Untitled

Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area

We are given the region bounded by the curves

$$y = x^2 \text{ and } y = 4$$

These curves intersect when $x^2 = 4$, i.e.,

$$x = -2 \quad \text{and} \quad x = 2$$

Thus, the region is the set of all points (x, y) satisfying

$$x^2 \leq y \leq 4 \quad \text{with} \quad -2 \leq x \leq 2$$

Step 1. Compute the Total Area of the Region

The total area A_{total} is given by

$$A_{\text{total}} = \int_{x=-2}^{2} \left[4 - x^2\right] dx$$

Since the integrand is even, we can write:

$$A_{\text{total}} = 2\int_0^2 (4 - x^2)\, dx.$$

Compute the integral:

$$\int_0^2 (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \left(4(2) - \frac{2^3}{3}\right) - 0 = 8 - \frac{8}{3} = \frac{24 - 8}{3} = \frac{16}{3}.$$

Thus,

$$A_{\text{total}} = 2 \cdot \frac{16}{3} = \frac{32}{3}.$$

Step 2. Determine the Area that Must Be Equal

We want the horizontal line $y = b$ to divide the region into two parts of equal area. Therefore, each part must have area

$$\frac{1}{2} A_{\text{total}} = \frac{1}{2}\left(\frac{32}{3}\right) = \frac{16}{3}.$$

Step 3. Express the Area of the Lower Region

The lower part of the region is bounded by the parabola $y = x^2$ below and the line $y = b$ above. For a given y between 0 and b, the horizontal cross-section of the region runs from

$$x = -\sqrt{y} \quad \text{to} \quad x = \sqrt{y},$$

so its length is $2\sqrt{y}$.

Thus, the area A_{lower} of the lower region is

$$A_{\text{lower}} = \int_{y=0}^{b} 2\sqrt{y}\, dy.$$

Evaluate the integral:

$$\int 2\sqrt{y}\, dy = 2\int y^{1/2}\, dy = 2 \cdot \frac{2}{3} y^{3/2} = \frac{4}{3} y^{3/2}.$$

So,

$$A_{\text{lower}} = \frac{4}{3} y^{3/2} \Big|_0^b = \frac{4}{3} b^{3/2}.$$

Step 4. Set Up the Equation and Solve for bb

We want the lower region's area to equal $\frac{16}{3}$:

$$\frac{4}{3} b^{3/2} = \frac{16}{3}.$$

Multiply both sides by 3 to eliminate the denominator:

$$4b^{3/2} = 16.$$

Divide both sides by 4:

$$b^{3/2} = 4.$$

Now, raise both sides to the power $\frac{2}{3}$ to solve for b:

$$b = 4^{2/3}$$

Noticing that $4 = 2^2$, we can also write:

$$b = (2^2)^{2/3} = 2^{4/3}.$$

Alternatively, since $(2^{4/3})^3 = 2^4 = 16$, we can express the answer as

$$b = \sqrt[3]{16}.$$

Final Answer

The number b such that the line $y = b$ divides the region into two regions of equal area is:

$$\boxed{2^{4/3}} \quad \text{or equivalently} \quad \boxed{\sqrt[3]{16}}.$$

Suppose $f(x) = x^{c-1}$ and $g(x) = \frac{x^c}{c}$ for some $c > 1$. Then the area under the curve $y = f(x)$ on the interval $[0, 2]$ is given by $g(0) - g(2)$.

For any continuous function $f(x)$ with an antiderivative $g(x)$, the Fundamental Theorem of Calculus tells us that

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

.

Here, we have

$$f(x) = x^{c-1} \quad \text{and} \quad g(x) = \frac{x^c}{c},$$

with $c > 1$.

To find the area under $f(x)$ on the interval $[0, 2]$, we calculate

$$\int_0^2 x^{c-1}\,dx = g(2) - g(0).$$

Now, evaluate g at the endpoints:

$$g(2) = \frac{2^c}{c}, \quad \text{and} \quad g(0) = \frac{0^c}{c} = 0.$$

Thus, the area is

$$g(2) - g(0) = \frac{2^c}{c} - 0 = \frac{2^c}{c}.$$

The statement in the problem claims the area is $g(0) - g(2)$, but that would give

$$g(0) - g(2) = 0 - \frac{2^c}{c} = -\frac{2^c}{c},$$

which is negative and does not represent an area (since area is nonnegative).

