Computational MRI

Fourier Image Reconstruction Basics

Based on a lecture by Ricardo Otazo

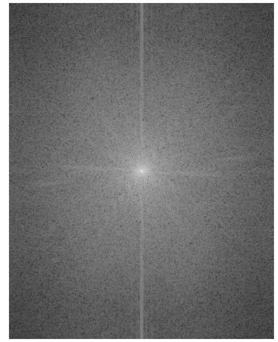
FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG

NYU

Imaging Process

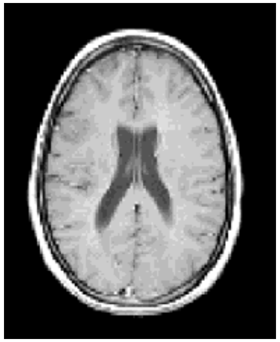
Encoding

k-space



Reconstruction

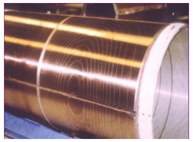
Image

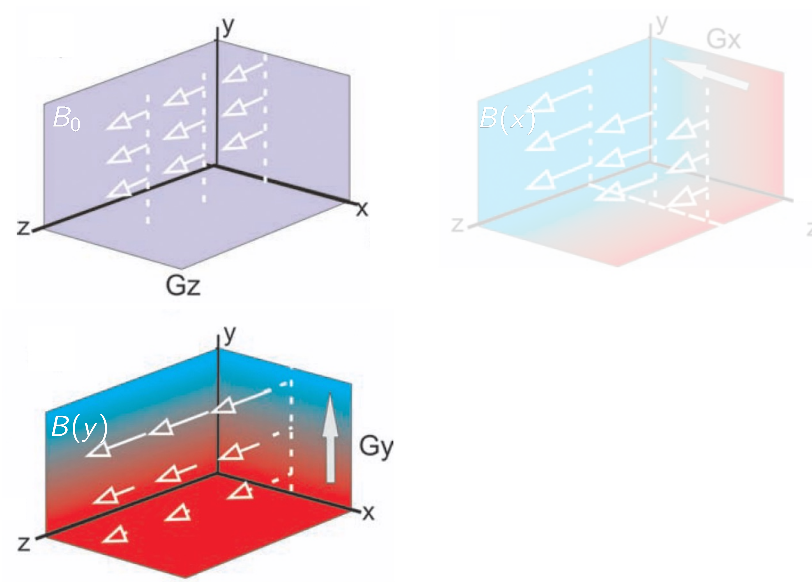


- Pulse sequence
- Spatial information about object is transformed into measured data
- Forward problem

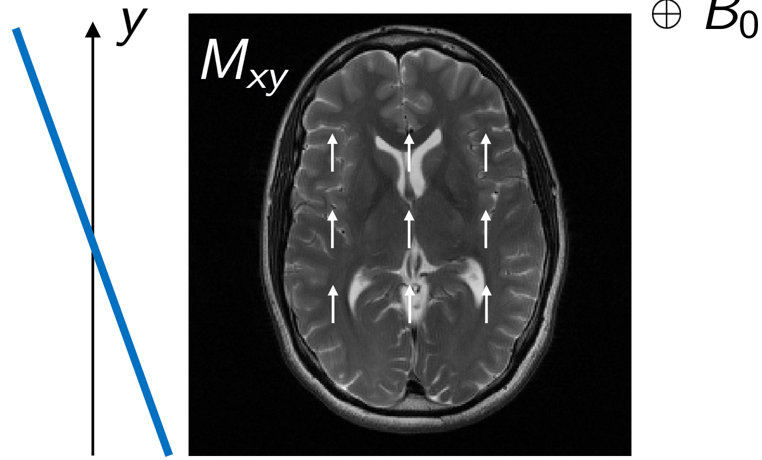
- Computer algorithm
- Measured data is transformed into spatial information about object
- Inverse problem

