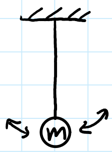
hw 4

Sunday, February 9, 2025 10:08 PM

1)



i) $E = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell(1-\cos(\theta))$

$$\frac{T(\theta_m)}{T_0} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m} \frac{1}{\sqrt{\cos(\theta)-\cos(\theta_m)}}\,\partial\theta$$

when $\theta = \theta_m$, $\dot{\theta} = 0$

$$E = \cancel{\frac{1}{2} m \ell (0)^2}^{\,0} + mg\ell(1-\cos(\theta_m))$$

$$E = mg\ell(1-\cos(\theta_m))$$

$$\frac{1}{2}\cancel{m}\ell^{\cancel{2}}\dot{\theta}^2 + \cancel{m}g\ell(1-\cos(\theta)) = \cancel{m}g\ell(1-\cos(\theta_m))$$

$$\frac{1}{2}\ell\dot{\theta}^2 + \cancel{g} - g\cos(\theta) = \cancel{g} - g\cos(\theta_m)$$

$$\frac{1}{2}\ell\dot{\theta}^2 = g\cos(\theta) - g\cos(\theta_m)$$

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos(\theta)-\cos(\theta_m))} = \frac{\partial\theta}{\partial t}$$

$$\int_0^{\frac{T}{4}} \partial t = \int_0^{\theta_m} \frac{1}{\sqrt{\frac{2g}{\ell}(\cos(\theta)-\cos(\theta_m))}}\,\partial\theta$$

$$\frac{T}{4} = \sqrt{\frac{\ell}{2g}}\int_0^{\theta_m} \frac{1}{\sqrt{\cos(\theta)-\cos(\theta_m)}}\,\partial\theta$$

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}}$$

$$\frac{T(\theta_m)}{T} = \underbrace{\frac{4}{2\pi\sqrt{\frac{\ell}{g}}}\sqrt{\frac{\ell}{2g}}}\int_0^{\theta_m} \frac{1}{\sqrt{\cos(\theta)-\cos(\theta_m)}}\,\partial\theta$$

$$\left(\frac{\cancel{4}2}{\cancel{2}\pi}\right)\left(\sqrt{\frac{\cancel{g}}{\cancel{\ell}}}\right)\left(\sqrt{\frac{\cancel{\ell}}{2\cancel{g}}}\right)$$

$$\left(\frac{2}{\pi}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{\pi}$$

$$\boxed{\frac{T(\theta_m)}{T} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m} \frac{1}{\sqrt{\cos(\theta)-\cos(\theta_m)}}\,\partial\theta}$$

ii) $\cos(\theta) = 1 - 2\sin^2\left(\frac{1}{2}\theta\right)$

$$\frac{T(\theta_m)}{} = \sqrt{2}\int^{\theta_m} \frac{1}{} \,\partial\theta$$

ii) $\cos(\theta) = 1 - 2\sin^2\left(\tfrac{1}{2}\theta\right)$

$$\frac{T(\theta_m)}{T} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m} \frac{1}{\sqrt{\cancel{1} - 2\sin^2(\frac{1}{2}\theta) - \cancel{1} + 2\sin^2(\frac{1}{2}\theta_m)}}\,\partial\theta$$

$$\frac{T(\theta_m)}{T} = \frac{1}{\pi}\int_0^{\theta_m} \frac{1}{\sqrt{\sin^2(\frac{1}{2}\theta_m) - \sin^2(\frac{1}{2}\theta)}}\,\partial\theta$$

iii) $\sin(\psi) = \dfrac{\sin(\frac{1}{2}\theta)}{\sin(\frac{1}{2}\theta_m)}$

$$\cos(\psi)\frac{\partial\psi}{\partial\theta} = \frac{\cos(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta_m)}$$

$$\partial\psi = \frac{\cos(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta_m)\cos(\psi)}\,\partial\theta \rightarrow \partial\theta = \frac{2\sin(\frac{1}{2}\theta_m)\cos(\psi)}{\cos(\frac{1}{2}\theta)}\,\partial\psi$$

bounds:

$\theta_m \rightarrow \sin(\psi) = 1 \rightarrow \psi = \frac{\pi}{2}$

$0 \rightarrow \sin(\psi) = 0 \rightarrow \psi = 0$

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin^2(\frac{1}{2}\theta_m) - \sin^2(\frac{1}{2}\theta)}}\underbrace{\left(\frac{2\sin(\frac{1}{2}\theta_m)\cos(\psi)}{\cos(\frac{1}{2}\theta)}\right)}\,\partial\psi$$

$$\frac{1}{\dfrac{\cos(\frac{1}{2}\theta)}{2\sin(\frac{1}{2}\theta_m)\cos(\psi)}}$$

