

Computational MRI

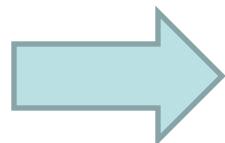
Fourier Image Reconstruction Basics

Based on a lecture by Ricardo Otazo

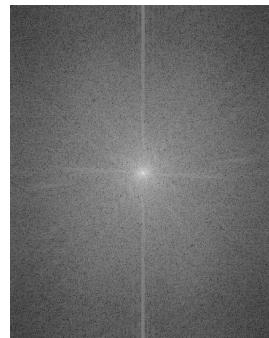
Imaging Process



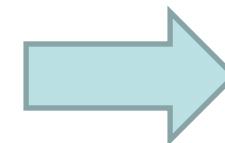
Encoding



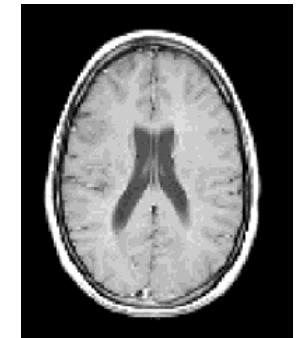
k-space



Reconstruction



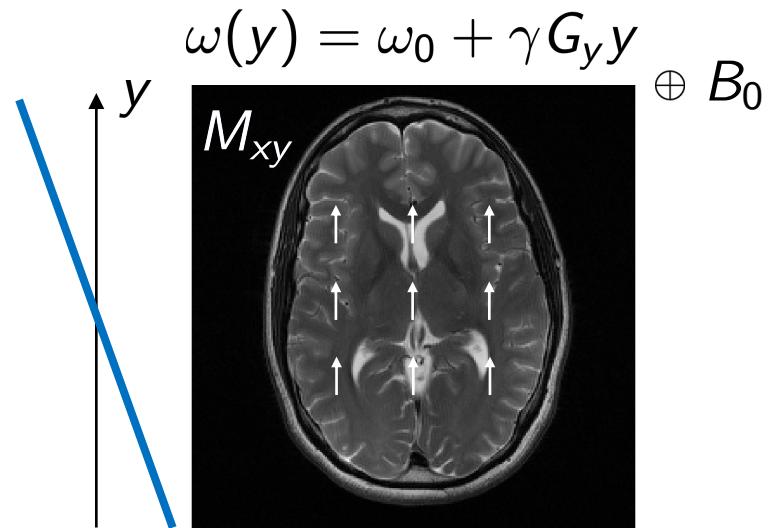
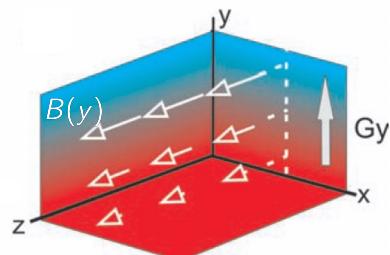
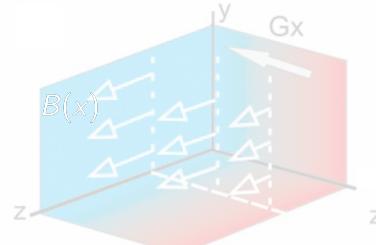
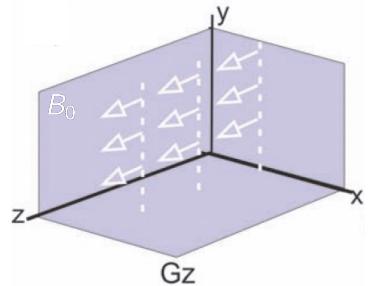
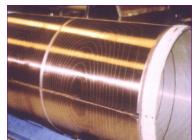
Image



- Pulse sequence
- Spatial information about object is transformed into measured data
- Forward problem

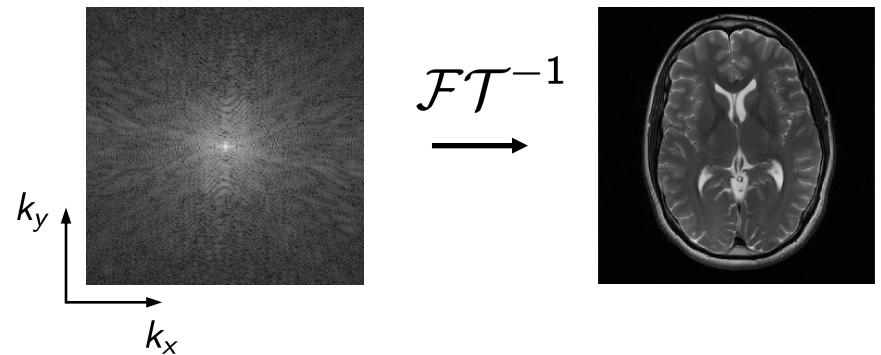
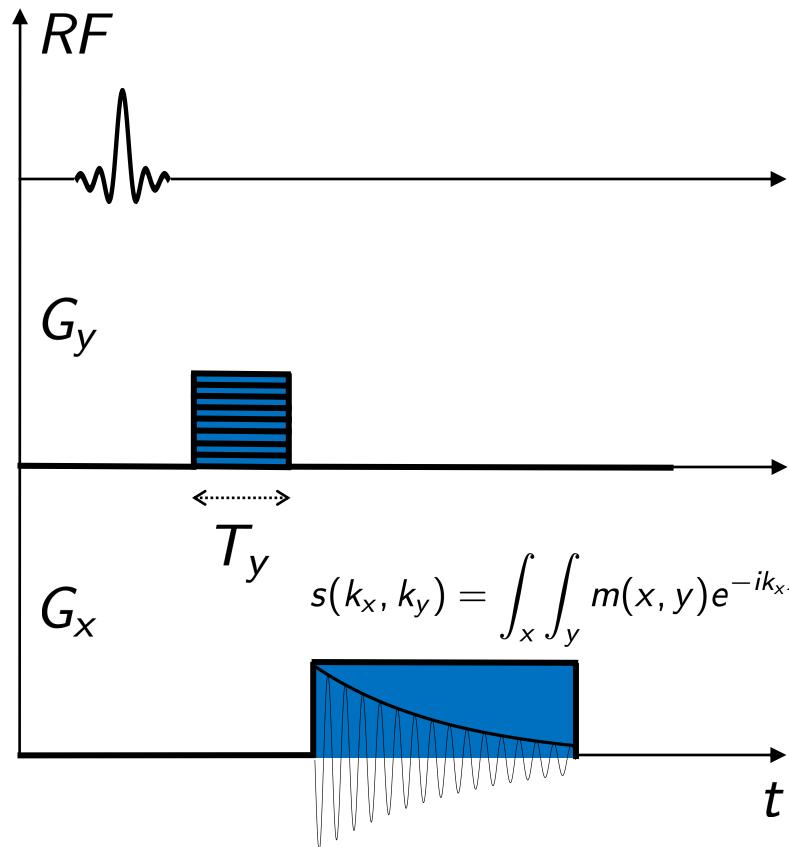
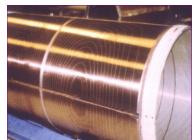
- Computer algorithm
- Measured data is transformed into spatial information about object
- Inverse problem

