Complexity of Reasoning in Kleene and Action Algebras

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Lecture 4

Divide and Conquer

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- The logic of *-continuous action lattices is infinitary action logic $\mathbf{ACT}_\omega$.

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- An action lattice is *-continuous, if $a^* = \sup_{\preceq}\{a^n \mid n \geq 0\}$.

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- In particular, this holds for finite action lattices.

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- Later on, Lambek (1958) introduced the Lambek calculus for defining natural language syntax.
- The Lambek calculus is a basic substructural logic; on the connection of substructural logics and residuated lattices see Galatos et al. (2007).

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- Namely, Kleene algebras do not form a finitely based variety (Redko 1964, Conway 1971), i.e., they cannot be axiomatised by a finite set of universally valid equations.
- In contrast, action algebras do. Namely, as shown by Pratt, the condition for Kleene star can be replaced by **"pure induction"**

$$(A \setminus A)^* = A \setminus A,$$

and monotonicity: $A^* \preceq (A + B)^*$.

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These action lattices are *distributive* (as lattices) and **-continuous.*

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- Extra properties: *commutativity* and *linearity* of $\preceq$.

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- This allows deriving the following sequent:

$$(p \wedge q \wedge (p \setminus q) \wedge (p \,/\, q))^+ \to p,$$

which can be falsified on a non-*-continuous action algebra.

Complexity of Action Logic

Theorem (Buszkowski & Palka 2007)

$\mathbf{ACT}_\omega$ *is* Π^0_1*-complete.*

Theorem (K. 2019–20)

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- For simplicity, in what follows we shift to the commutative situation, and consider **CommACT**$_\omega$ and **CommACT**.
- In the commutative case, we have only one division: $A \setminus B \equiv B \mathbin{/} A$.

Commutative Action Algebras/Lattices

A commutative action algebra is an action algebra satisfying $ab = ba$. Whereas action logic in general is neutral as to whether ab combines a and b sequentially or concurrently, commutative action logic in effect commits to concurrency.

Pratt (1991)

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Multiplicative-additive Lambek Calculus

- The core of **ACT** and **ACT**$_\omega$ is **MALC**, the inequational theory of residuated lattices.
- We consider its commutative modification **CommMALC**, that is, left-hand sides are multisets.
- Axioms and rules: $\dfrac{}{A \to A}\ Id$ $\qquad \dfrac{}{\Gamma, 0 \to B}\ 0L$

$$\frac{\Gamma \to B}{\Gamma, 1 \to B}\ 1L \qquad \frac{}{\to 1}\ 1R \qquad \frac{\Gamma, A, B \to C}{\Gamma, A \cdot B \to C}\ {\cdot}L \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ {\cdot}R$$

$$\frac{\Gamma, A \to C \quad \Gamma, B \to C}{\Gamma, A \vee B \to C}\ \vee L \qquad \frac{\Pi \to A}{\Pi \to A \vee B}\ \vee R_1 \qquad \frac{\Pi \to B}{\Pi \to A \vee B}\ \vee R_2$$

$$\frac{\Gamma, A \to C}{\Gamma, A \wedge B \to C}\ \wedge L_1 \qquad \frac{\Gamma, B \to C}{\Gamma, A \wedge B \to C}\ \wedge L_2 \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \wedge B}\ \wedge R$$

$$\frac{\Pi \to A \quad \Gamma, B \to C}{\Gamma, \Pi, A \setminus B \to C}\ \setminus L \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ / L$$

Rules for Kleene Star

- In $\mathbf{CommACT}_\omega$:

$$\frac{\left(\Gamma, A^n \to C\right)_{n=0}^{\infty}}{\Gamma, A^* \to C} \ {*L_\omega} \qquad \frac{\Gamma_1 \to A \quad \dots \quad \Gamma_n \to A}{\Gamma_1, \dots, \Gamma_n \to A^*} \ {*R_n},\ n \geq 0$$

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- Instructions of $\mathscr{M}$ can be of the following sorts:

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- ... thus, *non-halting* is Π^0_1-complete.
- Sometimes it is more convenient to use three counters.

Encoding Minsky Instructions

- Each instruction I of M is encoded by a formula A_I:

$$A_{\textsc{inc}(p,r,q)} = p \setminus (q \cdot r)$$
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- The encoding is due to Lincoln et al. 1992.

