Complexity of Reasoning in Kleene and Action Algebras

Stepan Kuznetsov

ESSLLI 2022 · Galway, Ireland · Gallimh, Éire

Lecture 2

Recap: $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

- We are studying infinitary logics for *-continuous Kleene algebras.
- If you don't remember what a *-continuous Kleene algebra means, just think about regular expressions, in their standard interpretation.
- Sequents are expressions of the form $\Pi \to B$, where B is a formula (reg. exp.) and Π is a sequence of formulae.
- Our calculi are sound and complete w.r.t. the standard interpretation, i.e., $\Pi \to B$ is derivable iff $v(\Pi) \subseteq v(B)$.

The first calculus, $\mathbf{KA}_\omega$, has the ω-rule, but proofs are required to be well-founded.

$$\frac{}{A \to A}\ Id \qquad \frac{}{\Gamma, 0, \Delta \to B}\ 0L$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, 1, \Delta \to B}\ 1L \qquad \frac{}{\to 1}\ 1R$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ {\cdot L} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ {\cdot R}$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A + B, \Delta \to C}\ {+L} \qquad \frac{\Pi \to A}{\Pi \to A + B}\ {+R_1} \quad \frac{\Pi \to B}{\Pi \to A + B}\ {+R_2}$$

$$\frac{\left(\Gamma, A^n, \Delta \to B\right)_{n=0}^{\infty}}{\Gamma, A^*, \Delta \to B}\ {*L_\omega} \qquad \frac{\Gamma_1 \to A \quad \dots \quad \Gamma_n \to A}{\Gamma_1, \dots, \Gamma_n \to A^*}\ {*R_n},\ n \geq 0$$

Recap: $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

- In contrast, proofs in $\mathbf{KA}_\infty$ has only binary branching, but are allowed to be non-well-founded.

Recap: $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

- In contrast, proofs in $\mathbf{KA}_\infty$ has only binary branching, but are allowed to be non-well-founded.
- The rules for * in $\mathbf{KA}_\infty$ are as follows:

$$\frac{\Gamma, \Delta \to B \quad \Gamma, A, A^*, \Delta \to B}{\Gamma, A^*, \Delta \to B} \ast L \qquad \frac{}{\to A^*} \ast R_0 \qquad \frac{\Gamma \to A \quad \Delta \to A^*}{\Gamma, \Delta \to A^*} \ast R$$

In $\ast R$ we require Γ to be non-empty.

Recap: $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

- In contrast, proofs in $\mathbf{KA}_\infty$ has only binary branching, but are allowed to be non-well-founded.
- The rules for * in $\mathbf{KA}_\infty$ are as follows:

$$\frac{\Gamma, \Delta \to B \quad \Gamma, A, A^*, \Delta \to B}{\Gamma, A^*, \Delta \to B}\ {*L} \qquad \frac{}{\to A^*}\ {*R_0} \qquad \frac{\Gamma \to A \quad \Delta \to A^*}{\Gamma, \Delta \to A^*}\ {*R}$$

  In $*R$ we require Γ to be non-empty.
- In the absence of *Cut*, no correctness condition on infinite paths is necessary.

Equivalence Between $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

Theorem (Das & Pous 2018)

$\mathbf{KA}_\omega$ *and* $\mathbf{KA}_\infty$ *(its cut-free version) derive the same set of theorems.*

Equivalence Between $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

Theorem (Das & Pous 2018)

$\mathbf{KA}_\omega$ *and* $\mathbf{KA}_\infty$ *(its cut-free version) derive the same set of theorems.*

- The direction from $\mathbf{KA}_\omega$ to $\mathbf{KA}_\infty$ is easy: we just model the ω-rule by an infinite branch:

$$
\dfrac{\Gamma, \Delta \to B \qquad \dfrac{\Gamma, A, \Delta \to B \qquad \dfrac{\Gamma, A, A, \Delta \to B \qquad \dfrac{\vdots}{\Gamma, A, A, A^*, \Delta \to B}}{\Gamma, A, A, A^*, \Delta \to B}\,{*L}}{\Gamma, A, A^*, \Delta \to B}\,{*L}}{\Gamma, A^*, \Delta \to B}\,{*L}
$$

Equivalence Between $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

Theorem (Das & Pous 2018)

$\mathbf{KA}_\omega$ *and* $\mathbf{KA}_\infty$ *(its cut-free version) derive the same set of theorems.*

- The direction from $\mathbf{KA}_\omega$ to $\mathbf{KA}_\infty$ is easy: we just model the ω-rule by an infinite branch:

$$
\dfrac{\Gamma, \Delta \to B \qquad \dfrac{\Gamma, A, \Delta \to B \qquad \dfrac{\Gamma, A, A, \Delta \to B \qquad \dfrac{\vdots}{\Gamma, A, A, A^*, \Delta \to B}}{\Gamma, A, A, A^*, \Delta \to B}\,{*L}}{\Gamma, A, A^*, \Delta \to B}\,{*L}}{\Gamma, A^*, \Delta \to B}\,{*L}
$$

- The $*R_n$ rules is several iterations of $*R$.

