

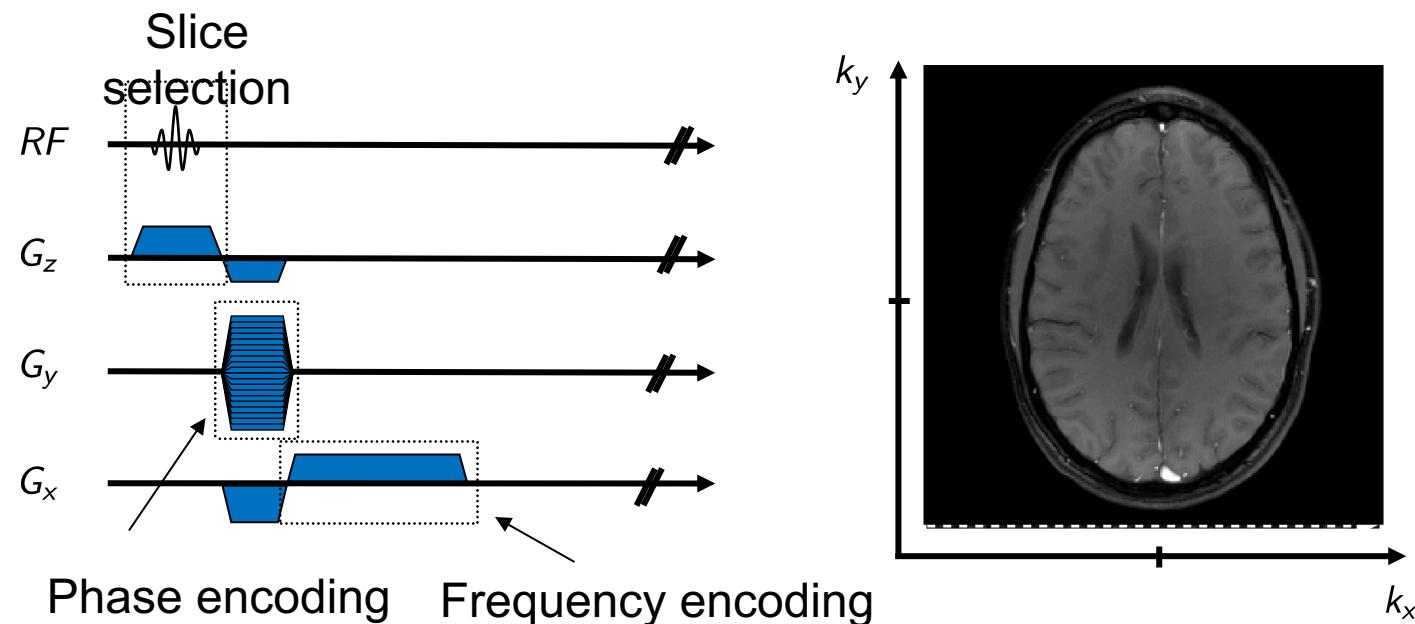
# Computational MRI

## Parallel imaging I: Image-domain methods

# k-space undersampling

# K-space encoding

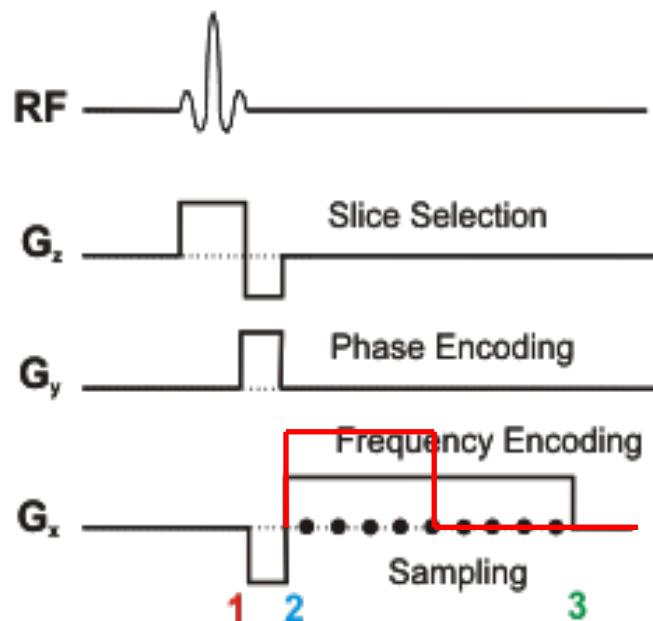
- Speed of k-space traversal
- Switching rate and amplitude of magnetic field gradients



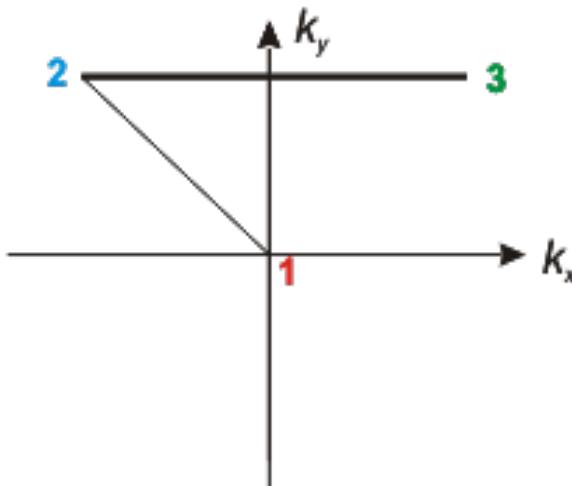
# Imaging speed in MRI

- Speed of k-space traversal
- Switching rate and amplitude of magnetic field gradients

Sequence

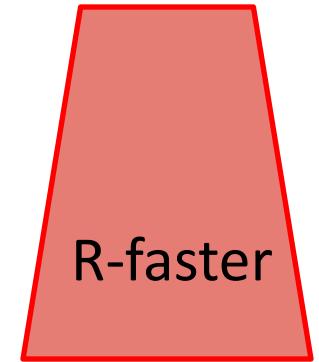
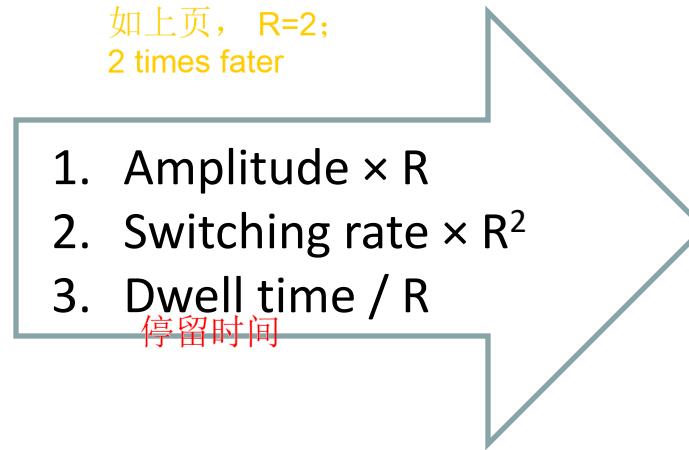


k-space trajectory



2X faster!

