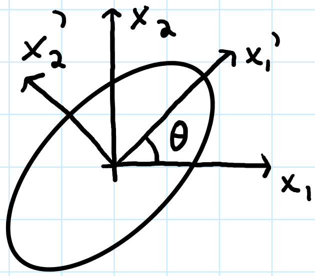
hw 8

Friday, April 4, 2025 8:45 PM

1)



$$14x_1^2 - 4x_1x_2 + 11x_2^2 = 25$$

i) $\vec{x}^T \underline{A} \vec{x} = (x_1 \;\; x_2)\begin{pmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 25$

$$\underline{A} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix}$$

$-4x_1x_2$ splits into x_1 and x_2 $\rightarrow -2, -2$

ii) $\lambda = ?$

$$\det(\underline{A} - \lambda \underline{1})$$

$$\begin{vmatrix} 14-\lambda & -2 \\ -2 & 11-\lambda \end{vmatrix} = 0$$

$$(14-\lambda)(11-\lambda) - 4 = 0$$

$$154 - 25\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 25\lambda + 150 = 0$$

$$(\lambda - 15)(\lambda - 10) = 0$$

$$\lambda = 10, 15$$

iii) $\lambda = 10$:

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$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4a = 2b$

$2a = b$

Unit vector condition: $a^2 + b^2 = 1$

$a^2 + (2a)^2 = 1$

$5a^2 = 1 \rightarrow a^2 = \frac{1}{5} \rightarrow a = \frac{1}{\sqrt{5}}$

$b = \frac{2}{\sqrt{5}}$

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$\lambda = 15$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-a - 2b = 0 \rightarrow a = -2b$

$a^2 + b^2 = 1$

$(-2b)^2 + b^2 = 1$

$5b^2 = 1 \rightarrow b = \frac{1}{\sqrt{5}}$

$a = \frac{-2}{\sqrt{5}}$

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$$v = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

iv) $$\underline{B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underline{B}^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underline{B}^T \underline{B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} + \frac{4}{5} & \frac{-2}{5} + \frac{2}{5} \\ \frac{-2}{5} + \frac{2}{5} & \frac{4}{5} + \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

v) $$\underline{B}^T \underline{A}\, \underline{B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{14}{\sqrt{5}} - \frac{4}{\sqrt{5}} & \frac{-28}{\sqrt{5}} - \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} + \frac{22}{\sqrt{5}} & \frac{4}{\sqrt{5}} + \frac{11}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{10}{\sqrt{5}} & \frac{-30}{\sqrt{5}} \\ \frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{5} + \frac{40}{5} & -\frac{30}{5} + \frac{30}{5} \\ -\frac{20}{5} + \frac{20}{5} & \frac{60}{5} + \frac{15}{5} \end{pmatrix}$$

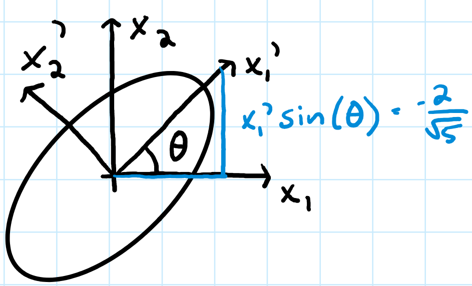
$$= \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix} \quad \checkmark$$

vi) $\vec{x}' = B^T \vec{x}$

we choose $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then

$$\vec{x}_1' = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$$



$$|\vec{x}_1'| = \sqrt{\frac{1}{5} + \frac{4}{5}} = 1$$

$$\sin(\theta) = -\frac{2}{\sqrt{5}} \rightarrow \arcsin\left(\frac{-2}{\sqrt{5}}\right) = \begin{matrix} -1.107 \text{ rad} \\ -63.43^\circ \end{matrix}$$

vii) $\left(\frac{x_1'}{\alpha_1}\right)^2 + \left(\frac{x_2'}{\alpha_2}\right)^2 = 1$

the general form of an ellipse is given by

$$\alpha_1 x_1^2 + 2\beta x_1 x_2 + \alpha_2 x_2^2 = \gamma$$

and the standard form is seen above

we set $\underline{B}^T \underline{A}\, \underline{B} \equiv \underline{D}$ and then rewrite our matrix for the ellipse:

$$\vec{x}^T \underline{A}\, \vec{x} = \gamma \longrightarrow \gamma = \left(\vec{x}^T (\underline{B})(\underline{B}^T)\, \underline{A}\, (\underline{B})(\underline{B}^T)\, \vec{x}\right)$$

which becomes

$$\gamma = \underbrace{\underbrace{\vec{x}^T \underline{B}}_{(\underline{B}^T \vec{x})^T}}_{\vec{x}'^T} \underbrace{(\underline{B}^T \underline{A}\, \underline{B})}_{\underline{D}} \underbrace{(\underline{B}^T \vec{x})}_{\text{linear transform to some new vector } \vec{x}'}$$

which simplifies to

$$\gamma = \vec{x}'^T \underline{D}\, \vec{x}'$$

$$25 = \begin{pmatrix} x_1' & x_2' \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1' & x_2' \end{pmatrix} \begin{pmatrix} 10x_1' \\ 15x_2' \end{pmatrix}$$

$$25 = 10x_1'^2 + 15x_2'^2$$

$$1 = \frac{10}{25} x_1'^2 + \frac{15}{25} x_2'^2$$

$$1 = \frac{2}{} x_1'^2 + \frac{3}{} x_2'^2$$

$$1 = \frac{2}{5} x_1'^2 + \frac{3}{5} x_2'^2$$

$$\boxed{\begin{aligned} \alpha_1 &= \sqrt{\frac{5}{2}} \\ \alpha_2 &= \sqrt{\frac{5}{3}} \end{aligned}}$$

2) $$K(A) = \frac{\max(|\lambda(A)|)}{\min(|\lambda(A)|)}$$

i) The condition number is 16024255940444.816, which is of the order 10^{13}.

ii) We find the inverse using `scipy.linalg.inv` and multiply it with our hilbert matrix using the `@` operator. We expect the result to be the identity matrix, but limited precision and an ill-conditioned matrix lead to errors. We subtract the identity matrix from our product and take the absolute value of the difference, and thus find a maximum absolute error of $6.30534834797885 \times 10^{-5}$.

iii) $\underline{D}$ is the diagonal matrix of EWs

$\underline{D}^{-1}$ is simply given by reciprocating the diagonal

Having found D^{-1} analytically, we then use the `@` operator to multiply it by the matrix of eigenvalues and that matrix's transpose. This yields the inverse of our hilbert matrix, and we then repeat the previous part with this inverse to find a maximum absolute error of $3.2833022640763834 \times 10^{-4}$. In both cases, we find that the error is several orders of magnitude greater than the expected numerical error on the order of 10^{-15}.