hw 10

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1) $h(t) \Leftrightarrow H(f)$

i) $t \to at$

$$h(at) \Leftrightarrow \frac{1}{|a|} H\left(\frac{f}{a}\right)$$

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} \, dt$$ ← definition of FT

$$\int_{-\infty}^{\infty} h(at) e^{-2\pi i f t} \, dt$$

we u-substitute $u = at$

$$\frac{1}{|a|} \int_{-\infty}^{\infty} h(u) e^{-2\pi i f \left(\frac{u}{a}\right)} \, du$$

which is the same as the definition, except for an $\frac{f}{a}$ frequency component

$$\boxed{\frac{1}{|a|} H\left(\frac{f}{a}\right) = \frac{1}{|a|} \int_{-\infty}^{\infty} h(at) e^{-2\pi i f t} \, dt}$$

ii) $t \to t - t_0$

$$\int_{-\infty}^{\infty} h(t - t_0) e^{-2\pi i f t} \, dt$$

$u = t - t_0 \to du = dt$

$$\int_{-\infty}^{\infty} h(u) e^{-2\pi i f (u + t_0)} \, du$$

$$\int_{-\infty}^{\infty} h(u) e^{-2\pi i f u} e^{-2\pi i f t_0} \, du$$

$$e^{-2\pi i f t_0} \int_{-\infty}^{\infty} h(u) e^{-2\pi i f u} \, du$$

$$e^{-2\pi i f t_0} \underbrace{\int_{-\infty}^{\infty} h(t - t_0) e^{-2\pi i f (t - t_0)} \, dt}_{H(f)}$$

$$h(t - t_0) \Leftrightarrow e^{-2\pi i f t_0} H(f)$$

iii) $h(t) = [h(t)]^*$

negative omitted for complex conjugate reasons

$$\int_{-\infty}^{\infty} [h(t)]^* e^{2\pi i f t} \, dt$$

so then we have

$$H^*(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} \, dt$$

for the equation to look how we need it to, we need

$$H^*(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i (-f) t} \, dt$$

so then

$$H^*(f) = H(-f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i (-f) t} \, dt$$

iv) $h(-t) = h(t)$

$$\underbrace{\int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} \, dt}_{H(f)} = \int_{-\infty}^{\infty} h(-t) e^{-2\pi i f (-t)} \, dt$$

we know this is equivalent to $h(t)$ — substituting that

$H(f)$

we know this is equivalent to $h(t)$ — substituting that in yields what we found for $H(-f)$ in (iii)

$$H(f) = H(-f)$$

2) $(g * h)_k \Leftrightarrow G_n H_n$

$$(g*h)_k \equiv \sum_{k'=0}^{N-1} g_{k-k'} \, h_{k'}$$

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i (n-n') k/N} = \delta_{nn'}$$

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the fourier transform of $(g*h)_k$ is given by

$$\sum_{k=0}^{N-1} h_{k'} \sum_{k=0}^{N-1} g_{k-k'} \, e^{-2\pi i k n/N}$$

and using inverse fourier transforms, we can find that

$$h_{k'} = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{2\pi i k n/N} \qquad g_{k-k'} = \frac{1}{N} \sum_{n=0}^{N-1} G_n e^{2\pi i k (n-n')/N}$$

this allows us to write the FT of $(g*h)_k$ as

$$\sum_{k'=0}^{N-1} \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{2\pi i k n/N} \sum_{k=0}^{N-1} G_n \underbrace{e^{2\pi i k (n-n')/N}}_{\delta_{nn'}} e^{-2\pi i k n/N}$$

we see the discrete completness relation emerge in that expression, and can use this to rewrite it as

$$\sum_{k'=0}^{N-1} \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{2\pi i k n/N} \sum_{k=0}^{N-1} \delta_{nn'} G_n e^{-2\pi i k n/N}$$

cancelling the exponential terms reduces this to

$$\sum_{k'=0}^{N-1} \frac{1}{N} \sum_{n=0}^{N-1} H_n \sum_{k=0}^{N-1} \delta_{nn'} G_n$$

$$\sum_{k'=0}^{N-1} \frac{1}{N} \sum_{n=0}^{N-1} H_n \sum_{k=0}^{N-1} \delta_{nn'} G_n$$

the kroniker delta tells us that

$$\delta_{kk'} = \begin{cases} 0, & k \neq k' \\ 1, & k = k' \end{cases}$$

so the above expression goes to 0 in the case that $n' \neq n$. we can now simplify the expression to