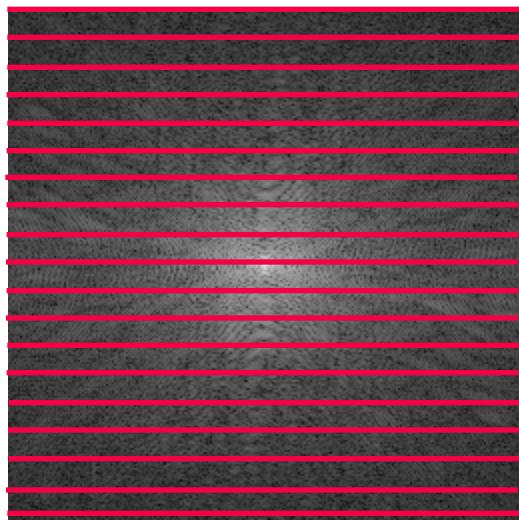
# Speed of conventional MRI is limited



- What is the effect on the gradient amplifier?
  - Power increases with  $R^3$
- What is the effect on SNR?
  - SNR decreases with  $\sqrt{R}$
- It can also cause peripheral nerve stimulation

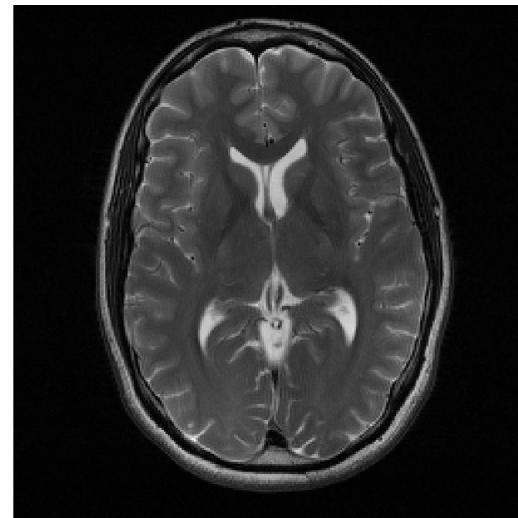


# k-space undersampling



$\Delta k_y$

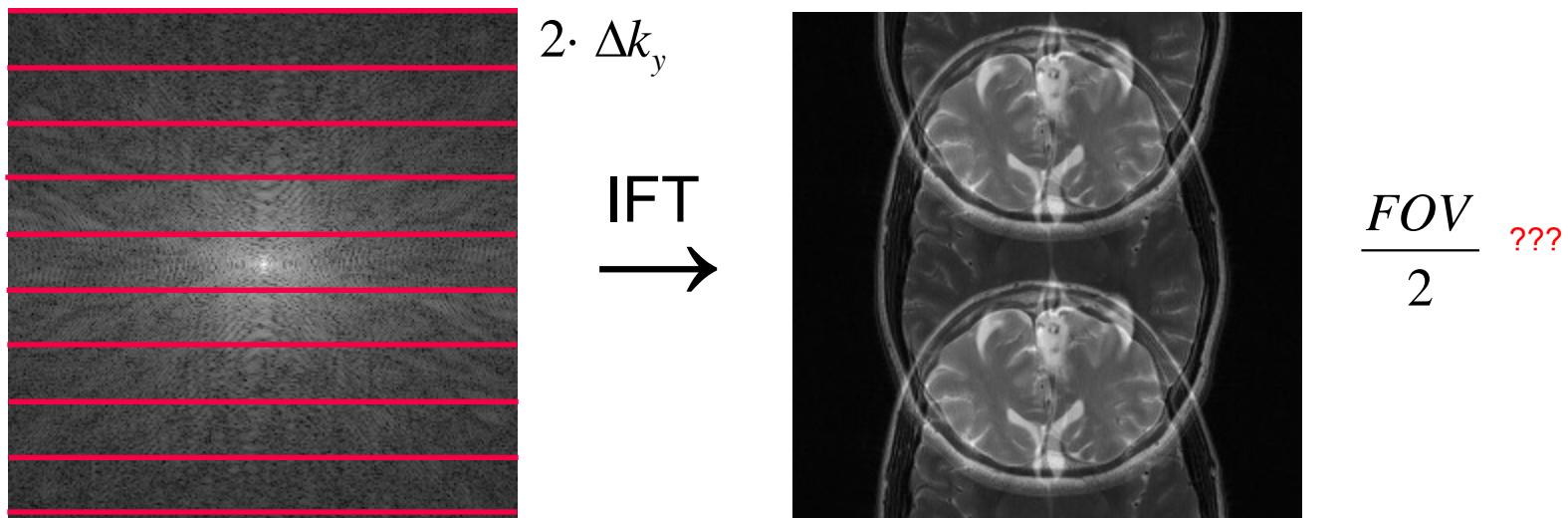
IFT  
→



*FOV*

# k-space undersampling

- Faster, no changes in gradient switching, but conventional Fourier reconstruction will result in aliasing artifacts



Question: Can we undersample in the readout dimension?

no need to do

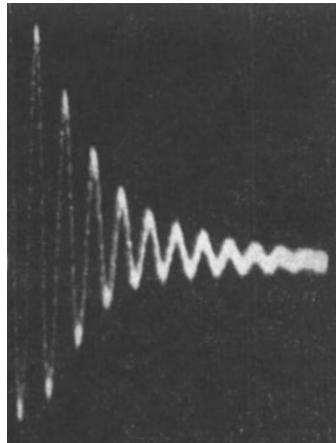
# k-space undersampling

- Reconstruction
  - Exploit data redundancies!
- Parallel imaging
  - Multiple coils with different spatial sensitivities (real data redundancy)
- Compressed sensing, constrained reconstruction, machine learning,...
  - Image compressibility/sparsity  
(inherent redundancy)  
固有的

