Using Circles to Approximate Riemann Sums

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1 Introduction

Riemann Sums are a method of approximating the area underneath a curve by adding up the areas of shapes, usually rectangles, whose height is bound by a point on the function. As the width of these shapes approaches zero, the summation approaches the Riemann integral, the true area underneath the curve.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right) = \int_a^b f(x)dx$$

Riemann sums can be approximated three different ways, all of which depend on where the rectangle intercepts the curve. When the rectangle's top right vertex is bounded by the curve, this is known as a RRAM, or Right-hand Rectangular Approximation Method. The same goes for a left-bounded rectangle (LRAM), and a rectangle whose midpoint intercepts the curve (MRAM). The formula changes slightly for each method, denoted by x_i^*. For LRAM, $x_i^* = x_{i-1}$; RRAM, $x_i^* = x_i$; and MRAM, $x_i^* = \frac{1}{2}(x_i + x_{i-1})$.

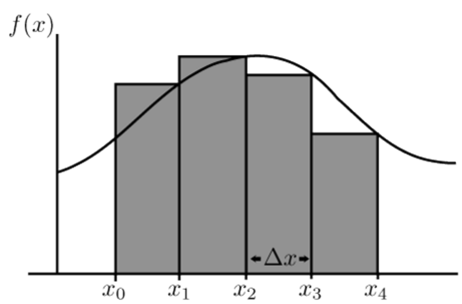

Figure 1: A function with Right-hand Riemann sum rectangles drawn underneath.

Riemann sums are not limited to rectangles, however. Another, more accurate method of approximation makes use of trapezoids, whose area formula is comparably simple to that of a rectangle's. Trapezoidal approximation is similar to the result of averaging a left and right approximation. This method makes use of the trapezoid area formula $A = \frac{1}{2}h(b_1 + b_2)$. A Riemann sum constructed with trapezoids takes this form:

$$\lim_{n\to\infty} \frac{1}{2}\Delta x \sum_{i=1}^{n} f(a + i\Delta x)$$

Historically, Riemann's definition was the first rigourous definition of the integral of a function on an interval. Thus, in Calculus classes today, it is often the first introduction students recieve to the idea of calculating the area under a curve; the AP Calculus Syllabus even specifies Riemann sums as the first material in the Integrals unit.[1]

As a student who was introduced to integration with Riemann sums, I often wondered: are there other shapes that you could use or approximations similar to Riemann's that would behave similarly? In this paper I intend to investigate just that: are circles a viable shape to use in the fashion of a Riemann sum? How accurate is such an approximation? Does this approximation behave like a Riemann sum in that its limit approaches the real area under the curve?

In this investigation, I'll derive this approximation (hereafter called the Circle sum) in a similar method to how Riemann sums can be derived. I'll then test each of my questions. I'll compare the efficacy of a Circle sum versus a similar Riemann sum, as well as examine the behavior of the Circle sum as it approaches infinity.

2 Derivation

The Riemann sum can be simply derived by finding the area of a rectangle under the curve, summing all such rectangles between two bounds, and finally taking the limit of this summation as the number of rectangles approaches infinity. In order to find a Circle sum, a similar process will be followed.

2.1 Riemann

We begin by being given a function $f(x)$ and a set of bounds a and b. The width of the workable area is defined as $b - a$. Thus, we can divide the workable area into n partitions, each with a width of $\Delta x = \frac{b-a}{n}$. The right-bound x-value for any such rectangle inside of these bounds can be found by adding the partition width n times for the nth partition. Such a rectangle on the curve has width Δx and height $f(a + n_i\Delta x)$. Thus, the area of this rectangle is $f(a + n_i\Delta x)\Delta x$. This rectangle would appear below the function in this manner:

The sum of the rectangles that exist from a to b can be expressed with summation notation:

[1]http://media.collegeboard.com/digitalServices/pdf/ap/ap-calculus-course-description.pdf

$$A_{total} = f(a + (0)\Delta x)\Delta x + f(a + (1)\Delta x)\Delta x + \ldots + f(a + n\Delta x)\Delta x$$

$$= \sum_{i=0}^{n} f(a + i\Delta x)\Delta x$$

A_{total} is represented by the shaded green areas in Figure 3.

When we take the limit of A_{total} as the number of partitions, n, approaches infinity, we find that A_{total} approaches A_{real}, given by integrating the function from a to b.

$$\lim_{n \to \infty} \sum_{i=0}^{n} f(a + i\Delta x)\Delta x = \int_{a}^{b} f(x)dx$$