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LME

Test Exam

You have 60 minutes for the exam. It contains three sections with 20, 24, and 16 points.

Preprocessing

Question 1: MRI-Inhomogenities

A common artifact in magnet-resonance imaging (MRI) are MRI-inhomogenities. Name three possible reasons for these artifacts and one possible solution.

Solution:

- Sources of MRI-inhomogeneities: Non-uniform radio-frequency pulses, inhomogeneity of the main static (B_0) field, patient motion.
- Possible solutions: Frequency domain filters, homomorphic filtering, homomorphic unsharp masking, Kullback-Leibler divergence minimization, Fuzzy C-means clustering.

4 P.

Question 2: Defect Pixel Interpolation

Defect pixels on detectors can be compensated by defect pixel interpolation in frequency domain. In a 1-D case a signal g can be represented by:

$$g(t) = f(t) \cdot w(t), \tag{1}$$

$f(t)$ is the ideal signal, $g(t)$ is a measured signal with missing pixels, $w(t)$ is a binary mask describing the missing pixels. The corresponding Fourier-transfroms are $G(\xi)$, $W(\xi)$ and $F(\xi)$. For this task we consider the signal $F(\xi)$ only at the two frequencies s and $N - s$:

$$F(\xi) = \widehat{F}(s)\delta(\xi - s) + \widehat{F}(N - s)\delta(\xi - N + s). \tag{2}$$

$\widehat{F}$ denotes an estimate of F and δ is the Dirac-delta function defined by:

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Find an estimator for $\widehat{F}(s)$ to interpolate the corrupted signal g in frequency domain.

Solution:

$$\begin{aligned}
G(s) = F(s) * W(s) &= \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, W(s-k) = \\
&= \frac{1}{N} (\widehat{F}(s)\, W(0) + \widehat{F}(N-s)\, W(s-N+s)) = \\
&= \frac{1}{N} (\widehat{F}(s)\, W(0) + \overline{\widehat{F}}(s)\, W(2s)).
\end{aligned}$$

$\boxed{\textbf{3P. + 3P. + 2P.}}$

$$\begin{aligned}
\widehat{G}(s) &= \frac{1}{N} (\widehat{F}(s)\, W(0) + \overline{\widehat{F}}(s)\, W(2s)) \qquad (I) \\
\overline{\widehat{G}}(s) &= \frac{1}{N} (\overline{\widehat{F}}(s)\, \overline{W}(0) + \widehat{F}(s)\, \overline{W}(2s)) \qquad (II) \\
\widehat{F}(s) &= N \left(\frac{\widehat{G}(s)\, \overline{W}(0) - \overline{\widehat{G}}(s)\, W(2s)}{|W(0)|^2 + |W(2s)|^2} \right). \quad (II \; in \; I)
\end{aligned}$$

$\boxed{\textbf{6P.}}$

Image Reconstruction

Question 3: Parallel-Beam Reconstruction

a) Describe shortly two possible alternative analytic parallel-beam reconstruction algorithms besides the "Filtered Backprojection".

Solution:

- Backprojection + 2D Ramp Filter with Fourier Transform
- Derivative + Hilbert Transform + Backprojection
- Backprojection + Derivative + Hilbert Transform
- ...

More examples are possible. See lecture slides (Parallel Beam Reconstruction (3/6)) or G.L. Zheng: "Medical Image Reconstruction: A Conceptual Tutorial", 2009, S.29.

$\boxed{\textbf{2 P.}}$

b) The CT reconstruction algorithm "Filtered Backprojection" consists of a ramp filter $h(s)$ and a backprojection. $h(s)$ is defined by

$$h(s) = \int_{-B}^{B} |\omega| e^{2\pi i\omega s} \, \mathrm{d}s, \tag{4}$$

with the bandwidth $B = \frac{1}{2\tau}$, the frequency ω, and τ the detector spacing. For this task we use a cut-off frequency $B = \frac{1}{2}$. In this case, the filter can be reformulated using the rectangular function $\mathrm{rect}(t)$:

$$h(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega| \, e^{2\pi i\omega s} \mathrm{d}\omega = \int_{-\infty}^{\infty} |\omega| \, \mathrm{rect}\,(\omega) \, e^{2\pi i\omega s} \mathrm{d}\omega \tag{5}$$

where $\mathrm{rect}(t)$ is defined by:

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For $B = \frac{1}{2}$, $|\omega|$ can be rewritten as $|\omega| = \frac{1}{2} - \mathrm{rect}\,(2\omega) * \mathrm{rect}\,(2\omega)$ on $[-\frac{1}{2}, \frac{1}{2}]$. Note that the inverse Fourier transform of $\mathrm{rect}(t)$ is $\mathrm{FT}^{-1}(\mathrm{rect}(C\,t)) = \frac{1}{|C|} \,\mathrm{sinc}(\frac{1}{C}\,t)$ with a constant C and $\mathrm{sinc(t)} = \frac{\sin(\pi t)}{\pi t}$.

Task: Derive the continuous form of the **Ramachandran-Lakshminarayanan** convolver by using the inverse Fourier transform.

