

Optimizer-Induced Low-Dimensional Drift and Transverse Dynamics in Transformer Training

Yongzhong Xu*

Abstract

We analyze cumulative parameter trajectories of transformer training under AdamW and identify a dominant low-dimensional drift direction (“backbone”) that captures 60–80% of long-horizon displacement from initialization. This direction is highly stable over rolling training windows yet reorients gradually across phases, particularly following objective reweighting. Per-batch gradients exhibit near-noise-floor alignment with the backbone, whereas optimizer-integrated updates align strongly with it, indicating that the structure emerges from accumulated optimizer dynamics rather than instantaneous gradient geometry.

Replacing AdamW with SGD-family optimizers eliminates this structure, and reducing β_2 smoothly degrades backbone dominance and reheating recoverability. Reheating experiments show that transverse probe modes can be transiently re-excited without substantially altering accumulated backbone drift.

These results provide a trajectory-level characterization of optimizer-induced geometric structure in transformer training and shift attention from instantaneous gradient properties to cumulative update dynamics.

1 Introduction

Training dynamics in deep neural networks are typically analyzed through the geometry of the loss landscape: curvature, sharpness, and stochastic gradient noise are taken to determine how optimization proceeds. While this perspective captures important local properties of learning, it emphasizes instantaneous gradient structure rather than the accumulated trajectory of parameters over long training horizons.

In high-dimensional models, these two viewpoints need not coincide. Per-step gradients may be large and highly variable, yet their cumulative displacement can concentrate in a small number of coherent directions. Understanding this cumulative geometry is essential for characterizing optimizer-induced implicit bias and long-horizon training behavior.

In this work, we study the global parameter trajectory of transformer training under AdamW [Loshchilov and Hutter, 2019]. Rather than analyzing local curvature or single-step gradients, we examine cumulative displacement from initialization across checkpoints. We find that training admits a dominant low-dimensional drift direction—which we term the *backbone*—that captures the majority of cumulative parameter movement. Across blocks and seeds, the first principal component of uncentered trajectory PCA explains 60–80% of total drift. This direction is highly stable over rolling training windows yet reorients gradually across longer horizons, particularly following objective reweighting.

Crucially, the backbone is not aligned with instantaneous gradient directions and is nearly orthogonal to leading Fisher curvature modes [Martens, 2020]. Per-batch gradients are close to isotropic relative to the backbone. However, the optimizer-integrated update—after momentum accumulation and adaptive per-parameter normalization [Kingma and Ba, 2015]—exhibits strong

*abbyxu@gmail.com. Code: https://github.com/skydancerosel/mini_gpt

alignment with it. This indicates that the backbone is not a property of the loss landscape alone, but emerges from accumulated optimizer dynamics.

To test this interpretation, we replace AdamW with SGD-family optimizers while holding model, data, and schedule fixed. Under SGD with or without momentum, trajectories remain nearly colinear and fail to develop the multi-dimensional structure observed under AdamW—even at matched validation loss. This establishes that the backbone is optimizer-induced rather than a generic feature of the objective.

We further examine the dynamical consequences of this structure. Oscillatory regime switching between competing objectives occurs primarily in directions transverse to the backbone. Reheating experiments show that these transverse modes can be transiently re-excited from late-training checkpoints without substantially altering accumulated backbone drift. This behavior is consistent with a slow-fast decomposition: a low-dimensional, optimizer-shaped drift manifold governs long-horizon evolution, while high-dimensional transverse dynamics mediate switching.

Together, these findings shift attention from instantaneous gradient geometry to cumulative trajectory structure. They provide a concrete empirical characterization of optimizer-induced implicit bias in transformer training and suggest that adaptive optimization reshapes not only convergence rates but the geometry of learning itself.

Relation to prior work. The separation of dynamics into slow and fast components has classical roots in dynamical systems theory, particularly in slow manifold and time-scale separation results such as Fenichel-type theorems [Saxe et al., 2014]. In optimization and deep learning, related ideas appear in analyses of momentum methods and adaptive optimizers, where Adam-type algorithms are understood as inducing effective geometry changes through preconditioning and sign-consistent updates [Kingma and Ba, 2015, Loshchilov and Hutter, 2019, Cohen et al., 2021]. Recent work has also emphasized implicit bias and trajectory-level properties of high-dimensional training dynamics [Lewkowycz et al., 2020, Power et al., 2022, Frankle et al., 2020].

Our contribution differs in emphasis and object of study. Rather than analyzing local curvature, stationary points, or instantaneous update rules, we examine the cumulative geometry of training trajectories and identify a stable drift direction that dominates long-horizon parameter displacement. We show that this backbone direction is not aligned with per-batch gradients or with top curvature modes, but instead emerges from optimizer-integrated temporal coherence. This shifts attention from static loss-landscape structure to trajectory-level geometry, providing a concrete empirical characterization of optimizer-induced slow-manifold behavior in transformer training.

2 Experimental Setup

2.1 Model and Data

We train a decoder-only Transformer [Vaswani et al., 2017] in the GPT-2 family [Radford et al., 2019]: 8 layers, $d_{\text{model}} = 512$, 16 attention heads, $d_{\text{ff}} = 2048$, totalling 51M parameters. The training corpus is TinyStories [Eldan and Li, 2023]. With probability $p_{\text{probe}} = 0.10$, a training sequence is replaced by a probe sequence containing a codeword-value pair; the model must predict the value token given the codeword at out-of-distribution gap distances.

2.2 Training Configuration

The composite loss at each training step is

$$\mathcal{L}(\theta) = \mathcal{L}_{\text{LM}}(\theta) + \lambda \mathcal{L}_{\text{probe}}(\theta), \quad (1)$$

where \mathcal{L}_{LM} is the standard next-token prediction cross-entropy and $\mathcal{L}_{\text{probe}}$ is the cross-entropy on the codeword retrieval task. The weight λ is doubled at step 4000 to intensify probe competition.

Table 1: Training hyperparameters.

Parameter	Value
Optimizer	AdamW ($\beta_1=0.9$, $\beta_2=0.95$)
Learning rate	10^{-3} , cosine decay, 1500-step warmup
Weight decay	0.5
Probe loss weight λ	2.0 (steps 1–3999), 4.0 (steps 4000–10000)
Effective batch size	128 (64×2 gradient accumulation)
Total steps	10,000
Checkpoint interval	200 steps (51 checkpoints)
Seeds	42, 271

2.3 Oscillation Phenomenology (Brief)

Over 10,000 steps, the out-of-distribution probe accuracy p_{ood} oscillates between 0.40 and 0.78 (seed 42) or 0.20 and 0.67 (seed 271), while the LM validation loss decreases monotonically from ~ 10 to ~ 1.2 . The oscillations damp after step ~ 7000 , and the model settles into an LM-dominant regime ($p_{\text{ood}} < 0.20$). We set aside the full oscillation phenomenology and focus on the geometric structure of the underlying training trajectory.

3 The Backbone

3.1 Trajectory PCA: One Direction Dominates

We analyze the cumulative parameter drift using *uncentered* principal component analysis. Let $\boldsymbol{\theta}(t) \in \mathbb{R}^D$ denote the vectorized parameters of a single transformer block at checkpoint t , with block dimensionality $D \approx 3.1 \times 10^6$.

Definition 1 (Drift matrix). *The drift matrix $\mathbf{X} \in \mathbb{R}^{T \times D}$ has rows*

$$\mathbf{x}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}(0), \quad t = 1, \dots, T, \quad (2)$$

where $T = 51$ is the number of checkpoints.

