Diagnostic Medical Image Processing
Prof. Dr.-Ing. Andreas Maier
Exercises (DMIP-E)
WS 2016/17

LEHRSTUHL FÜR MUSTER-ERKENNUNG

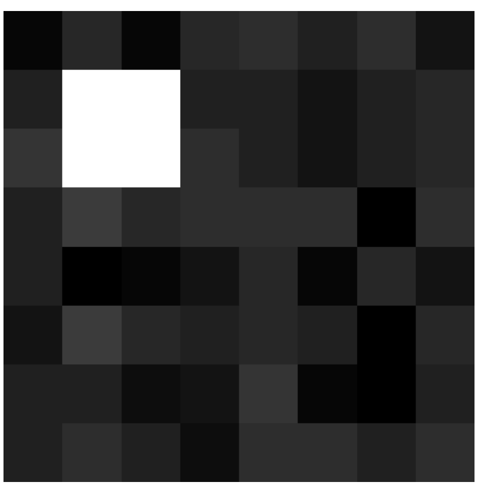
MR Intensity Inhomogeneities

Frank Schebesch, Tobias Würfl, Matthias Utzschneider,
Yixing Huang, Asmaa Khdeir, Houman Mirzaalian

Exercise Sheet 4

11 Histograms and KL Divergence

(i) Originally, the pixels in the following image were distributed by a nice and well-known distribution. Unfortunately, the image got somehow corrupted on the disk and the white area now shows the wrong pixel value (see matrix on the right). Can you guess what value it has originally shown? Can you also support your guess by an argument using histograms?
[The image data can also be found in the file `whatpixelvalue.csv`.]



$$\begin{bmatrix} 2 & 7 & 2 & 7 & 8 & 6 & 8 & 4 \\ 6 & 40 & 40 & 6 & 6 & 4 & 6 & 7 \\ 9 & 40 & 40 & 8 & 6 & 4 & 6 & 7 \\ 6 & 10 & 7 & 8 & 8 & 8 & 1 & 8 \\ 6 & 1 & 2 & 4 & 7 & 2 & 7 & 4 \\ 4 & 10 & 7 & 6 & 7 & 6 & 1 & 7 \\ 6 & 6 & 3 & 4 & 9 & 2 & 1 & 6 \\ 6 & 8 & 6 & 3 & 8 & 8 & 6 & 8 \end{bmatrix}$$

(ii) Write down the definition of the Kullback-Leibler (KL) divergence between two discrete probability density functions p and q. Show its relation to entropy.

(iii) Suppose we have four events a, b, c, d which are distributed as follows:

$$p(a) = \frac{3}{5},\ p(b) = \frac{1}{5},\ p(c) = \frac{1}{5},\ p(d) = 0.$$

From noisy measurements we determined the following relative frequencies:

$$q(a) = \frac{5}{9},\ q(b) = \frac{3}{9},\ \text{and}\ q(d) = \frac{1}{9}.$$

We want to use the KL divergence $\mathrm{KL}(p, q)$ to decide if the measurement approximates the actual distribution. Can you compute it? State the reason if not.
Hint:

$$\lim_{p \to 0} p \log p = 0, \quad \lim_{q \to 0} p \log \frac{p}{q} = \infty,\ p \neq 0$$

(iv) We now apply a smoothing trick for both distributions, i.e., we add a probability of $\epsilon = 10^{-3}$ to those events with zero probability/frequency and distribute the error made to equal parts on the events with nonzero probability/frequency:

$$p_\epsilon(a) = \frac{3}{5} - \frac{\epsilon}{3},\ p_\epsilon(b) = \frac{1}{5} - \frac{\epsilon}{3},\ p_\epsilon(c) = \frac{1}{5} - \frac{\epsilon}{3},\ p_\epsilon(d) = \epsilon,$$
$$q_\epsilon(a) = \frac{5}{9} - \frac{\epsilon}{3},\ q_\epsilon(b) = \frac{3}{9} - \frac{\epsilon}{3},\ q_\epsilon(c) = \epsilon,\ q_\epsilon(d) = \frac{1}{9} - \frac{\epsilon}{3}.$$

Compute the KL divergence $\mathrm{KL}(p_\epsilon, q_\epsilon)$. What is your conclusion?