Interaction with gradient fields G



$$B(y) = B_0 + G_y y$$

Interaction with gradient fields G

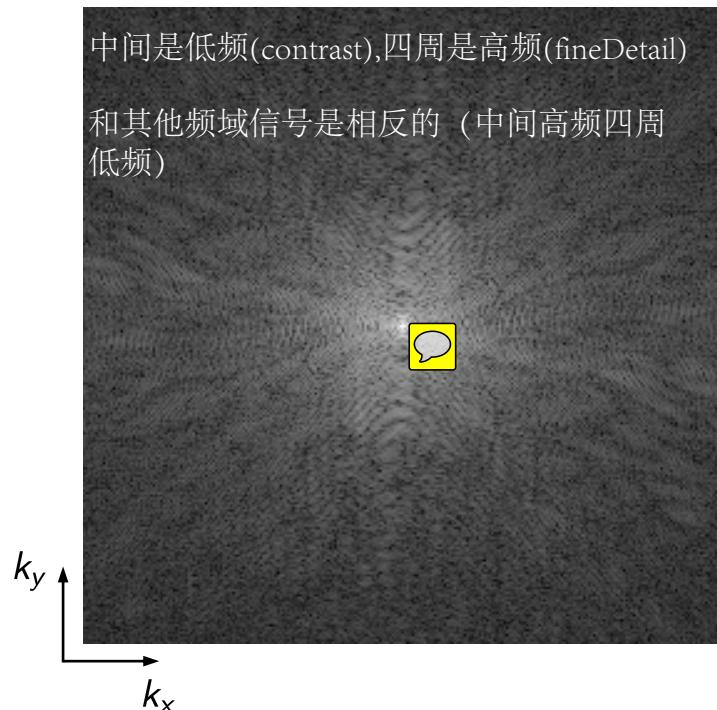


$$s(k_x, k_y) = \int_x \int_y m(x, y) e^{-ik_x x} e^{-ik_y y} dx dy$$

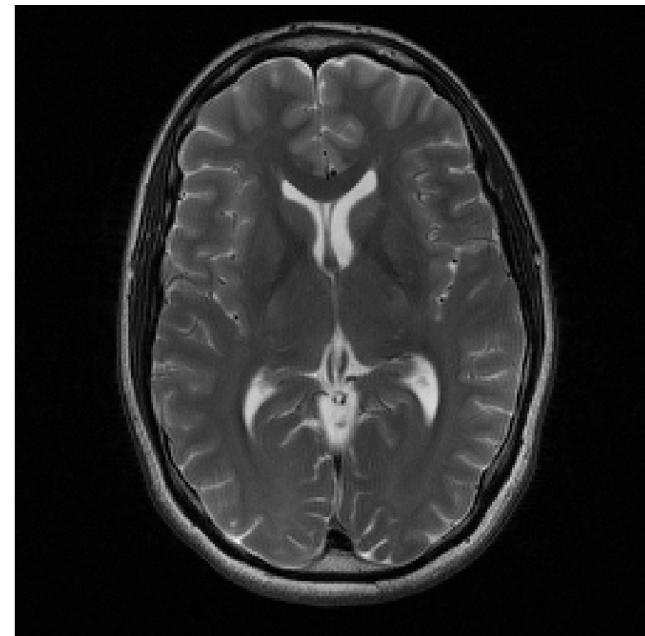
$$\phi(x, t) = \gamma x \int_0^t G_x d\tau \equiv k_x x$$

$$\phi(y, G_y) = \gamma y \int_0^{T_y} G_y d\tau \equiv k_y y$$

K-space

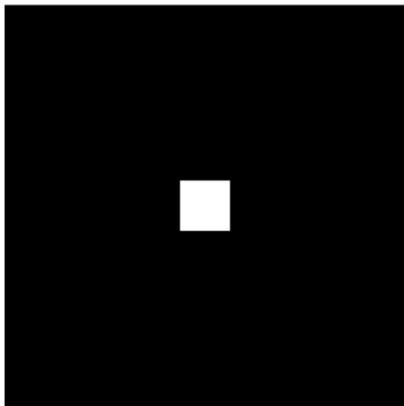


$$\mathcal{FT}^{-1}$$

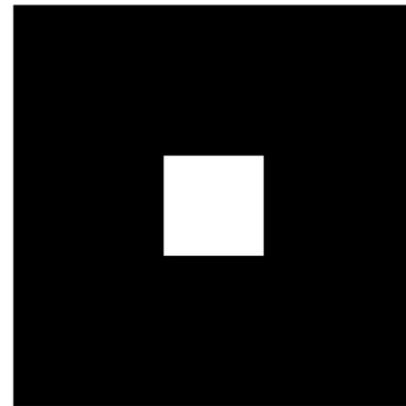


k-space and spatial frequencies

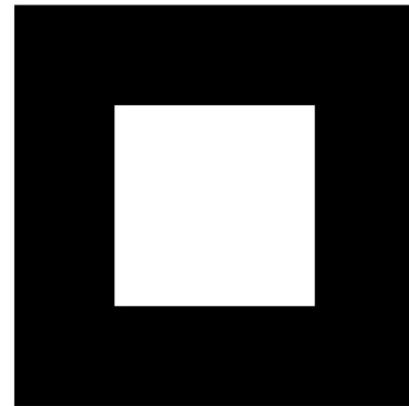
64x64



128x128

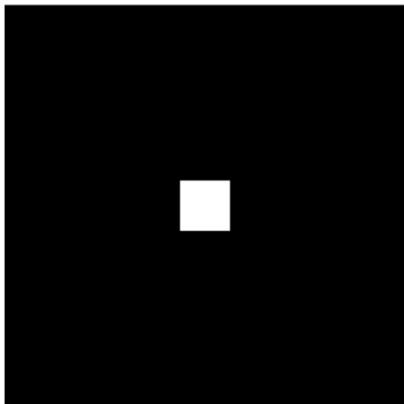


256x256

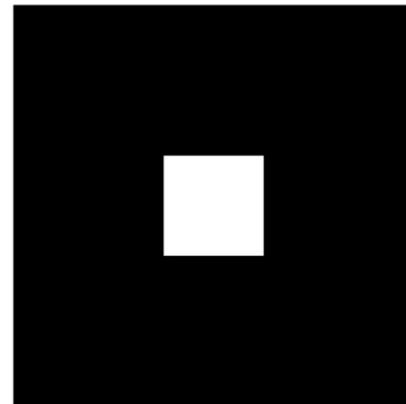


k-space and spatial frequencies

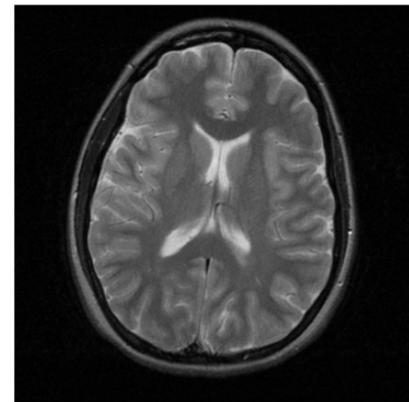
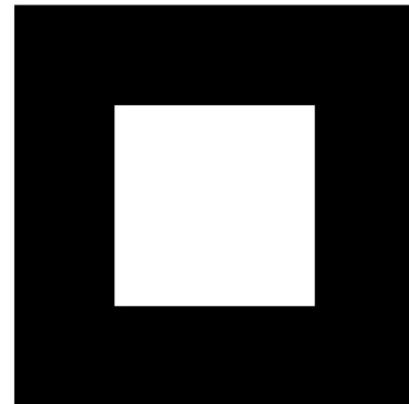
64x64



128x128

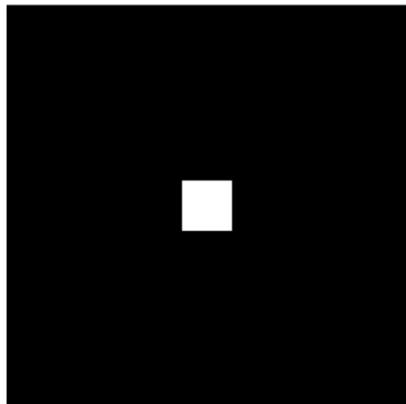


256x256

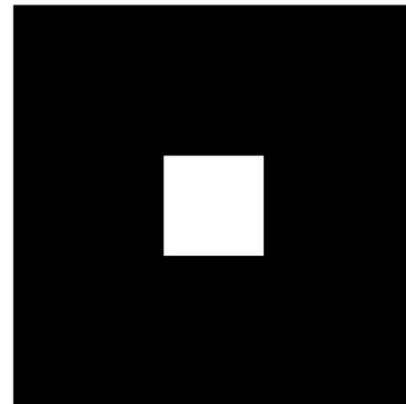


k-space and spatial frequencies

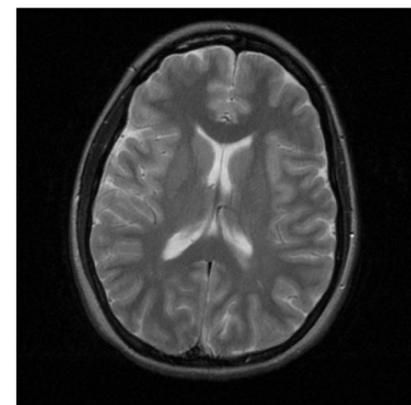
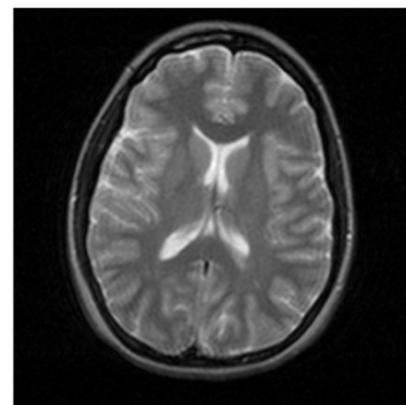
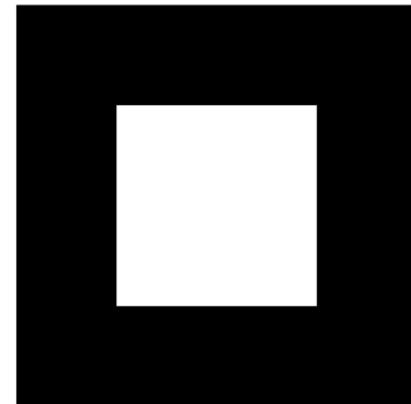
64x64