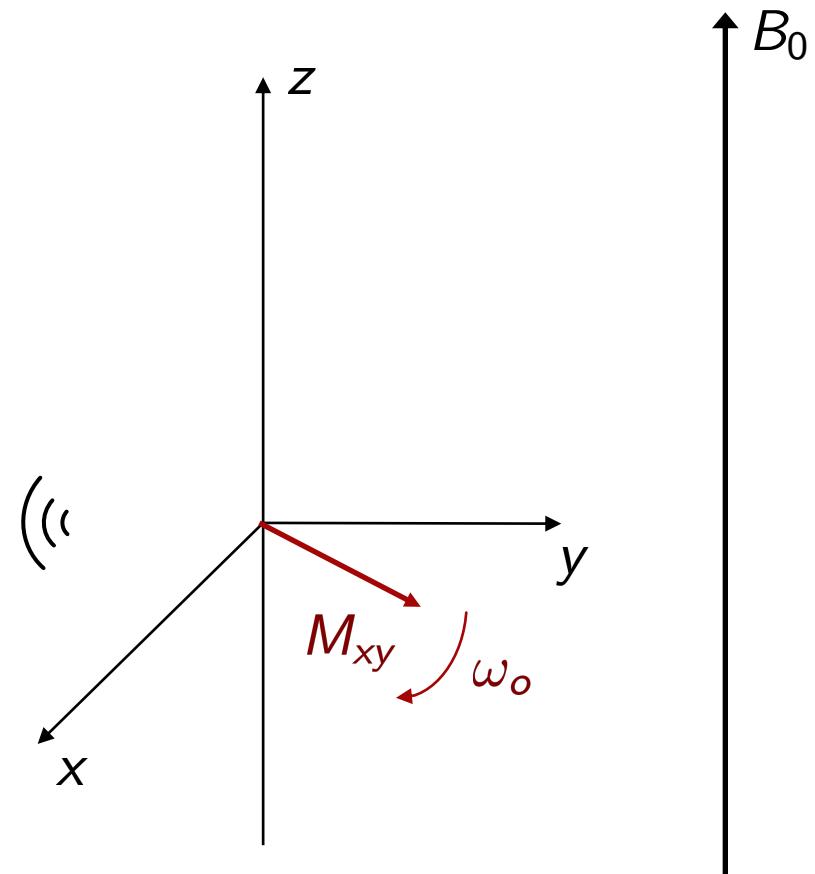
# Multi-channel receive coils

# Signal reception: MR receive coils

$$u_{ind} = -\frac{d\Phi}{dt}$$

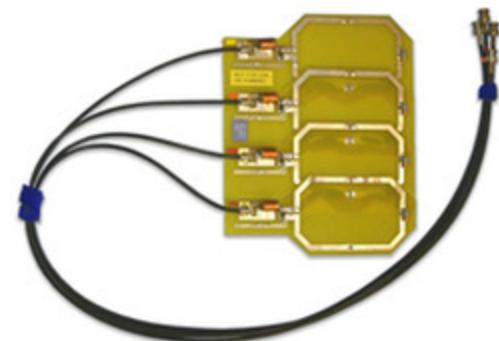
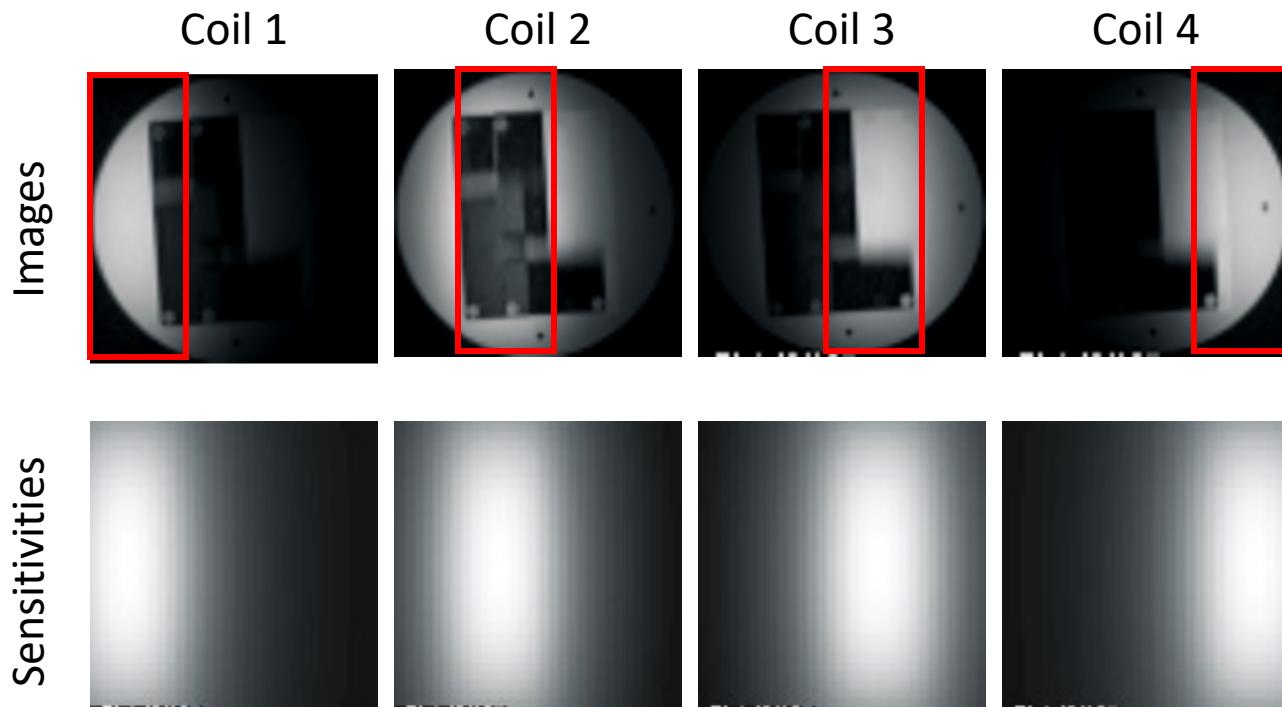


Hahn 1950



# Multiple receiver coils

- Different spatial sensitivities



# Multiple receiver coils

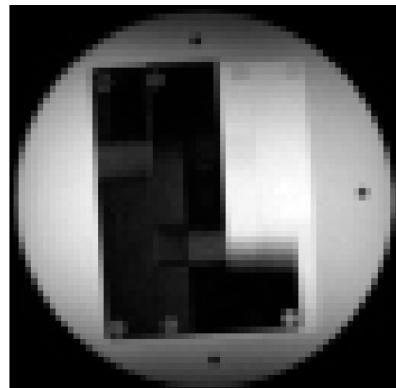
- Sensitivity-encoding equation

$$m_i(r)$$



Coil image

$$f(r)$$



Image

$$c_i(r)$$



Coil sensitivity

$$+ n_i(r)$$

Noise

# Multiple receiver coils

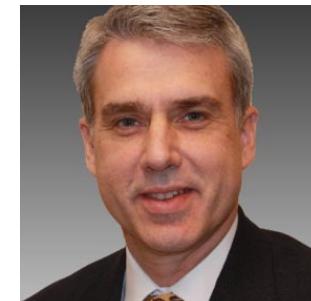
- First used to improve SNR
  - What is the optimal coil combination?
- Matched-filter or least-squares combination

$$f(r) = \frac{\sum_{i=1}^{N_c} c_i^*(r) m_i(r)}{\sqrt{\sum_{l=1}^{N_c} |c_l(r)|^2}}$$

$m_i(r)$ : single coil images

$c_i(r)$ : coil sensitivities

Roemer FB et al. Magn Reson Med. 1990; 16(2):192-225.



# Multiple receiver coils

- Matched-filter or least-squares combination
  - In matrix form (for each pixel)

$$\mathbf{m} = \mathbf{f} \times \mathbf{c}$$

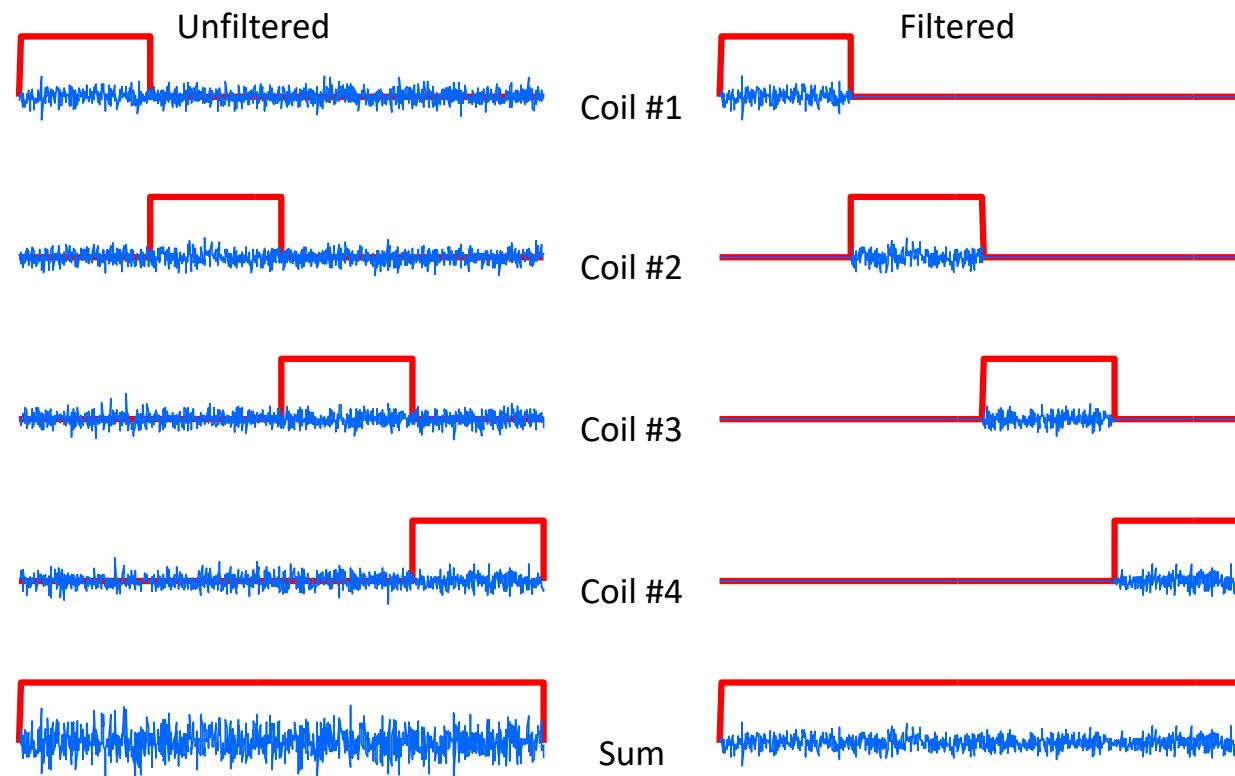
$$f = (\mathbf{C}^H \mathbf{C})^{-1} \mathbf{C}^H \mathbf{m}$$

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} \quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix}$$

Cn: diagnose matrix, 256×256行, 256×256列. img value存放在对角。

# Multiple receiver coils

- Matched-filter or sensitivity-weighted combination
  - Effects on noise



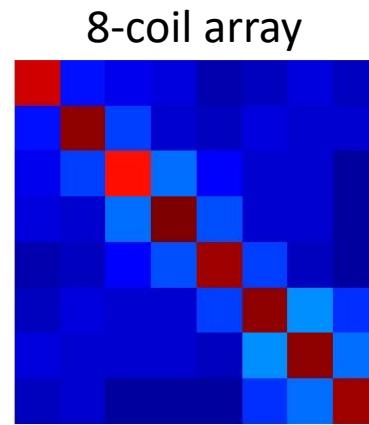
前提是noise均匀分布的

# Multiple receiver coils

- Noise signals from different coils are correlated

$\Psi$ : coil noise covariance matrix

noise不是均匀分布的，  
有noise covariance matrix



- Least-squares combination using the covariance matrix

covariance matrix : 8×8

$$f = (\mathbf{C}^H \boldsymbol{\Psi}^{-1} \mathbf{C})^{-1} \mathbf{C}^H \boldsymbol{\Psi}^{-1} \mathbf{m}$$

