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Ellipse Fitting and Parameter Assessment of Circular Object Targets for Robot Vision

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Abstract

The least squares fitting minimizes the squares sum of error-of-fit in predefined measures. By the geometric fitting, the error distances are defined with the shortest distances from the given points to the geometric feature to be fitted. For the geometric fitting of ellipse, a robust algorithm is proposed. This is based on the coordinate description of the corresponding point on the ellipse for the given point, where the connecting line of the two points is the shortest path from the given point to the ellipse. As a practical application example, we show the geometric ellipse fitting to the image of circular point targets, where the contour points are weighted with their image gradient across the boundary of the image ellipse.

1 Introduction

The fitting of geometric features to given 2D and 3D points is desired in various fields of science and engineering, e.g. astronomy[1], physics[2], biology[3], quality control and metrology[4-5]. In particular, ellipse is one of the most common geometric features for the application of image processing. In the past, fitting problems have usually been solved through the least squares method (LSM)[1] with respect to the effective implementation and acceptable computing costs. The main alternative methods for the detection and analysis of geometric features are, e.g. Hough transform[6] and the moment method[7].

In this paper, we suggest robust algorithm for *least squares orthogonal distances ellipse fitting* The LS fitting minimizes the squares sum of error-of-fit in predefined measures. There are two main categories of LS fitting problems for geometric features; algebraic and geometric. These categories are differentiated by their respective definition of the error distances involved[8-9].

By the *algebraic fitting* of a geometric feature, which is described by implicit equation $F(\mathbf{x}, \mathbf{a}) = 0$ and the parameters vector $\mathbf{a} = (a_1, \cdots, a_q)$, the error distances are defined with the deviations of the implicit equation from the expected value, i.e. zero, at each given point. The nonequality of the equation indicates that the given point does not lie on the geometric feature, i. e., there is some error-of-fit.

Most publications about the LS fitting of ellipse have been concerned with the squares sum of algebraic distances or their modifications [3,8,10-21]:

$$\sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} (ax_i^2 + 2bx_i y_i + cy_i^2 + 2dx_i + 2ey_i + f)^2 . \quad (1)$$

In spite of the advantages in implementation and computing costs, the algebraic fitting has drawbacks in accuracy, and in relation to the physical interpretation of the fitting parameters and errors[17,22]. The known disadvantages of the algebraic fitting are [9,14,18]:

- Its definition of error distances does not coincide with the measurement guideline.
- It is very difficult to test the reliability of the estimated fitting parameters.
- The fitting parameters are dependent on the coordinate transformation (e.g., a simple parallel translation of the given points causes changes not only in the center coordinates but also in the axis lengths and angle of the ellipse).
- The estimated fitting parameters are biased.
- The fitting errors are unwillingly weighted.
- The fitting procedure sometimes ends in an unintended geometric feature (e.g. a hyperbola instead of an ellipse).

By the *geometric fitting*, named also as the best fitting, the error distances are defined with the orthogonal, or shortest, distances from the given points to the geometric feature to be fitted. The geometric fitting is possibly the only solution to all of the above problems of the algebraic fitting.

The geometric fitting of ellipse is a nonlinear problem and must be solved through iteration. Gander et. al.[23] have proposed a geometric ellipse fitting algorithm in parametric form, which has a large number of fitting parameters, $m+5$ unknowns by $2m$ equations from m measurement points, where each point has an individual angular parameter to be simultaneously estimated with the 5 ellipse parameters. As a consequence, their linear system has a bulky Jacobian matrix and will cause disadvantages in computing costs and convergence performance. Besides, their algorithm can not directly take the parameters from a circle fitting as the reasonable initial values for the iterative ellipse fitting because of the singularity in their Jacobian, when the two axes of the ellipse have the same length.

In this paper, we propose a new algorithm of geometric fitting of ellipse. Our algorithm is based on the coordinate description of the corresponding point on the ellipse for the given point, where the connecting line of the two points is the shortest path from the given point to the ellipse. The corresponding point on the ellipse is described only implicitly through the orthogonal contacting conditions. Once the corresponding point on the ellipse for the given point is numerically determined, we can derive the Jacobian matrix at this point and apply the iterative nonlinear least squares method.

