math50dif

Theorem 12.1.1
The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If A is a 2×2 matrix applying a transformation on the unit square, then

- If $\det(A) > 0$, the unit square is not flipped.
- If $\det(A) < 0$, the unit square is flipped.
- The area of the unit square is scaled by $|\det(A)|$ (the absolute value of the determinant of A).

Definition 12.1.2
If A is a square matrix, then the **minor of entry $\mathbf{a_{ij}}$**, denoted M_{ij}, is the determinant of the matrix obtained by deleting the ith row and jth column of A.
The **cofactor of entry $\mathbf{a_{ij}}$**, denoted C_{ij}, is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Note that $(-1)^{i+j}$ is 1 if $i+j$ is even and -1 if $i+j$ is odd.

This can be visualized in the following sign patterns:

3×3 matrix	4×4 matrix	5×5 matrix
$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$

Definition 12.1.3: I
A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant** of A.
This gives us two ways to compute the determinant:

- Cofactor expansion along the ith row.

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

- Cofactor expansion along the jth column.

$$\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

Example. Let A be the upper triangular matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$. We can get $\det(A)$ using repeated cofactor expansion on the first column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} = a_{11} a_{22} a_{33} a_{44}$$

We can generalize the above pattern to lower triangular and diagonal matrices.

Theorem 12.1.5
The determinant of a triangular matrix is the product of the entries on the main diagonal.

Theorem 12.1.6
Let A and B be square matrices.

- When B is obtained from A by interchanging two rows of A, $\det(B) = -\det(A)$
- When B is obtained from A by multiplying a row of A by c, $\det(B) = c \cdot \det(A)$
- When B is obtained from A by adding a multiple of a row of A to another row, then $\det(B) = \det(A)$

Theorem 12.1.7
If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Theorem 12.1.8
If A is an $n \times n$ matrix and $c \in \mathbb{R}$, then $\det(cA) = c^n \det(A)$.

Theorem 12.1.9
If A is an $n \times n$ matrix then $\det(A) = \det(A^T)$.

Definition 12.2.1
If A is an $n \times n$ matrix, then a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if

$$A\vec{x} = \lambda\vec{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\vec{x}$ is said to be an eigenvector corresponding to λ.

Theorem 12.2.2
If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

This is called the **characteristic equation** of A.

To find their corresponding eigenvectors, we simply return to the definition and solve the linear system of equations induced by $(\lambda I - A)\,\vec{x} = \vec{0}$. For example, take $\lambda = 2$.

$$\begin{bmatrix} 2-7 & 15 \\ -2 & 2+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Definition 12.2.3: I
A and B are square matrices, then we say that B **is similar to** A if there is an invertible matrix P such that $B = P^{-1}AP$.

Definition 12.2.4
A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case, the matrix P is said to **diagonalize** A.

Theorem 12.2.5
Let A be a square diagonal matrix. Then, for any non-negative integer k,

$$\begin{bmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ 0 & 0 & a_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{bmatrix}^k = \begin{bmatrix} (a_{1,1})^k & 0 & 0 & \dots & 0 \\ 0 & (a_{2,2})^k & 0 & \dots & 0 \\ 0 & 0 & (a_{3,3})^k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (a_{n,n})^k \end{bmatrix}.$$

Theorem 12.2.6
Let A be diagonalizable, i.e. there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Then, for any non-negative integer k,

$$A^k = PD^kP^{-1}.$$

Theorem 12.2.7
If A is an $n \times n$ matrix, it is diagonalizable **if and only if** it has n linearly independent eigenvectors. In such a case, let $(\vec{p}_1, \lambda_1), (\vec{p}_2, \lambda_2), \dots, (\vec{p}_n, \lambda_n)$ be our n linearly independent eigenvectors with their corresponding eigenvalues. Then,

$$P = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \dots & \vec{p_n} \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

That is, the matrix P can be found by taking the n linearly independent eigenvectors as column vectors (in any order), and by putting their corresponding eigenvalues along the diagonal matrix's diagonal (in that same order).