$\mathbf{C}$ : vector of 8

The element in the vector is a diagonal matrix with 256×256 rows and 256×256 columns

# Multiple receiver coils

- Pre-whitening
  - Virtual coils with uncorrelated noise

$$\mathbf{m}_w = \boldsymbol{\Psi}^{-\frac{1}{2}} \mathbf{m}$$

$$\mathbf{C}_w = \boldsymbol{\Psi}^{-\frac{1}{2}} \mathbf{C}$$

- Solution

$$f = (\mathbf{C}_w^H \mathbf{C}_w)^{-1} \mathbf{C}_w^H \mathbf{m}_w$$

# Multiple receiver coils

- Sum of squares
  - Approximation to the optimal combination
  - Images as coil sensitivities
  - SNR penalty of about 10%

perfect received coil

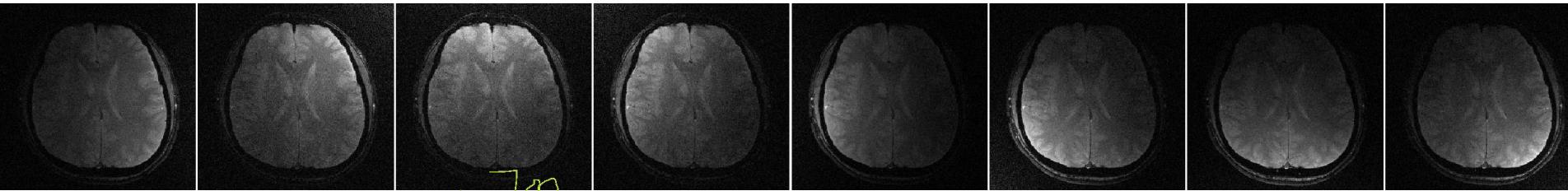
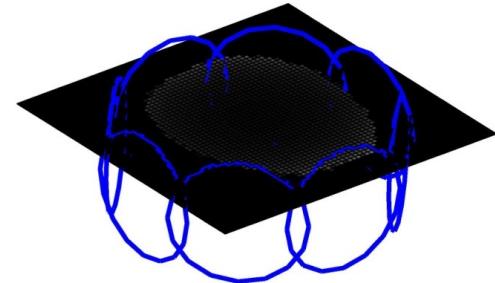
$$c_i(r) = m_i(r) \Rightarrow f(r) = \sqrt{\sum_{i=1}^{N_c} |m_i(r)|^2}$$

$$\mathbf{c} = \mathbf{m} \Rightarrow f = \sqrt{\mathbf{m}^H \mathbf{m}}$$

m: 8-vextor

# Signal combination example

- 8-coil circular array



直接complex sum会导致phase cancellation

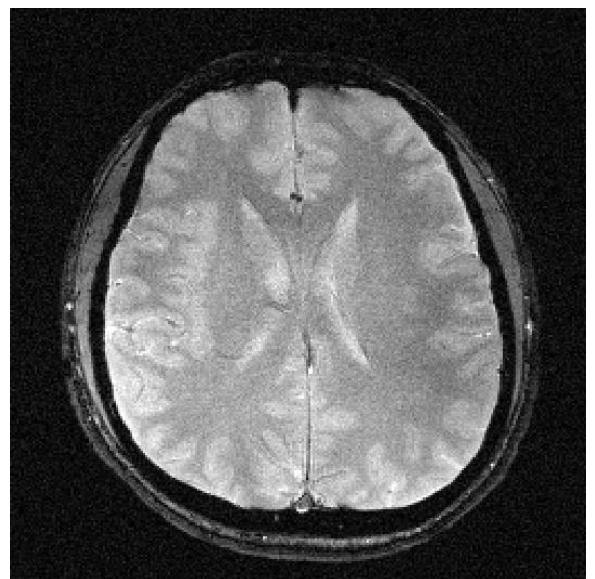
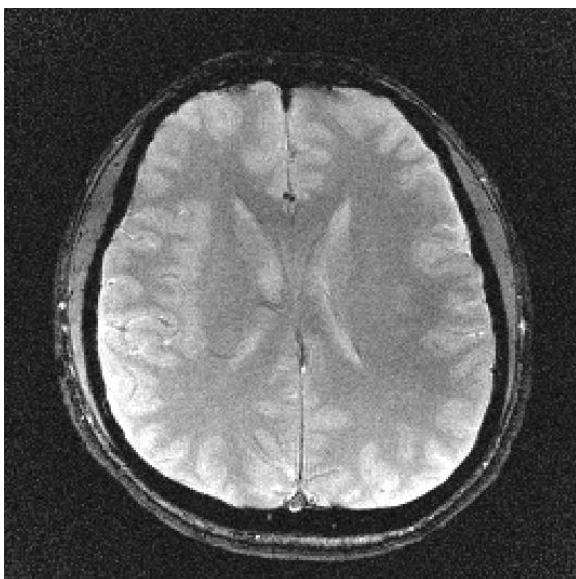
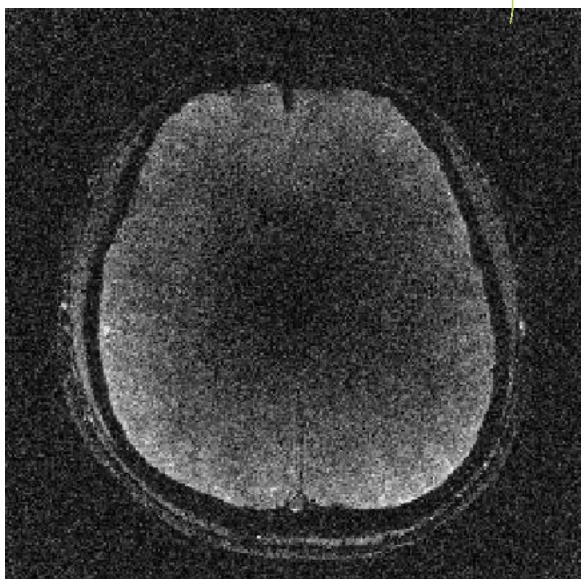
Complex sum



因此要用sum of squares

Sum of squares

Least-squares  
(matched-filter)



# Signal combination example

- Matched-filter combination   
 sequence中不给RFpulse(没有img的信号), 就能得到noise covariance matrix  
 sequence中给RFpulse,得到的就是img signal和noise

Without  $\Psi$



With  $\Psi$



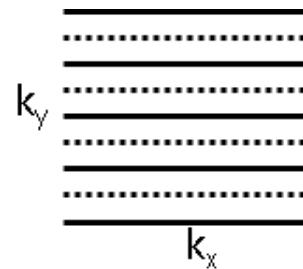
covariance matrix:  $8 \times 8$   
表示不同channel之间的corelation

# Parallel imaging

# Parallel imaging

- Multiple coils enable acceleration of MRI data acquisition
  - Multiple coil data are redundant!