Conclusion

The correct expression for the area under the curve $y = f(x)$ on $[0, 2]$ is

$$\boxed{\frac{2^c}{c}},$$

which comes from $g(2) - g(0)$, not $g(0) - g(2)$.

Let f and g be continuous functions on the interval $[0, 5]$. If $f > g$ on $[0, 3]$ and $f < g$ on $[3, 5]$, then the area bounded by the two curves on the interval $[2, 4]$ is $\int_2^3 (g - f)\, dx + \int_3^4 (f - g)\, dx$

FALSE

On any interval, the area between two curves is given by the integral of the absolute difference:

$$\text{Area} = \int |f(x) - g(x)|\, dx.$$

In this problem:

- On $[2, 3]$, since $f(x) > g(x)$, the difference $f(x) - g(x)$ is positive. Thus, the area on this interval is $\int_2^3 [f(x) - g(x)]\, dx$.
- On $[3, 4]$, since $f(x) < g(x)$, the difference $g(x) - f(x)$ is positive. Thus, the area on this interval is $\int_3^4 [g(x) - f(x)]\, dx$.

So the total area bounded by the curves on the interval $[2, 4]$ is

$$\int_2^3 [f(x) - g(x)]\, dx + \int_3^4 [g(x) - f(x)]\, dx.$$

However, the statement given in the problem is

$$\int_2^3 [g(x) - f(x)]\, dx + \int_3^4 [f(x) - g(x)]\, dx,$$

which reverses the differences on both intervals. This would result in negative values for each integral, rather than the correct positive area.

Thus, the correct expression for the area is:

$$\boxed{\int_2^3 (f(x) - g(x))\, dx + \int_3^4 (g(x) - f(x))\, dx.}$$

The expression given in the problem is incorrect because it reverses the order of the functions in the integrals.

Since $\int \frac{1}{x-5}\, dx = \ln|x - 5| + C$ for $C \in R$, it directly follows that $\int_4^8 \frac{1}{x-5}\, dx = \ln 3 - \ln 1 = \ln 3$
.

FALSE

The antiderivative of

$\frac{1}{x-5}$

is indeed

$\ln|x-5|+C.$

However, notice that the integrand has a vertical asymptote (a discontinuity) at $x=5$. Since the interval $[4,8]$ contains $x=5$, we must treat the integral as an improper integral.

Why the Direct Evaluation is Problematic

If we naïvely apply the Fundamental Theorem of Calculus, we might write

$$\int_4^8 \frac{1}{x-5}\,dx = \ln|8-5| - \ln|4-5| = \ln 3 - \ln 1 = \ln 3.$$

But this method ignores the fact that the integrand is not defined at x=5.

Proper Treatment as an Improper Integral

We must split the integral at the point of discontinuity and take limits:

$$\int_4^8 \frac{1}{x-5}\,dx = \lim_{a\to 5^-}\int_4^a \frac{1}{x-5}\,dx + \lim_{b\to 5^+}\int_b^8 \frac{1}{x-5}\,dx.$$

1. **For x from 4 to 5:**

$$\int_4^a \frac{1}{x-5}\,dx = \ln|a-5| - \ln|4-5| = \ln|a-5| - \ln 1 = \ln|a-5|.$$

   As $a\to 5^-$, $|a-5|\to 0$, so

$$\lim_{a\to 5^-}\ln|a-5| = -\infty.$$

2. **For x from 5 to 8:**

$$\int_b^8 \frac{1}{x-5}\,dx = \ln|8-5| - \ln|b-5| = \ln 3 - \ln|b-5|.$$

   As $b\to 5^+$, $|b-5|\to 0$, so

$$\lim_{b\to 5^+}\ln|b-5| = -\infty,$$

   and hence

$$\lim_{b\to 5^+}(\ln 3 - \ln|b-5|) = \infty.$$

Since one limit goes to $-\infty$ and the other to $+$\inftyy, the overall integral

$$\int_4^8 \frac{1}{x-5}\,dx$$

diverges.