128x128

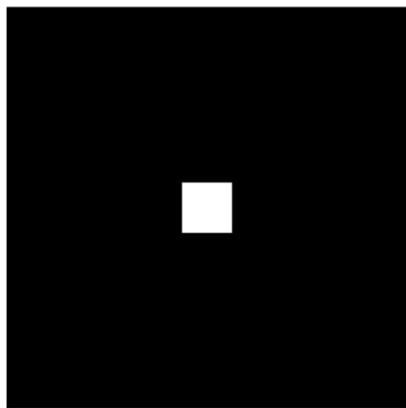


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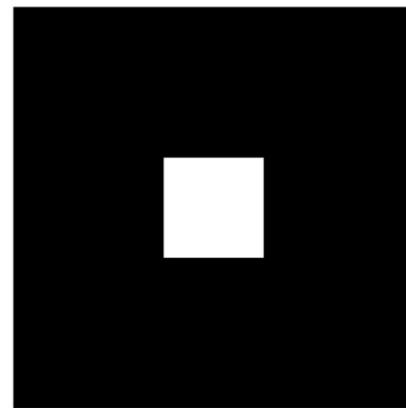


k-space and spatial frequencies

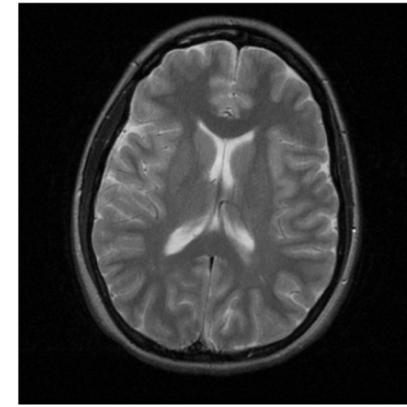
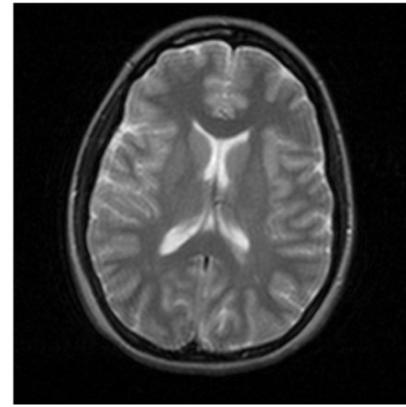
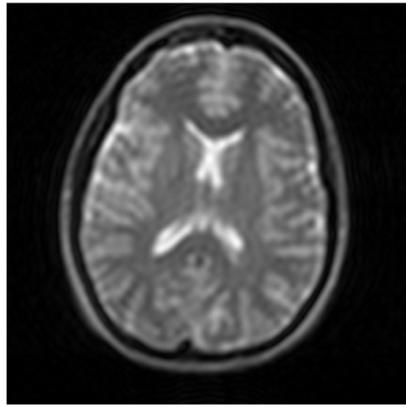
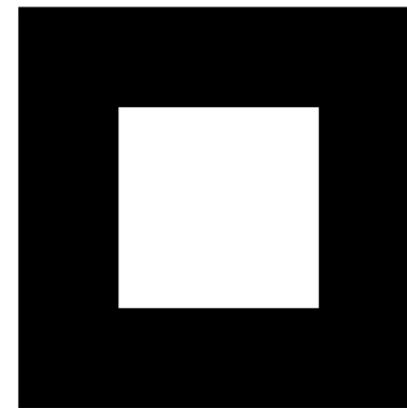
64x64



128x128



256x256

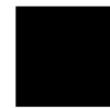


k-space and spatial frequencies

64x64



128x128



256x256

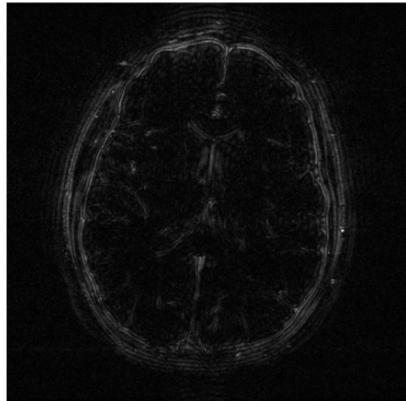


k-space and spatial frequencies

64x64

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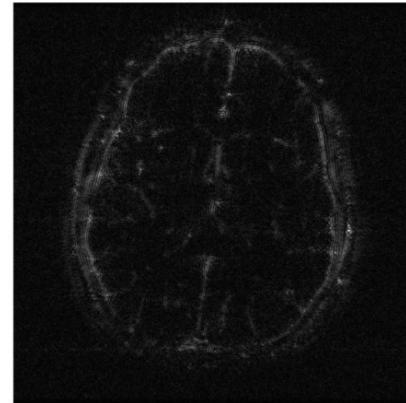
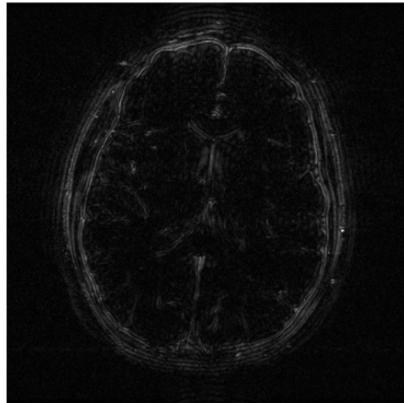


k-space and spatial frequencies

64x64

128x128

256x256



k-space and spatial frequencies

64x64



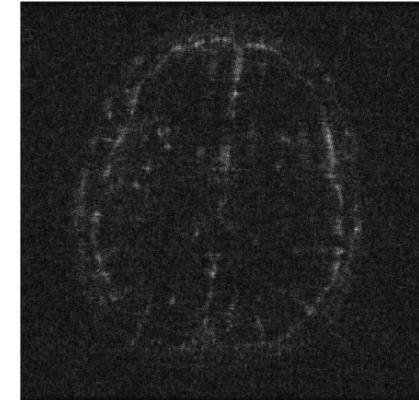
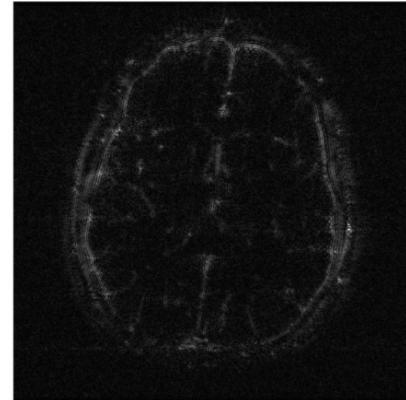
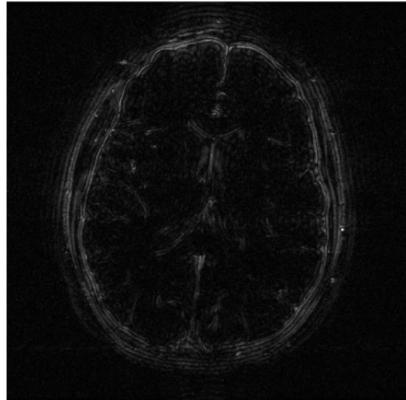
128x128