Equivalence Between $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

- The opposite direction is based on the following lemma:

Lemma

If $\mathbf{KA}_\infty \vdash \Gamma, A^*, \Delta \to B$, *then* $\mathbf{KA}_\infty \vdash \Gamma, A^n, \Delta \to B$ *for each* n.

- This lemma is proved by induction on n.
    - We replace A^* with A^n and go upwards the proof. At the points of $*L$ we refer to the induction hypothesis.
- Now we may eagerly apply the ω-rule, and translate finitary rules to $\mathbf{KA}_\omega$.

- As we have shown, infinite proofs correspond to the ω-rule.

Circular Proofs for KA_ω

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.

Circular Proofs for KA_ω

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.
- In a circular derivation, each infinite path at some point reaches **the same** sequent as already seen below.

Circular Proofs for KA_ω

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.
- In a circular derivation, each infinite path at some point reaches **the same** sequent as already seen below.
- At this point, there is no need to develop the derivation further, but one can rather refer backwards via a **backlink.**

Circular Proofs for KA_ω

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.
- In a circular derivation, each infinite path at some point reaches **the same** sequent as already seen below.
- At this point, there is no need to develop the derivation further, but one can rather refer backwards via a **backlink.**
- Alternatively, circular proofs can be defined as **regular** non-well-founded proofs, i.e., proofs with a finite number of non-isomorphic subtrees.

Circular Proofs for KA_ω

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.
- In a circular derivation, each infinite path at some point reaches **the same** sequent as already seen below.
- At this point, there is no need to develop the derivation further, but one can rather refer backwards via a **backlink.**
- Alternatively, circular proofs can be defined as **regular** non-well-founded proofs, i.e., proofs with a finite number of non-isomorphic subtrees.
- We consider circular proofs with cuts, obeying the correctness condition.

Circular Proofs for $\mathbf{KA}_\omega$

- As we have shown, infinite proofs correspond to the ω-rule.
- In contrast, **induction** (fixpoint rules) correspond to **circular** proofs.
- In a circular derivation, each infinite path at some point reaches **the same** sequent as already seen below.
- At this point, there is no need to develop the derivation further, but one can rather refer backwards via a **backlink.**
- Alternatively, circular proofs can be defined as **regular** non-well-founded proofs, i.e., proofs with a finite number of non-isomorphic subtrees.
- We consider circular proofs with cuts, obeying the correctness condition.
- Let us denote this system by $\mathbf{KA}_{\circlearrowright}$.

Circular Derivation: Example

$$
\cfrac{\to a^*(ba^*)^* \qquad \cfrac{\cfrac{\cfrac{a \to a \quad (a+b)^* \to a^*(ba^*)^*}{a, (a+b)^* \to aa^*(ba^*)^*} \qquad aa^*(ba^*)^* \to a^*(ba^*)^*}{a, (a+b)^* \to a^*(ba^*)^*} \qquad \cfrac{\to a^* \qquad \cfrac{\cfrac{b \to b \quad (a+b)^* \to a^*(ba^*)^*}{b, (a+b)^* \to ba^*(ba^*)^*} \qquad ba^*(ba^*)^* \to (ba^*)^*}{b, (a+b)^* \to (ba^*)^*}}{b, (a+b)^* \to a^*(ba^*)^*}}{a+b, (a+b)^* \to a^*(ba^*)^*}}{(a+b)^* \to a^*(ba^*)^*}\ {*L}
$$

$\mathbf{KA}_{\circlearrowright}$ and KA

- $\mathbf{KA}_{\circlearrowright}$ derives the same set of theorems as Kozen's **KA** does (we are not going to prove this).

$\mathbf{KA}_{\circlearrowright}$ and KA

- $\mathbf{KA}_{\circlearrowright}$ derives the same set of theorems as Kozen's **KA** does (we are not going to prove this).
- Unfortunately, cut elimination for $\mathbf{KA}_{\circlearrowright}$ again fails: examples include $(a+b)^* \to a^*(ba^*)^*$ (see above) and even $aa^* \to a^*a$.

$\mathbf{KA}_{\circlearrowright}$ and KA

- $\mathbf{KA}_{\circlearrowright}$ derives the same set of theorems as Kozen's **KA** does (we are not going to prove this).
- Unfortunately, cut elimination for $\mathbf{KA}_{\circlearrowright}$ again fails: examples include $(a+b)^* \to a^*(ba^*)^*$ (see above) and even $aa^* \to a^*a$.
- A sequent may have a circular proof with cuts, but unravelling it to an infinite proof and eliminating cuts yields a non-circular proof.

$\mathbf{KA}_{\circlearrowright}$ and KA

- $\mathbf{KA}_{\circlearrowright}$ derives the same set of theorems as Kozen's **KA** does (we are not going to prove this).
- Unfortunately, cut elimination for $\mathbf{KA}_{\circlearrowright}$ again fails: examples include $(a+b)^* \to a^*(ba^*)^*$ (see above) and even $aa^* \to a^*a$.
- A sequent may have a circular proof with cuts, but unravelling it to an infinite proof and eliminating cuts yields a non-circular proof.
- Furthermore, it is unclear how to reduce $\mathbf{KA}_{\infty}$ to $\mathbf{KA}_{\circlearrowright}$ (even with cuts).

