PHYS 250 Homework 9

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Problem 1

```
def linear_leastsq(x, y, sigma=None):
    """
    Perform a least squares fit to some line a1 + a2*x, given data points for x and y.
    Inputs:
      x: array: x (independent variable) data
      y: array: y (dependent variable) data
      sigma: array: optional uncertainties in dependent variable measurements
    Outputs:
      params: tuple of best fit parameters a1 and a2 for a line
      uncerts: tuple of uncertainties in best fit parameters
    """
    if sigma is None :
        sigma = np.ones_like(y) # This sets sigma=1 for each y

    S = np.sum(1/(sigma**2))
    Sx = np.sum(x/(sigma**2))
    Sy = np.sum(y/(sigma**2))
    Sxx = np.sum((x**2)/(sigma**2))
    Sxy = np.sum((x*y)/(sigma**2))

    delta = S*Sxx - Sx**2

    a1 = (Sxx*Sy - Sx*Sxy)/delta
    sig_a1 = np.sqrt(Sxx/delta)

    a2 = (S*Sxy - Sx*Sy)/delta
    sig_a2 = np.sqrt(S/delta)

    params=(a1, a2)
    uncerts=(sig_a1, sig_a2)

    return (params, uncerts)
```

Problem 2

2 i

Using the function above to fit a line to that data, we find that the best fit parameters a_1 and a_2 are given by

$$a_1 = -3.8 \pm 0.4472135954999579$$

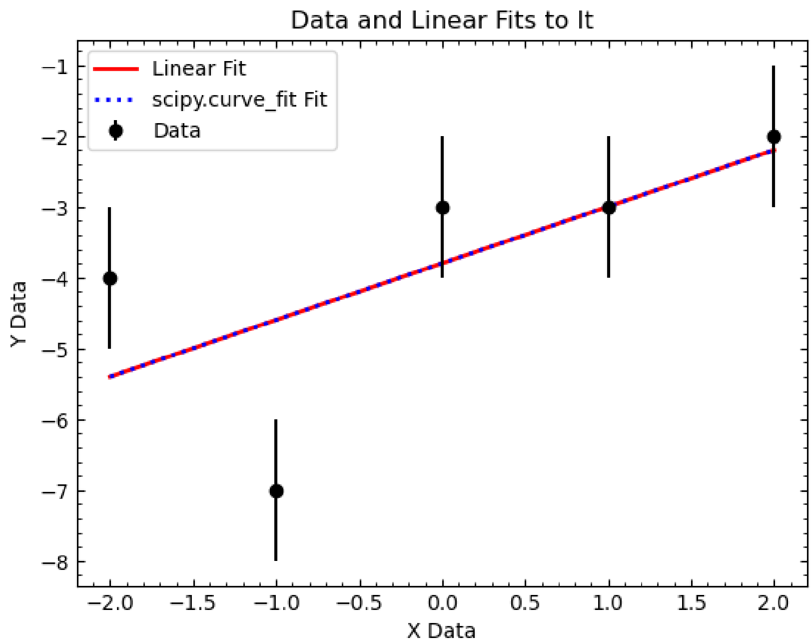
and

$$a_2 = 0.8 \pm 0.31622776601683794.$$

The χ^2 for this fit is 8.400000000000002, and the reduced χ^2 is 2.8000000000000007.

2 ii

The easiest way to determine this is via a plot. We overplot `scipy.optimize.curve_fit`'s fit to verify that our "homemade" fitting function is good.



We see that the point at $x = -1$ is an outlier from the rest.

2 iii

We remove the $x = -1$ point from the data set and re-fit a line. Now, we find that the best fit parameters a_1 and a_2 are given by