256x256



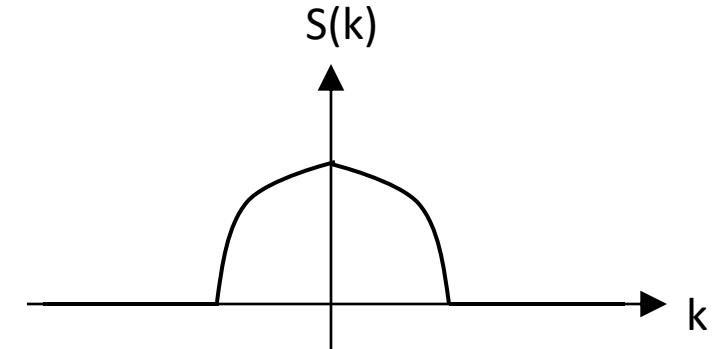
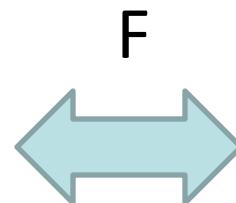
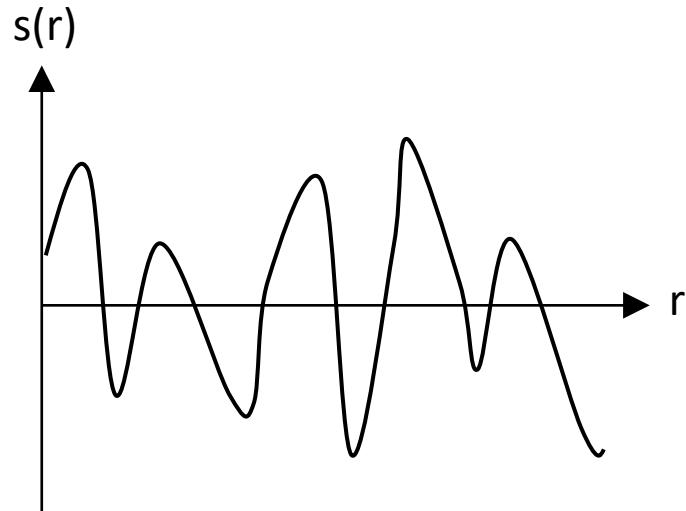
保留四周高频信息（edge），
丢失中间低频信息
所以重建的图片还有edge



Fourier transform

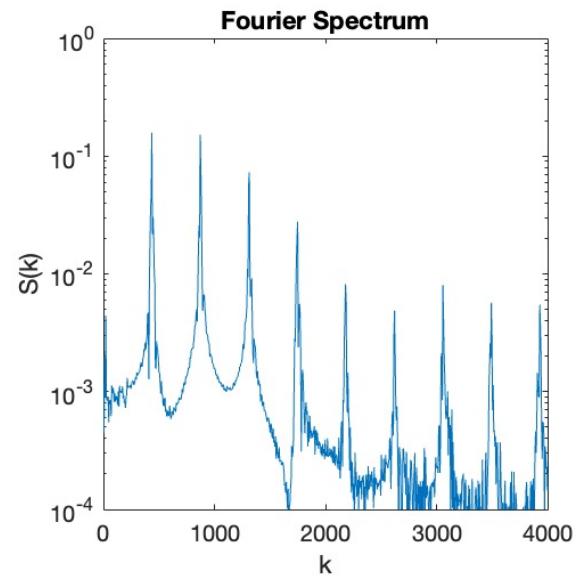
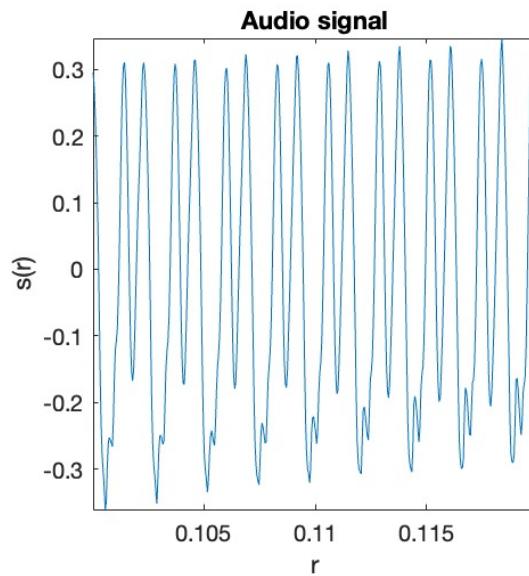
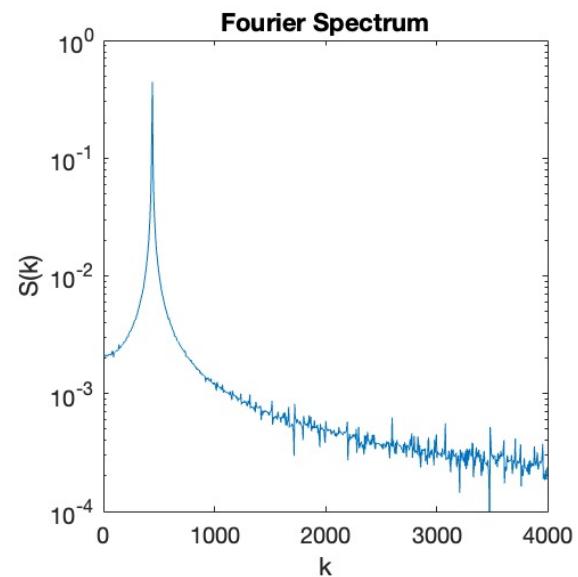
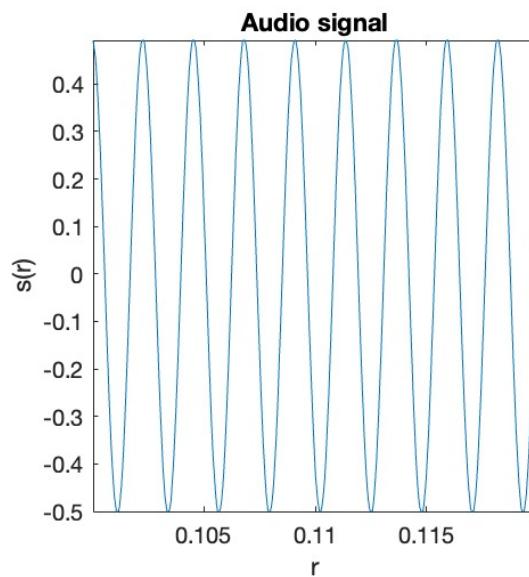
$$S(k) = \int_{-\infty}^{\infty} s(r) e^{-i2\pi kr} dr \quad (\text{forward})$$

$$s(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(k) e^{i2\pi kr} dk \quad (\text{inverse})$$



1D example

A4 (440Hz)



Fourier transform properties

- Linearity: $F\{as_1(r) + bs_2(r)\} = aS_1(k) + bS_2(k)$
- Shifting: $F\{s(r - r_0)\} = e^{-i2\pi k r_0} S(k)$ ← 可用于 motion correction
- Modulation: $F\{e^{i2\pi k_0 r} s(r)\} = S(k - k_0)$
- Conjugate symmetry: $s(r) \text{ real} \Rightarrow S(-k) = S^*(k)$
- Scaling: $F\{s(ar)\} = \frac{1}{|a|} S\left(\frac{k}{a}\right)$

Fourier transform properties

- Parseval's formula: $\int s_1(r)s_2(r) dr = \int S_1(k)S_2(k) dk$

$$s_1 = s_2 = s \quad \Rightarrow \quad \int |s(r)|^2 dr = \int |S(k)|^2 dk$$

- Convolution & multiplication

$$\mathcal{F}\{s_1(r) * s_2(r)\} = S_1(k)S_2(k)$$

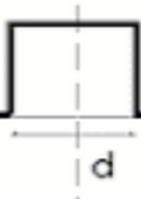
$$\mathcal{F}\{s_1(r)s_2(r)\} = S_1(k) * S_2(k)$$