$\mathbf{KA}_{\circlearrowright}$ and KA

- $\mathbf{KA}_{\circlearrowright}$ derives the same set of theorems as Kozen's **KA** does (we are not going to prove this).
- Unfortunately, cut elimination for $\mathbf{KA}_{\circlearrowright}$ again fails: examples include $(a+b)^* \to a^*(ba^*)^*$ (see above) and even $aa^* \to a^*a$.
- A sequent may have a circular proof with cuts, but unravelling it to an infinite proof and eliminating cuts yields a non-circular proof.
- Furthermore, it is unclear how to reduce $\mathbf{KA}_\infty$ to $\mathbf{KA}_{\circlearrowright}$ (even with cuts).
- However, it is possible to construct a cut-free circular calculus for **KA**, but with a more involved sequential syntax.

Hypersequential Calculus HKA_∞

- In a hypersequent $\Pi \to \mathcal{X}$ (Das & Pous 2017), the left-hand side Π is a sequence of formulae, and the right-hand side $\mathcal{X} = \Delta_1 \mid \ldots \mid \Delta_m$ is a multiset of sequences of formulae.

Hypersequential Calculus HKA_∞

- In a hypersequent $\Pi \to \mathscr{X}$ (Das & Pous 2017), the left-hand side Π is a sequence of formulae, and the right-hand side $\mathscr{X} = \Delta_1 \mid \dots \mid \Delta_m$ is a multiset of sequences of formulae.
- $\mid$ means $+$ (like , means $\cdot$)

Hypersequential Calculus HKA_∞

- In a hypersequent $\Pi \to \mathcal{X}$ (Das & Pous 2017), the left-hand side Π is a sequence of formulae, and the right-hand side $\mathcal{X} = \Delta_1 \mid \ldots \mid \Delta_m$ is a multiset of sequences of formulae.
- $\mid$ means $+$ (like , means $\cdot$)
- The left rules are the same, e.g.:

$$\frac{\Gamma_1, \Gamma_2 \to \mathcal{X} \quad \Gamma_1, A, A^*, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A^*, \Gamma_2 \to \mathcal{X}} \; {*L} \qquad \frac{\Gamma_1, A, B, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A \cdot B, \Gamma_2 \to \mathcal{X}} \; {\cdot L} \qquad \ldots$$

Hypersequential Calculus $\mathbf{HKA}_\infty$

- In a hypersequent $\Pi \to \mathcal{X}$ (Das & Pous 2017), the left-hand side Π is a sequence of formulae, and the right-hand side $\mathcal{X} = \Delta_1 \mid \ldots \mid \Delta_m$ is a multiset of sequences of formulae.
- $\mid$ means $+$ (like , means $\cdot$)
- The left rules are the same, e.g.:

$$\frac{\Gamma_1, \Gamma_2 \to \mathcal{X} \quad \Gamma_1, A, A^*, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A^*, \Gamma_2 \to \mathcal{X}} *L \qquad \frac{\Gamma_1, A, B, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A \cdot B, \Gamma_2 \to \mathcal{X}} \cdot L \qquad \ldots$$

- The right-hand side enjoys structural rules:

$$\frac{\Pi \to \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} WR \qquad \frac{\Pi \to \Delta \mid \Delta \mid \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} CR$$

Hypersequential Calculus HKA_∞

- The atomic case is handled as follows:

$$\frac{}{\varepsilon \to \varepsilon}\ Ax \qquad \frac{\Gamma \to \Delta_1 \mid \ldots \mid \Delta_m}{a, \Gamma \to a, \Delta_1 \mid \ldots \mid a, \Delta_m}\ K$$

- The atomic case is handled as follows:

$$\frac{}{\varepsilon \to \varepsilon}\ Ax \qquad \frac{\Gamma \to \Delta_1 \mid \dots \mid \Delta_m}{a, \Gamma \to a, \Delta_1 \mid \dots \mid a, \Delta_m}\ K$$

- Finally, the right rules are as follows:

$$\frac{\Pi \to \Delta_1, \Delta_2 \mid \Delta_1, A, A^*, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A^*, \Delta_2 \mid \mathcal{X}}\ {*R} \qquad \frac{\Pi \to \Delta_1, A, B, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A \cdot B, \Delta_2 \mid \mathcal{X}}\ {\cdot R}$$

$$\frac{\Pi \to \Delta_1, A, \Delta_2 \mid \Delta_1, B, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A + B, \Delta_2 \mid \mathcal{X}}\ {+R} \qquad \frac{\Pi \to \Delta_1, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, 1, \Delta_2 \mid \mathcal{X}}\ 1R$$

Hypersequential Calculus $\mathbf{HKA}_\infty$

- The atomic case is handled as follows:

$$\frac{}{\varepsilon \rightarrow \varepsilon}\ Ax \qquad \frac{\Gamma \rightarrow \Delta_1 \mid \ldots \mid \Delta_m}{a, \Gamma \rightarrow a, \Delta_1 \mid \ldots \mid a, \Delta_m}\ K$$

