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## Day 2

### Counting Techniques

(The  $mn$  Rule [Fundamental Counting Principle])

- First stage =  $m$  & Second stage =  $n$
- $mn$  ways to accomplish an experiment

(Extended  $mn$  Rule)

- $k$  stages with  $n_1$  ways for the first stage,  $n_2$  ways for the second stage, and  $n_k$  ways to for the  $k^{th}$  stage
- $\prod_{x=1}^k n_x$

### Permutation

#### Distinct Permutation

- Permutation of  $n$  objects is  $n!$
- Permutation of  $n$  objects  $r$  at a time

$$\begin{aligned}P(n, r) &= \frac{n!}{(n-r)!} \\&= nPr \\&= P_r^n\end{aligned}$$

- Note that  $P(n, n) = n!$

#### Repeating Permutation

- Permutation of  $n$  objects with  $k$  types

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

#### Example

How many ways can 9 beads (3 red, 2 yellow, 4 blue) lined up

$$\frac{9!}{3!2!4!} = 1260$$

#### Circular Permutation

- Permutation in a circle with one fixed object is  $(n-1)!$

#### Example

6 different colored beads (r b y p bl g) into a bracelet if red, blue, yellow are together, and purple, black must not be adjacent

$$\begin{array}{ll}3!(4-1)! = 36 & \text{rby together} \\3!2!(3-1)! = 24 & \text{rby together and pbl together} \\36 - 24 = 12 & \text{rby together, pbl not}\end{array}$$

## Combination

- Counting without arrangement

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

### Example

Possible combinations in 7 balls drawn from a set of 42 numbered balls

$$\begin{aligned} C(42, 7) &= \binom{42}{7} \\ &= \frac{42!}{7!(42-7)!} \\ &= 26\,978\,328 \end{aligned}$$

## Probability

### Uniform Probability Model

$$\begin{aligned} P(A) &= \frac{\text{no. of simple events in } A}{\text{no. of simple events in } S} \\ &= \frac{|A|}{|S|} \end{aligned}$$

- In cases where outcomes are not equally likely to occur:

$$P(A) = P(A_1) + P(A_2) + \dots + P(A_n)$$

- Note that:

- $0 \leq P(A) \leq 1$
- $P(\emptyset) = 0$
- $P(S) = 1$

- Union and Intersection

$$\begin{aligned} P(A \cup B) &= \frac{|A \cup B|}{|S|} \\ &= P(A) + P(B) \end{aligned}$$

$$P(A \cap B) = \frac{|A \cap B|}{|S|}$$

- Note the **Addition Rule**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Complement  $\longrightarrow P(A^C) = 1 - P(A)$

## Conditional Probability

### Dependent Events

- An event occurring affects the probability of the following event
- Probability of  $A$  given event  $B$  has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Note the **Multiplication Rule**

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A) \end{aligned}$$

## Independent Events

- Probability of one does not affect the other
- Independent if

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

or

$$P(A \cap B) = P(A) \cdot P(B)$$

- Mutual Independence
  - Events  $A_1, A_2, \dots, A_n$  are mutually independent if each pair of events  $A_i$  and  $A_j$  are independent.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n)$$

## Day 3

### Probability Distribution

- A formula, table, or graph that gives all the possible values  $k$  of the discrete random variable  $X$ , and the probability  $p_X(k) = P(X = k)$  associated with each value
- $p_X(k) \geq 0$
- $\sum_{\text{all } k} p_X(k) = 1$

### Example

Toss two fair coins and let  $X$  be the number of heads observed. Find the probability distribution for  $X$ .

Simple Event	Coin 1	Coin 2	Probability of Simple Event	Number of Heads Observed
	H	H		2
	H	T		1
	T	H		1
	T	T		0

## Probability Distribution Function (pdf) $p_X(k)$

$$p_X(k) = \begin{cases} \frac{1}{4} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{4} & \text{if } k = 2 \end{cases}$$

## Probability Distribution Table

0	
1	
2	

## Cumulative Distribution $F_X(k)$

- formula, table or graph that gives all the possible values  $k$  and  $F_X(k) = P(X \leq k)$ , the probability that  $X$  is at most  $k$

