# Lab Session 9

**MA-571 :** Numerical Linear Algebra Lab 2022 R. Alam

1. The purpose of this experiment is to illustrate that QR factorization by Householder reflector (MATLAB command `[Q, R] = qr(A)`) is better than modified Gram-Schmidt scheme (MGS) and classical Gram-Schmidt scheme (CGS).

   Consider the $n$-by-$n$ Hilbert matrix $H$ (use MATLAB command `H = hilb(n)` to generate $H$). Your task is to use different methods listed below to orthonormalize the columns of $H$ for $n = 7$ and $n = 12$.

   (a) Write a MATLAB function implementing classical Gram-Schmidt method (CGS).

```
function [Q, R] = cgsqr(A)
% [Q, R] = cgsqr(A) employs classical Gram-Schmidt scheme to compute
% an isometry Q, an upper triangular matrix R such that A=QR.

[m, n] = size(A); %  Assume that m >= n
Q = A; R = zeros(n);
for k = 1:n
 R(1:k-1,k) = Q(:,1:k-1)' * A(:,k);
 Q(:,k) = A(:,k)- Q(:,1:k-1) * R(1:k-1,k);
 R(k,k) = norm(Q(:,k));
 Q(:,k) = Q(:,k)/R(k,k);
end
```

   (b) Write a MATLAB function implementing modified Gram-Schmidt method (MGS).

```
function [Q, R] = mgsqr(A)
% [Q, R] = mgsqr(A) employs modified Gram-Schmidt scheme to compute
% an isometry Q, an upper triangular matrix R such that A=QR.

[m,n] = size(A);  % Assume that m >=n
Q = A; R = zeros(n);
for k = 1:n
R(k,k) = norm(Q(:,k));
Q(:,k) = Q(:,k)/R(k,k);
R(k,k+1:n) = Q(:,k)' * Q(:,k+1:n);
Q(:,k+1:n) = Q(:,k+1:n) - Q(:,k) * R(k,k+1:n);
end
```

   (c) QR decomposition with reflectors. Use MATLAB command `[Q,R] = qr(H, 0)`, which produces an 'economy size' $QR$ decomposition of $H$ with $Q$ being an isometry.

   Examine the deviation from orthonormality by computing $\|Q'*Q-\texttt{eye(n)}\|_2$ in each case (MATLAB command `norm(eye(n)-Q'*Q)`). Setting $E := QR - H$, we have $H + E = QR$. Check the residual norm $\|E\|_2 =$ `norm(H - Q*R)`. The small residual error of order $\mathcal{O}(\mathbf{u})$ as well as small deviation from orthonormality of order $\mathcal{O}(\mathbf{u})$ imply that the algorithm is backward stable. In other words, the algorithm computes QR factroization of a slightly perturbed matrix which is indistinguishable from $A$. Test the backward stability of CGS, MGS and Householder QR factorization.

   Find the condition number of $H$ and check whether or not the matrix $Q$ obtained from the MGS program satisfies $\|Q' * Q - \texttt{eye(n)}\|_2 \approx u * \texttt{cond(H)}$.

   Did you get what you would expect in light of the values of unit roundoff $\mathbf{u}$ and $\text{cond}(H)$? Which among all the above methods produces the smallest deviation from orthonormality?

2. Your task is to find the polynomial $p(t)$ of degree 18 that best fits the function $f(t) = \sin(2\pi t) + \dfrac{t}{5}$ for $t_1 = -5, t_2 = -4.5, \ldots, t_{23} = 6$. Determine the polynomial $p$ whose coefficients are given by $x$ (that is, $p(t) := \sum_{j=1}^{19} x_j t^{j-1}$) by solving LSP $Ax = b$ in two different ways: solve LSP $Ax = b$ using QR factorization of $A$ and QR factorization of the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$. Here are the details.

   Let `[Q, R] = cgsqr(A)` and `[Q, R] = mgsqr(A)` be MATLAB functions implementing classical Gram-Schmidt and modified Gram-Schmidt methods.

   (a) Compute `[Q, R] = cgsqr([A b])` and use `R` to solve the LSP $Ax = b$. Compute the residual `res1` $:= \|Ax - b\|_2$ from the matrix `R`. Call the polynomial $p_1(t)$.

   Next compute `[QC, QC] = cgsqr(A)` and use `QC` and `RC` to solve the LSP $Ax = b$. Compute the residual `res2` $:= \|Ax - b\|_2$. Call the polynomial $p_2(t)$. Which method gives a better fit (small residual error)? Plot $p_1(t), p_2(t)$ and $f(t)$ in a single plot and comment on the result.

   (b) Compute `[Q, R] = mgsqr([A b])` and use `R` to solve the LSP $Ax = b$. Compute the residual `res3` $:= \|Ax - b\|_2$ from the matrix `R`. Call the polynomial $p_3(t)$.

   Next compute `[QM, QM] = cgsqr(A)` and use `QM` and `RM` to solve the LSP $Ax = b$. Compute the residual `res4` $:= \|Ax - b\|_2$. Call the polynomial $p_4(t)$. Which method gives a better fit (small residual error)? Plot $p_3(t), p_4(t)$ and $f(t)$ in a single plot and comment on the result.

   (c) Compute `[Q, R] = qr([A b])` and use `R` to solve the LSP $Ax = b$. Compute the residual `res5` $:= \|Ax - b\|_2$ from the matrix `R`. Call the polynomial $p_5(t)$.

   Next solve the LSP $Ax = b$ using MATLAB command `x = A\b` and compute the residual `res6` $:= \|Ax - b\|_2$. Call the polynomial $p_6(t)$. Which method gives a better fit (small residual error)? Plot $p_5(t), p_6(t)$ and $f(t)$ in a single plot and comment on the result.

   Among the six methods, which methods provide the best fit? What is the impact of the condition number of $A$ on these methods?

*** End ***