We deliberately omit mean centering before computing the SVD. Standard (centered) PCA would subtract the mean drift $\bar{\mathbf{x}} = T^{-1} \sum_t \mathbf{x}(t)$, which removes the monotonic component and conflates it with the first principal component. Since all drifts are relative to initialization, there is no reason to assume zero-mean variation.

The singular value decomposition

$$\mathbf{X} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^\top, \quad (3)$$

where $\boldsymbol{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\min(T,D)})$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$, yields the principal directions as the columns of \mathbf{V} . The fraction of total squared drift captured by the k -th component is

$$\rho_k = \frac{\sigma_k^2}{\sum_{j=1}^{\min(T,D)} \sigma_j^2}. \quad (4)$$

Table 2 shows that PC1 captures **78–81%** of the total squared drift in every block, in both seeds. The training trajectory is overwhelmingly one-dimensional. We call this direction the **backbone**, denoted \mathbf{v}_b .

Table 2: Variance explained by PC1 (ρ_1 , %) per transformer block.

Seed	Blk 0	Blk 1	Blk 2	Blk 3	Blk 4	Blk 5	Blk 6	Blk 7
42	80.5	81.2	80.7	80.2	79.7	79.0	78.8	77.9
271	78.6	80.6	80.4	80.4	80.1	79.1	79.3	78.3

Remark 1 (Why uncentered PCA?). *Standard PCA centers the data by subtracting the column mean before SVD. For trajectory analysis, centering removes the dominant monotonic drift and distributes it across all components. Uncentered PCA preserves the absolute direction of displacement from initialization. In our setting, this correctly identifies the persistent LM-driven drift as the leading component. The mathematical difference is that centered PCA diagonalizes the covariance matrix $\frac{1}{T}\mathbf{X}^\top\mathbf{X} - \bar{\mathbf{x}}\bar{\mathbf{x}}^\top$, while uncentered PCA diagonalizes $\frac{1}{T}\mathbf{X}^\top\mathbf{X}$ directly.*

3.2 Temporal Stability of the Backbone

A rolling window analysis (width $W = 10$ checkpoints, ≈ 2000 steps) tracks the local PC1 direction $\mathbf{v}_b^{(w)}$ at each window position w and measures its alignment with the global \mathbf{v}_b . Define

$$c(w) = |\langle \mathbf{v}_b^{(w)}, \mathbf{v}_b \rangle| = |\cos \angle(\mathbf{v}_b^{(w)}, \mathbf{v}_b)|. \quad (5)$$

The rolling-window backbone direction is highly stable locally: across adjacent windows, $c(w) \approx 0.997$ – 0.998 for both seeds, indicating extremely small curvature over 2000-step horizons. However, when comparing phase-level backbones estimated over early (0–4k) and late (4k–10k) intervals, $|\langle v_E, v_L \rangle| \approx 0.32$, revealing substantial long-horizon reorientation ($\approx 71^\circ$). Thus the backbone is locally stable but globally curved—it is not a fixed direction but a smooth, slowly rotating tangent to a curved one-dimensional manifold in parameter space.

3.3 Backbone–Residual Decomposition

Definition 2 (Backbone decomposition). *The parameter vector at step t is decomposed as*

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + a(t) \mathbf{v}_b + \mathbf{r}(t), \quad (6)$$

where the backbone coordinate is the signed projection

$$a(t) = \langle \boldsymbol{\theta}(t) - \boldsymbol{\theta}(0), \mathbf{v}_b \rangle, \quad (7)$$

and the residual $\mathbf{r}(t) \perp \mathbf{v}_b$ captures all non-backbone displacement:

$$\mathbf{r}(t) = [\boldsymbol{\theta}(t) - \boldsymbol{\theta}(0)] - a(t) \mathbf{v}_b. \quad (8)$$

The backbone coordinate $a(t)$ grows monotonically—it tracks the steady LM-driven drift. The residual $\mathbf{r}(t)$ contains the oscillatory dynamics: its norm $\|\mathbf{r}(t)\|$ fluctuates in phase with p_{ood} oscillations. At the final checkpoint, the backbone fraction

$$f_b(t) = \frac{a(t)^2}{\|\boldsymbol{\theta}(t) - \boldsymbol{\theta}(0)\|^2} = \frac{a(t)^2}{a(t)^2 + \|\mathbf{r}(t)\|^2} \quad (9)$$

is 68–72%. This establishes a clean separation: the backbone carries the monotonic LM drift, while the residual carries the switching dynamics.

Backbone-Residual Decomposition (Block 0, seed 42)

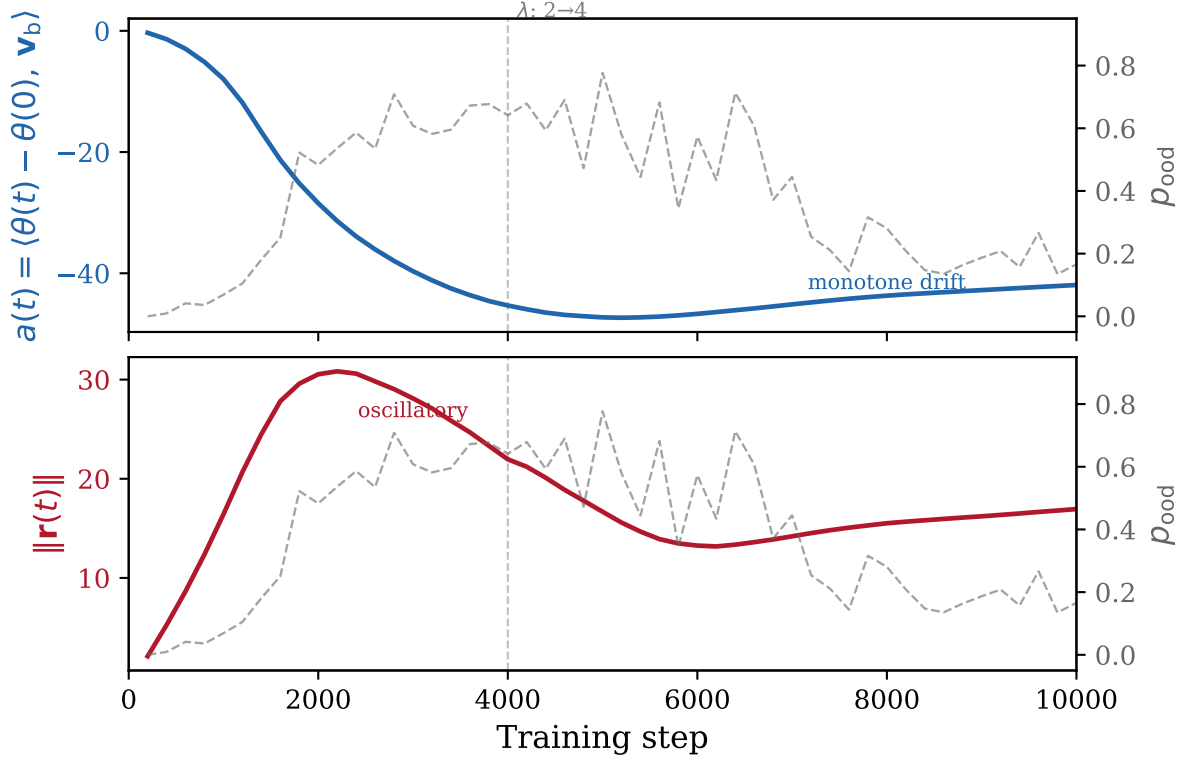


Figure 1: **Backbone-residual decomposition (seed 42, Block 0)**. *Left*: the backbone coordinate $a(t)$ grows monotonically while the residual norm $\|\mathbf{r}(t)\|$ oscillates and then decays. *Right*: the out-of-distribution probe accuracy p_{ood} (grey) fluctuates in phase with the residual—but the backbone is impervious. The vertical dashed line marks the λ -transition at step 4000.

