math50 lt4

Definition 11.4.1
Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Determine all vectors in $\mathbb{R}^2$ that are orthogonal to $\mathbf{u} = (2, 4)$.

Solution. Let $\mathbf{v} = (v_1, v_2)$. Then

$$\mathbf{u} \cdot \mathbf{v} = 2v_1 + 4v_2.$$

We want to find all values of v_1 and v_2 such that

$$2v_1 + 4v_2 = 0.$$

Rearranging this equation gives

$$v_1 = -2v_2$$

The solutions of this equation are given by

$$\left\{ \begin{bmatrix} -2r \\ r \end{bmatrix} \;\middle|\; r \in \mathbb{R} \right\}.$$

This is the set of all vectors which are orthogonal to $\mathbf{u}$.

Theorem 11.4.3
The **orthogonal projection** of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$

2. Let $\mathbf{u} = (2, 4, 1, -2)$ and $\mathbf{v} = (3, 3, 3, -3)$.
   (a) Compute the distance between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Compute the angle between $\mathbf{u}$ and $\mathbf{v}$ (in radians).
   (c) Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.
   (d) Find two vectors in $\mathbb{R}^4$ parallel to $\mathbf{v}$ with the same length as $\mathbf{u}$.

Answer:

(a)

$$||\vec{u} - \vec{v}|| = \sqrt{(2-3)^2 + (4-3)^2 + (1-3)^2 + (-2-(-3))^2} = \sqrt{7}.$$

(b)

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{27}{\sqrt{25}\sqrt{36}} = \frac{27}{30} = \frac{9}{10}.$$
$$\theta = \arccos\left(\frac{9}{10}\right) \approx 0.4510$$

(c)

$$\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{27}{36}(3, 3, 3, -3) = \frac{3}{4}(3, 3, 3, -3) = \left(\frac{9}{4}, \frac{9}{4}, \frac{9}{4}, \frac{-9}{4}\right)$$

(d) The answer will be of the form $k\vec{v}$ for some k. If we set

$$||k\vec{v}|| = ||\vec{u}||$$
$$|k| = \frac{||\vec{u}||}{||\vec{v}||} = \frac{5}{6}$$
$$k = \pm\frac{5}{6}.$$

Then, multiplying this to $(3, 3, 3, -3)$, we get the answers,

$$\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{-5}{2}\right),$$
$$\left(\frac{-5}{2}, \frac{-5}{2}, \frac{-5}{2}, \frac{5}{2}\right).$$

3. Consider the linear transformation $T(x, y) = (2x + 3y, x - y)$.

   (a) Give the standard matrix for T.

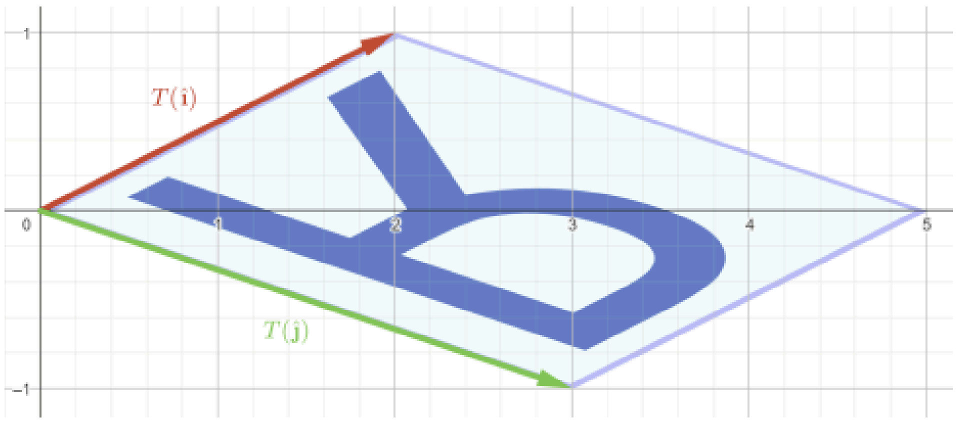
   (b) Sketch the effect of this linear transformation on the unit square.

   (c) Find x and y so that $T(x, y) = (11, 2)$.

   Answer:

   (a) $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

   (b) Image of unit square:



   (c) $A^{-1} \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \dfrac{1}{-2-3} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \begin{bmatrix} 17/5 \\ 7/5 \end{bmatrix}$

6. Let S be the set given by $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(a) Show that S spans $\mathbb{R}^2$.

(b) Explain why S does not form a basis for $\mathbb{R}^2$.

Answer:

(a) Any (x, y) can be expressed as $(1/2)(\vec{v_1} - \vec{v_2})x + \vec{v_2}y$. But many students will do the Gauss Jordan, so let's show that here.

We wish to show that for any x, y, there exist k_1, k_2, k_3 such that

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

So, perform the Gauss Jordan,

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x/2 \\ y \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -5/2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x/2 \\ y - x/2 \end{bmatrix}.$$

Let $k_3 = t$, for any $t \in \mathbb{R}$. Then,

$$k_1 + k_3(1/2) = x/2$$
$$k_2 - k_3(5/2) = y - x/2$$

can be rearranged into

$$k_1 = x/2 - t(1/2)$$
$$k_2 = y - x/2 + t(5/2)$$

which then gives us a formula for k_1 and k_2 which works for any $x, y \in \mathbb{R}$.

(b) Here are some possible arguments.

- There are $3 \neq 2$ vectors.
- By the Gauss Jordan, there are many nontrivial linear combinations which would yield $(0, 0)$, therefore the vectors are linearly dependent.

7. Let S be the set given by $\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\4\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\-2\end{bmatrix}$

(a) Show that S is linearly independent, and explain why S is a basis for $\mathbb{R}^3$.

(b) Find the coordinates of $(1, 0, -1)$ relative to S.

(c) If the coordinates of $\mathbf{x}$ relative to S are $(-5, 7, 8)$, find the coordinates of $\mathbf{x}$ relative to the standard basis.

Answer:

(a) Argue that $A = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 4 & -2\end{bmatrix}$ is invertible by performing Gauss-Jordan to show its RREF is the identity matrix (such as what will be shown in the next part). Then, because there $|S| = 3$ vectors, this is a basis.

(b) We are solving,

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}0\\1\\4\end{bmatrix} + z\begin{bmatrix}0\\0\\-2\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix},$$

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 4 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

which is straightforward with Gauss-Jordan or back-substitution.

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 4 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-2\\-4\end{bmatrix},$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-2\\4\end{bmatrix},$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-2\\-2\end{bmatrix}.$$

Thus, the answer is $(1, -2, -2)$.

(c) Just perform the matrix multiplication.

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 4 & -2\end{bmatrix}\begin{bmatrix}-5\\7\\8\end{bmatrix} = \begin{bmatrix}-5\\-3\\-3\end{bmatrix}$$

Answer:

Rotate $90°$ counterclockwise and then reflect across $x = 0$:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Reflect across $x = 0$ and then rotate $90°$ clockwise:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We conclude that the transformations are the same due to being represented by the same underlying matrix.