Section 2 describes the applied nonlinear LS fitting in a general form. Section 3 contains the geometric fitting of ellipse in greater detail. In section 4, we show the ellipse fitting for the image ellipses of circular point targets. This paper ends with a conclusion and summary of our discussion.

2 Nonlinear least squares fitting

Suppose that q parameters $\mathbf{a}$ are assumed to be related to the p ($>q$) measurements $\mathbf{X}$ according to

$$\mathbf{X}=\mathbf{F}(\mathbf{a})+\mathbf{e}, \tag{2}$$

where $\mathbf{F}$ represents some nonlinear continuously differentiable observation functions of $\mathbf{a}$, and $\mathbf{e}$ denotes errors with zero mean whose influence is to be eliminated. The nonlinear least squares estimate of $\mathbf{a}$ given $\mathbf{X}$ must minimize the performance index

$$\sigma_0^2=\left[\mathbf{X}-\mathbf{F}(\hat{\mathbf{a}})\right]^{\mathrm{T}}\left[\mathbf{X}-\mathbf{F}(\hat{\mathbf{a}})\right]. \tag{3}$$

For convenience, the weighting matrix or noise covariance matrix is chosen as an identity matrix, i. e. independent identically distributed errors are assumed.

The existence of the unbiased optimal solution, which is generally nonlinear in the estimate of $\mathbf{a}$, can be referenced from elsewhere[24]. There are various methods for the numerical solution of the above nonlinear estimation problem. In this paper, we have chosen the Gauss-Newton iteration with initial parameters $\mathbf{a}_k$ and step-size parameter λ

$$\left.\frac{\partial\mathbf{F}}{\partial\mathbf{a}}\right|_{\mathbf{a}_k}\Delta\mathbf{a}=\mathbf{X}-\mathbf{F}(\mathbf{a}_k)\,,\quad \mathbf{a}_{k+1}=\mathbf{a}_k+\lambda\Delta\mathbf{a} \tag{4}$$

with the terminating conditions [24]

$$\left\|\left.\frac{\partial\mathbf{F}}{\partial\mathbf{a}}\right|_{\mathbf{a}_k}^{\mathrm{T}}\left[\mathbf{X}-\mathbf{F}(\mathbf{a}_k)\right]\right\|\approx 0\,,\quad \|\Delta\mathbf{a}\|\approx 0\,,\quad \left.\sigma_0^2\right|_k-\left.\sigma_0^2\right|_{k+1}\approx 0\,. \tag{5}$$

The matrix of partial derivatives appearing in equation (4) is the Jacobian matrix $\mathbf{J}$

$$J_{ij}\equiv\frac{\partial F_i}{\partial a_j}\,. \tag{6}$$

For this type of numerical solution, the user must supply the function values vector $\mathbf{F}$ and the Jacobian matrix $\mathbf{J}$, of course, at the nearest corresponding points on the geometric feature from each given point. This is a necessary requirement for the least squares orthogonal distances fitting with the performance index of equation (3). Unfortunately, in the case of ellipse fitting, it is not easy to locate the nearest point on the ellipse from the given point.

When the function values vector $\mathbf{F}$ and the Jacobian matrix $\mathbf{J}$ are provided, the Jacobian matrix can be decomposed through SVD as below [25, 26]

$$\left.\frac{\partial\mathbf{F}}{\partial\mathbf{a}}\right|_{\mathbf{a}_k}=\mathbf{U}\mathbf{W}\mathbf{V}^{\mathrm{T}}\quad\text{with}\quad \mathbf{U}^{\mathrm{T}}\mathbf{U}=\mathbf{V}^{\mathrm{T}}\mathbf{V}=\mathbf{I}\quad\text{and} \tag{7}$$

$$\mathbf{W}=\left[diag(w_1,\cdots,w_q)\right].$$

After a successful termination of the iteration procedure, beside the performance index σ_0^2 of equation (3), the information about the quality of fitting will be provided.