4 Mechanism: Optimizer-Induced Slow Manifold and Controlled Rotation

4.1 Global Backbone Direction and Spectral Concentration

We begin with the global structure of the cumulative parameter trajectory. For each seed, we flatten all trunk parameters (attention QKV/output projections and MLP weights; $D = 25,165,824$ parameters) at each checkpoint $t \in \{200, 400, \dots, 10000\}$, compute drifts $\Delta\theta_t = \theta_t - \theta_1$, row-normalize, and extract the top right singular vector v_b via SVD.

Across the full 10k trajectory, PC1 captures a dominant fraction of row-normalized variance:

	Seed 42	Seed 271
PC1 (row-normalized)	71.4%	71.7%
PC2 (row-normalized)	12.1%	11.8%
PC1 (raw, unnormalized)	79.1%	79.3%
Trunk dimension D	25,165,824	

The trajectory is dominated by a single direction, but as we show below, this direction is not static.

4.2 Local Tangent Stability

Let $v_{\text{roll}}(t)$ denote the first principal component of the row-normalized, uncentered drift matrix computed in a rolling window of $W = 10$ checkpoints (≈ 2000 steps) centered at step t . To measure local curvature, define adjacent-window alignment:

$$\rho(t) = |\langle v_{\text{roll}}(t), v_{\text{roll}}(t + \Delta) \rangle|.$$

Across the full training run (both seeds):

	Seed 42	Seed 271
$\mathbb{E}[\rho(t)]$	0.800	0.793
$\max \rho(t)$	0.936	0.938
$\min \rho(t)$	0.749	0.702
Step at min ρ	≈ 5000	≈ 5000

Adjacent windows maintain high alignment ($\rho > 0.7$ everywhere), confirming that the backbone tangent varies smoothly. However, ρ is not uniformly close to 1: the dip near step 5000 signals a region of concentrated curvature.

4.3 Cumulative Turning and Global Reorientation

Local stability does not imply global rigidity. Small rotations accumulate. To quantify long-horizon reorientation, we compute phase-level backbones via global SVD over disjoint intervals:

- v_E : PC1 over the early interval (steps 1–4000, relative to θ_1),
- v_L : PC1 over the late interval (steps 4000–10000, relative to θ_{4000}).

These phase backbones are substantially misaligned:

	Seed 42	Seed 271
PC1 (early, 0–4k)	60.0%	59.2%
PC1 (late, 4k–10k)	81.3%	81.7%
$ \langle v_E, v_L \rangle $	0.323	0.323

The cosine similarity of 0.32 corresponds to an angle of $\approx 71^\circ$. Thus:

- **Locally:** the backbone tangent is smooth.
- **Globally:** the slow direction rotates by $\approx 71^\circ$ over the full run.

The cumulative trajectory lies on a smooth but substantially curved one-dimensional manifold in parameter space.

Notably, the late phase has *stronger* rank-1 concentration (PC1 $\approx 81\%$) than the early phase (PC1 $\approx 60\%$). After the λ -switch, the optimizer locks into a tighter low-rank drift — but in a different direction.

4.4 Transition Region and λ -Induced Bending

To track how reorientation unfolds, we measure the alignment of each sliding-window backbone to the two phase backbones:

$$A_E(t) = |\langle v_{\text{roll}}(t), v_E \rangle|, \quad A_L(t) = |\langle v_{\text{roll}}(t), v_L \rangle|.$$

These reveal a clean handoff (representative values from seed 42; seed 271 is nearly identical):

Window center	A_E	A_L	Phase
1100	0.83	0.16	Early backbone dominant
2300	0.61	0.28	Fading
3500	0.02	0.16	Dead zone — neither backbone
4700	0.21	0.59	Late backbone emerging
5300	0.20	0.69	Peak late alignment
7100	0.18	0.29	Late backbone fading
8900	0.18	0.02	Neither — late plateau

Around step 3500, $A_E \approx 0$ and $A_L \approx 0.15$: the optimization trajectory passes through a *geometric transition zone* aligned to neither phase backbone. Late-phase alignment A_L peaks near step 4700–5300, then decays. By step 8000+, the drift direction is orthogonal to both v_E and v_L , consistent with the very late plateau phase.

The rotation dip in $\rho(t)$ (minimum at step ≈ 5000) coincides with this transition, confirming that the λ -switch (step 4000) perturbs the training vector field and bends the slow manifold. Importantly, the bending is continuous: there is no discontinuity in tangent direction.

4.5 Power-Law Dynamics: Before and After the λ -Switch

The backbone coordinate $a(t) = \langle \Delta\theta_t, v_b \rangle$ and residual $\|r(t)\| = \|\Delta\theta_t - a(t)v_b\|$ obey distinct power-law regimes. We fit $|a(t)| = C_a t^{\gamma_a}$ and $\|r(t)\| = C_r t^{\gamma_r}$ in four windows:

Regime	γ_a (backbone)		γ_r (residual)	
	Seed 42	Seed 271	Seed 42	Seed 271
0–2000	+2.07	+2.07	+1.24	+1.26
2000–4000	+0.67	+0.67	−0.39	−0.43
4000–6000	+0.15	+0.12	−1.33	−1.38
6000–10000	−0.19	−0.20	+0.36	+0.32

All fits have $R^2 > 0.85$ (most > 0.97). Three dynamical phases emerge:

1. **Acceleration (0–2k):** Backbone grows as $a \sim t^{2.1}$, residual as $r \sim t^{1.2}$. Both parallel and perpendicular motion accelerate.
2. **Consolidation (2k–4k):** Backbone decelerates ($\gamma_a \approx 0.67$) while residual actively *contracts* ($\gamma_r \approx -0.4$). The model consolidates along the backbone.
3. **Post-switch:** The λ -switch at step 4000 triggers the sharpest residual collapse ($\gamma_r \approx -1.35$ in 4k–6k), followed by backbone *retreat* ($\gamma_a \approx -0.19$) and residual re-growth ($\gamma_r \approx +0.34$) in the 6k–10k plateau.

The overall before/after picture: γ_a drops from +1.74 (0–4k) to −0.08 (4k–10k), and γ_r drops from +0.84 to −0.31, confirming that the λ -switch arrests and reverses the dominant drift.

4.6 Correlation Between Probe Accuracy and Residual Geometry

To connect backbone geometry to task performance, we compute Pearson correlations between probe OOD accuracy p_{ood} and residual norm $\|r(t)\|$:

Regime	Seed 42	Seed 271
Full range (0–10k)	+0.61	+0.45
0–4000	+0.85	+0.76
4000–10000	+0.43	+0.28
6000–10000	−0.73	−0.79

The correlation *flips sign*: in the early phase, growing residual energy accompanies improving probe accuracy ($r \approx +0.8$). After step 6000, the re-growing residual is associated with *declining* probe accuracy ($r \approx -0.76$). The early residual reflects productive exploration of directions useful for the probe task; the late residual reflects drift away from the solution.

