Medical Image Processing for Interventional Applications

Epipolar Geometry

Online Course – Unit 30
Andreas Maier, Joachim Hornegger, Frank Schebesch
Pattern Recognition Lab (CS 5)

Topics

Magnetic Navigation

Epipolar Geometry

Summary

Take Home Messages

Further Readings

Motivation

- So far we have learned a lot about image preprocessing in this course.
- As seen in the last unit, multiple images provide additional information that can be merged to enhance the diagnostic value.
- Dependent on the used modality, this information could also be acquisition geometry or 3-D-structure.
- The following units will give an introduction into the topic of multiple image processing.

Magnetic Navigation: The Idea

Figure 1: The Stereotaxis Niobe® magnetic navigation system (left, taken from Stereotaxis webpage) / A C-arm for cardiac procedures is combined with magnets to adjust the orientation of the catheter tip (right, image courtesy of Siemens Medical Solutions).

Problem Setting

In this section we will **not** introduce the physics of magnetic navigation.
The problem we will focus on is as follows:

How can we provide a simple user interface to define the direction of the magnetic field?

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Solution:

- Use two projections from different viewpoints.
- Draw the orientation of the magnetic field in both projections (this should be done by the treating physician).
- Compute both the 3-D points from projections and the transform of the C-arm between two image acquisitions from different viewpoints.