- Regular k-space undersampling

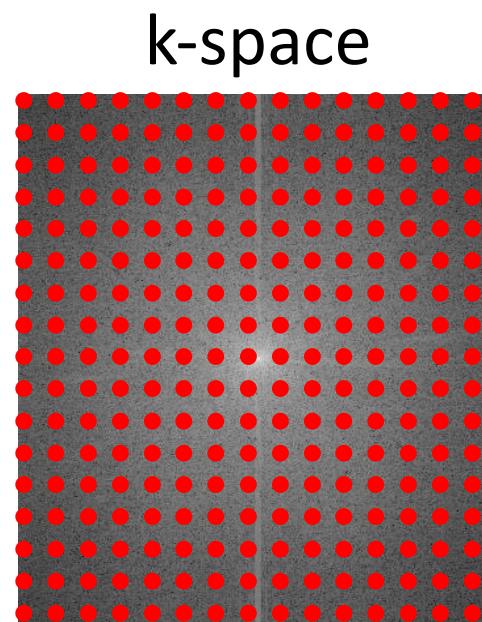
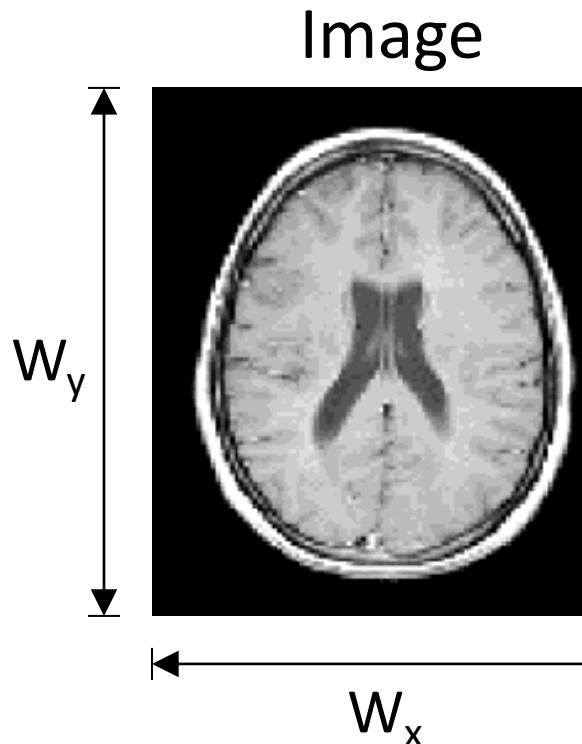


- Reconstruction using matrix inversion

Sodickson DK, Manning WJ. Magn Reson Med. 1997; 38: 591-603  
Pruessmann KP et al. Magn Reson Med 1999; 42: 952-962



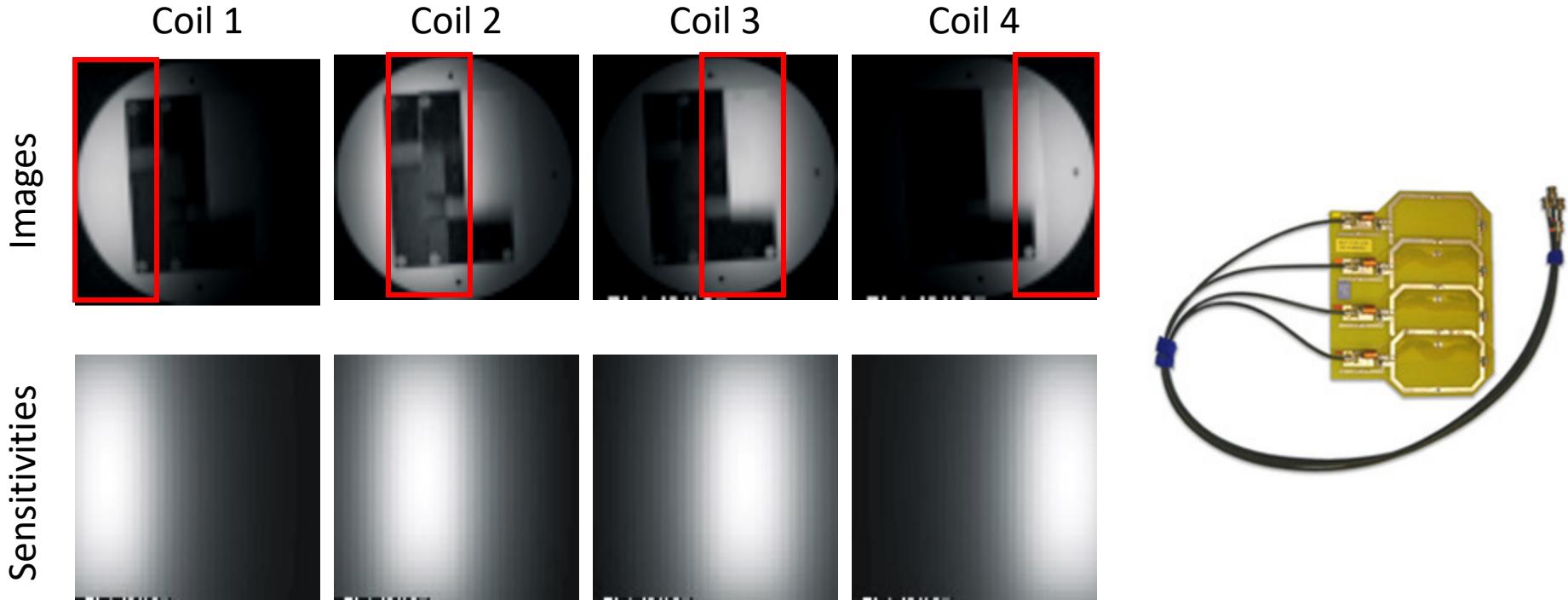
# Recap: k-space sampling density and image FOV



Nyquist rate:

