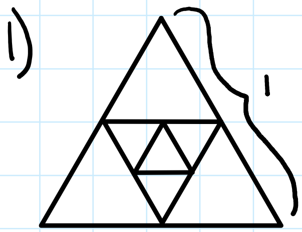
hw 1

Saturday, January 18, 2025 12:36 AM

1)




i) perimeter of 1st: 3

perimeter of 2nd: $3(\frac{1}{2})$

perimeter of 3rd: $3(\frac{1}{4}L)$

perimeter of Nth: $3(\frac{1}{2^{N-1}})$

for N triangles:

$$\sum_{i=0}^{N} 3\left(\frac{1}{2^i}\right)$$

the sum of a geometric series is given by:

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

where $a_1$: 1st term

$r$: common ratio

$n$: number of terms

$$\left.\begin{array}{l} a_1 = 3 \\ r = \frac{1}{2} \\ n = N \end{array}\right\} 3\left(\frac{1-(\frac{1}{2})^{N+1}}{1-\frac{1}{2}}\right) = 3\left(\frac{1-(\frac{1}{2})^{N}(\frac{1}{2})}{\frac{1}{2}}\right)$$

$$\boxed{6\left(1-\frac{1}{2^{N+1}}\right)}$$

for $N \to \infty$: $6\left(1-\cancel{\frac{1}{2^{N+1}}}^{\,0}\right) = 6$

ii) area of a triangle:

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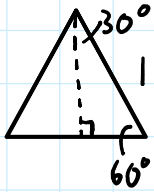
$$\frac{1}{2}bh$$




$$\sin(60°) = \frac{h}{1} \rightarrow h = \frac{\sqrt{3}}{2}$$

1st triangle: $\frac{1}{2}\left(\frac{\sqrt{3}}{2}L\right)L = \frac{\sqrt{3}}{4}L^2$

2nd: $\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right) = \frac{\sqrt{3}}{16}L^2$     where $L=1$

3rd: $\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{L}{4}\right)\left(\frac{L}{4}\right) = \frac{\sqrt{3}}{64}L^2$

for $N$ triangles:

$$\sum_{i=1}^{N} \sqrt{3}\left(\frac{1}{4^i}\right)$$

$$S_n = a_1\left(\frac{1-r^N}{1-r}\right)$$

$$= \frac{\sqrt{3}}{4}\left(\frac{1-\left(\frac{1}{4}\right)^N}{1-\frac{1}{4}}\right) = \frac{\sqrt{3}}{4}\left(\frac{1-\frac{1}{4^N}}{\frac{3}{4}}\right) = \boxed{\frac{\sqrt{3}}{3}\left(1-\frac{1}{4^N}\right)}$$

for $N \rightarrow \infty$

$$\frac{\sqrt{3}}{3}\left(1-\cancel{\frac{1}{4^N}}^{\,0}\right) = \frac{\sqrt{3}}{3}$$

iii) evaluating to an absolute accuracy $\varepsilon$

$$\varepsilon_p \geq 6 - 6\left(1 - \frac{1}{2^{N+1}}\right)$$

$$\varepsilon_p \geq \cancel{6 - 6} + \frac{6}{2^{N+1}}$$

$$\varepsilon_p \geq \cancel{6 - 6} + \frac{6}{2^{N+1}}$$

$$\varepsilon_p \geq \frac{6}{2^{N+1}} \rightarrow 2^{N+1} \geq \frac{6}{\varepsilon_p}$$

$$\log_2(2^{N+1}) \geq \log_2\left(\frac{6}{\varepsilon_p}\right)$$

$$(N+1) \geq \log_2\left(\frac{6}{\varepsilon_p}\right)$$

$$\boxed{N_p \geq \log_2\left(\frac{6}{\varepsilon_p}\right) - 1}$$

$$\varepsilon_A \geq \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\left(1 - \frac{1}{4^N}\right)$$