- Finally, the right rules are as follows:

$$\frac{\Pi \rightarrow \Delta_1, \Delta_2 \mid \Delta_1, A, A^*, \Delta_2 \mid \mathcal{X}}{\Pi \rightarrow \Delta_1, A^*, \Delta_2 \mid \mathcal{X}}\ {*R} \qquad \frac{\Pi \rightarrow \Delta_1, A, B, \Delta_2 \mid \mathcal{X}}{\Pi \rightarrow \Delta_1, A \cdot B, \Delta_2 \mid \mathcal{X}}\ {\cdot R}$$

$$\frac{\Pi \rightarrow \Delta_1, A, \Delta_2 \mid \Delta_1, B, \Delta_2 \mid \mathcal{X}}{\Pi \rightarrow \Delta_1, A + B, \Delta_2 \mid \mathcal{X}}\ {+R} \qquad \frac{\Pi \rightarrow \Delta_1, \Delta_2 \mid \mathcal{X}}{\Pi \rightarrow \Delta_1, 1, \Delta_2 \mid \mathcal{X}}\ 1R$$

- $\mathbf{HKA}_\omega$ allows non-well-founded proofs, with the following **fairness condition:** each infinite path should traverse $*L$ infinitely many times.

Hypersequential Calculus $\mathbf{HKA}_\infty$

- The atomic case is handled as follows:

$$\frac{}{\varepsilon \to \varepsilon}\ Ax \qquad \frac{\Gamma \to \Delta_1 \mid \ldots \mid \Delta_m}{a, \Gamma \to a, \Delta_1 \mid \ldots \mid a, \Delta_m}\ K$$

- Finally, the right rules are as follows:

$$\frac{\Pi \to \Delta_1, \Delta_2 \mid \Delta_1, A, A^*, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A^*, \Delta_2 \mid \mathcal{X}}\ {*R} \qquad \frac{\Pi \to \Delta_1, A, B, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A \cdot B, \Delta_2 \mid \mathcal{X}}\ {\cdot R}$$

$$\frac{\Pi \to \Delta_1, A, \Delta_2 \mid \Delta_1, B, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, A + B, \Delta_2 \mid \mathcal{X}}\ {+R} \qquad \frac{\Pi \to \Delta_1, \Delta_2 \mid \mathcal{X}}{\Pi \to \Delta_1, 1, \Delta_2 \mid \mathcal{X}}\ 1R$$

- $\mathbf{HKA}_\omega$ allows non-well-founded proofs, with the following **fairness condition:** each infinite path should traverse $*L$ infinitely many times.
- There is no cut rule in $\mathbf{HKA}_\infty$ (it is even hard to formulate).

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

Theorem (Das & Pous 2017)

Sequents of the form $\Pi \to B$ are equiderivable in $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

Theorem (Das & Pous 2017)

Sequents of the form $\Pi \rightarrow B$ *are equiderivable in* $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, *and* $\mathbf{KA}_\omega$.

- For the direction from $\mathbf{HKA}_\infty$ to $\mathbf{KA}_\omega$, we use the same argument as for $\mathbf{KA}_\infty$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

Theorem (Das & Pous 2017)

Sequents of the form $\Pi \to B$ are equiderivable in $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$.

- For the direction from $\mathbf{HKA}_\infty$ to $\mathbf{KA}_\omega$, we use the same argument as for $\mathbf{KA}_\infty$.
- Namely, we establish the following key lemma: if $\Gamma, A^*, \Delta \to \mathcal{X}$ is derivable in $\mathbf{HKA}_\infty$, then so is $\Gamma, A^n, \Delta \to \mathcal{X}$ for each n. (Induction on n.)

Equivalence of HKA_∞, KA_∞, and KA_ω

- Next, we define the ***-height** of a reg. exp. as the maximal nesting depth of *.

Equivalence of HKA_∞, KA_∞, and KA_ω

- Next, we define the ***-height** of a reg. exp. as the maximal nesting depth of *.
- For a sequence Γ, its **weighted *-height,** $\mathrm{wh}_*(\Gamma)$, is the multiset of *-heights of the elements of Γ.

Equivalence of HKA_∞, KA_∞, and KA_ω

- Next, we define the ***-height** of a reg. exp. as the maximal nesting depth of *.
- For a sequence Γ, its **weighted *-height,** $\mathrm{wh}_*(\Gamma)$, is the multiset of *-heights of the elements of Γ.
- These wh_*'s have the following well-founded partial order: multiset M (of natural numbers) is greater than multiset N, if N can be obtained from M by replacing some elements by multisets of smaller elements (Dershowitz & Manna 1979).

