Untitled 1

Theorem 10.3.1
If A, B, and C are matrices of the appropriate sizes such that all matrix multiplications are defined, and c and d are real numbers, then the following are true.

1. $A + B = B + A$ commutative property of addition
2. $A + (B + C) = (A + B) + C$ associative property of addition
3. $(cd)A = c(dA)$ associative property of scalar multiplication
4. $1A = A$ scalar multiplicative identity
5. $A(BC) = (AB)C$ associative property of matrix multiplication
6. $c(AB) = (cA)B$ associative property of scalar and matrix multiplication
7. $c(A + B) = cA + cB$ scalar multiplication distributes over addition
8. $(c + d)A = cA + dA$ scalar addition distributes over scalar multiplication
9. $A(B + C) = AB + AC$ matrix left-multiplication distributes over addition
10. $(A + B)C = AC + BC$ matrix right-multiplication distributes over addition

Definition 10.3.2
A **square matrix** is a matrix which has an equal number of rows and columns.

Definition 10.3.3
The **zero matrix** of size $m \times n$, denoted $O_{m\times n}$ is a matrix where all the entries are zero. The **zero vector**, denoted $\vec{0}$, is a vector where all the entries are zero. The dimensions of a zero vector depend on the context.

Definition 10.3.4
The **identity matrix** of order n, denoted I_n is a square matrix of size $n \times n$, where

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 10.3.5
If A is an $m \times n$ matrix and c is a real number, then the following properties are true.

1. $A + O_{m\times n} = A$.
2. $A + (-A) = O_{m\times n}$.
3. If $cA = O_{m\times n}$, then $c = 0$ or $A = O_{m\times n}$.
4. $AI_n = A$.
5. $I_m A = A$.

Theorem 10.3.7
If A and B are matrices with sizes such that the given matrix operations are defined, and c is a real number, then the following properties are true.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$
4. $(AB)^T = B^T A^T$

Definition 10.4.1
A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 10.4.2
Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix then the resulting matrix is given by the product EA.

Definition 10.4.3
Let A and B be $m \times n$ matrices. We say A is **row-equivalent** to B when there exists a finite number of elementary matrices $E_1, E_2, ..., E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = B.$$

Definition 10.4.4
An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called the **inverse** of A, and is denoted A^{-1}.
A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Theorem 10.4.7
Suppose A and B are invertible matrices of the same size, and suppose c is a nonzero real number. Then

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$.
5. $(AB)^{-1} = B^{-1}A^{-1}$

Example. Find a solution to the linear system

$$\begin{cases} x_1 - x_2 = 3 \\ x_1 - x_3 = 4 \\ -6x_1 + 2x_2 + 3x_3 = 5 \end{cases}$$

given that the inverse of $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$ is $\begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$.

Solution. The linear system can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Left-multiplying the inverse of the coefficient matrix gives

$$\begin{aligned} \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \\ I_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -23 \\ -26 \\ -27 \end{pmatrix} \end{aligned}$$

which gives the solution $x_1 = -23, x_2 = -26, x_3 = -27$.