mathVec30.24

Definition of a Vector

A vector in three-dimensional space is an ordered triple of real numbers $\langle x, y, z \rangle$. The numbers of x, y, z are called the components of the vector. We denote the set of all such ordered triple as V_3.

Definition of a Zero Vector

A zero vector is vector whose components are all zero and has no direction. Mathematically, $\langle 0, 0, 0 \rangle$.

Magnitude of a Vector

If $\mathrm{A} = \left\langle a_1, a_2, a_3 \right\rangle$, then the magnitude of the vector A is given by $|\mathrm{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Direction of a Two Dimensional Vector

The direction of a two dimensional vector is given by $\theta = \tan^{-1}\dfrac{a_2}{a_1}$.

If $a_1 = 0, a_2 > 0$ then $\theta = \dfrac{\pi}{2}$. If $a_1 = 0, a_2 < 0$ then $\theta = \dfrac{3\pi}{2}$.

Sum of Two Vectors

If $\mathrm{A} = \left\langle a_1, a_2, a_3 \right\rangle$ and $\mathrm{B} = \left\langle b_1, b_2, b_3 \right\rangle$, then the sum of the two vectors is given by $\mathrm{A} + \mathrm{B} = \left\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \right\rangle$.

Negative of a Vector

If A = $\langle a_1, a_2, a_3 \rangle$, then the negative of **A**, denoted by **−A**, is given by **−A** = $-\langle a_1, a_2, a_3 \rangle = \langle -a_1, -a_2, -a_3 \rangle$. The direction is opposite the direction of the original vector**A** (π radians or 180^0 degrees).

Difference of Two Vectors

If A = $\langle a_1, a_2, a_3 \rangle$and B = $\langle b_1, b_2, b_3 \rangle$, then

A − B = $\langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$.

Scalar Multiple of a Vector

If c is a scalar and A = $\langle a_1, a_2, a_3 \rangle$, then cA = $\langle ca_1, ca_2, ca_3 \rangle$.

Properties of Vector Operations

If **A, B, C** ∈ V_3, c, d ∈ R, then

(a) **A + B = B + A**

(b) **A + (B + C) = (A + B) + C**

(c) There exists a vector **0** ∈ V_3 such that **A + 0 = A**.

(d) There exists a vector **−A** ∈ V_3such that **A + (− A) = 0**.

(e) c(d**A**) = (cd)**A**

(f) c(**A + B**) = c**A** + c**B**

(g) (c + d)**A** = c**A** + d**A**

(h) 1(**A**) = **A**

Definition of a Real Vector Space

A real vector space V is a set of elements, called vectors, together with the set of real numbers, called scalars with two operations called 'addition' and 'multiplication' such that all properties above are satisfied.

The magnitude of each of the vectors ⟨1, 0, 0⟩, ⟨0, 1, 0⟩, ⟨0, 0, 1⟩is 1 unit. So we call these as unit vectors. We introduce the following notations: **i** = ⟨1, 0, 0⟩, **j** = ⟨0, 1, 0⟩, **k** = ⟨0, 0, 1⟩. We call **i**, **j**, **k** as basis vectors. They form the basis for the vector space V_3. The number of elements in a basis of a vector space is called the dimension of a vector space. Hence, $V_3$is a three dimensional vector space.

Unit Vector

Consider a nonzero vector **A** = a_1**i** + a_2**j** + a_3**k**. The unit vector having the same direction as **A** is given by:

$$\hat{\mathbf{A}} = \frac{a_1}{|\mathbf{A}|}\mathbf{i} + \frac{a_2}{|\mathbf{A}|}\mathbf{j} + \frac{a_3}{|\mathbf{A}|}\mathbf{k}$$

Definition of the Dot Product

If **A** = a_1**i** + a_2**j** + a_3**k** and **B** = b_1**i** + b_2**j** + b_3**k** then the dot product between **A** and **B**, denoted as **A** · **B** is given by
A · **B** = $a_1b_1 + a_2b_2 + a_3b_3$.