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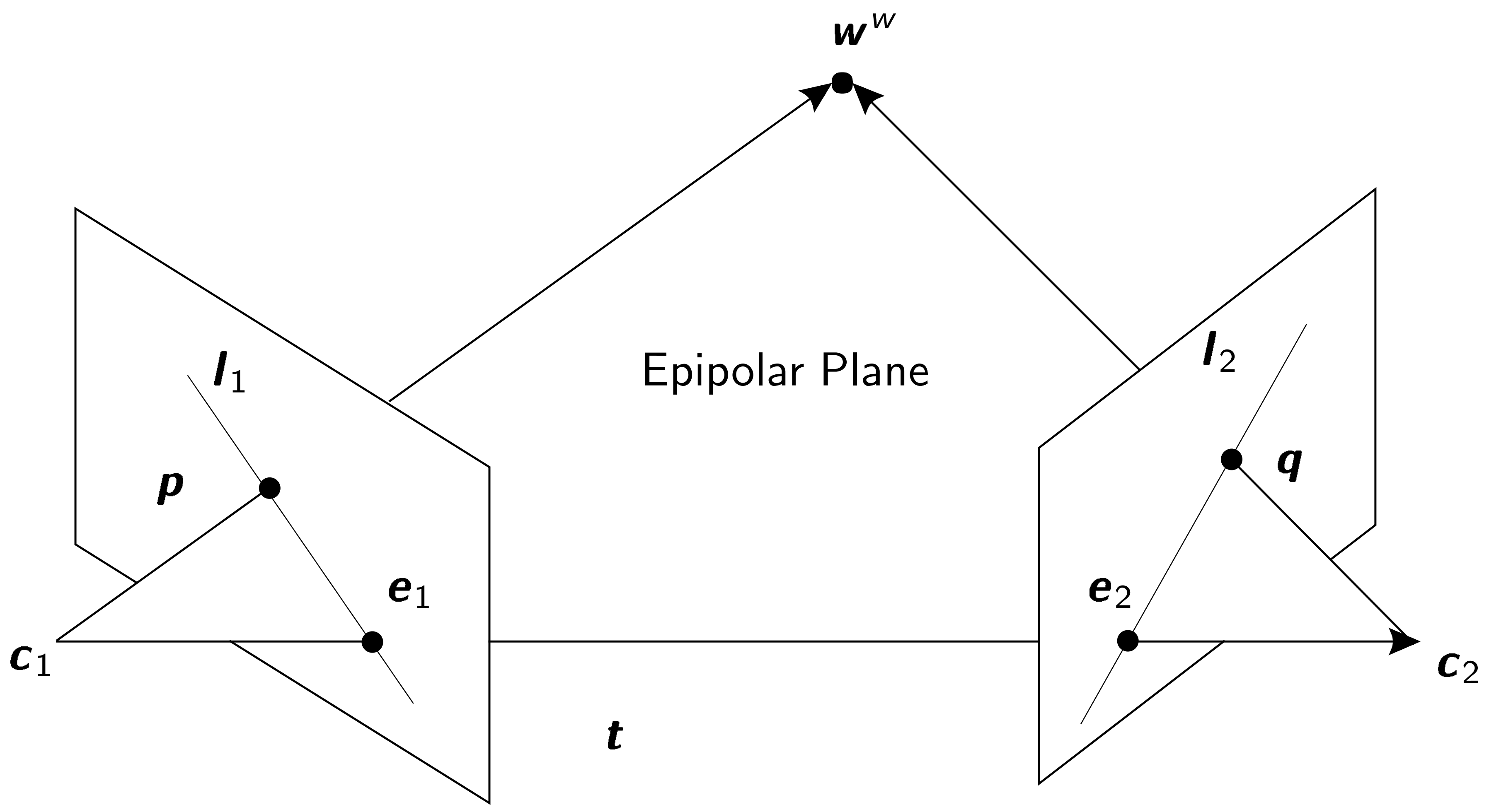
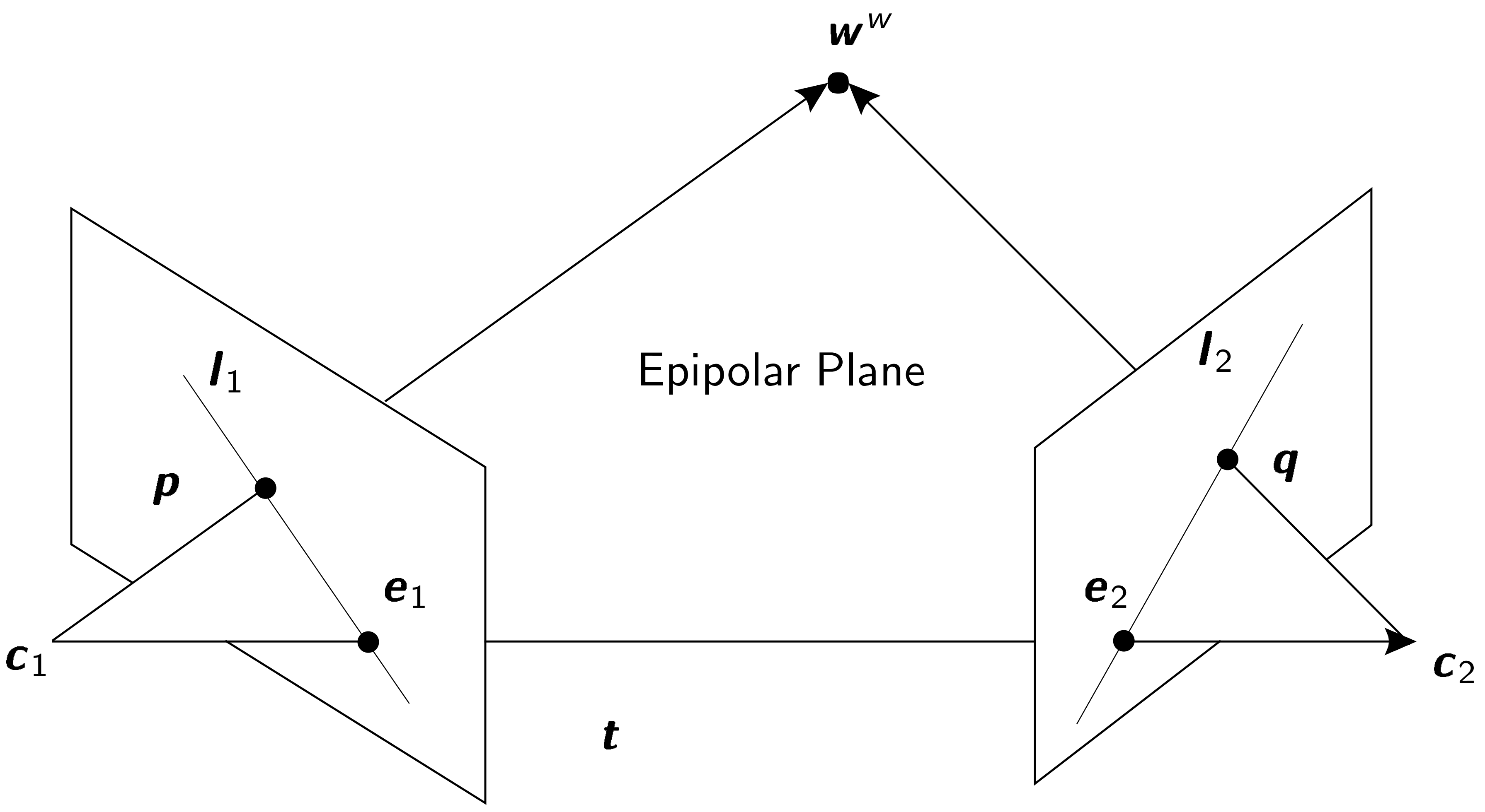


Figure 2: Epipolar geometry of two images

The involved variables denote the following:

- two cameras defined by their optical centers $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathbb{R}^3$,