The parameter covariance matrix:

$$\mathrm{Cov}(\mathbf{a})=(\mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}=(\mathbf{V}\mathbf{W}\mathbf{U}^{\mathrm{T}}\mathbf{U}\mathbf{W}\mathbf{V}^{\mathrm{T}})^{-1}=\mathbf{V}\mathbf{W}^{-2}\mathbf{V}^{\mathrm{T}},$$

$$\mathrm{Cov}(a_j,a_k)=\sum_{i=1}^{q}\left(\frac{V_{ji}V_{ki}}{w_i^2}\right),\quad j=1,\cdots,q\ ,\quad k=1,\cdots,q\,. \tag{8}$$

The variance of the estimated parameters:

$$\sigma^2(a_j)=\frac{\sigma_0^2}{p-q}\mathrm{Cov}(a_j,a_j)\quad,\ j=1,\cdots,q\,. \tag{9}$$

3 Geometric fitting of ellipse

An ellipse in a plane can be uniquely described with 5 parameters, the center coordinates X_c, Y_c, axis lengths a, b $(a\geq b)$ and pose angle α $(-\pi/2<\alpha\leq\pi/2)$(see fig. 1). For the least squares orthogonal distances fitting of ellipse, we must locate the nearest corresponding point $\mathbf{X}_i'$

(orthogonal contacting point) on the ellipse for the given point $\mathbf{X}_i$, and evaluate the Jacobian matrix at $\mathbf{X}_i'$.

We have solved these problems for the orthogonal contacting point and Jacobian matrix by introducing a temporary coordinate system xy positioned at (X_c, Y_c) and rotated with the angle α [14]. The coordinate transformation between the two coordinate systems xy and XY is (see fig. 1)

$$\mathbf{x}=\mathbf{R}(\mathbf{X}-\mathbf{X}_c) \quad \text{or} \quad \mathbf{X}=\mathbf{R}^{-1}\mathbf{x}+\mathbf{X}_c, \tag{10}$$

$$\text{with } \mathbf{R}=\begin{pmatrix} C & S \\ -S & C \end{pmatrix}, \quad C=\cos\alpha, S=\sin\alpha. \tag{11}$$

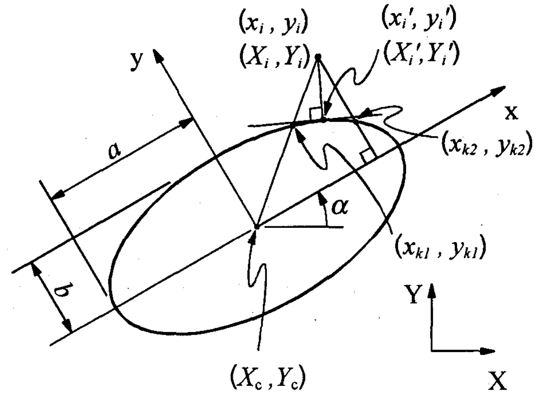


Fig. 1. An ellipse at (X_c, Y_c) with axis lengths a, b and angle α.

3.1 Orthogonal contacting point on ellipse

In the xy coordinate system, the 3 of 5 ellipse parameters disappear (X_c, Y_c and α) and the ellipse will be described only with the axis lengths a, b as below (standard position)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{12}$$

For a given point (x_i, y_i), the tangent line at the orthogonal contacting point (x, y) on the ellipse, and the connecting line of the two points are perpendicular to each other

$$\frac{dy}{dx}\cdot\frac{y_i-y}{x_i-x}=\frac{-b^2x}{a^2y}\cdot\frac{y_i-y}{x_i-x}=-1. \tag{13}$$

Rewrite the equations (12)-(13)

$$f_1(x,y)=\frac{1}{2}(a^2y^2+b^2x^2-a^2b^2)=0, \text{ and} \tag{14}$$

$$f_2(x,y)=b^2x(y_i-y)-a^2y(x_i-x)=0. \tag{15}$$

The orthogonal contacting point (x, y) on the ellipse must simultaneously satisfy equations (14)-(15) (orthogonal contacting conditions). Safaee-Rad et. al.[14] have combined the two equations (14)-(15) into one quartic equation and chosen one solution which has the shortest connecting line length among the maximum 4 real solutions. Solving this quartic equation shows a singularity, if $|x_i|\approx 0$ or $|y_i|\approx 0$. We have solved this nonlinear problem of orthogonal contacting point using the generalized Newton method as below:

