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Day 2

Counting Techniques

(The mn Rule [Fundamental Counting Principle])

- First stage = m & Second stage = n
- mn ways to accomplish an experiment

(Extended mn Rule)

- k stages with n_1 ways for the first stage, n_2 ways for the second stage, and n_k ways to for the k^{th} stage
- $\prod_{x=1}^k n_x$

Permutation

Distinct Permutation

- Permutation of n objects is $n!$
- Permutation of n objects r at a time

$$\begin{aligned}P(n, r) &= \frac{n!}{(n-r)!} \\&= nPr \\&= P_r^n\end{aligned}$$

- Note that $P(n, n) = n!$

Repeating Permutation

- Permutation of n objects with k types

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example

How many ways can 9 beads (3 red, 2 yellow, 4 blue) lined up

$$\frac{9!}{3!2!4!} = 1260$$

Circular Permutation

- Permutation in a circle with one fixed object is $(n-1)!$

Example

6 different colored beads (**r** **b** **y** **p** **bl** **g**) into a bracelet if **red**, **blue**, **yellow** are together, and **purple**, **black** must not be adjacent

$$\begin{aligned}3!(4-1)! &= 36 && \text{rby together} \\3!2!(3-1)! &= 24 && \text{rby together and pbl together} \\36 - 24 &= 12 && \text{rby together, pbl not}\end{aligned}$$

Combination

- Counting without arrangement

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Example

Possible combinations in 7 balls drawn from a set of 42 numbered balls

$$\begin{aligned}C(42, 7) &= \binom{42}{7} \\&= \frac{42!}{7!(42-7)!} \\&= 26\,978\,328\end{aligned}$$

Probability

Uniform Probability Model

$$\begin{aligned}P(A) &= \frac{\text{no. of simple events in } A}{\text{no. of simple events in } S} \\&= \frac{|A|}{|S|}\end{aligned}$$

- In cases where out comes are not equally likely to occur:

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

- Note that:
 - $0 \leq P(A) \leq 1$
 - $P(\emptyset) = 0$
 - $P(S) = 1$
- Union and Intersection

$$P(A \cup B) = \frac{|A \cup B|}{|S|}$$

$$= P(A) + P(B)$$

$$P(A \cap B) = \frac{|A \cap B|}{|S|}$$

- Note the **Addition Rule**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Compliment $\longrightarrow P(A^C) = 1 - P(A)$

Conditional Probability

Dependent Events

- An event occurring affects the probability of the following event
- Probability of A given event B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Note the **Multiplication Rule**

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A) \end{aligned}$$

Independent Events

- Probability of one does not affect the other
- Independent if

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

or

$$P(A \cap B) = P(A) \cdot P(B)$$

- Mutual Independence
 - Events A_1, A_2, \dots, A_n are mutually independent if each pair of events A_i and A_j are independent.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n)$$

Day 3

Probability Distribution

- A formula, table, or graph that gives all the possible values k of the discrete random variable X , and the probability

$p_X(k) = P(X = k)$ associated with each value

- $p_X(k) \geq 0$
- $\sum_{\text{all } k} p_X(k) = 1$

Example

Toss two fair coins and let X be the number of heads observed. Find the probability distribution for X .

Simple Event	Coin 1	Coin 2	Probability of Simple Event	Number of Heads Observed
			$P(E_i)$	X
E_1	H	H	$\frac{1}{4}$	2
E_2	H	T	$\frac{1}{4}$	1
E_3	T	H	$\frac{1}{4}$	1
E_4	T	T	$\frac{1}{4}$	0

Probability Distribution Function (pdf) $p_X(k)$

$$p_X(k) = \begin{cases} \frac{1}{4} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{4} & \text{if } k = 2 \end{cases}$$

Probability Distribution Table

k	$p_X(k)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Cumulative Distribution $F_X(k)$

- formula, table or graph that gives all the possible values k and $F_X(k) = P(X \leq k)$, the probability that X is at most k

k	$F_X(k)$
0	$p_X(0) = \frac{1}{4}$
1	$p_X(0) + p_X(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$
2	$p_X(0) + p_X(1) + p_X(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

Mean or Expected Value

- The average value of X in the population

$$\mu = E(X) = \sum_{\text{all } k} k \cdot p_X(k)$$

Standard Deviation and Variance

- Standard Deviation
 - Measures the spread or variability of the random variable

$$\begin{aligned}\sigma &= \sqrt{E((X - \mu)^2)} \\ &= \sqrt{\sum_{\text{all } k} (k - \mu)^2 \cdot p_X(k)}\end{aligned}$$

- Variance

$$\begin{aligned}\sigma^2 &= E((X - \mu)^2) \\ &= \sum_{\text{all } k} (k - \mu)^2 \cdot p_X(k)\end{aligned}$$

Binomial Distribution

- Experiment consists of n identical trials
- Each trial results in **one of two** outcomes
- The probability of success on a single trial is equal to p and remains from trial to trial. Failure, $q = 1 - p$
- Trials are independent
- Each trial is called a **Bernoulli Trial**

Example

33 distinguishable biased coins \rightarrow 0.60 heads.