Interaction with gradient fields G





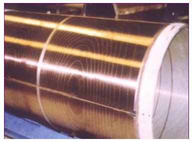
$$\omega(y) = \omega_0 + \gamma G_y y$$

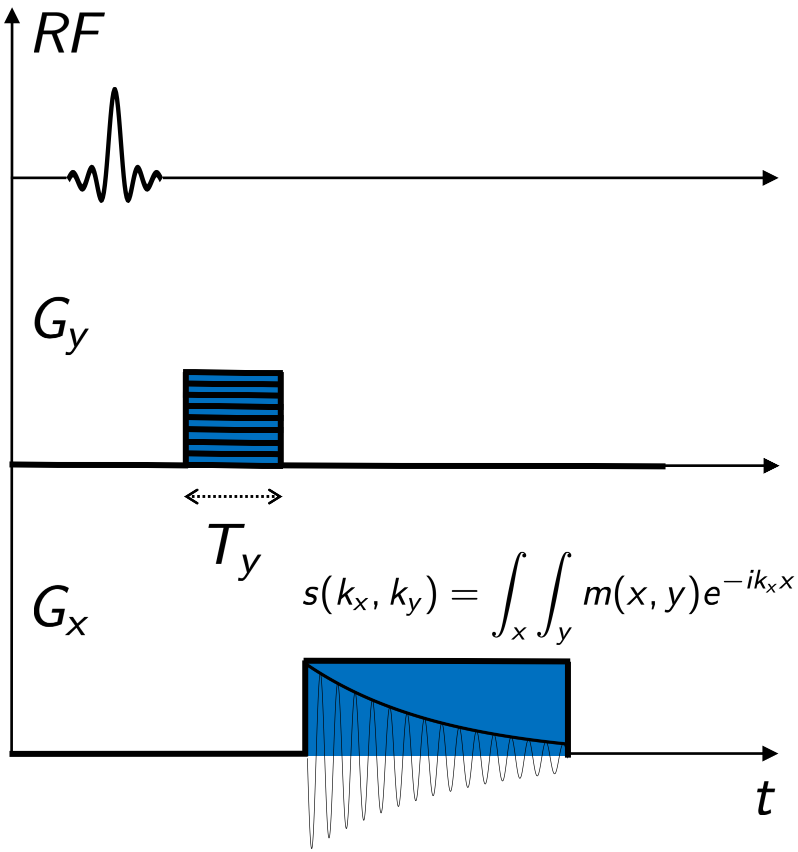


Plewes 2012

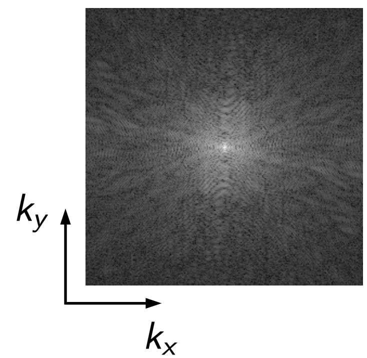
$$B(y) = B_0 + G_y y$$

Interaction with gradient fields G

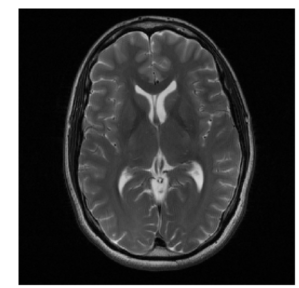




$$s(k_x, k_y) = \int_x \int_y m(x,y) e^{-ik_x x} e^{-ik_y y} \, dx dy$$



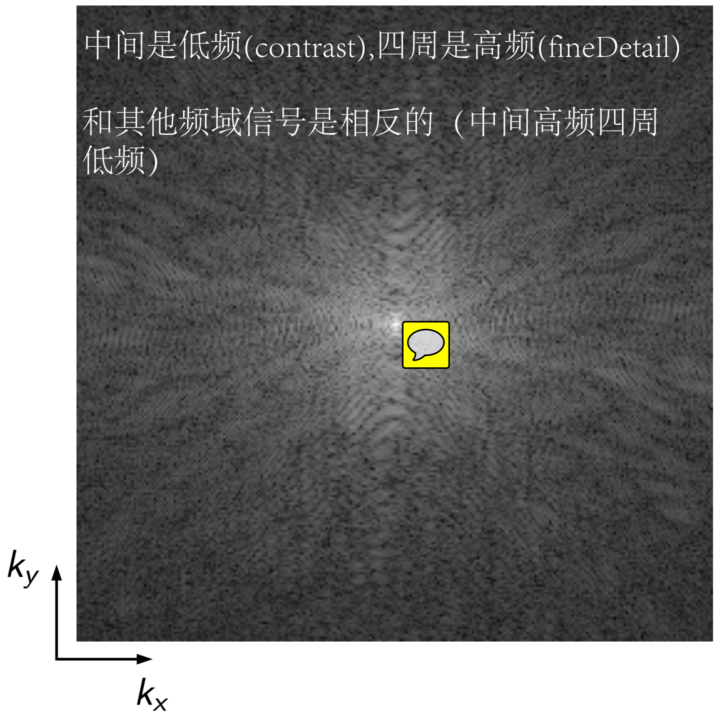
$\xrightarrow{\mathcal{FT}^{-1}}$



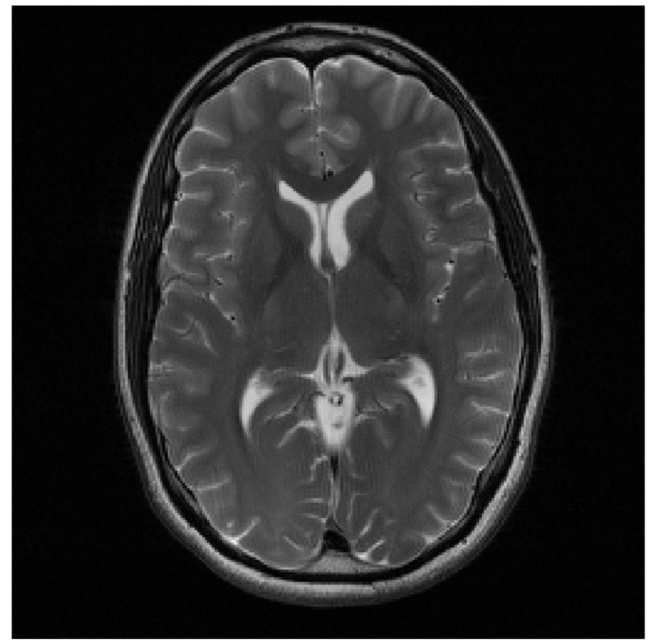
$$\phi(x,t) = \gamma x \int_0^t G_x d\tau \equiv k_x x$$

$$\phi(y, G_y) = \gamma y \int_0^{T_y} G_y d\tau \equiv k_y y$$

K-space

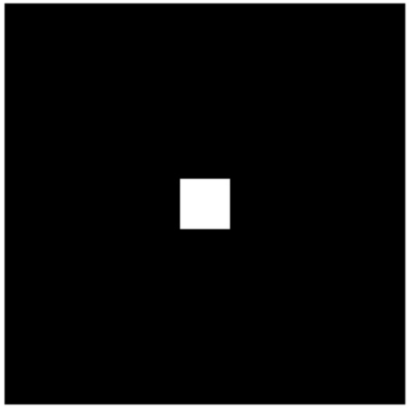


$$\xrightarrow{\mathcal{F}\mathcal{T}^{-1}}$$

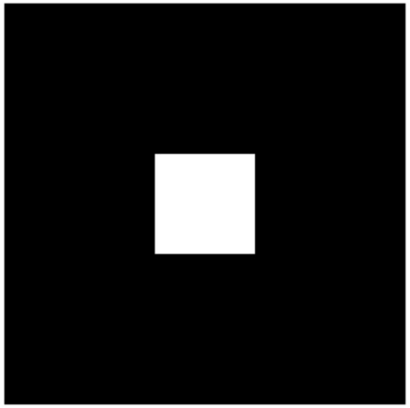


k-space and spatial frequencies

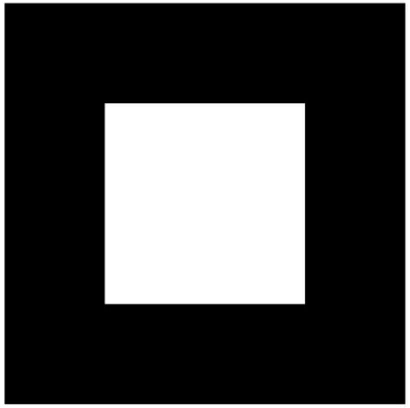
64x64



128x128



256x256



k-space and spatial frequencies

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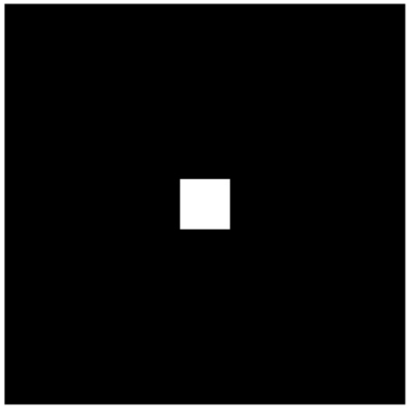
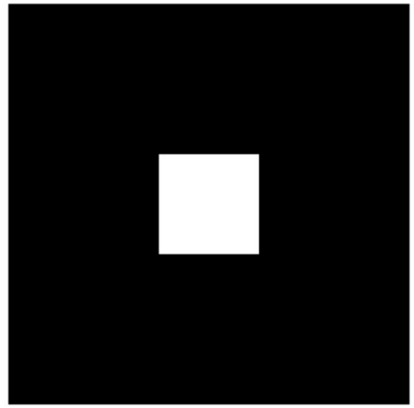
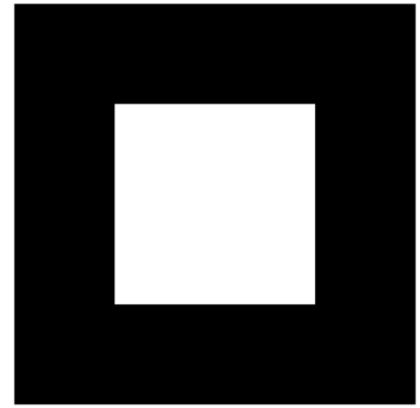
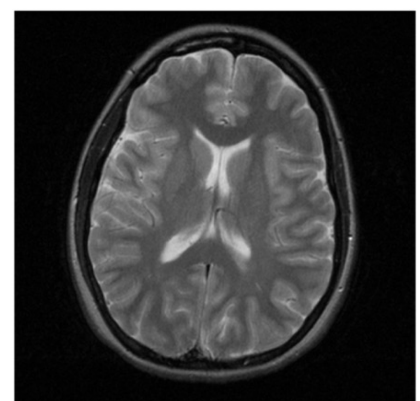


128x128



256x256





k-space and spatial frequencies

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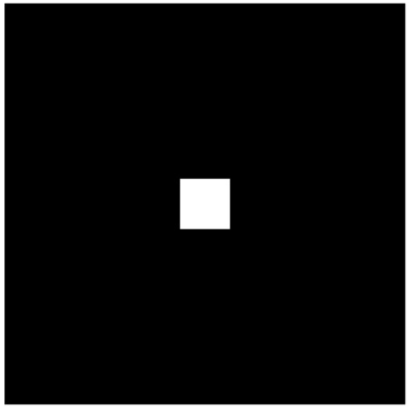
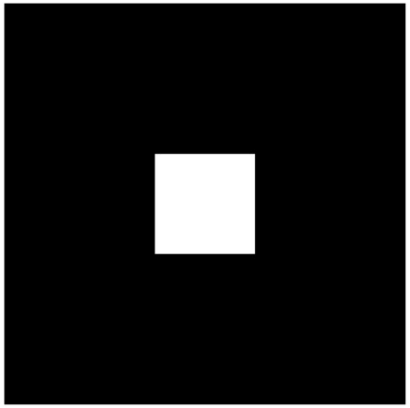
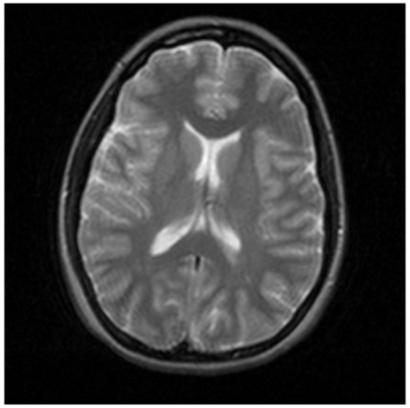
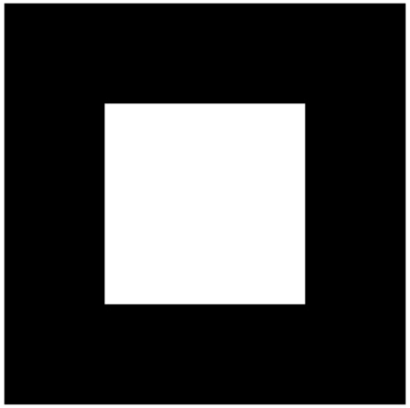