Properties of the Dot Product

If **A**, **B**, **C** ∈ V_3 and c ∈ R, then
(a) **A** · **B** = **B** · **A**.
(b) **A** · (**B** + **C**) = **A** · **B** + **A** · **C**.
(c) c(**A** · **B**) = (c**A**) · **B**
(d) **0** · **A** = **A** · **0** = 0
(e) **A** · **A** = $|\mathbf{A}|^2$

Theorem

Let A, B $\in V_3$ and α is the angle (measured in radians) formed by the two vectors, then $A \cdot B = |A|\,|B| \cos\alpha$.

that the vectors are perpendicular to each other. In most of engineering, we say that the vectors are *orthogonal* to each other.

Definition of Parallel Vectors

Two vectors are parallel to each other if they are a scalar multiple of the other.

Definition of Perpendicular/Orthogonal Vectors

Two vectors are perpendicular (or orthogonal) to each other if and only if their dot product is zero.

Definition of a Cross Product

Let $A = \langle a_1, a_2, a_3 \rangle$ and $B = \langle b_1, b_2, b_3 \rangle$, then the cross product, $A \times B$, is given by

$$A \times B = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

$$A \times B = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties of Vector Products

Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V_3$ and $c \in R$, then

(a)$\mathbf{A} \times \mathbf{A} = \mathbf{0}$. The determinant of a matrix with at least two equal rows/columns is zero.

(b) $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The determinant of a matrix whose rows or columns are switched is the negative to that of the original matrix. This means that vector products are not commutative.

(c) $\mathbf{0} \times \mathbf{A} = \mathbf{A} \times \mathbf{0} = \mathbf{0}$.

(d) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

(e) $(c\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (c\mathbf{B}) = c(\mathbf{A} \times \mathbf{B})$.

Product Identities

Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V_3$

(a) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$

(b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Theorem

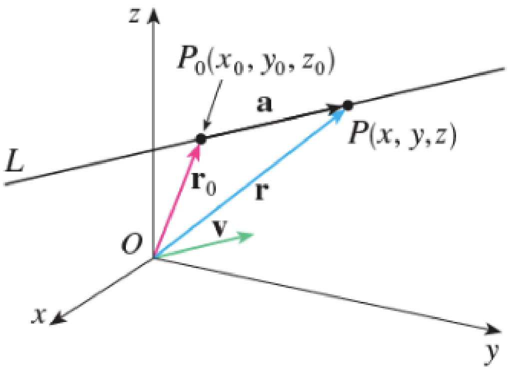
Let $\mathbf{A}, \mathbf{B} \in V_3$ and θ be the angle between these two vectors, then $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\sin\theta$.

Theorem: Parallel Vectors

Two nonzero vectors are parallel if and only if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Vector equation of the line

Let us consider a known point $P_0(x_0, y_0, z_0)$ and another (arbitrary) point $P(x, y, z)$ somewhere on the line. We can create a vector from the origin to the point P_0 and denote it as the vector $\mathbf{r}_0$. Likewise, we can create a vector from the origin to the arbitrary point and name it as $\mathbf{r}$. We can create a vector $\mathbf{a}$ with initial point P_0 to terminal point P, then by the parallelogram rule we have $\mathbf{r}_0 + \mathbf{a} = \mathbf{r}$.



Parametric equation of the line

Here, the vector v gives the direction of the line and is therefore called the direction vector. We can represent the components of v as $\langle a, b, c\rangle$. If we expand everything in terms of its components, the vector equation of the line can be expressed as $\left\langle x_0, y_0, z_0\right\rangle + t\langle a, b, c\rangle = \langle x, y, z\rangle$. Remember that two vectors are equal if and only if their corresponding components are equal, thus:

$$x_0 + ta = x \quad y_0 + tb = y \quad z_0 + tc = z$$

These are known as the parametric equations of the line.