Fourier transform of basic functions

"time" domain

frequency domain

Impulse, or "delta" function



Boxcar



Sync Function

T

Sin wave

$1/T$

T

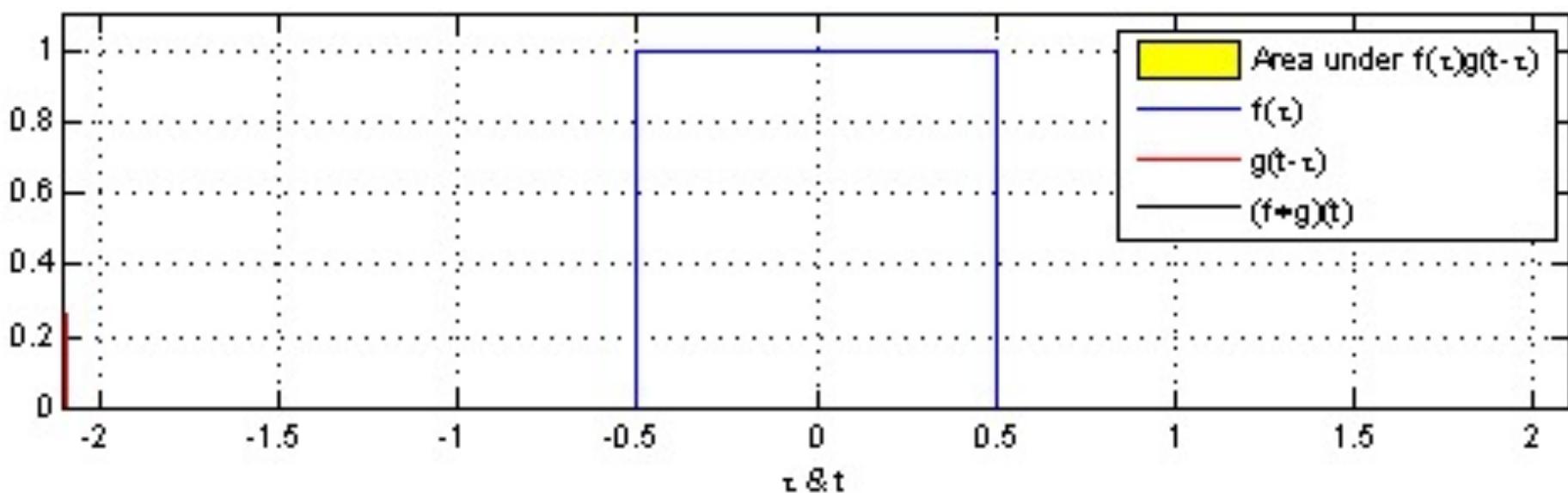
comb

$1/T$

sampling theory

What is the image space representation of sinc^2 ?

What is the image space representation of sinc^2 ?



Multidimensional Fourier transform

$$S(\mathbf{k}) = \int_{-\infty}^{\infty} s(\mathbf{r}) e^{-i2\pi\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \quad (\text{forward})$$

$$s(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\mathbf{k}) e^{i2\pi\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \quad (\text{inverse})$$

2D $\mathbf{r} = (x, y)$

$\mathbf{k} = (k_x, k_y)$

$$S(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(x, y) e^{-i2\pi(k_x x + k_y y)} dx dy$$

$$s(x, y) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y) e^{i2\pi(k_x x + k_y y)} dk_x dk_y$$

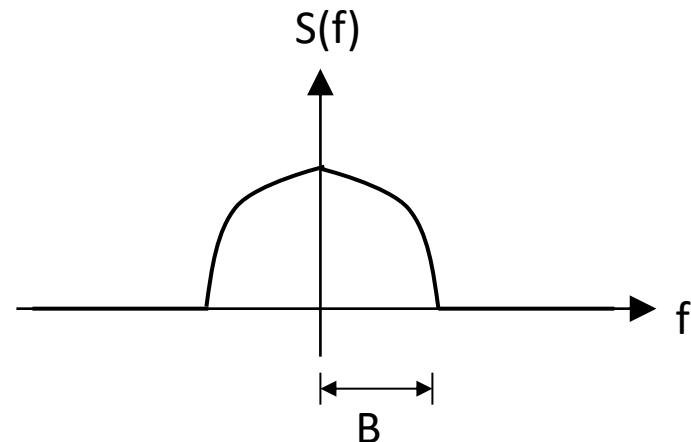
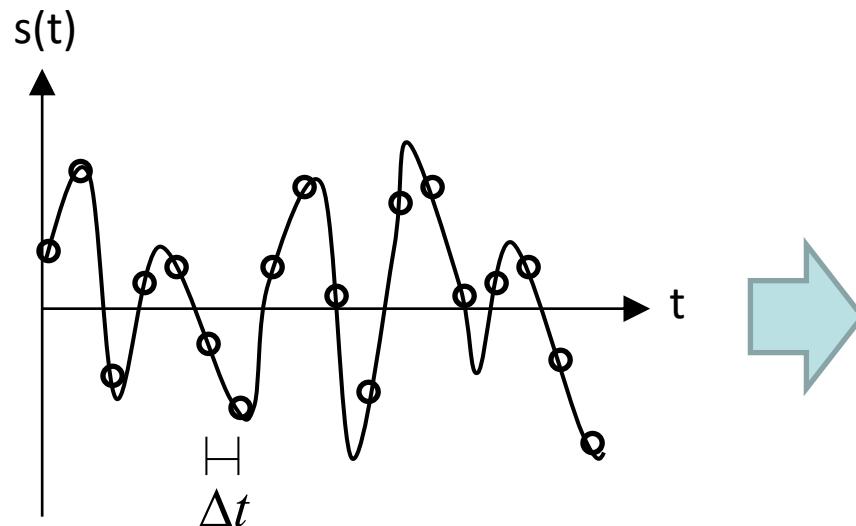
The multidimensional Fourier transform is separable

Sampling of continuous signals

- Nyquist/Shannon theorem
 - A signal with bandwidth B can be reconstructed from its samples if they are taken regularly with a period no larger than $1/2B$