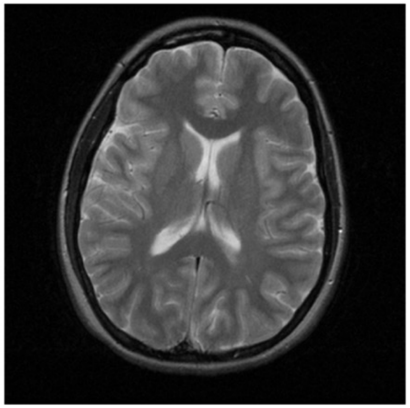
128x128





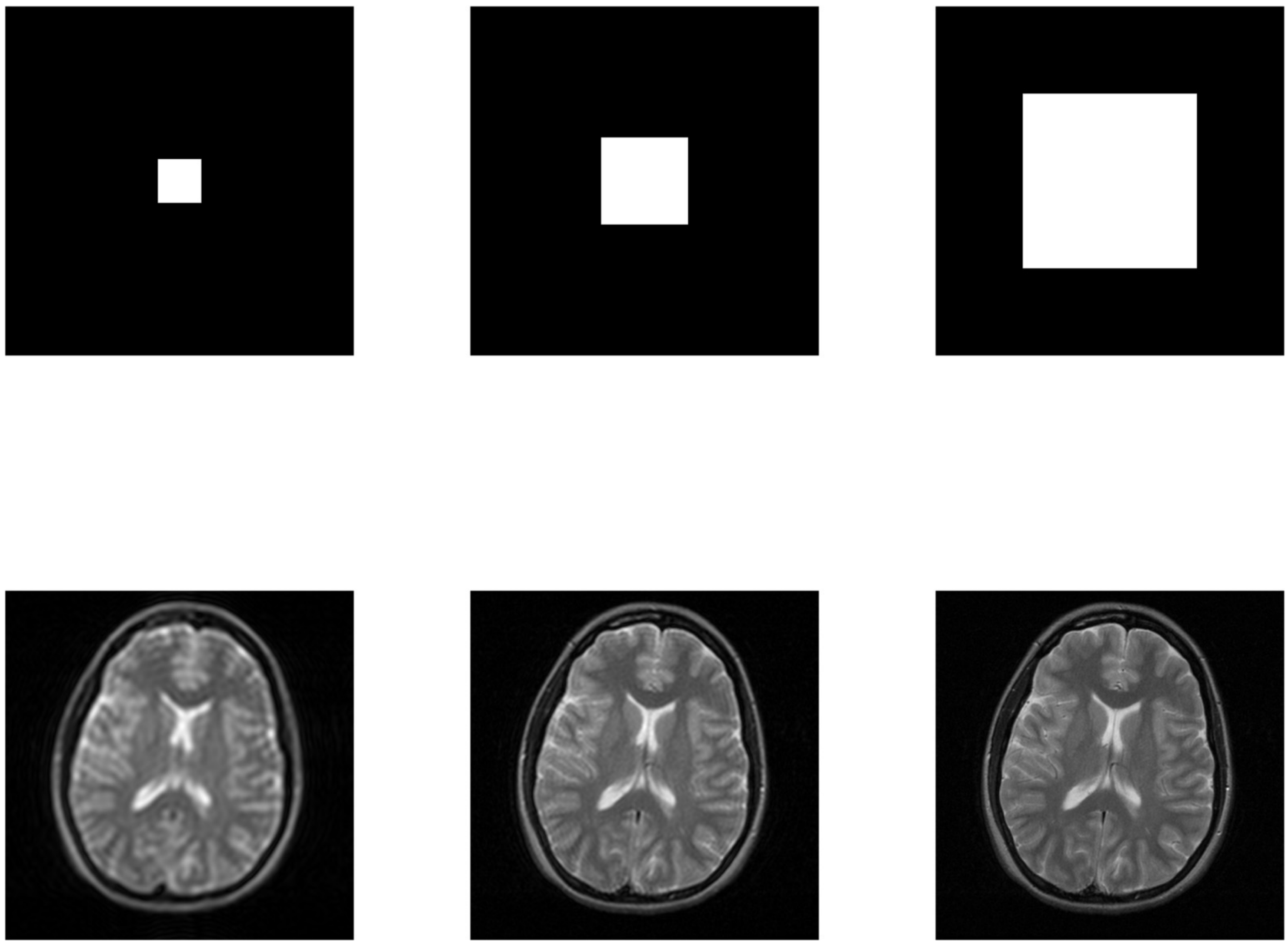
256x256





k-space and spatial frequencies

64x64 128x128 256x256



k-space and spatial frequencies

64x64

128x128

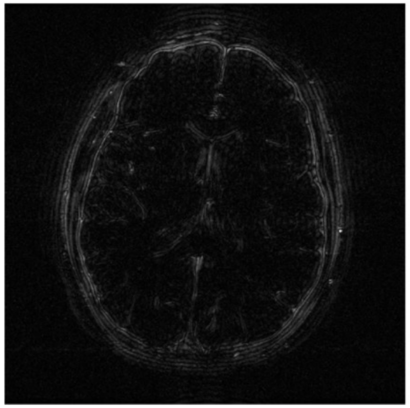
256x256

k-space and spatial frequencies

64x64

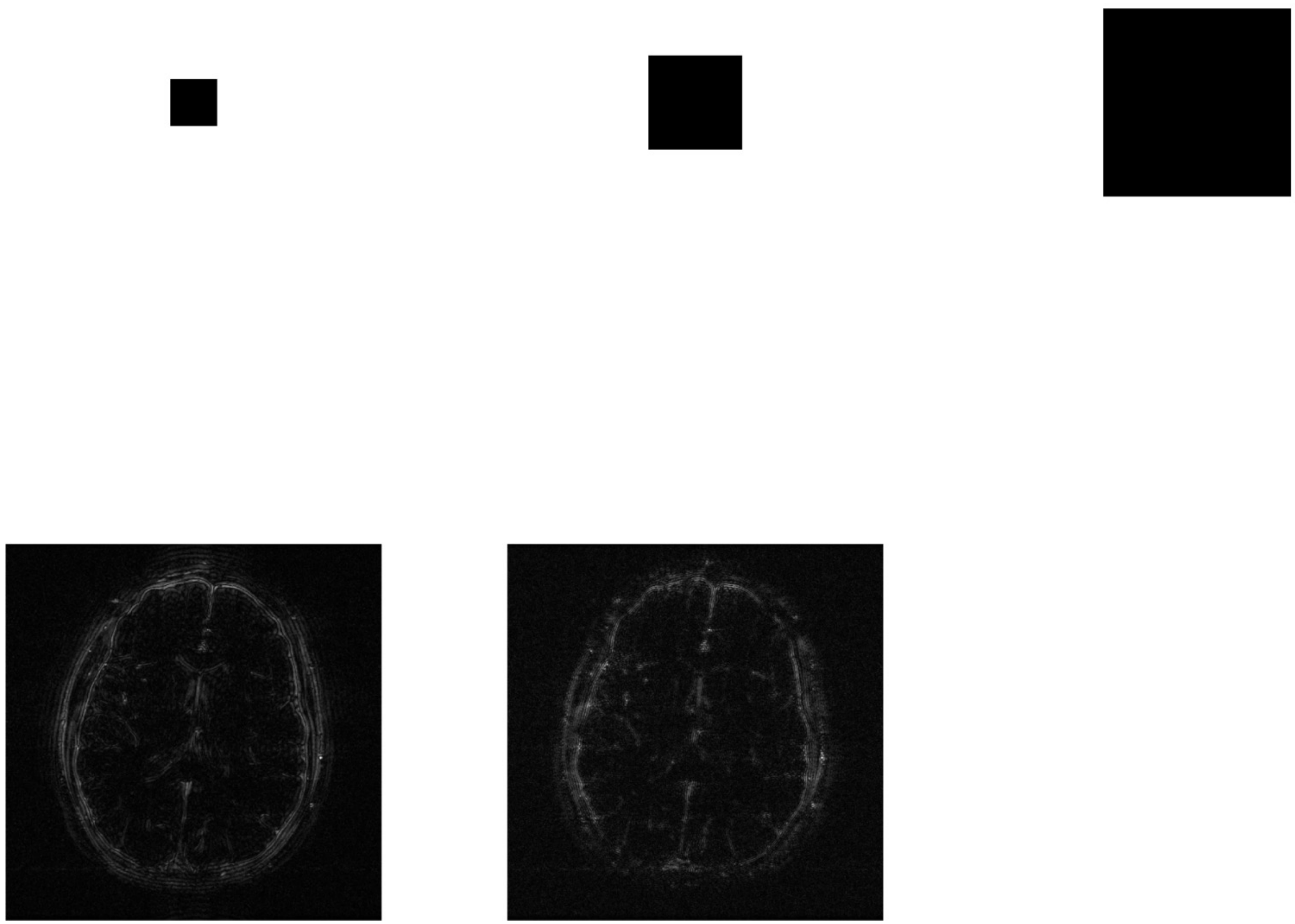
128x128

256x256



k-space and spatial frequencies

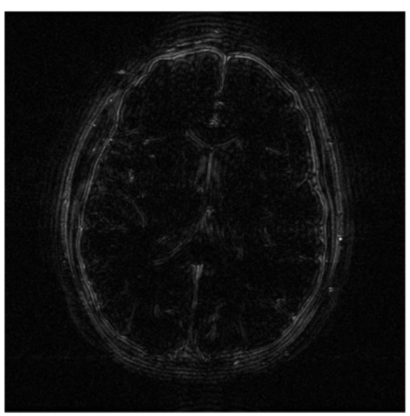
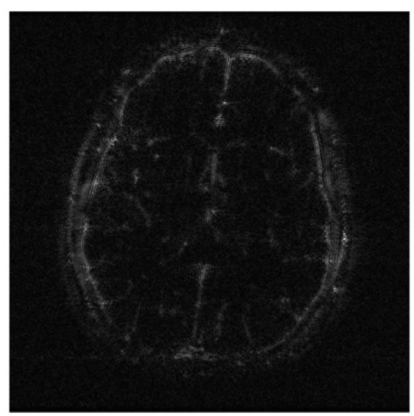
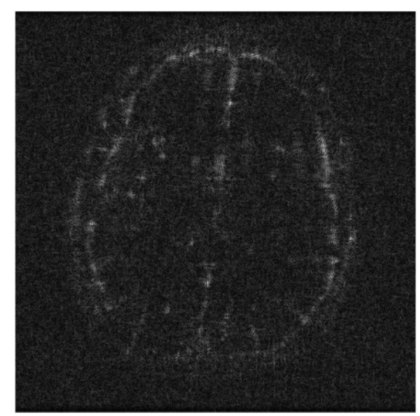
64x64 128x128 256x256