Epipolar Geometry



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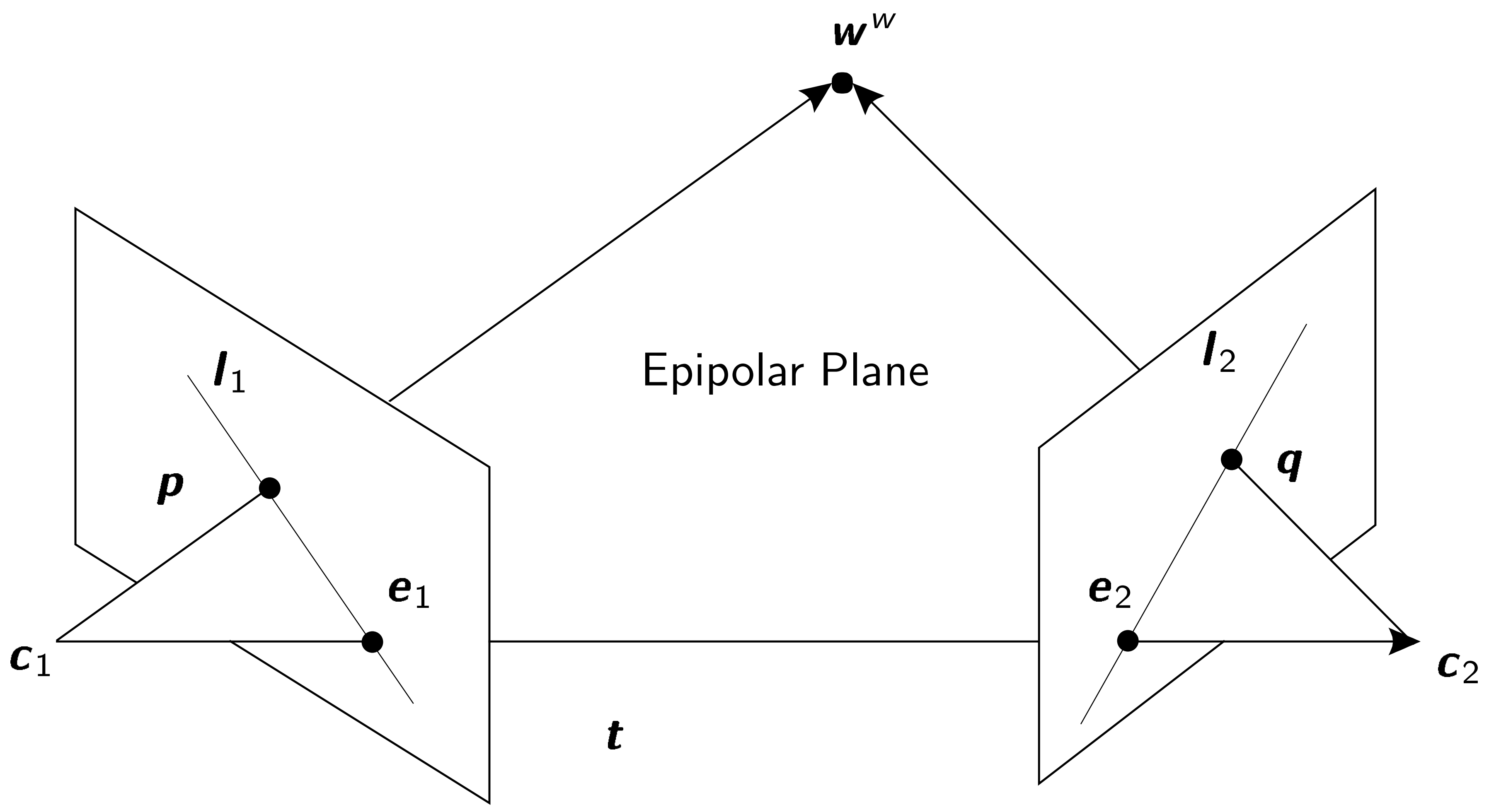
Epipolar Geometry



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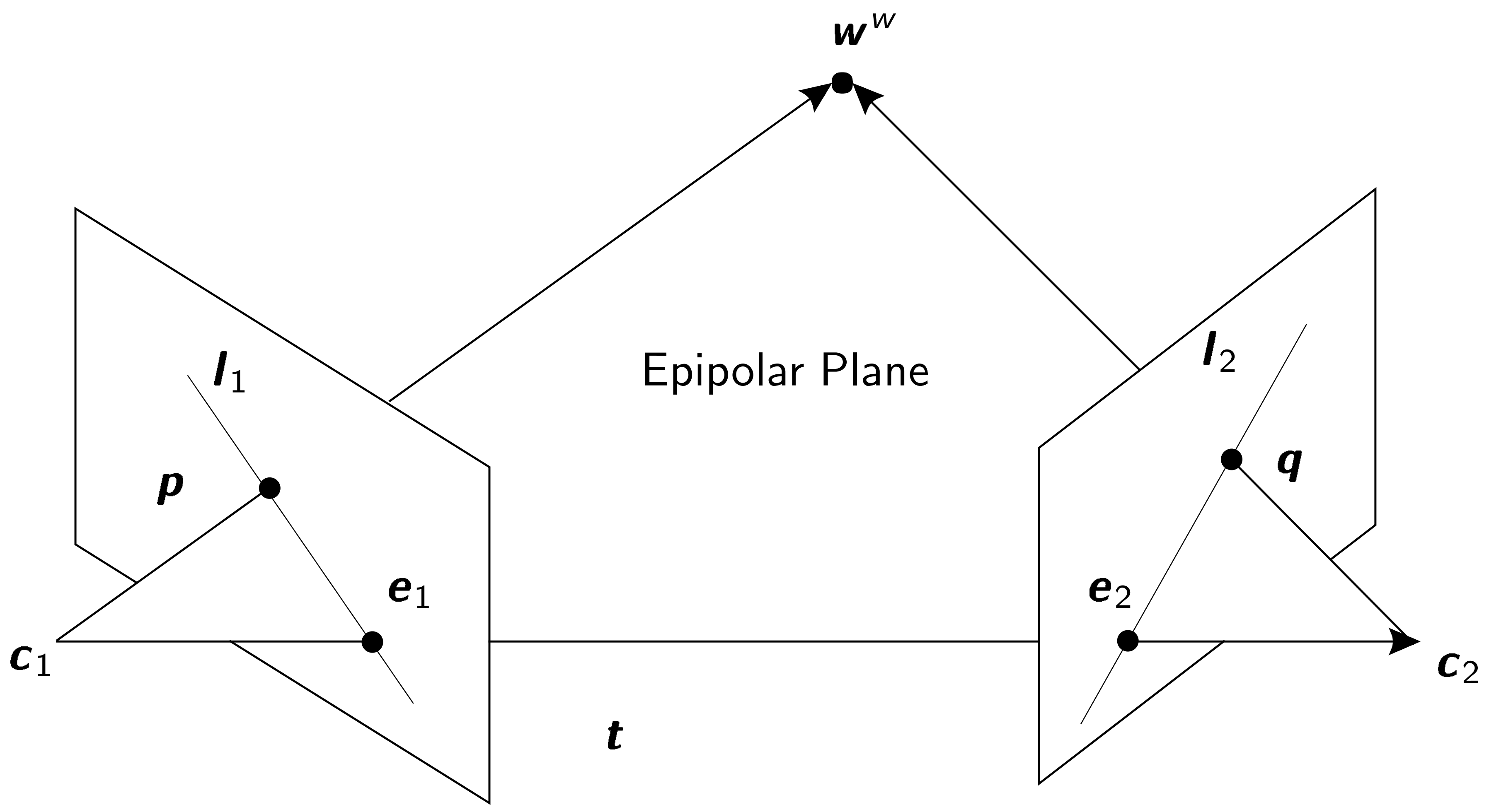
Epipolar Geometry