$$\frac{1}{N} \sum_{n=0}^{N-1} H_n G_n$$

the sum and $\frac{1}{N}$ term cancel, and we find that

$$(g*h)_k = H_n G_n$$

3) h_k sampled at N values; H_n is its DFT

i) $g_{k'} = \begin{cases} h_{k'}, & k' < N \\ 0, & N \leq k' < 2N \end{cases}$ $\qquad G_{2n} = H_n$

the discrete FT of $h_{k'}$ is given by

$$H_n = \sum_{k'=0}^{N-1} h_{k'} e^{-2\pi i n k'/N}$$

the DFT of $g_{k'}$ is given by

$$G_n = \sum_{k'=0}^{2N-1} g_{k'} e^{-2\pi i n k'/(2N)}$$

where we have substituted $2N$ in for N as needed to account for a signal of length $2N$

we can rewrite this as

$$G_{2n} = \sum_{k'=0}^{2N-1} g_{k'} e^{-2\pi i (\cancel{2}n) k'/\cancel{2}N}$$

which simplifies to

$$G_{2n} = \sum_{k'=0}^{2N-1} g_{k'} e^{-2\pi i n k'/N}$$

the argument of the sum is now identical to that in our expression for H

$G_{2n} \;\; \sum_{k'=0} g_{k'}$

the argument of the sum is now identical to that in our expression for H_n. we know also that $g_{k'} = 0$ for all $k' \geq N$, so those terms contribute nothing to the sum; only the terms from 0 to $N-1$ contribute. we can rewrite the sum to reflect this:

$$G_{2n} = \sum_{k'=0}^{N-1} g_{k'} e^{-2\pi i n k'/N} = H_n$$

so then

$$G_{2n} = H_n$$

ii) $g_{k'} = h_{2k'}$ for $k' = 0, 1, \ldots, \frac{N}{2} - 1 \longrightarrow G_m = \frac{1}{2}(H_m + H_{m+\frac{N}{2}})$

we begin by taking the DFT of $g_{k'}$ (and thereby of $h_{2k'}$):

$$G_m = \sum_{k'=0}^{\frac{N}{2}-1} h_{2k'} e^{-2\pi i m (2k')/(\frac{N}{2})}$$

since we are only summing over even values, we can rewrite this with that in mind for our N and k values (where $k = 2k'$)

$$G_m = \sum_{k=0}^{N-1} h_k e^{-2\pi i m k/N}$$

we can do the same thing but for the odd values:

$$G_{m,o} = \sum_{k=0}^{N-1} h_k e^{-2\pi i m k/N}$$

combining the even and odd expressions for G_m gives us H_m, since it includes both the even and odd values:

$$H_m = G_m + G_{m,o} = \underbrace{\sum_{k=0}^{N-1}}_{\text{even}} h_k e^{-2\pi i m k/N} + \underbrace{\sum_{k=0}^{N-1}}_{\text{odd}} h_k e^{-2\pi i m k/N}$$

substituting in $m + N/2$ for m in H_m, we find

$$H_{m+\frac{N}{2}} = \sum_{k=0}^{N-1} h_k e^{-2\pi i (m+\frac{N}{2}) k/N} = \sum_{k=0}^{N-1} h_k e^{-2\pi i m k/N} \underbrace{e^{-\pi i k}}_{(-1)^{-k} \text{ via euler's identity}}$$

$H_{m+\frac{N}{2}} \quad \sum_{k=0} h_k e \qquad \sum_{k=0} h_k e$

$(-1)^{-k}$ via euler's identity

we split this into even and odd cases, where $(-1)^{-k} = 1$ and $= -1$ respectively:

$$H_{m+\frac{N}{2}} = \sum_{k=0}^{N-1} h_k e^{-2\pi i m k/N} \quad \leftarrow \text{even case}$$

$$H_{m+\frac{N}{2}} = \sum_{k=0}^{N-1} -h_k e^{-2\pi i m k/N} \quad \leftarrow \text{odd case}$$

so we find that $H_{m+\frac{N}{2},even} = G_{m,even}$ and $H_{m+\frac{N}{2},odd} = -G_{m,odd}$. we now add H_m and $H_{m+\frac{N}{2}}$:

$$H_m + H_{m+\frac{N}{2}} = G_{m,even} + \cancel{G_{m,odd}} + G_{m,even} - \cancel{G_{m,odd}}$$

$$H_m + H_{m+\frac{N}{2}} = 2G_{m,even}$$

however, we only want the even values, so $G_{m,even}$ is simply G_m. we can then find that

$$2G_m = H_m + H_{m+\frac{N}{2}}$$

or,

$$G_m = \frac{1}{2}\left(H_m + H_{m+\frac{N}{2}}\right)$$