3.3 Theorem (Jensen's Inequality)

Jeawon Na

3.3 Theorem (Jensen's Inequality)

Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\phi \circ f) d\mu. \tag{1}$$

convex function $\varphi : (a, b) \to \mathbb{R}$이 있을 때, μ와 f는 구간 (a, b)의 가중평균을 나타낸다고 볼 수 있다. $\frac{1}{\mu(\Omega)} \int_\Omega f d\mu$는 Ω 위에서 f의 평균인데, $\mu(\Omega) = 1$이기 때문이다. 특히 f가 simple function일 때에는, 어떤 $c \in (a, b)$의 가중치를 $f^{-1}(c)$로 볼 수 있다.

증명요약

$t = \int_\Omega f d\mu$로 놓고, convex 식을 정리하면 된다. 이때, $a < s < b$인 모든 s에 대해 성립하는 식을 하나 만들어놓고, $s = f(x)$를 대입한 뒤 μ로 적분하면 식이 정리된다.

proof

note:1 Let $t = \int_\Omega f d\mu$. Then $a < t < b$[1]. Since φ is convex,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t} \tag{2}$$ eq:convex

whenever $a < s < t < u < b$. (a, t, b는 fixed point). Let $\beta = \sup_{s, a<s<t} \frac{\varphi(t)-\varphi(s)}{t-s}$. Then,

$$\beta \geq \frac{\varphi(t)-\varphi(s)}{t-s} \Rightarrow \beta(t-s) \geq \varphi(t) - \varphi(s) \tag{3}$$

`eq:convex1`

for any s with $a < s < t$, by the definition of supremum, and

$$\beta \leq \frac{\varphi(u)-\varphi(t)}{u-t} \Rightarrow \beta(u-t) \leq \varphi(u) - \varphi(t) \Rightarrow \beta(t-u) \geq \varphi(t) - \varphi(u) \tag{4}$$

`eq:convex2`

for any u with $t < u < b$, by (2). By (3) and (4), $\beta(t-s) \geq \varphi(t) - \varphi(s)$ for all s such that $a < s < b$. $a < f(x) < b$ for all $x \in \Omega$이므로, put $s = f(x)$.

$$\beta(t - f(x)) \geq \varphi(t) - \varphi(f(x)) \tag{5}$$

`eq:phi f`

$$\varphi(f(x)) \geq \varphi(t) - \beta(t - f(x)) \tag{6}$$

Since φ is continuous, $\varphi \circ f$ is measurable. Integrate both sides of (5) with respect to μ gives

$$\int_\Omega \varphi \circ f d\mu \geq \int_\Omega \varphi(t) d\mu - \beta \left(\int_\Omega t d\mu - \int_\Omega f d\mu \right) \tag{7}$$

$$= \varphi(t) \cdot \mu(\Omega) - \beta(t \cdot \mu(\Omega) - t) = \int_\Omega \varphi(t) d\mu - \beta(t - t) \tag{8}$$

`note:2`

$$= \varphi(t) = \varphi\left(\int_\Omega f d\mu \right). \tag{9}$$

note 1.2
$f(x) < g(x)$ for all $x \in X$일 때, $0 < g(x) - f(x)$ for all $x \in X$. 시발뭐야 $\therefore \int_X f d\mu < \int_X g d\mu$ if $f < g$ and $\mu(X) > 0$.

note 8
x에 대해 적분하는거임. 그리고 t는 상수임. 그러니까 상수함수(simple function)로 보고 적분하면 됨.