1.5+1+1+1

(i) Basically, this problem can be solved by generating the histogram of the image intensities. The original distribution is approximately a Gaussian distribution, and there is a gap at 5, and an outlier at 40. There is no 100 % correct solution, but at least the following answers should be considered:

- If we are looking for a value such that the distribution is close to a Gaussian distribution, it is likely that the gap was filled with a value of 5 at the four outlier positions.
- If we argue that we should take the most probable value due to the histrogram of the given data, we could guess the missing value to be 6.

Note: we assume that the error was a constant shift. When the exercise image was generated the actual original value was indeed 5.

(ii)

$$\mathrm{KL}(p, q) = \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{q(x)}\, \mathrm{d}x = -\int_{-\infty}^{\infty} p(x) \log q(x)\, \mathrm{d}x - \left(-\int_{-\infty}^{\infty} p(x) \log p(x)\, \mathrm{d}x \right)$$
$$= H(p, q) - H(p)$$

$H(p, q)$ is called *cross entropy*.

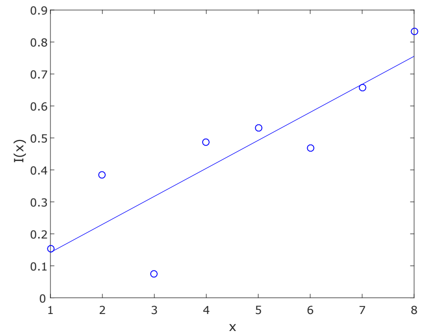


Figure 1: MRI-measurements with polynomial fitting for gain-field correction.

(iii)

$$q(c) = 1 - q(a) + q(b) + q(d) = 0,$$

$$\mathrm{KL}(p,q) = \lim_{\rho\to 0,\varphi\to 0}\left(\frac{3}{5}\log\frac{3\cdot 9}{5\cdot 5} + \frac{1}{5}\log\frac{1\cdot 9}{5\cdot 3} + \frac{1}{5}\log\frac{1}{5\cdot\rho} + \varphi\log\left(9\cdot\varphi\right)\right) = \infty$$

(iv)

$$\begin{aligned}\mathrm{KL}(p_\epsilon, q_\epsilon) &= \frac{9-\epsilon}{15}\log\frac{9(9-\epsilon)}{15(15-\epsilon)} + \frac{3-\epsilon}{15}\log\frac{9(3-\epsilon)}{15(9-\epsilon)} + \frac{3-\epsilon}{15}\log\frac{3-\epsilon}{15\epsilon} + \epsilon\log\frac{9\epsilon}{3-\epsilon}\\ &= \cdots = \frac{9+29\epsilon}{15}\log 3 + \frac{\epsilon-5}{5}\log 5 + \frac{16\epsilon-3}{15}\log\epsilon + \frac{6-17\epsilon}{15}\log(3-\epsilon)\\ &\quad + \frac{2}{5}\log(9-\epsilon) + \frac{\epsilon-9}{15}\log(15-\epsilon) \approx 0.1187\end{aligned}$$

12 Bias Field Correction

(i) What are the three major causes for intensity inhomogeneities in MR imaging?

(ii) MRI Inhomogenities are often modeled by a pixelwise gain field $b_{i,j}$. For b_{ij} different mathematical models can be used, where $g_{i,j}$ are the observed intensities and n_{ij} is additive Gaussian noise. Which of the following models are not commonly used to model MRI inhomogeneities?

- ☐ M_1: $g_{i,j} = f_{i,j}\cdot b_{i,j} + n_{i,j}$
- ■ M_2: $\log(g_{i,j}) = \log(f_{i,j})\cdot\log(b_{i,j})$
- ■ M_3: $g_{i,j} = f_{i,j} + b_{i,j}$
- ■ M_4: $\log(g_{i,j}) = \log(f_{i,j}\cdot b_{i,j} + n_{i,j})$
- ■ M_5: $g_{i,j} = f_{i,j} + n_{i,j}$
- ☐ M_6: $\log(g_{i,j}) = \log(f_{i,j}\cdot b_{i,j})$

(iii) Assume Fig. 1 shows data points of a 1-D MR-image. By a simple polynomial fitting the gain field is estimated by the *blue line*. What numerical problem can arise if the correction of the gain field is applied? How can we account for it, without changing the polynomial fitting?

