{PP}, p_{PP}) = f, \quad \left(A + \frac{1}{\varepsilon}B\right)(u_\varepsilon, p_\varepsilon) = f,$$

where $f = (F, G)$. *We have* $A + tB \in \operatorname{Isom}(X,Y)$ *for an arbitrary* $t \geq 0$ *by the analogy of Theorem 2.17 ($t = 0$) and Theorem 2.18 ($t = 1/\varepsilon$). Equation (4.22) states that*

$$A(v^{(i)}, q^{(i)}) = -B(v^{(i-1)}, q^{(i-1)})$$

for $i = 1, \cdots, k$, *i.e.*

$$\begin{aligned} (v^{(i)}, q^{(i)}) &= -A^{-1}B(v^{(i-1)}, q^{(i-1)}) = \cdots = (-A^{-1}B)^i(u_{PP}, p_{PP}) \\ &= A^{-1}(-BA^{-1})^i f. \end{aligned}$$

By Theorem 4.5, there exists a constant $c > 0$ *such that*

$$\left\| \left(A + \frac{1}{\varepsilon}B\right)^{-1} f - A^{-1} \sum_{i=0}^{k} \left(-\frac{1}{\varepsilon}BA^{-1}\right)^i f \right\|_X \leq \frac{c}{\varepsilon^{k+1}} \|(BA^{-1})^{k+1} f\|_Y$$

for all $\varepsilon > 0, f \in Y$.

5 Links between (ES) and (S)

In this section, we show that $(u_\varepsilon, p_\varepsilon)$ converges to (u_S, p_S) weakly in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ as $\varepsilon \to 0$. Moreover, if $p_S \in H^1(\Omega)$, then $(u_\varepsilon, p_\varepsilon)$ converges to (u_S, p_S) strongly in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ as $\varepsilon \to 0$.

The outline of the proof of our convergence results (Theorem 5.2, 5.3 and 5.4) is as follows. First, we prove the boundedness of the sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. By the reflexivity of $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$, the sequence has a subsequence converging weakly in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. In the end, we show that the limit pair of functions satisfies (S').

We start this section with a useful lemma.

Lemma 5.1. *If $v \in H^1(\Omega)^n, q \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)^n$ satisfy*

$$\int_\Omega \nabla v : \nabla\varphi + \langle \nabla q, \varphi\rangle = \langle f, \varphi\rangle \quad \text{for all } \varphi \in H_0^1(\Omega)^n,$$

then there exists a constant $c > 0$ such that

$$\|q\|_{L^2(\Omega)/\mathbb{R}} \leq c(\|\nabla v\|_{L^2(\Omega)^{n\times n}} + \|f\|_{H^{-1}(\Omega)^n}).$$

Proof. Let c be the constant from Theorem 2.9. Then we obtain

$$\begin{aligned}
\|q\|_{L^2(\Omega)/\mathbb{R}} \leq\ & c\|\nabla q\|_{H^{-1}(\Omega)^n} = c \sup_{\varphi\in S_n} |\langle \nabla q, \varphi\rangle| \\
\leq\ & c \sup_{\varphi\in S_n} \left(\left| \int_\Omega \nabla v : \nabla\varphi \right| + |\langle f, \varphi\rangle| \right) \\
\leq\ & c(\|\nabla v\|_{L^2(\Omega)^{n\times n}} + \|f\|_{H^{-1}(\Omega)^n}).
\end{aligned}$$

□

Theorem 5.2. *There exists a constant $c > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{H^1(\Omega)^n} \leq c, \quad \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq c \quad \text{for all } \varepsilon > 0.$$

Furthermore, if $C_0^\infty(\Omega) \subset Q$, then we obtain

$$u_\varepsilon \rightharpoonup u_S \text{ weakly in } H^1(\Omega)^n,\ [p_\varepsilon] \rightharpoonup p_S \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

See Appendix A for the proof.

If we add a regularity assumption of p_S, then $(u_\varepsilon, p_\varepsilon)$ converges strongly in $H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$

Theorem 5.3. *Suppose that $p_S \in H^1(\Omega)$. Then we obtain*

$$u_\varepsilon \to u_S \text{ strongly in } H^1(\Omega)^n,\ [p_\varepsilon] \to p_S \text{ strongly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

See Appendix A for the proof.

Theorem 5.3 does not give the convergence rate. If $Q = H^1(\Omega)/\mathbb{R}$ (corresponding to the Neumann boundary condition (1.2)), then the convergence rate becomes $\sqrt{\varepsilon}$.

Theorem 5.4. *Suppose that $Q = H^1(\Omega)/\mathbb{R}$ and $p_S \in H^1(\Omega)$. Then there exists a constant $c > 0$ independent of ε such that*

$$\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \le c\sqrt{\varepsilon},\ \ \|p_\varepsilon - p_S\|_{L^2(\Omega)} \le c\sqrt{\varepsilon}.$$

Proof. We obtain from (ES') and (S') that

$$\begin{cases} \displaystyle\int_\Omega \nabla(u_\varepsilon - u_S) : \nabla\varphi + \int_\Omega (\nabla(p_\varepsilon - p_S)) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon \int_\Omega \nabla p_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = \varepsilon\langle G, \psi\rangle & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}. \end{cases}$$

Putting $\varphi := u_\varepsilon - u_S \in H_0^1(\Omega)^n$ and $\psi := p_\varepsilon - p_S \in H^1(\Omega)/\mathbb{R}$, we get

$$\begin{aligned} &\|\nabla(u_\varepsilon - u_S)\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon \textstyle\int_\Omega \nabla p_\varepsilon \cdot \nabla(p_\varepsilon - p_S) \\ = \ & -\int_\Omega (\nabla(p_\varepsilon - p_S)) \cdot (u_\varepsilon - u_S) - \int_\Omega (\operatorname{div} u_\varepsilon)(p_\varepsilon - p_S) + \varepsilon\langle G, p_\varepsilon - p_S\rangle \\ = \ & \int_\Omega (\operatorname{div} u_\varepsilon - \operatorname{div} u_S)(p_\varepsilon - p_S) - \int_\Omega (\operatorname{div} u_\varepsilon)(p_\varepsilon - p_S) + \varepsilon\langle G, p_\varepsilon - p_S\rangle \\ = \ & \varepsilon\langle G, p_\varepsilon - p_S\rangle. \end{aligned} \tag{5.24}$$

Subtracting $\varepsilon \int_\Omega \nabla p_S \cdot \nabla(p_\varepsilon - p_S)$ from both sides of (5.24), we obtain

$$\begin{aligned} &\|\nabla(u_\varepsilon - u_S)\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla(p_\varepsilon - p_S)\|^2_{L^2(\Omega)^n} \\ = \ & -\varepsilon \int_\Omega \nabla p_S \cdot \nabla(p_\varepsilon - p_S) + \varepsilon\langle G, p_\varepsilon - p_S\rangle \\ \le \ & \varepsilon(\|\nabla p_S\|_{L^2(\Omega)^n} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*})\|\nabla(p_\varepsilon - p_S)\|_{L^2(\Omega)^n}. \end{aligned} \tag{5.25}$$

To clarify the following estimates, we set $\alpha := \|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}}$, $\beta := \|\nabla(p_\varepsilon - p_S)\|_{L^2(\Omega)^n}$, $a := \|\nabla p_S\|_{L^2(\Omega)^n} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*}$. The estimate (5.25) reads as

$$\alpha^2 + \varepsilon\beta^2 \le \varepsilon a\beta,\ \left(\frac{\alpha}{\sqrt{\varepsilon}}\right)^2 + \left(\beta - \frac{a}{2}\right)^2 \le \left(\frac{a}{2}\right)^2.$$

Hence, $\alpha \le a\sqrt{\varepsilon}/2$, i.e., $\|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}} \le (\sqrt{\varepsilon}/2)(\|\nabla p_S\|_{L^2(\Omega)^n} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*})$. By Lemma 5.1, we obtain

$$\begin{aligned} \|p_\varepsilon - p_S\|_{L^2(\Omega)} &\le c\|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}} = c\alpha \le c\frac{a\sqrt{\varepsilon}}{2} \\ &= c\frac{\sqrt{\varepsilon}}{2}(\|\nabla p_S\|_{L^2(\Omega)^n} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*}) \end{aligned}$$

for a constant $c > 0$ independent of ε. $\square$

6 Numerical examples

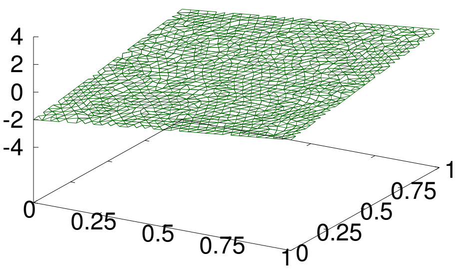
For our simulations, we consider $\Omega = (0,1) \times (0,1)$. We take the following boundary conditions:

$$u_b = (x(x-1), y(y-1))^T, \ g_b = (2,2)^T \cdot \nu$$

on Γ. The exact solutions for (PP$_1$) are $u_{PP} = (x(x-1), y(y-1))^T$ and $p_{PP} = 2x+2y-2$. We solve the problems (PP$_1$), (ES$_1$) and (S') numerically by using the finite element method with P2/P1 elements by the software FreeFem++ [18]. The numerical solutions $(u_{PP}, p_{PP}), (u_\varepsilon, p_\varepsilon)$ ($\varepsilon = 1, 10^{-2}$ or 10^{-4}) and (u_S, p_S) to the problems (PP$_1$), (ES$_1$) and (S'), respectively, are illustrated in Fig. 2–4. From these pictures we observe that $(u_\varepsilon, p_\varepsilon)$ seems to converge to (u_{PP}, p_{PP}) as $\varepsilon \to \infty$ and to (u_S, p_S) as $\varepsilon \to 0$ (as expected from Theorem 4.2 and 5.3.)

Next we compute the error estimate between the numerical solutions of (ES$_1$) and (PP$_1$). The numerical errors $\|u_\varepsilon - u_{PP}\|_{L^2(\Omega)^n}$, $\|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^{n\times n}}$, $\|p_\varepsilon - p_{PP}\|_{L^2(\Omega)}$ and $\|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n}$ are shown in Fig. 5 and Fig. 6. Based on these values, we have fitted a constant c such that $\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \sim c/\varepsilon$ and $\|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \sim c/\varepsilon$ for ε large. Fig. 5 and Fig. 6 indicate that there exists a constant c such that $\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \le c/\varepsilon$ and $\|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \le c/\varepsilon$, as expected from Theorem 4.2.

We also compute the error estimate between the problems (ES$_1$) and (S') by numerical calculation. The numerical error estimate $\|u_\varepsilon - u_S\|_{L^2(\Omega)^n}$, $\|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}}$, $\|p_\varepsilon - p_S\|_{L^2(\Omega)}$ and $\|\nabla(p_\varepsilon - p_S)\|_{L^2(\Omega)^n}$ are shown in Fig. 7 and Fig. 8. Based on these values, we have fitted a constant c such that $\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \sim c\varepsilon$ and $\|p_\varepsilon - p_S\|_{L^2(\Omega)} \sim c\varepsilon$ for ε small. Fig. 7 and Fig. 8 indicate that there exists a constant $\tilde{c}$ such that $\|u_\varepsilon - u_S\|_{H^1(\Omega)^n} \le \tilde{c}\sqrt{\varepsilon}$ and $\|p_\varepsilon - p_S\|_{L^2(\Omega)} \le \tilde{c}\sqrt{\varepsilon}$, as expected from Theorem 5.4.



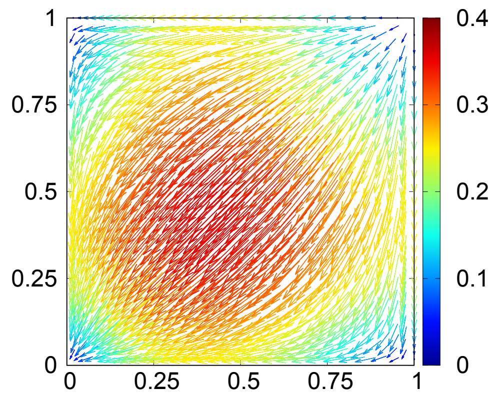


Figure 2: p_{PP} (left) and u_{PP} (right). The color scale indicates the length of $|u_{PP}(\xi)|$ at each node ξ.

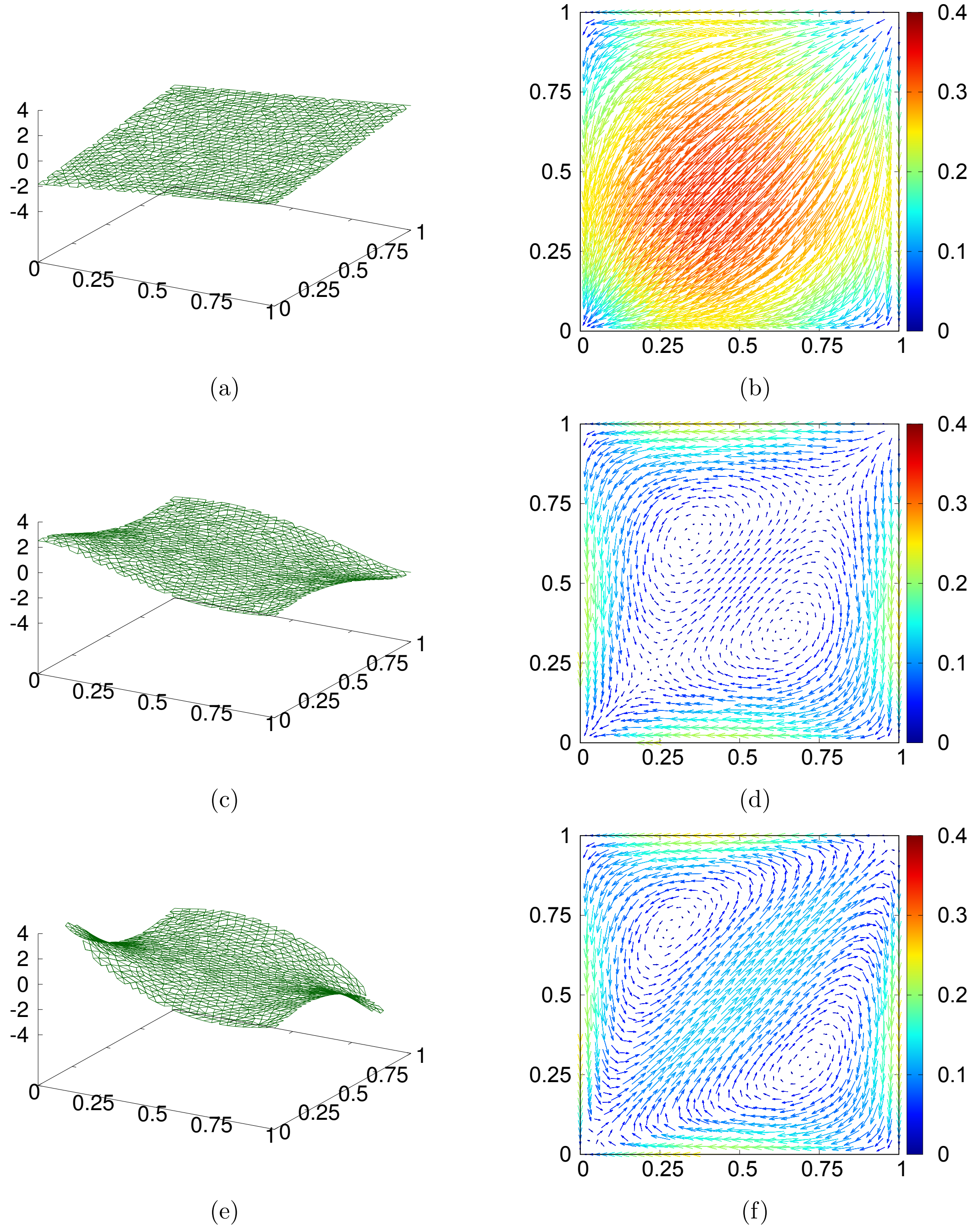


Figure 3: p_ε (a) and u_ε (b) with $\varepsilon = 1$. p_ε (c) and u_ε (d) with $\varepsilon = 10^{-2}$. p_ε (e) and u_ε (f) with $\varepsilon = 10^{-4}$. The color scales indicate the length of $|u_\varepsilon(\xi)|$ at each node ξ.