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$$A_{\mathrm{INC}(p,r,q)} = p \Big\backslash (q \cdot r)$$
$$A_{\mathrm{JZDEC}(p,r,q_0,q_1)} = ((p \cdot r) \Big\backslash q_1) \wedge (p \Big\backslash (q_0 \vee z_r)).$$

- Moreover, we add three extra formulae: $N_r = z_r \backslash z_r$ for each counter r (i.e., a, b, or c).
- The encoding is due to Lincoln et al. 1992.
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- The encoding is due to Lincoln et al. 1992.
- However, we now consider *non-halting* instead of halting, and model it using Kleene star instead of exponential.
- Also, in succedents of our sequents we now have to represent an *arbitrary* configuration of the Minsky machine being encoded, which is also implemented using Kleene star.

Encoding Infinite Execution

$$E = \bigwedge_I A_I \wedge N_{\mathrm{a}} \wedge N_{\mathrm{b}} \wedge N_{\mathrm{c}}$$

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Lemma

$E^*, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$ *is derivable in* $\mathbf{CommACT}_\omega$ *iff the machine runs infinitely starting from* (q, a, b, c).

Encoding Infinite Computation

- $E^*, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$ is derivable if and only if so is $E^n, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$ for any $n \geq 0$.

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- This corresponds to n steps of execution.
- Since our machine is deterministic, partial computations form an infinite one. (In the non-deterministic case, use Kőnig's lemma.)
- Base case: $n = 0$, and $\mathrm{a}^a, \mathrm{b}^b, \mathrm{b}^c, q \to D$ is derivable ((q, a, b, c) is a valid configuration).

Encoding $\textsc{inc}(p, a, q)$

$$\dfrac{p \to p \qquad \dfrac{E^{k-1}, \mathrm{a}^{a+1}, \mathrm{b}^{b}, \mathrm{c}^{c}, q \to D}{E^{k-1}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, q \cdot \mathrm{a} \to D}\ \cdot L}{\dfrac{E^{k-1}, A_{\textsc{inc}(p,\mathrm{a},q)}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to D}{E^{k}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to D}\ \wedge L \text{ several times}}\ \backslash L\ (A_{\textsc{inc}(p,\mathrm{a},q)} = p \backslash (q \cdot \mathrm{a}))$$

Encoding $\textsc{jzdec}(p, a, q)$

- $a \neq 0$

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \to p \quad \mathrm{a} \to \mathrm{a}}{p, \mathrm{a} \to p \cdot \mathrm{a}}\ {\cdot R} \qquad {\color{red}E^{k-1}}, \mathrm{a}^{a-1}, \mathrm{b}^{b}, \mathrm{c}^{c}, q_1 \to {\color{blue}D}}{{\color{red}E^{k-1}, (p \cdot \mathrm{a}) \setminus q_1}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to {\color{blue}D}}\ \setminus L}{{\color{red}E^{k-1}, A_{\textsc{jzdec}(p,\mathrm{a},q_0,q_1)}}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to {\color{blue}D}}\ \wedge L}{{\color{red}E^{k}}, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to {\color{blue}D}}\ \wedge L \text{ several times}
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$$

- $a = 0$

$$
\dfrac{\dfrac{\dfrac{p \to p \qquad \dfrac{E^{k-1}, \mathrm{b}^{b}, \mathrm{c}^{c}, q_0 \to D \qquad \dfrac{(z_{\mathrm{a}} \setminus z_{\mathrm{a}})^{k-1}, \mathrm{b}^{b}, \mathrm{c}^{c}, z_{\mathrm{a}} \to D}{E^{k-1}, \mathrm{b}^{b}, \mathrm{c}^{c}, z_{\mathrm{a}} \to D}\ \wedge L \text{ s.t.}}{E^{k-1}, q_0 \vee z_{\mathrm{a}}, \mathrm{b}^{b}, \mathrm{c}^{c} \to D}\ \vee L}{E^{k-1}, p \setminus (q_0 \vee z_{\mathrm{a}}), \mathrm{b}^{b}, \mathrm{c}^{c}, p \to D}\ \setminus L}{E^{k-1}, A_{\textsc{jzdec}(p,\mathrm{a},q_0,q_1)}, \mathrm{b}^{b}, \mathrm{c}^{c}, p \to D}\ \wedge L}{E^{k}, \mathrm{b}^{b}, p \to D}\ \wedge L \text{ several times}
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- As usual, proving the backwards direction, from derivation to computation, is technically more involved.

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Backwards Implication

- As usual, proving the backwards direction, from derivation to computation, is technically more involved.
- Again, we reduce to finite proofs of sequents $E^n, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$.
- In fact, it is possible that such a proof does not directly correspond to a counter machine computation: it could include "subprograms."

