Diagnostic Medical Image Processing
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Exercises (DMIP-E)
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LEHRSTUHL FÜR MUSTER-ERKENNUNG

Rigid Registration

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Exercise Sheet 8

Rigid registration only allows rotations and translations which means that the objects maintain their shape and size in the process. In 2-D rigid transformations between image points $\mathbf{p}_k$ and $\mathbf{q}_k$ can be described as

$$\mathbf{p}_k = \mathbf{R}\mathbf{q}_k + \mathbf{t}, \tag{1}$$

where $\mathbf{R}$ is a rotation matrix for the angle $\varphi \in [0, 2\pi)$

$$\mathbf{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and $\mathbf{t}$ is a translation vector $\mathbf{t} = (t_1, t_2)$, $t_1, t_2 \in \mathbb{R}$.

Solving the Correspondence Problem

Given N point correspondences $(\mathbf{p}_k, \mathbf{q}_k) \in \mathbb{R}^2$ from one of two images to the other, the associated optimization problem is

$$\left(\hat{\mathbf{R}}, \hat{\mathbf{t}}\right) = \underset{\varphi, t_1, t_2}{\operatorname{argmin}} \sum_{k=1}^{N} \|\mathbf{p}_k - \mathbf{R}\mathbf{q}_k - \mathbf{t}\|^2 .$$

To avoid solving this nonlinear problem, one can make use of complex numbers. The image points $\mathbf{p}_k = (p_{k,1}, p_{k,2})$ and $\mathbf{q}_k = (q_{k,1}, q_{k,2})$ are thereby identified with the complex numbers $p_{k,1}+ip_{k,2}$ and $q_{k,1}+iq_{k,2}$. The rigid transformation is then described by two complex numbers r_1+ir_2 and t_1+it_2 for rotation and translation, respectively, as follows:

$$p_{k,1}+ip_{k,2} = (r_1+ir_2)(q_{k,1}+iq_{k,2}) + t_1+it_2, \quad \text{for } k = 1, 2, \ldots, N. \tag{2}$$

Splitting real and imaginary part we get two equations

$$p_{k,1} = r_1 q_{k,1} - r_2 q_{k,2} + t_1 = (q_{k,1}, -q_{k,2}, 1, 0) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix},$$

$$p_{k,2} = r_1 q_{k,2} + r_2 q_{k,1} + t_2 = (q_{k,2}, q_{k,1}, 0, 1) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

A sufficient amount of correspondences provided, these equations can be used to determine the optimal rigid transform for these correspondences.

Mutual Information

Distance measures (or similarity measures) are used to calculate similarity between objects. Let F be a reference image and M be a moving image. Mutual information is defined as

$$\mathcal{D}_{MI} = H(F) + H(M) - H(F, M),$$

where $H(F)$ and $H(M)$ are the entropies of the images F and M, and $H(F, G)$ is the joint entropy. $\mathcal{D}_{MI}$ evaluates how much information is shared in both pictures. The mutual entropy should be maximized.

24 Rigid Registration in 2-D – Programming Exercise

Complete the gaps in the provided CONRAD classes that are marked with "**TODO**".

(i) Solving the correspondence problem:
Two point clouds are given. Calculate the translation and rotation using the information at the beginning of this sheet and what you have learned about quaternions in the lecture. Build the linear system resulting from the linear optimization problem and solve it by using SVD. Plot the result and discuss with your tutor.

(ii) Rigid registration with mutual information:
Load the images `Proton.png` and `T1.png`. The given code produces a blurred version of both images.

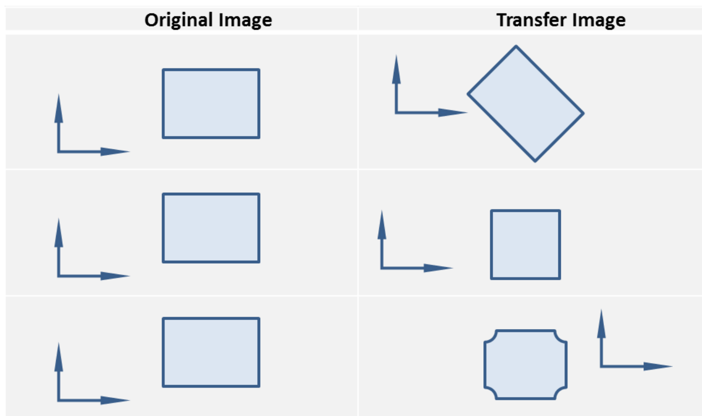
  (a) Calculate the joint histogram and a histogram for each image.
  (b) Calculate the joint and the marginal entropies.
  (c) Calculate the mutual information.
  (d) Show your result.

$\boxed{\textbf{4+4}}$

(code implementation)

25 Rigid Transformations

(i) Assume the images below are shown in 2-D. Which of them show rigid transformations? Now imagine you are just looking from the top view at a 3-D object (an ashlar for the originals). What is your answer now, which of the cases could represent a rigid transformation? (The arrows mark origin and direction of the coordinate system, in 3-D the third axis is orthogonal to the given ones.)



(ii) Compute the complex number $r = r_1 + ir_2$ (where $r_1^2 + r_2^2 = 1$) that corresponds to the rotation matrix $\mathbf{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.

(iii) What is the complex number for a rotation by 45°? What complex number do you get if you rotate $2+2i$ by 45°? And finally, consider rotating $3+3i$ about $2+2i$ by 45°, what do you get?

$\boxed{\mathbf{1+0.5+1.5}}$

(i) In 2-D only the first one is rigid, the second needs a non-equal scaling, the third is deformed at the corners. In 3-D similar arguments lead to the first one being rigid, the third one being non-rigid. However, the second one now can be rotated such that we see now another view of the ashlar (e.g., from side-view left to top view right).

