CHAPTER THREE : L^p- SPACES

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Convex Functions and Inequalities

3.1 Definition

A real function ϕ defined on a segment (a, b), where $-\infty \leq a < b \leq \infty$, is called *convex* if the inequality

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y) \tag{1}$$

holds whenever $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

3.2 Theorem

If φ is convex on (a, b) then φ is continuous on (a, b).
φ가 open set 위에서 convex일 때에만 continuous가 보장된다. 예를 들면, $\varphi(x) = 0$ for $x \in (0, 1)$ and $\varphi(x) = 1$ for $x = 1$인 함수 φ는 $(0, 1]$에서 convex이지만 continuous는 아니다.

3.3 Theorem (Jensen's Inequality)

Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\phi \circ f) d\mu. \tag{2}$$

3.4 Definition

If p and q are positive real numbers such that $p + q = pq$, or equivalently

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{3}$$

eq:conjugate ex

then we call p and q a pair of *conjugate exponents*. It is clear that (3) implies $1 < p < \infty$ and $1 < q < \infty$. An important special case is $p = q = 2$.
As $p \to 1$, (3) forces $q \to \infty$. Consequently 1 and ∞ are also regarded as a pair of conjugate exponents. Many analysis denote th exponents conjugate to p by p', often without saying so explicitly.

3.5 Theorem

Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure mu. Let f and g be measurable functions on X, with range in $[0, \infty]$. Then

$$\int_X fg d\mu \leq \left\{ \int_X f^p \right\}^{1/p} \left\{ \int_X g^q \right\}^{1/q} \tag{4}$$ `Holder's`

and

$$\left\{ \int_X (f+g)^p d\mu \right\}^{1/p} \leq \left\{ \int_X f^p \right\}^{1/p} + \left\{ \int_X g^p \right\}^{1/p} \tag{5}$$ `Minkowski's`

. The inequality (4) is Holder's; (5) is Minkowski's. If $p = q = 2$, (4) is known as the Schwarz inequality.