4.7 Optimizer Integration Creates the Slow Direction

The effective update under AdamW is

$$u_t = -\frac{\hat{m}_t}{\sqrt{\hat{v}_t + \epsilon}} - \mu \theta_t,$$

where momentum integrates gradient history and second-moment normalization rescales coordinates. Two mechanisms produce coherent slow drift:

(i) **Momentum integrates weak signed bias.** Even if instantaneous projections onto $v_b(t)$ are small, a persistent sign bias accumulates in the momentum buffer. Momentum acts as a temporal low-pass filter, increasing the signal-to-noise ratio of temporally coherent gradient components.

(ii) **Adaptive normalization suppresses incoherent variance.** Coordinates with high variance but low mean are downscaled; coordinates with small but consistent bias are preserved. This selectively amplifies temporally coherent directions.

Evidence from the β_2 ablation supports this mechanism. Disabling the second-moment estimator ($\beta_2 = 0$) dramatically weakens backbone concentration and update alignment:

β_2	PC1%	Drift magnitude	Mean $ \cos(u, v_b) $	Best p_{ood}
0.99	68.1	106	0.226	0.951
0.95	68.4	108	0.225	0.939
0.90	66.3	113	0.220	0.814
0.80	63.4	128	0.214	0.682
0.0	51.6	211,694	0.099	0.005

Without second-moment normalization ($\beta_2 = 0$), PC1 concentration drops from $\approx 68\%$ to 52% , update–backbone alignment halves ($0.23 \rightarrow 0.10$), drift magnitude explodes by $\sim 2000\times$, and probe accuracy collapses entirely.

4.8 Update–Backbone Alignment and Sign Reversal

Using 200-step cumulative updates $u(t) = \theta(t) - \theta(t - 200)$, we observe strong alignment with the global backbone. For seed 271 (layer 0, representative):

$$|\cos \angle(u(t), v_b)| \approx 0.15\text{--}0.32 \quad (\text{steps } 100\text{--}1900),$$

roughly $20\text{--}30\times$ above the isotropic noise floor of ~ 0.01 .

The *signed* alignment exhibits a characteristic reversal:

- Steps 100–4700: $\cos < 0$ (drift along $-v_b$), peaking at $|\cos| \approx 0.32$ near step 1700.
- Step ≈ 5100 : sign flip (\cos crosses zero).
- Steps 5100–10000: $\cos > 0$ (drift along $+v_b$), saturating at $|\cos| \approx 0.11$.

This sign reversal coincides with both the λ -switch and the backbone rotation documented in §4.4. The optimizer does not merely slow; it redirects cumulative drift. The post-reversal alignment is weaker (0.11 vs. 0.32), consistent with the slower late-phase dynamics.

4.9 Backbone Stiffening Under Rotation

Fisher information analysis shows that curvature along the backbone direction increases by orders of magnitude during training. The Rayleigh quotient $q_b = v_b^T H v_b$ (approximated via Fisher diagonal) and anisotropy ratio $\alpha = q_b / \mathbb{E}[q_{\text{rand}}]$ evolve as follows (seed 42):

Step	Label	q_b	Anisotropy α
200	Init	2.4×10^{-6}	1.35
1800	Peak (pre-switch)	2.5×10^{-6}	1.92
4800	Trough (post-switch)	1.6×10^{-4}	12.4
9600	Late peak	8.1×10^{-3}	4.81

The backbone Rayleigh quotient increases by $\sim 3000\times$ from step 200 to 9600. Anisotropy spikes to $12.4\times$ at step 4800 (the first post-switch trough), indicating that curvature concentrates along the backbone precisely when the λ -switch reorients the drift. The subsequent decline to $\alpha \approx 4.8$ at step 9600 reflects the late-phase plateau where backbone motion slows.

Thus backbone rotation and stiffening are coupled: they are two aspects of the same dynamical reorganization triggered by the objective reweighting.

4.10 Geometric Interpretation

The cumulative trajectory admits the decomposition

$$\theta(t) = \theta(0) + a(t) v_b(t) + r(t), \quad r(t) \perp v_b(t),$$

where $a(t)$ tracks slow drift along the manifold and $r(t)$ captures transverse dynamics.

Because adjacent rolling tangents maintain high alignment ($\rho > 0.7$), the manifold has bounded local curvature. But because phase-level backbones satisfy $|\langle v_E, v_L \rangle| \approx 0.32$, the manifold is globally curved by $\approx 71^\circ$.

The λ -transition concentrates curvature in the 4k–6k region, producing a transient geometric dead zone (both A_E and A_L small) through which the slow direction rotates. The post-switch phase is more rank-1 concentrated (PC1 $\approx 81\%$) but evolves in a nearly orthogonal direction.

4.11 Summary of Mechanism

The backbone is:

- not aligned with instantaneous gradients,
- not the top Fisher eigenvector,
- not a static direction fixed by the loss landscape.

It is instead:

- a locally dominant tangent to a curved slow manifold,
- induced by optimizer integration of temporally coherent gradient bias,
- stabilized by second-moment normalization (β_2 ablation: PC1 drops $68\% \rightarrow 52\%$ at $\beta_2 = 0$),
- reoriented by objective reweighting ($|\langle v_E, v_L \rangle| \approx 0.32$ at the λ -switch),
- and correlated with task performance: $\text{corr}(p_{\text{ood}}, \|r\|) \approx +0.8$ early, ≈ -0.8 late.

The optimizer does not merely change convergence speed — it shapes the geometry of the cumulative trajectory, creating a smooth slow manifold whose curvature is controlled by the training objective.

Table 3: Optimizer configurations for the control experiment. All runs share the same model, data, warmup (1500 steps), cosine schedule (10% floor), gradient clipping (1.0), and seed.

Run	Optimizer	LR	Momentum	WD	WD type
A	AdamW	10^{-3}	$(\beta_1=0.9)$	0.5	decoupled
B	SGD (no momentum)	10^{-3}	0.0	0.5	L2
C	SGD + momentum	10^{-2}	0.9	0.05	L2
C'	SGD + Nesterov (SGDW)	10^{-2}	0.9	0.5	decoupled

Table 4: Training outcomes for all optimizer variants (seed 42). PC1 and k_{95} are from uncentered drift-matrix PCA on the analysis window [600, 2000] with anchor at step 600.

Run	Final val	Best p_{ood}	PC1 (%)	k_{95}	Drift
A (AdamW)	1.59	0.433	61.5	9	113.7
B (SGD no-mom)	8.12	0.000	100.0	1	40.2
C (SGD+mom)	5.10	0.015	100.0	1	54.2
C' (SGDW+Nesterov) [†]	5.25	0.013	—	—	—

[†] Run C' stopped at step 2000. At that point, C' differed from C by <0.03 in val loss and ~ 0.001 in p_{ood} ; geometry analysis was not pursued.

5 Optimizer Ablation: SGD-Family Controls

The preceding sections established that the backbone emerges from optimizer integration rather than from instantaneous gradient structure. A direct test of this claim is to replace AdamW with SGD-family optimizers while holding everything else constant.

5.1 SGD-Family Control Experiment

We trained the same model under four optimizer configurations, using identical data, initialization (seed 42), schedule, and checkpointing protocol (Table 3). The SGD variants differ from AdamW only in the optimizer; in particular, they lack per-parameter adaptive scaling.