Equivalence of HKA_∞, KA_∞, and KA_ω

- Next, we define the ***-height** of a reg. exp. as the maximal nesting depth of *.
- For a sequence Γ, its **weighted *-height,** $\mathrm{wh}_*(\Gamma)$, is the multiset of *-heights of the elements of Γ.
- These wh_*'s have the following well-founded partial order: multiset M (of natural numbers) is greater than multiset N, if N can be obtained from M by replacing some elements by multisets of smaller elements (Dershowitz & Manna 1979).
    - For example, $\{3, 3, 4, 0\} \gg \{2, 2, 2, 3, 4, 0\} \gg \{2, 2, 2, 3, 2, 2, 1, 0\}$.

Equivalence of HKA_∞, KA_∞, and KA_ω

- Next, we define the ***-height** of a reg. exp. as the maximal nesting depth of *.
- For a sequence Γ, its **weighted *-height,** $\mathrm{wh}_*(\Gamma)$, is the multiset of *-heights of the elements of Γ.
- These wh_*'s have the following well-founded partial order: multiset M (of natural numbers) is greater than multiset N, if N can be obtained from M by replacing some elements by multisets of smaller elements (Dershowitz & Manna 1979).
  - For example, $\{3, 3, 4, 0\} \gg \{2, 2, 2, 3, 4, 0\} \gg \{2, 2, 2, 3, 2, 2, 1, 0\}$.
  - This is exactly what happens in our lemma, when A^* is replaced by $A^n = A, \dots, A$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.
- Rules of π_0 can be modelled in $\mathbf{KA}_\omega$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.
- Rules of π_0 can be modelled in $\mathbf{KA}_\omega$.
- For each cut-off point, we have $\Gamma, A^*, \Delta \to \mathscr{Y}$, where $\mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.
- Rules of π_0 can be modelled in $\mathbf{KA}_\omega$.
- For each cut-off point, we have $\Gamma, A^*, \Delta \to \mathscr{Y}$, where $\mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$.
- By our lemma, we get derivability of $\Gamma, A^n, \Delta \to \mathscr{X}$ for each n in $\mathbf{HKA}_\infty$, and since $\mathrm{wh}_*(\Gamma, A^n, \Delta) \ll \mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$, also in $\mathbf{KA}_\omega$ by induction hypothesis.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.
- Rules of π_0 can be modelled in $\mathbf{KA}_\omega$.
- For each cut-off point, we have $\Gamma, A^*, \Delta \to \mathscr{Y}$, where $\mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$.
- By our lemma, we get derivability of $\Gamma, A^n, \Delta \to \mathscr{X}$ for each n in $\mathbf{HKA}_\infty$, and since $\mathrm{wh}_*(\Gamma, A^n, \Delta) \ll \mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$, also in $\mathbf{KA}_\omega$ by induction hypothesis.
- Now we finalise our proof by applying $*L_\omega$.

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

- Now let π be a proof of $\Sigma \to \mathscr{X}$ in $\mathbf{HKA}_\infty$, and proceed by induction on $\mathrm{wh}_*(\Sigma)$.
- We cut off π at lowermost applications of $*L$, and this results in a finite proof π_0.
- Rules of π_0 can be modelled in $\mathbf{KA}_\omega$.
- For each cut-off point, we have $\Gamma, A^*, \Delta \to \mathscr{Y}$, where $\mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$.
- By our lemma, we get derivability of $\Gamma, A^n, \Delta \to \mathscr{X}$ for each n in $\mathbf{HKA}_\infty$, and since $\mathrm{wh}_*(\Gamma, A^n, \Delta) \ll \mathrm{wh}_*(\Gamma, A^*, \Delta) \underline{\ll} \mathrm{wh}_*(\Sigma)$, also in $\mathbf{KA}_\omega$ by induction hypothesis.
- Now we finalise our proof by applying $*L_\omega$.
- Notice that induction here is more involved than for $\mathbf{KA}_\infty$.

Hypersequents for Decidability

- For the opposite direction, a stronger result holds.

Hypersequents for Decidability

- For the opposite direction, a stronger result holds.

Theorem (Das & Pous 2017)

If a sequent is derivable in $\mathbf{KA}_\omega$*, then it can be derived in* $\mathbf{HKA}_\infty$ *by a circular derivation.*

Hypersequents for Decidability

- For the opposite direction, a stronger result holds.

Theorem (Das & Pous 2017)

If a sequent is derivable in $\mathbf{KA}_\omega$*, then it can be derived in* $\mathbf{HKA}_\infty$ *by a circular derivation.*

- Therefore, $\mathbf{KA}_\omega$ is enumerable, and by Post's theorem it is decidable.

Hypersequents for Decidability

- For the opposite direction, a stronger result holds.

Theorem (Das & Pous 2017)

If a sequent is derivable in $\mathbf{KA}_\omega$*, then it can be derived in* $\mathbf{HKA}_\infty$ *by a circular derivation.*

- Therefore, $\mathbf{KA}_\omega$ is enumerable, and by Post's theorem it is decidable.
- A more accurate analysis of circular proofs yields the PSPACE upper bound.

Hypersequents for Decidability

- For the opposite direction, a stronger result holds.

Theorem (Das & Pous 2017)

If a sequent is derivable in $\mathbf{KA}_\omega$, then it can be derived in $\mathbf{HKA}_\infty$ by a circular derivation.