The Cauchy Principal Value

It is worth noting that if one considers the Cauchy principal value, one defines

$$\text{p.v.} \int_4^8 \frac{1}{x-5}\,dx = \lim_{\epsilon \to 0^+} \left[\int_4^{5-\epsilon} \frac{1}{x-5}\,dx + \int_{5+\epsilon}^8 \frac{1}{x-5}\,dx \right].$$

In this symmetric approach,

$$\int_4^{5-\epsilon} \frac{1}{x-5}\,dx = \ln|5-\epsilon-5| - \ln|4-5| = \ln\epsilon - \ln 1 = \ln\epsilon,$$

and

$$\int_{5+\epsilon}^8 \frac{1}{x-5}\,dx = \ln|8-5| - \ln|5+\epsilon-5| = \ln 3 - \ln\epsilon.$$

Thus,

$$\text{p.v.} \int_4^8 \frac{1}{x-5}\,dx = \lim_{\epsilon \to 0^+} (\ln\epsilon + \ln 3 - \ln\epsilon) = \ln 3.$$

While the principal value is $\ln 3$, **this does not mean the original improper integral converges**. The actual integrals on $[4,5)$ and $(5,8]$ diverge.

Conclusion

The statement

$\int_4^8 \frac{1}{x-5}\,dx = \ln 3$

is misleading if interpreted in the usual sense of an improper integral. The correct interpretation is that the integral diverges, even though its Cauchy principal value is $\ln 3$.

Let f and f' be continuous on $[a,b]$. Show that

$$\int_a^b f(x)f'(x)\,dx = \frac{[f(b)]^2 - [f(a)]^2}{2}$$

We start by observing that the derivative of $[f(x)]^2$ is given by

$$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$$

Thus, we can write

$$f(x)f'(x) = \frac{1}{2}\frac{d}{dx}[f(x)]^2.$$

Now, integrate both sides from aa to bb:

$$\int_a^b f(x)f'(x)\,dx = \int_a^b \frac{1}{2}\frac{d}{dx}[f(x)]^2\,dx = \frac{1}{2}\int_a^b \frac{d}{dx}[f(x)]^2\,dx.$$

By the Fundamental Theorem of Calculus, we have:

$$\int_a^b \frac{d}{dx}[f(x)]^2\,dx = [f(b)]^2 - [f(a)]^2.$$

Thus,

$$\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\Big([f(b)]^2 - [f(a)]^2\Big).$$

This completes the proof:

$$\boxed{\int_a^b f(x)f'(x)\,dx = \frac{[f(b)]^2 - [f(a)]^2}{2}}$$

Let $f(x) = x^2 - c^2$ and $g(x) = c^2 - x^2$.
(a) Sketch the region A bounded by the graphs of $f(x)$ and $g(x)$ for any $c \in R$.
(b) Set-up the integral representing the area of region A.
(c) Solve for value(s) of c for which the area of region A is 576.

We are given the functions

$$f(x) = x^2 - c^2 \quad \text{and} \quad g(x) = c^2 - x^2,$$

and we wish to study the region A bounded by these curves.

(a) Sketching the Region

1. **Graphs of f and g:**
   - The graph of$ f(x)=x^2-c^2$ is an upward opening parabola with vertex at $(0, -c^2)$.
   - The graph of $g(x) = c^2 - x^2$ is a downward opening parabola with vertex at $(0, c^2)$.
2. **Intersection Points:** To find the intersections, set $f(x) = g(x)$:

   $$x^2 - c^2 = c^2 - x^2.$$

   Adding $x^2 + c^2$ to both sides gives:

   $$2x^2 = 2c^2 \quad \Longrightarrow \quad x^2 = c^2,$$

   so

   $$x = \pm c$$

   .

At $x = \pm c$, we have

$$f(\pm c) = (\pm c)^2 - c^2 = c^2 - c^2 = 0$$

,

and similarly $g(\pm c) = 0$.

