PHYS 250 Homework 3

Sophie von Coelln

16 February 2025

1

1 i

$h = 10^{-4}$	1.0239231684749939e-10
$h = 10^{-5}$	5.146993942162226e-13
$h = 10^{-6}$	2.0949397772085376e-10
$h = 10^{-7}$	7.319420625151452e-10

1 ii

We seek to minimize $e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6}M$. We do this by taking its derivative and setting it equal to 0:

$$e'(h) = \frac{-\varepsilon}{h^2} + \frac{h}{3}M = 0 \tag{1}$$

and then solving for h:

$$h = \sqrt[3]{\frac{3\varepsilon}{M}} \tag{2}$$

1 iii

Our function is given by $f(x) = e^{\frac{-x}{4}}$ so the magnitude of its third derivative is $\frac{1}{64}e^{\frac{-x}{4}}$. The exponential portion of the third derivative is never greater than one, so the highest value of the third derivative is $\frac{1}{64}$. We use this as our estimate for M. Plugging $M = \frac{1}{64}$ and $\varepsilon \approx 10^{-16}$ into (2), we find

$$h = 2.68 \times 10^{-5} \tag{3}$$

which is consistent with what we found in (i).

2

2 i

The derivative of

$$f(x) = 3^x \sin(x) \tag{4}$$

is given by

$$f'(x) = 3^x \cos(x) + 3^x \sin(x) \ln(3) \tag{5}$$

where $f'(1.15) = 4.992$.

2 ii

The magnitude of the fractional error in center differencing is $1.4432899320127035 \times 10^{-14}$ and in forward differencing, it is $6.439293542825908 \times 10^{-14}$.

2 iii

To calculate the derivative to an accuracy of at least 10^{-12} requires **5 iterations of center differencing and 7 of forward differencing**.

2 iv

The forward differencing algorithm is

$$f'(x) = \frac{f(x+h) - f(x)}{h} \tag{6}$$

and the center differencing algorithm is

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} \tag{7}$$

(where the error terms have been ignored). We can see that center differencing needs to evaluate the function twice for every new value of h. Forward differencing, on the other hand, needs to evaluate the function twice (at $f(x+h)$ and $f(x)$) for the first iteration, but the value of $f(x)$ does not change, so after that, it only has to evaluate the function once for each new value of h. Thus, **center differencing requires $2n$ calls for n iterations, but forward differencing only requires $n+1$ calls**.

2 v

In (iii), we found that calculating the derivative to an accuracy of 10^{-12} requires 5 iterations in center differencing, and 7 iterations in forward differencing. While it may seem like center differencing is thus the better algorithm, if we are concerned about minimizing the number of function evaluations, we are better off choosing forward differencing here: **an accuracy of 10^{-12} requires 8 function calls with forward differencing and 10 with center differencing**.

2 vi

Center differencing has its minimum error of $1.3322676295501878 \times 10^{-15}$ at 6 iterations. Forward differencing has its minimum error of $9.43689570931383 \times 10^{-15}$ at 9 iterations.

2 vii

Mathematically we can take the limit as h $\to 0$, but numerically, computers cannot work in infinitesimally small values the way calculus does, thanks to limited precision. As h gets very small, the error caused by limited precision grows much more significant relative to h. This means that past a certain point, rounding error determines the step size more than any intentional manipulation, which negatively affects the algorithm's outcome. In the worst case scenario, h would be stored as exactly 0 instead of a very small value, causing a division by 0 error, so these algorithms have a finite ability to accurately determine the derivative.