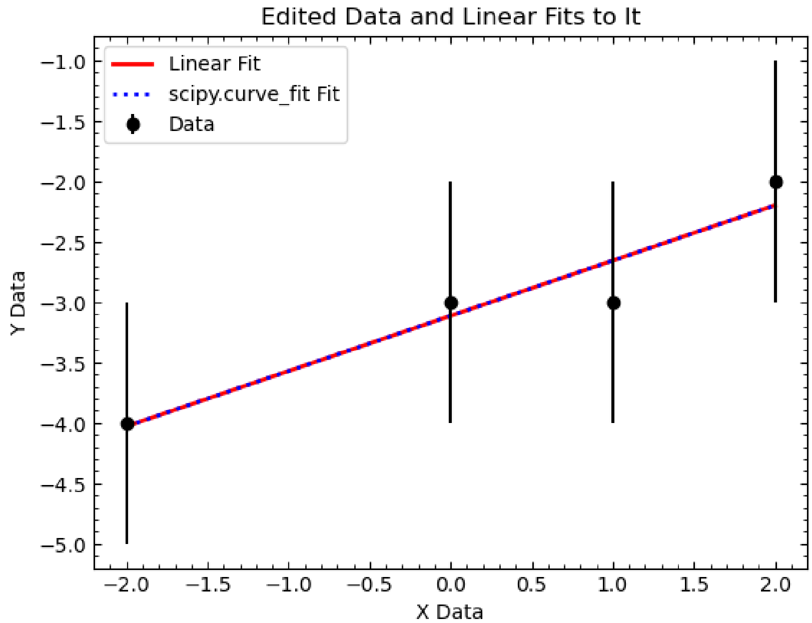
$$a_1 = -3.1142857142857143 \pm 0.50709255283711$$

and

$$a_2 = 0.45714285714285713 \pm 0.3380617018914066.$$

The χ^2 for this fit is 0.17142857142857157, and the reduced χ^2 is 0.08571428571428578.

We recall the prelab, where we found that a reduced χ^2 this small can be an indication that our uncertainties are too large. Looking at the plot (updated to remove the $x = -1$ point), we can visually confirm that the default uncertainty of 1 seems much too high.



Problem 3

3 i

To find the elements of $\mathbf{b}$, we begin by expanding out the sum in X^2 and approximating it to the second order, so that:

$$X^2 = \int_0^1 \left[g(x) - \sum_{i=0}^{n} a_i x^i \right]^2 dx \tag{1}$$

becomes

$$X^2 = \int_0^1 \left(g(x) - a_0 - a_1 x - a_2 x^2 \right)^2 dx.$$

We want to minimize X^2, so we differentiate it and set it equal to 0. Since we want to find the parameters a_i, we differentiate with respect to them. We move that derivative inside the integral, and iterating via an index variable j, which begins at 0, we find a system of equations:

$$\frac{\partial X^2}{\partial a_0} = \int_0^1 2 \left(g(x) - a_0 - a_1 x - a_2 x^2 \right) (1) \ dx \tag{2}$$

$$\frac{\partial X^2}{\partial a_1} = \int_0^1 2 \left(g(x) - a_0 - a_1 x - a_2 x^2 \right) (-x) \ dx \tag{3}$$

$$\frac{\partial X^2}{\partial a_2} = \int_0^1 2 \left(g(x) - a_0 - a_1 x - a_2 x^2 \right) \left(-x^2\right) \ dx \tag{4}$$

$$\tag{5}$$

at which point a pattern becomes clear:

$$\frac{\partial X^2}{\partial a_j} = \int_0^1 2 \left(g(x) - a_0 - a_1 x - a_2 x^2 \right) \left(-x^j\right) \ dx. \tag{6}$$

With this understanding, we rewrite our differentiated Eqn. 1 as

$$\int_0^1 \left(-2x^j g(x) + 2x^j \sum_{i=0}^{n} a_i x^i \right) dx = 0.$$

Using linearity to split this, and doing some basic manipulations, gives us the outline of our system of equations:

$$\int_0^1 x^j \sum_{i=0}^n a_i x^i dx = \int_0^1 x^j g(x)\ dx.$$

Leaving the left hand side for part ii of this problem, we see that the right hand side gives us $\mathbf{b}$ when the index j is related to the row of the vector. Thus,

$$\mathbf{b} = \begin{pmatrix} \int_0^1 g(x)dx \\ \int_0^1 xg(x)dx \\ \int_0^1 x^2 g(x)dx \\ \vdots \end{pmatrix}.$$

3 ii

We now examine the left hand side to find the matrix A. To do this, we again expand it to the second order, which yields

$$\int_0^1 x^j (a_0 + a_1 x + a_2 x^2)dx$$

for our same index variable j. We then multiply through and make use of exponent rules to get

$$\int_0^1 a_0 x^j + a_1 x^{j+1} + a_2 x^{j+2} dx.$$

We can take this integral:

$$\frac{a_0}{j+1} + \frac{a_1}{j+2} + \frac{a_2}{j+3}.$$

Again, the pattern becomes clear, and we find that our matrix A is given by

$$\mathsf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$