k-space and spatial frequencies

64x64	128x128	256x256
		

保留四周高频信息（edge），
丢失中间低频信息
所以重建的图片还有edge

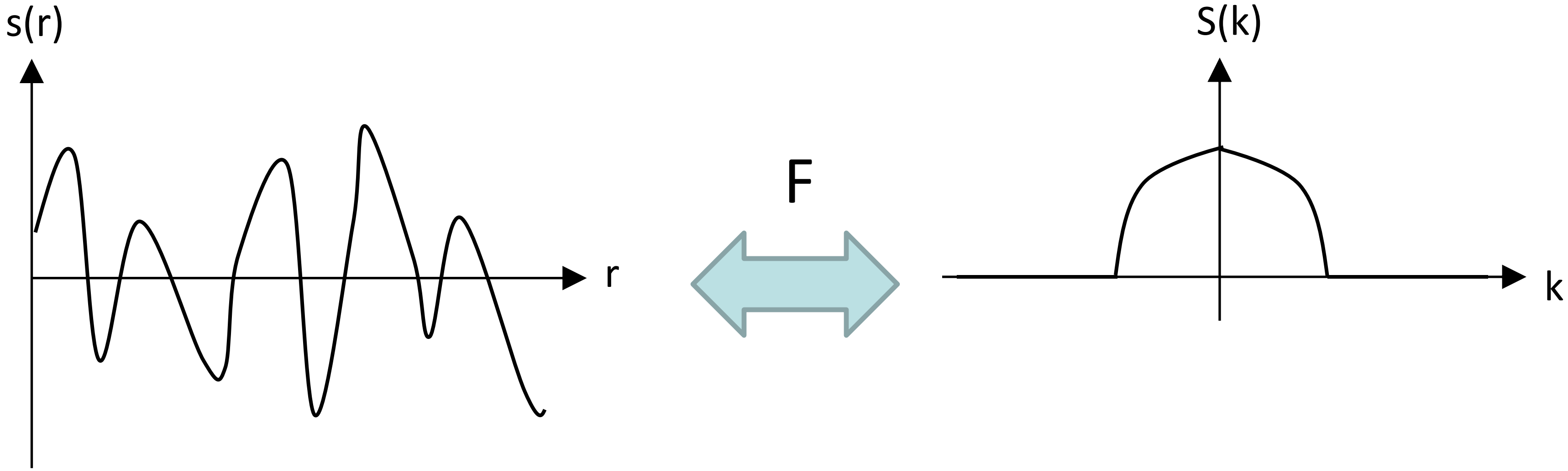




Fourier transform

kspace image space

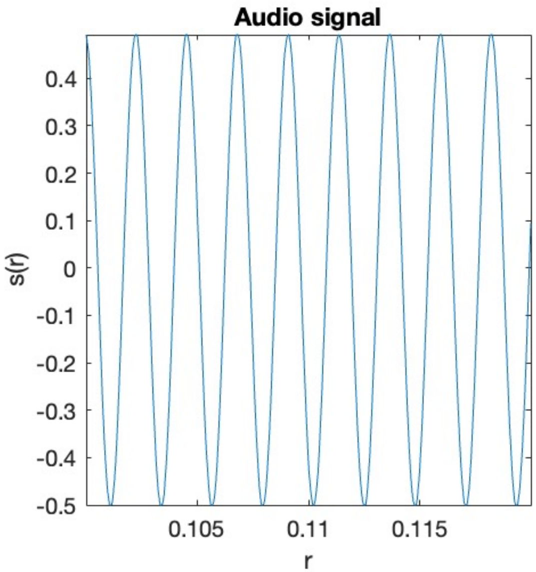
$$S(k) = \int_{-\infty}^{\infty} s(r)\, e^{-i2\pi kr}\, dr \qquad \text{(forward)}$$

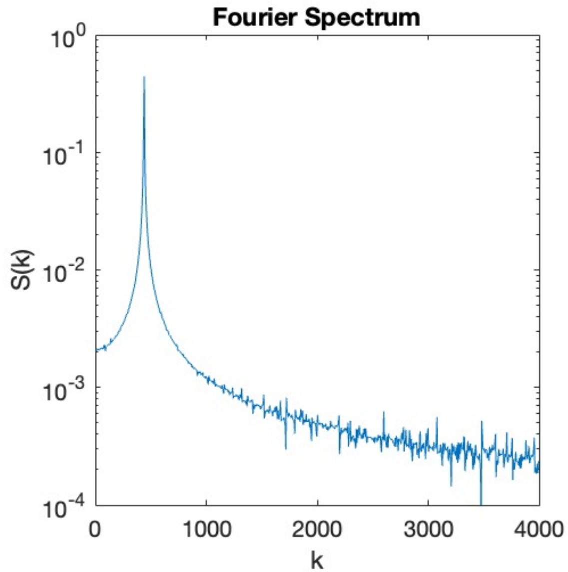
$$s(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(k)\, e^{i2\pi kr}\, dk \qquad \text{(inverse)}$$

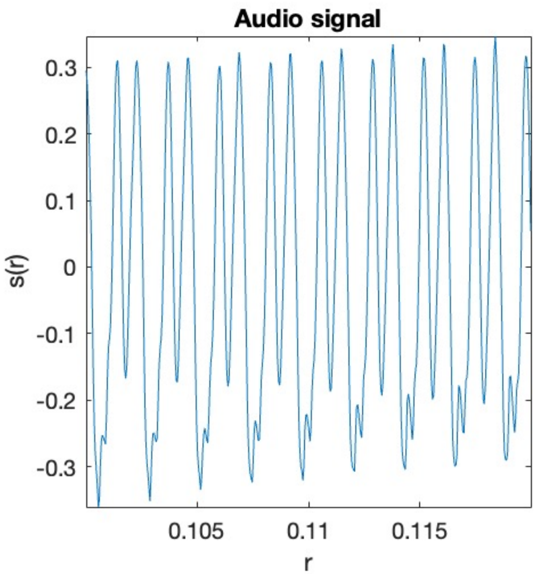


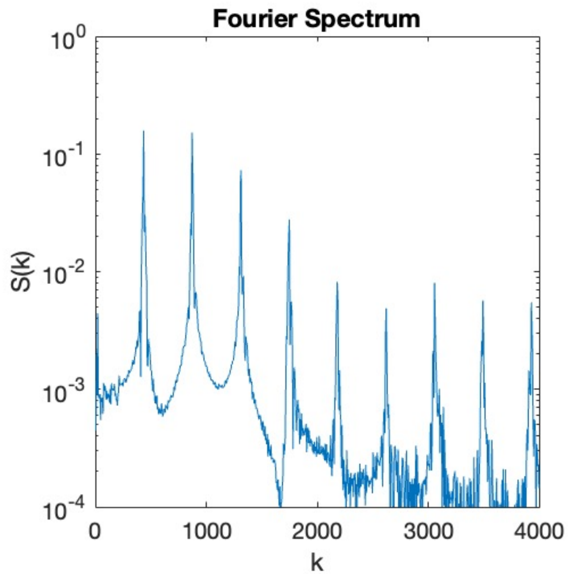
1D example

A4 (440Hz)









Fourier transform properties

- Linearity: $F\{as_1(r)+bs_2(r)\}=aS_1(k)+bS_2(k)$
- Shifting: $F\{s(r-r_0)\}= e^{-i2\pi k r_0}S(k)$ ←可用于motion correction
- Modulation: $F\{e^{i2\pi k_0 r}s(r)\}=S(k-k_0)$
- Conjugate symmetry: $s(r) real \Rightarrow S(-k)=S^*(k)$
- Scaling: $F\{s(ar)\}=\frac{1}{|a|}S\left(\frac{k}{a}\right)$