$$\varepsilon_A \geq \cancel{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}} + \frac{\sqrt{3}}{3 \cdot 4^N}$$

$$\varepsilon_A \geq \frac{\sqrt{3}}{3 \cdot 4^N} \rightarrow 4^N \geq \frac{\sqrt{3}}{3 \cdot \varepsilon_A}$$

$$\log_4(4^N) \geq \log_4\left(\frac{\sqrt{3}}{3\varepsilon_A}\right)$$

$$\boxed{N_A \geq \log_4\left(\frac{\sqrt{3}}{3\varepsilon_A}\right)}$$

iv) a) $\varepsilon = 10^{-7}$

$$N_p \geq \log_2\left(\frac{6}{10^{-7}}\right) - 1$$

$$\geq 24.8$$

$\boxed{\text{perimeter: 25 terms}}$

$$N_A \geq \log_4\left(\frac{\sqrt{3}}{3 \times 10^{-7}}\right)$$

$$\geq 11.2$$

$E = 10^{-7}$

$= 1 \times 10^{-7}$

$n_p = \log_2\left(\frac{6}{E}\right) - 1$

$= 24.8384591649$

$N_p = \mathrm{ceil}(n_p)$

$= 25$

$6 - 6\left(1 - \frac{1}{2^{N_p+1}}\right)$

$= 8.9406967163 \times 10^{-8}$

$n = \log\left(\frac{\sqrt{3}}{\ldots}\right)$

$\geq 11.2$

area: 12 terms

yes, the results agree

b) $\varepsilon = 10^{-15}$

$N_p \geq \log_2\left(\frac{6}{10^{-15}}\right) - 1$

$\geq 51.4$

perimeter: 52 terms

$N_A \geq \log_4\left(\frac{\sqrt{3}}{3\times10^{-15}}\right)$

$\geq 24.5$

area: 25 terms

yes, the results agree

$= 8.9406967163 \times 10$

$n_a = \log_4\left(\frac{\sqrt{3}}{3E}\right)$

$= 11.2305077069$

$N_a = \mathrm{ceil}(n_a)$

$= 12$

$\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\left(1 - \frac{1}{4^{N_a}}\right)$

$= 3.4412757755 \times 10^{-8}$

$E = 10^{-15}$

$= 1 \times 10^{-15}$

$n_p = \log_2\left(\frac{6}{E}\right) - 1$

$= 51.413883924$

$N_p = \mathrm{ceil}(n_p)$

$= 52$

$6 - 6\left(1 - \frac{1}{2^{N_p+1}}\right)$

$= 8.881784197 \times 10^{-16}$

$n_a = \log_4\left(\frac{\sqrt{3}}{3E}\right)$

$= 24.5182200865$

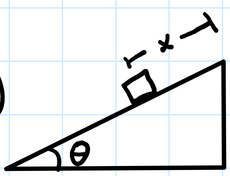
$N_a = \mathrm{ceil}(n_a)$

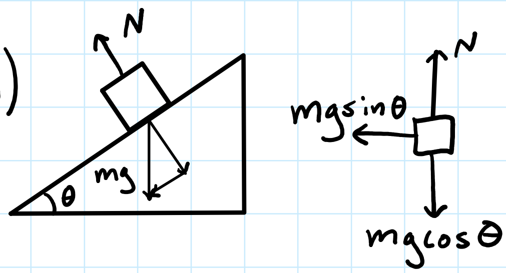
$= 25$

$\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\left(1 - \frac{1}{4^{N_a}}\right)$

