

Continuous, Lipschitz continuous, Hölder continuous

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1 Continuous

1.1 엡실론 델타 논법

Let X, Y be metric spaces, $f : X \rightarrow Y$ be a function.

f is continuous at $x \in X$ if

$\forall \epsilon > 0, \exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon \quad (1)$$

1.2 Topological Space

Let X, Y be topological spaces, $f : X \rightarrow Y$ be a function.

f is continuous at $x \in X$ if

for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

$f(U) \subset V$ 를 풀어 쓰면 $x \in U \Rightarrow f(x) \in V$ 이다.

이 정의는 거리공간에서 정의했던 연속에 대한 정의를 위상공간으로 확장한 정의이다.

거리공간에서의 정의를 완벽하게 유지하면서도, 위상공간에서의 연속성에 대해 논하고 있다.

X, Y 가 거리공간일 때, 위 정의는 엡실론 델타 논법과 동치이다.

proof

Let X, Y be metric spaces, $f : X \rightarrow Y$ be a function.

(i) Assume that for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

Then $\forall \epsilon > 0$, $\{y \in Y \mid d_Y(f(x), y) < \epsilon\}$ is a neighborhood of $f(x)$, and by the assumption, there exists a neighborhood U of x such that $f(U) \subset \{y \in Y \mid d_Y(f(x), y) < \epsilon\}$. And we can find δ such that $\{z \in X \mid d_X(x, z) < \delta\} \subset U$, since U is open.

(ii) Assume that $\forall \epsilon > 0, \exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon \quad (2)$$

Let V be a neighborhood of $f(x)$. And Let E be a preimage of V . That is, $E := f^{-1}(V)$.

Then, for each $x \in E$, we can find $\epsilon_x > 0$ such that $\mathcal{B}_Y(f(x), \epsilon_x) \subset V$, since V is open.

And by assumption, we can find $\delta_x > 0$ such that $f(\mathcal{B}_X(x, \delta_x)) \subset \mathcal{B}_Y(f(x), \epsilon_x)$. Define

$U := \bigcup_{x \in E} \mathcal{B}_X(x, \delta_x)$. Then

$$f(U) = f\left(\bigcup_{x \in E} \mathcal{B}_X(x, \delta_x)\right) = \bigcup_{x \in E} f(\mathcal{B}_X(x, \delta_x)) \subset V \quad (3)$$

1.3 Continuous on X

연속의 정의를 다음과 같이 확장할 수 있다.

Let X, Y be topological spaces, $f : X \rightarrow Y$ be a function.

f is continuous on X if for every open set V in Y , $f^{-1}(V)$ is open in X .

f is continuous on $E \subseteq X$ if for every open set V in Y , $f^{-1}(V)$ is open in the subspace topology on E .

위와 같은 정의는 아래와 동치이다.

Let X, Y be topological spaces, $f : X \rightarrow Y$ be a function.

f is continuous on X if f is continuous at $\forall x \in X$.

f is continuous on $E \subseteq X$ if f is continuous at $\forall x \in E$.

proof

”추가 바람”.

2 Lipschitz Continuous

어떤 점을 중심으로 f 의 제한이 생기면 그 제한 안으로 끌인시키는 x 가 존재해야 그 점에서 continuous라고 하는데, 그 끌인시키는 x 의 구간이 f 의 제한에 대하여 linear하면 그게바로 lipschitz.