Fourier transform properties

- Parseval's formula: $\int s_1(r)s_2(r)\,dr = \int S_1(k)S_2(k)\,dk$

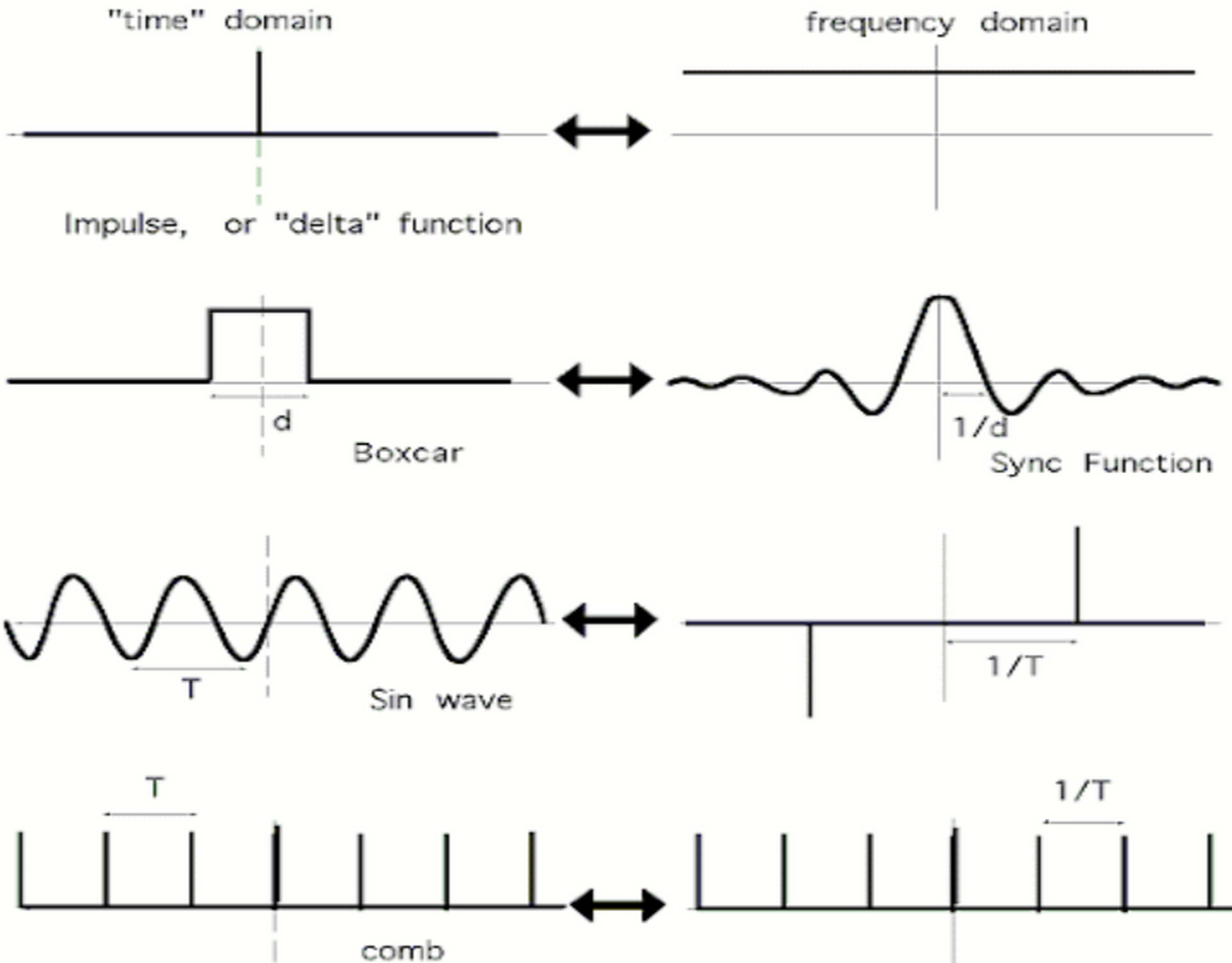
$$s_1 = s_2 = s \quad \Rightarrow \quad \int |s(r)|^2\,dr = \int |S(k)|^2\,dk$$

- Convolution & multiplication

$$F\{s_1(r) * s_2(r)\} = S_1(k)S_2(k)$$

$$F\{s_1(r)s_2(r)\} = S_1(k) * S_2(k)$$

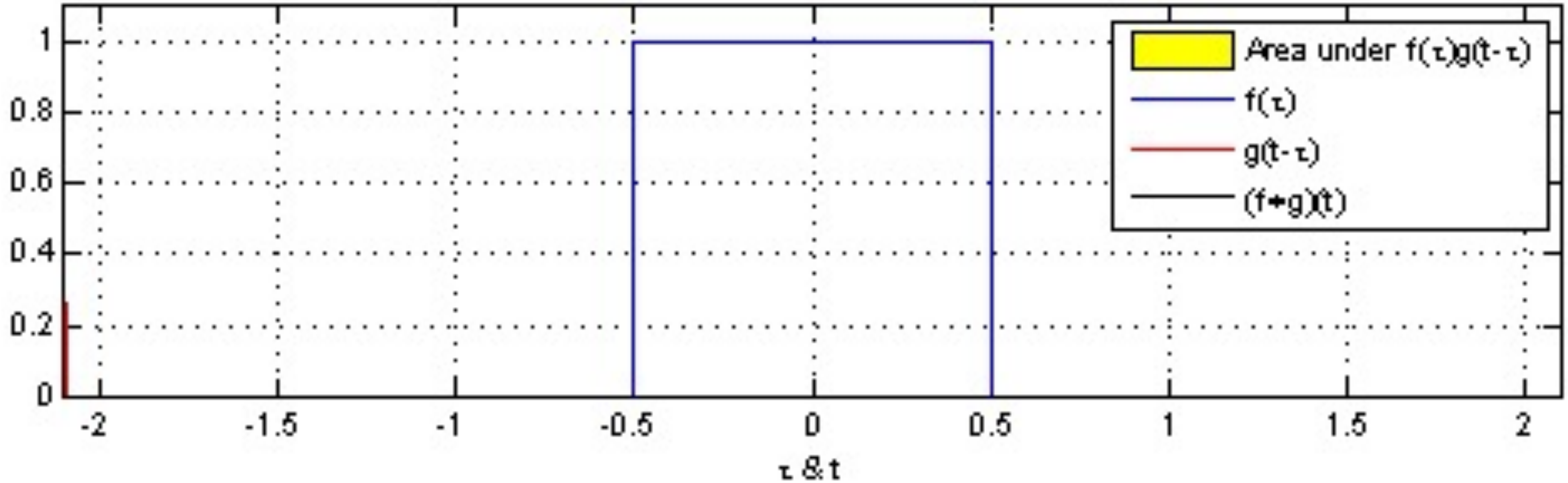
Fourier transform of basic functions



samping theory

What is the image space representation of sinc^2?

What is the image space representation of sinc^2?



Wikipedia

Multidimensional Fourier transform

$$S(\mathbf{k}) = \int_{-\infty}^{\infty} s(\mathbf{r})\, e^{-i2\pi\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \qquad \text{(forward)}$$

$$s(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\mathbf{k})\, e^{i2\pi\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \qquad \text{(inverse)}$$

2D $\Rightarrow$ $\mathbf{r} = (x, y)$, $\mathbf{k} = (k_x, k_y)$ $\Rightarrow$

$$S(k_x . k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x, y)\, e^{-i2\pi(k_x x + k_y y)} dxdy$$

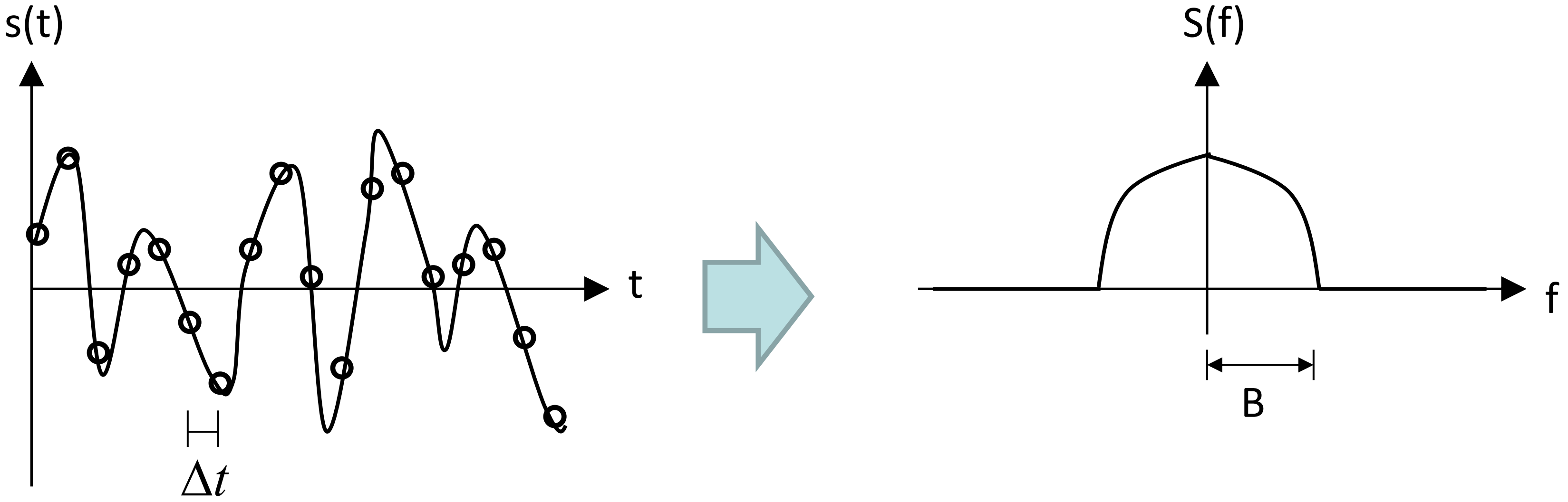
$$s(x, y) = \frac{4}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(k_x, k_y)\, e^{i2\pi(k_x x + k_y y)} dk_x dk_y$$