$$\mathbf{Q}=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}=\begin{pmatrix} b^2x & a^2y \\ (a^2-b^2)y+b^2y_i & (a^2-b^2)x-a^2x_i \end{pmatrix} \tag{16}$$

$$\mathbf{Q}_k\Delta\mathbf{x}=-\mathbf{f}(\mathbf{x}_k), \quad \mathbf{x}_{k+1}=\mathbf{x}_k+\Delta\mathbf{x}. \tag{17}$$

We supply the initial values as below in equation (18), from the fact that the given point and its nearest corresponding point on the ellipse lie in the same quadrant of the standard position (see fig. 1).

$$\mathbf{x}_k=\frac{1}{2}(\mathbf{x}_{k1}+\mathbf{x}_{k2}), \text{ where} \tag{18}$$

$$\mathbf{x}_{k1}=\begin{pmatrix} x_i \\ y_i \end{pmatrix}\cdot ab\Big/\sqrt{b^2x_i^2+a^2y_i^2}, \text{ and}$$

$$\mathbf{x}_{k2}=\begin{cases} \begin{pmatrix} x_i \\ sign(y_i)\cdot\frac{b}{a}\sqrt{a^2-x_i^2} \end{pmatrix} & \text{if } |x_i|<a \\ \begin{pmatrix} sign(x_i)\cdot a \\ 0 \end{pmatrix} & \text{if } |x_i|\geq a. \end{cases}$$

The iteration in equations (16)-(17) with initial values from equation (18) ends after 3-4 cycles by providing adequately accurate coordinates for the orthogonal contacting point. The Jacobian matrix $\mathbf{Q}$ in equation (16) is singular, if the given point $\mathbf{x}_i$ lies in the ellipse center and if the two axes of the ellipse have the same length. In other words, there is no unique orthogonal contacting point on a circle for the circle center.

After the given point $\mathbf{X}_i$ in XY system is transformed into $\mathbf{x}_i$ in xy system using equation (10), the orthogonal contacting point $\mathbf{x}_i'$ will be found by the above generalized Newton method. Finally, we will have the point $\mathbf{X}_i'$ through a backward transformation of $\mathbf{x}_i'$ into XY system using equation (10).

And the orthogonal error distances vector will be

$$\mathbf{X}_i''=\mathbf{X}_i-\mathbf{X}_i'. \tag{19}$$

3.2 Jacobian matrix at the orthogonal contacting point on ellipse

If we define the parameters vector $\mathbf{a}$

$$\mathbf{a}=(X_c, Y_c, a, b, \alpha)^{\mathrm{T}}, \tag{20}$$

we get from the derivatives of equation (10)

$$\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{a}}\right)=\begin{pmatrix} -C & -S & 0 & 0 & y_i \\ S & -C & 0 & 0 & -x_i \end{pmatrix}, \text{ and} \tag{21}$$

$$\mathbf{J}_{\mathbf{X}_i',\mathbf{a}} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{a}}\right)\bigg|_{\mathbf{X}=\mathbf{X}_i'} = \mathbf{R}^{-1}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}}\right)\bigg|_{\mathbf{x}=\mathbf{x}_i'} + \begin{pmatrix} 1 & 0 & 0 & 0 & -xS-yC \\ 0 & 1 & 0 & 0 & xC-yS \end{pmatrix}\bigg|_{\mathbf{x}=\mathbf{x}_i'}. \qquad (22)$$

The derivatives matrix appearing in the right side of equation (22) is to be derived from equations (14)-(15) and (21), since the point $\mathbf{x}_i'$ is only implicitly known through the orthogonal contacting conditions. We differentiate f_1 and f_2 of equations (14)-(15) relatively to the parameters vector $\mathbf{a}$