$$\Delta k_x = \frac{1}{W_x}; \quad \Delta k_y = \frac{1}{W_y}$$

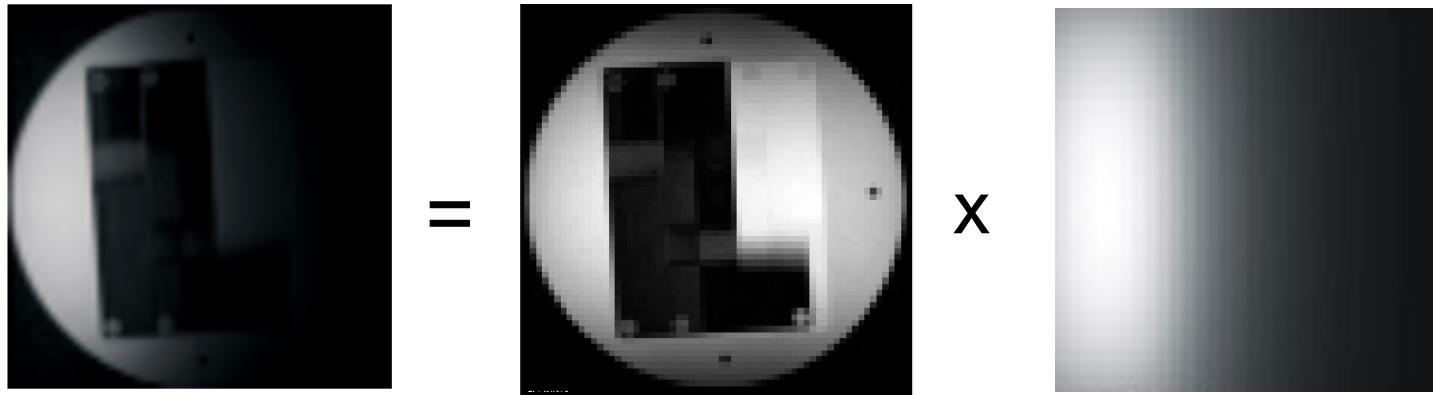
# Spatial encoding of receive coils



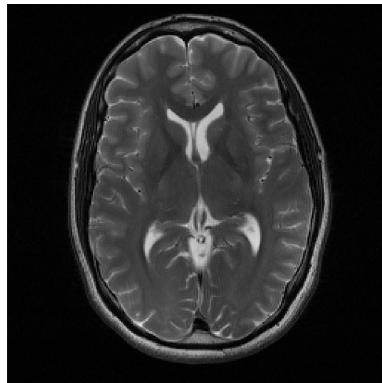
Coils also perform spatial encoding

- Pixels close to the coil are bright
- Pixels far from the coil are dark

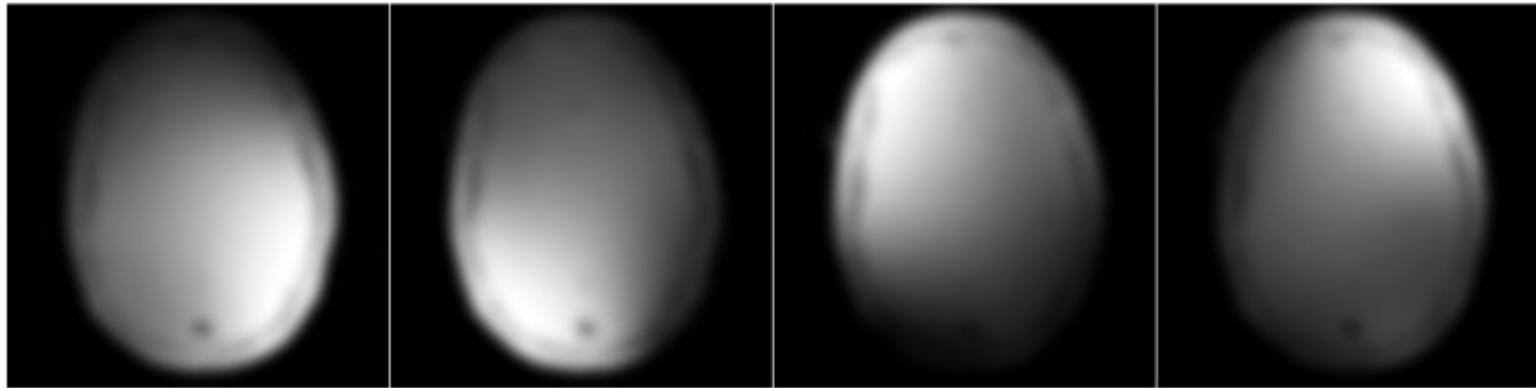
# Back to sensitivity-encoding equation

$$m_i(r) = f(r) \times c_i(r)$$
A diagram illustrating the sensitivity-encoding equation. It shows three grayscale images arranged horizontally. The first image on the left is labeled  $m_i(r)$  above it. The second image in the middle is labeled  $f(r)$  above it. The third image on the right is labeled  $c_i(r)$  above it. Between the first and second images is an equals sign (=). Between the second and third images is a multiplication sign (×).

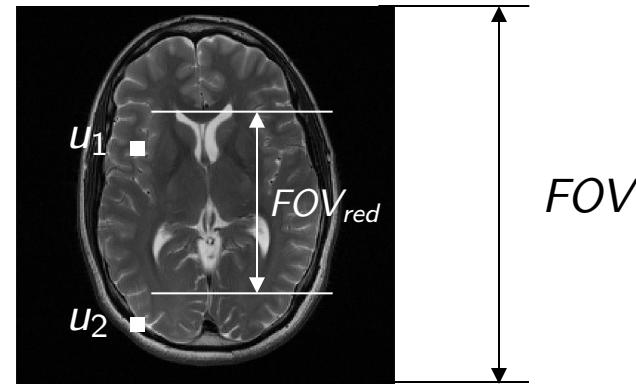
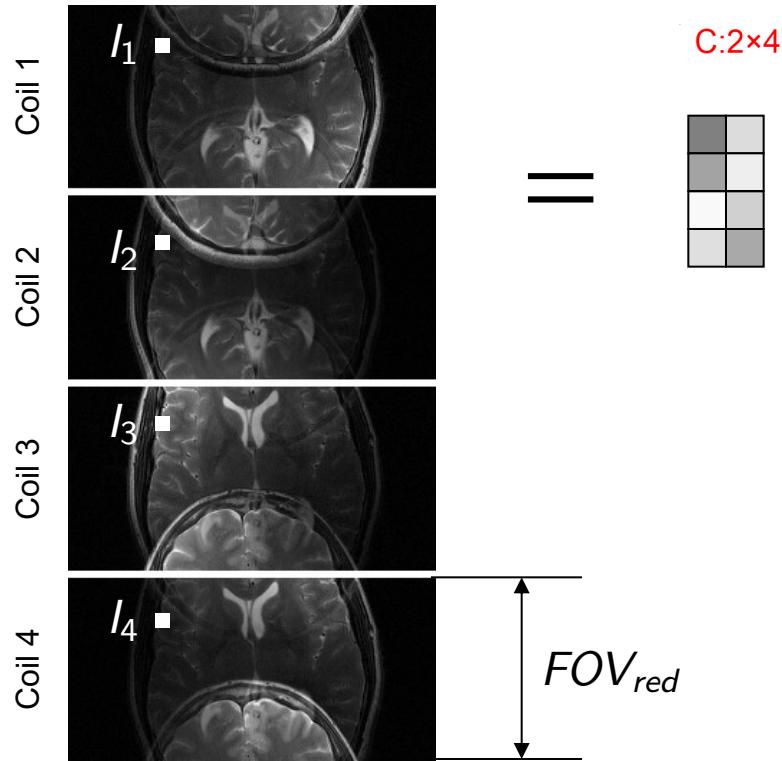
# Cartesian SENSE



- T2 weighted brain scan
- 4-Channel receive coil
- Sensitivities are known



# Cartesian SENSE



C11: coil1's sensitivity in position  $u_1$

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \\ C_{41} & C_{42} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

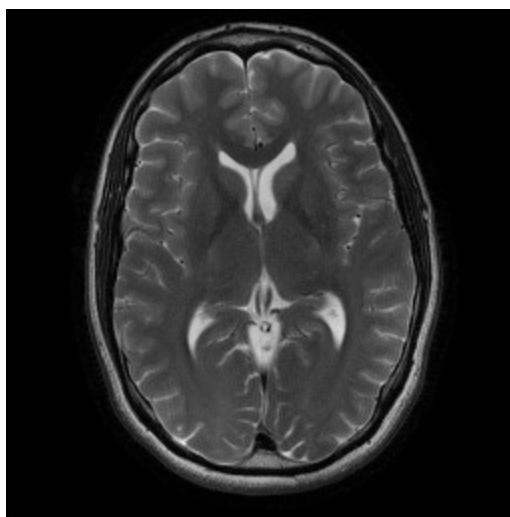
C needs rank of 2;  
2 independent equation  
上式有4 independent  
equation

求 $u_1, u_2$ ,  
这是重建的  
图

# Parallel Imaging: SENSE

$$u = (C^H C)^{-1} C^H I$$

SENSE, R=2



# SNR penalty in parallel imaging

SNR降低

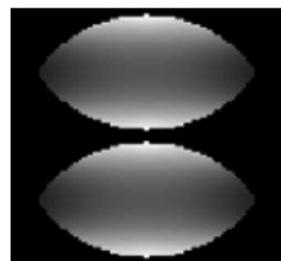
$$SNR_{acc} = \frac{SNR_{no-acc}}{g\sqrt{R}}$$

- g-factor: noise amplification due to ill-conditioning of the encoding matrix

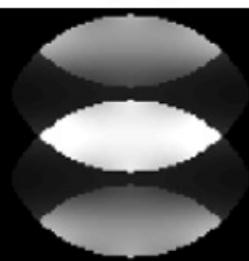
rho:取对角元素(np.diagonal)

$$g(r) = \sqrt{\left(\mathbf{E}^H \boldsymbol{\Psi}^{-1} \mathbf{E}\right)^{-1} \underset{rho}{\mathbf{E}^H \boldsymbol{\Psi}^{-1} \mathbf{E}} \underset{(rho)}{}}$$

