hw 5

Thursday, February 27, 2025 12:36 PM

1) $\frac{\partial y}{\partial t} = -f(y,t)$

i) center differencing: $f'(x) = \frac{f(x+h) - f(x-h)}{2h}$

sub in to euler method:

$$f(y(t),t) = \frac{-y(t+\Delta t) + y(t-\Delta t)}{2\Delta t}$$

$$y(t+\Delta t) = y(t-\Delta t) - 2\Delta t f(y,t)$$

we need y_0 and y_1 → evaluate at t_2 using t_0 and t_1

define $t_n \equiv t_0 + n\Delta t \rightarrow t_{n+1} = t_n + \Delta t$

so starting from t_0, $t_1 = t_0 + \Delta t$?

euler step to $y(t_2) = y(t_0) - 2\Delta t\, f(y(t_1), t_1)$

or, in shorthand: $y_2 = y_0 - 2\Delta t\, f(y_1, t_1)$

generalized: $y_{n+1} = y_{n-1} - 2\Delta t\, f(y_n, t_n)$

ii) $\delta y_{n-1} = ?$ $\delta y_n = \beta \delta y_{n-1}$ $\beta = ?$

forward differencing has error $O(\Delta t) \rightarrow$ EM has $O(\Delta t^2)$

center diff has error $O(h^2) \rightarrow$ leap-frog has $O(\Delta t^3)$

$$y_{n+1} = y_{n-1} - 2\Delta t\, f(y_n, t_n)$$

rewrite including error:

$$y_{n+1,t} + \delta y_{n+1} = y_{n-1,t} + \delta y_{n-1} - 2\Delta t f(y_n + \delta y_n, t_n)$$

δy is small; taylor expand around $\delta y_n = 0$

$$f(y_{n,t} + \delta_n, t_n) = f(y_{n,t}, t_n) + \frac{\partial f}{\partial y}\Big|_{(y_{n,t}, t_n)} \delta y_n + \ldots$$

small! go away!

$$y_{n+1,t} + \delta y_{n+1} = y_{n-1,t} + \delta y_{n-1} - 2\Delta t\left[f(y_{n,t}, t_n) + \frac{\partial f}{\partial y}\Big|_n \delta y_n\right]$$

all involve true values

all have SMALL error

$y_{n+1,t} = y_{n,t} + \Delta t f(y_{n,t}, t_n)$ satisfies our expression → these cancel

this leaves us with $\delta y_{n+1} = \delta y_{n-1} - 2\Delta t \frac{\partial f}{\partial y}\Big| \delta y_n$

$y_{n+1,t} = y_{n,t} + \Delta t f(y_{n,t}, t_n)$ satisfies our expression $\rightarrow$ these cancel

this leaves us with $\delta y_{n+1} = \delta y_{n-1} - 2\Delta t \frac{\partial f}{\partial y}\Big|_n \underbrace{\delta y_n}_{\beta \delta y_{n-1}}$

$\delta y_n = \beta \delta y_{n-1}$

then $\delta y_{n+1} = \beta \delta y_n = \beta^2 \delta y_{n-1}$

$$\beta^2 \cancel{\delta y_{n-1}} = \cancel{\delta y_{n-1}} - 2\Delta t \frac{\partial f}{\partial y}\Big|_n \beta \cancel{\delta y_{n-1}}$$

$$\beta^2 = 1 - 2\Delta t \frac{\partial f}{\partial y}\Big|_n \beta$$

$$\beta^2 + 2\Delta t \frac{\partial f}{\partial y}\Big|_n - 1 = 0$$

quadratic formula:

$$\beta = \frac{-2\Delta t \frac{\partial f}{\partial y}\Big|_n \pm \sqrt{4\Delta t^2 \left(\frac{\partial f}{\partial y}\Big|_n\right) + 4}}{2}$$

$$\beta = -\Delta t \frac{\partial f}{\partial y}\Big|_n \pm \sqrt{\Delta t^2 \left(\frac{\partial f}{\partial y}\Big|_n\right) + 1}$$

iii) is the method stable?