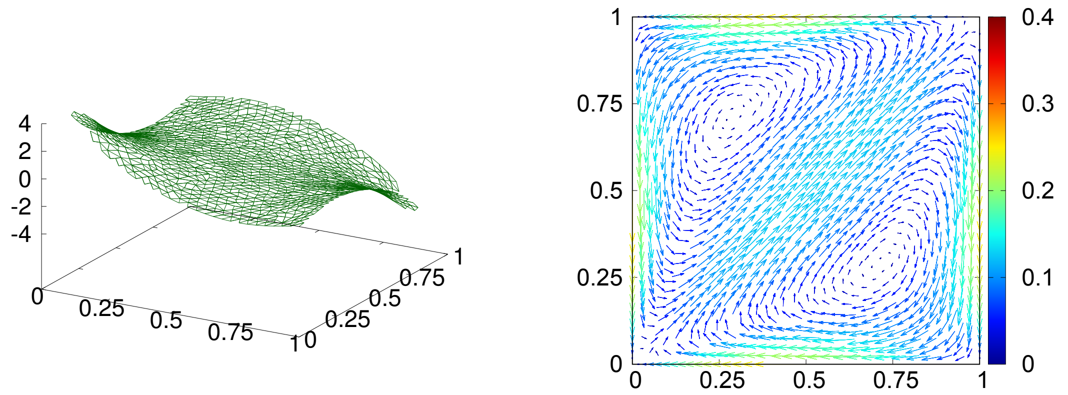


Figure 4: p_S (left) and u_S (right). The color scale indicates the length of $|u_S(\xi)|$ at each node ξ.

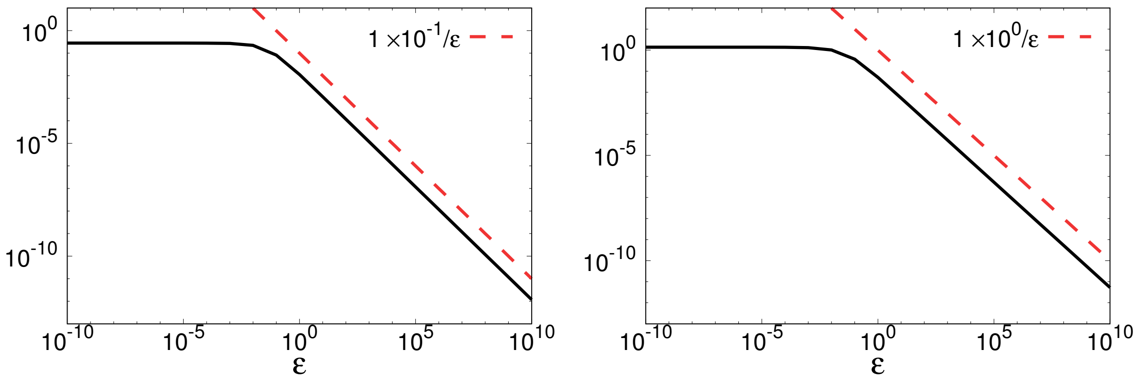


Figure 5: $\|u_\varepsilon - u_{PP}\|_{L^2(\Omega)^n}$ (left, solid line) and $\|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^{n\times n}}$ (right, solid line) as functions of ε.

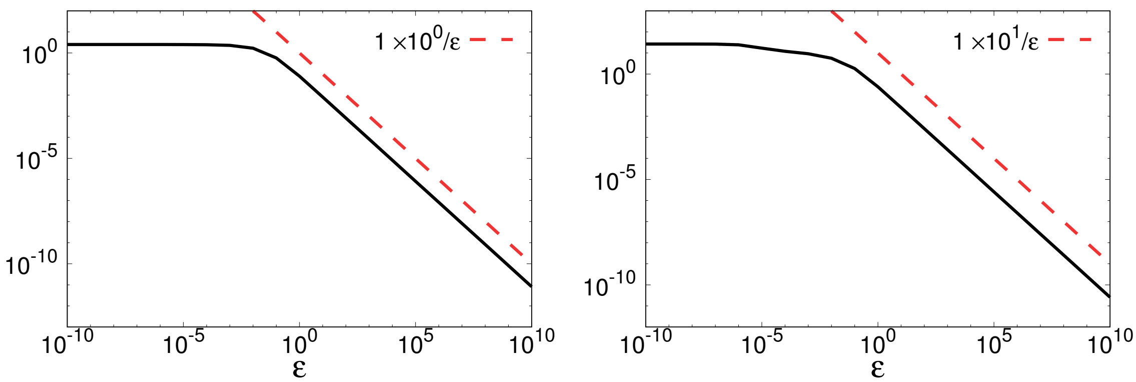


Figure 6: $\|p_\varepsilon - p_{PP}\|_{L^2(\Omega)}$ (left, solid line) and $\|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n}$ (right, solid line) as functions of ε.

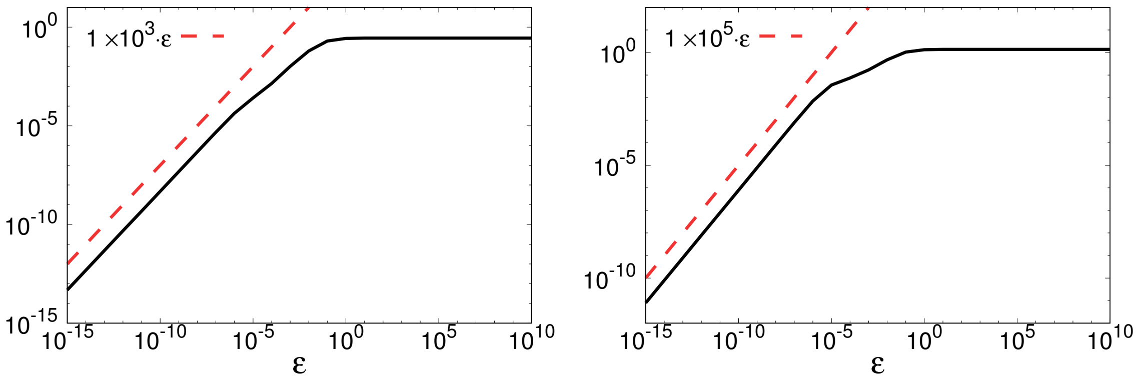


Figure 7: $\|u_\varepsilon - u_S\|_{L^2(\Omega)^n}$ (left, solid line) and $\|\nabla(u_\varepsilon - u_S)\|_{L^2(\Omega)^{n\times n}}$ (right, solid line) as functions of ε.

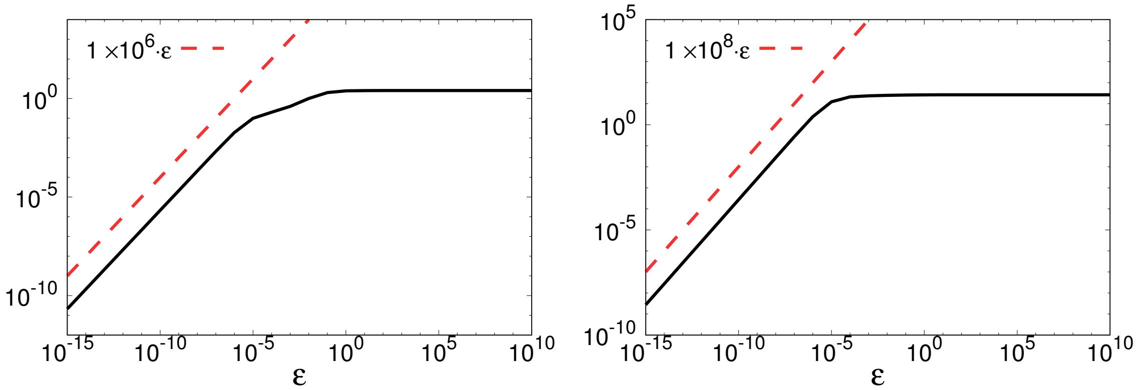


Figure 8: $\|p_\varepsilon - p_S\|_{L^2(\Omega)}$ (left, solid line) and $\|\nabla(p_\varepsilon - p_S)\|_{L^2(\Omega)^n}$ (right, solid line) as functions of ε.

7 Conclusion

We introduced the ε-Stokes problem (ES) connecting the Stokes problem (S) and corresponding pressure-Poisson problem (PP). For any fixed $\varepsilon > 0$, the ε-Stokes problem has a unique weak solution $(u_\varepsilon, p_\varepsilon)$ (Theorem 2.18) and u_ε is a good approximation as the solution to (S) while the solution to (S) and (PP) are close in the following sense;

$$\begin{aligned} \|u_S - u_{PP}\|_{H^1(\Omega)^n} &\leq c\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}, \\ \|u_S - u_\varepsilon\|_{H^1(\Omega)^n} &\leq c\|\gamma_0 p_S - \gamma_0 p_{PP}\|_{H^{1/2}(\Gamma)}, \end{aligned}$$

see Theorems 3.2, 3.6 and Corollary 3.7 for details. In other words, if we have a good prediction for the boundary data, then (PP) and (ES) are good approximations for (S).

We proved in Theorem 4.2 that a sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ converges strongly in $H^1(\Omega)^n \times H^1(\Omega)$ to the solution to (PP) as $\varepsilon \to \infty$ with convergence rate $O(1/\varepsilon)$. We also treated the case of the regular perturbation asymptotics by exploring the structure of the lower order terms and their effect on the convergence rate.

We proved in Theorem 5.2 that $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ converges weakly in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to the solution (u_S, p_S) to (S) as $\varepsilon \to 0$. If we add an assumption $p_S \in H^1(\Omega)$, then strongly convergence holds from Theorem 5.3. By the numerical examples, we observed the expected convergences as $\varepsilon \to \infty$ or $\varepsilon \to 0$.

We summarize our results as follows:

- We introduce the ε-Stokes problem (ES) as an interpolation between the Stokes problem (S) and the pressure-Poisson problem (PP).
- The solution $(u_\varepsilon, p_\varepsilon)$ to (ES) strongly converges in $H^1(\Omega)^n \times H^1(\Omega)$ to (u_{PP}, p_{PP}) as $\varepsilon \to \infty$ with convergence rate $O(1/\varepsilon)$.
- The solution $(u_\varepsilon, p_\varepsilon)$ to (ES) weakly converges in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to (u_S, p_S) as $\varepsilon \to 0$. If $p_S \in H^1(\Omega)$, then strongly convergence of $(u_\varepsilon, p_\varepsilon)$ to (u_S, p_S) as $\varepsilon \to 0$ holds. Furthermore, if $Q = H^1(\Omega)/\mathbb{R}$ and $p_S \in H^1(\Omega)$, then the convergence rate is $O(\sqrt{\varepsilon})$.

In this thesis, the domain of the numerical examples is in $\mathbb{R}^2$. Numerical comparison of (ES), (PP) and (S) in 3D is one of our interesting future works, for example the convergence rates and numerical instability. As another extension of our research, generalization of our results to the Navier—Stokes problem is important but is still completely open.

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A Nečas inequality on a bounded Lipschitz domain

The Nečas inequality (Lemma 2.8) is important for the proof of the existence of the solution to the Stokes problem and the Korn inequality, cf. [12, 14, 26]. The Nečas inequality on a bounded Lipschitz domain was proved by Nečas [24]. The Nečas inequality also holds on a Jorn domain that is a weaker condition than Lipschitz domain [1].

Nečas proceed with the proof in two steps:

1 Interior of Ω. Here, the proof follows the case $\Omega = \mathbb{R}^n$.

2 Neighborhood V near the boundary Γ. Here, the proof follows the case Ω is the half space $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

There are other methods for the proof of the Nečas inequality [2]. In [12], the authors prove that

$$\left\{ p \in H^{-1}(\Omega) \mid \frac{\partial p}{\partial x_i} \in H^{-1}(\Omega) \text{ for all } i = 1, \cdots, n \right\} = L^2(\Omega)$$

holds with $C^{1,1}$-class boundary. The equation is equivalent to the Nečas inequality. See also [4].

The purpose of this appendix is to provide the Nečas's style proof. In A.1 we introduce the notations and symbols used in this appendix. We prove the case $\Omega = \mathbb{R}^n$ in A.2 and the case Ω is a subset K of the half space in A.3. In A.4 we define mollifiers and show several properties of their properties. We also make the mapping $T : K \to V$ using the mollifiers. In A.5 we prove Nečas inequality and some corollaries.

A.1 Preliminaries

We introduce the notation used in this appendix.

For $\alpha, \beta > 0$ and an open set $U \subset \mathbb{R}^{n-1}$, we set

$$\begin{aligned}
B(\alpha) &:= \left\{ x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \;\middle|\; |x'| < \alpha \right\}, \\
K(\alpha, \beta) &:= B(\alpha) \times (0, \beta), \\
M(\alpha, \beta) &:= B(\alpha) \times (-\beta, \beta), \\
L^1(U) &:= \left\{ f : U \to \mathbb{R} : \text{measurable} \;\middle|\; \int_U |f| < \infty \right\}, \\
L^\infty(U) &:= \left\{ f : U \to \mathbb{R} : \text{measurable} \;\middle|\; \begin{array}{l} \text{there exists a constant } M \text{ such that} \\ |f(x')| \le M \text{ a.e. } x' \in U \end{array} \right\},
\end{aligned}$$

where $|x'| := \sqrt{x_1^2 + \cdots + x_{n-1}^2}$. A function $g : \overline{U} \to \mathbb{R}$ is called Lipschitz continuous if there exists a constant $c > 0$ such that

$$|g(x') - g(y')| \le c|x' - y'| \quad \text{for all } x', y' \in \overline{U}.$$

The constant c is called a Lipschitz constant c_g for g. The spaces $L^1(U)$ and $L^\infty(U)$ are Banach spaces with the norms

$$\begin{aligned}
\|f\|_{L^1(U)} &:= \int_U |f|, \\
\|f\|_{L^\infty(U)} &:= \inf \left\{ M \in \mathbb{R} \;\middle|\; |f(x')| \le M \text{ a.e. } x' \in U \right\},
\end{aligned}$$

respectively. In this appendix, we use the following norm of the space $H^{-1}(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} := \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \langle f, \psi \rangle \qquad \text{for all } f \in H^{-1}(\Omega).$$

This norm is equivalent to the norm defined in Section 2 by Poincaré inequality.

