Using Circles to Approximate Riemann Sums

Evan Derby

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1 Introduction

Riemann sums are a method of approximating the area underneath a curve by adding up the areas of shapes, often rectangles, whose height is restricted to a point on the function. As the width Δx of these shapes approaches zero, the summation approaches the true area underneath the curve.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

Normally, Riemann sums can be approximated three different ways, all of which depend on where the rectangle is being defined to intercept the curve. When the rectangle's top right vertex is bounded by the curve, this is known as the Right-hand Rectangular Approximation Method (RRAM). The same goes for a left-bounded rectangle (LRAM), and a rectangle whose midpoint intercepts the curve (MRAM). The formula changes slightly for each method, denoted by x_i^*. For LRAM, $x_i^* = x_{i-1}$; RRAM, $x_i^* = x_i$; and MRAM, $x_i^* = \frac{1}{2}(x_i + x_{i-1})$.

Riemann sums are not limited to rectangles, however. Another, more accurate method of approximation makes use of trapezoids, whose area formula is comparably simple to that of a rectangle's. Trapezoidal approximation is equal to the result of averaging a left and right approximation. This method makes use of the trapezoid area formula $A = \frac{1}{2}h(b_1 + b_2)$. A Riemann sum constructed with trapezoids takes this form:

$$\lim_{n\to\infty}\frac{1}{2}\Delta x\sum_{i=1}^{n} f(a + i\Delta x)$$

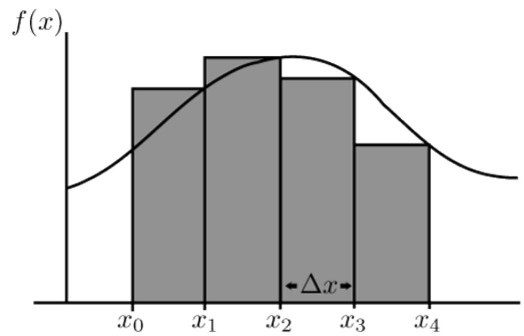

Figure 1: A function with Right-hand Riemann sum rectangles drawn underneath.

Historically, Riemann's definition was the first rigourous definition of the integral of a function on an interval.[1] Thus, in Calculus classes today, it is often the first introduction students recieve to the idea of calculating the area under a curve; the AP Calculus Syllabus even specifies Riemann sums as the first material in the Integrals unit.[1] The IB Math SL textbook follows a similar pattern, with Upper and Lower rectangles giving a brief introduction to the idea of area under a curve.[2]

As a student who was introduced to integration with Riemann sums, I often wondered: are there other shapes that you could use or approximations similar to Riemann's that would behave similarly? In this paper I intend to investigate just that: are circles a viable shape to use in the fashion of a Riemann sum? How accurate is such an approximation? Does this approximation behave like a Riemann sum in that its limit approaches the real area under the curve?

In this investigation, I'll derive this approximation (hereafter called the circle sum) in a similar method to how Riemann sums can be derived. I'll then test each of my questions. I'll compare the efficacy of a circle sum versus a similar Riemann sum, as well as examine the behavior of the circle sum as its number of partitions approaches infinity.

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[1] http://media.collegeboard.com/digitalServices/pdf/ap/ap-calculus-course-description.pdf

[2] https://www.tacoma.k12.wa.us/foss/IB%20Guides/IB-MathSLGuide2014.pdf

2 Derivation

The Riemann sum can be derived by finding the area of a shape under the curve, summing all such shapes between two bounds, and finally taking the limit of this summation as the number of rectangles approaches infinity. In order to find a circle sum, a similar process will be followed.

2.1 Riemann

We begin with a function $f(x)$ and a set of bounds a and b. The width of the workable area is defined as $b - a$. Thus, we can divide the workable area into n partitions, each with a width of $\Delta x = \frac{b-a}{n}$. The right-bound x-value for any such rectangle inside of these bounds can be bound by adding the partition width n times for the nth partition. Such a rectangle on the curve has width Δx and height $f(a + n_i \Delta x)$. Thus, the area of this rectangle is $f(a + n_i \Delta x)\Delta x$. This rectangle would appear below the function in the manner shown in Figure 2.