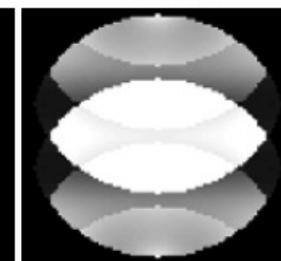
R=2



R=3



R=4



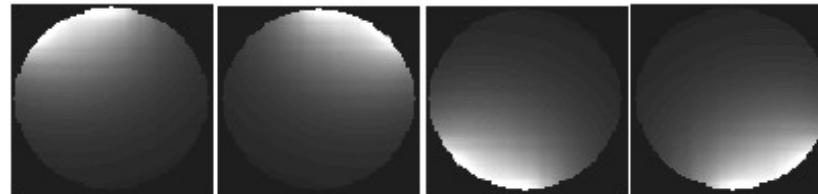
g-factor

$1 < g < 1.2$   
mean=1.07  
max=1.2

$1 < g < 3$   
mean=2.2  
max=3.1

$1 < g < 5$   
mean=5.4  
max=8.6

Coil sensitivities

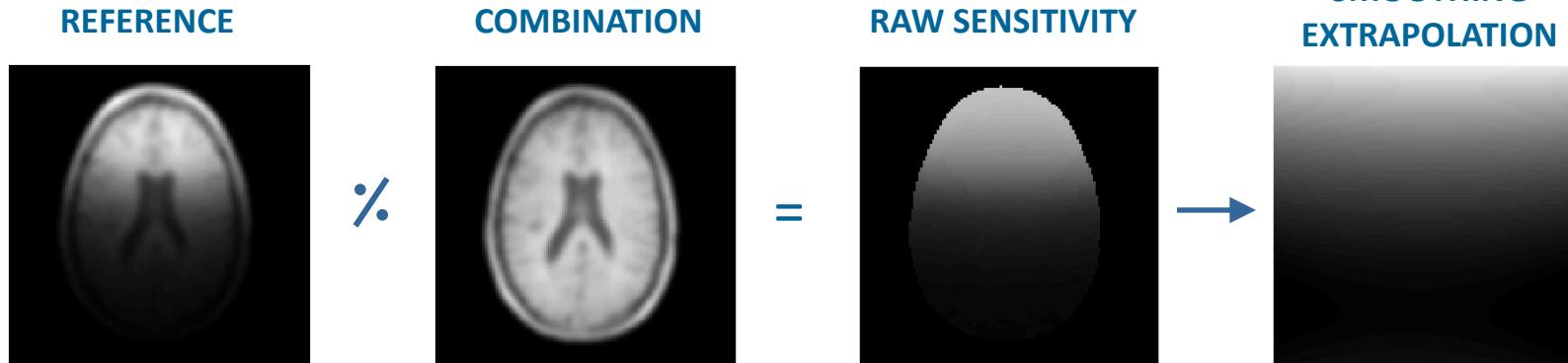


R 增大, noise增大  
因为R  
增加 (更多coil), 更容易有o  
verlap, overlap会导致not  
independent equation

# Coil sensitivity estimation for SENSE

- Estimation of pure coil sensitivities (Pruessmann et al. MRM 1999).
  - Separate low resolution image for each coil.

$$m = f \times c \quad \text{所以: } c = m/f$$



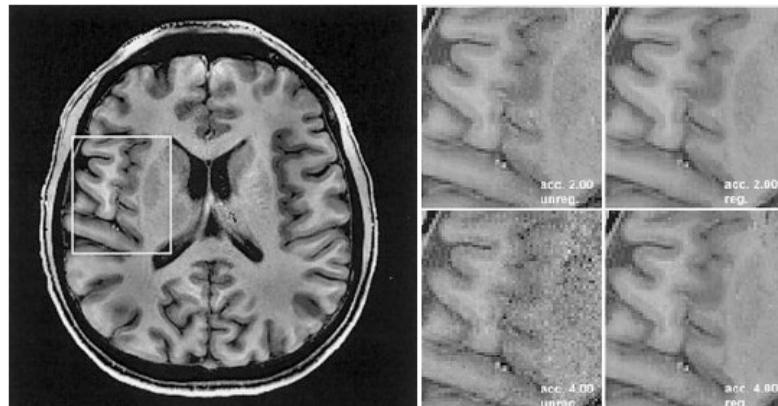
# How to reduce noise amplification?

- Use more coils
- Improve coil array design
- Regularization of the inverse reconstruction  
if the coils are not strictly independent
- 2D acceleration instead of 1D acceleration (3D imaging)

# Regularization of the inverse reconstruction

- Constrain the inverse problem to reduce noise amplification and control numerical instabilities
- Method 1: Tikhonov regularization
  - Constrain the power of the solution

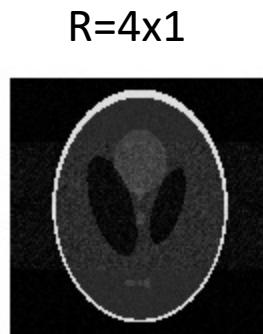
$$\hat{\mathbf{m}} = \min_{\mathbf{m}} \left\{ \|\mathbf{E}\mathbf{m} - \mathbf{s}\|_2^2 + \lambda \|\mathbf{m}\|_2 \right\} = (\mathbf{E}^H \mathbf{E} + \lambda \mathbf{I})^{-1} \mathbf{E}^H \mathbf{s}$$



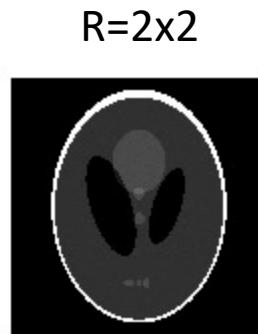
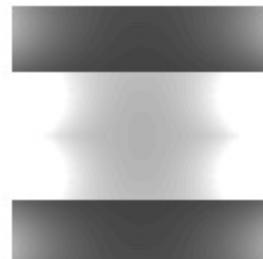
Lin FH et al, Magn Reson Med 2004; 51:559-567

# 2D acceleration Vs. 1D acceleration

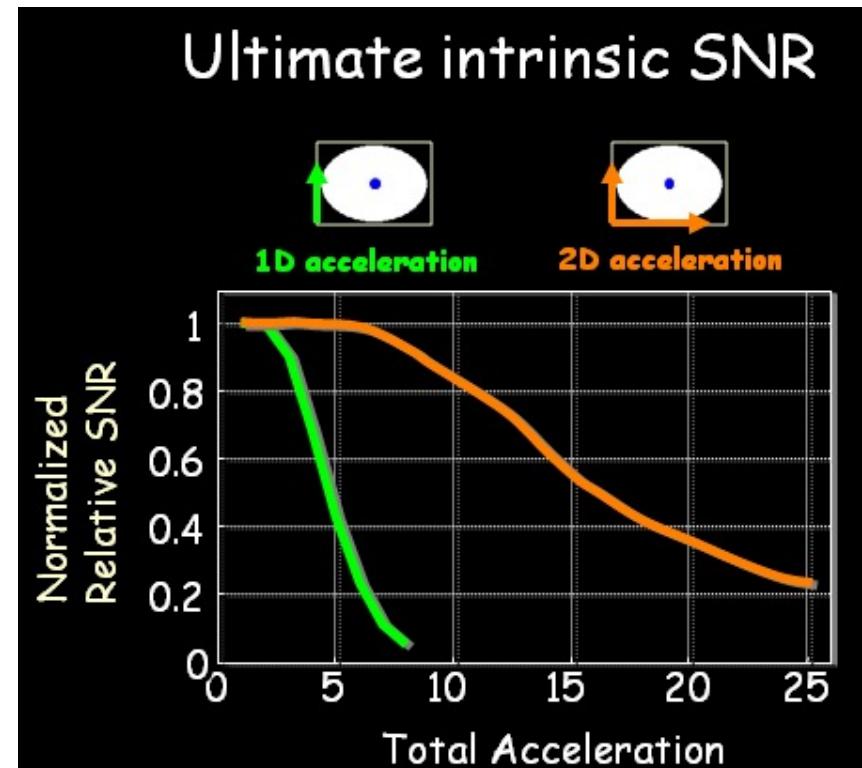
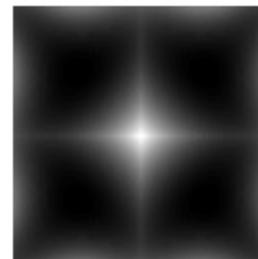
- 2D acceleration reduces g-factor