We use the following theorems and lemmas.

Theorem A.1 (partition of unity)**.** [5, Lemma 9.3] *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and let open subsets $U_0, U_1, \cdots, U_m \subset \mathbb{R}^n$ satisfy $\overline{\Omega} \subset \bigcup_{r=0}^m U_r$. Then there exists functions $\eta_0, \eta_1, \cdots, \eta_m \in C^\infty(\mathbb{R}^n)$ such that*

$$\begin{array}{ll} \eta_r \in C_0^\infty(U_r) & \text{for all } r = 0, 1, \cdots, m, \\ 0 \le \eta_r(x) \le 1 & \text{for all } r = 0, 1, \cdots, m, x \in U_r, \\ \displaystyle\sum_{r=0}^m \eta_r(x) = 1 & \text{for all } x \in \overline{\Omega}. \end{array}$$

Lemma A.2. *Let $U \subset \mathbb{R}^n$ be a open set. We have*

$$\|\nabla p\|_{H^{-1}(U)^n} \le \sum_{i=1}^n \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(U)} \le \sqrt{n} \|\nabla p\|_{H^{-1}(U)^n}$$

for all $p \in L^2(U)$.

Proof.

$$\begin{aligned} \sum_{i=1}^n \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(U)} &= \sum_{i=1}^n \sup_{\varphi_i \in H^1(U),\ \|\varphi_i\|_{H^1(U)}=1} \left\langle \frac{\partial p}{\partial x_i}, \varphi_i \right\rangle \\ &= \sup_{\varphi \in H^1(U)^n,\ \|\varphi_1\|_{H^1(U)}=\cdots=\|\varphi_n\|_{H^1(U)}=1} \langle \nabla p, \varphi \rangle \\ &\le \sup_{\varphi \in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=\sqrt{n}} \langle \nabla p, \varphi \rangle \\ &= \sqrt{n} \sup_{\varphi \in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=\sqrt{n}} \langle \nabla p, \frac{1}{\sqrt{n}} \varphi \rangle \\ &= \sqrt{n} \sup_{\varphi \in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=1} \langle \nabla p, \varphi \rangle \\ &= \sqrt{n} \|\nabla p\|_{H^{-1}(U)^n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla p\|_{H^{-1}(U)^n} &= \sup_{0 \ne \varphi \in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n} \le 1} \langle \nabla p, \varphi \rangle \\ &= \sup_{0 \ne \varphi \in H^1(U)^n,\ \|\varphi_1\|_{H^1(U)}^2 + \cdots + \|\varphi_n\|_{H^1(U)}^2 \le 1} \langle \nabla p, \varphi \rangle \\ &\le \sum_{i=1}^n \sup_{0 \ne \varphi_i \in H^1(U),\ \|\varphi_i\|_{H^1(U)} \le 1} \left\langle \frac{\partial p}{\partial x_i}, \varphi_i \right\rangle \\ &= \sum_{i=1}^n \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(U)}. \end{aligned}$$

□

Lemma A.3. *Let Ω be an open set in $\mathbb{R}^n$. If there exists a constant $c > 0$ such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in C_0^\infty(\Omega),$$

then it holds that

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in L^2(\Omega).$$

Proof. For $p \in L^2(\Omega)$, we have

$$\begin{aligned}
\|p\|_{H^{-1}(\Omega)} &= \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \int_\Omega p\psi \\
&\leq \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \|p\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\
&\leq \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \|p\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} \\
&= \|p\|_{L^2(\Omega)}, \\
\|\nabla p\|_{H^{-1}(\Omega)^n} &= \sup_{\varphi \in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1} \int_\Omega p \operatorname{div} \varphi \\
&\leq \sup_{\varphi \in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1} \|p\|_{L^2(\Omega)} \|\operatorname{div} \varphi\|_{L^2(\Omega)} \\
&\leq \sqrt{n} \sup_{\varphi \in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1} \|p\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)^{n\times n}} \\
&\leq \sqrt{n} \|p\|_{L^2(\Omega)}.
\end{aligned}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we obtain the result. □

A.2 Total space

If $p, q : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with compact support, its Fourier transform $\hat{p}$ and inverse Fourier transform $\check{q}$ is defined by

$$\begin{aligned}
\hat{p}(\xi) &:= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} p(x) dx \quad \text{for all } \xi \in \mathbb{R}^n, \\
\check{q}(x) &:= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(\xi) d\xi \quad \text{for all } x \in \mathbb{R}^n,
\end{aligned}$$

where $x \cdot \xi = \sum_{k=1}^n x_k \xi_k$. It is easy to see that $\hat{q}(x) = \check{q}(-x)$ for all $x \in \mathbb{R}^n$. It is well known property of the Fourier transform that

$$\widehat{\frac{\partial p}{\partial x_k}}(\xi) = i\xi_k \hat{p}(\xi) \quad \text{for all } k = 1, \cdots, n.$$

One proves (Plancherel theorem [13, Theorem 4.3.2 (ii)]) that if $p \in L^2(\mathbb{R}^n)$ then $\hat{p} \in L^2(\mathbb{R}^n)$ and $\|\hat{p}\|_{L^2(\mathbb{R}^n)} = \|p\|_{L^2(\mathbb{R}^n)}$. By continuous extension, one can therefore define $\mathcal{F} : L^2(\mathbb{R}^n) \ni p \to \hat{p} \in L^2(\mathbb{R}^n)$ and $\mathcal{F}^* : L^2(\mathbb{R}^n) \ni q \to \check{q} \in L^2(\mathbb{R}^n)$. The linear isometric mapping $\mathcal{F}$ is an unitary map and has the inverse map:

Theorem A.4. [13, Theorem 4.3.2 (i)]

$$\int_{\mathbb{R}^n} p(x)\overline{q(x)}dx = \int_{\mathbb{R}^n} \hat{p}(\xi)\overline{\hat{q}(\xi)}d\xi \quad \textit{for all } p, q \in L^2(\mathbb{R}^n),$$

Theorem A.5. [13, Theorem 4.3.2 (iv)]

$$\mathcal{F}\mathcal{F}^* p = \mathcal{F}^*\mathcal{F} p = p \quad \textit{for all } p \in L^2(\mathbb{R}^n).$$

For Sobolev space $H^1(\mathbb{R}^n)$, we have a lemma:

Theorem A.6. [13, Theorem 5.8.8]

$$p \in H^1(\mathbb{R}^n) \Leftrightarrow \int_{\mathbb{R}^n} |\hat{p}(\xi)|(1+|\xi|^2)d\xi < +\infty$$

and we have

$$\|p\|_{H^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\hat{p}(\xi)|(1+|\xi|^2)d\xi\right)^{1/2} \quad \textit{for all } p \in H^1(\mathbb{R}^n).$$

By using Fourier transform, we prove Theorem 2.8 when $\Omega = \mathbb{R}^n$.

Lemma A.7. *There exists a constant $c > 0$ such that*

$$\|p\|_{L^2(\mathbb{R}^n)} \leq \sqrt{n}(\|p\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla p\|_{H^{-1}(\mathbb{R}^n)^n})$$

for all $p \in L^2(\mathbb{R}^n)$.

Proof. Using Fourier transform, we get

$$\begin{aligned} \|p\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2(1+|\xi|^2)(1+|\xi|^2)^{-1}d\xi \\ &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2(1+|\xi|^2)^{-1}d\xi + \sum_{j=1}^n \int_{\mathbb{R}^n} |\xi_j\hat{p}(\xi)|^2(1+|\xi|^2)^{-1}d\xi. \end{aligned} \tag{A.1}$$

Here it follows that

$$\|p\|^2_{H^{-1}(\mathbb{R}^n)} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} p\varphi\right)^2}{\|\varphi\|^2_{H^1(\mathbb{R}^n)}} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} \hat{p}\ \overline{\hat{\varphi}}\right)^2}{\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2(1+|\xi|^2)d\xi}. \tag{A.2}$$

Putting $\varphi := \mathcal{F}^{-1}[(1+|\xi|^2)^{-1}\hat{p}(\xi)]$, we obtain

$$\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2(1+|\xi|^2)d\xi = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2(1+|\xi|^2)^{-1}d\xi \leq \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 d\xi < +\infty.$$

By Lemma A.6, $\varphi \in H^1(\mathbb{R}^n)$, hence, it follows from (A.2) that

$$\|p\|^2_{H^{-1}(\mathbb{R}^n)} \geq \frac{\left(\displaystyle\int_{\mathbb{R}^n} \hat{p}(\xi)\overline{(1+|\xi|^2)^{-1}\hat{p}(\xi)}d\xi\right)^2}{\displaystyle\int_{\mathbb{R}^n} |(1+|\xi|^2)^{-1}\hat{p}(\xi)|^2(1+|\xi|^2)d\xi} = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2(1+|\xi|^2)^{-1}d\xi. \tag{A.3}$$

On the other hand, for $j = 1, \cdots, n$, it holds that

$$\left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\displaystyle\int_{\mathbb{R}^n} \frac{\partial p}{\partial x_j}\varphi\right)^2}{\|\varphi\|^2_{H^1(\mathbb{R}^n)}} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\displaystyle\int_{\mathbb{R}^n} i\xi_j\hat{p}(\xi)\overline{\hat{\varphi}(\xi)}d\xi\right)^2}{\displaystyle\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2(1+|\xi|^2)d\xi}$$

Putting $\varphi := \mathcal{F}^{-1}[i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)]$, we have

$$\|\varphi\|^2_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2(1+|\xi|^2)d\xi = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2\frac{\xi_j^2}{1+|\xi|^2}d\xi \leq \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2d\xi < +\infty.$$

By Lemma A.6, we obtain $\varphi \in H^1(\mathbb{R}^n)$, hence,

$$\left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)} \geq \frac{\left(\displaystyle\int_{\mathbb{R}^n} i\xi_j\hat{p}(\xi)\overline{i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)}d\xi\right)^2}{\displaystyle\int_{\mathbb{R}^n} |i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)|^2(1+|\xi|^2)d\xi} = \int_{\mathbb{R}^n} |\xi_j\hat{p}(\xi)|^2(1+|\xi|^2)^{-1}d\xi. \tag{A.4}$$

By (A.1), (A.3) and (A.4),

$$\|p\|^2_{L^2(\mathbb{R}^n)} \leq \|p\|^2_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^n \left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)}.$$

By Lemma A.2, we obtain the result;

$$\begin{aligned}
\|p\|_{L^2(\mathbb{R}^n)} &\leq \sqrt{\|p\|^2_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^n \left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)}} \\
&\leq \|p\|_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^n \left\|\frac{\partial p}{\partial x_j}\right\|_{H^{-1}(\mathbb{R}^n)} \\
&\leq \sqrt{n}\left(\|p\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla p\|_{H^{-1}(\mathbb{R}^n)^n}\right).
\end{aligned}$$

□

Using Lemma A.7, we prove the following lemma.

Lemma A.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let a bounded open set $U \subset \Omega$ satisfy that $\overline{U} \subset \Omega$. There exists a constant $c = c(\Omega, U) > 0$ depending only on U such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \textit{for all } p \in C_0^\infty(U)\ (\subset C_0^\infty(\Omega)). \tag{A.5}$$

Proof. For $p \in C_0^\infty(\Omega)$, we set

$$\tilde{p}(x) = \begin{cases} p(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

It is easy to see that $\tilde{p} \in L^2(\mathbb{R}^n)$ and $\|\tilde{p}\|_{L^2(\mathbb{R}^n)} = \|p\|_{L^2(\Omega)}$. One can make a function $\rho \in C_0^\infty(\mathbb{R}^n)$ such that

$$\rho(x) = 0 \ (x \notin \Omega), \qquad 0 \leq \rho(x) \leq 1 \ (x \in \Omega), \qquad \rho(x) = 1 \ (x \in U).$$

Since $\rho \in H^1(\Omega) \subset H^1(\mathbb{R}^n)$, there exists a constant $c > 0$ such that

$$\|\rho\psi\|_{H^1(\Omega)} \leq c\|\rho\|_{H^1(\Omega)}\|\psi\|_{H^1(\mathbb{R}^n)} \qquad \text{for all } \psi \in H^1(\mathbb{R}^n).$$

Thus it follows that for $p \in C_0^\infty(U)$ $(\subset C_0^\infty(\Omega))$

$$\begin{aligned} \|\tilde{p}\|_{H^{-1}(\mathbb{R}^n)} &= \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} \tilde{p}\psi}{\|\psi\|_{H^1(\mathbb{R}^n)}} \\ &= \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\Omega} p\rho\psi}{\|\psi\|_{H^1(\mathbb{R}^n)}} \qquad (\text{by } \operatorname{supp}(p) \subset U) \\ &\leq c\|\rho\|_{H^1(\Omega)} \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\Omega} p\rho\psi}{\|\rho\psi\|_{H^1(\Omega)}} \\ &\leq c\|\rho\|_{H^1(\Omega)} \sup_{0 \neq \tilde{\psi} \in H_0^1(\Omega)} \frac{\displaystyle\int_{\Omega} p\tilde{\psi}}{\|\tilde{\psi}\|_{H^1(\Omega)}} \\ &\leq c\|\rho\|_{H^1(\Omega)}\|p\|_{H^{-1}(\Omega)}, \end{aligned}$$

and

$$\|\nabla\tilde{p}\|_{H^{-1}(\mathbb{R}^n)^n} \leq c\|\rho\|_{H^1(\Omega)}\|\nabla\tilde{p}\|_{H^{-1}(\Omega)^n}.$$

By Lemma A.7, we obtain the result;

$$\begin{aligned} \|p\|_{L^2(\Omega)} &= \|\tilde{p}\|_{L^2(\mathbb{R}^n)} \\ &\leq \sqrt{n}(\|\tilde{p}\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla\tilde{p}\|_{H^{-1}(\mathbb{R}^n)^n}) \\ &\leq c\sqrt{n}\|\rho\|_{H^1(\Omega)}(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}). \end{aligned}$$

□

A.3 Extension of $K(\alpha, \beta)$ to $M(\alpha, \beta)$

We shall consider Lemma A.8 on the subset $K(\alpha, \beta)$ of the half space $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. We prepare the following lemma.

Lemma A.9. *Let $\alpha, \beta > 0$, $K = K(\alpha, \beta)$ and $M = M(\alpha, \beta)$. There exist an extension operator $C_0^\infty(K) \ni p \mapsto \tilde{p} \in C_0^\infty(M)$ and a constant $c > 0$ independent of α and β such that*

$$\begin{aligned} \|p\|_{L^2(K)} &\leq \|\tilde{p}\|_{L^2(M)}, \\ \|\tilde{p}\|_{H^{-1}(M)} &\leq c\|p\|_{H^{-1}(K)}, \\ \|\nabla\tilde{p}\|_{H^{-1}(M)^n} &\leq c\|\nabla p\|_{H^{-1}(K)^n}, \end{aligned}$$

for all $p \in C_0^\infty(K)$, and if $p \in C_0^\infty(K(\alpha', \beta'))$ $(0 < \alpha' < \alpha, 0 < \beta' < \beta)$ then $\tilde{p} \in C_0^\infty(M(\alpha', \beta'))$.