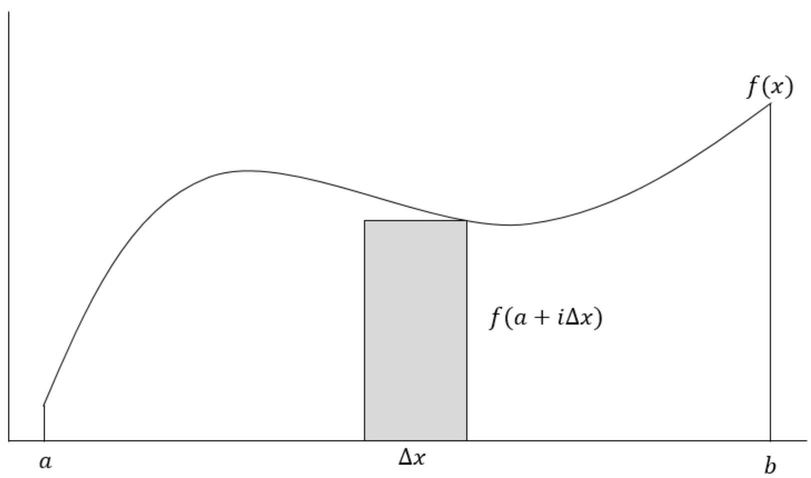


Figure 2: A Riemann-style rectangle placed below a curve.

The sum of the rectangles that exist from a to b can be expressed with summation notation:

$$A_{total} = f(a + (0)\Delta x)\Delta x + f(a + (1)\Delta x)\Delta x + \ldots + f(a + n\Delta x)\Delta x$$

$$= \sum_{i=0}^{n} f(a + i\Delta x)\Delta x$$

A_{total} is represented by the shaded green areas in Figure 3.

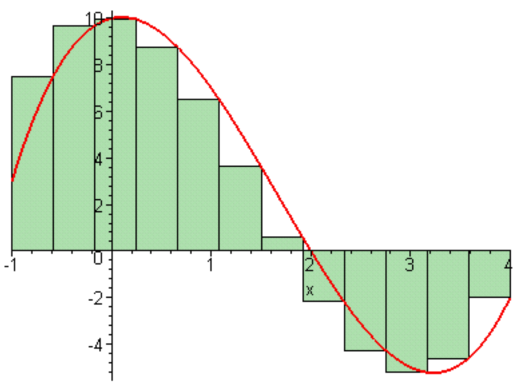


Figure 3: A function with Riemann rectangles placed underneath it.

When we take the limit of A_{total} as the number of partitions, n, approaches infinity, we find that A_{total} approaches A_{real}, given by integrating the function from a to b.

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(a + i\Delta x)\Delta x = \int_{a}^{b} f(x)dx$$

2.2 Circle

There are two concievable methods for using circles to determine the area underneath a function. Both will be examined here.

2.2.1 Filled Circles

The Filled Circle method finds the largest possible circle which is tangential to the curve at a given x and sums the areas of the circles that do not overlap and which fit inside the bounds.

However, the Filled Circle method has some failings. Because circles cannot overlap, they cannot appropriatley grow to fit the area underneath

