Complexity of Reasoning in Kleene and Action Algebras

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Lecture 1

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- Due to its inductive nature, theories involving Kleene star, even in a purely “propositional” (atomic) language, share some features with much more powerful theories like Peano arithmetic.
- The methods used will be mostly proof-theoretic: thus, we shall discuss proof theory for algebraic substructural logics including Kleene star.

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- The **Computation** side is connected to complexity results.
- The **Logic** side is connected to substructural proof theory used for logics with Kleene star.
- There is also connection to **Language,** as Kleene star is one of the natural operations on formal languages.

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- However, if you see a notion which you are not familar with, please feel free to ask for definition. In fact, we mostly do not use really advanced concepts, and many basic things could be explained in a minute, if needed.
- We shall try to give mathematical proofs for most of the results, but due to time limitations some of them will be just sketches of proofs, without deep details.

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- Complexity range will be huge: from PSPACE up to Π^1_1-hard.

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$$v(a) = \{a\} \qquad v(0) = \varnothing \qquad v(1) = \{\varepsilon\}$$

$$v(A \cdot B) = v(A) \cdot v(B) = \{xy \mid x \in v(A), y \in v(B)\}$$

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- On the other hand, $A = (ab^*)^*$ and $B = (a+b)^*$ are not equivalent: $aba \in \nu(B)$, but $aba \notin \nu(A)$.
- The equivalence problem, of course, has standard solutions (e.g., using finite automata), but we shall focus on the logic behind it.

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    - Inequation is more like implication, and it will be easier to axiomatise.

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- Let us formulate an infinitary axiomatization as a **sequent calculus** $\mathbf{KA}_\omega$, with sequents of the form $\Pi \to B$, where Π is a sequence of formulae (reg. exp.), B is a formula.
- "**KA**" stands for "Kleene algebra," and "ω" is due to the infinitary rule (ω-rule).

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- If we wish to prove $A \preceq C$ using *Trans*, we have no idea what B could be.
- We shall avoid using such rules.

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- If Π is empty, we have $1 \preceq B$.
- Thus, the sequential comma in Π is a *meta-syntactic* counterpart of product (multiplication); $\rightarrow$ corresponds to $\preceq$.
- Π and B are called, respectively, the **antecedent** and the **succedent** of the sequent.

Axioms and rules of inference of $\mathbf{KA}_\omega$ are as follows:

$$\frac{}{A \to A}\ Id \qquad \frac{}{\Gamma, 0, \Delta \to B}\ 0L$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, 1, \Delta \to B}\ 1L \qquad \frac{}{\to 1}\ 1R$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ {\cdot}L \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ {\cdot}R$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A + B, \Delta \to C}\ {+}L \qquad \frac{\Pi \to A}{\Pi \to A + B}\ {+}R_1 \quad \frac{\Pi \to B}{\Pi \to A + B}\ {+}R_2$$

$$\frac{\left(\Gamma, A^n, \Delta \to B\right)_{n=0}^{\infty}}{\Gamma, A^*, \Delta \to B}\ {*}L_\omega \qquad \frac{\Gamma_1 \to A \quad \dots \quad \Gamma_n \to A}{\Gamma_1, \dots, \Gamma_n \to A^*}\ {*}R_n,\ n \geq 0$$

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- Equivalently (see Aczel 1977), the set of sequents derivable in $\mathbf{KA}_\omega$ can be defined as the minimal (w.r.t. set inclusion) set of sequents including all axioms and closed under inference rules.
- This set is the least fixpoint of the immediate derivability operator, and it is reached by its transfinite iteration up to the *closure ordinal* for $\mathbf{KA}_\omega$.

Cut Rule

- $\mathbf{KA}_\omega$ admits the cut rule, which is a generalisation of *Trans*:

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ \mathit{Cut}$$

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- However, already today we shall obtain a semantic proof of cut admissibility.

Substructurality

- Let us concentrate on the *-free fragment of $\mathbf{KA}_{\omega}$ and compare it with more well-known syntactic system.