The multidimensional Fourier transform is separable

Sampling of continuous signals

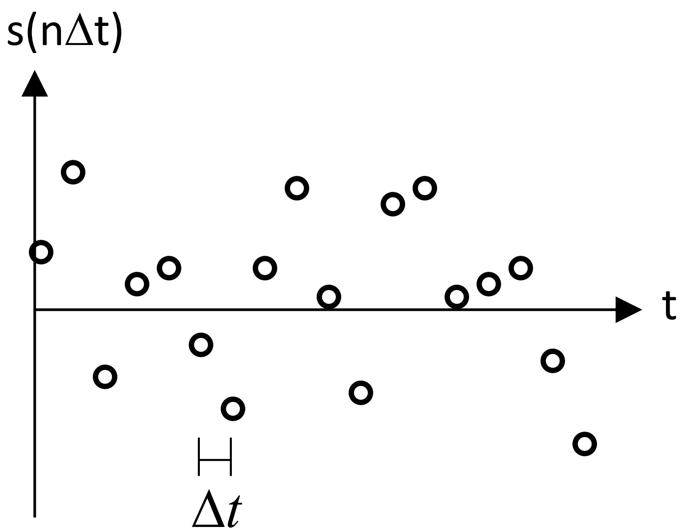
- Nyquist/Shannon theorem
  - A signal with bandwidth B can be reconstructed from its samples if they are taken regularly with a period no larger than 1/2B

$$\text{Nyquist rate}: \Delta t = \frac{1}{2B}$$

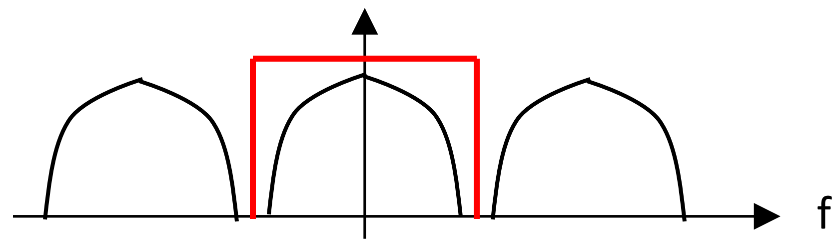


Sampling of continuous signals

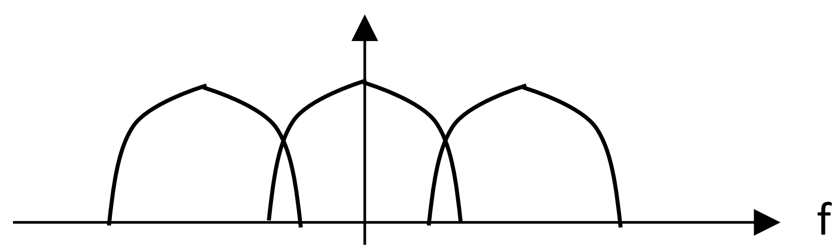
- Nyquist/Shannon theorem



$$\Delta t \le \frac{1}{2B} \quad \Rightarrow \text{ no aliasing}$$

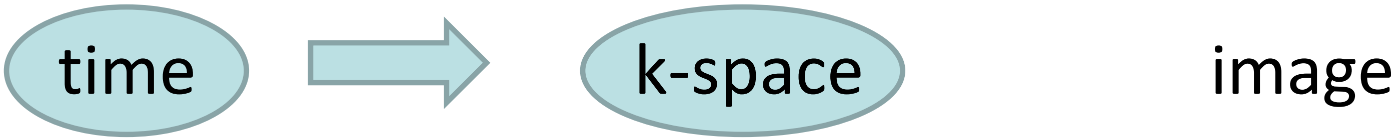


$$\Delta t > \frac{1}{2B} \quad \Rightarrow \text{ aliasing}$$



Sampling of MRI signals

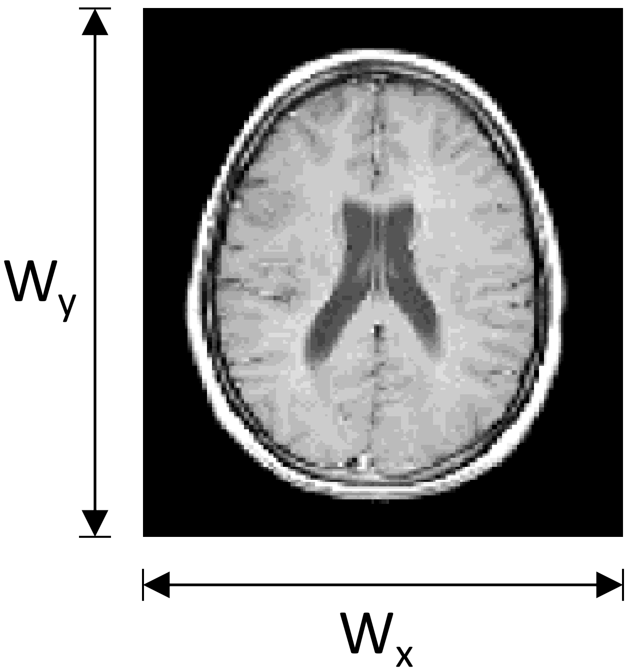
- Where do we sample?



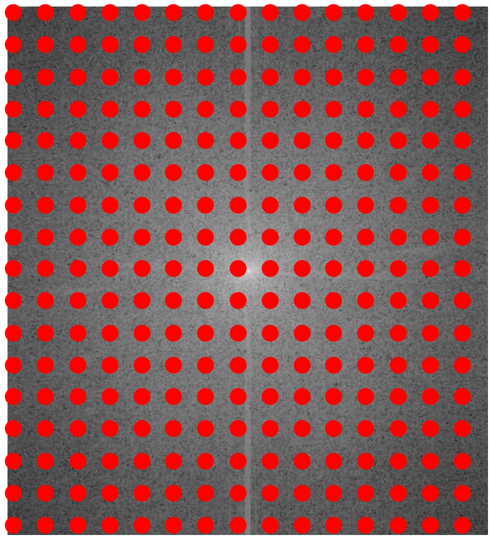
- How do we apply the sampling theorem?
  - bandwidth: image
  - sampling rate: k-space

Cartesian sampling of k-space

Image



k-space

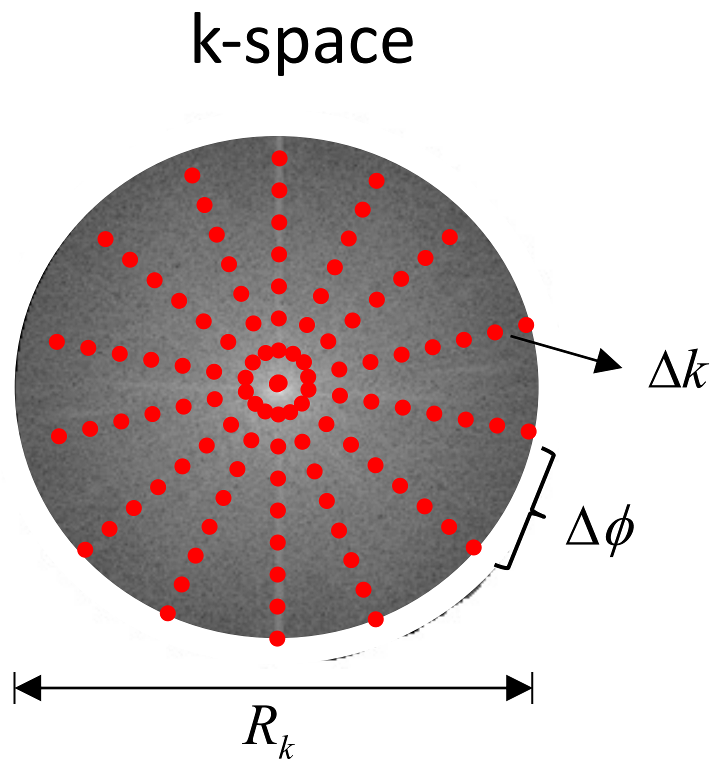


Nyquist rate:

$$\Delta k_x = \frac{1}{W_x}; \quad \Delta k_y = \frac{1}{W_y}$$

comb
function

Radial sampling of k-space



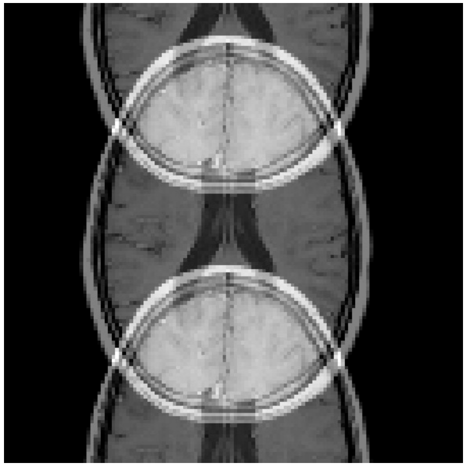
Nyquist rate: $N_{radial} = \frac{\pi}{2} N_{Cartesian}$

