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Day 2

Counting Techniques

(The mn Rule [Fundamental Counting Principle])

- First stage = m & Second stage = n
- mn ways to accomplish an experiment

(Extended mn Rule)

- k stages with n_1 ways for the first stage, n_2 ways for the second stage, and n_k ways to for the k^{th} stage
- $\prod_{x=1}^{k} n_x$

Permutation

Arrangement of Objects

(Distinct Permutation)

- Permutation of n objects is $n!$
- Permutation of n objects r at a time

$$
\begin{aligned}
P(n,r) &= \frac{n!}{(n-1)!} \\
&= nPr \\
&= P_r^n
\end{aligned}
$$

- Note that $P(n,n) = n!$

(Repeating Permutation)

- Permutation of n objects with k types

$$
\frac{n!}{n_1!n_2!\cdots n_k!}
$$

(Circular Permutation)

- Permutation in a circle with one fixed object is $(n-1)!$

Combination

- Counting without arrangement

$$
C(n,r) = \frac{n!}{r!(n-r)!}
$$

Probability

(Uniform Probability Model)

$$
\begin{aligned}
P(A) &= \frac{\text{no. of simple events in } A}{\text{no. of simple events in } S} \\
&= \frac{|A|}{|S|}
\end{aligned}
$$

- In cases where out comes are <u>not equally likely to occur</u>:

$$
P(A) = P(A_1) + P(A_2) + \cdots + P(A_n)
$$

- Note that:
  1. $0 \le P(A) \le 1$
  2. $P(\varnothing) = 0$
  3. $P(S) = 1$
- Union and Intersection

$$
\begin{aligned}
P(A \cup B) &= \frac{|A \cup B|}{|S|} \\
&= P(A) + P(B)
\end{aligned}
$$

$$
P(A \cap B) = \frac{|A \cap B|}{|S|}
$$

- Note the **Addition Rule**

$$
P(A \cup B) = P(A) + P(B) - P(A \cap B)
$$

- Compliment $\longrightarrow$ $P(A^C) = 1 - P(A)$

Conditional Probability

(Dependent Events)

- An event occurring affects the probability of the following event
- Probability of A given event B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Note the **Multiplication Rule**

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A) \end{aligned}$$

(Independent Events)

- Probability of one does not affect the other
- Independent if

$$\begin{gathered} P(A|B) = P(A) \\ \text{or} \\ P(B|A) = P(B) \\ \\ \text{or} \\ P(A \cap B) = P(A) \cdot P(B) \end{gathered}$$

- Mutual Independence
  - Events $A_1, A_2, \cdots, A_n$ are mutually independent if each pair of events A_i and A_j are independent.

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \cdots \cdot P(A_n)$$

Day 3

Probability Distribution

- A formula, table, or graph that gives all the possible values k of the discrete random variable X, and the probability $p_X(k) = P(X = k)$ associated with each value
- $p_X(k) \geq 0$
- $\sum_{\text{all } k} p_x(k) = 1$

(Example)
Toss two fair coins and let X be the number of heads observed. Find the probability distribution for X.

Simple Event	**Coin 1**	**Coin 2**	Probability of Simple Event	**Number of Heads Observed**
	H	H		2

Simple Event	**Coin 1**	**Coin 2**	Probability of Simple Event	**Number of Heads Observed**
	H	T		1
	T	H		1
	T	T		0

Probability Distribution Function (pdf) $p_X(k)$

$$p_X(k) = \begin{cases} \frac{1}{4} \text{ if } k = 0 \\ \frac{1}{2} \text{ if } k = 1 \\ \frac{1}{4} \text{ if } k = 2 \end{cases}$$

Probability Distribution Table

0	
1	
2	

Cumulative Distribution $F_X(k)$

- formula, table or graph that gives all the possible values k and $F_X(k) = P(X \leq k)$, the probability that X is at most k

0	
1	
2	

Mean or Expected Value

- The average value of X in the population

$$\mu = E(X) = \sum_{\text{all } k} k \cdot p_X(k)$$

Standard Deviation and Variance

(Standard Deviation)

- Measures the spread or variability of the random variable

$$\begin{aligned} \sigma &= \sqrt{E((X-\mu)^2)} \\ &= \sqrt{\sum_{\text{all } k} (k-\mu)^2 \cdot p_X(k)} \end{aligned}$$

(Variance)

$$\begin{aligned}\sigma^2 &= E((X-\mu)^2)\\ &= \sum_{\text{all } k}(k-\mu)^2 \cdot p_X(k)\end{aligned}$$

Binomial Distribution

- Experiment consists of n identical trials
- Each trial results in one of two outcomes
- The probability of success on a single trial is equal to p and remains from trial to trial. Failure, $q = 1 - p$
- Trials are independent
- Each trial is called a Bernoulli Trial

(Example)
3 distinguishable biased coins $\longrightarrow$ 0.60 heads.

Coin 1	Coin 2	Coin 3	Number of Heads	Probability
H	H	H	3	
H	H	T	2	
H	T	H	2	
H	T	T	1	
T	H	H	2	
T	H	T	1	
T	T	H	1	
T	T	T	0	

Probability Distribution Function

- If p is the probability of success in n Bernoulli Trials, then the probability of k successes:

$$p_X(k) = P(X = k) = nCp \cdot p^k \cdot (1-p)^{n-k}$$

$$\text{for } k = 0, 1, \cdots, n$$
$$\text{aka. } X \sim B(n, p)$$

Mean, Variance, Standard Deviation

$$\mu = np$$

$$\sigma^2 = np(1-p)$$

$$\sigma = \sqrt{np(1-p)}$$

Excel

(Probability Dist Func)

```
=BINOM.DIST(X,N,p,FALSE)
```

(Cumulative Dist Func)

```
=BINOM.DIST(X,N,p,TRUE)
```

Probability Distribution Continuous Variable

Probability Density Function $f_X(x)$

- For all values x of X
  - $f_X(x) \geq 0$
  - $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \quad \text{(the total area under the curve)}$
- Integration is actually done over all values x that X can assume

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

- Rule Satisfaction
  - $P(X = a) = 0$
  - $P(X \geq a) = P(X > a)$
  - $P(X \leq a) = P(X < a)$
  - $P(X > a) = 1 - F_X(a)$
  - $\lim_{x \to \infty} F_X(x) = 1$
  - $\lim_{x \to -\infty} F_X(x) = 0$

Cumulative Distribution Function $F_X(x)$

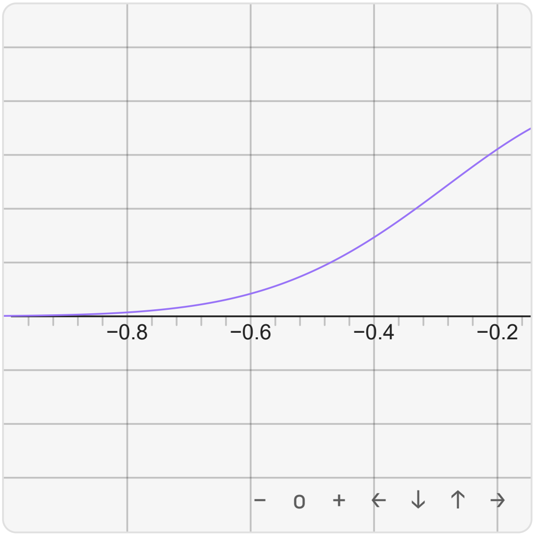
$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$\sigma = \sqrt{E((X-\mu)^2)} = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx}$$

$$\sigma^2 = \text{Var}(X) = E((X-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx$$

Normal Distribution



$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \; -\infty < x < \infty$$