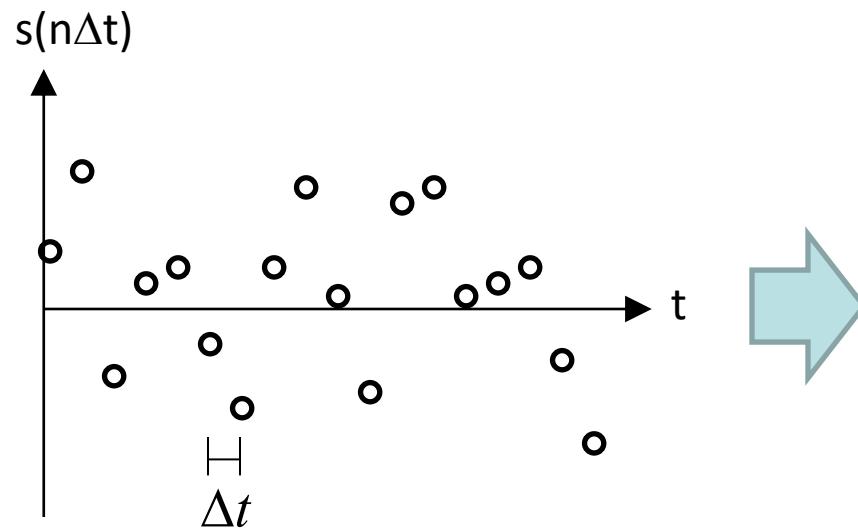


$$\text{Nyquist rate : } \Delta t = \frac{1}{2B}$$

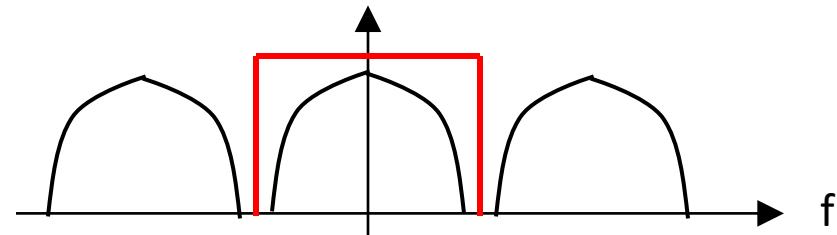


Sampling of continuous signals

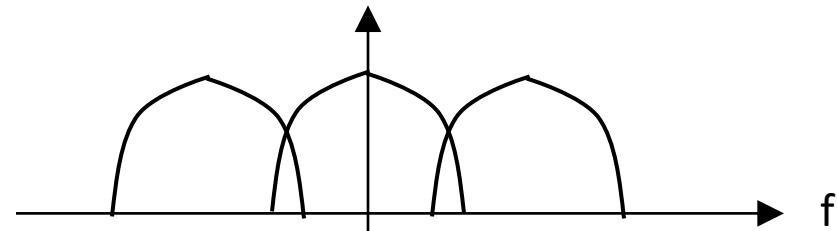
- Nyquist/Shannon theorem



$$\Delta t \leq \frac{1}{2B} \Rightarrow \text{no aliasing}$$

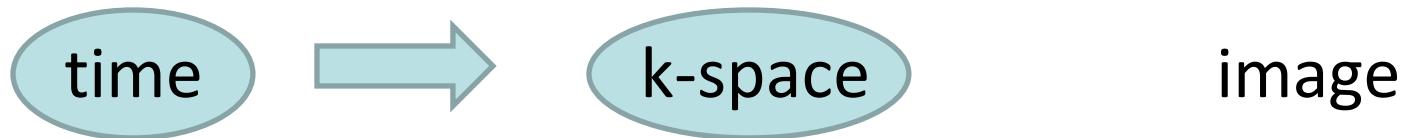


$$\Delta t > \frac{1}{2B} \Rightarrow \text{aliasing}$$



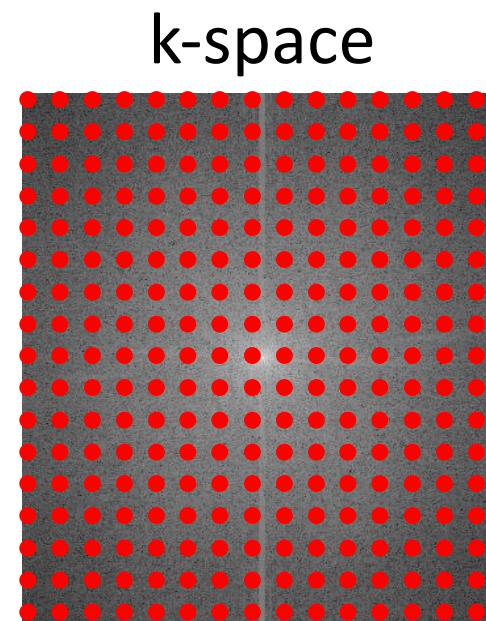
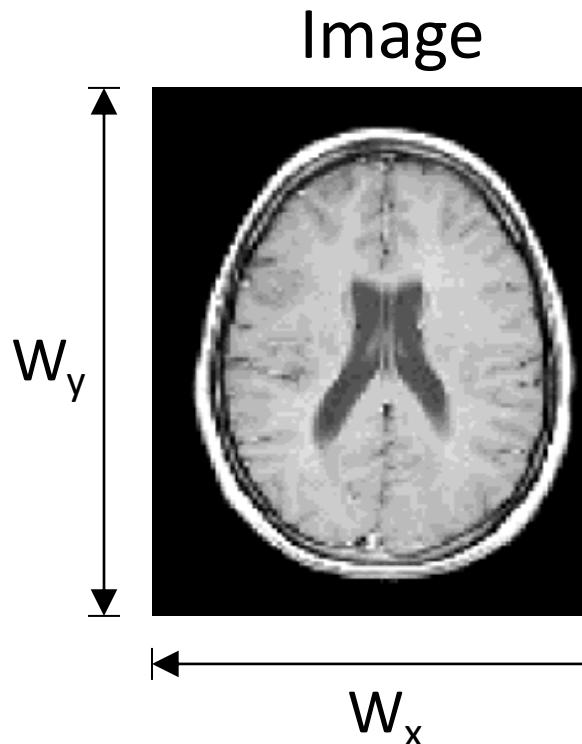
Sampling of MRI signals

- Where do we sample?



- How do we apply the sampling theorem?
 - bandwidth: image
 - sampling rate: k-space

Cartesian sampling of k-space

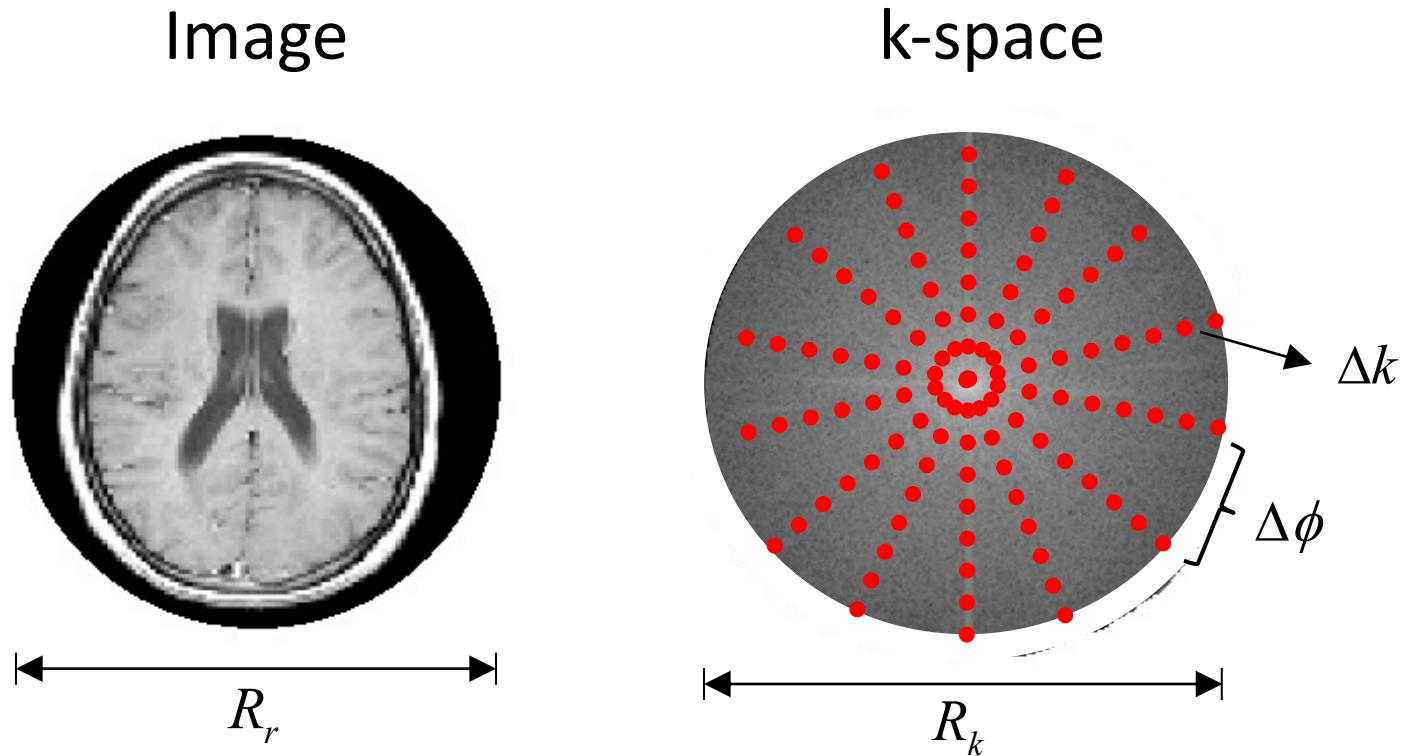


Nyquist rate:

$$\Delta k_x = \frac{1}{W_x}; \quad \Delta k_y = \frac{1}{W_y}$$

comb
function

Radial sampling of k-space

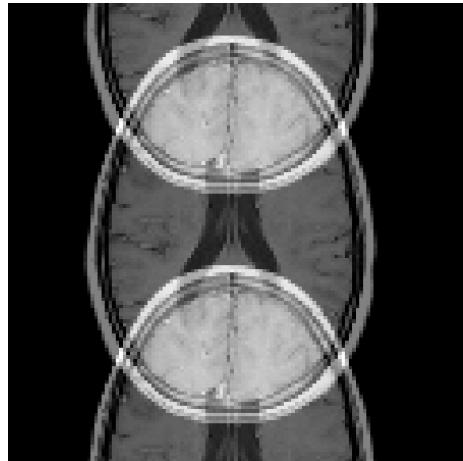


Nyquist rate: $N_{radial} = \frac{\pi}{2} N_{Cartesian}$

Aliasing examples

Cartesian

$$\Delta k_y = \frac{2}{W_y}$$

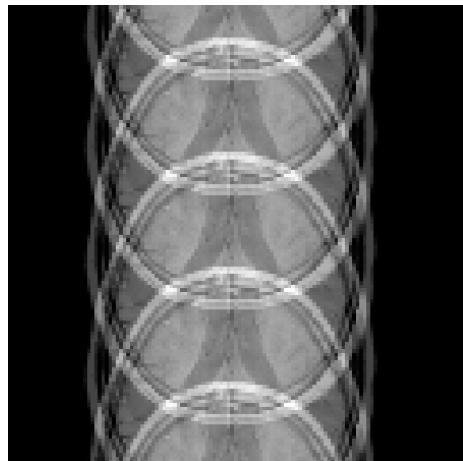


Radial

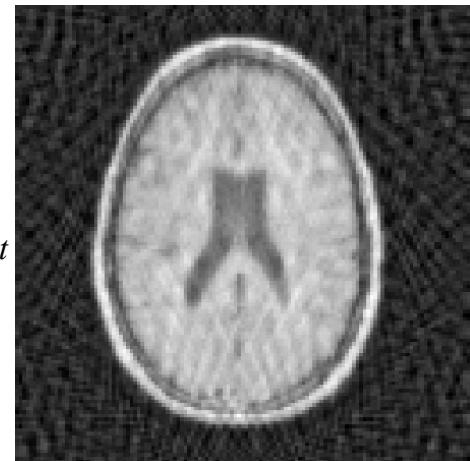
$$\Delta\phi = 2\Delta\phi_{Nyquist}$$



$$\Delta k_y = \frac{4}{W_y}$$



$$\Delta\phi = 4\Delta\phi_{Nyquist}$$



Discrete Fourier transform (DFT)