- For example, let the machine include the following instructions: $\textsc{inc}(p, \mathrm{a}, q)$ and $\textsc{jzdec}(q, \mathrm{a}, p, p)$, and consider a 4-step execution starting from $(p, 0, 0, 0)$.
- This execution has the following "non-canonical" representation:

$$
\dfrac{\underline{\dfrac{
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p \to p \qquad
\dfrac{\underline{\dfrac{q, \mathrm{a} \to q \cdot \mathrm{a} \qquad p \to p}{(q \cdot \mathrm{a}) \setminus p, \mathrm{a}, q \to p}\ \setminus L}}{E, q, \mathrm{a} \to p}\ \wedge L
}{E, p \setminus (q \cdot \mathrm{a}), p \to p}\ \cdot L, \setminus L}}{E^2, p \to p}\ \wedge L
\qquad
\dfrac{\dfrac{\underline{\dfrac{q, \mathrm{a} \to q \cdot \mathrm{a} \qquad p \to D}{(q \cdot \mathrm{a}) \setminus p, q, \mathrm{a} \to D}\ \setminus L}}{E, q, \mathrm{a} \to D}\ \wedge L}{E, q \cdot \mathrm{a} \to D}\ \cdot L
}{E^3, p \setminus (q \cdot \mathrm{a}), p \to D}\ \setminus L}}{E^4, p \to D}\ \wedge L
$$

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- Here we perform the INC step, and then start a “subroutine” which performs INC and JZDEC, returning to the same state. Finally, we perform JZDEC.

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Backwards Implication

- Here we perform the INC step, and then start a "subroutine" which performs INC and JZDEC, returning to the same state. Finally, we perform JZDEC.
- The crucial point is that such a "subroutine" could not use the zero branch of JZDEC.
- This is due to the fact that D is available only on the main branch.

Backwards Implication

Let $\widetilde{E}_i$ denote any formula in the conjunction E or a conjunction of such formulae. The backwards implication is proved by joint induction of 6 statements:

1. Sequents of the form $\widetilde{E}_1, \dots, \widetilde{E}_k, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to t$, where $t \in Q \cup Z$, are never derivable, neither are sequents of the form $\widetilde{E}_1, \dots, \widetilde{E}_k, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to t \cdot r$, where $r \in R$.
2. Sequents of the form $\widetilde{E}_1, \dots, \widetilde{E}_k, z_r, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to t$, where $r \in R$ and $t \in Q \cup Z_{\bar{r}}$, are never derivable, neither are sequents of the form $\widetilde{E}_1, \dots, \widetilde{E}_k, z_r, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to t \cdot r'$, where $r, r' \in R$ and $t \in Q \cup Z_{\bar{r}}$.
3. If $\widetilde{E}_1, \dots, \widetilde{E}_k, z_{\mathrm{a}}, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to D$ is derivable, then $a = 0$. Similarly for b and c.

Backwards Implication

4. If $\widetilde{E}_1, \ldots, \widetilde{E}_k, q, \mathrm{a}^{a'}, \mathrm{b}^{b'}, \mathrm{c}^{c'} \to p$ is derivable, where $p, q \in Q$, then the machine can move from $\langle q, a' + a, b' + b, c' + c\rangle$ to $\langle p, a, b, c\rangle$ in k steps for any a, b, c.
5. If $\widetilde{E}_1, \ldots, \widetilde{E}_k, q, \mathrm{a}^{a'}, \mathrm{b}^{b'}, \mathrm{c}^{c'} \to p \cdot \mathrm{a}$, where $p, q \in Q$, is derivable, then the machine can move from $\langle q, a' + a, b' + b, c' + c\rangle$ to $\langle p, a + 1, b, c\rangle$ in k steps for any a, b, c. Similarly for b and c.
6. If $\widetilde{E}_1, \ldots, \widetilde{E}_k, p, \mathrm{a}^{a}, \mathrm{b}^{b}, \mathrm{c}^{c} \to D$ is derivable ($p \in Q$), then the machine can perform k steps, starting from $\langle p, a, b, c\rangle$.

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- Next, we aim to prove undecidability of **CommACT**.
- This will be done by encoding circular computations by circular proofs.