(iv) Suppose we have the following minimalistic MRI image $\mathbf{g} = (g_{i,j})$ inherently with bias $\mathbf{b} = (b_{i,j})$ and we know the gain field:

$$\mathbf{g} = \begin{pmatrix} 4 & 6 & 9 \\ 9 & 24 & 12 \\ 7 & 15 & 22 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 8 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

If we assume the multiplicative model for the gain field, and the noise to be zero, correct the image matrix by removing the gain field effect.

$\boxed{\mathbf{0.5+1+1+1}}$

(i) non-uniform radio-frequency, inhomogeneity of the static main field, patient motion

(ii) See marks

(iii) The correction of the dataset using the estimation can result in negative intensity values, which cannot be displayed. Adding the overall mean of the dataset will compensate for this problem.
Note: This exercise is not completely consistent with the lecture content. The polynomial fitting method in the lecture was introduced for the bias field, that is in the logarithmic domain. In principal, one can try to estimate the gain/bias field in both domains. However, in the multiplicative model subtraction of the bias is of course only valid in the logarithmic domain.
[If such a task was given in the exam, its formulation and solution would be more straightforward.]

(iv)

$$\begin{pmatrix} 4 & 3 & 3 \\ 9 & 3 & 3 \\ 7 & 15 & 22 \end{pmatrix}$$

13 Fuzzy C-means Clustering

Complete the following partition matrix related to fuzzy c-means:

0.25	1	0.1	0.6	0.75
$\square$	$\square$	0.85	0.2	$\square$
0.25	$\square$	$\square$	0.1	0.1
0.5	$\square$	0	$\square$	0.1

How many clusters does it have? And how many data points?

$\boxed{\mathbf{2}}$

number of clusters is 4, number of data points is 5, each column has to add up to 1

14 Random Sample Consensus – Programming Exercise

Model estimations on noisy data are usually error-prone. Even a small number of outliers will influence the estimation such that the resulting model can produce large errors. The goal is to identify models that minimize the error and ignore outliers.
In the RANSAC algorithm we assume that a model built with a minimum number of data points does not contain outliers. If we imagine the minimum number of points for a line, the generated line will exactly fit through those two points. To consider every point, the model error is evaluated on the whole data set. In a scenario where we expect the majority of the data points to be in a valid range we can use this strategy to find a model fitting the inliers and ignoring the outliers.
We want to complete the gaps in `exercise41.java`. It implements a RANSAC algorithm to fit a line through a point plot. Sample points are assumed to be obtained on a line where outliers occur due to measurement errors.

(i) Compute the correct number of iterations needed to satisfy a given probability that only inliers are picked.

(ii) An error function considering the whole dataset should be implemented. The error should measure how many points lie within a certain range of the estimated line. Implement this function in the following method:
`lineError(SimpleVector line_params, SimpleMatrix points)`.

RANSAC algorithm

1. Determine the minimum number N of data points required to build the model.
2. FOR n iterations DO
   i. Choose randomly N points out of your data to estimate the model.
   ii. Determine the error of the current model using all data points.
3. Choose the model with the lowest error.

1+4

(i) code implementation

(ii) code implementation

15 Unsharp Masking – Programming Exercise

In the lecture several approaches to correct MR-images for the bias field are introduced. In this exercise we have a look at both methods which are referred to as "unsharp masking". Basically, in these methods you want to subtract a low frequency field from the (hopefully) high frequency content that has actual diagnostic value.
Therefore, have a look at the java code in `exercise42.java`.

(i) First, we implement homomorphic unsharp masking. In this method, we attempt to reweight the image by the quotient of the global mean μ and the local means $\mu_{i,j}$, assuming the bias field is approximated by $\frac{\mu_{i,j}}{\mu}$.

(ii) Second, the low frequencies are reduced or cut-off in frequency domain directly. Use the internal Conrad methods to transform the image and try both, the filter given in the lecture and a hard cut-off. For the later, just review the code and find the fitting method.

Use the windowing tool of ImageJ to analyze the corrected images and discuss your results.

2+3

(i) code implementation

(ii) code implementation

Total: 20