PHYS 250 Homework 5

3 March 2025

Problem 1. (10 points) The Leap-Frog method for solving the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -f(y,t)$$

is derived using center differencing to approximate the derivative instead of using forward differencing (as we discussed in class for the Euler method).

(i) Derive the leap-frog approximation to y_{n+1}. Notice that to use the leap-frog method we need the value of $y(t)$ at two steps, y_0 and y_1, not just one.

(ii) Derive an estimate for the error δy_{n+1} for the leap-frog method. To do so use the fact that $\delta y_n = \beta \delta y_{n-1}$ and find an expression for the constant, β.

(iii) Determine whether the method is stable or unstable when applied to decaying, growing, and oscillating differential equations. If it is stable determine what condition must be satisfied for stability.

Problem 2. (10 points) Write python functions that implement the second and fourth order Runge-Kutta algorithm to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y).$$

The functions should be of the form `rk2(f,y0,t,args=())` (and similarly for `rk4`) where the function, `f`, is called as `f(t,y,*args)`. Here `t` is the *array of times* at which to calculate $y(t)$. You may assume that these functions will only be called for a single differential equation (that is, `f()` returns a single number, not an array). The functions should return the array of y-values evaluated at the specified times. You will use these functions in the next problem. Include the code for the functions in your solutions.

Problem 3. (10 points) Consider the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 12y = 6t^2 - t,$$

with the initial condition $y(0) = 1/24$.

(i) Find the analytic solution for $y(t)$. In case you have forgotten how to solve such differential equations one way to proceed is as follows. (You can use your own approach if you prefer.) First find the general *homogeneous* solution, $y_h(t)$, that is the solution to the differential equation

$$\frac{\mathrm{d}y_h}{\mathrm{d}t} + 12y_h = 0.$$

Next find the *particular* solution, $y_p(t)$, to the original differential equation. Here we can make a good guess at the form for this solution,

$$y_p(t) = \alpha_2 t^2 + \alpha_1 t + \alpha_0,$$

and find the values of the constants. Finally the complete solution is

$$y(t) = y_h(t) + y_p(t)$$

and we can now apply the initial condition to find the remaining constant.

(ii) Use your `rk2` function to solve the differential equation with $\Delta t = 0.005$. What value do you find for $y(0.2\text{ s})$ and what is its error?

(iii) Repeat the previous part using your `rk4` function.