Lemma

If the machine runs **circularly** *starting from* (q, a, b, c)*, then* $E^*, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$ *admits a circular proof, thus, a proof in* **CommACT**.

$$\dfrac{q \to D \qquad \dfrac{\dfrac{\dfrac{p, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to D \qquad \dfrac{\dfrac{E^*, p, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to D}{\vdots}}{E^*, E, p, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to D}}{E^*, p, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c \to D}\ *L}{\vdots}}{E^*, E, q \to D}}{E^*, q \to D}\ *L$$

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- This will be needed for modelling computation before the cycle.

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- For simplicity, let $F = (p \cdot \mathrm{a}^a \cdot \mathrm{b}^b \cdot \mathrm{c}^c) \setminus D$. Then our sequent is equivalent to $E^* \to F$.
- From the circular proof, we can easily extract $E^i \to F$ for $0 \leq i < k$ (by replacing E^* with E^i).

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- Due to lack of time, we omitted the zero-checks, which would also involve circular reasoning (or using "pure induction," $(z_\mathrm{a} \setminus z_\mathrm{a})^* = z_\mathrm{a} \setminus z_\mathrm{a}$).
- In fact, circular proofs can always be rebuilt into inductive ones, but proving this in a general setting is much harder.

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- For example, if the machine just increases one counter, $\textsc{inc}(q, \mathrm{a}, q)$, then $E^*, \mathrm{a}^a, \mathrm{b}^b, \mathrm{c}^c, q \to D$ is also derivable in **CommACT**.

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- In this case $E = q \backslash (q \cdot \mathrm{a})$, and the circular derivation, for $a = b = c = 0$, is as follows:

$$
\dfrac{\dfrac{q_S \to \mathrm{a}^*\cdot q_S \quad \dfrac{\dfrac{q_S \to q_S \quad \dfrac{\dfrac{E^*, q_S \to \mathrm{a}^*\cdot q_S \quad \mathrm{a}\to\mathrm{a}}{E^*, q_S, \mathrm{a} \to \mathrm{a}\cdot(\mathrm{a}^*\cdot q_S)}\,{\cdot R} \quad \mathrm{a}\cdot(\mathrm{a}^*\cdot q_S)\to \mathrm{a}^*\cdot q_S}{E^*, q_S, \mathrm{a}\to \mathrm{a}^*\cdot q_S}\,\mathit{Cut}}{E^*, q_S \backslash (q_S\cdot \mathrm{a}), q_S \to \mathrm{a}^*\cdot q_S}\,{\cdot L}, \backslash L}{\overline{E^*, E, q_S \to \mathrm{a}^*\cdot q_S}}\,\wedge L}{E^*, q_S \to \mathrm{a}^* \cdot q_S}\,{*L} \quad \mathrm{a}^*\cdot q_S \to D}{E^*, q_S \to D}\,\mathit{Cut}
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Inseparability

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- Let $\mathscr{C}$ be the set of machines and input data such that the machine works circularly, and let $\mathscr{H}$ be the one where the machine halts.
- There is no decidable set $\mathscr{K}$ which separates $\mathscr{C}$ from $\mathscr{H}$ (i.e., $\mathscr{C} \subseteq \mathscr{K}$ and $\mathscr{H} \cap \mathscr{K} = \emptyset$), therefore **CommACT** is undecidable; but we do not know its complexity yet.

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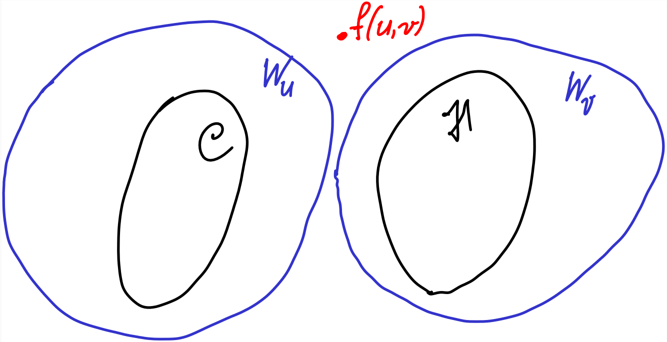
- **Folklore:** $\mathscr{C}$ and $\mathscr{H}$ are *effectively* inseparable.

Effective Inseparability



(Here W_u is the u-th r.e. set; f is computable.)

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CommACT *is* Σ^1_0*-complete.*

Myhill's Theorem

- (Corollary of) **Myhill's theorem:** if A separates $\mathscr{C}$ and $\mathscr{H}$ and A is r.e., then A is Σ_1^0-complete.
- Therefore, we obtain the necessary result:

Theorem

CommACT *is* Σ_0^1*-complete.*

- The reasoning in the non-commutative case is similar, however, the encoding of computations is more involved.