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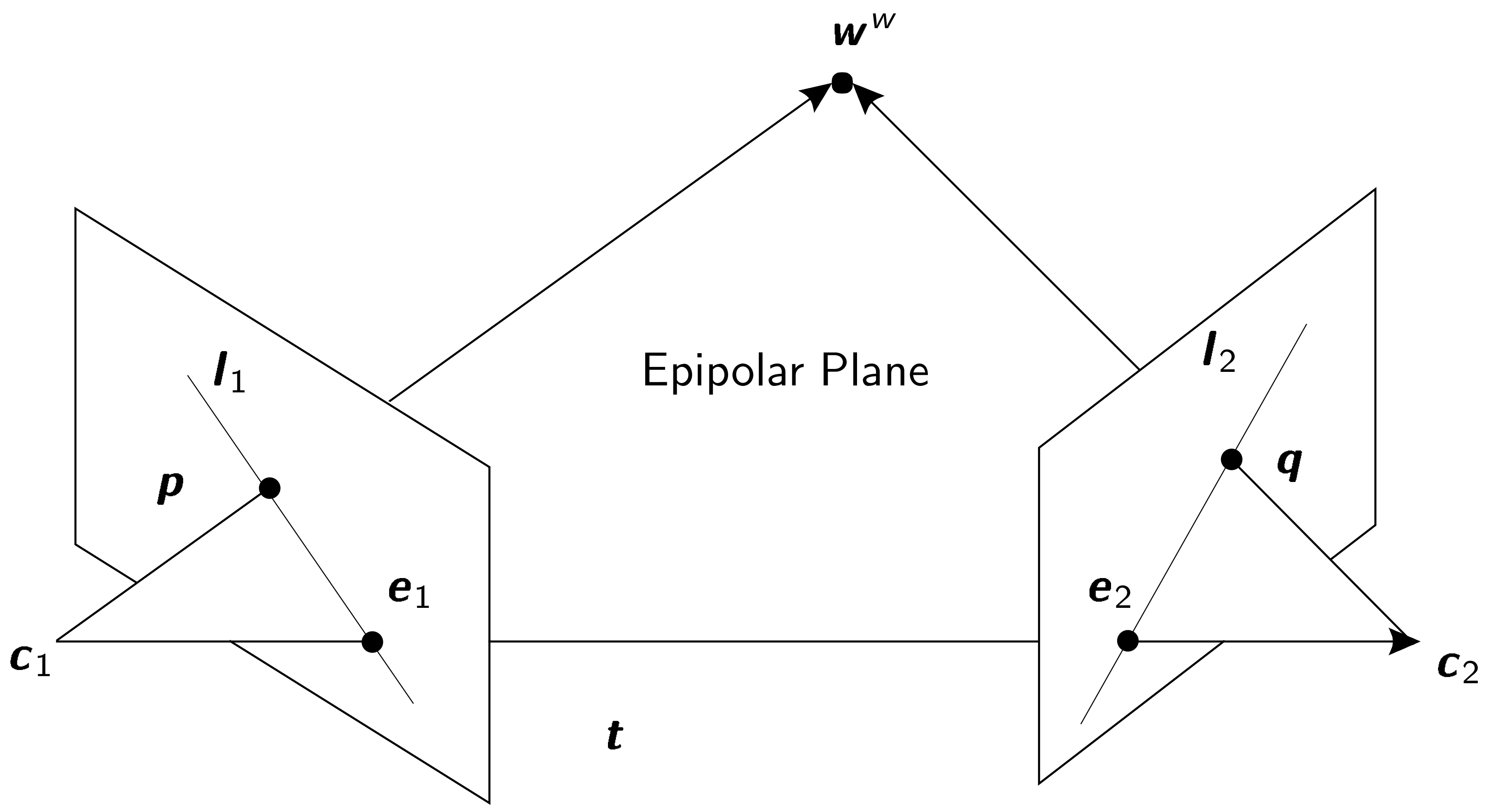