Proof. Let λ_1, λ_2 are solutions of the linear system:

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 + 2\lambda_2 = -1, \tag{A.6}$$

(i.e. $\lambda_1 = 3, \lambda_2 = -2$). For $p \in C_0^\infty(K)$, we define $\tilde{p} \in C_0^\infty(M)$ as follows:

$$\tilde{p}(x', x_n) = \begin{cases} p(x', x_n) & \text{if } 0 < x_n < \beta, \\ 0 & \text{if } x_n = 0, \\ \lambda_1 p(x', -x_n) + \lambda_2 p\left(x', -\dfrac{x_n}{2}\right) & \text{if } -\beta < x_n < 0, \end{cases}$$

for $x' \in B(\alpha)$ and $x_n \in (-\beta, \beta)$. It is easy to see that

$$\|p\|_{L^2(K)} \leq \|\tilde{p}\|_{L^2(M)}$$

and if $p \in C_0^\infty(K(\alpha', \beta'))$ then $\tilde{p} \in C_0^\infty(M(\alpha', \beta'))$. Moreover, if $p \in C_0^\infty(K(\alpha', \beta'))$ then $\mathrm{supp}(\tilde{p}) \subset M(\alpha', \beta')$.

For $p \in C_0^\infty(K), v \in C_0^\infty(M)$ and $i = 1, 2, \cdots, n-1$, we have

$$\begin{aligned} & \int_M \frac{\partial \tilde{p}}{\partial x_i}(x', x_n) v(x', x_n) dx \\ = & \int_K \frac{\partial p}{\partial x_i}(x', x_n) v(x', x_n) dx + \int_{K^-} \left(\lambda_1 \frac{\partial p}{\partial x_i}(x', -x_n) + \lambda_2 \frac{\partial p}{\partial x_i}\left(x', -\frac{x_n}{2}\right)\right) v(x', x_n) dx \\ = & \int_K \frac{\partial p}{\partial x_i}(x', x_n) \left(v(x', x_n) + \lambda_1 v(x', -x_n) + 2\lambda_2 v(x', -2x_n)\right) dx \\ = & \int_K \frac{\partial p}{\partial x_i}(x', x_n) P_1 v(x', x_n) dx \end{aligned}$$

where $K^- := B(\alpha) \times (-\beta, 0)$ and $P_1 v(x', x_n) := v(x', x_n) + \lambda_1 v(x', -x_n) + 2\lambda_2 v(x', -2x_n)$. By (A.6), we obtain that for $x' \in B(\alpha)$

$$P_1 v(x', 0) = v(x', 0) + \lambda_1 v(x', 0) + 2\lambda_2 v(x', 0) = (1 + \lambda_1 + 2\lambda_2) v(x', 0) = 0.$$

It implies that $P_1 v \in H_0^1(K)$. Moreover, it follows that

$$\|P_1 v\|_{H^1(K)} \leq c_1 \|v\|_{H^1(M)} \qquad \text{for all } v \in C_0^\infty(M).$$

Thus we can extend P_1 as a bounded operator on $H_0^1(M)$, which satisfies that

$$\int_M \frac{\partial \tilde{p}}{\partial x_i}(x) v(x) dx = \int_K \frac{\partial p}{\partial x_i}(x) P_1 v(x) dx$$

for all $v \in H_0^1(M)$ and $i = 1, 2, \cdots, n-1$. The same argument works for $\int_M \tilde{p} v dx$. It implies that

$$\|\tilde{p}\|_{H^{-1}(M)} \le c_1 \|p\|_{H^{-1}(K)}, \qquad \left\| \frac{\partial \tilde{p}}{\partial x_i} \right\|_{H^{-1}(M)} \le c_1 \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(K)}$$

for $i = 1, 2, \cdots, n-1$. For $v \in C_0^\infty(M)$, we have

$$\begin{aligned}
&\int_M \frac{\partial \tilde{p}}{\partial x_n}(x', x_n) v(x', x_n) dx \\
= &\int_K \frac{\partial p}{\partial x_n}(x', x_n) v(x', x_n) dx + \int_{K^-} \left(-\lambda_1 \frac{\partial p}{\partial x_n}(x', -x_n) - \frac{\lambda_2}{2} \frac{\partial p}{\partial x_n}\left(x', -\frac{x_n}{2}\right) \right) v(x', x_n) dx \\
= &\int_K \frac{\partial p}{\partial x_n}(x', x_n) \left(v(x', x_n) - \lambda_1 v(x', -x_n) - \lambda_2 v(x', -2x_n) \right) dx \\
= &\int_K \frac{\partial p}{\partial x_n}(x', x_n) P_2 v(x', x_n) dx
\end{aligned}$$

where $P_2 v(x', x_n) := v(x', x_n) - \lambda_1 v(x', -x_n) - \lambda_2 v(x', -2x_n)$. By (A.6), we obtain that for $x' \in B(\alpha)$

$$P_2 v(x', 0) = v(x', 0) - \lambda_1 v(x', 0) - \lambda_2 v(x', 0) = (1 - \lambda_1 - \lambda_2) v(x', 0) = 0.$$

It implies that $P_2 v \in H_0^1(K)$. Moreover, it follows that

$$\|P_2 v\|_{H^1(K)} \le c_2 \|v\|_{H^1(M)} \qquad \text{for all } v \in C_0^\infty(M).$$

Thus we can extend P_2 as a bounded operator on $H_0^1(M)$, which satisfies that

$$\int_M \frac{\partial \tilde{p}}{\partial x_n}(x) v(x) dx = \int_K \frac{\partial p}{\partial x_n}(x) P_2 v(x) dx$$

for all $v \in H_0^1(M)$. It implies

$$\left\| \frac{\partial \tilde{p}}{\partial x_n} \right\|_{H^{-1}(M)} \le c_2 \left\| \frac{\partial p}{\partial x_n} \right\|_{H^{-1}(K)}.$$

□

By Lemma A.8 and Lemma A.9, the following lemma holds.

Lemma A.10. *Let $0 < \alpha' < \alpha$, $0 < \beta' < \beta$ and $K = K(\alpha, \beta)$. There exists a constant $c = c(\alpha, \beta, \alpha', \beta') > 0$ such that*

$$\|p\|_{L^2(K)} \le c(\|p\|_{H^{-1}(K)} + \|\nabla p\|_{H^{-1}(K)^n}) \qquad \textit{for all } p \in C_0^\infty(K(\alpha', \beta')).$$

Proof. By Lemma A.9, we have $\tilde{p} \in C_0^\infty(M(\alpha', \beta'))$. Here, it is immediate to check that $\overline{M(\alpha', \beta')} \subset M = M(\alpha, \beta)$. By Lemma A.8 and Lemma A.9, it follows that

$$\|p\|_{L^2(K)} \le \|\tilde{p}\|_{L^2(M)} \le c_1(\|\tilde{p}\|_{H^{-1}(M)} + \|\nabla \tilde{p}\|_{H^{-1}(M)^n}) \le c_2(\|p\|_{H^{-1}(K)} + \|\nabla p\|_{H^{-1}(K)^n}).$$

□

A.4 Local Lipschitz boundary

We shall consider a neighborhood of the boundary Γ. For $0 < \alpha' \le \alpha$ and $0 < \beta' \le \beta$, let a function $g : \overline{\Delta(\alpha)} \to \mathbb{R}$ be Lipschitz continuous and let

$$U_g^+(\alpha', \beta') := \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in B(\alpha'), g(x') < x_n < g(x') + \beta'\}.$$

In this subsection, we make a mapping $K(\alpha', \beta') \to U_g^+(\alpha', \beta)$ and extend Lemma A.10 to $U_g^+(\alpha, \beta)$. The simple mapping $K(\alpha, \beta) \ni (y', y_n) \mapsto (y', g(y') + y_n) \in U_g^+(\alpha, \beta)$ is not smooth enough to prove the lemma if g is not sufficient smooth, thus we define a mapping $K(\alpha', \beta') \to U_g^+(\alpha', \beta)$ using mollifiers according to the Nečas's proof.

For $f \in L^1(B(h))$ and $g \in L^\infty(B(\alpha))$ with $0 < \alpha' < \alpha$ and $h = \alpha - \alpha'$, one can define the convolution product of f and g;

$$(f * g)(x') := \int_{B(h)} f(y') g(x' - y') dy' \quad \text{for a.e. } x' \in B(\alpha').$$

It is easy to see that

$$\|f * g\|_{L^\infty(B(\alpha'))} \le \|f\|_{L^1(B(h))} \|g\|_{L^\infty(B(\alpha))}. \tag{A.7}$$

Let $\rho_1 \in C_0^\infty(\mathbb{R}^{n-1})$ satisfy that $\mathrm{supp}(\rho_1) \subset \overline{B(1)}$, $\rho_1 \ge 0$ on $\mathbb{R}^{n-1}$ and $\int_{B(1)} \rho_1 = 1$; for example the function

$$\rho_1(x') := \begin{cases} P_0 \exp\left(\dfrac{1}{|x'|^2 - 1}\right) & \text{if } |x'| < 1, \\ 0 & \text{if } |x'| \ge 1, \end{cases}$$

where $P_0 = 1 / \int_{\mathbb{R}^{n-1}} \exp(\frac{1}{|x'|^2 - 1}) dx'$. For $h > 0$, we set

$$\rho_h(x') := \frac{1}{h^{n-1}} \rho_1\left(\frac{x'}{h}\right)$$

for $x' \in \mathbb{R}^{n-1}$.

We show some properties of the mollifiers.

Proposition A.11. *For $h > 0$, we have*

$$\int_{B(h)} \rho_h = 1. \tag{A.8}$$

Furthermore, there exists a constant $c > 0$ independent of h such that

$$\left|\frac{\partial \rho_h}{\partial x_i}(x')\right| < \frac{c}{h^n}, \quad \left|\frac{\partial \rho_h}{\partial h}(x')\right| < \frac{c}{h^n} \qquad \textit{for all } h > 0, x' \in \mathbb{R}^{n-1}, i = 1, \cdots, n-1.$$

Proof. We compute

$$\int_{B(h)} \rho_h = \frac{1}{h^{n-1}} \int_{B(h)} \rho_1 \left(\frac{x'}{h}\right) dx' = \frac{1}{h^{n-1}} \int_{B(1)} \rho_1(x') h^{n-1} dx' = \int_{B(1)} \rho_1(x') dx' = 1.$$

For $x' = (x_1, \cdots, x_{n-1}) \in B(h)$ and $i = 1, \cdots, n-1$, we have

$$\begin{aligned}
\frac{\partial \rho_h}{\partial x_i}(x') &= \frac{\partial}{\partial x_i}\left(\frac{1}{h^{n-1}}\rho_1\left(\frac{x'}{h}\right)\right) = \frac{1}{h^n}\frac{\partial \rho_1}{\partial x_i}\left(\frac{x'}{h}\right), \\
\frac{\partial \rho_h}{\partial h}(x') &= -\frac{n-1}{h^n}\rho_1\left(\frac{x'}{h}\right) + \frac{1}{h^{n-1}}\sum_{j=1}^{n-1}\left(-\frac{x_j}{h^2}\right)\frac{\partial \rho_1}{\partial x_j}\left(\frac{x'}{h}\right) \\
&= \frac{1}{h^n}\left\{(1-n)\rho_1\left(\frac{x'}{h}\right) - \sum_{j=1}^{n-1}\frac{x_j}{h}\frac{\partial \rho_1}{\partial x_j}\left(\frac{x'}{h}\right)\right\}.
\end{aligned}$$

Since $\rho_1 \in C^\infty(\mathbb{R}^{n-1})$ and $\mathrm{supp}(\rho_1) \subset \overline{B(1)}$, functions $\frac{\partial \rho_1}{\partial x_i}\left(\frac{x'}{h}\right)$ and $\rho_1\left(\frac{x'}{h}\right)$ are bounded on $B(h)$. Therefore, there exists a constant $c > 0$ such that

$$\left|\frac{\partial \rho_h}{\partial x_i}(x')\right| < \frac{c}{h^n}, \quad \left|\frac{\partial \rho_h}{\partial h}(x')\right| < \frac{c}{h^n} \qquad \text{for all } h > 0, x' \in \mathbb{R}^{n-1}, i = 1, \cdots, n-1.$$

□

Lemma A.12. *Let $0 < \alpha' < \alpha$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exists a constant $M = M(\alpha, g) > 0$ such that*

$$\left|\frac{\partial}{\partial x_i}(\rho_h * g)(x')\right| < M, \quad \left|\frac{\partial}{\partial h}(\rho_h * g)(x')\right| < M$$

for all $0 < h < \alpha - \alpha'$ and $x' \in B(\alpha')$.

Proof. For a Lipschitz constant c_g for g, it follows that

$$\left|\frac{\partial g}{\partial x_i}(x')\right| \le c_g \qquad \text{a.e. for } x' \in B(\alpha)$$

with $i = 1, \cdots, n-1$. For $h > 0, x' \in B(\alpha')$ and $i = 1, \cdots, n-1$,

$$\begin{aligned}
\left|\frac{\partial}{\partial x_i}(\rho_h * g)(x')\right| &= \left|\frac{\partial}{\partial x_i}\int_{B(h)} \rho_h(y') g(x' - y') dy'\right| \\
&= \left|\int_{B(h)} \rho_h(y') \frac{\partial g}{\partial x_i}(x' - y') dy'\right| \\
&\le \int_{B(h)} \left|\rho_h(y') \frac{\partial g}{\partial x_i}(x' - y')\right| dy' \\
&\le c_g \int_{B(h)} |\rho_h(y')| dy' = c_g,
\end{aligned}$$

$$
\begin{aligned}
\left|\frac{\partial}{\partial h}(\rho_h * g)(x')\right| &= \left|\frac{\partial}{\partial h}\int_{B(h)} \rho_h(y')g(x'-y')dy'\right| \\
&= \left|\frac{\partial}{\partial h}\int_{B(h)} \frac{1}{h^{n-1}}\rho_1\left(\frac{y'}{h}\right) g(x'-y')dy'\right| \\
&= \left|\frac{\partial}{\partial h}\int_{B(1)} \frac{1}{h^{n-1}}\rho_1(z')g(x'-hz')h^{n-1}dz'\right| \\
&= \left|\int_{B(1)} \frac{\partial}{\partial h}\{\rho_1(z')g(x'-hz')\}dz'\right| \\
&= \left|\int_{B(1)} \rho_1(z')\sum_{j=1}^{n-1} z_j\frac{\partial g}{\partial z_j}(x'-hz')dz'\right| \\
&\le \int_{B(1)} |\rho_1(z')|\sum_{j=1}^{n-1}|z_j|\left|\frac{\partial g}{\partial z_j}(x'-hz')\right|dz' \\
&\le (n-1)c_g\int_{B(1)}|\rho_1(z')|dz' = (n-1)c_g.
\end{aligned}
$$

□

Lemma A.13. *Let $0 < \alpha' < \alpha$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exists a constant $c = c(\alpha, g) > 0$ such that*

$$
\begin{aligned}
\left\|\frac{\partial^2}{\partial x_i\partial x_j}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}, \\
\left\|\frac{\partial^2}{\partial x_i\partial h}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}, \\
\left\|\frac{\partial^2}{\partial h^2}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}
\end{aligned}
$$

for all $i, j = 1, \cdots, n-1$ and $0 < h < \alpha - \alpha'$.