Coin 1	Coin 2	Coin 3	Number of Heads	Probability
H	H	H	3	0.216
H	H	T	2	0.144
H	T	H	2	0.144
H	T	T	1	0.096
T	H	H	2	0.144
T	H	T	1	0.096
T	T	H	1	0.096
T	T	T	0	0.064

Probability Distribution Function

- If p is the probability of success in n Bernoulli Trials, then the probability of k successes:

$$p_X(k) = P(X = k) = nCk \cdot p^k \cdot (1 - p)^{n-k}$$

for $k = 0, 1, \dots, n$
 aka. $X \sim B(n, p)$

Mean, Variance, Standard Deviation

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

$$\sigma = \sqrt{np(1 - p)}$$

Excel

- Probability Dist Func

```
=BINOM.DIST(X,N,p,FALSE)
```

- Cumulative Dist Func

```
=BINOM.DIST(X,N,p,TRUE)
```

Examples

6 chips (N Total), 2 defective (D Defective). $X = \text{defective} = 0, 1, 2$.
Find probability distribution

1. $\binom{D}{k} \rightarrow$ ways to choose k defective chips
2. $\binom{N-D}{n-k} \rightarrow$ ways to choose the rest $(n-k)$
3. $\binom{N}{n} \rightarrow$ total number of possible ways to choose any n chips

$$\frac{\binom{2}{k} \binom{4}{3-k}}{\binom{6}{3}}$$

k	0	1	2
$p_X(k)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

4 laptops, probability distribution

k	0	1	2	3	4	5
$p_X(k)$	0.1	0.4	0.2	0.15	0.1	0.05

Find mean, variance, standard deviation

$$\begin{aligned}\mu &= \sum_{k=0}^5 k \cdot p_X(k) \\ &= 0(0.10) + 1(0.40) + 2(0.20) + 3(0.15) + 4(0.10) + 5(0.05) \\ &= 1.90\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \sum_{k=0}^5 (k - 1.90)^2 p_X(k) \\ &= (0 - 1.90)^2(0.10) + (1 - 1.90)^2(0.40) + \dots + (5 - 1.90)^2(0.05) \\ &= 1.79 \\ \sigma &= \sqrt{1.79} \\ &\approx 1.34\end{aligned}$$

5000 tickets @ Php 100 each. Jackpot = Php 25,00. Expected gain of 4 tickets?

$$\begin{aligned}
\mu = E(X) &= \sum_{k \in \{-400, 24\,600\}} k \cdot p_X(k) \\
&= (-400) \cdot \frac{4\,996}{5\,000} + 24\,600 \cdot \frac{4}{5\,000} \\
&= -380
\end{aligned}$$

Probability Distribution Continuous Variable

Probability Density Function $f_X(x)$

- For all values x of X
 - $f_X(x) \geq 0$
 - $\int_{-\infty}^{\infty} f_X(x) dx = 1$ (the total area under the curve)
- Integration is actually done over all values x that X can assume

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

- Rule Satisfaction
 - $P(X = a) = 0$
 - $P(X \geq a) = P(X > a)$
 - $P(X \leq a) = P(X < a)$
 - $P(X > a) = 1 - F_X(a)$
 - $\lim_{x \rightarrow \infty} F_X(x) = 1$
 - $\lim_{x \rightarrow -\infty} F_X(x) = 0$

Cumulative Distribution Function $F_X(x)$

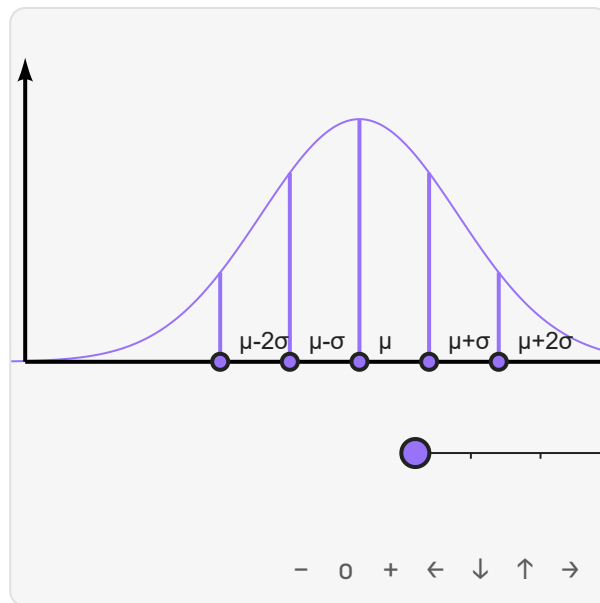
$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$\sigma = \sqrt{E((X - \mu)^2)} = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 \cdot f_X(x) dx}$$

$$\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f_X(x) dx$$

Normal Distribution



$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

- Large values of σ reduce the height of the curve and increase the spread
- $X \sim N(\mu, \sigma)$

Gaussian Curve

$$X \sim N(0, 1)$$

or

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

- $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.6827$
- $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545$
- $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.9973$

Standardization

- Expressing a normal random variable $X \sim N(\mu, \sigma)$ as the number of standard deviations it lies to the left or the right of its mean

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$