$= 5.5511151231 \times 10^{-16}$

2)



i) 


$$\Sigma F_x = mg\sin(\theta)$$

$$\cancel{m}a = \cancel{m}g\sin(\theta)$$

$$a = g\sin(\theta)$$

$$V_0 = 0\ ,\ X_0 = 0$$

$$\Delta x = \cancel{V_0 t}^{0} + \frac{1}{2}at^2$$

$$x_f - \cancel{x_0}^{0} = \frac{1}{2}at^2$$

$$X = \frac{1}{2}g\sin(\theta)t^2$$

ii) $x = 2m \quad t = 1s \quad \theta = ?$

$$2x = g\sin(\theta)t^2$$

$$\sin(\theta) = \frac{2x}{gt^2} \rightarrow \theta = \arcsin\left(\frac{2x}{gt^2}\right)$$

$$\theta = \arcsin\left(\frac{4}{g}\right)$$

$$\theta = .42\,\text{rad}$$

iii) $\dot{\theta} = \omega$

$$x(t) = \frac{g}{4\omega^2}\left(e^{\omega t} - e^{-\omega t} - 2\sin(\omega t)\right)$$

$$\varepsilon = 10^{-12},\ x = 2m,\ t = 1s$$

$\omega, \theta_f = ?$

$\omega = 1.22125479731 5$ rad

$\omega = 69.972745596274°$

$\theta = 69.972745596274°$

I calculated these by constructing a function for $x(t)$ which took $\omega$ and $t$ as arguments, and solved it for 0 by moving $x$ (2 here) to the other side. I then ran scipy.optimize.brentq() on it, where I passed 1 and 2 as brackets and 1e-12 as x tolerance. I multiplied $\omega$ by $t$ (1s here) to get $\theta$.

iv) $\omega = 135.289782760917°/s$

$\theta = 135.289782760917°$

this would not make physical sense; the block would be upside-down by then.

3) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

i) $x_{n+1} - a = -(x_n - a)$

$x_{n+1} - a = -x_n + a$

$x_{n+1} = -x_n + 2a$

$x_n - \frac{f(x_n)}{f'(x_n)} = -x_n + 2a$

$2x_n - \frac{f(x_n)}{f'(x_n)} = 2a$

$$2x_n - \frac{f(x_n)}{f'(x_n)} = 2a$$

$$\frac{f(x_n)}{f'(x_n)} = 2x_n - 2a = \frac{y}{y'}$$

$$2(x_n - a) = \frac{y}{y'}$$

$$y' = \frac{y}{2(x_n - a)}$$

$$\int \frac{1}{y} y' = \int \frac{1}{2(x_n - a)}$$

$$\ln(|y|) = \frac{1}{2}\ln(|x_n - a|) + c$$

$$\ln(|y|) = \ln\left(\sqrt{|x_n - a|}\right) + c$$

$$e^{\ln(|y|)} = e^{\ln(\sqrt{|x_n - a|}) + c}$$

$$y = \pm c\sqrt{|x_n - a|}$$

for the integrals to be equal, they must have the same sign, so we convert the $\pm$ to the sign function to ensure that $y$ and $(x_n - a)$ will always have the same sign.