- Therefore, $\mathbf{KA}_\omega$ is enumerable, and by Post's theorem it is decidable.
- A more accurate analysis of circular proofs yields the PSPACE upper bound.
- Circular proofs in $\mathbf{HKA}_\infty$ can be translated to $\mathbf{KA}_\circlearrowright$ proofs (but with cuts), and therefore to $\mathbf{KA}$ proofs. Thus, $\mathbf{KA} = \mathbf{KA}_\omega$. (We shall not go into this.)

Leftmost Proofs

- We are going to construct proofs of a specific form, which will force them to become circular

Leftmost Proofs

- We are going to construct proofs of a specific form, which will force them to become circular
- A proof in $\mathbf{HKA}_\infty$ is called a **leftmost** one, if for each logical rule its active formula is the first formula in a sequence.

Leftmost Proofs

- We are going to construct proofs of a specific form, which will force them to become circular
- A proof in $\mathbf{HKA}_\infty$ is called a **leftmost** one, if for each logical rule its active formula is the first formula in a sequence.
- This holds for both left and right rules.

Leftmost Proofs

- We are going to construct proofs of a specific form, which will force them to become circular
- A proof in $\mathbf{HKA}_\infty$ is called a **leftmost** one, if for each logical rule its active formula is the first formula in a sequence.
- This holds for both left and right rules.
- Restricting ourselves to leftmost proofs means dropping left contexts in sequent rules.

Leftmost Proofs

- We are going to construct proofs of a specific form, which will force them to become circular
- A proof in $\mathbf{HKA}_\infty$ is called a **leftmost** one, if for each logical rule its active formula is the first formula in a sequence.
- This holds for both left and right rules.
- Restricting ourselves to leftmost proofs means dropping left contexts in sequent rules.
- Now we have the following left logical rules:

$$\frac{\Gamma \to \mathcal{X} \qquad A, A^*, \Gamma \to \mathcal{X}}{A^*, \Gamma \to \mathcal{X}} \ {*L} \qquad \frac{A, B, \Gamma \to \mathcal{X}}{A \cdot B, \Gamma \to \mathcal{X}} \ {\cdot L}$$

$$\frac{A, \Gamma \to \mathcal{X} \qquad B, \Gamma \to \mathcal{X}}{A + B, \Gamma \to \mathcal{X}} \ {+L} \qquad \frac{\Gamma \to \mathcal{X}}{1, \Gamma \to \mathcal{X}} \ 1L \qquad \frac{}{0, \Gamma \to \mathcal{X}} \ 0L$$

Leftmost Proofs

- ... and the following right rules:

$$\frac{\Pi \to \Delta \mid A, A^*, \Delta \mid \mathscr{X}}{\Pi \to A^*, \Delta \mid \mathscr{X}} \ *R \qquad \frac{\Pi \to A, B, \Delta \mid \mathscr{X}}{\Pi \to A \cdot B, \Delta \mid \mathscr{X}} \ \cdot R$$

$$\frac{\Pi \to A, \Delta \mid B, \Delta \mid \mathscr{X}}{\Pi \to A + B, \Delta \mid \mathscr{X}} \ +R \qquad \frac{\Pi \to \Delta \mid \mathscr{X}}{\Pi \to 1, \Delta \mid \mathscr{X}} \ 1R$$

Leftmost Proofs

- ... and the following right rules:

$$\frac{\Pi \to \Delta \mid A, A^*, \Delta \mid \mathscr{X}}{\Pi \to A^*, \Delta \mid \mathscr{X}} \; *R \qquad \frac{\Pi \to A, B, \Delta \mid \mathscr{X}}{\Pi \to A \cdot B, \Delta \mid \mathscr{X}} \; \cdot R$$

$$\frac{\Pi \to A, \Delta \mid B, \Delta \mid \mathscr{X}}{\Pi \to A + B, \Delta \mid \mathscr{X}} \; +R \qquad \frac{\Pi \to \Delta \mid \mathscr{X}}{\Pi \to 1, \Delta \mid \mathscr{X}} \; 1R$$

- Other rules are the same:

$$\frac{}{\varepsilon \to \varepsilon} \; Ax \qquad \frac{\Gamma \to \Delta_1 \mid \ldots \mid \Delta_m}{a, \Gamma \to a, \Delta_1 \mid \ldots \mid a, \Delta_m} \; K$$

$$\frac{\Pi \to \mathscr{X}}{\Pi \to \Delta \mid \mathscr{X}} \; WR \qquad \frac{\Pi \to \Delta \mid \Delta \mid \mathscr{X}}{\Pi \to \Delta \mid \mathscr{X}} \; CR$$

Some Speculations

- The usage of an ω-rule is arguably not really a mechanism of "proof" in the desirable sense.