Proof. For $i, j = 1, \cdots, n-1$ and $0 < h < \alpha - \alpha'$, by (A.7), we obtain

$$
\begin{aligned}
\left\|\frac{\partial^2}{\partial x_i\partial x_j}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &= \frac{1}{h}\left\|\left(h\frac{\partial\rho_h}{\partial x_j}\right) * \frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha'))} \\
&\le \frac{1}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}\left\|\frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha))} \\
&\le \frac{c_g}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}, \\
\left\|\frac{\partial^2}{\partial x_i\partial h}(\rho_h * f)\right\|_{L^\infty(B(\alpha'))} &\le \frac{1}{h}\left\|h\frac{\partial\rho_h}{\partial h}\right\|_{L^1(B(h))}\left\|\frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha))} \\
&\le \frac{c_g}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}.
\end{aligned}
$$

By Proposition A.11, it follows that

$$\left\| h\frac{\partial \rho_h}{\partial x_j} \right\|_{L^1(B(h))} = h\int_{B(h)} \left| \frac{\partial \rho_h}{\partial x_j}(x') \right| dx \leq \frac{c}{h^{n-1}} \int_{B(h)} dx = c|B(1)|,$$

where $|B(1)|$ is the volume of an $(n-1)$-dimensional unit ball. Hence, we get

$$\begin{aligned} \left\| \frac{\partial^2}{\partial x_i \partial x_j}(\rho_h * g) \right\|_{L^\infty(B(\alpha'))} &\leq \frac{cc_g|B(1)|}{h}, \\ \left\| \frac{\partial^2}{\partial x_i \partial h}(\rho_h * g) \right\|_{L^\infty(B(\alpha'))} &\leq \frac{cc_g|B(1)|}{h}. \end{aligned}$$

For all $x' \in B(\alpha')$, we have

$$\frac{\partial}{\partial h}(\rho_h * g)(x') = \int_{B(1)} \rho_1(z') \sum_{k=1}^{n-1} z_i \frac{\partial g}{\partial z_k}(x' - hz')dz' = \sum_{k=1}^{n-1} \int_{B(h)} \frac{y_k}{h} \rho_h(y') \frac{\partial g}{\partial y_k}(x' - y')dy'.$$

Thus, it holds that

$$\frac{\partial^2}{\partial h^2}(\rho_h * g)(x') = \frac{1}{h} \sum_{k=1}^{n-1} \int_{B(h)} \left(-\frac{y_k}{h}\rho_h(y') + \frac{y_k}{h} h \frac{\partial \rho_h}{\partial h}(y') \right) \frac{\partial g}{\partial y_k}(x' - y')dy'.$$

By (A.7) and Proposition A.11, it follows that

$$\begin{aligned} &\left\| \frac{\partial^2}{\partial h^2}(\rho_h * g) \right\|_{L^\infty(B(\alpha'))} \\ \leq\ & \frac{1}{h} \sum_{k=1}^{n-1} \left\| \frac{\partial g}{\partial y_k} \right\|_{L^\infty(B(\alpha))} \left\{ \int_{B(h)} \left| \frac{y_k}{h} \rho_h(y') \right| dy' + \int_{B(h)} \left| \frac{y_k}{h} h \frac{\partial \rho_h}{\partial h}(y') \right| dy' \right\} \\ \leq\ & \frac{c_g}{h} \sum_{k=1}^{n-1} \left\{ \int_{B(h)} |\rho_h(y')|dy' + \int_{B(h)} \frac{c}{h^{n-1}} dy' \right\} \\ =\ & \frac{(n-1)c_g(1 + c|B(1)|)}{h}. \end{aligned}$$

□

For $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$, we make a mapping $K(\alpha', \beta') \to U_g^+(\alpha', \beta)$ using the mollifiers according to the Nečas's proof.

Lemma A.14. *Let $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exist two constants $\delta = \delta(\alpha, \beta, \alpha', \beta', g), M = M(\alpha, g) > 0$ such that the mapping $T : K(\alpha', \beta') \to U_g^+(\alpha', \beta)$;*

$$T(y) := (y', G(y', \delta y_n) + (1 + \delta M)y_n) \quad \textit{for } y = (y', y_n) \in K(\alpha', \beta'),$$

where

$$G(y', h) := (\rho_h * g)(y') = \int_{B(h)} \rho_h(\xi')g(y' - \xi')d\xi',$$

satisfies the following statements:

- *The mapping* $T : K(\alpha', \beta') \to V(\alpha', \beta') := T(K(\alpha', \beta'))$ *is* C^∞*-diffeomorphism.*
- $U_g^+(\alpha', \beta') \subset V(\alpha', \beta') \subset U_g^+(\alpha', \beta)$ *(Fig. 9).*
- *Let* $(x_1, \cdots, x_n) = T(y_1, \cdots, y_n)$ *for* $(y_1, \cdots, y_n) \in K(\alpha', \beta')$*. The Jacobian* $\mathrm{Jac}(T) := \det(\frac{\partial x_i}{\partial y_j})_{1 \le i,j \le n}$ *satisfies*
$$1 \le \mathrm{Jac}(T) \le 1 + 2\delta M.$$

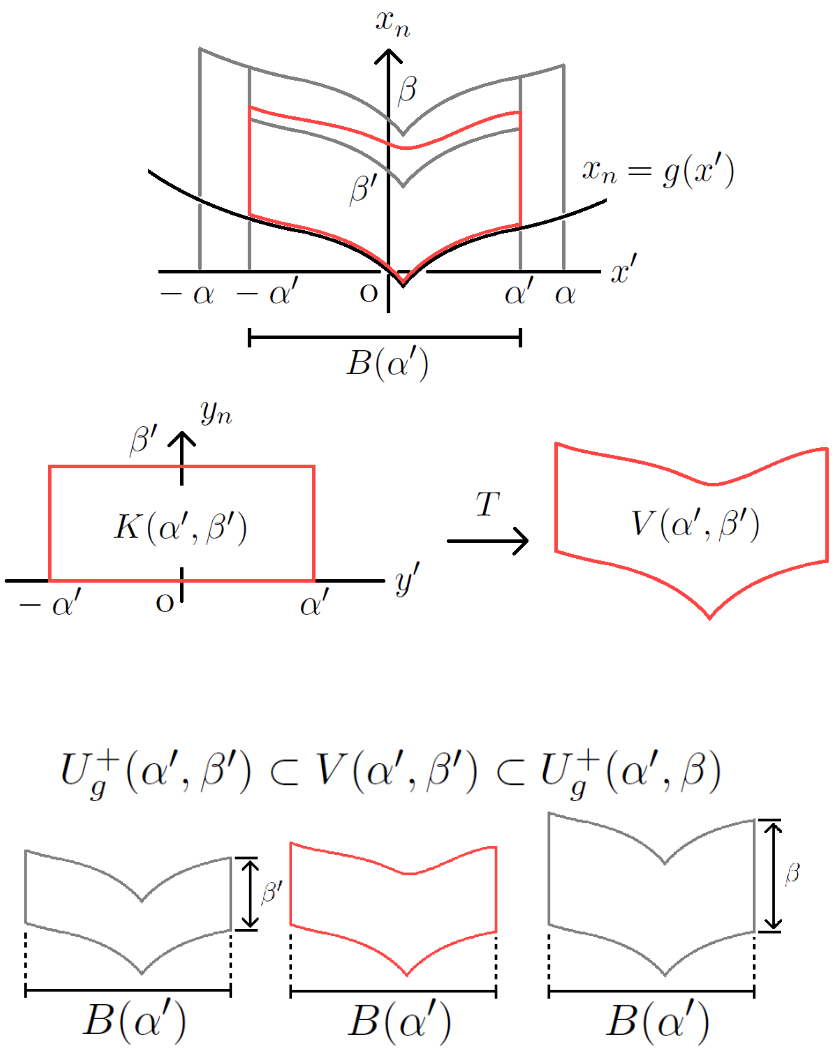

Figure 9: Sketch of the mapping T

Proof. Since $\rho_h \in C_0^\infty(\mathbb{R}^{n-1})$ and g is a Lipschitz continuous function, the function G is infinitely differentiable on $B(\alpha') \times (0, \alpha - \alpha')$. By Lemma A.12, there exists a constant $M > 0$ such that
$$-M \le \frac{\partial G}{\partial h}(y', h) \le M \tag{A.9}$$

for all $y' \in B(\alpha')$ and $0 < h < \alpha - \alpha'$. Let

$$\delta := \min\left\{\frac{1}{2M}\left(\frac{\beta}{\beta'} - 1\right), \frac{\alpha - \alpha'}{\beta'}\right\}.$$

It is easy to see that $\delta\beta' \leq \alpha - \alpha'$, hence, T is well-defined. Since G is infinitely differentiable on $B(\alpha') \times (0, \alpha - \alpha')$, T is also infinitely differentiable on $K(\alpha', \beta')$.

For $y = (y', y_n) \in K(\alpha', \beta')$, by equation (A.9), it follows that

$$1 = -\delta M + 1 + \delta M \leq \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M \leq \delta M + 1 + \delta M = 1 + 2\delta M.$$

Hence, $y_n \mapsto T(y', y_n)$ is strictly increasing for all $y' \in B(\alpha')$;

$$1 \leq \frac{\partial}{\partial y_n}\{G(y', \delta y_n) + (1 + \delta M)y_n\} \leq 1 + 2\delta M \quad \text{for all } (y', y_n) \in K(\alpha', \beta'). \tag{A.10}$$

Therefore, T is a bijective mapping from $K(\alpha', \beta')$ to $V(\alpha', \beta') = T(K(\alpha', \beta'))$. Moreover, $T : K(\alpha', \beta') \to V(\alpha', \beta')$ is C^∞-diffeomorphism.

Integrating with respect to y_n from 0 to β', we get

$$\beta' \leq G(y', \delta\beta') - g(y') + (1 + \delta M)\beta' \leq (1 + 2\delta M)\beta' \leq \beta$$

for all $y' \in B(\alpha')$. Hence, we obtain $U_g^+(\alpha', \beta') \subset V(\alpha', \beta') \subset U_g^+(\alpha', \beta)$.

Let $(x_1, \cdots, x_n) = T(y_1, \cdots, y_n)$ for $(y', y_n) = (y_1, \cdots, y_n) \in K(\alpha', \beta')$. For $i = 1, \cdots, n-1$ and $j = 1, \cdots, n$, it follows that

$$\begin{aligned}
\frac{\partial x_i}{\partial y_j}(y', y_n) &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\
\frac{\partial x_n}{\partial y_i}(y', y_n) &= \frac{\partial G}{\partial y_i}(y', \delta y_n), \\
\frac{\partial x_n}{\partial y_n}(y', y_n) &= \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M.
\end{aligned} \tag{A.11}$$

Thus, the Jacobian of T;

$$\mathrm{Jac}(T)(y', y_n) = \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M$$

satisfies

$$0 < 1 \leq \mathrm{Jac}(T)(y', y_n) \leq 1 + 2\delta M.$$

□

We remind the following theorem.

Theorem A.15. [24, Lemma 3.2] *Let $U, V \subset \mathbb{R}^n$ be two bounded open sets. If $\Phi : U \to V$ is bijective Lipschitz continuous mapping satisfies that Φ^{-1} is also Lipschitz continuous mapping, then the mapping $H^1(V) \ni f \mapsto f \circ \Phi \in H^1(U)$ is homeomorphism between Banach spaces. Furthermore, the mapping $H_0^1(V) \ni f \mapsto f \circ \Phi \in H_0^1(U)$ is also homeomorphism between Banach spaces.*

By Lemma A.14 and Theorem A.15, we obtain the following lemma.

Lemma A.16. *For the mapping $T : K(\alpha', \beta') \to V(\alpha', \beta')$ defined in Lemma A.14, the mapping $H_0^1(V(\alpha', \beta')) \ni \chi \mapsto \chi \circ T \in H_0^1(K(\alpha', \beta'))$ is isomorphic between Banach spaces. In particular, there exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$\frac{1}{c}\|\chi\|_{H^1(V(\alpha',\beta'))} \le \|\psi\|_{H^1(K(\alpha',\beta'))} \le c\|\chi\|_{H^1(V(\alpha',\beta'))}$$

for all $\psi \in H_0^1(K(\alpha', \beta'))$ and $\chi = \psi \circ T^{-1}$.

Proof. By (A.11), $\frac{\partial x_i}{\partial y_j}$ is bounded in $K(\alpha', \beta')$ and $\frac{\partial y_i}{\partial x_j}$ is bounded in $V(\alpha', \beta')$ for all $i, j = 1, \cdots, n$. By Theorem A.15, $H_0^1(V(\alpha', \beta')) \ni \chi \mapsto \chi \circ T \in H_0^1(K(\alpha', \beta'))$ is isomorphic. $\square$

We give the proof of a Hardy-type inequality.

Lemma A.17. *Let $\alpha, \beta > 0$ and $\varphi \in H_0^1(K(\alpha, \beta))$. Then $\varphi(x_1, \cdots, x_n)/x_n \in L^2(K(\alpha, \beta))$ and*

$$\left\|\frac{\varphi}{x_n}\right\|_{L^2(K(\alpha,\beta))} \le 2\left\|\frac{\partial\varphi}{\partial x_n}\right\|_{L^2(K(\alpha,\beta))}.$$

Proof. For $\varphi \in C_0^\infty(K(\alpha, \beta))$, we have

$$\begin{aligned}
\left\|\frac{\varphi}{x_n}\right\|^2_{L^2(K(\alpha,\beta))} &= \int_{B(\alpha)} dx' \int_0^\beta dx_n \left|\frac{\varphi(x', x_n)}{x_n}\right|^2 \\
&= -\int_{B(\alpha)} dx' \int_0^\beta dx_n |\varphi(x', x_n)|^2 \frac{d}{dx_n}\left(\frac{1}{x_n}\right) \\
&= 2\int_{B(\alpha)} dx' \int_0^\beta dx_n \frac{\partial\varphi}{\partial x_n}(x', x_n)\frac{\varphi(x', x_n)}{x_n} \\
&= 2\left\|\frac{\partial\varphi}{\partial x_n}\right\|_{L^2(K(\alpha,\beta))}\left\|\frac{\varphi}{x_n}\right\|_{L^2(K(\alpha,\beta))}.
\end{aligned}$$

Hence,

$$\left\|\frac{\varphi}{x_n}\right\|_{L^2(K(\alpha,\beta))} \le 2\left\|\frac{\partial\varphi}{\partial x_n}\right\|_{L^2(K(\alpha,\beta))} \qquad \text{for all } \varphi \in C_0^\infty(K(\alpha, \beta)).$$

Since $C_0^\infty(K(\alpha, \beta))$ is dense in $H_0^1(K(\alpha, \beta))$, we obtain the result. $\square$

We use the following lemmas.