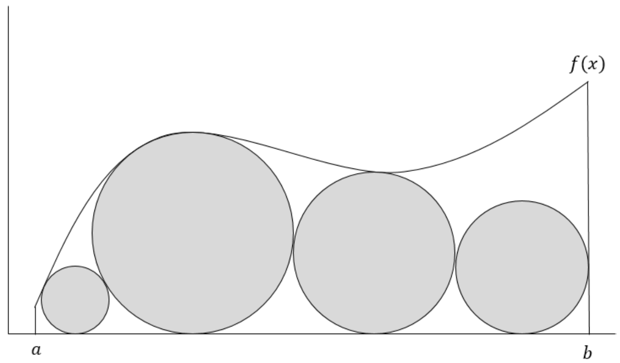
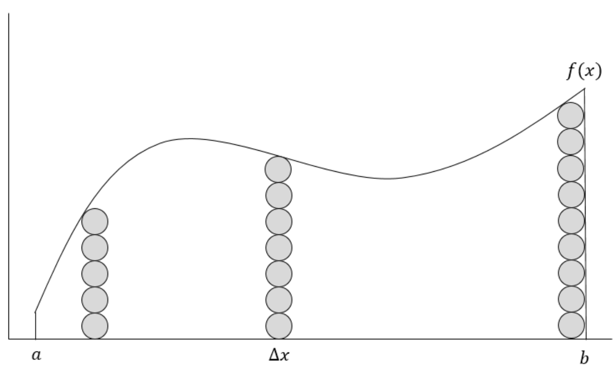


Figure 4: A filled-circle style approximation.

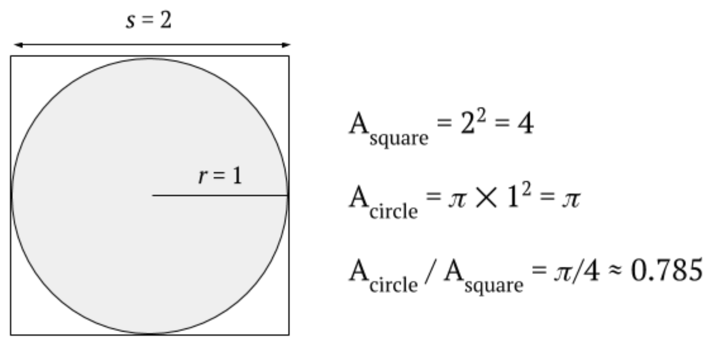
the curve. This means that the approximation will have huge failings when it comes to estimating the real area underneath the curve. In order to examine this method as we would a Riemann sum, by examining a variable which approaches infinity, we would need to continue to add circles into the empty space, a process which is most likely too drawn-out to fit into this 12-page report. Finding the optimal number of circles that will fit under the curve is a problem of *circle packing*, which examines how to fit circles into an enclosed space. [2]

2.2.2 Stacked Circles

This method stacks circles with a diameter equal to the partition width Δx up to the curve, and sums their areas, and is the most likely to behave like a Riemann sum because it creates smaller and smaller shapes to fit under the curve.



However it might suffer in accuracy, since circle packing immediately introduces a problem with computing a Riemann-style circle sum: a circle has approximatley 78% the area of an similarly-sized square. It is to be discovered if this discrepancy will disappear as the number of circles increases to infinity.



We can begin with the same set of givens for a Riemann sum: a function, $f(x)$, and a set of bounds, a and b. This workable area, with n partitions, has a partition width of $\Delta x = \frac{b-a}{n}$. Each circle will have diameter Δx. Therefore each will have radius $r = \frac{\Delta x}{2}$ and area $\pi \left(\frac{\Delta x}{2}\right)^2$. We can find the number of circles required for a given x by dividing the function value by the diameter of a circle. Therefore, for a given x in partition n, the area of the circles beneath it is:

$$A_{stack} = \frac{f(a + n_i \Delta x)}{\Delta x} \times \frac{\pi \Delta x^2}{4}$$

$$= \frac{f(a + n_i \Delta x)\pi \Delta x}{4}$$

Substituting $\Delta x = \frac{b-a}{n}$ back in, this simplifies to

$$A_{stack} = \frac{f(a + n_i(\frac{b-a}{n}))(b-a)\pi}{4n}$$

Therefore, the sum of these circles over a function from a to b partitioned n times can be expressed in summation notation as

$$A_{total} = \sum_{i=0}^{n} \frac{f(a + i(\frac{b-a}{n}))(b-a)\pi}{4n}$$

3 Accuracy

Now that we have a general equation for the circle sum, we can test it. Let $f(x) = x^2 + 1; [a, b] = [0, 1]; n = 10$. A circle sum appears thusly:

$$A_{total} = \sum_{i=0}^{10} \frac{f(\frac{i}{10})\pi}{40} = \frac{\pi}{40} + \frac{1.01\pi}{40} + \ldots + \frac{2\pi}{40} = \frac{297}{800}\pi \approx 1.166$$