Some Speculations

- The usage of an ω-rule is arguably not really a mechanism of "proof" in the desirable sense.
- To compare, let us consider the arithmetical statement of Fermat's Last Theorem:

$$\forall x, y, z, n\,(x, y, z > 0 \mathbin{\&} n > 2 \supset x^n + y^n \neq z^n).$$

Some Speculations

- The usage of an ω-rule is arguably not really a mechanism of "proof" in the desirable sense.
- To compare, let us consider the arithmetical statement of Fermat's Last Theorem:

$$\forall x, y, z, n\, (x, y, z > 0 \,\&\, n > 2 \supset x^n + y^n \neq z^n).$$

- Using the arithmetical ω-rule, we can reduce it to proving the statement for each **concrete** tuple (x, y, z, n), which is an obvious check.

Some Speculations

- The usage of an ω-rule is arguably not really a mechanism of "proof" in the desirable sense.
- To compare, let us consider the arithmetical statement of Fermat's Last Theorem:

$$\forall x, y, z, n\,(x, y, z > 0 \mathbin{\&} n > 2 \supset x^n + y^n \neq z^n).$$

- Using the arithmetical ω-rule, we can reduce it to proving the statement for each **concrete** tuple (x, y, z, n), which is an obvious check.
- This, of course, does not constitute a mathematical proof of the theorem in a reasonable sense.

Some Speculations

- The same happens in $\mathbf{KA}_\omega$. With the ω-rule, as we say yesterday, one proves $A \to B$, i.e., $v(A) \subseteq v(B)$, just by brute-force over all words in $v(A)$.

Some Speculations

- The same happens in $\mathbf{KA}_\omega$. With the ω-rule, as we say yesterday, one proves $A \to B$, i.e., $v(A) \subseteq v(B)$, just by brute-force over all words in $v(A)$.
- Non-well-founded proofs in general ($\mathbf{KA}_\infty$) do not resolve this issue, since they are capable of modelling the ω-rule directly.

Some Speculations

- The same happens in $\mathbf{KA}_\omega$. With the ω-rule, as we say yesterday, one proves $A \to B$, i.e., $v(A) \subseteq v(B)$, just by brute-force over all words in $v(A)$.
- Non-well-founded proofs in general ($\mathbf{KA}_\infty$) do not resolve this issue, since they are capable of modelling the ω-rule directly.
- This simulation is performed by eager application of $*L$ to one and the same instance of A^*.

Some Speculations

- The same happens in $\mathbf{KA}_\omega$. With the ω-rule, as we say yesterday, one proves $A \to B$, i.e., $v(A) \subseteq v(B)$, just by brute-force over all words in $v(A)$.
- Non-well-founded proofs in general ($\mathbf{KA}_\infty$) do not resolve this issue, since they are capable of modelling the ω-rule directly.
- This simulation is performed by eager application of $*L$ to one and the same instance of A^*.
- Leftmost proofs disallow this:

$$\frac{\Gamma \to \mathcal{X} \quad A, A^*, \Gamma \to \mathcal{X}}{A^*, \Gamma \to \mathcal{X}} *L$$

since now we are forced to decompose A before we are allowed to do the next iteration of A^*.

Some Speculations

- The sophisticated hypersequent structure on the right is needed to postpone the choice ($+R_{1,2}$ or $*R$) upwards the proof.

Some Speculations

- The sophisticated hypersequent structure on the right is needed to postpone the choice ($+R_{1,2}$ or $*R$) upwards the proof.
- An example is $a, a^* \to a^* \cdot a$.

Some Speculations

- The sophisticated hypersequent structure on the right is needed to postpone the choice ($+R_{1,2}$ or $*R$) upwards the proof.
- An example is $a, a^* \to a^* \cdot a$.
- As the proof should be leftmost, we are not allowed to decompose a^* using $*L$ on the left.

Some Speculations

- The sophisticated hypersequent structure on the right is needed to postpone the choice ($+R_{1,2}$ or $*R$) upwards the proof.
- An example is $a, a^* \to a^* \cdot a$.
- As the proof should be leftmost, we are not allowed to decompose a^* using $*L$ on the left.
- We think of decomposing a^* on the right, but this would yield a choice: $a^*a \equiv a + (aa^*a)$, and this choice should actually be made **after** applying $*L$.

Some Speculations

- The sophisticated hypersequent structure on the right is needed to postpone the choice ($+R_{1,2}$ or $*R$) upwards the proof.
- An example is $a, a^* \to a^* \cdot a$.
- As the proof should be leftmost, we are not allowed to decompose a^* using $*L$ on the left.
- We think of decomposing a^* on the right, but this would yield a choice: $a^*a \equiv a + (aa^*a)$, and this choice should actually be made **after** applying $*L$.
    - On the left branch of $*L$ we choose $a \to a$, and on the right one we choose $aaa^* \to aa^*a$.

Some Speculations

This is handled by the hypersequential version of the $*R$ rule, which keeps both choices in the sequent, and postpones the decision to applications of WR above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varepsilon \to \varepsilon}{\varepsilon \to \varepsilon \mid a^*, a}\,WR \qquad \dfrac{a, a^* \to a^*, a}{a, a^* \to \varepsilon \mid a^*, a}\,WR}{a^* \to \varepsilon \mid a^*, a}\,{*L}}{a, a^* \to a \mid a, a^*, a}\,K}{a, a^* \to a^*, a}\,{*R}}{a, a^* \to a^* \cdot a}\,{\cdot R}$$

Now we are going to prove the following two statements, which give the necessary result:

1. If $\nu(A) \subseteq \nu(B)$, then $A \to B$ has a leftmost proof in $\mathbf{HKA}_\infty$.
2. Each leftmost proof can be transformed into a circular one.