Lemma A.18. *For the mapping $T : K(\alpha', \beta') \ni (y_1, \cdots, y_n) \mapsto (x_1, \cdots, x_n) \in V(\alpha', \beta')$ defined in Lemma A.14, there exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$\left\|\frac{\partial x_n}{\partial y_i}\frac{\psi}{\mathrm{Jac}(T)}\right\|_{H^1(K)} \le c\left\|\frac{\partial\psi}{\partial y_n}\right\|_{L^2(K)}, \qquad \left\|\frac{\psi}{\mathrm{Jac}(T)}\right\|_{H^1(K)} \le c\left\|\frac{\partial\psi}{\partial y_n}\right\|_{L^2(K)}$$

for all $\psi \in H_0^1(K)$ and $i = 1, \cdots, n-1$, where $K := K(\alpha', \beta')$.

Proof. We compute

$$
\begin{aligned}
&\left\| \frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \\
\leq\ & c_1 \left(\left\| \frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right\|_{L^2(K)} + \sum_{j=1}^{n} \left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right) \right\|_{L^2(K)} \right) \\
\leq\ & c_1 \left(c_2 \|\psi\|_{L^2(K)} + \sum_{j=1}^{n} \left(\left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n/\partial y_i}{\mathrm{Jac}(T)} \right) \psi \right\|_{L^2(K)} + \left\| \frac{\partial x_n/\partial y_i}{\mathrm{Jac}(T)} \frac{\partial \psi}{\partial y_j} \right\|_{L^2(K)} \right) \right) \\
\leq\ & c_1 \left(c_2 \|\psi\|_{H^1(K)} + \sum_{j=1}^{n} \left(\left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n} \right) \psi \right\|_{L^2(K)} + c_2 \|\psi\|_{H^1(K)} \right) \right) \\
\leq\ & c_1 \left(c_2 (n+1) \|\psi\|_{H^1(K)} + \sum_{j=1}^{n} \left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n} \right) \psi \right\|_{L^2(K)} \right)
\end{aligned}
$$

for two constants $c_1, c_2 > 0$. Here, we have

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n} \right) \psi \right\|_{L^2(K)} \\
=\ & \left\| \frac{ \frac{\partial^2 x_n}{\partial y_j \partial y_i} \frac{\partial x_n}{\partial y_n} - \frac{\partial x_n}{\partial y_i} \frac{\partial^2 x_n}{\partial y_j \partial y_n} }{ \left(\frac{\partial x_n}{\partial y_n} \right)^2 } \psi \right\|_{L^2(K)} \\
\leq\ & c_3 \left(\left\| \frac{\partial^2 x_n}{\partial y_j \partial y_i} \psi \right\|_{L^2(K)} + \left\| \frac{\partial^2 x_n}{\partial y_j \partial y_n} \psi \right\|_{L^2(K)} \right) \\
=\ & c_3 \left(\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i} \frac{\psi}{y_n} \right\|_{L^2(K)} + \left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_n} \frac{\psi}{y_n} \right\|_{L^2(K)} \right) \\
\leq\ & c_3 \left(\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i} \right\|_{L^\infty(K)} \left\| \frac{\psi}{y_n} \right\|_{L^2(K)} + \left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_n} \right\|_{L^\infty(K)} \left\| \frac{\psi}{y_n} \right\|_{L^2(K)} \right)
\end{aligned}
$$

for a constant $c_3 > 0$. By Lemma A.13, there exists a constant $c_4 > 0$ such that

$$
\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i} \right\|_{L^\infty(K)} \leq c_4
$$

for all $i, j = 1, \cdots, n$. By Lemma A.17, it holds that

$$
\left\| \frac{\psi}{y_n} \right\|_{L^2(K)} \leq 2 \left\| \frac{\partial \psi}{\partial y_n} \right\|_{L^2(K)} \leq 2 \|\psi\|_{H^1(K)}.
$$

Hence, we obtain that

$$\begin{aligned}\left\|\frac{\partial x_n}{\partial y_i}\frac{\psi}{\mathrm{Jac}(T)}\right\|_{H^1(K)} &\le c_1\left(c_2(n+1)\|\psi\|_{H^1(K)}+\sum_{j=1}^n\left\|\frac{\partial}{\partial y_j}\left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n}\right)\psi\right\|_{L^2(K)}\right)\\ &\le c_1\left(c_2(n+1)\|\psi\|_{H^1(K)}+\sum_{j=1}^n(2c_4\|\psi\|_{H^1(K)}+2c_4\|\psi\|_{H^1(K)})\right)\\ &= c_1(c_2(n+1)+4c_4n)\|\psi\|_{H^1(K)}\end{aligned}$$

for all $\psi\in H_0^1(K)$ and $i=1,\cdots,n-1$. The following inequality can be proved in the same way:

$$\left\|\frac{\psi}{\mathrm{Jac}(T)}\right\|_{H^1(K)}\le c_5\|\psi\|_{H^1(K)}$$

for all $\psi\in H_0^1(K)$. □

Lemma A.19. *For the mapping $T:K(\alpha',\beta')\to V(\alpha',\beta')$ defined in Lemma A.14, there exists a constant $c=c(\alpha,\beta,\alpha',\beta',g)>0$ such that*

$$\|q\|_{H^{-1}(K)}+\|\nabla q\|_{H^{-1}(K)^n}\le c\left(\|p\|_{H^{-1}(V)}+\|\nabla p\|_{H^{-1}(V)^n}\right)$$

for all $p\in C_0^\infty(V)$, where $q:=p\circ T(\in C_0^\infty(K))$, $K:=K(\alpha',\beta')$ and $V:=V(\alpha',\beta')$.

Proof. For $\psi\in H_0^1(K)$ and $\chi=\psi\circ T^{-1}$, we obtain

$$\begin{aligned}\int_K q(y)\psi(y)dy &= \int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx,\\ \int_K\frac{\partial q}{\partial y_i}(y)\psi(y)dy &= \int_V\sum_{j=1}^n\frac{\partial p}{\partial x_j}(x)\frac{\partial x_j}{\partial y_i}(T^{-1}(x))\ \chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx\\ &= \int_V\left\{\frac{\partial p}{\partial x_i}(x)+\frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\right\}\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx,\\ \int_K\frac{\partial q}{\partial y_n}(y)\psi(y)dy &= \int_V\sum_{j=1}^n\frac{\partial p}{\partial x_j}(x)\frac{\partial x_j}{\partial y_n}(T^{-1}(x))\ \chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx\\ &= \int_V\frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_n}(T^{-1}(x))\frac{\chi(x)}{\dfrac{\partial x_n}{\partial y_n}(T^{-1}(x))}dx\\ &= \int_V\frac{\partial p}{\partial x_n}(x)\chi(x)dx\end{aligned}$$

for $i=1,\cdots,n-1$. By Lemma A.16 and Lemma A.18, there exist three constants $c_1,c_2,c_3>0$ such that

$$\begin{aligned}\|\chi\|_{H^1(V)} &\le c_1\|\psi\|_{H^1(K)}\\ \left\|\frac{\partial x_n}{\partial y_i}\circ T^{-1}\frac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\right\|_{H^1(V)} &\le c_2\|\chi\|_{H^1(V)},\\ \left\|\frac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\right\|_{H^1(V)} &\le c_3\|\chi\|_{H^1(V)}\end{aligned}\tag{A.12}$$

for all $\chi \in H_0^1(V)$ and $i = 1, \cdots, n-1$. Thus we have

$$
\begin{aligned}
\|q\|_{H^{-1}(K)} &\leq \sup_{0\neq\psi\in H_0^1(K)} \frac{\displaystyle\int_K q(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\leq c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\leq c_1c_3 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\dfrac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\right\|_{H^1(V)}} \\
&\leq c_1c_3\|p\|_{H^{-1}(V)},
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial q}{\partial y_i}\right\|_{H^{-1}(K)} &\leq \sup_{\psi\in H_0^1(K)} \frac{\displaystyle\int_K \frac{\partial q}{\partial y_i}(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\leq c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \left\{\frac{\partial p}{\partial x_i}(x) + \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\right\}\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\leq c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_i}(x)\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\quad + c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\leq c_1c_3 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_i}(x)\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\dfrac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\right\|_{H^1(V)}} \\
&\quad + c_1c_2 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\dfrac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\dfrac{\partial x_n}{\partial y_i}\circ T^{-1}\right\|_{H^1(V)}} \\
&\leq c_4\left(\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(V)} + \left\|\frac{\partial p}{\partial x_n}\right\|_{H^{-1}(V)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\| \frac{\partial q}{\partial y_n} \right\|_{H^{-1}(K)} &\leq \sup_{0 \neq \psi \in H_0^1(K)} \frac{\displaystyle\int_K \frac{\partial q}{\partial y_n}(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\leq c_1 \sup_{0 \neq \chi \in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\chi(x)dx}{\|\chi\|_{H^1(V)}} \\
&= c_1 \left\| \frac{\partial p}{\partial x_n} \right\|_{H^{-1}(V)},
\end{aligned}
$$

for $i = 1, \cdots, n-1$, where $c_4 := c_1 \max\{c_2, c_3\}$. Finally, by Lemma A.2, it follows that

$$
\begin{aligned}
\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n} &\leq \|q\|_{H^{-1}(K)} + \sum_{i=1}^n \left\| \frac{\partial q}{\partial y_i} \right\|_{H^{-1}(K)} \\
&\leq c_5 \left(\|p\|_{H^{-1}(V)} + \sum_{i=1}^n \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(V)} \right) \\
&\leq \sqrt{n} c_5 \left(\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n} \right)
\end{aligned}
$$

for all $p \in C_0^\infty(V)$ and $q := p \circ T (\in C_0^\infty(K))$, where $c_5 := c_1 + (n-1)c_4$. $\square$

Lemma A.20. *Let $0 < \alpha' < \alpha$, $0 < \beta' < \beta$ and let $g : \overline{B(\alpha)} \to \mathbb{R}$ be a Lipschitz continuous function. There exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$
\|p\|_{L^2(U)} \leq c(\|p\|_{H^{-1}(U)} + \|\nabla p\|_{H^{-1}(U)^n}) \qquad \text{for all } p \in C_0^\infty(U_g^+(\alpha', \beta')),
$$

where $U = U_g^+(\alpha, \beta)$.

Proof. Let $\alpha_1', \alpha_2', \beta_1'$ and β_2' satisfy $\alpha_1' := \alpha' < \alpha_2' < \alpha, \beta_1' := \beta' < \beta_2' < \beta$ and let $K := K(\alpha_2', \beta_2')$. By Lemma A.12, there exists a C^∞-diffeomorphic map $T : K \to V := V(\alpha_2', \beta_2')$. Let us denote $q = p \circ T$ for $p \in C_0^\infty(U_g^+(\alpha_1', \beta_1'))$. We have $\mathrm{supp}(p) \subset U_g^+(\alpha_2', \beta_2') \subset V$, thus,

$$
\|p\|_{L^2(U)} = \|p\|_{L^2(V)}. \tag{A.13}
$$

Since $L^2(V) \ni p \mapsto p \circ T \in L^2(K)$ is isomorphic between Banach spaces, we get the following inequality:

$$
\|p\|_{L^2(V)} \leq c_1 \|q\|_{L^2(K)}. \tag{A.14}
$$

By Lemma A.14, we have $U_g^+(\alpha_1', \beta_1') \subset T(K(\alpha_1', \beta_1'))$, hence, $\mathrm{supp}(q) \subset K(\alpha_1', \beta_1')$. By Lemma A.10, there exists a constant $c_2 > 0$ such that

$$
\|q\|_{L^2(K)} \leq c_2(\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n}). \tag{A.15}
$$

Moreover, by Lemma A.19, there exists a constant $c_3 > 0$ such that

$$
\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n} \leq c_3(\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}). \tag{A.16}
$$

Therefore, by (A.13), (A.14), (A.15) and (A.16), it follows that

$$
\begin{aligned}
\|p\|_{L^2(U)} &\leq c_1 c_2 c_3 (\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}) \\
&\leq c_1 c_2 c_3 (\|p\|_{H^{-1}(U)} + \|\nabla p\|_{H^{-1}(U)^n})
\end{aligned}
$$

for all $p \in C_0^\infty(U_g^+(\alpha_1', \beta_1'))$. $\square$

A.5 Original Nečas inequality

Definition A.1. *A bounded open set Ω is called Lipschitz domain if there exist two real numbers $\alpha, \beta > 0$, an integer $m \in \mathbb{N}$, systems of local charts $(x_{r1}, \cdots, x_{rn})$ $(r = 1, 2, \cdots, m)$ and Lipschitz continuous functions $g_r : \overline{B_r(\alpha)} := \{x'_r \in \mathbb{R}^{n-1} \mid |x_{ri}| \leq \alpha, i = 1, 2, \cdots, n-1\} \to \mathbb{R}$ such that*

$$\Gamma = \bigcup_{r=1}^{m} \{x_r \in \mathbb{R}^n \mid x_r = (x'_r, x_{rn}), x'_r \in B_r(\alpha), x_{rn} = g_r(x'_r)\}$$

and it follows that for all $y'_r \in \overline{B_r(\alpha)}$

$$\begin{array}{ll} g_r(y'_r) < y_{rn} < g_r(y'_r) + \beta & \Rightarrow (y'_r, y_{rn}) \in \Omega, \\ g_r(y'_r) - \beta < y_{rn} < g_r(y'_r) & \Rightarrow (y'_r, y_{rn}) \in \mathbb{R}^n \backslash \Omega. \end{array}$$

By using systems of local charts in Definition A.1, we define subsets;

$$\begin{array}{rcl} B_r(\alpha') & := & \{x'_r \in \mathbb{R}^{n-1} \mid |x'_r| < \alpha'\}, \\ U_r(\alpha', \beta') & := & \{x_r = (x'_r, x_{rn}) \in \mathbb{R}^n \mid x'_r \in B_r(\alpha'), g_r(x'_r) - \beta' < x_{rn} < g_r(x'_r) + \beta'\}, \\ U_r^+(\alpha', \beta') & := & \{x_r = (x'_r, x_{rn}) \in \mathbb{R}^n \mid x'_r \in B_r(\alpha'), g_r(x'_r) < x_{rn} < g_r(x'_r) + \beta'\} \end{array}$$

for $0 < \alpha' \leq \alpha, 0 < \beta' \leq \beta$ and $r = 1, 2, \cdots, m$.

We prove Lemma 2.8 which is the goal of this appendix.