$$\frac{1}{\sqrt{\dfrac{\cos^2(\frac{1}{2}\theta)}{4\sin^2(\frac{1}{2}\theta_m)\cos^2(\psi)}}}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\left(\sin^2(\frac{1}{2}\theta_m) - \sin^2(\frac{1}{2}\theta)\right)\left(\dfrac{\cos^2(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta_m)\cos^2(\psi)}\right)}}\,\partial\psi$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\left(1 - \sin^2(\psi)\right)\left(\dfrac{\cos^2(\frac{1}{2}\theta)}{\cos^2(\psi)}\right)}}\,\partial\psi$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cancel{\left(1 - \sin^2(\psi)\right)}\left(\dfrac{\cos^2(\frac{1}{2}\theta)}{\cancel{1 - \sin^2(\psi)}}\right)}}\,\partial\psi$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \sin^2(\frac{1}{2}\theta)}}\,\partial\psi$$

$$\pi \int_0 \sqrt{1-\sin^2(\tfrac{1}{2}\theta)}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2(\frac{1}{2}\theta_m)\sin^2(\psi)}}\, d\psi$$

iv) at $\theta_m = \pi$, the top of the pendulum's swing would be directly vertical, at which point it would either loop around for a complete rotation or stand still, neither of which are consistent with the behavior of a pendulum.

v) $$\frac{T(\theta_m)}{T_0} = 1 + \sum_{n=1}^{\infty} a_{2n}\sin^{2n}\left(\frac{1}{2}\theta_m\right)$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2(\frac{1}{2}\theta_m)\sin^2(\psi)}}\, d\psi$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-x}}\, d\psi \quad \text{where } x = \sin^2(\tfrac{1}{2}\theta_m)\sin^2(\psi)$$

taylor expand $\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n \rightarrow \frac{f^{(n)}(0)}{n!}x^n$$

$$\frac{1}{2}(1-x)^{3/2} \rightarrow \frac{3}{4}(1-x)^{-5/2} \rightarrow \frac{15}{8}(1-x)^{-7/2} \longrightarrow \frac{105}{16}(1-x)^{-9/2}$$

$$\frac{\frac{1}{2}}{1!} \rightarrow \frac{\frac{3}{4}}{2!} \rightarrow \frac{\frac{15}{8}}{3!} \rightarrow \frac{\frac{105}{16}}{4!}$$

$$\frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2(\tfrac{1}{2}\theta_m)\sin^2(\psi) + \frac{3}{8}\sin^4(\tfrac{1}{2}\theta_m)\sin^4(\psi) + \frac{5}{16}\sin^6(\tfrac{1}{2}\theta_m)\sin^6(\psi) + \frac{35}{128}\sin^8(\tfrac{1}{2}\theta_m)\sin^8(\psi)\, d\psi$$

$$\frac{2}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{8}\sin^2(\tfrac{1}{2}\theta_m) + \frac{9\cancel{\pi}}{128}\sin^4(\tfrac{1}{2}\theta_m) + \frac{25\cancel{\pi}}{512}\sin^6\left(\tfrac{1}{2}\theta_m\right) + \frac{1225\cancel{\pi}}{32768}\sin^8(\tfrac{1}{2}\theta_m)\right)$$

$$\frac{1}{4}\sin^2\left(\frac{1}{2}\theta_m\right) + \frac{9}{64}\sin^4\left(\frac{1}{2}\theta_m\right) + \frac{25}{256}\sin^6\left(\frac{1}{2}\theta_m\right) + \frac{1225}{16384}\sin^8\left(\frac{1}{2}\theta_m\right)$$

$a_2 = \frac{1}{4}$

$a_4 = \frac{9}{64}$

$a_6 = \frac{25}{256}$

$$\boxed{\frac{T(\theta_m)}{T_0} = 1 + \sum_{n=1}^{\infty} a_{2n}\sin^{2n}\left(\frac{1}{2}\theta_m\right)}$$

$$a_6 = \frac{25}{256}$$

$$a_8 = \frac{1225}{16384}$$

2) i) newton-côtes integration: integrate the lagrange interpolating polynomial

lagrange interp poly:

$$L_{n,j}(x_j) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \left(\frac{x - x_i}{x_j - x_i} \right)$$

simpson's rule: $n=2$

$$L_{2,j}(x_j) = \prod_{\substack{i=0 \\ i \neq j}}^{2} \left(\frac{x - x_i}{x_j - x_i} \right)$$

$$a_0 = \int_{x_0}^{x_2} L_{2,0} = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \partial x = \int_{x_0}^{x_2} \frac{x^2 - x_1 x - x_2 x + x_1 x_2}{(x_0 - x_1)(x_0 - x_2)} \partial x = \frac{1}{(-h)(-2h)} \int_{x_0}^{x_2} x^2 - x_1 x - x_2 x + x_1 x_2 \, \partial x$$

$$\frac{1}{2h^2}\left[\frac{1}{3}x^3 - \frac{x_1}{2}x^2 - \frac{x_2}{2}x^2 + x_1 x_2 x\right]_{x_0}^{x_2} = \frac{1}{2h^2}\left[\frac{1}{3}x_2^3 - \frac{x_1}{2}x_2^2 - \frac{x_2}{2}x_2^2 + x_1 x_2 x_2 - \frac{1}{3}x_0^3 + \frac{x_1}{2}x_0^2 + \frac{x_2}{2}x_0^2 - x_1 x_2 x_0\right]$$

