Diagnostic Medical Image Processing
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Exercises (DMIP-E)
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LEHRSTUHL FÜR MUSTER-ERKENNUNG

Fan Beam Reconstruction

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Exercise Sheet 6

19 Fan Beam Reconstruction

(i) Please explain graphically how the point spread function (PSF) for fan beam looks like and show its relation to the parallel beam PSF.
Using this insight, what trick have you learned in the lecture to derive an analytical reconstruction algorithm for fan beam reconstruction? State its name and a very short description of the general idea.

(ii) The parallel beam sinogram is denoted by $p(s,\theta)$ and the equal angle fan beam sinogram is denoted by $g(\gamma,\beta)$. The distance from the X-ray source to the isocenter is denoted by D. The filtered backprojection (FBP) algorithm for parallel beam reconstruction is given as follows:

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s,\theta)h(x\cos\theta + y\sin\theta - s)\mathrm{d}s\mathrm{d}\theta.$$

The FBP algorithm for fan beam reconstruction is derived from the parallel beam reconstruction. Before we apply the fan beam filter, we apply the cosine weighting $\cos\gamma$ first. Why do we need the cosine weighting?
Hint: We used a transformation to polar coordinates for the derivation of the algorithm.

(iii) In order to write the final fan beam algorithm in form of a convolution, we used a specific property of the ramp filter h:

$$h(D'\sin\gamma) = \left(\frac{\gamma}{D'\sin\gamma}\right)^2 h(\gamma).$$

Show that this relation is correct.

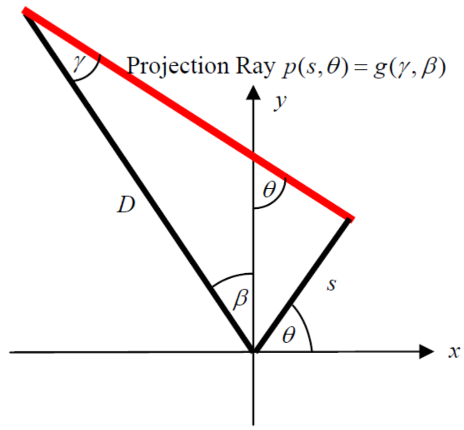

Figure 1: Geometry and notation for the fan beam system

(iv) Assume there is a point object at position (x_0, y_0) and we use the fan beam system as shown in **Fig.** 1. When the X-ray source and the curved detector rotate over 360°, the projection of the point will be a curve in the sinogram. Please give the function of the sinogram curve, i.e., the relation of γ and β.

$\boxed{4 \times 0.5\mathbf{+1+1+1}}$

(i) The PSF tells us how a single point is "smeared" by the backprojection process.

- For parallel beam geometry, draw a line through the reconstruction point that is perpendicular to the detector and repeat for every detector position.
  → In this case, the point spread function is shift-invariant, i.e., the PSF shows the same pattern for every reconstruction point (see unit 28 "Reconstruction Basics").
- For fan beam geometry, draw a line through the reconstruction point and the source position and repeat for every source position.
  → For a complete circle, the pattern is also shift-invariant and it can be shown that the full circle PSF is equivalent to the parallel beam PSF.

The trick is called *rebinning*, which is a resorting of the sinogram → estimate parallel beam data from fan beam sinogram and use parallel beam filtered backprojection. This way we get a modified version of FBP specifically for fan beam data.

(ii) For fan beam backprojection, we need to backproject the sinogram with respect to γ and β, i.e., the double integral should become

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ldots \, \mathrm{d}\gamma \, \mathrm{d}\beta.$$

The required coordinate transformation is:

$$\theta = \gamma + \beta,$$
$$s = D \sin\gamma,$$

which yields the Jacobian matrix:

$$\begin{vmatrix} \frac{\partial\theta}{\partial\gamma} & \frac{\partial\theta}{\partial\beta} \\ \frac{\partial s}{\partial\gamma} & \frac{\partial s}{\partial\beta} \end{vmatrix} = D\cos\gamma.$$

Thus, it needs the cosine weight.