$g_{avg} = 13.7$   
 $g_{max} = 20.0$



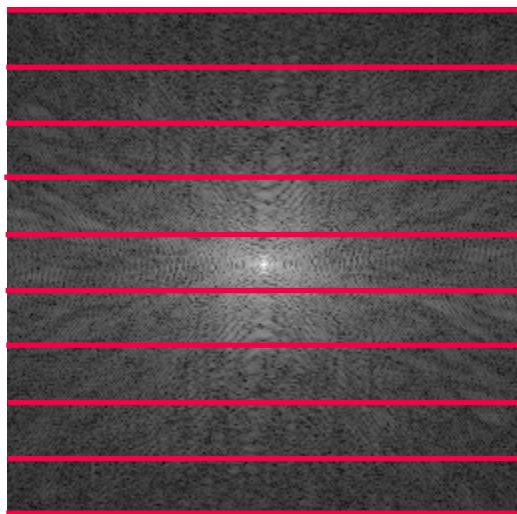
$g_{avg} = 1.9$   
 $g_{max} = 2.1$



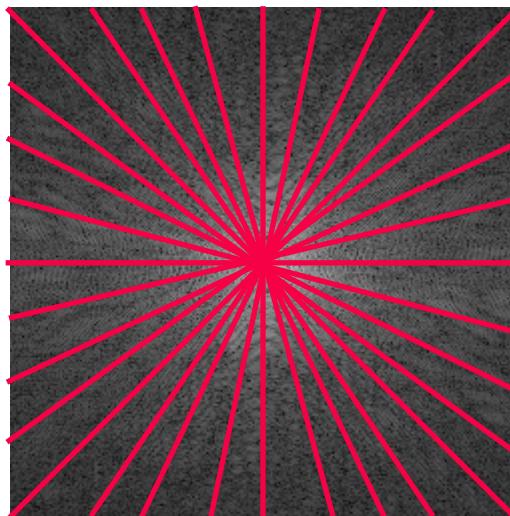
Ohliger MA et al. MRM 2003;50:1018-30

# Non-Cartesian undersampling

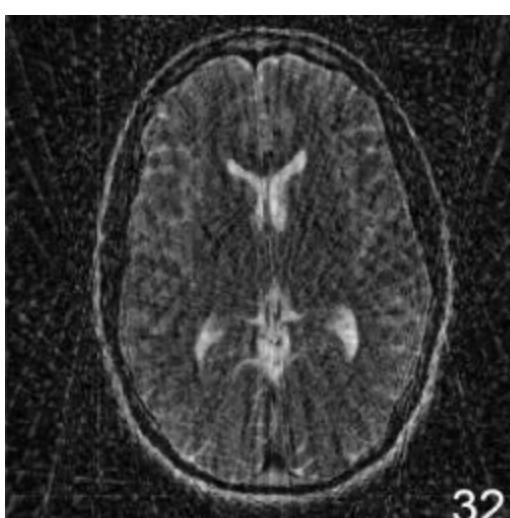
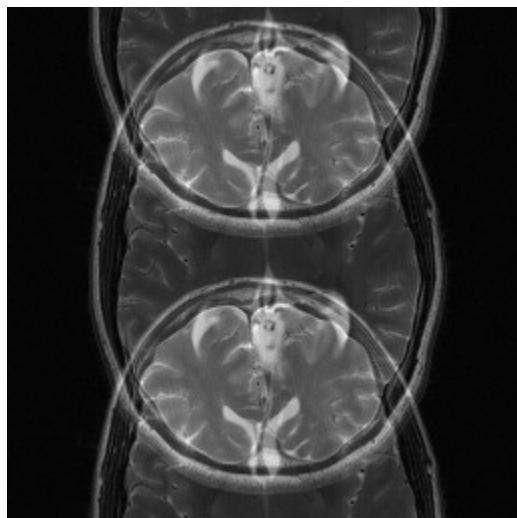
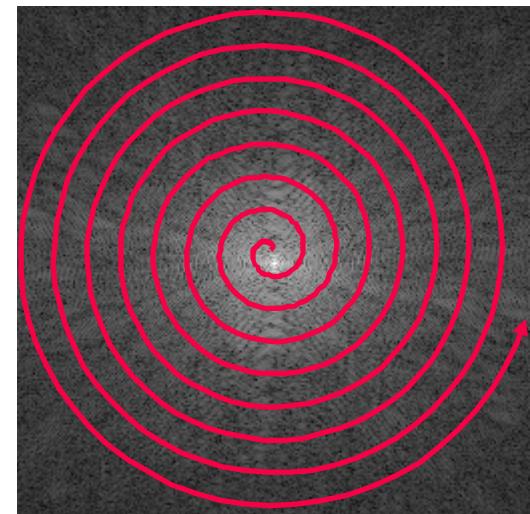
Cartesian



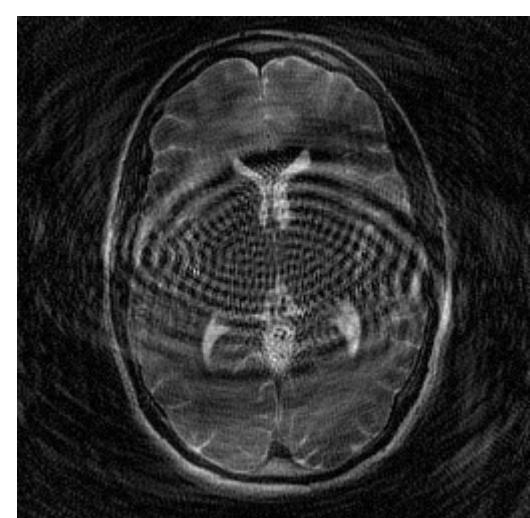
Radial



Spiral



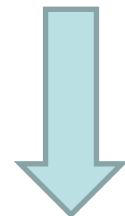
32



# Non-Cartesian SENSE

- Decoupling is lost  
解耦
- Each pixel is aliased with all other pixels
  - e.g. streaks in undersampled radial imaging
- Need to invert the full encoding equation
- Calls for an iterative algorithm
  - No explicit matrix inverse
  - Matrix-vector multiplications only

Cartesian



Radial



# Summary

- Speed of gradient encoding is limited
  - Physical and physiological constraints on the gradient amplitude and switching rate
- Fast, rapid or accelerated MRI
  - k-space undersampling  
用undersampling(multi-coild来加速)
  - Reconstruction is more challenging, but more fun
  - Exploit redundancies in the acquired data

# Summary

- Parallel imaging
  - Exploit additional encoding provided by multiple receiver coils with different sensitivities
  - SNR penalty
  - SENSE (image-domain)
    - Unfolding images using coil sensitivities
    - Matrix inversion
  - SMASH, GRAPPA (k-space)
    - Next lecture

# Outlook lab exercise

## Computational MR imaging Laboratory 5: Image space parallel imaging

Report is due on Wednesday the week after the lab session at 23:59. Send your report by email to Bruno Riemenschneider (bruno.riemenschneider@fau.de) and Florian Knoll (florian.knoll@fau.de).

### Learning objectives

- Combine multicoil images
- Reconstruct undersampled multicoil data using SENSE algorithm
- Compute g-factor and SNR

**Before the lab:** Get familiar with the functions `inv` (matrix inverse) and `pinv` (matrix pseudo-inverse), and the operators `'` (conjugate transpose) and `*` (matrix multiplication).

1. **Multicoil combination:** Load the file `data_brain_8coils.mat`. The variable `d` is the fully-sampled k-space data ( $256 \times 256 \times 8$ ), the dimensions of the data are `[PE,FE,channels]`, `c` is the coil sensitivity maps ( $256 \times 256 \times 8$ ) and `n` is the noise-only scan ( $256 \times 8$ ). Combine the multicoil images using sum-of-squares and matched-filter (least-squares) algorithms. You might want to create a function for each combination, so that you can use it again. Comment of the effect of using the noise correlation matrix.

```
function [mc] = sos_comb(m,Psi)
% Input:
% m: multicoil images [nPE,nFE,nCh]
% Psi: noise correlation matrix [nCh, nCh]
% Output:
% mc: combined image [nPE,nFE,nCh]
```

```
function [mc] = ls_comb(m,c,Psi)
% Input:
% m: multicoil images [nPE,nFE,nCh]
% c: coil sensitivity maps [nPE,nFE,nCh]
% Psi: noise correlation matrix [nCh, nCh]
% Output:
% mc: combined image [nPE,nFE,nCh]
```

2. **Cartesian SENSE reconstruction and g-factor:** Write a function that reconstructs regularly undersampled data along the phase-encoding dimension using the SENSE method and computes the corresponding g-factor. The function will unfold multicoil aliased images using coil sensitivity maps in the image domain.

```
function [ir,g] = sense1d(ia,c,Psi,R)
% Input:
% ia: multicoil aliased images [Nx, Ny/R,Nc]
```

```
% c: coil sensitivity maps [Nx,Ny,Nc]
% Psi: noise correlation matrix [Nc,Nc]
% R: acceleration factor
% Output:
% ir: unaliased image [Nx,Ny]
% g: g-factor map [Nx,Ny]
```

Simulate acceleration factors of  $R= 2, 3, 4$  along the phase-encoding dimension for the 8-coil data set from exercise 1. Reconstruct each undersampled data set using your SENSE implementation; compute the average g-factor and SNR loss (make sure to exclude the pixels outside the brain). Compute the RMSE with respect to the matched-filter combination of the fully-sampled data in exercise 1. Plot the reconstructed image, reconstruction error and g-factor map for each  $R$ .