3. **The Region A:**
   - For x between $-c$ and c (assuming $c > 0$ for clarity), note that: $f(x) = x^2 - c^2 \leq 0 \quad \text{and} \quad g(x) = c^2 - x^2 \geq 0.$
   - Thus, $g(x)$ lies above $f(x)$ on the interval $[-c, c]$.
   - The region A is the set of points between these curves for x from $-c\$to$ c$.

A sketch would show two parabolas symmetric about the y-axis: one opening upward with its vertex below the x-axis and the other opening downward with its vertex above the$ x $-axis, meeting at the points$(-c, 0)$ and $(c, 0)$.

(b) Setting Up the Integral for the Area

Since $g(x) \geq f(x)$ for $x \in [-c, c]$, the vertical distance between the curves is

$$g(x) - f(x) = \left[c^2 - x^2\right] - \left[x^2 - c^2\right] = 2c^2 - 2x^2.$$

Thus, the area A is given by

$$\text{Area} = \int_{-c}^{c} \Big(g(x) - f(x)\Big)\, dx = \int_{-c}^{c} \left(2c^2 - 2x^2\right) dx.$$

It is often convenient to factor out the constant:

$$\text{Area} = 2\int_{-c}^{c} \left(c^2 - x^2\right) dx.$$

(c) Solving for c When the Area is 576

1. **Evaluate the Integral:**

   The function $c^2 - x^2$ is even, so we can use symmetry:

   $$\int_{-c}^{c} (c^2 - x^2)\, dx = 2\int_{0}^{c} (c^2 - x^2)\, dx.$$

   Compute the integral on $[0, c]$:

$$\int_0^c (c^2 - x^2)\,dx = \left[c^2x - \frac{x^3}{3}\right]_0^c = c^2 \cdot c - \frac{c^3}{3} = c^3 - \frac{c^3}{3} = \frac{2c^3}{3}.$$

Then,

$$\int_{-c}^{c} (c^2 - x^2)\,dx = 2 \cdot \frac{2c^3}{3} = \frac{4c^3}{3}.$$

2. **Multiply by the Constant Factor:**

   Recall that

$$\text{Area} = 2\int_{-c}^{c} (c^2 - x^2)\,dx = 2 \cdot \frac{4c^3}{3} = \frac{8c^3}{3}.$$

3. **Set Equal to**

$$576$$

   and Solve:

   We set

$$\frac{8c^3}{3} = 576.$$

   Multiply both sides by 3:

$$8c^3 = 576 \cdot 3 = 1728.$$

   Now divide by $:

   $c^3 = \frac{1728}{8} = 216.$

   Taking the cube root:

   $c = \sqrt[3]{216} = 6.$

   Note: Although the functions are defined for any real c, the geometry of the region (with intersection points at $x = \pm c$) is most natural when $c > 0$. If one were to allow $c < 0$, the area would depend on $|c|$. In that case, the condition becomes $|c| = 6$, so $c = 6$ or $c = -6$. However, by convention we take $c = 6$.

Final Answers

- **(a)** The region A is bounded by the parabolas

$$f(x) = x^2 - c^2 \quad \text{(opening upward with vertex at } (0, -c^2))$$

and

$$g(x) = c^2 - x^2 \quad \text{(opening downward with vertex at } (0, c^2)\text{)},$$

which intersect at $(x, y) = (-c, 0)$ and $(c, 0)$. The region is the set of points between these curves for $x \in [-c, c]$.

- **(b)** The area of region A is given by

$$\text{Area} = \int_{-c}^{c} \Big[g(x) - f(x)\Big]\, dx = \int_{-c}^{c} \Big(2c^2 - 2x^2\Big)\, dx = 2\int_{-c}^{c} (c^2 - x^2)\, dx.$$

- **(c)** Evaluating the integral, we found

$$\text{Area} = \frac{8c^3}{3}$$

  Setting this equal to 576 gives

$$\frac{8c^3}{3} = 576 \quad \Longrightarrow \quad c^3 = 216 \quad \Longrightarrow \quad c = 6.$$

Thus, the value of c for which the area of region A is 576 is

$$\boxed{6}$$