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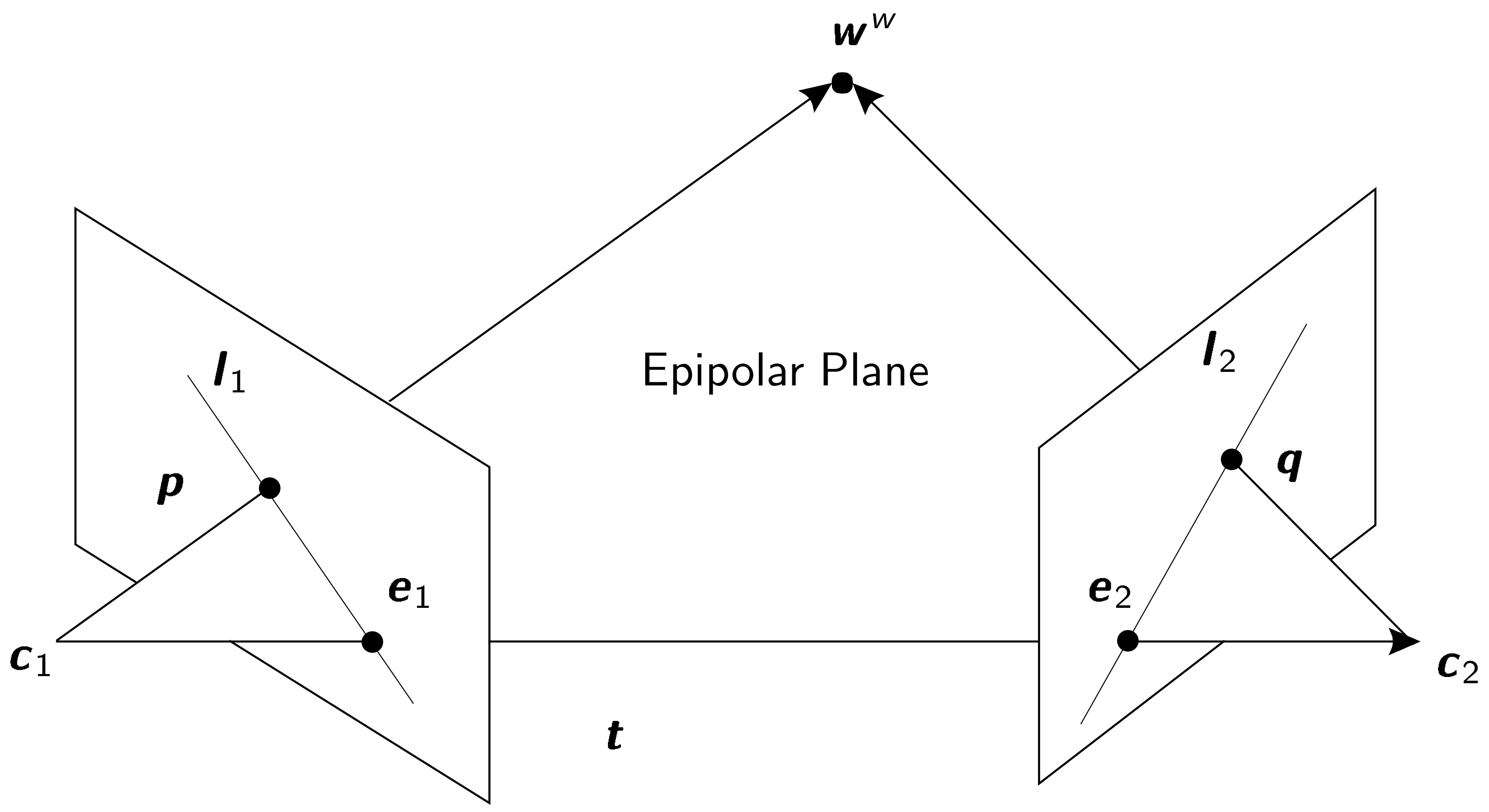