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- The intuitionistic propositional calculus (**IPC**) in it sequent form (see, e.g., Takeuti 1975) also uses sequents of the form $\Pi \to A$, but now formulae are built using intuitionistic connective: $\&$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and constants $\bot$ (falsity) and $\top$ (truth).

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- The intuitionistic propositional calculus (**IPC**) in it sequent form (see, e.g., Takeuti 1975) also uses sequents of the form $\Pi \to A$, but now formulae are built using intuitionistic connective: $\&$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and constants $\perp$ (falsity) and $\top$ (truth).
- In usual formulations of **IPC**, antecedents (Π) are multisets or even sets of formulae. That is, order does not matter: **IPC** is commutative.

Substructurality

Axioms and rules of **IPC** are as follows:

$$\frac{}{A \to A}\ Id \qquad \frac{}{\Gamma, \bot, \Delta \to B}\ \bot L$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, \top, \Delta \to B}\ \top L \qquad \frac{}{\to \top}\ \top R$$

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$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \supset B, \Delta \to C}\ \supset L \qquad \frac{A, \Pi \to B}{\Pi \to A \supset B}\ \supset R$$

$$\frac{\Gamma, \Delta \to C}{\Gamma, A, \Delta \to C}\ Weak \qquad \frac{\Gamma, A, A, \Delta \to C}{\Gamma, A, \Delta \to C}\ Contr \qquad \frac{\Gamma, B, A, \Delta \to C}{\Gamma, A, B, \Delta \to C}\ Perm$$

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    2. $\mathbf{KA}_\omega$ is **substructural,** i.e., it does not include structural rules Weak, Contr, Perm.
    3. **IPC** does not have Kleene star. In fact, A^* trivialises to 1 in the presence of Weak. Thus, it is important for logics with Kleene star to be substructural.

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  4. We shall add implications (divisions) to Kleene algebras, yielding action algebras. Here we prove complexity results for commutative action algebras.
  5. Here we sketch complexity proofs for the non-commutative case, and also consider action logic with exponentials.

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- The key to the PSPACE decidability procedure is the possibility to make proofs in **IPC** having a polymodal bound on depth (Dyckhoff 1992).
- Such a proof could be exponential in size, but there is a non-deterministic depth-first search procedure for finding it.

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- And, by Savitch's theorem, NPSPACE is PSPACE.
- For $\mathbf{KA}_\omega$, all of this will not work, due to the presence of the ω-rule, $*L_\omega$.
- Proofs are infinite, no "out-of-the-box" finite proof search is possible.

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- One could expect $\exists Y\ \xi(Y, a)$, where Y is the infinite proof (i.e., Σ^1_1). However, the key issue here is checking that the proof is well-founded, and this is non-arithmetical.
- There is, however, a much better complexity bound for $\mathbf{KA}_\omega$, namely, Π^0_1.

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- A syntactic proof of the Π^0_1 upper bound will be, again, given later for the larger system $\mathbf{ACT}_\omega$.
- However, semantically it is obvious: given soundness and completeness (to appear), we disprove $A \preceq B$ by presenting a word w such that $w \in v(A)$ and $w \notin v(B)$.

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- Exactly one of the searches succeeds.
- If we also manage to impose a polynomial boundary on the proof depth, we get PSPACE.

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- This is an infinitary brute-force procedure.

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- Finally, each of these sequents is derived using exhaustive applications of $+L$.

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- For example, if we wish to prove $a, a, a \to b + a^*$, we first use $+R_2$ and then $*R_3$.

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- Soundness still holds, and completeness gets inherited from the standard interpretation (which is a specific case of language model).
- Further generalisation leads to the abstract class of ***-continuous Kleene algebras.**
- Basically, a *-continuous Kleene algebra is any algebraic structure $(\mathscr{A}; \preceq, \cdot, +, ^*, 0, 1)$ in which axioms and rules of $\mathbf{KA}_\omega$ are valid.

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- Another example of *-continuous KAs is the algebra of binary relations over a set, product being composition, $+$ being union, and * being the reflexive-transitive closure.
- $\mathbf{KA}_\omega$ is sound and complete w.r.t. interpretations on *-continuous KAs.

