$f: X_{n\times m} \rightarrow \mathbb{R}$ 性质 $d|X| = tr(|X|X^{-1}dX)$

求 $\frac{\partial f}{\partial X}$?

$$df = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial f}{\partial X_{ij}} dX_{ij}$$
$$= tr\left(\left(\frac{\partial f}{\partial X}\right)^T dX\right)$$

性质: $tr(A^TB) = \sum_i \sum_j A_{ij}B_{ij}$

目前矩阵对矩阵的导数是未定义的，故

性质: $d(XY) = \lim\limits_{\substack{\Delta X \to 0 \\ \Delta Y \to 0}} (X+\Delta X)(Y+\Delta Y) - XY$

$$= \lim XY + \Delta XY + X\Delta Y + \Delta X\Delta Y - XY$$
$$= \quad + dXY + XdY$$

$dX^T = (dX)^T$ $\quad d\,tr(X) = tr(dX)$

$dXX^{-1} = dI = 0 = (dX)X^{-1} + X(dX^{-1})$

$\therefore dX^{-1} = -X^{-1}dXX^{-1}$

例: $f(x) = a^Txb$ 求 $\frac{\partial f}{\partial x}$

$$df = da^Txb + a^Tdx\,b + a^Txdb$$
$$= a^Tdxb = tr\left(\left(\frac{\partial f}{\partial x}\right)^T dx\right)$$

$$df = tr(df) = tr(a^T dX b) = tr((\frac{\partial f}{\partial X})^T dX)$$

$$= tr(dX\, b a^T) = tr((ab^T)^T dX)$$

$$\therefore \frac{\partial f}{\partial X} = ab^T$$

若 $Y = AXB$ 求 $\frac{\partial f}{\partial X}$

$$df = tr((\frac{\partial f}{\partial Y})^T dY)$$

$$= tr[(\frac{\partial f}{\partial Y})^T (A\, dX\, B)]$$

$$= tr((\frac{\partial f}{\partial Y})^T A\, dX\, B)$$

$$= tr(B(\frac{\partial f}{\partial Y})^T A\, dX)$$

$$= tr((A^T \frac{\partial f}{\partial Y} B^T)^T dX)$$

$$\therefore \frac{\partial f}{\partial X} = A^T \frac{\partial f}{\partial Y} B^T$$

例3. $f = a^T \exp(Xb)$ $\quad a: m\times 1 \quad X: m\times n \quad b: n\times 1$

$$df = a^T(\exp(Xb) \odot d(Xb))$$

$A^T(B \odot C)$

$$= a^T(\exp(Xb) \odot (dX \cdot b))$$

$(A \odot B)^T$

$$=tr(a^T exp(xb) \odot (dx \cdot b))$$

$$= tr([a \odot exp(xb)]^T dx\, b)$$

$$= tr[b[a \odot exp(xb)]^T dx]$$

$$\therefore \frac{\partial f}{\partial x} = [a \odot exp(xb)] b^T$$

例13. $f = tr(Y^T M Y)$, $Y = \sigma(WX)$ 求 $\frac{\partial f}{\partial X}$

$W: l \times m \quad X: m \times n \quad Y: l \times n \quad M: l \times l$

$$df = d\,tr(Y^T M Y) = tr[d(Y^T M Y)]$$

$$= tr[dY^T M Y + Y^T d(MY)]$$

$$= tr[dY^T M Y + Y^T M dY]$$

$$= tr[dY^T M Y] + tr(Y^T M dY)$$

$$= tr[(dY)^T M Y] + tr(Y^T M dY)$$

$$= tr[(MY)^T dY] + tr(Y^T M dY)$$

$$+ tr[(M^T Y)^T$$

$$\therefore \frac{\partial f}{\partial Y} = (M + M^T) Y \quad = 2MY \qquad Y = \sigma(WX)$$

$$df = tr\left(\left(\frac{\partial f}{\partial X}\right)^T dX\right) = tr\left(\left(\frac{\partial f}{\partial Y}\right)^T [\sigma(WX) \odot dWX]\right)$$

$$= tr((2MY)^T [\sigma(WX) \odot (W dX)]$$

$$= tr([(2mY) \odot \sigma'(WX)]^T W dX]$$

$$\therefore \frac{\partial f}{\partial X} = W^T[(2mY) \odot \sigma'(WX)]$$

$$= W^T[[2m\,\sigma(WX)] \odot \sigma'(WX)]$$

$$E_X \frac{\partial \log f(x|\theta)}{\partial X}$$

$$= \frac{\partial (E_X \log f(x|\theta))}{\partial X}$$

$$= \frac{\partial (\int_x \log f(x|\theta)\, q(x)\, dx}{\partial X}$$