$$= \frac{1}{2h^2}\left[\frac{x_2^3}{3} - \frac{x_1 x_2^2}{2} - \frac{x_2^3}{2} + x_1 x_2^2 - \frac{x_0^3}{3} + \frac{x_1 x_0^2}{2} + \frac{x_2 x_0^2}{2} - x_1 x_2 x_0\right] = \frac{(x_0 - x_2)(2x_0 - 3x_1 - x_2)}{6h}$$

$$= \frac{-2h}{6h}\left(\cancel{2x_0} - \cancel{3x_0} - 3h + \cancel{x_0} + 2h\right) = \frac{h}{3}$$

$$a_1 = \int_{x_0}^{x_2} L_{2,1} = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \partial t = \frac{4}{3}h$$

$$a_2 = \int_{x_0}^{x_2} L_{2,2} = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \partial x = \frac{h}{3}$$

$$a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) = \frac{h}{3} f(x_0) + \frac{4}{3}h f(x_1) + \frac{h}{3} f(x_2)$$

$$\boxed{\int_{x_0}^{x_2} f(x)\partial x = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)}$$

ii) complete range split into N evenly spaced intervals; $N \% 2 = 0$

try with $N = 4$

region 1 is simply the result in (i): $\frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$

then region 2 begins at x_2, so $\frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right)$

region 1 is simply the result in (i): $\frac{h}{3}(f(x_0)+4f(x_1)+f(x_2))$

then region 2 begins at x_2, so $\frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)$

region 3: $\frac{h}{3}\left(f(x_4)+4f(x_5)+f(x_6)\right)$

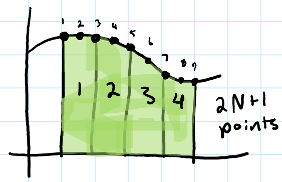
region 4: $\frac{h}{3}\left(f(x_6)+4f(x_7)+f(x_8)\right)$

so combining, we get:

$$\frac{h}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+2f(x_6)+4f(x_7)+f(x_8)\right)$$

so, generalized:

$$I=\frac{h}{3}\left(f(x_0)+f(x_{2N+1})+2\sum_{j=1}^{N-1}f(x_{2j})+4\sum_{j=1}^{N}f(x_{2j-1})\right)$$



iii) $\int_{\frac{1}{4}}^{4}\ln(x)\,dx$

true value: $\int_{\frac{1}{4}}^{4}\ln(x)\,dx$ $\quad u=\ln(x)\quad v=x$ $\quad du=\frac{1}{x}\quad dv=dx$

$$x\ln(x)-\int_{\frac{1}{4}}^{4}1\,dx = x\ln(x)-x\Big|_{\frac{1}{4}}^{4}$$

$$4\ln(4)-4-\frac{1}{4}\ln\left(\frac{1}{4}\right)+\frac{1}{4}=2.1418$$

trapezoid rule: $\frac{1}{2}(b-a)\left(f(a)+f(b)\right)$

$$\frac{1}{2}(3.75)\left(\ln\left(\frac{1}{4}\right)+\ln(4)\right)=0$$

$$\left|1-\frac{0}{2.1418}\right|=1$$

value: 0
error: 100%

simpson's rule: $\frac{b-a}{3}\left(f(a)+f\left(\frac{b-a}{2}\right)+f(b)\right)$

$$\frac{1}{3}(3.75)\left(\ln\left(\frac{1}{4}\right)+\ln\left(\frac{17}{8}\right)+\ln(4)\right)=.942215$$

$$\left|1-\frac{.942215}{2.1418}\right|=.56$$

value: .942215
error: 56%

iv) trapezoid: $\frac{1}{2}\left(\frac{b-a}{2}\right)\left(f(a)+2f\left(\frac{b-a}{2}+a\right)+f(b)\right)$

$$\frac{1}{4}\left(\frac{15}{8}\right)\left(\ln\left(\frac{1}{4}\right)+2\ln\left(\frac{17}{8}\right)+\ln(4)\right)=.7067$$

$$\left|1-\frac{.7067}{2.1418}\right|=.67$$

$4(\frac{[illegible]}{8})(\ln(\frac{[illegible]}{4}) + 2\ln(\frac{[illegible]}{8}) + \ln([illegible])) = .7067$

$\left|1 - \frac{.7067}{2.1418}\right| = .67$

value: .7067
error: 67%

simpson's: $\frac{1}{3}\left(\frac{b-a}{2}\right)\left(f(a) + f(b) + 4f\left(\frac{b-a}{2} + a\right)\right)$

$\frac{1}{3}\left(\frac{15}{8}\right)\left(\ln\left(\frac{1}{4}\right) + \ln(4) + 4\ln\left(\frac{17}{8}\right)\right) = 1.8844$

$\left|1 - \frac{1.8844}{2.1418}\right| = .12$

value: 1.8844
error: 12%