Finite KA theory

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- Axioms and rules for other operations are the same as in $\mathbf{KA}_\omega$.

Finite KA Theory

Theorem (Krob 1991, Kozen 1994)

KA *and* $\mathbf{KA}_\omega$ *have the same set of theorems.*

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- Thus a^* should be the left and the right Kleene star at the same time.
  - There exist left and right KAs.

Kozen's Completeness

- $\mathbf{KA}_\omega$ has the same set of theorems as **KA**, and they both axiomatise the inequational of each of the following classes:
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- Moreover, one can take only “left” versions of Kozen’s rules (left-handed completeness).

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- The lecturer is not aware of a concrete example at the moment, but we shall see this from complexity results further in the course.
- In this course, we are not going to prove equivalence of $\mathbf{KA}_\omega$ and $\mathbf{KA}$. Instead of that, we discuss alternative finitary formulations of the logic of Kleene algebras.

Circular and Non-Well-Founded Proofs for $\mathbf{KA}_\omega$

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- Replace the rules for * with the following ones:

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and allow infinite (non-well-founded) paths in derivations, provided the following **correctness condition** holds.

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- Correctness condition: for every infinite branch, there is a trace of A^*, which undergoes ${*L}$ infinitely many times.

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- We shall not need this result, but rather consider only the cut-free version of $\mathbf{KA}_\infty$.

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- Das & Pous (2018) prove that $\mathbf{KA}_\infty$ enjoys cut elimination.
- We shall not need this result, but rather consider only the cut-free version of $\mathbf{KA}_\infty$.
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- Under these assumptions, no correctness condition is needed!
    - Indeed, in all rules, except $*L$, the premises are strictly simpler than the conclusion.
    - Thus, each infinite branch traverses $*L$ infinitely many times.
    - Moreover, the number of A^*'s is finite, so by pigeonhole principle there will be a trace.

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$$
\dfrac{\Gamma, \Delta \to B \qquad \dfrac{\Gamma, A, \Delta \to B \qquad \dfrac{\Gamma, A, A, \Delta \to B \qquad \dfrac{\vdots}{\Gamma, A, A, A^*, \Delta \to B}}{\Gamma, A, A, A^*, \Delta \to B}\ {*L}}{\Gamma, A, A^*, \Delta \to B}\ {*L}}{\Gamma, A^*, \Delta \to B}\ {*L}
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$$

- The $*R_n$ rules is several iterations of $*R$.

- The opposite direction is based on the following lemma:

Lemma

If $\mathbf{KA}_\infty \vdash \Gamma, A^*, \Delta \to B$, *then* $\mathbf{KA}_\infty \vdash \Gamma, A^n, \Delta \to B$ *for each* n.

- This lemma is proved by induction on n.
  - We replace A^* with A^n and go upwards the proof. At the points of $*L$ we refer to the induction hypothesis.
- Now we may eagerly apply the ω-rule, and translate finitary rules to $\mathbf{KA}_\omega$.

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- Let us denote this system by $\mathbf{KA}_{\circlearrowright}$.

Circular Derivation: Example

$$
\dfrac{
  \to a^*(ba^*)^*
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{a \to a \quad (a+b)^* \to a^*(ba^*)^*}{a, (a+b)^* \to aa^*(ba^*)^*}
      \quad
      aa^*(ba^*)^* \to a^*(ba^*)^*
    }{a, (a+b)^* \to a^*(ba^*)^*}
    \qquad
    \dfrac{
      \to a^*
      \quad
      \dfrac{
        \dfrac{b \to b \quad (a+b)^* \to a^*(ba^*)^*}{b, (a+b)^* \to ba^*(ba^*)^*}
        \quad
        ba^*(ba^*)^* \to (ba^*)^*
      }{b, (a+b)^* \to (ba^*)^*}
    }{b, (a+b)^* \to a^*(ba^*)^*}
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- However, it is possible to construct a cut-free circular calculus for **KA**, but with a more involved sequential syntax.
- We shall discuss this system tomorrow.