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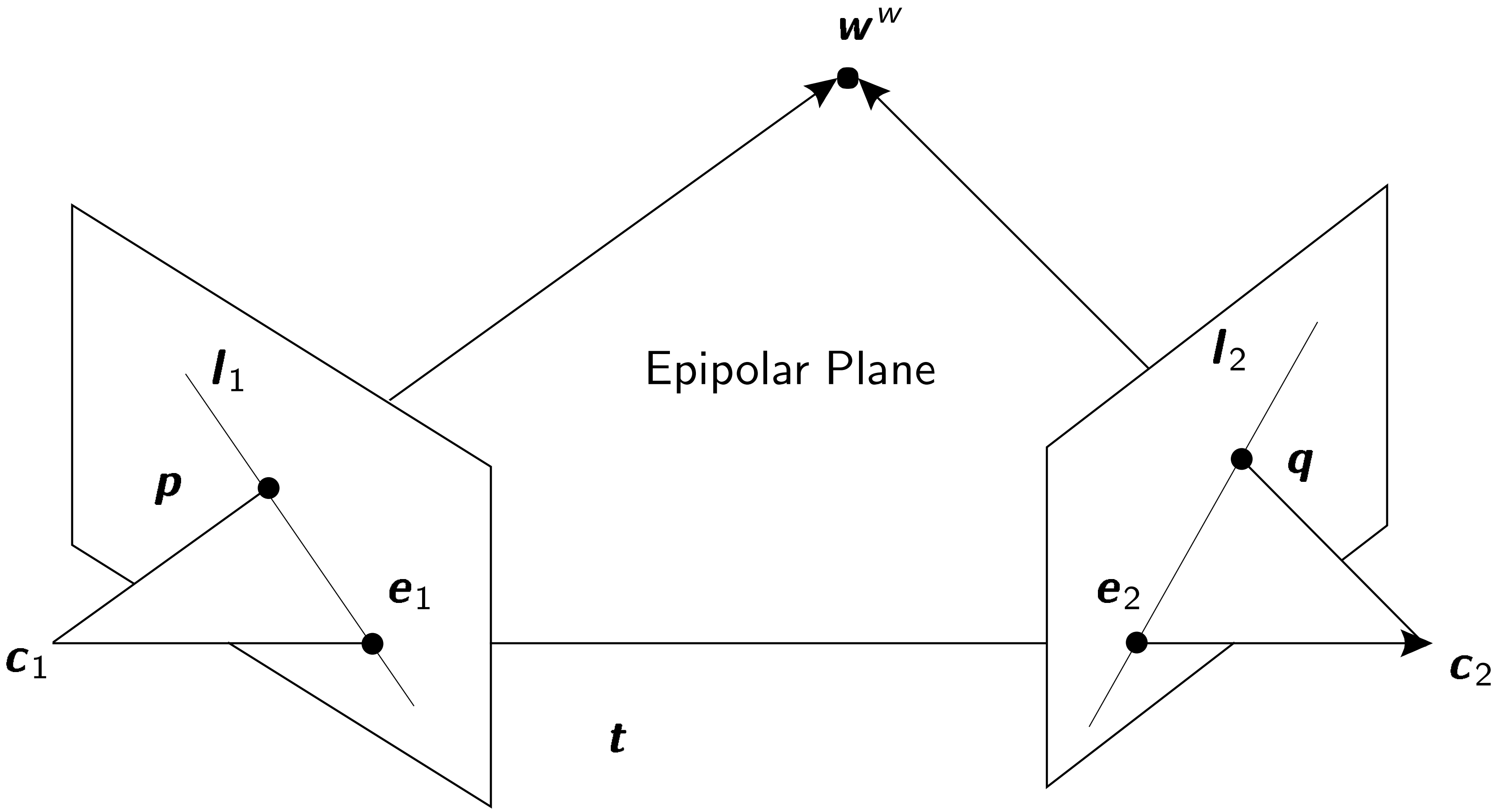

