math

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. Then the **sum** $A + B$ is given by
$$(A + B)_{ij} = a_{ij} + b_{ij}.$$
The sum of two matrices of different sizes is undefined.
- Let $A = [a_{ij}]$ be an $m \times n$ matrix and let c be a real number. Then the **scalar multiple** of A by c, denoted cA, is given by
$$(cA)_{ij} = ca_{ij}.$$
- Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. The **matrix product** AB is the $m \times p$ matrix given by
$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Theorem 10.3.1
If A, B, and C are matrices of the appropriate sizes such that all matrix multiplications are defined, and c and d are real numbers, then the following are true.

1. $A + B = B + A$ — commutative property of addition
2. $A + (B + C) = (A + B) + C$ — associative property of addition
3. $(cd)A = c(dA)$ — associative property of scalar multiplication
4. $1A = A$ — scalar multiplicative identity
5. $A(BC) = (AB)C$ — associative property of matrix multiplication
6. $c(AB) = (cA)B$ — associative property of scalar and matrix multiplication
7. $c(A + B) = cA + cB$ — scalar multiplication distributes over addition
8. $(c + d)A = cA + dA$ — scalar addition distributes over scalar multiplication
9. $A(B + C) = AB + AC$ — matrix left-multiplication distributes over addition
10. $(A + B)C = AC + BC$ — matrix right-multiplication distributes over addition

Definition 10.3.2
A **square matrix** is a matrix which has an equal number of rows and columns.

Definition 10.3.3
The **zero matrix** of size $m \times n$, denoted $O_{m\times n}$ is a matrix where all the entries are zero. The **zero vector**, denoted $\vec{0}$, is a vector where all the entries are zero. The dimensions of a zero vector depend on the context.

Definition 10.3.4
The **identity matrix** of order n, denoted I_n is a square matrix of size $n \times n$, where
$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 10.3.5
If A is an $m \times n$ matrix and c is a real number, then the following properties are true.
1. $A + O_{m\times n} = A$.
2. $A + (-A) = O_{m\times n}$.
3. If $cA = O_{m\times n}$, then $c = 0$ or $A = O_{m\times n}$.
4. $AI_n = A$.
5. $I_m A = A$.

Theorem 10.3.7
If A and B are matrices with sizes such that the given matrix operations are defined, and c is a real number, then the following properties are true.
1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$

Definition 10.4.1
A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 10.4.2
Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix then the resulting matrix is given by the product EA.

Definition 10.4.3
Let A and B be $m \times n$ matrices. We say A is **row-equivalent** to B when there exists a finite number of elementary matrices $E_1, E_2, ..., E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = B.$$

Definition 10.4.4
An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called the **inverse** of A, and is denoted A^{-1}.
A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Theorem 10.4.7
Suppose A and B are invertible matrices of the same size, and suppose c is a nonzero real number. Then
1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \dfrac{1}{c} A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$.
5. $(AB)^{-1} = B^{-1}A^{-1}$

Example. Find a solution to the linear system

$$\begin{cases} x_1 - x_2 = 3 \\ x_1 - x_3 = 4 \\ -6x_1 + 2x_2 + 3x_3 = 5 \end{cases}$$

given that the inverse of $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$ is $\begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$.

Solution. The linear system can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Left-multiplying the inverse of the coefficient matrix gives

$$\begin{aligned} \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \\ I_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -23 \\ -26 \\ -27 \end{pmatrix} \end{aligned}$$

which gives the solution $x_1 = -23, x_2 = -26, x_3 = -27$.

Theorem 10.4.9

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 10.5.2: The RREF of a square matrix

Let A be an $n \times n$ matrix. The RREF of A is I_n if and only if A is invertible.

Theorem 10.5.3: Solving general linear systems

Let A be an $n \times n$ matrix and consider the system $A\vec{x} = \vec{b}$.
A is invertible if and only if the system has exactly one solution, given by $\vec{x} = A^{-1}\vec{b}$.
A is not invertible if and only if the system has no solution or infinitely many solutions.

$$\begin{cases} 2x+3y+z=-1 \\ 3x+3y+z=1 \\ 2x+4y+z=-2 \end{cases} \qquad \begin{cases} 2x+3y+z=4 \\ 3x+3y+z=8 \\ 2x+4y+z=5 \end{cases} \qquad \begin{cases} 2x+3y+z=0 \\ 3x+3y+z=1 \\ 2x+4y+z=0 \end{cases} \qquad \begin{cases} 2x+3y+z=3 \\ 3x+3y+z=0 \\ 2x+4y+z=1 \end{cases}$$

Solution. If we form a combined augmented matrix for this problem, we would have to reduce a 3×7 matrix to RREF. But if we instead get the inverse of the coefficient matrix, we only need to reduce a 3×6 matrix to RREF.

We can determine that $A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}$.

Since A^{-1} exists, all of the linear systems have unique solutions. We can multiply $A^{-1}\vec{b}$ for each system to obtain the solution to each system. We can do this efficiently by multiplying by a partitioned matrix.

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix} \left(\begin{array}{c|c|c|c} -1 & 4 & 0 & 3 \\ 1 & 8 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c|c|c} 2 & 4 & 1 & -3 \\ -1 & 1 & 0 & -2 \\ -2 & -7 & 2 & 15 \end{array}\right)$$

Definition 10.5.4

A system of linear equations is **homogeneous** if all the constant terms are 0; that is, if the system is of the form

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n &= 0 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n &= 0 \end{aligned}$$

Every homogeneous system of linear equations has a solution, obtained by setting all the variables to 0. This solution is called the **trivial solution**. If a homogeneous system of equations has other solutions, these are called **nontrivial solutions**.

Theorem 10.5.5: Solving homogeneous linear systems

If a homogeneous system has fewer equations than variables, then it must have infinitely many solutions.

And if we have a homogeneous linear systems involving n equations in n variables, then the invertibility of the coefficient matrix can once again be used to characterise the solutions.

Theorem 10.5.6: Solving homogeneous linear systems (n equations in n unknowns)

Let A be an $n \times n$ matrix.

A is invertible if and only if the system $A\vec{x} = \vec{0}$ has only the trivial solution.

If A is not invertible, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.