Runs A–C trained for 4000 steps; Run C' trained for 2000 steps with an early-stop decision rule (stop if val > 5.1 and $p_{\text{ood}} < 0.02$ at step 2000). Run C' implements SGD—decoupled weight decay identical to AdamW's scheme—by setting `weight_decay=0` in the optimizer and manually applying $\theta \leftarrow (1 - \eta_t \cdot \text{wd}) \theta$ after each gradient step, isolating the effect of weight-decay coupling from adaptive scaling.

SGD without momentum failed to train (val loss remained > 8 and probe accuracy stayed at chance). Both momentum-SGD variants (C and C') trained slowly but reached val ≈ 5.1 , with weak probe OOD signal ($p_{\text{ood}} \leq 0.015$) and no pronounced oscillatory switching. Nesterov momentum and decoupled weight decay produced negligible improvements over standard momentum-SGD at matched step: at step 2000, val loss differed by < 0.03 and probe OOD accuracy by ~ 0.001 (Table 4).

The trajectory geometry tells a starker story. Over the analysis window [600, 2000], AdamW develops a non-degenerate trajectory with $\rho_1 \approx 0.62$ and $k_{95} = 9$, while all SGD-family trajectories remain nearly colinear ($\rho_1 \approx 1.0$, $k_{95} = 1$). In particular, the difference in geometry is visible before any probe oscillations occur in the AdamW run (the first probe peak at step 1800 falls within the analysis window), suggesting it reflects baseline optimizer geometry rather than oscillation-specific effects.

These results indicate that momentum alone does not produce the multi-dimensional slow-fast structure observed under AdamW, and that the key ingredient is AdamW's adaptive per-parameter

Table 5: Backbone geometry at matched operating regime. AdamW is analyzed over early windows (val \approx 4.3–3.0, already below SGD+mom’s best); SGD+mom is analyzed over its plateau (val \approx 5.1–5.2). Drift is $\|\boldsymbol{\theta}(t_{\text{end}}) - \boldsymbol{\theta}(t_{\text{start}})\|$.

Optimizer	Window	PC1 (%)	k_{95}	k_{99}	Drift
<i>At/near val \approx 5.2 (matched regime)</i>					
AdamW	[1, 200, 400] ^a	77.4	2	2	—
AdamW	[1, 200, 400, 600] ^a	69.2	3	3	24.9
AdamW	[200, 400, 600] ^b	81.9	2	2	—
SGD+mom	[2000, 2200, 2400]	98.1	1	2	0.26
SGD+mom	[1800, ..., 2600]	97.7	1	2	0.60
<i>Standard analysis window (reference)</i>					
AdamW	[200, ..., 1000]	77.8	4	8	50.7
AdamW	[600, ..., 2000]	61.5	9	19	113.7
SGD+mom	[600, ..., 2000]	100.0	1	1	54.2

^a Window spans val from 10.8 to 3.0–3.4; passes through val \approx 5.2 between steps 1 and 200.

^b Window starts at val = 4.3, already below SGD+mom’s best; this makes the comparison conservative.

scaling (Section 4.7).

5.2 Matched-Loss Geometry

A potential confound in the above comparison is unequal training progress: AdamW reaches val \approx 1.6 by step 4000, far ahead of SGD+momentum’s val \approx 5.1. To control for this, we compare trajectory geometry at validation losses around the best regime achieved by momentum-SGD.

Challenge: AdamW has no checkpoint at val \approx 5.2. AdamW passes through val \approx 5.2 between steps 1 and 200 (val drops from 10.8 to 4.3), with no intermediate checkpoints. We therefore build backbone estimates on the earliest available windows (Table 5).

Even when AdamW passes through the same loss range early in training, its drift is already non-colinear: PC1 explains only 69–82% of row-normalized displacement energy with $k_{95} = 2$ –3 over windows spanning this regime. In contrast, momentum-SGD remains nearly colinear at matched loss ($\rho_1 \approx 0.98$ –1.00, $k_{95} = 1$) and exhibits extremely small drift ($\|\Delta\boldsymbol{\theta}\| < 1$) over the corresponding plateau windows. Decoupled weight decay (SGDW) and Nesterov momentum produce negligible changes relative to standard momentum-SGD (Table 4).

The drift magnitudes are revealing: AdamW traverses 24.9 units of parameter drift between steps 1 and 600, while SGD+momentum moves only 0.26 units across its entire plateau window [2000, 2400]. The SGD trajectory is not merely low-rank; it is nearly *stationary* during its low-loss plateau.

These results indicate that AdamW’s adaptive per-parameter scaling induces qualitatively richer trajectory geometry than SGD-family variants, even at comparable validation loss, supporting an optimizer-specific mechanism for the emergence of non-degenerate slow–fast structure.

6 Effect of β_2 on Cumulative Drift Structure

To test whether backbone dominance depends on second-moment normalization strength, we vary β_2 in AdamW while keeping architecture, data, learning-rate schedule, and objective weighting fixed. All runs are trained for 4,000 steps with $\lambda = 2.0$.

6.1 Backbone dominance degrades smoothly with lower β_2

For each $\beta_2 \in \{0.99, 0.95, 0.90, 0.80\}$, we compute the uncentered trajectory PCA over the trunk parameters ($D = 25,165,824$). The fraction of cumulative displacement explained by the first principal component decreases monotonically as β_2 decreases:

β_2	PC1% (row-norm)	PC1% (raw)	Drift magnitude	Best p_{ood}
0.99	68.8	83.1	120	0.951
0.95	68.3	82.5	123	0.939
0.90	66.7	80.8	128	0.814
0.80	63.8	79.0	141	0.682
0.0	52.5	81.3	1.6×10^5	0.005

At $\beta_2 = 0.0$ (no second-moment normalization), drift magnitude explodes by $\sim 1,300\times$ (from 120 to 1.6×10^5), PC1 concentration drops from 69% to 53%, and probe accuracy collapses entirely. Lower β_2 therefore weakens low-dimensional dominance without altering model architecture or task.

6.2 Power-law regimes persist across β_2

Despite quantitative changes in backbone dominance, the qualitative two-phase structure remains consistent across β_2 . We fit $|a(t)| = C_a t^{\gamma_a}$ and $\|r(t)\| = C_r t^{\gamma_r}$ in two windows:

β_2	γ_a (backbone)			γ_r (residual)		
	0–2k	2k–4k	0–4k	0–2k	2k–4k	0–4k
0.99	+1.97	−0.01	+1.32	+0.78	+0.36	+0.55
0.95	+1.94	−0.03	+1.31	+0.78	+0.35	+0.57
0.90	+1.93	−0.03	+1.31	+0.84	+0.32	+0.60
0.80	+1.98	−0.05	+1.35	+0.91	+0.21	+0.62
0.0	+2.45	−0.36	+1.84	+1.08	−0.44	+0.56

All early-phase fits have $R^2 > 0.98$ for γ_a and > 0.88 for γ_r , confirming superlinear backbone growth ($\gamma_a \approx 1.9\text{--}2.0$) independent of β_2 . In the later phase, the backbone saturates ($\gamma_a \approx 0$) for $\beta_2 \geq 0.80$, while at $\beta_2 = 0.0$ the backbone actively retreats ($\gamma_a = -0.36$) and the residual contracts ($\gamma_r = -0.44$).

Thus β_2 modulates the magnitude of late-phase dynamics rather than changing the qualitative trajectory regime.