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial \mathbf{a}} \\ \dfrac{\partial f_2}{\partial \mathbf{a}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial X_c} & \dfrac{\partial f_1}{\partial Y_c} & \dfrac{\partial f_1}{\partial a} & \dfrac{\partial f_1}{\partial b} & \dfrac{\partial f_1}{\partial \alpha} \\ \dfrac{\partial f_2}{\partial X_c} & \dfrac{\partial f_2}{\partial Y_c} & \dfrac{\partial f_2}{\partial a} & \dfrac{\partial f_2}{\partial b} & \dfrac{\partial f_2}{\partial \alpha} \end{pmatrix} = \mathbf{0}. \qquad (23)$$

After a series of substitutions and reductions in equations (21)-(23), we will get,

$$\mathbf{J}_{\mathbf{X}_i',\mathbf{a}} = (\mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{B})\big|_{\mathbf{x}=\mathbf{x}_i'} \qquad (24)$$

where, $\mathbf{Q}$ is the Jacobian matrix as in equation (16), and $\mathbf{B} = (\mathbf{B}_1 \ \mathbf{B}_2 \ \mathbf{B}_3 \ \mathbf{B}_4 \ \mathbf{B}_5)$, with

$$\mathbf{B}_1 = \begin{pmatrix} b^2xC - a^2yS \\ b^2(y_i-y)C + a^2(x_i-x)S \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} a(b^2-y^2) \\ 2ay(x_i-x) \end{pmatrix},$$

$$\mathbf{B}_2 = \begin{pmatrix} b^2xS + a^2yC \\ b^2(y_i-y)S - a^2(x_i-x)C \end{pmatrix}, \quad \mathbf{B}_4 = \begin{pmatrix} b(a^2-x^2) \\ -2bx(y_i-y) \end{pmatrix},$$

$$\mathbf{B}_5 = \begin{pmatrix} (a^2-b^2)xy \\ (a^2-b^2)(x^2-y^2-xx_i+yy_i) \end{pmatrix}. \qquad (25)$$

3.3 Orthogonal distances fitting of ellipse

With the Jacobian matrix $\mathbf{J}_{\mathbf{X}_i',\mathbf{a}}$ of equation (24), and the error distances vector $\mathbf{X}_i''$ of equation (19) at each point $\mathbf{X}_i'$, we construct $p\,(=2m)$ linear equations for the m given 2-dimensional points. The linear equations (4) look like

$$\begin{pmatrix} J_{X_1',X_c} & J_{X_1',Y_c} & J_{X_1',a} & J_{X_1',b} & J_{X_1',\alpha} \\ J_{Y_1',X_c} & J_{Y_1',Y_c} & J_{Y_1',a} & J_{Y_1',b} & J_{Y_1',\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{X_m',X_c} & J_{X_m',Y_c} & J_{X_m',a} & J_{X_m',b} & J_{X_m',\alpha} \\ J_{Y_m',X_c} & J_{Y_m',Y_c} & J_{Y_m',a} & J_{Y_m',b} & J_{Y_m',\alpha} \end{pmatrix} \begin{pmatrix} \Delta X_c \\ \Delta Y_c \\ \Delta a \\ \Delta b \\ \Delta \alpha \end{pmatrix} = \begin{pmatrix} X_1'' \\ Y_1'' \\ \vdots \\ X_m'' \\ Y_m'' \end{pmatrix}. \qquad (26)$$

The initial parameters vector starting the Gauss-Newton iteration may be supplied from an algebraic ellipse fitting, or from a circle fitting. The use of the circle parameters as the initial parameters for iterative ellipse fitting is also recommended, since:

- The algebraic fitting of ellipse sometimes delivers obviously incorrect parameters.
- Circle is a special case of ellipse.
- Circle fitting is more robust than ellipse fitting, thus it guarantees a reasonable initial parameters set starting iterative ellipse fitting.

When the parameters from a circle fitting are to be used as the initial values for ellipse fitting, we set

$$(X_c, Y_c)_{\text{ellipse}} = (X_c, Y_c)_{\text{circle}}, \ a = b = R \ \text{ and } \ \alpha = 0. \qquad (27)$$

If $a = b$ during iteration, the elements of $\mathbf{B}_5$ in equation (25) are all zero, and consequently, the last column of the Jacobian matrix in equation (26) will be filled with zero. In other word, the parameter α is redundant for a circle. In this case, $\Delta\alpha$ will have the solution $\Delta\alpha = 0$ from SVD and subsequent backsubstitution[25-26]. When $a < b$ after an updating of the parameters, we simply exchange the two values and set $\alpha \leftarrow \alpha - sign(\alpha)\cdot\pi/2$.

