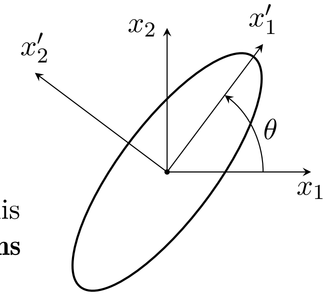
# PHYS 250 Homework 8

**Due:** 7 April 2025




**Problem 1.** (20 points) Consider the ellipse given by

$$14x_1^2 - 4x_1x_2 + 11x_2^2 = 25.$$

This ellipse is shown at the right. We want to find the principle axes of this ellipse represented by $x_1'$ and $x_2'$ in the figure. In this problem **all calculations can and must be done by hand** unless otherwise noted.

**(i)** We may write the equation for the ellipse as a matrix equation of the form

$$\boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 25.$$

Notice that I have written $\mathsf{A}$ as a *symmetric* matrix. Write down the matrix $\mathsf{A}$.

**(ii)** Calculate the eigenvalues, $\lambda_n$, of $\mathsf{A}$. Clearly show how you calculate the eigenvalues. [*Note*: Despite how it may appear, the coefficients of the ellipse have been chosen to lead to simple eigenvalues.]

**(iii)** Calculate the eigenvectors, $\boldsymbol{v}_n$, of $\mathsf{A}$. Remember that eigenvectors are normalized so that $|\boldsymbol{v}_n| = 1$. [*Note*: Again, the coefficients have been chosen so that the eigenvectors will have "simple" values.]

**(iv)** Construct $\mathsf{B}$ from the eigenvectors as the columns in this matrix. Since $\mathsf{A}$ is symmetric, $\mathsf{B}$ should be orthogonal. This means $\mathsf{B}^{-1} = \mathsf{B}^{\mathsf{T}}$. To show this calculate the product $\mathsf{B}^{\mathsf{T}}\mathsf{B}$. [*Note*: What should the answer be? Make sure your answer agrees with this.]

**(v)** Show that $\mathsf{B}$ diagonalizes the matrix $\mathsf{A}$. To do this, calculate the product $\mathsf{B}^{\mathsf{T}}\mathsf{A}\mathsf{B}$. [*Note*: What diagonal matrix should this product produce? Make sure your answer agrees with this.]

**(vi)** The matrix $\mathsf{B}$ is a rotation matrix that transforms between the two coordinate systems shown in the figure. Recall that $\boldsymbol{x}' = \mathsf{B}^{\mathsf{T}}\boldsymbol{x}$. Determine the angle, $\theta$, between the original $x_1$-axis and the new $x_1'$-axis.

**(vii)** Finally, write the ellipse in standard form in terms of the principle axes

$$\left(\frac{x_1'}{\alpha_1}\right)^2 + \left(\frac{x_2'}{\alpha_2}\right)^2 = 1.$$

In other words, determine $\alpha_1$ and $\alpha_2$.

**Problem 2.** (10 points) [**Ill conditioned matrices.**] Even when a matrix is not singular it can still be difficult to work with numerically. The condition number of a matrix, $\mathsf{A}$, is denoted by $\kappa(\mathsf{A})$. If $\kappa$ is large we call the matrix *ill conditioned*. Roughly, if the condition number is of the form $\kappa(\mathsf{A}) \sim 10^k$ then we expect to lose up to $k$ digits of accuracy in addition to the normal loss of accuracy in an algorithm involving $\mathsf{A}$. We will calculate the condition number as[†]

$$\kappa(\mathsf{A}) = \frac{\max\left(|\lambda(\mathsf{A})|\right)}{\min\left(|\lambda(\mathsf{A})|\right)},$$

[†]This is not the most general definition but is sufficient for our purposes.

where $\lambda(\mathsf{A})$ are the eigenvalues of $\mathsf{A}$. The Hilbert matrix is an example of a non-singular but ill conditioned matrix. It has elements $A_{ij} = 1/(i + j + 1)$ so that $A_{00} = 1$, $A_{01} = 1/2$, *etc.* In python we may easily construct this matrix by hand but we do not need to! We can instead use `scipy.linalg.hilbert()`. [*Note*: A number of other, standard matrices are defined too.]

**(i)** Calculate the condition number $\kappa(A)$ for the $10 \times 10$ Hilbert matrix. Since $\mathsf{A}$ is symmetric you should use `scipy.linalg.eigh` for this purpose.

**(ii)** Calculate the inverse of $\mathsf{A}$ and determine the maximum absolute error in $\mathsf{A}^{-1}\mathsf{A} - \mathsf{I}$.

**(iii)** An alternative way to calculate the inverse is to use the matrix of eigenvectors, $\mathsf{B}$. Since $\mathsf{A}$ is symmetric we know that $\mathsf{B}^\mathsf{T}\mathsf{A}\mathsf{B} = \mathsf{D}$ is the diagonal matrix of the eigvenvalues. With this we may calculate the inverse as

$$\mathsf{A}^{-1} = \mathsf{B}\mathsf{D}^{-1}\mathsf{B}^\mathsf{T}.$$

Using this expression repeat the previous part. We should find in these two parts that the maximum error is huge compared to the expected numerical error. [*Hint*: Calculating the inverse of a diagonal matrix is very easy to do analytically. Do not calculate it numerically!]