Aliasing examples

Cartesian

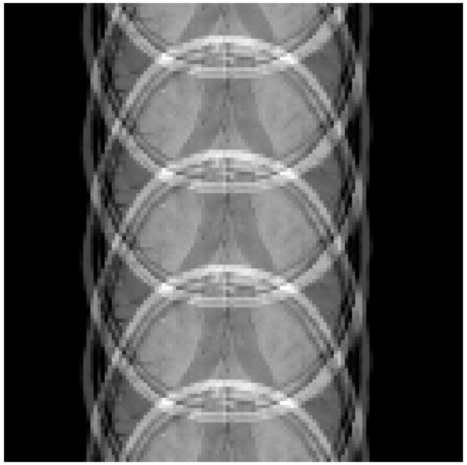
Radial

$$\Delta k_y = \frac{2}{W_y}$$

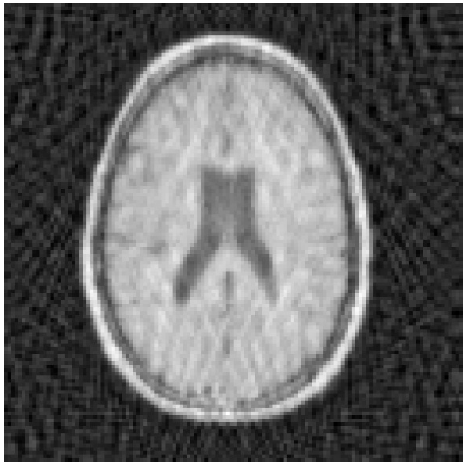


$$\Delta\phi = 2\Delta\phi_{Nyquist}$$

$$\Delta k_y = \frac{4}{W_y}$$



$$\Delta\phi = 4\Delta\phi_{Nyquist}$$



Discrete Fourier transform (DFT)

- Discrete signals (sequence of numbers)
- Fast implementation: FFT

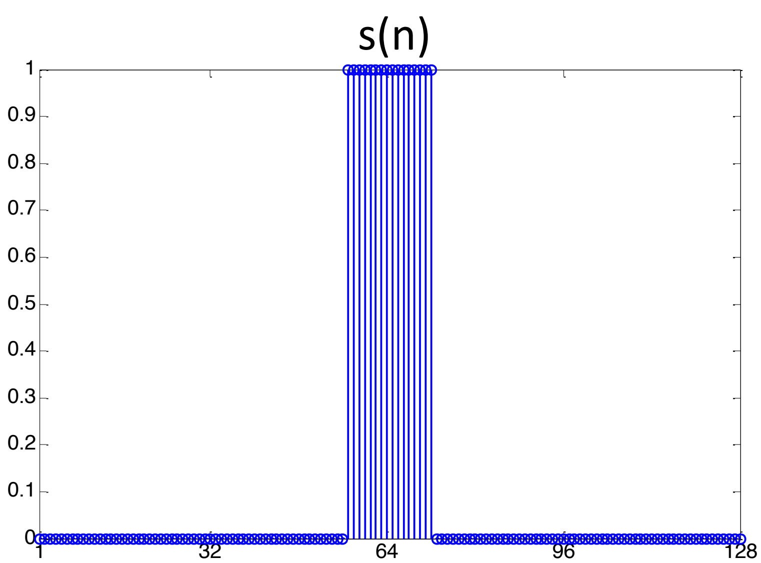
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-i\frac{2\pi}{N}nk} \qquad \text{(forward)}$$

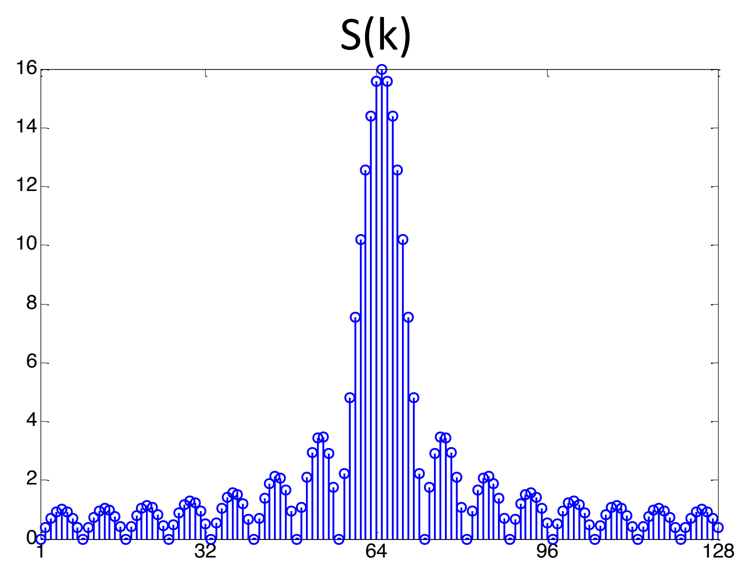
$$s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k) e^{i\frac{2\pi}{N}nk} \qquad \text{(inverse)}$$

Matlab/Python

```
S=fft(s)

s=ifft(S)
```



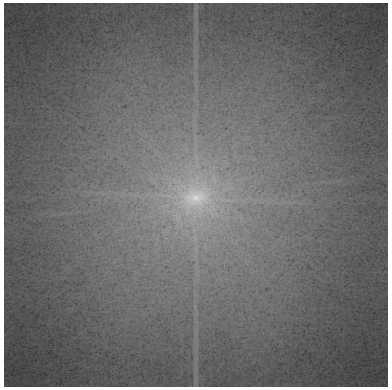


DFT reconstruction of Cartesian k-space data

- S(k) is known at $k = n\Delta k$ $\left(-\frac{N}{2} \le n \le \frac{N}{2}\right)$

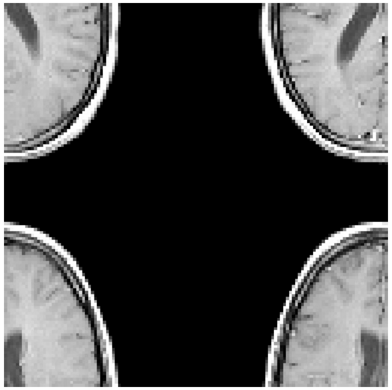
编程里原点在左下
MR里原点在中心

NxN k-space data



$-\frac{N}{2}$ $\frac{N}{2}$

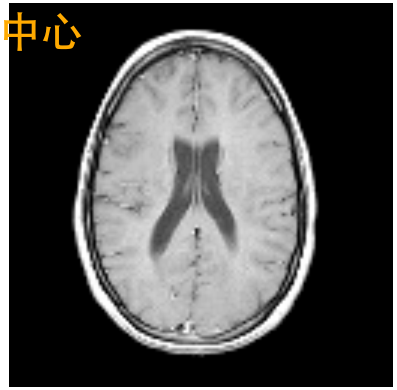
`s=ifft2(S)`



Solution: r and k from 0 to N-1

```
s=sqrt(length(S)*fftshift(ifft2(if
ftshift(S))
```

先移动原点到图中心



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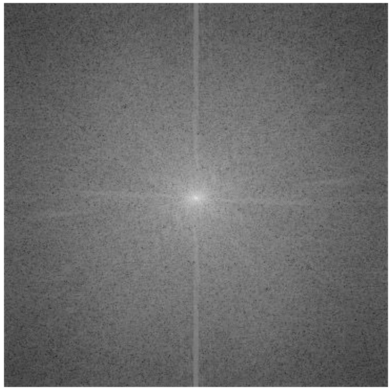
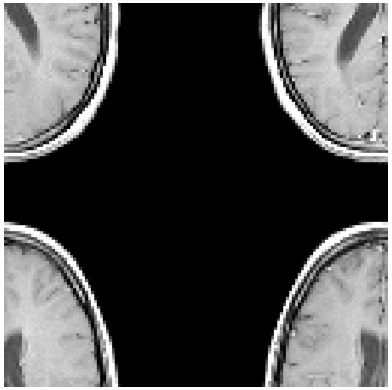
NxN k-space data



$-\frac{N}{2}$ $\frac{N}{2}$

`S=fft2(s)`



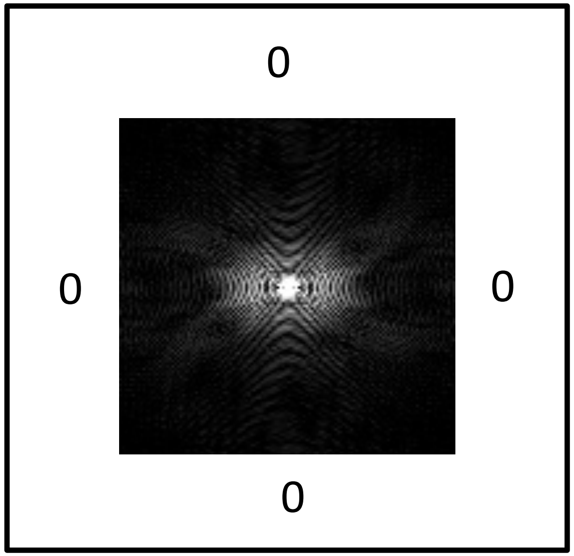
Solution: r and k from 0 to N-1

```
S=1/sqrt(length(s)*fftshift(fft2(
ifftshift(s))
```

