math50 q5

Definition 11.2.1

If T is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, we say that T is a **transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$, or that T **maps** from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted by

$$T : \mathbb{R}^n \to \mathbb{R}^m.$$

In the special case where $m = n$, a transformation is sometimes called an **operator** on $\mathbb{R}^n$.

Transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ often arise from linear systems. Suppose we have the linear system

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \qquad \vdots \\ y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{cases} \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or, as a matrix equation,

$$\mathbf{y} = A\mathbf{x}.$$

We have previously viewed this as a problem where we are given A and $\mathbf{y}$, and we are to solve for $\mathbf{x}$. But an equation of this form looks like a regular *function* (such as $y = 2x$).

If $A \in \mathbb{R}^{m\times n}$, we can think of the equation $\mathbf{y} = A\mathbf{x}$ as a function that maps an input vector $\mathbf{x} \in \mathbb{R}^n$ to an output vector $\mathbf{y} \in \mathbb{R}^m$.

Definition 11.2.2
Let A be an $m \times n$ matrix. Then we denote with T_A the function or transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$

where

$$T_A(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$. The matrix A is called the **standard matrix** for T_A.
If $\mathbf{x} = (x_1, x_2, ..., x_n)$, we may alternatively write

$$T_A(\mathbf{x}) = T_A(x_1, x_2, ..., x_n)$$

The functions above are called **linear transformations** because of the following properties:

Theorem 11.2.3
Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$. Then for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$,
(a) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$
(b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$

Example. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Then, T_A is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Describe the output of T_A with respect to its input algebraically.

Solution.

The value of $T_A(x, y)$ is $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$.

Therefore, we can describe T_A by $T_A(x, y) = (2x, -y)$.

Example. Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Then, T_A is a function from $\mathbb{R}^3$ to $\mathbb{R}^2$. Describe the output of T_A with respect to its input algebraically.

Solution.

The value of $T_A(x, y, z)$ is $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y + z \\ x + 2z \end{bmatrix}$.

Therefore, we can describe T_A by $T_A(x, y, z) = (z - y, x + 2z)$.

To see why, consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Notice what happens if we multiply A to the standard basis vectors.

$$A\mathbf{e_1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad A\mathbf{e_2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

In general, multiplying A by the standard basis vector $\mathbf{e}_i$ will yield the ith column of A.

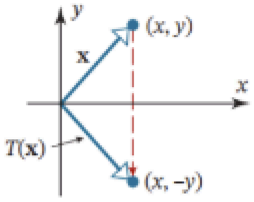
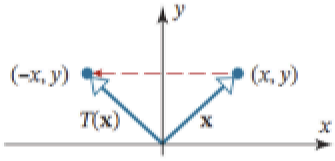
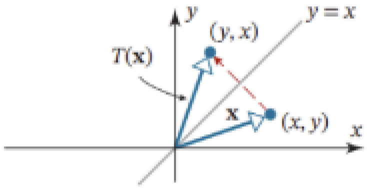
Since matrix multiplication by A is the same as applying the transformation T_A, we now have a procedure for obtaining the matrix for a given transformation.

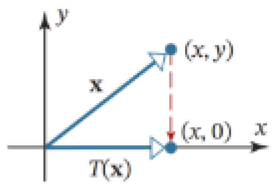
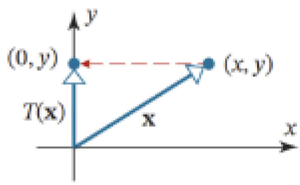
Theorem 11.2.4
Let A be an $m \times n$ matrix, and consider $T_A : \mathbb{R}^n \to \mathbb{R}^m$. Then,

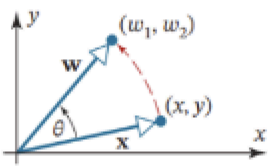
$$A = \left[\ T_A(\mathbf{e}_1) \ \middle|\ T_A(\mathbf{e}_2) \ \middle|\ \cdots \ \middle|\ T_A(\mathbf{e}_n)\ \right]$$

Theorem 11.2.5
A function $f : \mathbb{R}^n \to \mathbb{R}^m$ has a standard matrix if and only if f is a linear transformation, i.e. it satisfies

(a) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$
(b) $f(k\mathbf{u}) = kf(\mathbf{u})$

