P250 Homework 4 Solutions

1) i) $E = \frac{1}{2} m l^2 \dot\theta^2 + mgl(1-\cos\theta)$

At its maximum, $\theta = \theta_m$, the pendulum is at rest, $\dot\theta = 0$, so conservation of energy gives

$$E(\theta_m) = \cancel{m}g\cancel{l}(\cancel{1}-\cos\theta_m) = E(\theta) = \tfrac{1}{2}\cancel{m}l^{\cancel{2}}\dot\theta^2 + \cancel{m}g\cancel{l}(\cancel{1}-\cos\theta)$$

$$\Rightarrow \tfrac{1}{2} l \dot\theta^2 = g(\cos\theta - \cos\theta_m) \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_m}.$$

~~Sen~~ Since the pendulum travels a time $T/4$ for $\theta \to 0$ to θ_m we can integrate to find

$$\int_0^{T/4} dt = T/4 = \int_0^{\theta_m} \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_m}}.$$

Using $T_0 = 2\pi\sqrt{l/g}$ we find

$$T(\theta_m) = {}^2\cancel{4}\frac{1}{\sqrt{2}}\left(\frac{T_0}{2\pi}\right)\int_0^{\theta_m}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_m}} \Rightarrow \boxed{\frac{T(\theta_m)}{T_0} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_m}}}$$

ii) Consider the transformation $\cos\theta = 1 - 2\sin^2(\theta/2)$.

Then the term in the denominator is

$$\sqrt{\cos\theta - \cos\theta_m} = \sqrt{1 - 2\sin^2(\tfrac{\theta}{2}) - 1 + 2\sin^2(\theta_m/2)} = \sqrt{2}\sqrt{\sin^2(\theta_m/2) - \sin^2(\theta/2)}$$

Since θ does not change this is the only effect of the transformation.

So we have $$\boxed{\frac{T(\theta_m)}{T_0} = \frac{1}{\pi}\int_0^{\theta_m}\frac{d\theta}{\sqrt{\sin^2(\theta_m/2) - \sin^2(\theta/2)}}}$$

iii) Now consider the substitution $\sin\varphi = \frac{\sin(\theta/2)}{\sin(\theta_m/2)} \Rightarrow \sin(\frac{\theta}{2}) = \sin(\frac{\theta_m}{2})\sin\varphi$.

The interval transforms to $\underline{\theta = 0} \Rightarrow \sin\varphi = 0 \Rightarrow \underline{\varphi = 0}$,

$\underline{\theta = \theta_m} \Rightarrow \sin\varphi = 1 \Rightarrow \underline{\varphi = \pi/2}$

Also ~~the~~ we have the relation from the derivative:

$$\tfrac{1}{2}\cos(\tfrac{\theta}{2})\, d\theta = \sin(\tfrac{\theta_m}{2})\cos\varphi\, d\varphi$$

Notice $\cos(\frac{\theta}{2}) = \sqrt{1 - \sin^2(\frac{\theta}{2})} = \sqrt{1 - \sin^2(\frac{\theta_m}{2})\sin^2\varphi}$

Thus $$d\theta = \frac{2\sin(\frac{\theta_m}{2})\cos\varphi}{\sqrt{1 - \sin^2(\frac{\theta_m}{2})\sin^2\varphi}}\, d\varphi$$

iii) con't

Finally the denominator from the previous part becomes

$$\sqrt{\sin^2\left(\frac{\theta_m}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} = \sqrt{\sin^2\left(\frac{\theta_m}{2}\right)\left(1 - \sin^2\phi\right)} = \sin\left(\frac{\theta_m}{2}\right)\cos\phi$$

Putting this all together we have

$$\frac{T(\theta_m)}{T_0} = \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{\cancel{\sin\left(\frac{\theta_m}{2}\right)\cos\phi}} \frac{2\,\cancel{\sin\left(\frac{\theta_m}{2}\right)\cos\phi}}{\sqrt{1-\sin^2\left(\frac{\theta_m}{2}\right)\sin^2\phi}}\, d\phi$$

$$\Rightarrow \boxed{\frac{T(\theta_m)}{T_0} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2\left(\frac{\theta_m}{2}\right)\sin^2\phi}}}$$

iv) When $\theta_m = \pi$ the pendulum is vertical. This is an unstable equilibrium point. Classically the pendulum can stay here forever, so it is not surprising that $T(\pi) \to \infty$. This is a singularity we cannot get rid of.

v) To find the expansion we begin with $\frac{1}{\sqrt{1-x^2}}$ where we have $x \equiv \sin\left(\frac{\theta_m}{2}\right)\sin\phi$.