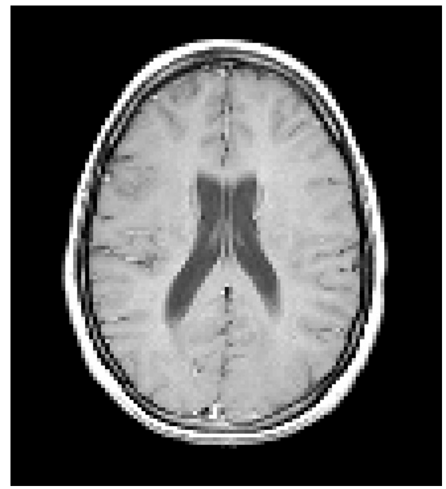
https://groups.google.com/g/comp.soft-sys.matlab/c/rUcc0bRRZf4/m/gxPvAgn8BgAJ

DFT reconstruction of Cartesian k-space data

- Zero-padding in k-space (Fourier interpolation)
  - Decreases the pixel size but does not increase resolution



Original 128x128



Zero-padded 256x256

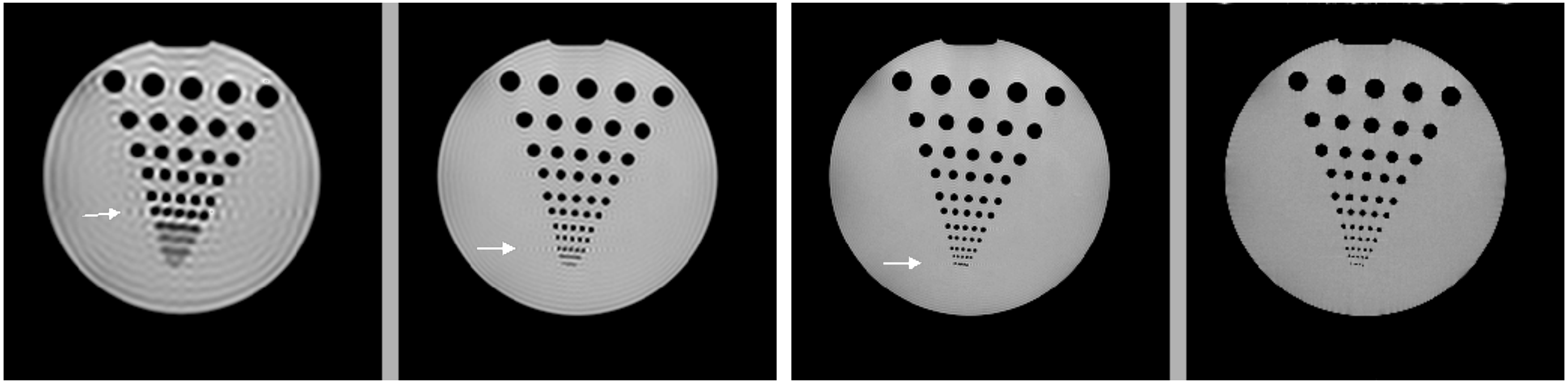
频域×rectangular
等于时域卷积sinc，会有

$$p_x = \frac{W_x}{N_{x,padded}};\ p_y = \frac{W_y}{N_{y,padded}}$$

DFT reconstruction of Cartesian k-space data

- Gibbs ringing
  - Spurious ringing around sharp edges
  - Caused by k-space truncation
  - Gets stronger for decreasing N)

64x64 128x128 256x256 512x512



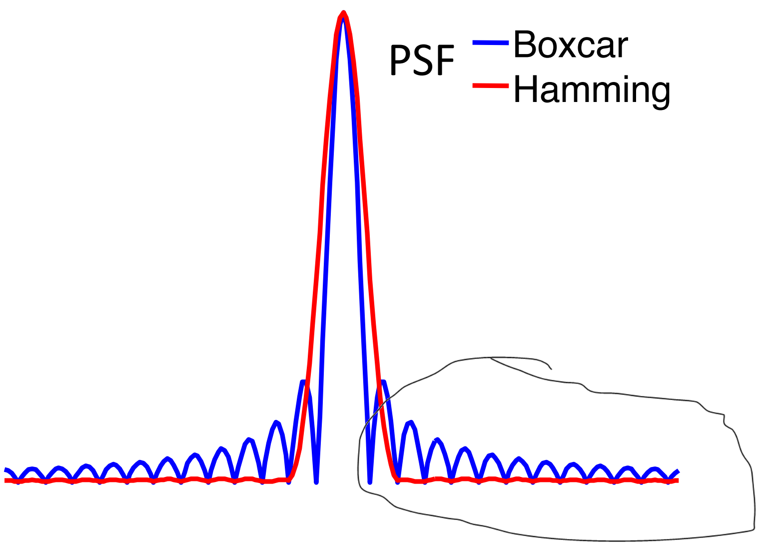
DFT reconstruction of Cartesian k-space data

- k-space filtering or windowing
  - Reduce Gibbs ringing at the expense of resolution loss

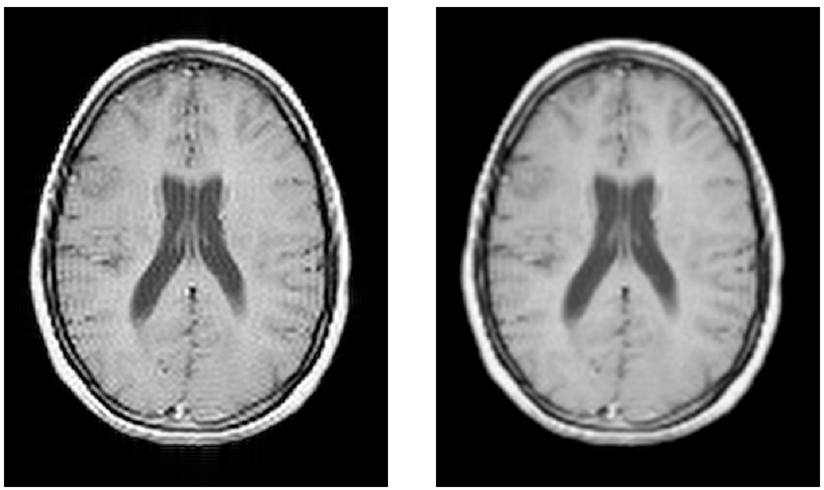
$$S_W(k) = S(k)W(k)$$

- Hamming filter

$$W(k) = 0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$$



Unfiltered Hamming filter

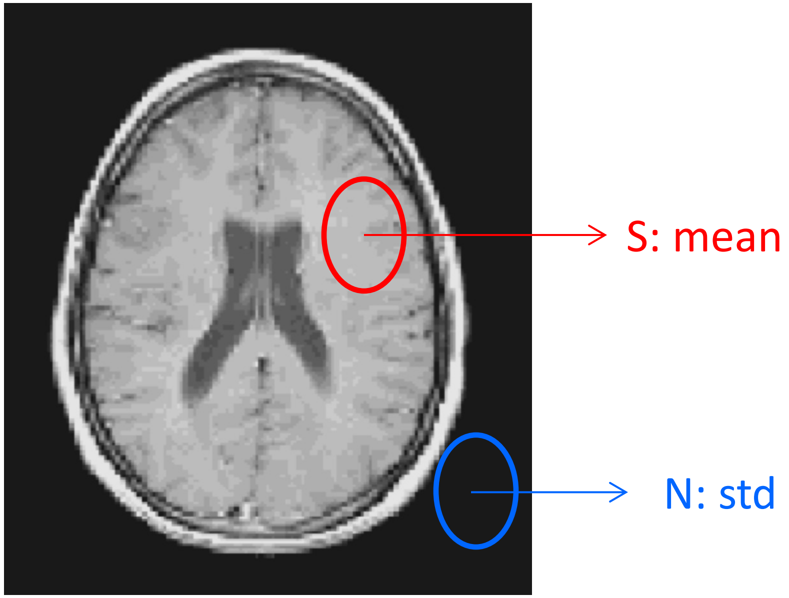


hamming 过滤周边信号，能去除伪影
但是也丢失图片信息

DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR)

$$SNR = \frac{S}{N} = \frac{\text{Pixel signal amplitude}}{\text{Standard deviation of background}}$$



Not entirely correct for many real world measurements!

DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR) (simplified)

$$SNR \propto V\sqrt{T}$$ (Al Macovski)

V: voxel volume

T: cumulated readout duration

have more voxel or scanning longer
will have larger SNR
but V变大的话resolution变小