Theorem A.21 (Reshown, see Lemma 2.8)**.** *If Ω is a bounded Lipschitz domain, then there exists a constant $c = c(\Omega) > 0$ such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in L^2(\Omega).$$

Proof. By Lemma A.3, it is enough to prove that there exists a constant $c_1 > 0$ such that

$$\|p\|_{L^2(\Omega)} \leq c_1(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in C_0^\infty(\Omega).$$

By Definition A.1, we have $\Gamma \subset \bigcup_{r=1}^m U_r(\alpha, \beta)$. We can choose two real numbers $0 < \alpha' < \alpha, 0 < \beta' < \beta$ and an open subset $U_0(\alpha', \beta')$ $(\overline{U_0(\alpha', \beta')} \subset \Omega)$ such that $\overline{\Omega} \subset \bigcup_{r=0}^m U_r(\alpha', \beta')$. By Lemma A.1, there exist functions $\eta_0, \cdots, \eta_m \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{array}{ll} \eta_r \in C_0^\infty(U_r(\alpha', \beta')) & \text{for all } r = 0, 1, \cdots, m, \\ 0 \leq \eta_r(x) \leq 1 & \text{for all } r = 0, 1, \cdots, m, x \in U_r(\alpha', \beta'), \\ \displaystyle\sum_{r=0}^m \eta_r(x) = 1 & x \in \overline{\Omega}. \end{array}$$

Let $p_r := p\eta_r \in C_0^\infty(U_r^+(\alpha', \beta'))$ for $r = 0, 1, \cdots, m$. Then it follows that

$$\|p\|_{L^2(\Omega)} = \left\| \sum_{r=0}^m p_r \right\|_{L^2(\Omega)} \leq \sum_{r=0}^m \|p_r\|_{L^2(\Omega)}$$

for all $p \in C_0^\infty(\Omega)$. On the other hand, for $\psi \in H_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega p\eta_r\psi &\leq \|p\|_{H^{-1}(\Omega)}\|\eta_r\psi\|_{H^1(\Omega)} \\ &= c_2\|p\|_{H^{-1}(\Omega)}\|\eta_r\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)},\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial x_i}(p\eta_r)\psi \\ \leq\ & \left|\frac{\partial p}{\partial x_i}\eta_r\psi\right| + \left|\int_\Omega p\frac{\partial \eta_r}{\partial x_i}\psi\right| \\ \leq\ & \left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(\Omega)}\|\eta_r\psi\|_{H^1(\Omega)} + \|p\|_{H^{-1}(\Omega)}\left\|\frac{\partial \eta_r}{\partial x_i}\psi\right\|_{H^1(\Omega)} \\ \leq\ & c_2\left(\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(\Omega)}\|\eta_r\|_{H^1(\Omega)} + \|p\|_{H^{-1}(\Omega)}\left\|\frac{\partial \eta_r}{\partial x_i}\right\|_{H^1(\Omega)}\right)\|\psi\|_{H^1(\Omega)}\end{aligned}$$

for all $i = 1, 2, \cdots, n$ and a constant $c_2 > 0$. Hence, we obtain

$$\begin{aligned}&\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n} \\ \leq\ & \|p\eta_r\|_{H^{-1}(\Omega)} + \sum_{i=1}^n\left\|\frac{\partial}{\partial x_i}(p\eta_r)\right\|_{H^{-1}(\Omega)} \\ =\ & \sup_{\psi\in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1}\int_\Omega p\eta_r\psi + \sum_{i=1}^n \sup_{\psi\in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1}\int_\Omega \frac{\partial}{\partial x_i}(p\eta_r)\psi \\ \leq\ & c_{2,r}\left(\|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^n\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(\Omega)}\right) \\ \leq\ & \sqrt{n}c_{2,r}(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}),\end{aligned}$$

where $c_{2,r} := c_2\left(\|\eta_r\|_{H^1(\Omega)} + n\sum_{i=1}^n\left\|\frac{\partial \eta_r}{\partial x_i}\right\|_{H^1(\Omega)}\right)$. Thus it suffices to show that for $r = 0, 1, \cdots, m$,

$$\|p_r\|_{L^2(\Omega)} \leq c_3(\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p_r \in C_0^\infty(U_r^+(\alpha', \beta')). \tag{A.17}$$

(i) The case $r = 0$.

We have $\mathrm{supp}(p_0) \subset U_0(\alpha', \beta')$ and $\overline{U_0(\alpha', \beta')} \subset \Omega$. By Lemma A.8, the inequality (A.17) holds with $r = 0$.

(i) The case $r = 1, 2, \cdots, m$.

Let $U_r := U_r^+(\alpha, \beta)$. By Lemma A.20, we obtain (A.17);

$$\begin{aligned}\|p_r\|_{L^2(\Omega)} &= \|p_r\|_{L^2(U_r)} \\ &\leq c_4(\|p_r\|_{H^{-1}(U_r)} + \|\nabla p_r\|_{H^{-1}(U_r)^n}) \\ &\leq c_4(\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n})\end{aligned}$$

for all $p_r \in C_0^\infty(U_r^+(\alpha', \beta'))$. □

Using Theorem 2.8, we obtain the following corollary which is important for existence and uniqueness of the solution to the Stokes problem.

Corollary A.22 (Reshown, see Theorem 2.9)**.** *If Ω is a bounded Lipschitz domain, then there exists a constant $c = c(\Omega) > 0$ such that*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla p\|_{H^{-1}(\Omega)^n} \qquad \textit{for all } p \in L^2(\Omega).$$

Proof. It suffices to show that there exists a constant $c > 0$ such that

$$\|p\|_{L^2(\Omega)} \leq c\|\nabla p\|_{H^{-1}(\Omega)^n} \qquad \text{for all } p \in L^2(\Omega)/\mathbb{R}.$$

Assume that this property does not hold. Then there exists a sequence of functions $(p_k)_{k\in\mathbb{N}} \subset L^2(\Omega)/\mathbb{R}$ such that $\|p_k\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$ and

$$\|\nabla p_k\|_{H^{-1}(\Omega)^n} \to 0 \qquad \text{as } k \to \infty.$$

Since the sequence $(p_k)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, there exists a subsequece $(p_l)_{l\in\mathbb{N}}$ that converges weakly in $L^2(\Omega)$. Let $\varphi \in L^2(\Omega)$ and $\delta > 0$ be arbitrary. Then there exists a constant $N_{\varphi,\delta} \in \mathbb{N}$ such that

$$l_1, l_2 \geq N_{\varphi,\delta} \Rightarrow |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| < \delta.$$

On the other hand, since $S := \{\varphi \in H^1(\Omega) \mid \|\varphi\|_{H^1(\Omega)} = 1\}$ is bounded in $H^1(\Omega)$, for every fixed $\delta' > 0$, there exist functions $\varphi_1, \cdots, \varphi_m \in S$ such that

$$S \subset \bigcup_{i=1}^{m} \{\varphi \in L^2(\Omega) \mid \|\varphi - \varphi_i\|_{L^2(\Omega)} < \delta'\}$$

by the Rellich–Kondrashov Theorem, i.e. there exists a number $i \in \{1, \cdots, m\}$ such that

$$\|\varphi - \varphi_i\|_{L^2(\Omega)} < \delta'$$

for all $\varphi \in S$. Thus we obtain

$$\begin{aligned} |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| &\leq |(\varphi - \varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| + |(\varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \\ &\leq \|\varphi - \varphi_i\|_{L^2(\Omega)}\|p_{l_1} - p_{l_2}\|_{L^2(\Omega)} + |(\varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \\ &\leq 2\delta' + \delta \end{aligned}$$

for all $\varphi \in S$ and $l_1, l_2 \geq N_\delta := \max\{N_{\varphi_1,\delta}, \cdots, N_{\varphi_m,\delta}\}$, and then

$$\|p_{l_1} - p_{l_2}\|_{H^{-1}(\Omega)} = \sup_{\varphi\in S} |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \leq 2\delta' + \delta.$$

It satisfies that

$$\limsup_{l_1,l_2\to\infty} \|p_{l_1} - p_{l_2}\|_{H^{-1}(\Omega)} \leq 2\delta'$$

for every $\delta' > 0$, which implies that $(p_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1}(\Omega)$. Besides, by the assumption, $(\nabla p_l)_{l\in\mathbb{N}}$ is also a Cauchy sequence in $H^{-1}(\Omega)$.

Theorem 2.8 leads that $(p_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. Thus, there exists a function $p \in L^2(\Omega)$ such that

$$\|p_l - p\|_{L^2(\Omega)} \to 0 \qquad \text{as } l \to \infty.$$

We have

$$\|\nabla p_l - \nabla p\|_{H^{-1}(\Omega)^n} \to 0 \qquad \text{as } l \to \infty,$$

since the operator $\nabla : L^2(\Omega) \to H^{-1}(\Omega)^n$ is continuous. Indeed, it holds that

$$\begin{aligned} |\langle \nabla\omega, u\rangle| = |(\omega, -\operatorname{div} u)_{L^2(\Omega)}| \leq \|\omega\|_{L^2(\Omega)} \|\operatorname{div} u\|_{L^2(\Omega)} \quad & \text{for all } \omega \in L^2(\Omega), u \in H^1(\Omega)^n, \\ |\langle \nabla\omega, u\rangle| \leq \sqrt{n}\|\omega\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)^n} \quad & \text{for all } \omega \in L^2(\Omega), u \in H^1(\Omega)^n, \\ \|\nabla\omega\|_{H^{-1}(\Omega)^n} \leq \sqrt{n}\|\omega\|_{L^2(\Omega)} \quad & \text{for all } \omega \in L^2(\Omega), \\ \|\nabla * \|_{\mathcal{L}(L^2(\Omega), H^{-1}(\Omega))} \leq \sqrt{n}. & \end{aligned}$$

By connectivity of the open set Ω, the function p is a constant, and this constant is 0 since $\int_\Omega p_l = 0$ for all $l \in \mathbb{N}$. But this contradicts the relation $\|p_l\|_{L^2(\Omega)} = 1$ for all $l \in \mathbb{N}$. □

Corollary A.23 (Reshown, see Theorem 2.10)**.** *If Ω is a bounded Lipschitz domain, then the divergence operator* div *maps $H_0^1(\Omega)^n$ onto $L^2(\Omega)/\mathbb{R}$.*

Proof. The operator $\nabla : L^2(\Omega) \to H^{-1}(\Omega)^n$ satisfies

$$\langle \nabla\omega, u\rangle = (\omega, -\operatorname{div} u)_{L^2(\Omega)} \quad \text{for all } \omega \in H^1(\Omega), u \in H^1(\Omega)^n, \tag{A.18}$$

and thus $-\operatorname{div} : H^1(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ is the dual operator of ∇. By the proof of Theorem 2.9, ∇ is continuous and thus closed. By Theorem 2.9, we deduce that the image of ∇ is closed, and so, the image of div is $(\mathrm{Ker}\nabla)^\perp = \mathbb{R}^\perp = L^2(\Omega)/\mathbb{R}$. □

B Proofs

Theorem 2.18, 5.2, 5.3, and Proposition 3.2 are generalizations of several theorems stated in [22]. In this appendix, however, we give their proofs for the readers' convenience. We define a continuous coercive bilinear form depending on ε and prove Theorem 2.18 by the Lax–Milgram Theorem.

Proof of Theorem 2.18. We take arbitrary $u_1 \in H^1(\Omega)^n$ with $\gamma_0 u_1 = u_b$. Since $\operatorname{div} : H^1_0(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ is surjective [14, Corollary 2.4, 2°], there exists $u_2 \in H^1_0(\Omega)^n$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$. We put

$$u_0 := u_1 - u_2, \tag{B.19}$$

and note that $\gamma_0 u_0 = u_b$ and $\operatorname{div} u_0 = 0$. To simplify the notation, we set $u := u_\varepsilon - u_0 \in H^1_0(\Omega)^n, p := p_\varepsilon - p_b \in Q$, and define $f \in H^{-1}(\Omega)^n$ and $g \in Q^*$ by

$$\begin{aligned} \langle f, v\rangle &:= \int_\Omega Fv - \int_\Omega \nabla u_0 : \nabla v - \int_\Omega (\nabla p_b)\cdot v \quad \text{for all } v \in H^1_0(\Omega)^n, \\ \langle g, q\rangle &:= \langle G, q\rangle - \int_\Omega \nabla p_b \cdot \nabla q \quad \text{for all } q \in Q. \end{aligned} \tag{B.20}$$

Then, $(u_\varepsilon, p_\varepsilon)$ satisfies (ES') if and only if (u, p) satisfies

$$\begin{cases} \displaystyle\int_\Omega \nabla u : \nabla\varphi + \int_\Omega (\nabla p)\cdot\varphi = \langle f, \varphi\rangle & \text{for all } \varphi \in H^1_0(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla p\cdot\nabla\psi + \int_\Omega (\operatorname{div} u)\psi = \varepsilon\langle g, \psi\rangle & \text{for all } \psi \in Q. \end{cases} \tag{B.21}$$

Adding the equations in (B.21), we get

$$\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)_\varepsilon := \int_\Omega \nabla u : \nabla\varphi + \varepsilon\int_\Omega \nabla p\cdot\nabla\psi + \int_\Omega (\nabla p)\cdot\varphi + \int_\Omega (\operatorname{div} u)\psi = \langle f, \varphi\rangle + \varepsilon\langle g, \psi\rangle.$$

We check that $(\cdot,\cdot)_\varepsilon$ is a continuous coercive bilinear form on $H^1_0(\Omega)^n \times Q$. The bilinearity and continuity of $(\cdot,\cdot)_\varepsilon$ are obvious. The coercivity of $(\cdot,\cdot)_\varepsilon$ is obtained in the following way. Take $(v, q)^T \in H^1_0(\Omega)^n \times Q$. We have the following sequence of inequalities:

$$\begin{aligned} \left(\begin{pmatrix} v \\ q \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right)_\varepsilon &= \int_\Omega \nabla v : \nabla v + \varepsilon\int_\Omega \nabla q\cdot\nabla q + \int_\Omega v\cdot\nabla q + \int_\Omega (\operatorname{div} v) q \\ &= \|\nabla v\|^2_{L^2(\Omega)} + \varepsilon\|\nabla q\|^2_{L^2(\Omega)} \\ &\ge \min\{1, \varepsilon\}\left(\|\nabla v\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{L^2(\Omega)}\right) \\ &\ge c\,\min\{1, \varepsilon\}\left(\|v\|^2_{H^1(\Omega)^n} + \|q\|^2_{H^1(\Omega)}\right). \end{aligned}$$

Summarizing, $(\cdot,\cdot)_\varepsilon$ is a continuous coercive bilinear form and $H^1_0(\Omega)^n \times Q$ is a Hilbert space. Therefore, the conclusion of Theorem 2.18 follows from the Lax–Milgram Theorem. □

Let $(u_S, p_S), (u_{PP}, p_{PP})$ and $(u_\varepsilon, p_\varepsilon)$ be the solutions of (S'), (PP') and (ES'), respectively, as guaranteed by Theorem 2.16, 2.17 and 2.18. We show that the subtract $p_S - p_{PP}$ satisfies

$$\Delta(p_S - p_{PP}) = 0$$

in distributions sense. The weak harmonicity is the key ingredient to proving Proposition 3.2.