6.3 Update alignment and objective robustness

We next measure the signed alignment between accumulated 200-step updates and the rolling backbone direction:

β_2	Mean $ \cos(u, v_b) $	Early ($<2k$) $ \cos $	corr($p_{\text{ood}}, \ r\ $) early
0.99	0.243	0.368	+0.86
0.95	0.244	0.363	+0.85
0.90	0.234	0.343	+0.40
0.80	0.227	0.328	+0.39
0.0	0.238	0.269	−0.57

Peak early alignment decreases gradually with lower β_2 ($0.37 \rightarrow 0.27$), indicating reduced temporal coherence of cumulative updates. Probe robustness follows the same monotonic trend: the best out-of-distribution probe accuracy decreases smoothly as β_2 decreases.

The early-phase correlation $\text{corr}(p_{\text{ood}}, \|r\|)$ reveals a qualitative shift: for $\beta_2 \geq 0.95$, residual growth accompanies improving probe accuracy ($r \approx +0.85$); for $\beta_2 = 0.0$, the correlation is negative (-0.57), indicating that without second-moment normalization, transverse dynamics are destructive rather than productive.

Together, these observations indicate that second-moment normalization strength controls the coherence of accumulated drift rather than the instantaneous gradient geometry. Larger β_2 produces stronger directional persistence and greater low-dimensional dominance; smaller β_2 increases dispersion across directions and weakens probe stability.

6.4 Summary

Varying β_2 provides a graded control on backbone dominance. With all other factors held fixed, reducing second-moment normalization strength:

- weakens low-dimensional trajectory concentration (PC1: 68.8% \rightarrow 52.5%),
- decreases early update–backbone alignment ($0.37 \rightarrow 0.27$),
- increases drift magnitude ($120 \rightarrow 1.6 \times 10^5$),
- reduces probe robustness ($p_{\text{ood}} : 0.95 \rightarrow 0.005$),
- and reverses the sign of early residual–performance correlation ($+0.86 \rightarrow -0.57$).

These results support the interpretation that backbone structure arises from optimizer-integrated update dynamics rather than architectural constraints alone.

7 Reheating: Re-Entering the Probe Basin

7.1 Protocol

From the endpoint of training (step 10,000; $p_{\text{ood}} \approx 0.16$, deep in the LM-dominant regime), we resume training with doubled probe loss weight $\lambda = 4.0$ and a *fresh* AdamW optimizer (zeroed momentum and second-moment buffers). Three learning rates are tested: $\eta \in \{10^{-3}, 6 \times 10^{-4}, 3 \times 10^{-4}\}$. Each reheating run lasts 2,000 steps with a cosine learning rate schedule.

The rationale is straightforward: if the probe attractor still exists in the loss landscape, a sufficiently strong gradient signal should be able to push the model back into it. The fresh optimizer ensures that momentum from the original training run does not confound the result.

Importantly, reheating primarily perturbs the transverse residual $\|\mathbf{r}(t)\|$ while leaving the accumulated backbone coordinate $a(t)$ largely unchanged. This indicates that the slow manifold persists even when the model temporarily re-enters the probe regime.

7.2 Results

At the optimal learning rate (6×10^{-4}), the model reaches $p_{\text{ood}} = 0.782$ —exceeding the training-time peak of 0.777—within 1000 reheating steps (Table 6). The probe basin remains geometrically present in the late-training landscape. Reheating reveals that it is not erased but becomes dynamically suppressed by backbone stiffening.

But re-entry is unstable. After peaking, p_{ood} decays to 0.28 by step 2000. The probe basin has become a transient saddle: reachable but not sustainable under these dynamics.

$\eta = 3 \times 10^{-4}$ is below the threshold for full re-entry—it provides insufficient gradient drive to overcome the curvature barrier separating the LM and probe basins.

Table 6: Reheating results (seed 42). The model starts from $p_{\text{ood}} = 0.16$.

Learning Rate	Peak p_{ood}	At Step	First ≥ 0.60	Final (step 2000)
10^{-3}	0.705	900	step 600	0.221
6×10^{-4}	0.782	1000	step 700	0.279
3×10^{-4}	0.578	1500	—	0.421

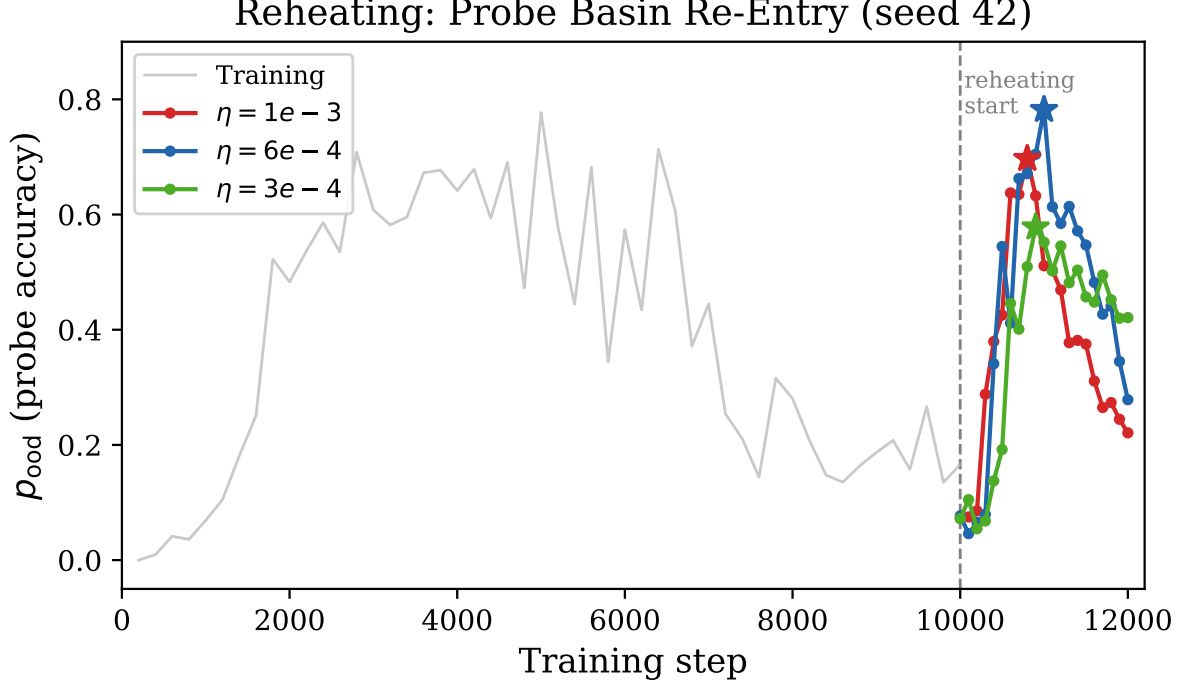


Figure 2: **Reheating trajectories (seed 42)**. Three learning rates are tested from the step-10,000 endpoint (grey background: original training). All three achieve probe re-entry, but the effect is transient: p_{ood} peaks and then decays as the cosine schedule reduces η_t . The optimal LR (6×10^{-4} , orange) exceeds the original training peak.

7.3 Connection to Backbone Stiffening

The transient nature of reheating correlates with increased Fisher curvature along the backbone direction, documented in Section 4.9. During training, $q(\mathbf{v}_b)$ increases by three orders of magnitude from initialization to step 10,000.