Performing a Taylor expansion we find

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \dots = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^n n!} x^{2n}.$$

This gives $$\frac{T(\theta_m)}{T_0} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 + \frac{1}{2}\sin^2\left(\frac{\theta_m}{2}\right)\sin^2\phi + \frac{3}{8}\sin^4\left(\frac{\theta_m}{2}\right)\sin^4\phi + \dots\, d\phi$$

So we need $\int_0^{\frac{\pi}{2}} \sin^{2n}\phi\, d\phi$.

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\phi\, d\phi = \underset{n=0}{\frac{\pi}{2}},\ \underset{n=1}{\frac{\pi}{4}},\ \underset{n=2}{\frac{3\pi}{16}},\ \underset{n=3}{\frac{5\pi}{32}},\ \underset{n=4}{\frac{35\pi}{256}},\ \dots,\ \frac{\pi}{2}\left[\frac{(2n)!}{4^n(n!)^2}\right]$$

Plugging back in this gives

$$\boxed{\frac{T(\theta_m)}{T_0} = 1 + \frac{1}{4}\sin^2\left(\frac{\theta_m}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_m}{2}\right) + \frac{25}{256}\sin^6\left(\frac{\theta_m}{2}\right) + \frac{1225}{16384}\sin^8\left(\frac{\theta_m}{2}\right) + \dots}$$

or in general $$\boxed{\frac{T(\theta_m)}{T_0} = \sum_{n=0}^{\infty}\left[\frac{(2n)!}{4^n(n!)^2}\right]^2 \sin^{2n}\left(\frac{\theta_m}{2}\right)}$$

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2) i) For Newton-Cotes integration with $n=2$ we need to evaluate

$$a_0 = \int_{x_0}^{x_2} L_{2,0}(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx = \frac{1}{2h^2}\left[\frac{1}{3}x^3 - \frac{1}{2}(x_1+x_2)x^2 + x_1 x_2 x\right]_{x_0}^{x}$$

$$= \frac{(x_0-x_2)(2x_0-3x_1+x_2)}{6h} = \frac{(-2h)}{6h}\left[2\cancel{x_0} - 3\cancel{x_0} - 3h + \cancel{x_0} + 2h\right] = \frac{h}{3}.$$

So $\boxed{a_0 = h/3}$ Here I have used $x_2 = x_0 + 2h$, $x_1 = x_0 + h$.

Similarly, $\underline{a_1} = \int_{x_0}^{x_2} L_{2,1}(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx = \cdots = \underline{\frac{4}{3}h}$

$\underline{a_2} = \int_{x_0}^{x_2} L_{2,2}(x)\,dx = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx = \cdots = \underline{\frac{1}{3}h}.$

Thus Simpson's rule is

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}$$ for equally spaced points

ii) The composite Simpson's rule makes sense for N even.
The simplest case is $N=4$ Here we have $h = \frac{b-a}{4}$, $x_j = a + jh$.

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{h}{3}\left[f(x_0) + f(x_4) + 2f(x_2) + 4f(x_1) + 4f(x_3)\right].$$

We could subdivide again to confirm the pattern if desired.
From this we see that

$$\boxed{I = \frac{h}{3}\left[f(x_0) + f(x_N) + 2\sum_{j=1}^{\frac{N}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{N/2} f(x_{2j-1})\right]}$$

2 con't

iii) $I = \int_0^4 e^x dx = e^x \Big|_0^4 = e^4 - 1 = 53.598...$

Trapezoid: $\boxed{I = \frac{b-a}{2}\left[f(a) + f(b)\right] = 111.96 \qquad \text{Error} = 57.598 = 107\%}$

A horrible estimate!

Simpson's: $\boxed{I = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right] = 56.770 \qquad \text{Error} = 3.171 = 5.9\%}$

where $h = \frac{b-a}{2}$

Much better.

iv): $\boxed{\text{Composite trapezoid: } h = \frac{b-a}{2}, \quad I = 70.376, \quad \text{Error} = 16.778 = 31.3\%}$

$\boxed{\text{Composite Simpson: } h = \frac{b-a}{4}, \quad I = 53.864, \quad \text{Error} = 0.266 = 0.5\%}$

Overall the Simpson's rule does MUCH better.