# Continuous, Lipschitz continuous, Hölder continuous

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## 1 Continuous

### 1.1 엡실론 델타 논법

Let $X, Y$ be metric spaces, $f : X \to Y$ be a function.

$f$ is continuous at $x \in X$ if

$\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon \tag{1}$$

### 1.2 Topological Space

Let $X, Y$ be topological spaces, $f : X \to Y$ be a function.

$f$ is continuous at $x \in X$ if

for every neighborhood $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subset V$.

$f(U) \subset V$를 풀어 쓰면 $x \in U \Rightarrow f(x) \in V$이다.

이 정의는 거리공간에서 정의했던 연속에 대한 정의를 위상공간으로 확장한 정의이다. 거리공간에서의 정의를 완벽하게 유지하면서도, 위상공간에서의 연속성에 대해 논하고 있다. $X, Y$가 거리공간일 때, 위 정의는 엡실론 델타 논법과 동치이다.

proof

Let $X, Y$ be metric spaces, $f : X \to Y$ be a function.

(i) Assume that for every neighborhood $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Then $\forall \epsilon > 0$, $\{y \in Y \mid d_Y(f(x), y) < \epsilon\}$ is a neighborhood of $f(x)$, and by the assumption, there exists a neighborhood $U$ of $x$ such that $f(U) \subset \{y \in Y \mid d_Y(f(x), y) < \epsilon\}$. And we can find $\delta$ such that $\{z \in X \mid d_X(x, z) < \delta\} \subset U$, since $U$ is open.

(ii) Assume that $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon \tag{2}$$

Let $V$ be a neighborhood of $f(x)$. And Let $E$ be a preimage of $V$. That is, $E := f^{-1}(V)$. Then, for each $x \in E$, we can find $\epsilon_x > 0$ such that $\mathcal{B}_Y(f(x), \epsilon_x) \subset V$, since $V$ is open. And by assumption, we can find $\delta_x > 0$ such that $f(\mathcal{B}_X(x, \delta_x)) \subset \mathcal{B}_Y(f(x), \epsilon_x)$. Define $U := \bigcup\limits_{x \in E} \mathcal{B}_X(x, \delta_x)$. Then

$$f(U) = f(\bigcup_{x \in E} \mathcal{B}_X(x, \delta_x)) = \bigcup_{x \in E} f(\mathcal{B}_X(x, \delta_x)) \subset V \tag{3}$$

### 1.3 Continuous on X

연속의 정의를 다음과 같이 확장할 수 있다.

Let $X, Y$ be topological spaces, $f : X \to Y$ be a function.

$f$ is continuous on $X$ if for every open set $V$ in $Y$, $f^{-1}(V)$ is open in $X$.

$f$ is continuous on $E \subseteq X$ if for every open set $V$ in $Y$, $f^{-1}(V)$ is open in the subspace topology on $E$.

위와 같은 정의는 아래와 동치이다.

Let $X, Y$ be topological spaces, $f : X \to Y$ be a function.

$f$ is continuous on $X$ if $f$ is continuous at $\forall x \in X$.

$f$ is continuous on $E \subseteq X$ if $f$ is continuous at $\forall x \in E$.

proof

"추가 바람".

## 2 Lipschitz Continuous

어떤 점을 중심으로 f의 제한이 생기면 그 제한 안으로 골인시키는 x가 존재해야 그 점에서 continuous라고 하는데, 그 골인시키는 x의 구간이 f의 제한에 대하여 linear하면 그게바로 lipschitz.