When curvature along the backbone is large, larger learning rates are required to perturb accumulated drift; as the learning rate decays, the system returns toward the LM-dominant regime. This is consistent with a picture in which transverse modes respond rapidly to increased probe weighting, while the backbone coordinate resists perturbation proportionally to accumulated curvature.

The sign flip in update–backbone alignment (Section 4.8) provides a complementary observation. In early training, the optimizer drifts in the $-\mathbf{v}_b$ direction (toward the probe basin). After the sign flip ($t^* \approx 5000$), the drift reverses—the optimizer moves along $+\mathbf{v}_b$, away from the probe basin. Reheating temporarily reverses this via the strong probe gradient, but once η_t decays, the accumulated drift direction reasserts itself.

7.4 Two-Seed Comparison

Seed 271 reheating shows the same qualitative pattern—transient probe re-entry followed by decay—though the peak p_{ood} is lower (0.36–0.42 vs. 0.78). Both seeds use identical hyperparameters (Table 1); the quantitative difference likely reflects seed-dependent differences in the late-training loss landscape geometry—in particular, the depth and width of the probe basin at the reheating start point—rather than any difference in the reheating protocol itself.

Thus reheating does not contradict backbone dominance; it demonstrates that switching dynamics are transverse excursions around a persistent slow manifold shaped by optimizer integration.

8 Switching Lives in the Transverse Subspace

With the backbone established as the dominant geometric feature, what can we say about the oscillatory dynamics? They occur primarily in the transverse subspace.

The switching direction between a peak and adjacent trough of p_{ood} is

$$\mathbf{v}_{\text{sw}} = \frac{\boldsymbol{\theta}_{\text{peak}} - \boldsymbol{\theta}_{\text{trough}}}{\|\boldsymbol{\theta}_{\text{peak}} - \boldsymbol{\theta}_{\text{trough}}\|}. \quad (10)$$

Its alignment with the backbone is:

Seed	$ \langle \mathbf{v}_{\text{sw}}, \mathbf{v}_{\text{b}} \rangle $ (per-block range)
42	0.20–0.25
271	0.28–0.31

We can decompose the switch direction into backbone and transverse components:

$$\mathbf{v}_{\text{sw}} = \underbrace{\langle \mathbf{v}_{\text{sw}}, \mathbf{v}_{\text{b}} \rangle \mathbf{v}_{\text{b}}}_{\text{backbone component}} + \underbrace{\mathbf{v}_{\text{sw}} - \langle \mathbf{v}_{\text{sw}}, \mathbf{v}_{\text{b}} \rangle \mathbf{v}_{\text{b}}}_{\text{transverse component}}. \quad (11)$$

The backbone component accounts for $\langle \mathbf{v}_{\text{sw}}, \mathbf{v}_{\text{b}} \rangle^2 \approx 0.04$ –0.10 of the switching direction’s variance. Switching is approximately **80% transverse** to the backbone.

Furthermore, different switching events use near-orthogonal directions (pairwise $|\cos| < 0.08$), and the switching manifold spans at least 10 independent dimensions in the 25M-dimensional trunk space.

To quantify how much of the transverse switching direction is captured by the leading residual PCs, we project out the backbone component to obtain $\mathbf{v}_{\text{sw}}^\perp = \mathbf{v}_{\text{sw}} - \langle \mathbf{v}_{\text{sw}}, \mathbf{v}_{\text{b}} \rangle \mathbf{v}_{\text{b}}$ (renormalized to unit length) and compute the energy fraction captured by PCs 2–6:

$$E_{2:6} = \sum_{k=2}^6 \langle \hat{\mathbf{v}}_{\text{sw}}^\perp, \mathbf{v}_k \rangle^2, \quad (12)$$

where \mathbf{v}_k are the k -th right singular vectors from the uncentered trajectory PCA. Per block, $E_{2:6}$ ranges from 15–22% in seed 42 and 60–67% in seed 271. In seed 42, the transverse switching direction is largely orthogonal to the low-rank PC subspace, meaning it lives in the high-dimensional tail of the trajectory variance. In seed 271, PCs 2–6 capture the majority of the transverse switch. In both cases the switching dynamics are distributed across multiple residual dimensions rather than concentrated on a single transverse mode.

The picture is: **the optimizer rides a one-dimensional backbone rail while oscillating in a high-dimensional transverse cloud.**

9 Discussion

9.1 Empirical Decomposition of Training Dynamics

We presented an empirical decomposition of transformer training under AdamW into two geometric components:

1. A dominant cumulative drift direction (the rolling backbone), defined operationally via uncentered trajectory PCA in rolling windows.
2. Transverse residual dynamics associated with probe switching.

This decomposition is descriptive rather than assumed. It follows directly from measured drift geometry. Across two seeds and 10,000 training steps, the backbone is:

- **Low-dimensional:** PC1 captures 71–72% of row-normalized variance over the full trajectory, and 60–81% in phase-specific windows (§4.1, §4.3).
- **Locally smooth:** Adjacent-window cosine $\rho(t) > 0.7$ everywhere, with mean ≈ 0.80 (§4.2).
- **Globally reoriented:** Early–late phase cosine $|\langle v_E, v_L \rangle| = 0.32$ ($\approx 71^\circ$ rotation), with curvature concentrated in the 4k–6k transition region (§4.3–4.4).

The quantitative consistency across seeds (all reported values agree to within 1–2% between seed 42 and seed 271) indicates that these geometric features are robust properties of the training dynamics rather than seed-specific artifacts.

9.2 Instantaneous Gradients vs Accumulated Updates

Per-batch gradients exhibit near-noise-floor projection onto the rolling backbone direction ($|\cos| \approx 0.008\text{--}0.012$). In contrast, 200-step accumulated updates align strongly with it ($|\cos| \approx 0.15\text{--}0.32$ in early training; §4.8). This 20–30 \times gap demonstrates that backbone dominance is not explained by instantaneous gradient alignment. It arises at the level of optimizer-integrated updates.

We do not claim a formal dynamical-systems separation. We only observe that accumulated update geometry differs qualitatively from per-step gradient geometry, and that this difference is controlled by optimizer configuration (§4.7).

9.3 Optimizer Dependence

Optimizer ablations provide causal support for the claim that backbone geometry is optimizer-induced:

- **β_2 ablation:** Reducing β_2 from 0.95 to 0.0 degrades PC1 concentration from 68% to 52%, halves update–backbone alignment ($0.23 \rightarrow 0.10$), explodes drift magnitude by $\sim 2000\times$, and collapses probe accuracy from 0.94 to 0.005 (§4.7, Table 4).
- **SGD control:** SGD-family variants fail to reproduce the same learning outcome. AdamW with the same architecture achieves PC1 $\approx 81\%$ on the same trajectory interval.
- **Monotonic degradation:** Across $\beta_2 \in \{0.99, 0.95, 0.90, 0.80, 0.0\}$, backbone concentration, update alignment, and best probe accuracy all degrade monotonically.

These effects occur under matched architecture, data, and schedules, isolating optimizer configuration as the primary variable. We therefore conclude that backbone geometry is optimizer-induced rather than a generic property of the loss landscape.