(iii) (compare Zeng's book "Medical Image Reconstruction" p. 58)

$$\begin{aligned} h(D'\sin\gamma) &= \int_{-\infty}^{\infty} |\omega| e^{2\pi i\omega D'\sin\gamma}\,\mathrm{d}\omega \\ &= \left(\frac{\gamma}{D'\sin\gamma}\right)^2 \int_{-\infty}^{\infty} \left|\omega\frac{D'\sin\gamma}{\gamma}\right| e^{2\pi i\omega\frac{D'\sin\gamma}{\gamma}\gamma}\,\mathrm{d}\left(\omega\frac{D'\sin\gamma}{\gamma}\right) \\ &= \left(\frac{\gamma}{D'\sin\gamma}\right)^2 \int_{-\infty}^{\infty} |\hat{\omega}| e^{2\pi i\hat{\omega}\gamma}\,\mathrm{d}\hat{\omega} \\ &= \left(\frac{\gamma}{D'\sin\gamma}\right)^2 h(\gamma) \end{aligned}$$

(iv) The sinogram curve for the parallel beam will be:

$$x_0\cos\theta + y_0\sin\theta = s.$$

We have,

$$\begin{aligned} \theta &= \gamma + \beta, \\ s &= D\sin\gamma \end{aligned}$$

Thus, the sinogram curve of the point object in the fan beam system is implicitly defined by:

$$x_0\cos(\beta+\gamma) + y_0\sin(\beta+\gamma) = D\sin\gamma.$$

Bonus: For the explicit formula we need to use some trigonometric identities:

$$\begin{aligned} &\Leftrightarrow && (x_0\cos\beta + y_0\sin\beta)\cos\gamma + (y_0\cos\beta - x_0\sin\beta - D)\sin\gamma = 0, \\ &\overset{|\gamma|<\frac{\pi}{2}}{\Leftrightarrow} && \tan\gamma = \frac{(x_0\cos\beta + y_0\sin\beta)}{(D + x_0\sin\beta - y_0\cos\beta)}, \end{aligned}$$

from which we find $\gamma(\beta) = \arctan(\dots)$.

20 Short Scan

(i) A CT acquisition of a Christmas tree (see **Fig.** 2) is reconstructed by a fan beam reconstruction with $\delta = 11°$. It is performed by a short-scan. Projections are taken with $\Delta\gamma = 2°$. How many projections have to be acquired to reconstruct the image?

(ii) In **Fig.** 3, you find a short-scan and three other trajectories. The object is assumed to occupy the circular shaded region of radius R_m, and the trajectory lies concentrically on a circle of radius R_0. Assume that the detector is always large enough to image the full object, thereby accommodating a maximum fan angle of $\gamma_m (= \delta)$. Explain for each figure if we can get a whole object from the reconstruction process?

(iii) What problem arises with short scans and how can it be fixed?

$\boxed{\mathbf{1+2+1}}$

(i) For a complete short-scan of the image $180° + 2\delta$ have to be acquired. Therefore $\frac{180°+22°}{2°} + 1 = 102$ projections would have to be acquired.

(ii) (a) A conventional short-scan of $\pi + 2\gamma_m$ allows reconstruction of the whole object. (b)-(c) A continuous scan of less than $\pi + 2\gamma_m$ allows reconstruction of all object points inside the convex hull of the scan. (d) A scan of three equally spaced segments of 80° each allows reconstruction of a triangular ROI in the center of the object.

(iii) In the case of parallel beam geometry, it is obvious that after an 180°-degree rotation the rays gets the same, just the direction is switched. So, with a full 360°-degree measurement we measure twice. In the case of a short scan in fan beam geometry this is different. There are cases where you have the ray two times and others where not. In short scans, projections with angles from 0° to 4δ and from $180° - 2\delta$ to $180° + 2\delta$ contain redundant information (cf. **Fig.** 4). This issue can be fixed by the weighting scheme called Parker weighting.

21 FBP for a Fan Beam Short Scan – Programming Exercise

The goal for this exercise is to reconstruct short scan data using the fan beam FBP algorithm with Parker weights.

Fan Beam FBP Algorithm

Given the fan beam opening angle δ and the flat panel 'fanogram' data $g(s, \beta)$, we use the following reconstruction steps from the lecture (see **Fig.** 4(a) for the notation):

