1) i) $p(y) = \frac{N}{y} \quad y \in [1, a)$

we first find the normalization constant for this distribution:

$$\int_1^a \frac{N}{y} dy = 1 \rightarrow N\left[\ln|y|\right]_1^a = 1 \rightarrow N\ln|a| - \cancel{N\ln|1|}^{0} = 1$$

$N\ln|a| = 1$ (the range is from 1 to a, so we take a to be positive)

$$N = \frac{1}{\ln(a)}$$

we then map our uniform distribution to our desired one:

our uniform distribution is given by

$$\int_0^u p_u(u')du' = u$$

we want the area under the two curves to be the same:

$$\int_0^u p_u(u')\,du' = u = \int_1^y p(y')\,dy = \int_1^y \frac{1}{y'\ln(a)}\,dy'$$

$$= \frac{\ln|y'|}{\ln(a)}\Bigg|_1^y = \frac{\ln|y|}{\ln(a)} - \cancel{\frac{\ln|1|}{\ln(a)}}^{0} = u$$

$$u\ln(a) = \ln|y| \rightarrow y = e^{u\ln(a)} \rightarrow y = a^u$$

ii) $p(y) = \frac{N}{y^2} \quad y \in [1, a)$

we again begin by normalizing:

$$\int_1^a \frac{N}{y^2} dy = \int_1^a N y^{-2} dy = 1 \rightarrow N\left[-y^{-1}\right]_1^a = 1$$

$$\frac{-N}{y}\Bigg|_1^a = \frac{-N}{a} + \frac{N}{1} = N\left(1 - \frac{1}{a}\right) = 1 \rightarrow N = \frac{1}{1 - \frac{1}{a}} = \frac{a}{a-1}$$

the next step is mapping:

$$\int_0^u p_u(u')\,\partial u' = u = \int_1^y p(y')\,dy' = \int_1^y \left(\frac{a}{a-1}\right)\left(\frac{1}{y'^2}\right) dy' = \frac{a}{a-1}\int_1^y y'^{-2}\,dy$$

$$\frac{a}{a-1}\left[\frac{-1}{y'}\right]_1^y = \frac{a}{a-1}\left(\frac{-1}{y}+1\right) = u \rightarrow \frac{u(a-1)}{a} - 1 = \frac{-1}{y}$$

$$\frac{ua-u-a}{a} = \frac{-1}{y} \rightarrow y = \frac{a}{u+a-ua}$$

2) ∂V in spherical coordinates is given by

$$\partial V = r^2 \sin(\theta)\,\partial r\,\partial\theta\,\partial\varphi$$

a uniform distribution in a sphere means that the probability of any given point having

$$0 \le r' < r \qquad 0 \le \theta' < \theta \qquad 0 \le \varphi' < \varphi$$

is given by

$$\int_0^r r'^2\,\partial r' \int_0^\theta \sin(\theta')\partial\theta' \int_0^\varphi \partial\varphi' = 1$$

we first examine φ, which we see is already uniformly distributed. we know in spherical coordinates $0 \le \varphi < 2\pi$, but $0 \le x_2 < 1$, so we can adjust the range by multiplying:

$$\varphi = 2\pi x_3$$

we next turn to θ, which we transform. we again begin by normalizing:

$$\int_0^\pi N\sin(\theta)\partial\theta = 1 \rightarrow -N\cos(\theta)\Big|_0^\pi = 1 \rightarrow -N[-1-1] = 1 \rightarrow 2N = 1 \rightarrow N = \tfrac{1}{2}$$

mapping:

$$\int_0^u p_u(u')\,\partial u' = u = \int_0^\theta \frac{1}{2}\sin(\theta')\,\partial\theta' = \frac{-1}{2}\left[\cos(\theta')\right]_0^\theta = \frac{-1}{2}\left(\cos(\theta)-1\right) = u$$

$$-2u = \cos(\theta) - 1 \rightarrow \theta = \arccos(1-2u) \rightarrow \theta = \arccos(1-2x_2)$$

$$-2u = \cos(\theta) - 1 \rightarrow \theta = \arccos(1-2u) \rightarrow \theta = \arccos(1-2x_2)$$

finally, we transform x_1 to r. we once again begin by normalizing:

$$\int_0^1 N r^2 \partial r = 1 \rightarrow \frac{N}{3} r^3 \Big|_0^1 = 1 \rightarrow \frac{N}{3} = 1 \rightarrow N = 3$$

$\leftarrow$ unit sphere

mapping:

$$\int_0^r 3r'^2 \partial r' = x_1 \rightarrow r'^3 \Big|_0^r = x_1 \rightarrow r^3 = x_1 \rightarrow r = \sqrt[3]{x_1}$$