- Discrete signals (sequence of numbers)
- Fast implementation: FFT

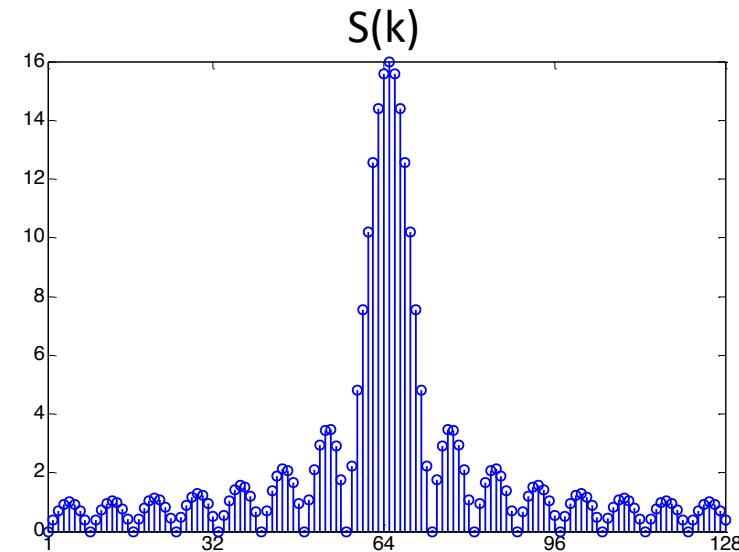
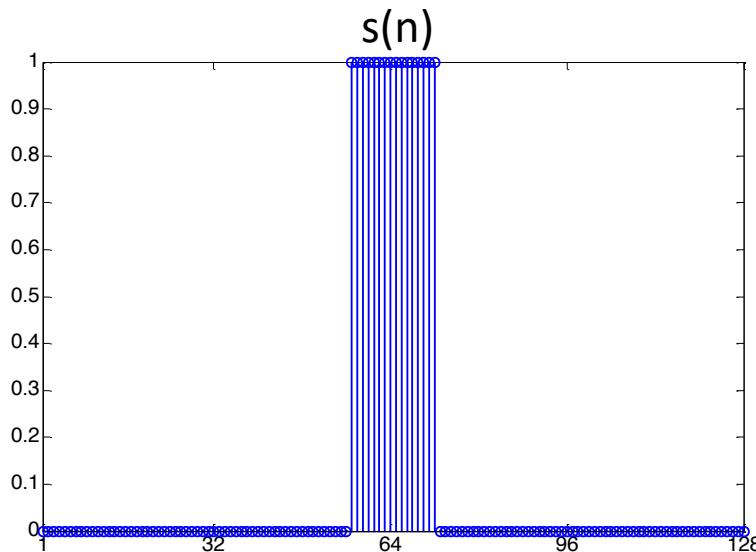
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-i \frac{2\pi}{N} nk} \quad (\text{forward})$$

$$s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k) e^{i \frac{2\pi}{N} nk} \quad (\text{inverse})$$

Matlab/Python

`S=fft (s)`

`s=ifft (S)`



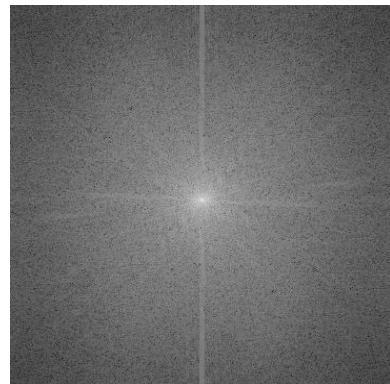
DFT reconstruction of Cartesian k-space data

- S(k) is known at $k=n\Delta k \quad \left(-\frac{N}{2} \leq n \leq \frac{N}{2} \right)$

`s=ifft2(S)`

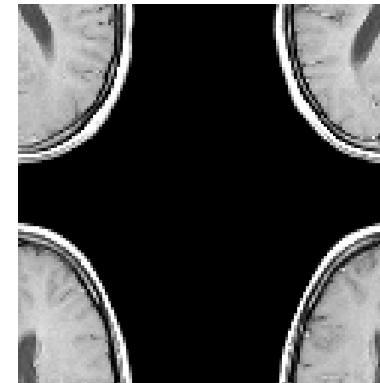
编程里原点在左下
MR里原点在中心

NxN k-space data



$-\frac{N}{2}$

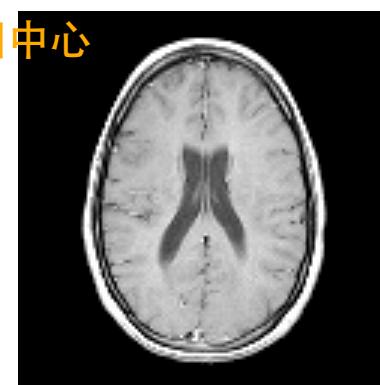
$\frac{N}{2}$



Solution: r and k from 0 to N-1

`s=sqrt(length(S))*fftshift(ifft2(ifftshift(S)))`

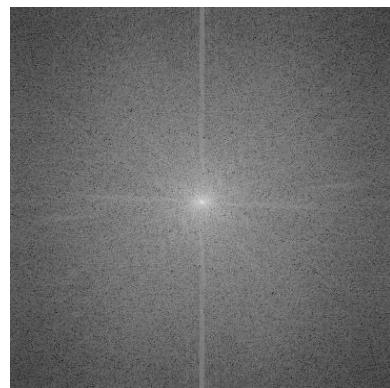
先移动原点到图中心



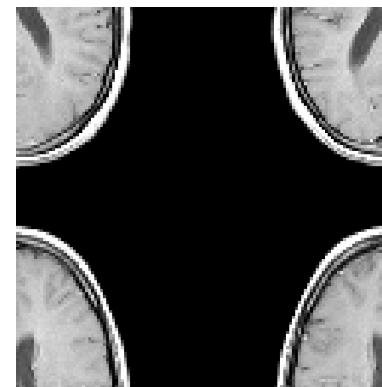
DFT reconstruction of Cartesian k-space data

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NxN k-space data

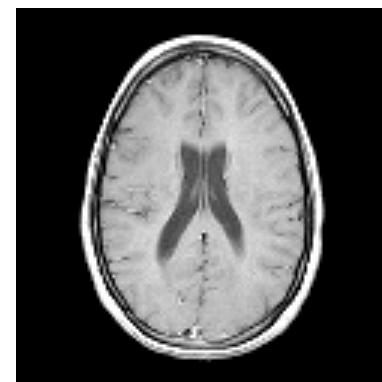


$S = \text{fft2}(s)$



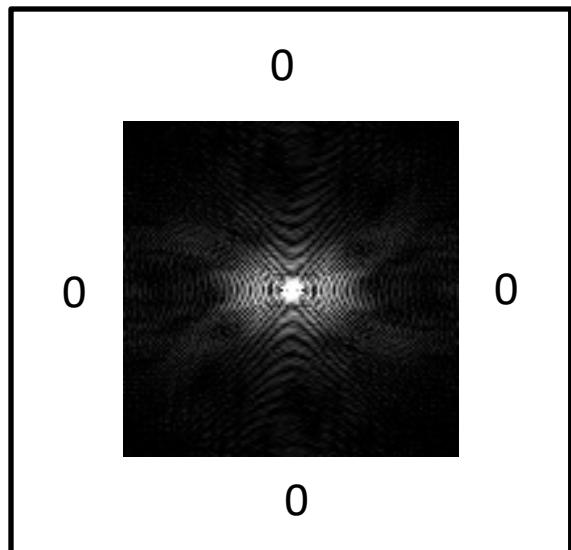
Solution: r and k from 0 to N-1

$S = 1/\sqrt{\text{length}(s)} * \text{fftshift}(\text{fft2}(\text{ifftshift}(s)))$