$\delta y_{n+1} = \beta \delta y_n$ so error scales with β

the algorithm is stable if $|\beta| \leq 1$

$$-1 \leq -\Delta t \frac{\partial f}{\partial y}\Big|_n \pm \sqrt{\Delta t^2 \left(\frac{\partial f}{\partial y}\Big|_n\right)^2 + 1} \leq 1$$

$$-1 + \Delta t \frac{\partial f}{\partial y}\Big|_n \leq \pm \sqrt{\left(\Delta t \frac{\partial f}{\partial y}\Big|_n\right)^2 + 1}$$

square both sides:

$$\cancel{\left(\Delta t \frac{\partial f}{\partial y}\Big|_n\right)^2} - 2\Delta t \frac{\partial f}{\partial y}\Big|_n \cancel{+1} \leq \cancel{\left(\Delta t \frac{\partial f}{\partial y}\Big|_n\right)^2} \cancel{+1}$$

$-2\Delta t \frac{\partial f}{\partial y}\Big|_n \leq 0 \rightarrow$ true for positive derivative (growing)

$$\frac{\partial y}{\,}\Big|_n \geq 0 \rightarrow \text{true for positive derivative (growing)}$$

$$-\Delta t \frac{\partial f}{\partial y}\Big|_n \pm \sqrt{\Delta t^2\left(\frac{\partial f}{\partial y}\Big|_n\right)^2+1} \leq 1$$

$$\pm\sqrt{\left(\Delta t \ \frac{\partial f}{\partial y}\Big|_n\right)^2+1} \leq 1 + \Delta t \frac{\partial f}{\partial y}\Big|_n$$

square both sides

$$\cancel{\left(\Delta t \frac{\partial f}{\partial y}\Big|_n\right)^2} \cancel{+1} \leq \cancel{\left(\Delta t \frac{\partial f}{\partial y}\Big|_n\right)^2} + 2\Delta t \frac{\partial f}{\partial y}\Big|_n \cancel{+1}$$

$$2\Delta t \frac{\partial f}{\partial y}\Big|_n \geq 0 \rightarrow \text{true for positive derivative (growing)}$$

We now examine oscillating solutions:

$$f = \pm i\omega y \rightarrow \frac{\partial f}{\partial y} = \pm i\omega$$

to account for imaginary values, we require $|\beta|^2 \leq 1$

$$\beta = -\Delta t \frac{\partial f}{\partial y}\Big|_n \pm \sqrt{\Delta t^2\left(\frac{\partial f}{\partial y}\Big|_n\right)+1}$$

$$= -\Delta t\left(\pm i\omega\right) \pm \sqrt{\Delta t^2(\pm i\omega)^2+1}$$

$$|\beta|^2 = \beta^*\beta \rightarrow \left((\mp\Delta t i\omega) \pm \sqrt{-\Delta t^2\omega^2+1}\right)\left(\pm\Delta t i\omega \pm \sqrt{-\Delta t^2\omega^2+1}\right)$$

$$-\Delta t^2(-1)\omega^2 \pm (-\Delta t^2\omega^2+1)$$

$$\Delta t^2\omega^2 \mp \left[\Delta t^2\omega^2+1\right] = \begin{cases} - : -1 & \rightarrow \text{stable} \\ + : 2\Delta t^2\omega^2+1 & \rightarrow 2\Delta t^2\omega^2+1 \leq 1 \rightarrow \text{unstable} \\ & \quad 2\Delta t^2\omega^2 \leq 0 \end{cases}$$

stable for growing solutions and for oscillatory solutions given $\beta = -\Delta t \frac{\partial f}{\partial y}\Big|_n + \sqrt{\Delta t^2\left(\frac{\partial f}{\partial y}\Big|_n\right)+1}$ (the + case of the ±)