Productivity Lemma

Lemma

If there is a subderivation

$$\dfrac{\dfrac{\dfrac{\Gamma \to A^*, \Delta \mid \mathscr{X}}{\vdots\ \pi}}{\Gamma \to \Delta \mid A, A^*, \Delta \mid \mathscr{Y}}}{\Gamma \to A^*, \Delta \mid \mathscr{Y}} *R$$

where π consists only of right logical rules, and the sequent $\Gamma \to A^*, \Delta \mid \mathscr{Y}$ is valid in the standard interpretation, then so is $\Gamma \to \mathscr{X}$ (i.e., A^*, Δ there is redundant).

Productivity Lemma

- All right rules are invertible, so $\Gamma \to A^*, \Delta \mid \mathcal{X}$ is also valid.
- It is sufficient to prove $A^*, \Delta \to \mathcal{X}$ in $\mathbf{HKA}_\infty$.
- By soundness, this means $v(A^*, \Delta) \subseteq v(\mathcal{X})$, and thus $v(\Gamma) \subseteq v(A^*, \Delta) + v(\mathcal{X}) = v(\mathcal{X})$.
- Proving $A^*, \Delta \to \mathcal{X}$ is performed by using the left rules corresponding to the rules from π:

$$\dfrac{\dfrac{\vdots\ \pi'}{\Delta \to \mathcal{X}} \qquad \dfrac{\vdots\ \pi''}{A, A^*, \Delta \to \mathcal{X}}}{A^*, \Delta \to \mathcal{X}}\ {*L}$$

Productivity Lemma

- In π' and π'', antecedents will be the corresponding sequences from the succedents in π. For example, if π had a step of the form

$$\frac{\Gamma \to B, \Delta' \mid C, \Delta' \mid \mathscr{Z}}{\Gamma \to B + C, \Delta' \mid \mathscr{Z}} \ +R$$

where $B + C, \Delta'$ was the antecedent in π' or π'', then we proceed as

$$\frac{B, \Delta' \to \mathscr{X} \quad C, \Delta' \to \mathscr{X}}{B + C, \Delta' \to \mathscr{X}} \ +L$$

Productivity Lemma

- In π' and π'', antecedents will be the corresponding sequences from the succedents in π. For example, if π had a step of the form

$$\frac{\Gamma \to B, \Delta' \mid C, \Delta' \mid \mathscr{Z}}{\Gamma \to B + C, \Delta' \mid \mathscr{Z}} \; +R$$

  where $B + C, \Delta'$ was the antecedent in π' or π'', then we proceed as

$$\frac{B, \Delta' \to \mathscr{X} \quad C, \Delta' \to \mathscr{X}}{B + C, \Delta' \to \mathscr{X}} \; +L$$

- Thus, a linear derivation π with rich succedents is decomposed into branching derivations π' and π'' with simple antecedents.

Productivity Lemma

- In π' and π'', antecedents will be the corresponding sequences from the succedents in π. For example, if π had a step of the form

$$\frac{\Gamma \to B, \Delta' \mid C, \Delta' \mid \mathscr{Z}}{\Gamma \to B + C, \Delta' \mid \mathscr{Z}} \; +R$$

  where $B + C, \Delta'$ was the antecedent in π' or π'', then we proceed as

$$\frac{B, \Delta' \to \mathscr{X} \qquad C, \Delta' \to \mathscr{X}}{B + C, \Delta' \to \mathscr{X}} \; +L$$

- Thus, a linear derivation π with rich succedents is decomposed into branching derivations π' and π'' with simple antecedents.
- In π'', on top we shall have either $\Phi \to \mathscr{X}$ where $\Phi \in \mathscr{X}$ (derivable by WR), or $A^*, \Delta \to \mathscr{X}$, which is derived by backlink to the goal.

Productivity Lemma

- Fairness is maintained by the lowermost $*L$.

Productivity Lemma

- Fairness is maintained by the lowermost $*L$.
- For π', we do not have this guarantee of fairness.

Productivity Lemma

- Fairness is maintained by the lowermost $*L$.
- For π', we do not have this guarantee of fairness.
- However, we claim that if a leaf of π' is of the form $A^*, \Delta \to \mathscr{X}$, then the corresponding path should have traversed $*L$.

Productivity Lemma

- Fairness is maintained by the lowermost $*L$.
- For π', we do not have this guarantee of fairness.
- However, we claim that if a leaf of π' is of the form $A^*, \Delta \to \mathscr{X}$, then the corresponding path should have traversed $*L$.
- Indeed, all rules in π' are left logical rules, and such rules other than $*L$ decrease the size of the antecedent. On the other hand, A^*, Δ is bigger than Δ.