$$\boxed{f(x) = A\,\text{sign}(x-a)\sqrt{|x-a|}}$$

ii) secant method: $A = 1$ $a = 2$; $x_0 = 0$ $x_1 = 4$

$$f(x) = \text{sign}(x-2)\sqrt{|x-2|}$$

$$f(x_0) = \text{sign}(0-2)\sqrt{|0-2|} = -\sqrt{2}$$

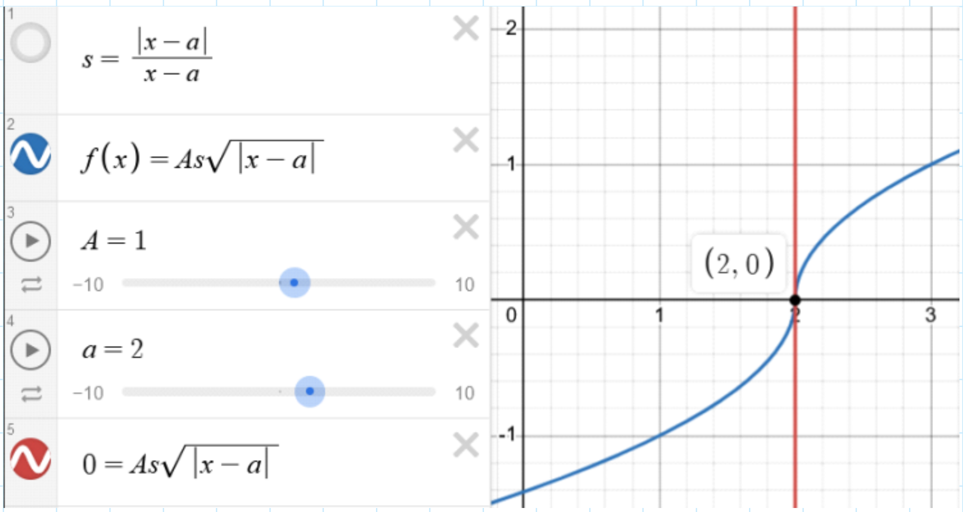
$$f(x_1) = \text{sign}(4-2)\sqrt{|4-2|} = \sqrt{2}$$

$f(x_1) = \text{sign}(4-2)\sqrt{|4-2|} = \sqrt{2}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{2}+\sqrt{2}}{4-0} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

$$y = mx + b \rightarrow -\sqrt{2} = \cancel{\frac{\sqrt{2}}{2}(0)} + b \rightarrow b = -\sqrt{2}$$

$$0 = \frac{\sqrt{2}}{2}x - \sqrt{2} \rightarrow \cancel{\sqrt{2}} = \frac{\cancel{\sqrt{2}}}{2}x \rightarrow 1 = \frac{1}{2}x \rightarrow x = 2$$




confirmation that $x=2$ is the correct solution

iii) $x_0 = 0 \quad x_1 = 3$

$$f(x) = \text{sign}(x-2)\sqrt{|x-2|}$$

$$f(x_0) = \text{sign}(0-2)\sqrt{|0-2|} = -\sqrt{2}$$

$$f(x_1) = \text{sign}(3-2)\sqrt{|3-2|} = 1$$

$$m = \frac{1+\sqrt{2}}{3}$$

$$-\sqrt{2} = \cancel{\left(\frac{1+\sqrt{2}}{3}\right)(0)} + b \qquad b = -\sqrt{2}$$

$$0 = \left(\frac{1+\sqrt{2}}{3}\right)(x) - \sqrt{2}$$

$$0 = \left(\frac{1+\sqrt{2}}{3}\right)(x) - \sqrt{2}$$

$$\sqrt{2} = \left(\frac{1+\sqrt{2}}{3}\right)x$$

$$x = \frac{\sqrt{2}}{\frac{1+\sqrt{2}}{3}} = 1.757$$

$$x_0 = x_1 = 3$$

$$x_1 = 1.757$$

$$f(x_0) = \text{sign}(3-2)\sqrt{|3-2|} = 1$$

$$f(x_1) = \text{sign}(1.757-2)\sqrt{|1.757-2|} = -.49$$

$$m = \frac{-.49-1}{1.757-3} = 1.201$$

$$1 = 1.201(3) + b$$

$$-2.603 = b$$

$$0 = 1.201x - 2.603$$

$$\frac{2.603}{1.201} = \frac{\cancel{1.201}x}{\cancel{1.201}} \rightarrow x = 2.167$$

$$x_0 = x_1 = 1.757$$

$$x_1 = 2.167$$

$$f(x_0) = \text{sign}(1.757-2)\sqrt{|1.757-2|} = -.493$$

$$f(x_1) = \text{sign}(2.167-2)\sqrt{|2.167-2|} = .409$$

$$m = \frac{.409+.493}{2.167-1.757} = 2.198$$

$$m = \frac{.409 + .493}{2.167 - 1.757} = 2.198$$

$$.409 = 2.198(2.167) + b$$

$$b = -4.357$$

$$0 = 2.198x - 4.357$$

$$x = 1.981$$

it appears to be converging, but slowly