Complexity of Reasoning in Kleene and Action Algebras

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ESSLLI 2022 · Galway, Ireland · Gallimh, Éire

Lecture 2

Recap: KA_ω and KA_∞

- We are studying infinitary logics for *-continuous Kleene algebras.
- If you don't remember what a *-continuous Kleene algebra means, just think about regular expressions, in their standard interpretation.
- Sequents are expressions of the form $\Pi \to B$, where B is a formula (reg. exp.) and Π is a sequence of formulae.
- Our calculi are sound and complete w.r.t. the standard interpretation, i.e., $\Pi \to B$ is derivable iff $v(\Pi) \subseteq v(B)$.

Recap: $\mathbf{KA}_\omega$ and $\mathbf{KA}_\infty$

The first calculus, $\mathbf{KA}_\omega$, has the ω-rule, but proofs are required to be well-founded.

$$\frac{}{A \to A}\ Id \qquad \frac{}{\Gamma, 0, \Delta \to B}\ 0L$$

$$\frac{\Gamma, \Delta \to B}{\Gamma, 1, \Delta \to B}\ 1L \qquad \frac{}{\to 1}\ 1R$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ \cdot L \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ \cdot R$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A + B, \Delta \to C}\ +L \qquad \frac{\Pi \to A}{\Pi \to A + B}\ +R_1 \quad \frac{\Pi \to B}{\Pi \to A + B}\ +R_2$$

$$\frac{\left(\Gamma, A^n, \Delta \to B\right)_{n=0}^{\infty}}{\Gamma, A^*, \Delta \to B}\ *L_\omega \qquad \frac{\Gamma_1 \to A \quad \dots \quad \Gamma_n \to A}{\Gamma_1, \dots, \Gamma_n \to A^*}\ *R_n,\ n \geq 0$$

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- The rules for * in $\mathbf{KA}_\infty$ are as follows:

$$\frac{\Gamma, \Delta \to B \quad \Gamma, A, A^*, \Delta \to B}{\Gamma, A^*, \Delta \to B}\ {*L} \qquad \frac{}{\to A^*}\ {*R_0} \qquad \frac{\Gamma \to A \quad \Delta \to A^*}{\Gamma, \Delta \to A^*}\ {*R}$$

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  In $*R$ we require Γ to be non-empty.
- In the absence of *Cut*, no correctness condition on infinite paths is necessary.

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- The $*R_n$ rules is several iterations of $*R$.

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- The opposite direction is based on the following lemma:

Lemma

If $\mathbf{KA}_\infty \vdash \Gamma, A^*, \Delta \to B$, *then* $\mathbf{KA}_\infty \vdash \Gamma, A^n, \Delta \to B$ *for each* n.

- This lemma is proved by induction on n.
  - We replace A^* with A^n and go upwards the proof. At the points of $*L$ we refer to the induction hypothesis.
- Now we may eagerly apply the ω-rule, and translate finitary rules to $\mathbf{KA}_\omega$.

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- Let us denote this system by $\mathbf{KA}_{\circlearrowright}$.

$$\dfrac{\to a^*(ba^*)^* \qquad \dfrac{\dfrac{\dfrac{a \to a \quad (a+b)^* \to a^*(ba^*)^*}{a, (a+b)^* \to aa^*(ba^*)^*} \quad aa^*(ba^*)^* \to a^*(ba^*)^*}{a, (a+b)^* \to a^*(ba^*)^*} \qquad \dfrac{\to a^* \quad \dfrac{\dfrac{b \to b \quad (a+b)^* \to a^*(ba^*)^*}{b, (a+b)^* \to ba^*(ba^*)^*} \quad ba^*(ba^*)^* \to (ba^*)^*}{b, (a+b)^* \to (ba^*)^*}}{b, (a+b)^* \to a^*(ba^*)^*}}{a+b, (a+b)^* \to a^*(ba^*)^*}}{(a+b)^* \to a^*(ba^*)^*}\ {*L}$$

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- Furthermore, it is unclear how to reduce $\mathbf{KA}_\infty$ to $\mathbf{KA}_{\circlearrowright}$ (even with cuts).
- However, it is possible to construct a cut-free circular calculus for **KA**, but with a more involved sequential syntax.