Proof of Proposition 3.2. First, we prove that there exists a constant $c > 0$ independent of ε such that $\|u_S - u_{PP}\|_{H^1(\Omega)^n} \le c\|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}$, and if $(p_S - p_{PP}) = 0$, then $p_{PP} = p_S$. Taking the divergence of the first equation of (S'), we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u_S + \nabla p_S) = -\Delta(\operatorname{div} u_S) + \Delta p_S = \Delta p_S.$$

in distributions sense. Since $p_S \in H^1(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that

$$\int_\Omega \nabla p_S \cdot \nabla\psi = -\int_\Omega (\operatorname{div} F)\psi$$

for all $\psi \in H_0^1(\Omega)$. Together with (S'), (PP') and $H_0^1(\Omega) \subset Q$, we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla(u_S - u_{PP}) : \nabla\varphi = -\int_\Omega (\nabla(p_S - p_{PP})) \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla(p_S - p_{PP}) \cdot \nabla\psi = 0 & \text{for all } \psi \in H_0^1(\Omega) \end{cases} \tag{B.22}$$

from the assumption $\langle G, \psi\rangle = \int_\Omega \nabla F \cdot \psi$. Putting $\varphi := u_S - u_{PP} \in H_0^1(\Omega)^n$ in (B.22), we get

$$\begin{aligned} \|\nabla(u_S - u_{PP})\|^2_{L^2(\Omega)^{n\times n}} &= -\int_\Omega (\nabla(p_S - p_{PP})) \cdot (u_S - u_{PP}) \\ &\le \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} \|u_S - u_{PP}\|_{L^2(\Omega)^n}. \end{aligned}$$

Hence,

$$\|u_S - u_{PP}\|_{H^1(\Omega)^n} \le c_1 \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} \tag{B.23}$$

holds. From the second equation of (B.22) and Lemma 3.4 (with $\Gamma_N = \emptyset$ and $\Gamma_D = \Gamma$ i.e. $H^1_{0,D}(\Omega) = H_0^1(\Omega)$), we obtain

$$\|p_S - p_{PP}\|_{H^1(\Omega)} \le c_2 \|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}. \tag{B.24}$$

Together with (B.23), we obtain $\|u_S - u_{PP}\|_{H^1(\Omega)^n} \le c_1 c_2 \|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}$. Moreover, if $\gamma_0(p_S - p_{PP}) = 0$ then $p_{PP} = p_S$.

Next, we prove that there exists a constant $c > 0$ independent of ε such that $\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} \le c\|p_S - p_\varepsilon\|_{H^{1/2}(\Gamma)}$, and if $\gamma_0(p_S - p_{PP}) = 0$, then $p_{PP} = p_\varepsilon$. Let $w_\varepsilon := u_S - u_\varepsilon \in H_0^1(\Omega)^n$ and $r_\varepsilon := p_{PP} - p_\varepsilon \in Q$. By (S'), (PP') and (ES'), we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla w_\varepsilon : \nabla\varphi + \int_\Omega (\nabla r_\varepsilon) \cdot \varphi = -\int_\Omega (\nabla(p_S - p_{PP})) \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla r_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} w_\varepsilon)\psi = 0 & \text{for all } \psi \in Q. \end{cases} \tag{B.25}$$

Putting $\varphi := w_\varepsilon$ and $\psi := r_\varepsilon$ and adding the two equations of (B.25), we get

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla r_\varepsilon\|^2_{L^2(\Omega)^n} \le \|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}\|w_\varepsilon\|_{L^2(\Omega)^n} \tag{B.26}$$

from $\int_\Omega (\nabla r_\varepsilon)\cdot w_\varepsilon = -\int_\Omega (\operatorname{div} w_\varepsilon) r_\varepsilon$. Thus we find

$$\|w_\varepsilon\|_{H^1(\Omega)^n} \le c_3\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}.$$

Together with (B.24), we obtain

$$\|u_S - u_\varepsilon\|_{H^1(\Omega)^n} = \|w_\varepsilon\|_{H^1(\Omega)^n} \le c_2c_3\|p_S - p_{PP}\|_{H^{1/2}(\Gamma)}.$$

Moreover, by (B.26), we obtain

$$\varepsilon\|p_{PP} - p_\varepsilon\|^2_{L^2(\Omega)} = \varepsilon\|r_\varepsilon\|^2_{L^2(\Omega)} \le c_4\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n}\|w_\varepsilon\|_{L^2(\Omega)^n}.$$

Hence, if $\gamma_0(p_S - p_{PP}) = 0$, then $p_{PP} = p_\varepsilon$. □

We show that the sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ is bounded in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. By the reflexivity of $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$, the sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ has a subsequence converging weakly to somewhere in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$. It is sufficient to check that the limit satisfies (S'). Since the solution of (S') is unique, the sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ converges weakly.

Proof of Theorem 5.2. We take $u_b \in H^1(\Omega)^n$, $f \in H^{-1}(\Omega)^n$ and $g \in Q^*$ as (B.19) and (B.20) in the proof of Theorem 2.18. We put $\tilde{u}_\varepsilon := u_\varepsilon - u_b \in H^1_0(\Omega)^n$, $\tilde{p}_\varepsilon := p_\varepsilon - p_b \in Q$. Then we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla\tilde{u}_\varepsilon : \nabla\varphi + \int_\Omega (\nabla\tilde{p}_\varepsilon)\cdot\varphi = \langle f, \varphi\rangle & \text{for all } \varphi \in H^1_0(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla\tilde{p}_\varepsilon\cdot\nabla\psi + \int_\Omega (\operatorname{div}\tilde{u}_\varepsilon)\psi = \varepsilon\langle g, \psi\rangle & \text{for all } \psi \in Q. \end{cases} \tag{B.27}$$

Putting $\varphi := \tilde{u}_\varepsilon, \psi := \tilde{p}_\varepsilon$ and adding the two equations of (B.27), we get

$$\|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla\tilde{p}_\varepsilon\|^2_{L^2(\Omega)^n} \le \|f\|_{H^{-1}(\Omega)^n}\|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)^{n\times n}} + \varepsilon\|g\|_{Q^*}\|\nabla\tilde{p}_\varepsilon\|_{L^2(\Omega)^n}$$

since $\int_\Omega (\nabla\tilde{p}_\varepsilon)\cdot\tilde{u}_\varepsilon = -\int_\Omega (\operatorname{div}\tilde{u}_\varepsilon)\tilde{p}_\varepsilon$. Hence, $(\|\tilde{u}_\varepsilon\|_{H^1(\Omega)^n})_{0<\varepsilon<1}$ and $(\|\sqrt{\varepsilon}\tilde{p}_\varepsilon\|_{H^1(\Omega)})_{0<\varepsilon<1}$ are bounded. Moreover, by Lemma 5.1, we obtain

$$\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \le c(\|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)^{n\times n}} + \|f\|_{H^{-1}(\Omega)^n}),$$

i.e., $(\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{0<\varepsilon<1}$ is bounded. By Theorem 4.2, $(\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon\ge1}$ and $(\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon\ge1}$ are bounded, and thus $(\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon>0}$ and $(\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon>0}$ are bounded.

Since $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ is reflexive and $(\tilde{u}_\varepsilon, [\tilde{p}_\varepsilon])_{0<\varepsilon<1}$ is bounded in $H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$, there exist $(u, p) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ and a subsequence of pairs $(\tilde{u}_{\varepsilon_k}, \tilde{p}_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1_0(\Omega)^n \times Q$ such that

$$\tilde{u}_{\varepsilon_k} \rightharpoonup u \text{ weakly in } H^1(\Omega)^n,\ [\tilde{p}_{\varepsilon_k}] \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } k \to \infty.$$

Hence, from (B.27) with $\varepsilon := \varepsilon_k$, taking $k \to \infty$, we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla u : \nabla\varphi + \langle \nabla p, \varphi\rangle = \langle f, \varphi\rangle & \text{for all } \varphi \in H_0^1(\Omega)^n \\ \displaystyle\int_\Omega (\operatorname{div} u)\psi = 0 & \text{for all } \psi \in Q, \end{cases} \tag{B.28}$$

where we have used that

$$|\varepsilon_k \int_\Omega \nabla\tilde{p}_{\varepsilon_k} \cdot \nabla\psi| \le \sqrt{\varepsilon_k}\|\sqrt{\varepsilon}\tilde{p}_\varepsilon\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)} \to 0,$$

$$\int_\Omega \nabla\tilde{p}_{\varepsilon_k} \cdot \varphi = -\int_\Omega [\tilde{p}_{\varepsilon_k}] \operatorname{div}\varphi \to -\int_\Omega p \operatorname{div}\varphi = \langle \nabla p, \varphi\rangle$$

as $k \to \infty$. By (B.20), the first equation of (B.28) implies that

$$\int_\Omega \nabla(u + u_b) : \nabla\varphi + \langle \nabla(p + p_b), \varphi\rangle = \int_\Omega F \cdot \varphi$$

for all $\varphi \in H_0^1(\Omega)^n$. From the second equation of (B.28) and $C_0^\infty(\Omega) \subset Q$, $\operatorname{div}(u + u_b) = 0$ follows. Hence, we obtain that $(u + u_b, p + [p_b])$ satisfies (S'), i.e., $u_S = u + u_b$ and $p_S = p + [p_b]$. Then we have

$$u_{\varepsilon_k} - u_S = u_{\varepsilon_k} - u - u_b = \tilde{u}_{\varepsilon_k} - u_S \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^n,$$

$$[p_{\varepsilon_k}] - p_S = [p_{\varepsilon_k} - p - p_b] = [\tilde{p}_{\varepsilon_k}] - p \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}$$

as $k \to \infty$. Since any arbitrarily chosen subsequence of $((u_\varepsilon, [p_\varepsilon]))_{0<\varepsilon<1}$ has a subsequence which converges to (u_S, p_S), we can conclude the proof. □

Using Theorem 5.2 and the Rellich-Kondrachov Theorem, it is easy to prove Theorem 5.3.

Proof of Theorem 5.3. We have from (ES') and (S') that

$$\begin{cases} \displaystyle\int_\Omega \nabla(u_\varepsilon - u_S) : \nabla\varphi + \int_\Omega (\nabla(p_\varepsilon - p_S)) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla p_\varepsilon \cdot \nabla\psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = \varepsilon\langle G, \psi\rangle & \text{for all } \psi \in Q. \end{cases}$$

Putting $\varphi := u_\varepsilon - u_S \in H_0^1(\Omega)^n$ and $\tilde{p}_S := p_S - p_b \in H^1(\Omega)$, we get

$$\begin{aligned} \|\nabla(u_\varepsilon - u_S)\|^2_{L^2(\Omega)^{n\times n}} &= -\int_\Omega (\nabla(p_\varepsilon - p_S)) \cdot (u_\varepsilon - u_S) \\ &= -\int_\Omega (\nabla(p_\varepsilon - p_b)) \cdot (u_\varepsilon - u_S) + \int_\Omega (\nabla(p_S - p_b)) \cdot (u_\varepsilon - u_S) \\ &= \int_\Omega (p_\varepsilon - p_b) \operatorname{div}(u_\varepsilon - u_S) + \int_\Omega (\nabla\tilde{p}_S) \cdot (u_\varepsilon - u_S) \\ &= \int_\Omega (p_\varepsilon - p_b) \operatorname{div} u_\varepsilon + \int_\Omega (\nabla\tilde{p}_S) \cdot (u_\varepsilon - u_S), \end{aligned}$$

since $-\int_\Omega(\nabla(p_\varepsilon - p_b))\cdot(u_\varepsilon - u_S) = \int_\Omega(p_\varepsilon - p_b)\operatorname{div}(u_\varepsilon - u_S)$ and $\operatorname{div} u_S = 0$. Thus,

$$\|\nabla(u_\varepsilon - u_S)\|^2_{L^2(\Omega)^{n\times n}} = \int_\Omega (p_\varepsilon - p_b)\operatorname{div} u_\varepsilon + \int_\Omega (\nabla\tilde{p}_S)\cdot(u_\varepsilon - u_S). \tag{B.29}$$

Putting $\psi := p_\varepsilon - p_b \in Q$, we have

$$\varepsilon\int_\Omega \nabla p_\varepsilon\cdot\nabla(p_\varepsilon - p_b) + \int_\Omega(\operatorname{div} u_\varepsilon)(p_\varepsilon - p_b) = \varepsilon\langle G, p_\varepsilon - p_b\rangle.$$

Hence,

$$\varepsilon\|\nabla(p_\varepsilon - p_b)\|^2_{L^2(\Omega)^n} = -\varepsilon\int_\Omega \nabla(p_\varepsilon - p_b)\cdot\nabla p_b - \int_\Omega (p_\varepsilon - p_b)\operatorname{div} u_\varepsilon + \varepsilon\langle G, p_\varepsilon - p_b\rangle. \tag{B.30}$$

Together with (B.29) and (B.30), we obtain

$$\begin{aligned}
& \|\nabla(u_\varepsilon - u_S)\|^2_{L^2(\Omega)^{n\times n}} + \varepsilon\|\nabla(p_\varepsilon - p_b)\|^2_{L^2(\Omega)^n} \\
= \; & \int_\Omega \nabla\tilde{p}_S\cdot(u_\varepsilon - u_S) - \varepsilon\int_\Omega \nabla(p_\varepsilon - p_b)\cdot\nabla p_b + \varepsilon\langle G, p_\varepsilon - p_b\rangle \\
\leq \; & \|\nabla\tilde{p}_S\|_{L^2(\Omega)^n}\|u_\varepsilon - u_S\|_{L^2(\Omega)^n} + \varepsilon(\|\nabla p_b\|_{L^2(\Omega)^n} + \|G\|_{Q^*})\|\nabla(p_\varepsilon - p_b)\|_{L^2(\Omega)^n}.
\end{aligned}$$

By Theorem 5.2 and the Rellich-Kondrachov Theorem, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$u_{\varepsilon_k} \to u_S \text{ strongly in } L^2(\Omega)^n \text{ as } k\to\infty.$$

Therefore,

$$\begin{aligned}
& \|\nabla(u_{\varepsilon_k} - u_S)\|^2_{L^2(\Omega)^{n\times n}} \\
\leq \; & \|\nabla\tilde{p}_S\|_{L^2(\Omega)^n}\|u_{\varepsilon_k} - u_S\|_{L^2(\Omega)^n} + \varepsilon_k(\|\nabla\tilde{p}_b\|_{L^2(\Omega)^n} + \|G\|_{Q^*})\|\nabla(p_{\varepsilon_k} - p_S)\|_{L^2(\Omega)^n} \\
\to \; & 0
\end{aligned}$$

as $k\to\infty$. This implies that

$$\|[p_{\varepsilon_k}] - p_S\|_{L^2(\Omega)/\mathbb{R}} = \|p_{\varepsilon_k} - p_S\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla(u_{\varepsilon_k} - u_S)\|_{L^2(\Omega)^{n\times n}} \to 0 \text{ as } k\to\infty$$

by Lemma 5.1. Since any arbitrarily chosen subsequence of $((u_\varepsilon, [p_\varepsilon]))_{0<\varepsilon<1}$ has a subsequence which converges to (u_S, p_S), we can conclude the proof. □