2)

```
def rk2(f, y0, t, args=()):
    y=np.zeros_like(t)
```

2)

```python
def rk2(f, y0, t, args=()):
    y=np.zeros_like(t)
    y[0]=y0
    delt=np.diff(t)
    for i in range(len(t)-1):
        k1 = delt[i]*f(t[i], y[i], *args)
        k2=delt[i]*f(t[i]+delt[i]/2, y[i]+k1/2, *args)
        y[i+1]=y[i]+k2
    return y

def rk4(f, y0, t, args=()):
    y=np.zeros_like(t)
    y[0]=y0
    delt=np.diff(t)
    for i in range(len(t)-1):
        k1 = delt[i]*f(t[i], y[i], *args)
        k2=delt[i]*f(t[i]+delt[i]/2, y[i]+k1/2, *args)
        k3=delt[i]*f(t[i]+delt[i]/2, y[i]+k2/2, *args)
        k4=delt[i]*f(t[i]+delt[i], y[i]+k3, *args)
        y[i+1] = y[i] + k1/6 + k2/3 + k3/3 + k4/6
    return y
```

3) $\frac{dy}{dt} + 12y = 6t^2 - t \qquad y(0) = \frac{1}{24}$

i) $y(t) = ?$

$$\frac{dy_h}{dt} + 12y_h = 0 \rightarrow \frac{dy_h}{dt} = -12y_h \rightarrow dy_h = -12y_h\,dt$$

$$\int \frac{1}{y_h}\,dy_h = \int -12\,dt \rightarrow \ln|y_h| = -12t \rightarrow y_h = c_1 e^{-12t}$$

$$y_p = \alpha_2 t^2 + \alpha_1 t + \alpha_0$$

$$\frac{dy_p}{dt} = 2\alpha_2 t + \alpha_1$$

$$2\alpha_2 t + \alpha_1 + 12\alpha_2 t^2 + 12\alpha_1 t + 12\alpha_0 = 6t^2 - t$$

$$12\alpha_2 t^2 + (2\alpha_2 + 12\alpha_1)t + (12\alpha_0 + \alpha_1) = 6t^2 - t$$

$$t^2 : 12\alpha_2 = 6 \rightarrow \alpha_2 = \frac{1}{2}$$

$$t : 2\alpha_2 + 12\alpha_1 = -1 \rightarrow 2\left(\tfrac{1}{2}\right) + 12\alpha_1 = -1 \rightarrow 12\alpha_1 = -2 \rightarrow \alpha_1 = -\frac{1}{6}$$

$$1 : 12\alpha_0 + \alpha_1 = 0 \rightarrow 12\alpha_0 - \frac{1}{6} = 0 \rightarrow 12\alpha_0 = \frac{1}{6} \rightarrow \alpha_0 = \frac{1}{72}$$

$$y_p = \frac{1}{2}t^2 - \frac{1}{6}t + \frac{1}{72}$$

$$1: 12\alpha_0 + \alpha_1 = 0 \rightarrow 12\alpha_0 - \frac{1}{6} = 0 \rightarrow 12\alpha_0 = \frac{1}{6} \rightarrow \alpha_0 = \frac{1}{72}$$

$$y_p = \frac{1}{2}t^2 - \frac{1}{6}t + \frac{1}{72}$$

$$y = \frac{1}{2}t^2 - \frac{1}{6}t + \frac{1}{72} + c_1 e^{-12t}$$

$$\frac{1}{24} = \cancel{\tfrac{1}{2}(0)^2}^{\,0} - \cancel{\tfrac{1}{6}(0)}^{\,0} + \frac{1}{72} + c_1 \cancel{e^0}^{\,1} \rightarrow \frac{1}{24} = \frac{1}{72} + c_1$$

$$c_1 = \frac{1}{36}$$

$$y = \frac{1}{2}t^2 - \frac{1}{6}t + \frac{1}{72} + \frac{1}{36}e^{-12t}$$

$$y(.2) = .0030755$$

ii, iii)

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2nd Order:
Approximated value: 0.003082226434529983
True value: 0.003075498702483683
Fractional error: 0.0021875255680865457

4th Order:
Approximated value: 0.003075500267447452
True value: 0.003075498702483683
Fractional error: 5.088487819993048e-07
```