Lab Session 4

MA-571 : Numerical Linear Algebra Lab 2022 R. Alam

Finite precision arithmetic and stability of GE

1. The purpose of this exercise is to illustrate anomaly in automatic computation. On my computer, MATLAB produces

$$
\begin{aligned}
(\frac{4}{3} - 1) * 3 - 1 &= -2.2204 \times 10^{-16} \\
5 \times \frac{(1 + \exp(-50)) - 1}{(1 + \exp(-50)) - 1} &= \mathbf{NaN} \\
\frac{\log(\exp(750))}{100} &= \mathbf{Inf}
\end{aligned}
$$

Try on your machine. Can you explain the reason behind these anomalies?

2. Suppose that $F(\beta, t, L, U) = (10, 8, -99, 99)$. Consider the normalized floating-point numbers

$$
\begin{aligned}
x &= 0.23371258 \times 10^{-4} \\
y &= 0.33678429 \times 10^{2} \\
z &= -0.33677811 \times 10^{2}
\end{aligned}
$$

Using round to nearest rounding mode (MATLAB command `round(x,n)` rounds x to n-digits), calculate (i) $\mathrm{round}(x + \mathrm{round}(y + z))$ and (ii) $\mathrm{round}(\mathrm{round}(x + y) + z)$. Calculate $x + y + z$ in exact arithmetic and determine the relative errors in calculating $\mathrm{round}(x + \mathrm{round}(y + z))$ and $\mathrm{round}(\mathrm{round}(x + y) + z)$.

3. In Treehouse of Horror VI of The Simpsons, Homer has a nightmare in which the following equation flies past him:

$$1782^{12} + 1841^{12} = 1922^{12}. \tag{1}$$

Note that this equation, if true, would contradict Fermat's last theorem, which states: For $n \geq 3$, there do not exist any natural numbers x, y and z that satisfy the equation $x^n + y^n = z^n$. Did Homer dream up a counterexample to Fermat's last theorem?

(a) Compute $(1782^{12} + 1841^{12})^{1/12}$ by typing the following into MATLAB:

```
>> format short
>> (1782^12 + 1841^12)^(1/12)
```

What result does MATLAB report? Now look at the answer using `format long`.

(b) Determine the absolute and relative error in the approximation

$$1782^{12} + 1841^{12} \simeq 1922^{12}.$$

(Such an example is called a Fermat near miss because of the small relative error.)

(c) Note that the right-hand side of equation (1) is even. Use this to prove that the equation cannot be true.

(d) In a later episode entitled The Wizard of Evergreen Terrace, Homer writes the equation

$$3987^{12} + 4365^{12} = 4472^{12}.$$

Can you debunk this equation?

4. Wilkinson's matrix is defined as follows: 1 on the diagonal, -1 everywhere below the main diagonal, 1 in the last column, and 0 everywhere else. Write a MATLAB function `W = Wilkinson(n)` that generates Wilkinson's matrix W of size n using MATLAB functions `eye, tril` and `ones`.

    (a) For $n = 32$, pick a random x and then compute $b := W * x$. Solve $Ax = b$ using MATLAB backslash command and compute the error $\|x - \hat{x}\|_\infty / \|x\|_\infty$ (type `help norm` for more info about computing norm). Does the size of the error confirm that GEPP is unstable for this system? Also compute $\mathrm{cond}(A)$. Can the poor answer be attributed to ill-conditioning of the matrix W? Repeat the test for $n = 64$.

    (b) Repeat the experiment in part (a) using QR decomposition. It is easy in matlab. The command `[Q,R] =qr(A)` gives unitary Q and upper triangular R such that $A = QR$. Solve $Wx = b$ using QR decomposition and compare the results with those in part(a). Which of the two methods appear to give a better answer?

5. Pivot growth of Gaussian elimination with partial pivoting (GEPP) is given by $PG(A) = \max_{ij} |U(i,j)|/\max_{ij} |A(i,j)|$, which influences the accuracy of computed solution. Use MATLAB function `[L, U, p] = lu(A)` for computing LU decomposition of a nonsingular matrix A and compute the pivot growth `rho` $= PG(A)$. Use commands `max` and `abs`.

    It is well known that the pivot growth factor for GEPP satisfies $PG(A) \leq 2^{n-1}$ which is attained by the Wilkinson matrix. Verify this graphically by doing the following:

    First plot the graph of 2^{n-1} in $\log 10$ scale for $n = 10 : .5 : 505$ by setting `X = 2.^(n-1)` and then typing `semilogy(n,X,'r')`. Hold this plot by typing `hold on` and type the following sequence of commands (which assumes that the Wilkinson matrix of size n is generated by the function `W = Wilkinson(n)`).

    ```
    n = 10:20:500;  m = length(n); G = zeros(m,1);
    for i = 1:m
    W = Wilkinson(n(i)); [L,U,p] = lu(W);
    G(i) = max(max(abs(U)))/(max(max(abs(W))));
    end
    semilogy(n,G,'b*')
    ```

    The second plot should come in the form of blue dots that fall on the red curve produced by the earlier plot.

    However, statistics suggest that for most practical examples, $PG(A) \leq n^{2/3}$ for GEPP. Verify this graphically by generating random matrices instead of Wilkinson matrices in the sequence of commands given above.

6. There is no strong correlation between pivot growth and the ill-conditioning of a matrix. This is illustrated by a Golub matrix. A Golub matrix A of size n is an ill-conditioned integer matrix whose LU factorization without pivoting fails to reveal that A is ill-conditioned. The matrix A is given by $A := LU$, where L unit lower triangular with random integer entries and U is unit upper triangular with random integer entries. The function `golub` given below generates a Golub matrix of size n :

```
function A = golub(n)
s = 10;
L = tril(round(s*randn(n)),-1)+eye(n);
U = triu(round(s*randn(n)),1)+eye(n);
A = L*U;
```

Compute LU factorization of A using your function `[L, U] = GENP(A)`. Also, compute the pivot growth $PG(A)$ and the condition number $\text{cond}(A) = \|A\|_2 \|A^{-1}\|_2$ using MATLAB command `cond(A)`. If $\text{cond}(A)$ is large then the system $Ax = b$ is ill-conditioned and in such a case A is called ill-conditioned. Does $PG(A)$ reflect the ill-conditioning of A?

*** End ***