For an experimental example of the ellipse fitting, we have taken 8 coordinate pairs in table 1 [23], initial parameters $\mathbf{a}_k = (4.84,\ 4.80,\ 3.39,\ 3.39,\ 0.00)^{\mathrm{T}}$ from the geometric circle fitting, and step size $\lambda = 1.2$. After 19 Gauss-Newton steps with the norm of the terminal correction vector $\|\Delta\mathbf{a}\| = 4.17\times10^{-6}$, we have obtained $\hat{\mathbf{a}} = (2.69961,\ 3.81596,\ 6.51872,\ 3.03189,\ 0.35962)^{\mathrm{T}}$, $\sigma(\hat{\mathbf{a}}) = (3.2719,\ 1.6488,\ 3.9751,\ 0.4750,\ 0.1547)^{\mathrm{T}}$ and $\sigma_0 = 1.1719$ (see fig. 2).

Table 1. 8 coordinate pairs used for ellipse fitting.[23]

X	1	2	5	7	9	3	6	8
Y	7	6	8	7	5	7	2	4

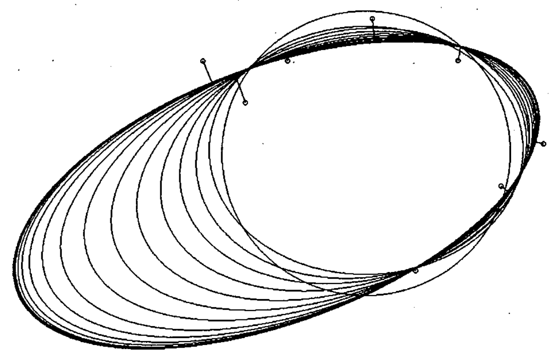


Fig. 2. Geometric ellipse fitting for a set of points. The parameter values from a geometric circle fitting are used as the initial values for the ellipse parameters.

The comparable algorithm of Gander et. al.[23] delivers the same estimations after 71 Gauss-Newton steps. Besides, they could not directly use the parameters from a circle fitting because of the singularity in their Jacobian matrix by $a = b$. To bypass the singularity problem, they have used $a = R$, $b = R/2$. But an arbitrary modification of one parameter without proper adjustments to the others can cause a divergence!

4 Ellipse fitting to circular point targets

The use of circular point targets is very common for an optical 3D measurement and camera calibration. An object circle is projected on the image plane as an ellipse, if the object target plane and the camera image plane are not parallel to each other. The image coordinates of the ellipse centers from a numbers of circular point targets are then used as the observations for the camera calibration and for the determination of the 3D target point coordinates. The accurate determination of the center of the image ellipse is very important, because the overall measuring performance and the resulting accuracy levels of the 3D measurement are directly linked to the quality of this image point determination.

In this paper, we have applied three steps procedure to the detection and fitting of the target contour. In the first step, we extract the target contour points from binary image[27] and apply an ellipse fitting to these pixel accurate contour points(fig. 4a, fig. 5a). In next step, subpixel accurate contour points will be obtained through a forced edge detection in gray image along the orthogonal path to the estimated ellipse[28](fig. 3). As a result, the succeeding ellipse fitting will be more reliable and accurate than the preceding one from the pixel accurate contour points. The length constraint on the image interpolation path keeps the edge detection from straying away in an image of low quality. In the final step, ellipse fitting to these subpixel accurate contour points will be applied, where the image gradient information across the boundary of the image ellipse can be used as the weighting values in equation (26) row by row.

We see the improvement in residuals of the subpixel accurate contouring upon the pixel accurate one, and of the geometric ellipse fitting upon the algebraic one(fig. 4, fig. 5, and table 2).