Hypersequential Calculus HKA_∞

- In a hypersequent $\Pi \to \mathcal{X}$ (Das & Pous 2017), the left-hand side Π is a sequence of formulae, and the right-hand side $\mathcal{X} = \Delta_1 \mid \ldots \mid \Delta_m$ is a multiset of sequences of formulae.

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- The left rules are the same, e.g.:

$$\dfrac{\Gamma_1, \Gamma_2 \to \mathcal{X} \quad \Gamma_1, A, A^*, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A^*, \Gamma_2 \to \mathcal{X}}\ {*L} \qquad \dfrac{\Gamma_1, A, B, \Gamma_2 \to \mathcal{X}}{\Gamma_1, A \cdot B, \Gamma_2 \to \mathcal{X}}\ {\cdot L} \qquad \dots$$

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- The right-hand side enjoys structural rules:

$$\frac{\Pi \to \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} WR \qquad \frac{\Pi \to \Delta \mid \Delta \mid \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} CR$$

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- $\mathbf{HKA}_\omega$ allows non-well-founded proofs, with the following **fairness condition:** each infinite path should traverse $*L$ infinitely many times.
- There is no cut rule in $\mathbf{HKA}_\infty$ (it is even hard to formulate).

Equivalence of $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$

Theorem (Das & Pous 2017)

Sequents of the form $\Pi \to B$ are equiderivable in $\mathbf{HKA}_\infty$, $\mathbf{KA}_\infty$, and $\mathbf{KA}_\omega$.

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- For the direction from $\mathbf{HKA}_\infty$ to $\mathbf{KA}_\omega$, we use the same argument as for $\mathbf{KA}_\infty$.
- Namely, we establish the following key lemma: if $\Gamma, A^*, \Delta \to \mathcal{X}$ is derivable in $\mathbf{HKA}_\infty$, then so is $\Gamma, A^n, \Delta \to \mathcal{X}$ for each n. (Induction on n.)

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    - For example, $\{3,3,4,0\} \gg \{2,2,2,3,4,0\} \gg \{2,2,2,3,2,2,1,0\}$.
    - This is exactly what happens in our lemma, when A^* is replaced by $A^n = A, \ldots, A$.

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- Notice that induction here is more involved than for $\mathbf{KA}_\infty$.

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- A more accurate analysis of circular proofs yields the PSPACE upper bound.
- Circular proofs in $\mathbf{HKA}_\infty$ can be translated to $\mathbf{KA}_\circlearrowright$ proofs (but with cuts), and therefore to $\mathbf{KA}$ proofs. Thus, $\mathbf{KA} = \mathbf{KA}_\omega$. (We shall not go into this.)

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- This holds for both left and right rules.
- Restricting ourselves to leftmost proofs means dropping left contexts in sequent rules.
- Now we have the following left logical rules:

$$\frac{\Gamma \to \mathcal{X} \quad A, A^*, \Gamma \to \mathcal{X}}{A^*, \Gamma \to \mathcal{X}} \; {*L} \qquad \frac{A, B, \Gamma \to \mathcal{X}}{A \cdot B, \Gamma \to \mathcal{X}} \; {\cdot L}$$

$$\frac{A, \Gamma \to \mathcal{X} \quad B, \Gamma \to \mathcal{X}}{A + B, \Gamma \to \mathcal{X}} \; {+L} \qquad \frac{\Gamma \to \mathcal{X}}{1, \Gamma \to \mathcal{X}} \; 1L \qquad \frac{}{0, \Gamma \to \mathcal{X}} \; 0L$$

Leftmost Proofs

- ... and the following right rules:

$$\frac{\Pi \to \Delta \mid A, A^*, \Delta \mid \mathscr{X}}{\Pi \to A^*, \Delta \mid \mathscr{X}} *R \qquad \frac{\Pi \to A, B, \Delta \mid \mathscr{X}}{\Pi \to A \cdot B, \Delta \mid \mathscr{X}} \cdot R$$

$$\frac{\Pi \to A, \Delta \mid B, \Delta \mid \mathscr{X}}{\Pi \to A + B, \Delta \mid \mathscr{X}} +R \qquad \frac{\Pi \to \Delta \mid \mathscr{X}}{\Pi \to 1, \Delta \mid \mathscr{X}} 1R$$

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- Other rules are the same:

$$\frac{}{\varepsilon \to \varepsilon} \; Ax \qquad \frac{\Gamma \to \Delta_1 \mid \dots \mid \Delta_m}{a, \Gamma \to a, \Delta_1 \mid \dots \mid a, \Delta_m} \; K$$

$$\frac{\Pi \to \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} \; WR \qquad \frac{\Pi \to \Delta \mid \Delta \mid \mathcal{X}}{\Pi \to \Delta \mid \mathcal{X}} \; CR$$

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- This, of course, does not constitute a mathematical proof of the theorem in a reasonable sense.