(ii) Compare the real and imaginary parts of (2) and the two components of (1) and solve for r_1 and r_2. The result should be $r_1 = \cos\varphi$, $r_2 = \sin\varphi$, or $r = e^{i\varphi}$.

(iii) rotation by 45°: $\cos 45° + i \sin 45° = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$$\Rightarrow \qquad \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) * (2+2i) = 2\sqrt{2}i$$

for the last question we first need to translate:

$$\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) * (3+3i - (2+2i)) + (2+2i) = \ldots = 2 + (2+\sqrt{2})i$$

26 Quaternions

(i) Which geometric degrees of freedom are typically described using quaternions?

(ii) Is there a difference between a four dimensional vector space and quaternions? Describe this difference.

(iii) Let the quaternion $\mathbf{r}$ be defined as $\mathbf{r} = w+ix+jy+kz = (w, \mathbf{q})$, where 1, i, j, and k denote the imaginary units and $\mathbf{q} = (x, y, z)^\top$. Regarding $\mathbf{r} = (w, \mathbf{q})$ as a rotation, please give its rotation axis $\mathbf{u}$ and rotation angle θ in the axis-angle representation of rotations.

(iv) Given $\mathbf{r}_1 = i+2j+3k$ and $\mathbf{r}_2 = 1+2i+3j+4k$, calculate the product of $\mathbf{r}_1$ and $\mathbf{r}_2$.

(v) Compute the inverse of $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e., $\mathbf{r}_1^{-1}$ and $\mathbf{r}_2^{-1}$.

0.5+0.5+0.5+0.5+1

(i) Rotations around all three coordinate axis

(ii) Yes, there is a difference. Quaternions are a divison ring/skew field instead of a vectorspace. Multiplication is not commutative.

(iii)

$$\theta = 2\tan^{-1}(||\mathbf{q}||/w), \qquad \mathbf{u} = \mathbf{q}/||\mathbf{q}||$$

(iv)

$$\begin{aligned} r_1 * \mathbf{r}_2 &= (w_1 w_2 - \mathbf{q}_1^\top \mathbf{q}_2, w_1\mathbf{q}_2 + w_2\mathbf{q}_1 + \mathbf{q}_1 \times \mathbf{q}_2) \\ &= (0 - (1,2,3)(2,3,4)^\top, \mathbf{0} + (1,2,3)^\top + (1,2,3)^\top \times (2,3,4)^\top) \\ &= (-20, (1,2,3)^\top + (-1,2,-1)^\top) = (-20, (0,4,2)^\top) = -20 + 0i + 4j + 2k. \end{aligned}$$

(v)

$$\mathbf{r}_1^{-1} = \bar{\mathbf{r}}_1/||\mathbf{r}_1||^2 = (-i - 2j - 3k)/(1+4+9) = (-\tfrac{1}{14}i - \tfrac{1}{7}j - \tfrac{3}{14}k)$$

$$\mathbf{r}_2^{-1} = \bar{\mathbf{r}}_2/||\mathbf{r}_2||^2 = (1 - 2i - 3j - 4k)/(1+4+9+16) = (1 - 2i - 3j - 4k)/30$$

27 Iterative Closest Point (ICP)

Here is a simple example to show the idea of the ICP algorithm. There are two curves P and Q. On the first curve P, four points are observed: $\mathbf{p}_1 = (0,1)$, $\mathbf{p}_2 = (2,2)$, $\mathbf{p}_3 = (4,1)$, and $\mathbf{p}_4 = (6,2)$. On the second curve Q, eight points are observed: $\mathbf{q}_1 = (-1.5,-1)$, $\mathbf{q}_2 = (0,0)$, $\mathbf{q}_3 = (0.5,0.5)$, $\mathbf{q}_4 = (2,1)$, $\mathbf{q}_5 = (3,0.5)$, $\mathbf{q}_6 = (4,0)$, $\mathbf{q}_7 = (5,0.5)$, and $\mathbf{q}_8 = (6,1)$.

(i) Determine the four closest of the given points on curve Q for the four given points on curve P w.r.t. Euclidean distance (graphical solution accepted).

(ii) With these point correspondences, compute the translation $\mathbf{t} = (t_x, t_y)$ of curve Q that makes the average square distance of the corresponding points minimal when no additional rotation is used.

(iii) Describe a procedure to compute an estimate of the translation between two point clouds which you want to register.

0.5+1+0.5

(i) By drawing the points, we can easily get the closest points for $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ are $\mathbf{q}_3$, $\mathbf{q}_4$, $\mathbf{q}_6$, and $\mathbf{q}_8$, respectively.

(ii) The translation of a point of $\mathbf{q}_i$ is $\mathbf{q}'_i = \mathbf{q}_i + \mathbf{t}$. The distance is given by

$$d(t_x, t_y) = \sum_{i=1}^{4} \left\| p_i - \mathbf{q}'_i \right\|^2 = ... = 4t_x^2 + 4t_y^2 + t_x - 7t_y + \frac{7}{2}$$

Then, finding the zeros of the partial derivatives

$$\frac{\partial d}{\partial t_x} = 8t_x + 1 \stackrel{!}{=} 0, \quad \frac{\partial d}{\partial t_y} = 8t_y - 7 \stackrel{!}{=} 0$$

yields $\mathbf{t} = (-\frac{1}{8}, \frac{7}{8})$.

(iii) Procrustes analysis for translation: compute the mean coordinates of both clouds and subtract them.

Total: 16