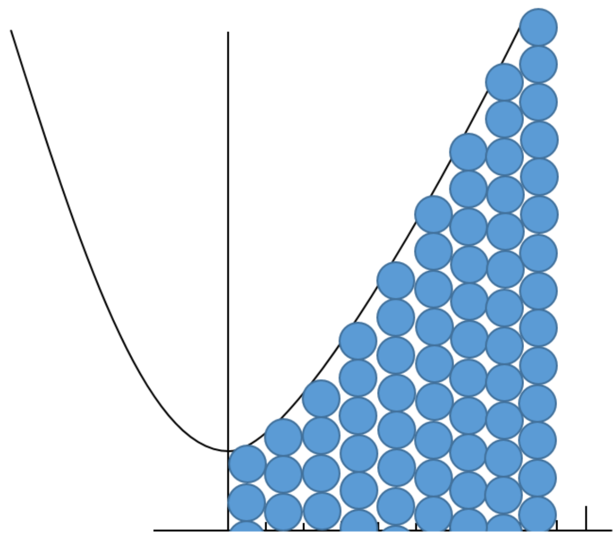


Figure 5: A graphical representation of a circle sum for $f(x) = x^2 + 1$.

Comparatively, a rectangular right-hand Riemann sum for these givens produces the following output:

$$A_{total} = \sum_{i=0}^{10} \frac{f(\frac{i}{10})}{10} = \frac{1}{10} + \frac{1.01}{10} + \ldots + \frac{2}{10} = 1.485$$

Interestingly, the effects of circle packing are already visible. The approximate area acquired through the circle sum is 78% of the area acquired through the rectangular sum.

$$\frac{1.166}{1.485} \approx 0.7851$$

One characteristic of right-hand Riemann sums is that they tend to underestimate while handling concave curves, and overestimate while handling

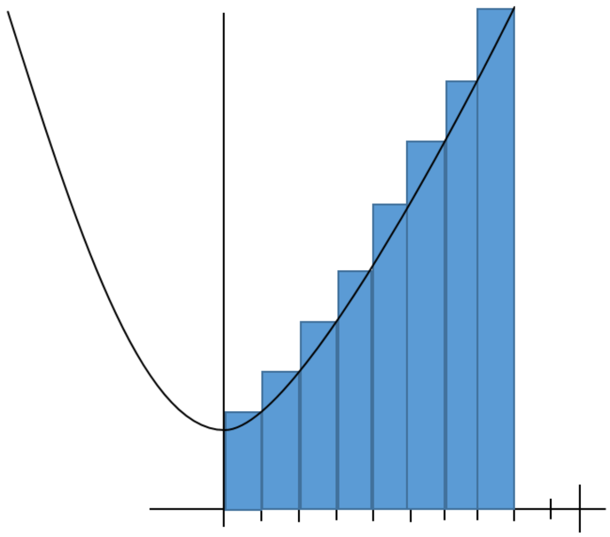


Figure 6: A graphical representation of a Riemann sum for $f(x) = x^2 + 1$.

convex curves. It is possible that our circle sum would exhibit the same tendency. Let's test the circle sum again, this time modifying our $f(x)$ to be concave.

$$f(x) = -x^2 + 1$$

$$A_{total} = \sum_{i=0}^{10} \frac{f(\frac{i}{10})\pi}{40} = \frac{\pi}{40} + \frac{0.99\pi}{40} + \ldots + \frac{0\pi}{40} = \frac{143\pi}{800} \approx 0.562$$

$$\int_0^1 (-x^2 + 1)dx = \frac{2}{3} \approx 0.66667$$

It would appear that the circle sum underestimates, no matter the shape of the curve. This makes sense, because of the lost space created by circle packing. However, we now must examine the circle sum's behavior at infinity, to discover if infinitley small circles will perfectly pack to equal the integral of the function.