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- The same happens in $\mathbf{KA}_\omega$. With the ω-rule, as we say yesterday, one proves $A \to B$, i.e., $v(A) \subseteq v(B)$, just by brute-force over all words in $v(A)$.

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- Non-well-founded proofs in general ($\mathbf{KA}_\infty$) do not resolve this issue, since they are capable of modelling the ω-rule directly.
- This simulation is performed by eager application of $*L$ to one and the same instance of A^*.
- Leftmost proofs disallow this:

$$\frac{\Gamma \to \mathcal{X} \qquad A, A^*, \Gamma \to \mathcal{X}}{A^*, \Gamma \to \mathcal{X}} \; *L$$

since now we are forced to decompose A before we are allowed to do the next iteration of A^*.

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- We think of decomposing a^* on the right, but this would yield a choice: $a^*a \equiv a + (aa^*a)$, and this choice should actually be made **after** applying $*L$.
    - On the left branch of $*L$ we choose $a \to a$, and on the right one we choose $aaa^* \to aa^*a$.

This is handled by the hypersequential version of the $*R$ rule, which keeps both choices in the sequent, and postpones the decision to applications of WR above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varepsilon \to \varepsilon}{\varepsilon \to \varepsilon \mid a^*, a}\,WR \qquad \dfrac{a, a^* \to a^*, a}{a, a^* \to \varepsilon \mid a^*, a}\,WR}{a^* \to \varepsilon \mid a^*, a}\,{*L}}{a, a^* \to a \mid a, a^*, a}\,K}{a, a^* \to a^*, a}\,{*R}}{a, a^* \to a^* \cdot a}\,{\cdot R}$$

Now we are going to prove the following two statements, which give the necessary result:

1. If $\nu(A) \subseteq \nu(B)$, then $A \to B$ has a leftmost proof in $\mathbf{HKA}_\infty$.
2. Each leftmost proof can be transformed into a circular one.

Productivity Lemma

Lemma

If there is a subderivation

$$\cfrac{\cfrac{\Gamma \to A^*, \Delta \mid \mathscr{X}}{\vdots\ \pi}}{\cfrac{\Gamma \to \Delta \mid A, A^*, \Delta \mid \mathscr{Y}}{\Gamma \to A^*, \Delta \mid \mathscr{Y}}\ {*R}}$$

where π *consists only of right logical rules, and the sequent* $\Gamma \to A^*, \Delta \mid \mathscr{Y}$ *is valid in the standard interpretation, then so is* $\Gamma \to \mathscr{X}$ *(i.e.,* A^*, Δ *there is redundant).*

Productivity Lemma

- All right rules are invertible, so $\Gamma \to A^*, \Delta \mid \mathcal{X}$ is also valid.
- It is sufficient to prove $A^*, \Delta \to \mathcal{X}$ in $\mathbf{HKA}_\infty$.
- By soundness, this means $v(A^*, \Delta) \subseteq v(\mathcal{X})$, and thus $v(\Gamma) \subseteq v(A^*, \Delta) + v(\mathcal{X}) = v(\mathcal{X})$.
- Proving $A^*, \Delta \to \mathcal{X}$ is performed by using the left rules corresponding to the rules from π:

$$\dfrac{\dfrac{\vdots\ \pi'}{\Delta \to \mathcal{X}} \qquad \dfrac{\vdots\ \pi''}{A, A^*, \Delta \to \mathcal{X}}}{A^*, \Delta \to \mathcal{X}}\ {*L}$$

Productivity Lemma

- In π' and π'', antecedents will be the corresponding sequences from the succedents in π. For example, if π had a step of the form

$$\frac{\Gamma \to B, \Delta' \mid \mathscr{Z} \quad \Gamma \to C, \Delta' \mid \mathscr{Z}}{\Gamma \to B + C, \Delta' \mid \mathscr{Z}} \; +R$$

where $B + C, \Delta'$ was the antecedent in π' or π'', then we proceed as

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- Thus, a linear derivation π with rich succedents is decomposed into branching derivations π' and π'' with simple antecedents.
- In π'', on top we shall have either $\Phi \to \mathscr{X}$ where $\Phi \in \mathscr{X}$ (derivable by WR), or $A^*, \Delta \to \mathscr{X}$, which is derived by backlink to the goal.