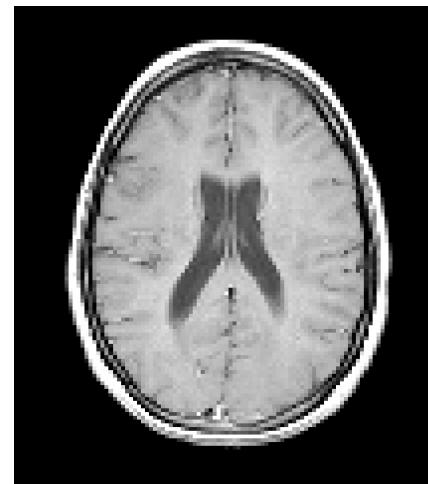


DFT reconstruction of Cartesian k-space data

- Zero-padding in k-space (Fourier interpolation)
 - Decreases the pixel size but does not increase resolution



Original 128x128



Zero-padded 256x256

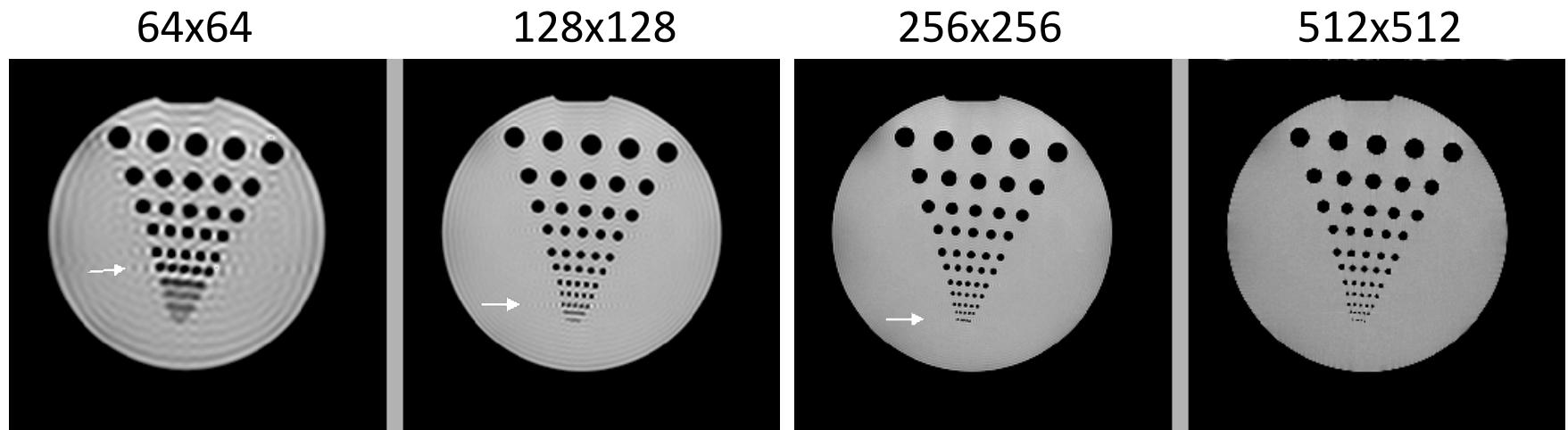


频域 \times rectangular
等于时域卷积sinc，会有

$$p_x = \frac{W_x}{N_{x,padded}} ; p_y = \frac{W_y}{N_{y,padded}}$$

DFT reconstruction of Cartesian k-space data

- Gibbs ringing
 - Spurious ringing around sharp edges
 - Caused by k-space truncation
 - Gets stronger for decreasing N)



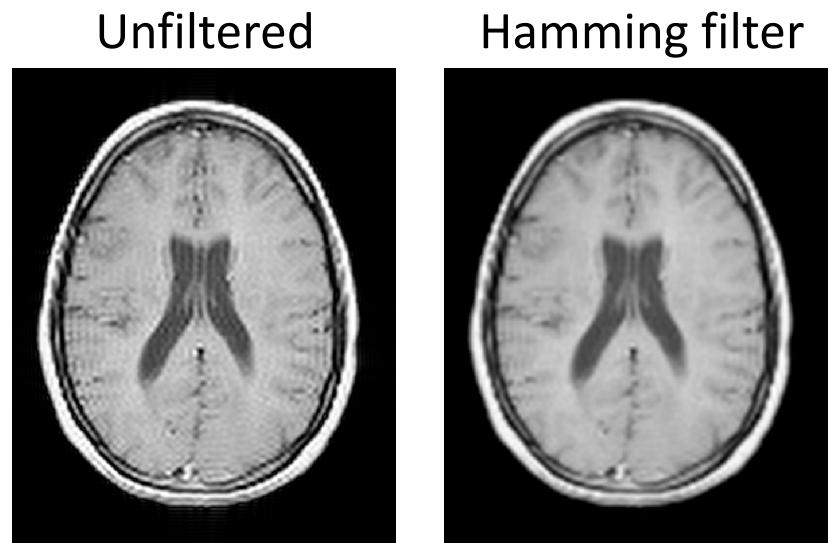
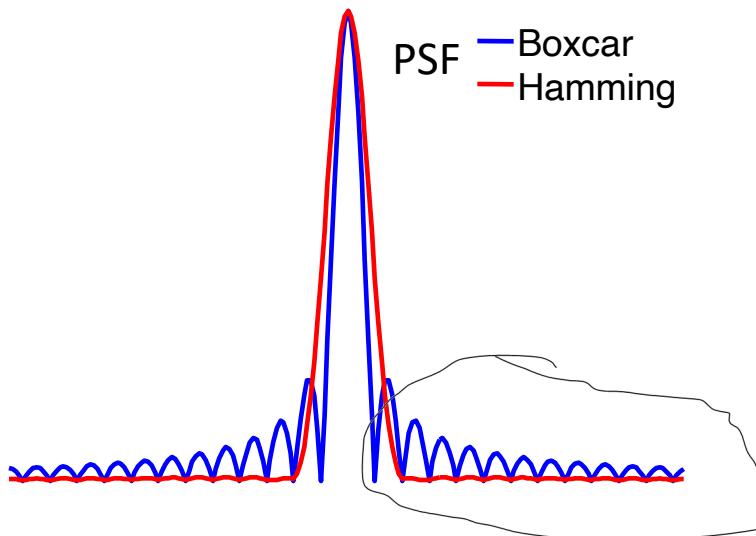
DFT reconstruction of Cartesian k-space data

- k-space filtering or windowing
 - Reduce Gibbs ringing at the expense of resolution loss

$$S_W(k) = S(k)W(k)$$

- Hamming filter

$$W(k) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N}\right)$$

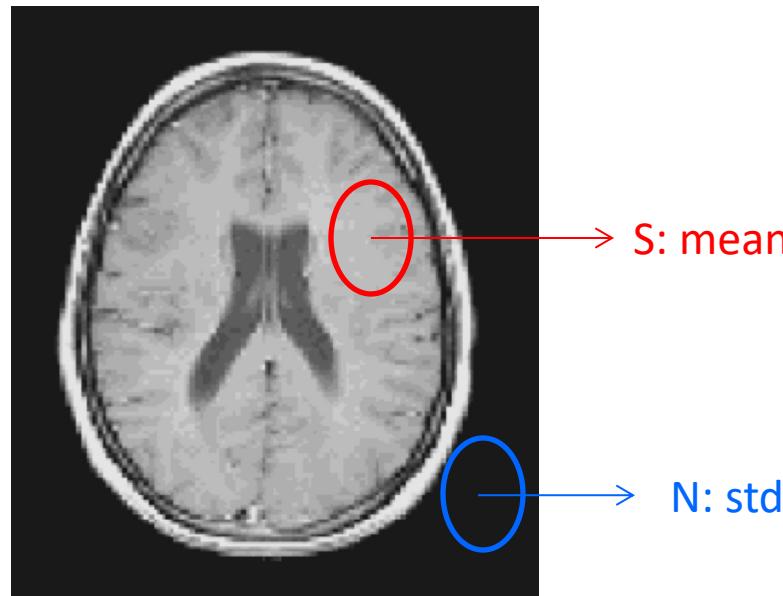


hamming 过滤周边信号，能去除伪影
但是也丢失图片信息

DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR)

$$SNR = \frac{S}{N} = \frac{\text{Pixel signal amplitude}}{\text{Standard deviation of background}}$$



Not entirely correct for many real world measurements!

DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR) (simplified)

$$SNR \propto V \sqrt{T} \quad (\text{AI Macovski})$$



V: voxel volume

T: cumulated readout duration

have more voxel or scanning longer
will have larger SNR
but V变大的话resolution变小