4 Sums at Infinity

Riemann sums converge to the real value of the integral when we let the number of partitions go to infinity. This is expressed as

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

There are two methods of coming to this conclusion. The first is through graphical analysis. If we let x be the number of partitions, increasing x to a very large number can initially confirm a limit. The second confirmed by finding a closed-form version of a Riemann sum, where n is left undefined. We then examine that form by taking the limit as n goes to infinity. The result is the real value of the integral over those bounds.

4.1 Graphical Analysis

We can intitally confirm a limit's value before we find its closed form by examining the curve graphically. By using a logarithmic scale on the x-axis, we can show the function value for very large values of x, which allow us to, in this case, make fairly accurate predictions about the sum's behavior when we allow n to go to infinity in the next section.

x	Riemann	circle
1	3	2.3561945
10	1.485	1.1663163
100	1.34853	1.0589916
1000	1.3348335	1.0483758
10000	1.3334833	1.0473154
100000	1.3333483	1.0472093

The graph and table show us that we can expect a circle sum to approach the value $1.0472\ldots$ as $n \to \infty$. This value is smaller than the true value of $\frac{4}{3}$ given by integration. However, this value does follow the pattern established when examining finite sums: the value given by the circle sum is 78% of the one given by the Riemann sum:

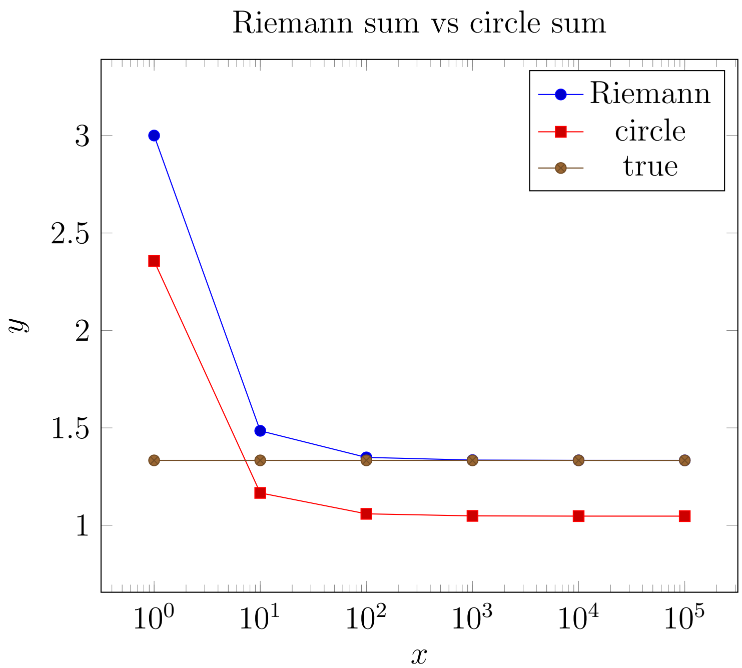


Figure 7: The Riemann and circle sum for $f(x) = x^2 + 1$ are plotted against x, where x is the number of partitions in the summation.

$$\frac{1.0472}{1.3333} \approx 0.7854$$

We now need to confirm this prediction by finding the closed form of each of the sums.