9.4 Phase Reorientation Under Objective Reweighting

The λ -switch at step 4000 ($\lambda : 2 \rightarrow 4$) coincides with a measurable dynamical phase transition:

- **Backbone rotation:** The slow direction reorients by $\approx 71^\circ$, passing through a geometric dead zone ($A_E \approx 0.02$, $A_L \approx 0.16$) near step 3500 (§4.4).
- **Power-law regime change:** Backbone growth exponent drops from $\gamma_a = +1.74$ (pre-switch) to -0.08 (post-switch); residual exponent drops from $\gamma_r = +0.84$ to -0.31 (§4.5).
- **Sign reversal:** Update-backbone alignment flips from $\cos \approx -0.32$ to $+0.11$ near step 5100 (§4.8).
- **Fisher anisotropy spike:** Backbone Rayleigh quotient increases $\sim 64\times$ from step 1800 to 4800, with anisotropy peaking at $12.4\times$ (§4.9).
- **Correlation flip:** $\text{corr}(p_{\text{ood}}, \|r\|)$ reverses from $+0.85$ (0–4k) to -0.76 (6k–10k) (§4.6).

We do not claim that curvature causes rotation. Rather, we document that objective reweighting produces correlated, measurable changes across five independent geometric diagnostics. The rolling tangent remains locally smooth throughout ($\rho > 0.7$); the phase transition reflects gradual global reorientation concentrated in the 4k–6k region.

9.5 Reheating as a Geometric Probe

Reheating experiments reveal that:

- Probe performance can be transiently restored from late checkpoints.
- Re-entry magnitude depends on learning rate and β_2 .
- Accumulated backbone drift remains largely unchanged during reheating.

These findings are consistent with the backbone-residual decomposition: reheating perturbs transverse components while leaving dominant cumulative drift largely intact. The correlation flip documented in §4.6 provides additional context: after step 6000, residual re-growth is associated with *declining* probe accuracy ($r \approx -0.76$ – -0.79), suggesting that late residual dynamics drive the model away from the probe solution rather than toward it.

9.6 The Three Dynamical Phases

The power-law analysis (§4.5) reveals that training decomposes into three phases with distinct geometric signatures:

1. **Acceleration (0–2k):** Both backbone and residual grow rapidly ($a \sim t^{2.1}$, $r \sim t^{1.2}$). The trajectory explores broadly, with productive residual energy ($\text{corr}(p_{\text{ood}}, \|r\|) \approx +0.8$).
2. **Consolidation (2k–6k):** Backbone growth slows ($\gamma_a : 2.1 \rightarrow 0.67 \rightarrow 0.15$) while the residual contracts ($\gamma_r \approx -0.4$ to -1.4). The λ -switch at step 4000 sharpens the contraction. The model consolidates along the slow direction.
3. **Plateau (6k–10k):** The backbone retreats ($\gamma_a \approx -0.19$), the residual re-grows ($\gamma_r \approx +0.34$), and probe accuracy declines. The drift direction becomes orthogonal to both phase backbones ($A_E \approx A_L \approx 0.02$ – 0.18). The late-phase rank-1 concentration (PC1 $\approx 81\%$) indicates tight low-rank drift, but in a direction no longer aligned with either learning phase.

These three phases are not assumed *a priori*; they emerge from piecewise power-law fits with high R^2 values (> 0.85 , most > 0.97) and are consistent across both seeds.

9.7 Scope and Limitations

This study uses a 51M-parameter transformer and a synthetic probe objective. The backbone phenomenon may scale differently in larger models or natural multi-objective settings.

The Fisher analysis relies on mini-batch approximations ($n = 32$ batches). Reported anisotropy ratios may underestimate full curvature structure.

The rotation curve analysis uses $W = 10$ checkpoint windows (≈ 2000 steps). Smaller windows would provide finer temporal resolution but lower statistical stability; the choice of W affects the absolute values of $\rho(t)$ but not the qualitative pattern (dip location, phase handoff structure).

All main results replicate across two seeds with quantitative agreement typically within 1–2%. Broader seed variation, dataset variation, and scaling to larger models remain future work.

9.8 Summary

This work provides a trajectory-level characterization of transformer training under AdamW:

- Cumulative drift concentrates in a locally smooth, globally evolving direction (PC1 $\approx 71\%$; early–late rotation $\approx 71^\circ$).
- Instantaneous gradients do not explain this direction; accumulated optimizer updates do ($20\text{--}30\times$ alignment gap).
- Optimizer configuration controls drift geometry (β_2 ablation: PC1 $68\% \rightarrow 52\%$, alignment $0.23 \rightarrow 0.10$).
- Objective reweighting reorients the dominant drift direction, with curvature concentrated in a ~ 2000 -step transition zone.
- Backbone and residual dynamics obey distinct power-law regimes that change sharply at the λ -switch ($\gamma_a : +1.74 \rightarrow -0.08$; $\gamma_r : +0.84 \rightarrow -0.31$).
- Residual geometry correlates with task performance, with the sign of the correlation reversing across training phases ($+0.85$ early, -0.79 late).

These findings shift attention from instantaneous gradient geometry to cumulative trajectory structure as a measurable, optimizer-dependent, and objective-sensitive feature of training dynamics.

10 Methods

10.1 Trunk Parameters

All geometric analyses use trunk-only parameters: weight matrices in attention (query, key, value, output projection) and MLP (up-projection, down-projection) across all 8 blocks. This excludes tied embeddings, causal masks, positional embeddings, and layer normalization parameters. Total trunk dimensionality: $\sim 25\text{M}$ parameters ($\sim 3.1\text{M}$ per block).

10.2 Uncentered PCA

For each transformer block $\ell \in \{0, \dots, 7\}$, the drift matrix $\mathbf{X}^{(\ell)} \in \mathbb{R}^{T \times D_\ell}$ has rows

$$\mathbf{x}^{(\ell)}(t) = \text{flatten}_\ell(\boldsymbol{\theta}(t)) - \text{flatten}_\ell(\boldsymbol{\theta}(0)). \quad (13)$$

SVD of $\mathbf{X}^{(\ell)}$ (no mean centering) yields the block backbone $\mathbf{v}_b^{(\ell)} \in \mathbb{R}^{D_\ell}$ as the first right singular vector.

10.3 Update-Direction Alignment

The 200-step update $\mathbf{u}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}(t - 200)$ is computed from consecutive checkpoints. This captures the net effect of AdamW (preconditioner, momentum, weight decay, gradient clipping). Alignment is reported as $C(t) = \langle \mathbf{u}(t), \mathbf{v}_b \rangle / \|\mathbf{u}(t)\|$.

10.4 Rayleigh Quotient Computation

Given a direction $\mathbf{v} \in \mathbb{R}^D$ and a gradient matrix $\mathbf{G} \in \mathbb{R}^{M \times D}$:

$$q(\mathbf{v}) = \frac{1}{M} \|\mathbf{G}\mathbf{v}\|^2 = \frac{1}{M} \sum_{i=1}^M \langle \mathbf{g}_i, \mathbf{v} \rangle^2. \quad (14)$$

This requires one matrix–vector product ($O(MD)$ operations, $O(M)$ storage for the result), avoiding construction of the $D \times D$ Fisher. Anisotropy uses $K = 10$ random orthogonal directions generated by Gram–Schmidt orthogonalization of Gaussian random vectors projected orthogonal to \mathbf{v}_b .

10.5 Reheating Protocol

Resume from step 10,000 checkpoint. Fresh AdamW optimizer (zeroed $\mathbf{m}_0, \mathbf{v}_0$). Composite loss weight $\lambda = 4.0$. Cosine learning rate schedule over 2,000 steps. Evaluate every 100 steps. Three learning rate values: $\{10^{-3}, 6 \times 10^{-4}, 3 \times 10^{-4}\}$.

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