Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – ML-EM Algorithms

Online Course – Unit 59

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

virtuelle hochschule bayern

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Maximum-Likelihood Expectation-Maximization Methods

Summary

- Take Home Messages
- Further Readings

Maximum Likelihood (ML) Methods

Idea:

- Formulate the objective function as a likelihood function.
- Find the optimum of the likelihood function, i. e., the most likely solution.

Commonly solved with the “**Expectation Maximization Algorithm**” (EM).

ML-EM Reconstruction: Poisson Distribution

- In the following we consider an *emission tomography* problem.
- The probability density function for the random variable r, that describes the emission of a certain amount of energy, follows the Poisson distribution:

$$P(r|\lambda) = e^{-\lambda} \frac{\lambda^r}{r!}.$$

- The expected value of this random variable is λ.
- The observed value p_i at each detector bin i is

$$p_i = \sum_j c_{ij},$$

  where each c_{ij} is a random variable distributed by the Poisson distribution.
- Therefore, we have $\lambda_{ij} = E(c_{ij}) = a_{ij} x_j$.

ML-EM Reconstruction: Objective Function

We set the following likelihood function:

$$L = \prod_{i,j} P(c_{ij}|\lambda_{ij}) = \prod_{i,j} e^{-\lambda_{ij}} \frac{{\lambda_{ij}}^{c_{ij}}}{c_{ij}!} = \prod_{i,j} e^{-a_{ij}x_j} \frac{(a_{ij}x_j)^{c_{ij}}}{c_{ij}!}.$$

Taking the logarithm yields:

$$\ln(L) = \sum_{i,j} (c_{ij} \ln(a_{ij}x_j) - a_{ij}x_j) - \sum_{i,j} \ln(c_{ij}!).$$

Note that $\sum_{i,j} \ln(c_{ij}!)$ is independent of x_j. Hence, it is valid to optimize with

$$\ln(L) = \sum_{i,j} c_{ij} \ln(a_{ij}x_j) - a_{ij}x_j.$$

ML-EM Reconstruction: Expectation Maximization

Compute the expected value of c_{ij}:

$$E(c_{ij}|p_i, \boldsymbol{X}^k) = \frac{a_{ij}x_j^k}{\sum_l a_{il}x_l^k} p_i.$$

Set c_{ij} to its expected value (E-step):

$$E(L|p_i, \boldsymbol{X}^k) = \sum_{i,j} \left(\frac{a_{ij}x_j^k}{\sum_l a_{il}x_l^k} p_i \ln(a_{ij}x_j) - a_{ij}x_j \right).$$

Maximize the expected value of the objective function (M-step):

$$\frac{\partial E(L|p_i, \boldsymbol{X}^k)}{\partial x_j} = 0.$$

ML-EM Reconstruction: Expectation Maximization

Compute the derivative of $E(L|p_i, \boldsymbol{X}^k)$:

$$\frac{\partial E(L|p_i, \boldsymbol{X}^k)}{\partial x_j} = \sum_i \left(\frac{a_{ij} x_j^k}{\sum_l a_{il} x_l^k} p_i \frac{a_{ij}}{a_{ij} x_j} - a_{ij} \right) = \frac{1}{x_j} \sum_i \frac{a_{ij} x_j^k}{\sum_l a_{il} x_l^k} p_i - \sum_i a_{ij} \stackrel{!}{=} 0.$$

Solving for x_j yields the ML-EM update rule:

$$x_j^{k+1} = \frac{x_j^k}{\sum_i a_{ij}} \sum_i a_{ij} \frac{p_i}{\sum_l a_{il} x_l^k}.$$

This can be interpreted as follows:

$$x_j^{k+1} = \frac{x_j^k}{\text{backproject}(1)} \text{backproject} \left(\frac{p_i}{\text{project}(x_l^k)} \right).$$

Topics

Maximum-Likelihood Expectation-Maximization Methods

Summary

Take Home Messages

Further Readings

Take Home Messages

- The ML-EM algorithm is based on likelihoods and the maximization of the expected value of the objective function.
- We have learned how emission events are modeled by random variables.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. “Angenäherte Auflösung von Systemen linearer Gleichungen”. In: *Bulletin International de l’Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. “Adaptive Iterative Reconstruction”. In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953