4.2 Riemann

In order to find the closed form of a Riemann sum we must leave n undefined. For this example, we will use the same givens as used in Section 2.2.2: $f(x) = x^2 + 1$ on $[a, b] = [0, 1]$. Therefore $\Delta x = \frac{1}{n}$.

$$A_{total} = \sum_{i=0}^{n} \frac{f(\frac{i}{n})}{n} = \frac{(\frac{0}{n})^2 + 1}{n} + \frac{(\frac{1}{n})^2 + 1}{n} + \ldots + \frac{(\frac{n}{n})^2 + 1}{n}$$

$$= \frac{1}{n}\left[\left(\left(\frac{0}{n}\right)^2 + 1\right) + \left(\left(\frac{1}{n}\right)^2 + 1\right) + \ldots + \left(\left(\frac{n}{n}\right)^2 + 1\right)\right]$$

$$= \frac{1}{n}\left[\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2\right] + \frac{n}{n}$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + 1 = \frac{(n+1)(2n+1) + 6n^2}{6n^2}$$

$$= \frac{8n^2 + 3n + 1}{6n^2} = \frac{4}{3} + \frac{1}{3n} + \frac{1}{6n^2}$$

$$\lim_{n\to\infty}\left(\frac{4}{3} + \frac{1}{3n} + \frac{1}{6n^2}\right) = \frac{4}{3} = \int_0^1 (x^2+1)dx$$

4.3 Circle

Now let's do the same thing for a circle sum. $f(x) = x^2 + 1, [a,b] = [0,1]$.

$$A_{total} = \sum_{i=0}^{n} \frac{f\left(\frac{i}{n}\right)\pi}{4n} = \frac{(\left(\frac{0}{n}\right)^2 + 1)\pi}{4n} + \ldots + \frac{(\left(\frac{n}{n}\right)^2 + 1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\left(\left(\frac{0}{n}\right)^2 + 1\right) + \left(\left(\frac{1}{n}\right)^2 + 1\right) + \ldots + \left(\left(\frac{n}{n}\right)^2 + 1\right)\right]$$

$$= \frac{\pi}{4n}\left[\left(\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2\right) + n\right]$$

$$= \frac{\pi}{4n}\left[\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2\right] + \frac{\pi}{4}$$

$$= \frac{\pi}{4n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{\pi}{4}$$

$$= \frac{8\pi n^2 + 3\pi n + \pi}{24n^2} = \frac{8\pi}{24} + \frac{3\pi}{24n} + \frac{\pi}{24n^2}$$

$$= \lim_{n\to\infty}\left(\frac{8\pi}{24} + \frac{3\pi}{24n} + \frac{\pi}{24n^2}\right) = \frac{1}{3}\pi \approx 1.0472$$

The sum does, in fact, approach a real value which is 78% of a Riemann sum's value.

5 Conclusion

The circle sum used by this investigation was derived similarly to how a rectangular Riemann sum is derived. As a summation, it behaves similarly to a rectangular sum but creates outputs which are smaller than those of a rectangular sum. This was found to be due to an inherent property of the areas of circles and rectangles- circles have 78% of the area of a comparable rectangle.

When the number of partitions is taken to infinity, the circle sum behaves like a Riemann sum in that it approaches a real value. However, the circle packing property persists when examined at an infinite number of partitions. No matter how many circles you attempt to fit underneath the curve, the total area will always be at a loss compared to that of a sum using rectangles, simply because of the area differences inherent in using circles.

There is one substantial explanation for this phenomenon. Because the circles are simply placed next to each other in each stack and no attempt is made to interlock the shapes, the lost packing space persists.

The point of this investigation was to examine the method itself, not to find a new, accurate way to estimate the area underneath a circle. Supplemental time will not be spent attempting to correct the error from circle packing.

In a future investigation, I'd like to dive into methods of summation that would relieve these sources of error, as well as examine the feasability of the Filled Circle method.

References

[1] Thomas, George B. Jr, and Ross L. Finney: *Calculus and Analytical Geometry (9th ed.)*. New York: Addison Wesley, 1996.

[2] Heppes, Aldar: "Some Densest Two-Size Disc Packings in the Plane", *Discrete and Computational Geometry*, 30. 2003.