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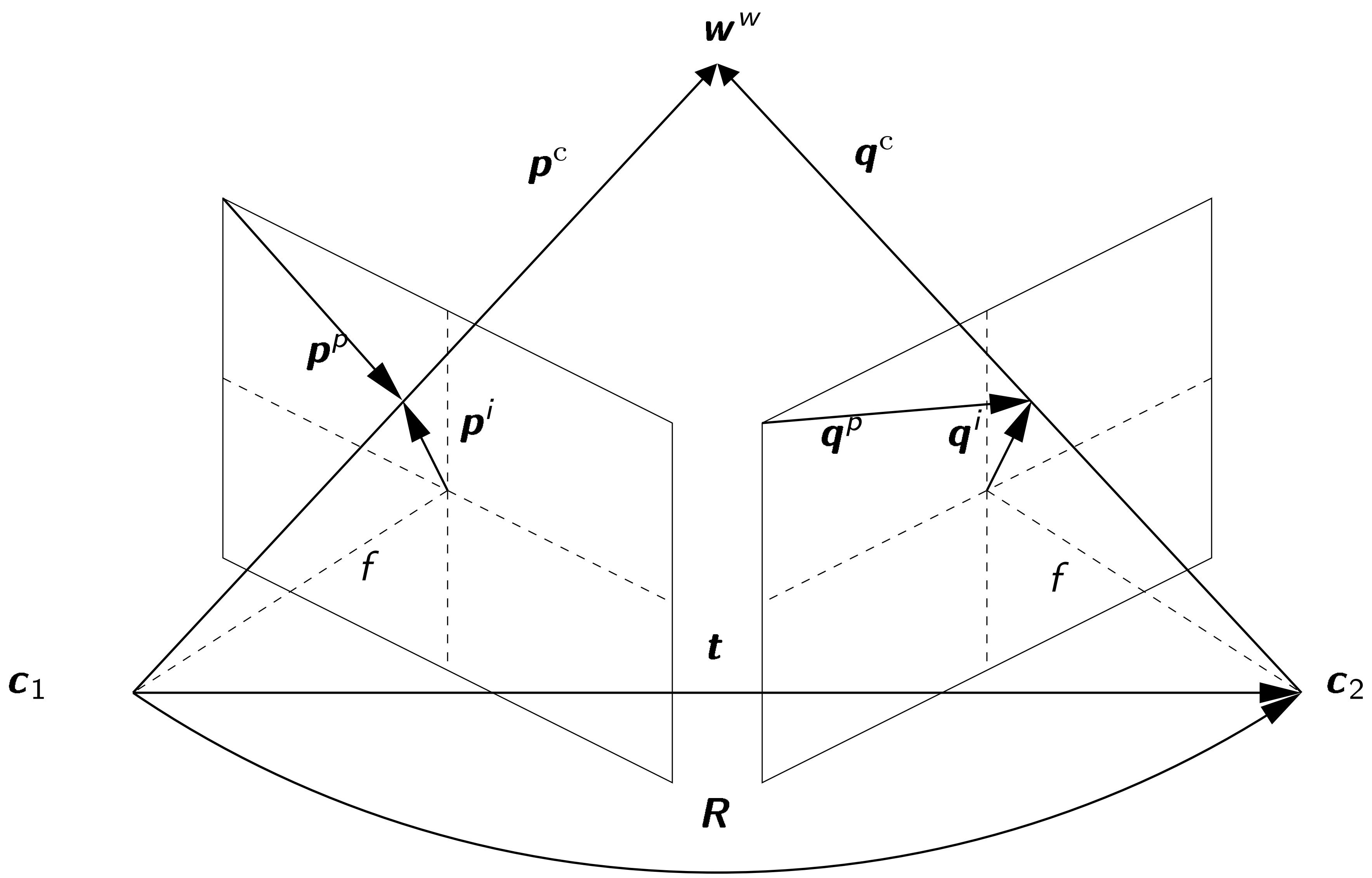
- ***epipolar lines*** are denoted by $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

Epipolar Geometry



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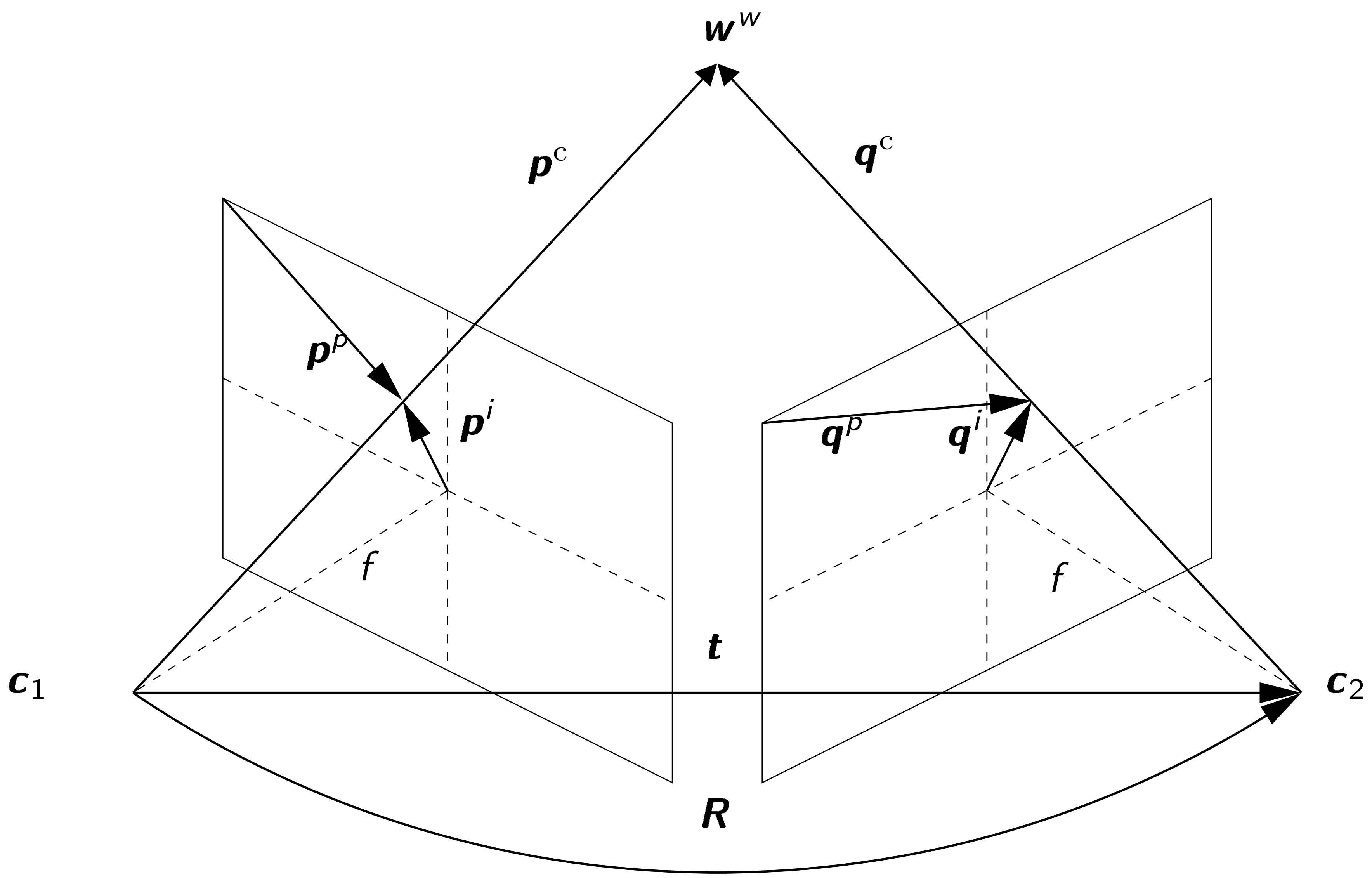