Solution:

$$
\begin{aligned}
h(s) &= \mathrm{FT}^{-1}\left(\left(\frac{1}{2} - \operatorname{rect}(2\omega)\operatorname{rect}(2\omega)\right)\operatorname{rect}(\omega)\right) \\
&= \mathrm{FT}^{-1}\left(\frac{1}{2}\operatorname{rect}(\omega)\right) - \mathrm{FT}^{-1}\left(\underbrace{(\operatorname{rect}(2\omega)\operatorname{rect}(2\omega))}_{\text{support on } [-\frac{1}{2},\frac{1}{2}]}\ \underbrace{\operatorname{rect}(\omega)}_{=1 \text{ on } [-\frac{1}{2},\frac{1}{2}]}\right) \\
&= \mathrm{FT}^{-1}\left(\frac{1}{2}\operatorname{rect}(\omega)\right) - \mathrm{FT}^{-1}(\operatorname{rect}(2\omega))\,\mathrm{FT}^{-1}(\operatorname{rect}(2\omega)) \\
&= \frac{1}{2}\operatorname{sinc}(s) - \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}s\right)\frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}s\right) \\
&= \frac{1}{2}\operatorname{sinc}(s) - \frac{1}{4}\operatorname{sinc}^2\left(\frac{1}{2}s\right).
\end{aligned}
$$

$\boxed{\textbf{4P. + 4P. + 4P.}}$

c) Use your result to derive the discrete form of the convolver. [Alternatively, use the following substitutional result: $h(s) = 2B^2\operatorname{sinc}(2Bs) - B^2\operatorname{sinc}^2(Bs)$, where $B = \frac{1}{2\tau}$.]

Solution:

$$
\begin{aligned}
h(s) &= \frac{1}{2}\operatorname{sinc}(s) - \frac{1}{4}\operatorname{sinc}^2\left(\frac{1}{2}s\right) \\
&= \frac{1}{2}\frac{\sin(\pi s)}{\pi s} - \frac{1}{4}\left(\frac{\sin(\frac{\pi s}{2})}{\frac{\pi s}{2}}\right)^2.
\end{aligned}
$$

With $s = n, n \in \mathbb{Z}$, the discrete filter in spatial domain is defined by:

$$
h(n) = \begin{cases} \frac{1}{4} & n = 0, \\ 0 & n \text{ even}, \\ -\frac{1}{n^2\pi^2} & n \text{ odd}. \end{cases}
$$

$\boxed{\textbf{3 P. + 3 P.}}$

Question 4: 3-D Reconstruction

a) What does Orlov's condition state for trajectories for 3-D CT reconstruction?

Solution:

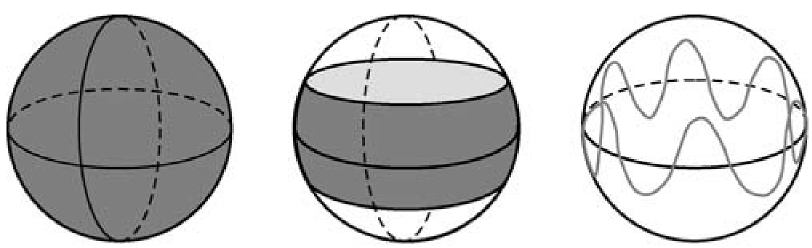
A complete data set can be obtained if every great circle intersects the trajectory of the unit vector θ, which is the direction of the parallel rays (see G.L. Zheng: "Medical Image Reconstruction: A Conceptual Tutorial", 2009, S.89).

2 P.

b) Draw two different trajectories around the unit sphere, which fulfill Orlov's condition.

Solution:

Three examples for trajectories fulfilling Orlov's condition (see G.L. Zheng: "Medical Image Reconstruction: A Conceptual Tutorial", 2009, S.90):



2 P.

Rigid Registration

Question 5: Quaternions

Marker positions $\mathbf{q}_k \in \mathbb{R}^3, k = 1, 2, ..., N$, are observed on a rendered view of a 3-D CT image I_1, and on a 3-D PET image I_2 markers are observed at the positions $\mathbf{p}_k \in \mathbb{R}^3$. The rotation $\mathbf{R} \in \mathbb{R}^3$ of all markers occurring in the rigid transformation from I_1 to I_2 can be described by a quaternion $\mathbf{r} = w + xi + yj + zk$.

a) Show how an arbitrary point $\mathbf{p} \in \mathbb{R}^3$ is rotated by a quaternion $\mathbf{r}_\mathrm{s}$.

Solution:

$$\begin{aligned} \mathbf{p}' &= (0, p_x, p_y, p_z), \\ \mathbf{p}'_{\mathrm{rot}} &= \mathbf{r} \cdot \mathbf{p}' \cdot \overline{\mathbf{r}}\,. \end{aligned}$$

2 P. + 2 P.

b) Show a way to estimate the rotation quaternion $\mathbf{r}$ such that the estimation is linear in the components of $\mathbf{r}$.

Solution:

$$\forall i: \; \mathbf{p}'_{\mathrm{rot,i}} = \mathbf{r} \cdot \mathbf{p}'_i \cdot \overline{\mathbf{r}}$$

$$\widehat{\mathbf{r}} = \arg\min_{\mathbf{r}} \sum_{i=0}^{n} ||\mathbf{p}'_{\mathrm{rot},i} - \mathbf{r} \cdot \mathbf{p_i}' \cdot \overline{\mathbf{r}}||$$

$$\widehat{\mathbf{r}} = \arg\min_{\mathbf{r}} \sum_{i=0}^{n} ||\mathbf{p}'_{\mathrm{rot},i} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{p_i}'||$$

$$\sum_{i=0}^{n} ||\mathbf{p}'_{\mathrm{rot},i} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{p_i}'|| \rightarrow \min\,.$$

12 P.