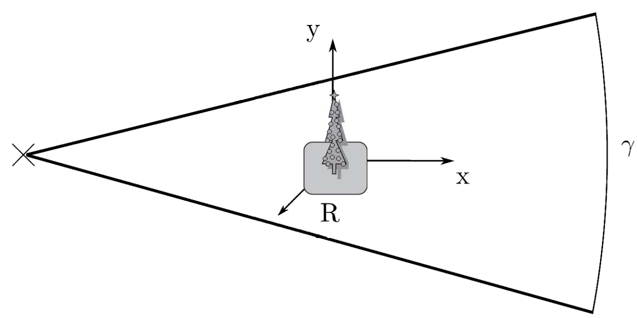

Figure 2: CT image for Christmas

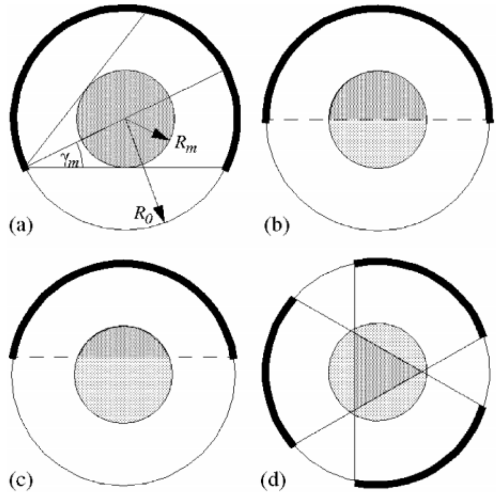

Figure 3: Different scan trajectories, how much of the object do they cover completely? (Source: `http://iopscience.iop.org/article/10.1088/0031-9155/47/14/311/`)

1. Cosine weighting of projection data to obtain $g_1(s,\beta)$:

$$g_1(s,\beta) = \frac{D}{\sqrt{D^2+s^2}}\, g(s,\beta)\ ,$$

2. Perform fan beam filtering:

$$g_F(s,\beta) = \int_{-\infty}^{\infty} h_R(s-s')\, g_1(s',\beta)\, ds'\,,$$

   where $h_R(s)$ is the filter kernel.

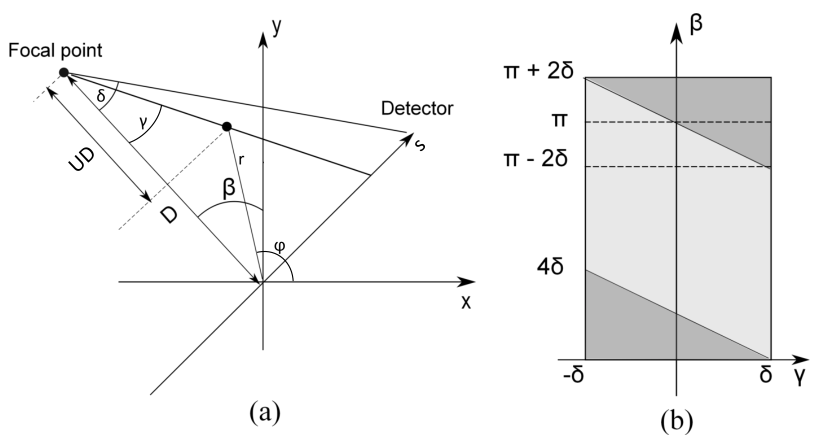

Figure 4: (a) Fan beam imaging geometry, (b) fan beam sinogram and data redundancy.

3. Backprojection with a weighting function of object to focal point distance U:

$$f\left(r,\varphi\right)=\int_{0}^{2\pi}\frac{1}{U^2}g_F\left(s,\beta\right)d\beta,\qquad U=\frac{D+r\sin(\beta-\varphi)}{D}$$

Parker Weights

A short scan measures some redundant rays at the beginning and at the end of data acquisition (see the two dark triangles in **Fig.** 4(b)). Corresponding redundant rays determined by the relation $g\left(\gamma,\beta\right)=g\left(-\gamma,\beta+\pi+2\gamma\right)$ are commonly by the Parker weighting function:

$$\omega\left(\gamma,\beta\right)=\begin{cases}\sin^2\left(\frac{\pi}{4}\frac{\beta}{\delta-\gamma}\right), & 0\leq\beta\leq 2\delta-2\gamma\\ 1, & 2\delta-2\gamma\leq\beta\leq\pi-2\gamma\\ \sin^2\left(\frac{\pi}{4}\frac{\pi+2\delta-\beta}{\delta+\gamma}\right), & \pi-2\gamma\leq\beta\leq\pi+2\delta\end{cases}$$

Projection rays measured twice are normalized to unity while guaranteeing smooth transitions between non-redundant and redundant data.

Implementation Tasks

Complete the gaps in the provided CONRAD class that are marked with "**TODO**"

(i) Initialize the parameters. Compute the fan angle and the short scan range.

(ii) Compute direction and position of the detector border at a rotation angle β.

(iii) Transform the pixel coordinates to world coordinates.

(iv) Compute the intersection point of a ray with the detector.

(v) Compute the distance weights for this point at the given rotation angle β.

(vi) Complete the cosine weights computation.

(vii) Complete the Parker weights implementation.

(viii) Test all available sinograms `Sinogram0.tif`, `Sinogram1.tif`, and `Sinogram2.tif`.

1+0.5+0.5+1+1+0.5+1+0.5

(code implementation)

Total: 15