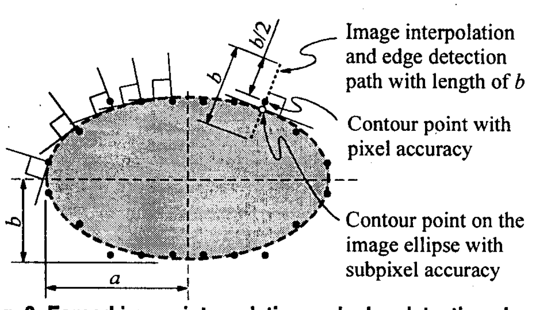
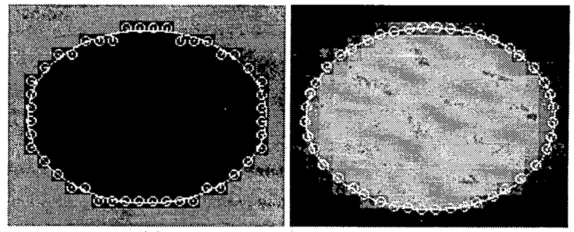
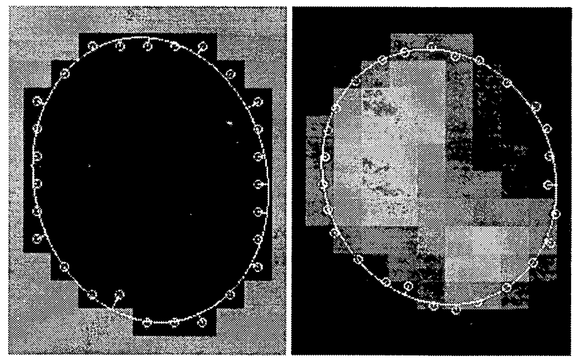


Fig. 3. Forced image interpolation and edge detection along the orthogonal path to the image ellipse[28].



(a) (b)

Fig. 4. Ellipse fitting to the image of a circular point target with high image quality: (a) Pixel accurate contour from binary image; (b) Subpixel accurate contour from gray image.



(a) (b)

Fig. 5. Ellipse fitting to the image of a circular target with low image quality: (a) Pixel accurate contour from binary image; (b) Subpixel accurate contour from gray image.

Table 2. Results of the ellipse fitting to the images of the circular point targets in fig. 4 and fig. 5.

		fig. 4a	fig. 4b	fig. 5a	fig. 5b
alg.	σ_{rms}	0.2479	0.0379	0.2579	0.1294
geo.	σ_{rms}	0.2470	0.0377	0.2543	0.1289
	σ_{Xc}	0.0437	0.0067	0.0479	0.0254
	σ_{Yc}	0.0345	0.0053	0.0560	0.0269
	σ_a	0.0518	0.0079	0.0674	0.0333
	σ_b	0.0418	0.0064	0.0575	0.0307
	σ_α	0.0172	0.0027	0.0537	0.0439

5 Discussion and conclusions

We have proposed a new algorithm for the least squares orthogonal distances fitting of ellipse. Although necessary for the iteration, the nearest point on the ellipse and the Jacobian matrix are not directly available. We have overcome these difficulties through the transformation of the ellipse into the standard position and by utilizing the orthogonal contacting conditions. The novelty of our algorithm is based on the derivatives of the coordinate transformation equations and of the orthogonal contacting equations as in equations (21)-(23). The proposed ellipse fitting is robust and the parameters from a circle fitting without modification can be used as the initial parameter values. Our comparison with other geometric ellipse fitting shows that it has a superior performance in convergence. The Gander's algorithm treated the position of the nearest point from a given point as an unknown, and assigned to each given point an additional angular parameter, which will be estimated together with the 5 ellipse parameters. But strictly speaking, the angular parameter is a function of the ellipse parameters and of the position of the given point. These additional parameters in Gander's algorithm demand a larger Jacobian matrix and computing costs, and will inherently deteriorate the convergence performance. Other two main advantages of our algorithm is that we can weight the given points, e.g. using the image gradient across the image ellipse contour, and the parameter covariance matrix is immediately available after the ellipse fitting. In summary, an accurate and robust ellipse fitting algorithm for image ellipses is proposed, and furthermore, the accuracy of the estimated ellipse parameters can be tested very simply.

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