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x,y) = (x, -y)$		$T(\mathbf{e}_1) = T(1,0) = (1,0)$ $T(\mathbf{e}_2) = T(0,1) = (0,-1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x,y) = (-x, y)$		$T(\mathbf{e}_1) = T(1,0) = (-1,0)$ $T(\mathbf{e}_2) = T(0,1) = (0,1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x,y) = (y, x)$		$T(\mathbf{e}_1) = T(1,0) = (0,1)$ $T(\mathbf{e}_2) = T(0,1) = (1,0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x,y) = (x, 0)$		$T(\mathbf{e}_1) = T(1,0) = (1,0)$ $T(\mathbf{e}_2) = T(0,1) = (0,0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x,y) = (0, y)$		$T(\mathbf{e}_1) = T(1,0) = (0,0)$ $T(\mathbf{e}_2) = T(0,1) = (0,1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta)$ $T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Definition 11.5.1
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$. We say that S **spans** $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ can be expressed as a linear combination of vectors in S.

It turns out that in order to determine if a set of vectors S spans $\mathbb{R}^n$, we simply need to solve a linear system.

Example. Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

Solution. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector in $\mathbb{R}^3$. We want to find constants k_1, k_2, k_3 such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

which gives a linear system in terms of k_1, k_2, k_3.

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We can verify that the coefficient matrix is invertible, hence the system has a unique solution. Since the system *has* a solution, we can say that S spans $\mathbb{R}^3$.

Example. Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ does not span $\mathbb{R}^2$.

Solution. To show that a set does not span $\mathbb{R}^2$, we just need to find one vector $\mathbf{v}$ which cannot be expressed as a linear combination of the vectors in S. You can verify that the vector $(-1, 0)$ cannot be expressed as a linear combination of the vectors in S.

"Is $\boldsymbol{b}$ in $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$?"

1. Set up the vector equation $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \boldsymbol{b}$
2. Construct and row-reduce the augmented matrix corresponding to this vector equation
3. If the vector equation is consistent, then **yes**, $\boldsymbol{b}$ is in $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$. If the vector equation is inconsistent, then **no**, $\boldsymbol{b}$ is **not** in $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$

"Does the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$ span $\mathbb{R}^n$?"

1. Using the Spanning Columns Theorem, we need to know whether the matrix whose columns are $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p$ has a pivot in every row
2. Construct and row-reduce this matrix
3. If there is a pivot in every row, then **yes**, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$ spans $\mathbb{R}^n$. If there is not a pivot in every row, then **no**, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$ does not span $\mathbb{R}^n$.

Definition 11.6.1
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$. We say that S is **linearly independent** when the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

has only the trivial solution

$$k_1 = 0, k_2 = 0, ..., k_r = 0.$$

If there are other solutions, then S is **linearly dependent**.

Checking if a set of vectors is linearly independent amounts to setting up a *homogeneous linear system* and checking if the solution is unique. If the system has a unique solution, then the set is linearly independent.

Example. Determine if the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

Solution.

We set up the vector equation

$$k_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the linear system

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Gauss-Jordan or attempting to find the inverse of the coefficient matrix shows that the system has a unique solution, so S is linearly independent.

Definition 11.6.3
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$. We say S is a **basis** for $\mathbb{R}^n$ if S is a linearly independent set which spans $\mathbb{R}^n$.
Let S is a basis whose vectors are ordered $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r$. If a vector $\mathbf{w} \in \mathbb{R}^n$ is written as a linear combination

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_r\mathbf{v}_r$$

then $k_1, k_2, ..., k_r$ are called the **coordinates** of $\mathbf{w}$ relative to the basis S. We can represent $\mathbf{w}$ with a **coordinate vector relative to S**:

$$[\mathbf{w}]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix}$$

Theorem 11.6.4
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$. S is a basis for $\mathbb{R}^n$ if at least two of the following conditions hold.

1. $|S| = n$.
2. S spans $\mathbb{R}^n$.
3. S is linearly independent.

Furthermore, if any of the above conditions do not hold, S is not a basis for $\mathbb{R}^n$.

Solution.

Note that S contains 2 vectors.

We now show S is linearly independent. We set up the vector equation

$$k_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives the linear system

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix is invertible, this system has a unique solution. Therefore S is linearly independent.

Since S contains 2 linearly independent vectors in $\mathbb{R}^2$, it forms a basis for $\mathbb{R}^2$.

To find the coordinates of $\mathbf{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ relative to S, we set up the vector equation

$$k_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

which gives the linear system

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

This system has the solution

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, $[\mathbf{w}]_S = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.