Symmetric equation of the line

Another way to represent a line is to solve all these equations for t: $t = \frac{x - x_0}{a}$, $t = \frac{y - y_0}{b}$, and $t = \frac{z - z_0}{c}$. Since all of these are equal to t, then we can say:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

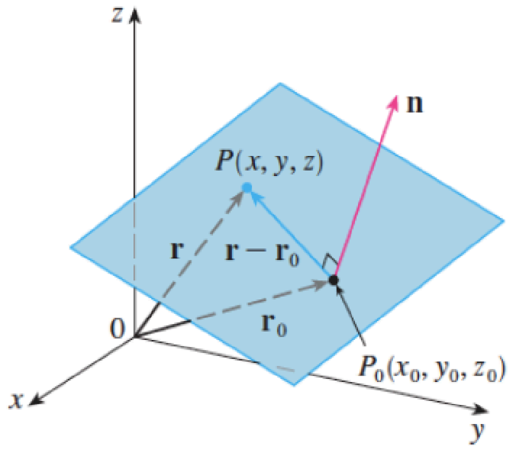
This is known as the symmetric equation of the line.

You might be asking: how is this similar to the equation of the line that we know in two dimensions? (The form is quite different as of now.) But, for two dimensions, we can set $z = z_0 = c = 0$. From the symmetric equation, we have: $\frac{x - x_0}{a} = \frac{y - y_0}{b} \Leftrightarrow \frac{b}{a}\left(x - x_0\right) = y - y_0$. Notice that this is exactly the equation of the line in two dimensions, with slope $\frac{b}{a}$. It makes perfect sense, too, to call $\langle a, b, c\rangle$ as the direction numbers they determine the slope of the line.

In two dimensions, we talk about lines that are parallel (same slope), intersecting, or two lines that one and the same (equation is a scalar multiple of the other). In three dimensions, parallel lines are those that have the same direction vector and do not intersect. Skew lines are lines whose points are in different planes: they have different direction vectors, but the lines do not intersect. Two lines whose equations are scalar multiples of the other are one and the same line (and intersect at infinitely many points). Finally, the last category would be intersecting lines at a point.

Equation of a plane in space

Consider a known point $P_0(x_0, y_0, z_0)$ and an arbitrary point $P(x, y, z)$. We let the vectors $\mathbf{r}_0$ and $\mathbf{r}$ be the vectors with initial points at the origin and terminal points at P_0 and P , respectively. In lines, the direction vector describes where the line is directed at, then in planes, its direction is defined as the normal vector. The normal vector is defined as the vector orthogonal to every point on the plane. We denote this as $\mathbf{n}$. So the vector equation of the plane is given by $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$.



If we write $\mathbf{n} = \langle a, b, c \rangle$, then we can talk about the scalar equation of a plane which we can write as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Two planes are parallel if their normal vectors are parallel. Two planes are perpendicular if their normal vectors are perpendicular.

A plane has two normal vectors. For example, for a plane lying horizontally, the normal vectors are pointing up and pointing down. You can use whichever, since one is just the negative of the other.

Example 3

Find the equation of the line formed by intersecting planes $z = 2x - y - 5$ and $z = 4x + 3y - 5$.

When two distinct planes intersect, they form a line. We need to find a point on the line first before we can make an equation out of it. To do this, we can set arbitrarily to $x = 0$, then solve for the remaining variables. This means that $z = -y - 5$ and $z = 3y - 5$. From here, we see that $y = 0$ and $z = -5$. The point $(0, 0, -5)$ satisfies both equations, hence it must be on the line of intersection.

Next, we need to find a direction vector. Picture two intersecting planes, each with their own normal vectors. The direction vector must be 'parallel' to the line (because it is the one giving direction to the line). This direction vector must be perpendicular to both the planes. So to get this direction vector, we need to take the cross product of the two normal vectors.

The normal vector to $z = 2x - y - 5$ is $\mathbf{n}_1 = \langle 2, -1, -1 \rangle$, while the normal vector to $z = 4x + 3y - 5$ is $\mathbf{n}_2 = \langle 4, 3, -1 \rangle$. The direction vector is given by

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 4 & 3 & -1 \end{vmatrix} = 4\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}.$$

Thus, the equation of the line is given by $\mathbf{r} = \langle 0, 0, -5 \rangle + t\langle 4, -2, 10 \rangle$.