0	
1	
2	

## Mean or Expected Value

- The average value of  $X$  in the population

$$\mu = E(X) = \sum_{\text{all } k} k \cdot p_X(k)$$

## Standard Deviation and Variance

- Standard Deviation
  - Measures the spread or variability of the random variable

$$\begin{aligned} \sigma &= \sqrt{E((X - \mu)^2)} \\ &= \sqrt{\sum_{\text{all } k} (k - \mu)^2 \cdot p_X(k)} \end{aligned}$$

- Variance

$$\begin{aligned} \sigma^2 &= E((X - \mu)^2) \\ &= \sum_{\text{all } k} (k - \mu)^2 \cdot p_X(k) \end{aligned}$$

## Binomial Distribution

- Experiment consists of  $n$  identical trials

- Each trial results in one of two outcomes
- The probability of success on a single trial is equal to  $p$  and remains from trial to trial. Failure,  $q = 1 - p$
- Trials are independent
- Each trial is called a **Bernoulli Trial**

## Example

33 distinguishable biased coins  $\rightarrow$  0.60 heads.

Coin 1	Coin 2	Coin 3	Number of Heads	Probability
H	H	H	3	
H	H	T	2	
H	T	H	2	
H	T	T	1	
T	H	H	2	
T	H	T	1	
T	T	H	1	
T	T	T	0	

## Probability Distribution Function

- If  $p$  is the probability of success in  $n$  Bernoulli Trials, then the probability of  $k$  successes:

$$p_X(k) = P(X = k) = {}^nC_k \cdot p^k \cdot (1 - p)^{n-k}$$

for  $k = 0, 1, \dots, n$   
 aka.  $X \sim B(n, p)$

## Mean, Variance, Standard Deviation

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

$$\sigma = \sqrt{np(1 - p)}$$

## Excel

- Probability Dist Func

**=BINOM.DIST(X,N,p,FALSE)**

- Cumulative Dist Func

=BINOM.DIST(X,N,p,TRUE)

## Probability Distribution Continuous Variable

### Probability Density Function $f_X(x)$

- For all values  $x$  of  $X$ 
  - $f_X(x) \geq 0$
  - $\int_{-\infty}^{\infty} f_X(x) dx = 1$  (the total area under the curve)
- Integration is actually done over all values  $x$  that  $X$  can assume

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

- Rule Satisfaction
  - $P(X = a) = 0$
  - $P(X \geq a) = P(X > a)$
  - $P(X \leq a) = P(X < a)$
  - $P(X > a) = 1 - F_X(a)$
  - $\lim_{x \rightarrow \infty} F_X(x) = 1$
  - $\lim_{x \rightarrow -\infty} F_X(x) = 0$

### Cumulative Distribution Function $F_X(x)$

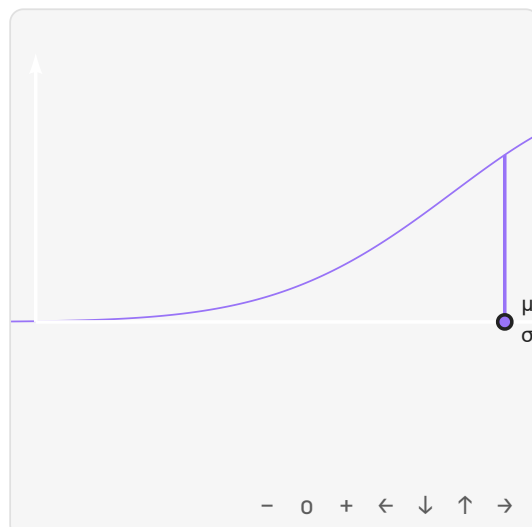
$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$\sigma = \sqrt{E((X - \mu)^2)} = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 \cdot f_X(x) dx}$$

$$\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f_X(x) dx$$

## Normal Distribution



$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

- Large values of  $\sigma$  reduce the height of the curve and increase the spread
- $X \sim N(\mu, \sigma)$

## Gaussian Curve

$$X \sim N(0, 1)$$

or

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

- $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.6827$
- $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545$
- $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.9973$

## Standardization

- Expressing a normal random variable  $X \sim N(\mu, \sigma)$  as the number of standard deviations it lies to the left or the right of its mean

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$