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- For π', we do not have this guarantee of fairness.
- However, we claim that if a leaf of π' is of the form $A^*, \Delta \to \mathscr{X}$, then the corresponding path should have traversed $*L$.
- Indeed, all rules in π' are left logical rules, and such rules other than $*L$ decrease the size of the antecedent. On the other hand, A^*, Δ is bigger than Δ.

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- Each leaf is either $\to \mathscr{X}$ or $a, \Gamma \to \mathscr{X}$.
- An important thing to mention here is that Stage 1 could generate infinite paths, e.g.:

$$\dfrac{\Gamma \to \mathscr{X} \qquad \dfrac{\dfrac{\vdots}{1^*, \Gamma \to \mathscr{X}}}{1, 1^*, \Gamma \to \mathscr{X}}\ 1L}{1^*, \Gamma \to \mathscr{X}}\ *L$$

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- These rules are also invertible.
- However, here we cannot go eagerly, since this would result in proofs violating fairness:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Pi \to 1^*, \Delta \mid \mathcal{X}}}{\Pi \to 1, 1^*, \Delta \mid \mathcal{X}}\ 1R}{\Pi \to 1^*, \Delta \mid \mathcal{X}}\ *R$$

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- The latter get weakened out by Productivity Lemma (weakenings form **Stage 3**).
- Recall that our leaves are $\to \mathscr{X}$ or $a, \Gamma \to \mathscr{X}$.
- In the first case, by validity, we have $\varepsilon \in v(\mathscr{X})$. Thus, $\mathscr{X}$ contains the empty sequence. Everything else gets weakened out.

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- Above K, we have new valid sequents, on which we again start with Stage 1.
- Notice that there is not (transfinite) induction here, we do not care for reducing any sort of parameter.
- The only thing we should care about is that the proof never gets stuck, and fairness is maintained.

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- To prove fairness, for each infinite path we consider two possibilities.
- Case 1: the branch eventually ends in one stage. Then it should be Stage 1, and fairness is maintained (see above).
- Case 2: the branch traverses Stages 1–4 infinitely.
- In this case, consider the size of the antecedent. This size gets strictly reduced at each Stage 4, thus, it should be restored, which can be done only by $*L$.

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Lemma

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- Corollary: the length of such a sequence is bounded by the size of the goal sequent.

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- A more accurate consideration shows that each path reaches its cycle in polynomial number of steps, which makes $\mathbf{KA}_\omega$ PSPACE.

Example

$$
\begin{prooftree}
\AxiomC{$\varepsilon \to \varepsilon$}
\RightLabel{WR}
\UnaryInfC{$\varepsilon \to \varepsilon \mid aa, (aa)^* \mid a(aa)^*$}
\RightLabel{$*R$}
\UnaryInfC{$\varepsilon \to (aa)^* \mid a(aa)^*$}
\RightLabel{$+R$}
\UnaryInfC{$\varepsilon \to (aa)^* + a(aa)^*$}
\AxiomC{$a^* \to a, (aa)^* \mid (aa)^*$}
\UnaryInfC{$\vdots$}
\UnaryInfC{$a^* \to a, (aa)^* \mid (aa)^*$}
\RightLabel{K}
\UnaryInfC{$a, a^* \to a, a, (aa)^* \mid a, (aa)^*$}
\RightLabel{WR}
\UnaryInfC{$a, a^* \to \varepsilon \mid a, a, (aa)^* \mid a, (aa)^*$}
\RightLabel{$\cdot R$}
\UnaryInfC{$a, a^* \to \varepsilon \mid aa, (aa)^* \mid a(aa)^*$}
\RightLabel{$*R$}
\UnaryInfC{$a, a^* \to (aa)^* \mid a(aa)^*$}
\RightLabel{$+R$}
\UnaryInfC{$a, a^* \to (aa)^* + a(aa)^*$}
\RightLabel{$*L$}
\BinaryInfC{$a^* \to (aa)^* + a(aa)^*$}
\end{prooftree}
$$