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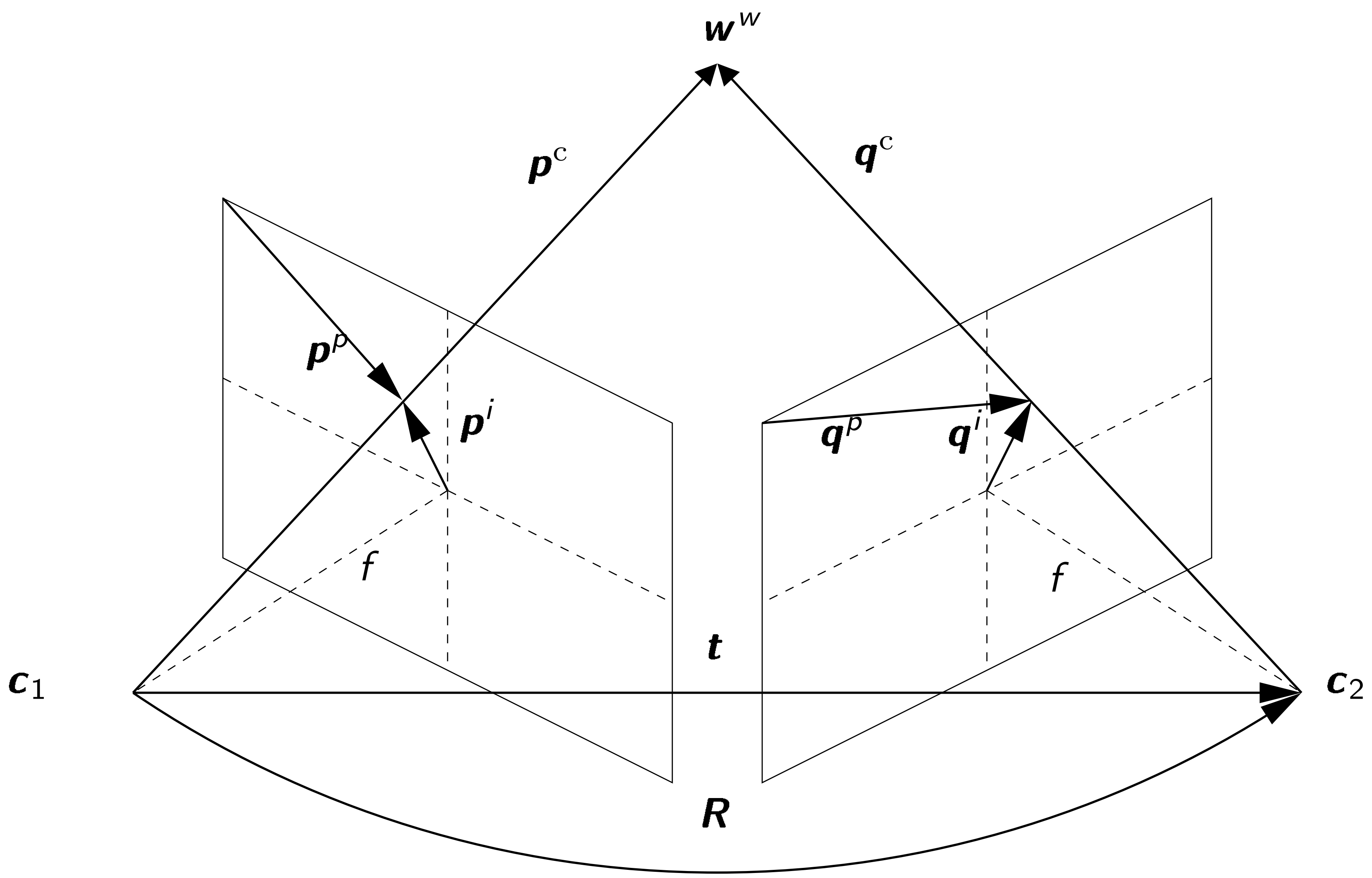

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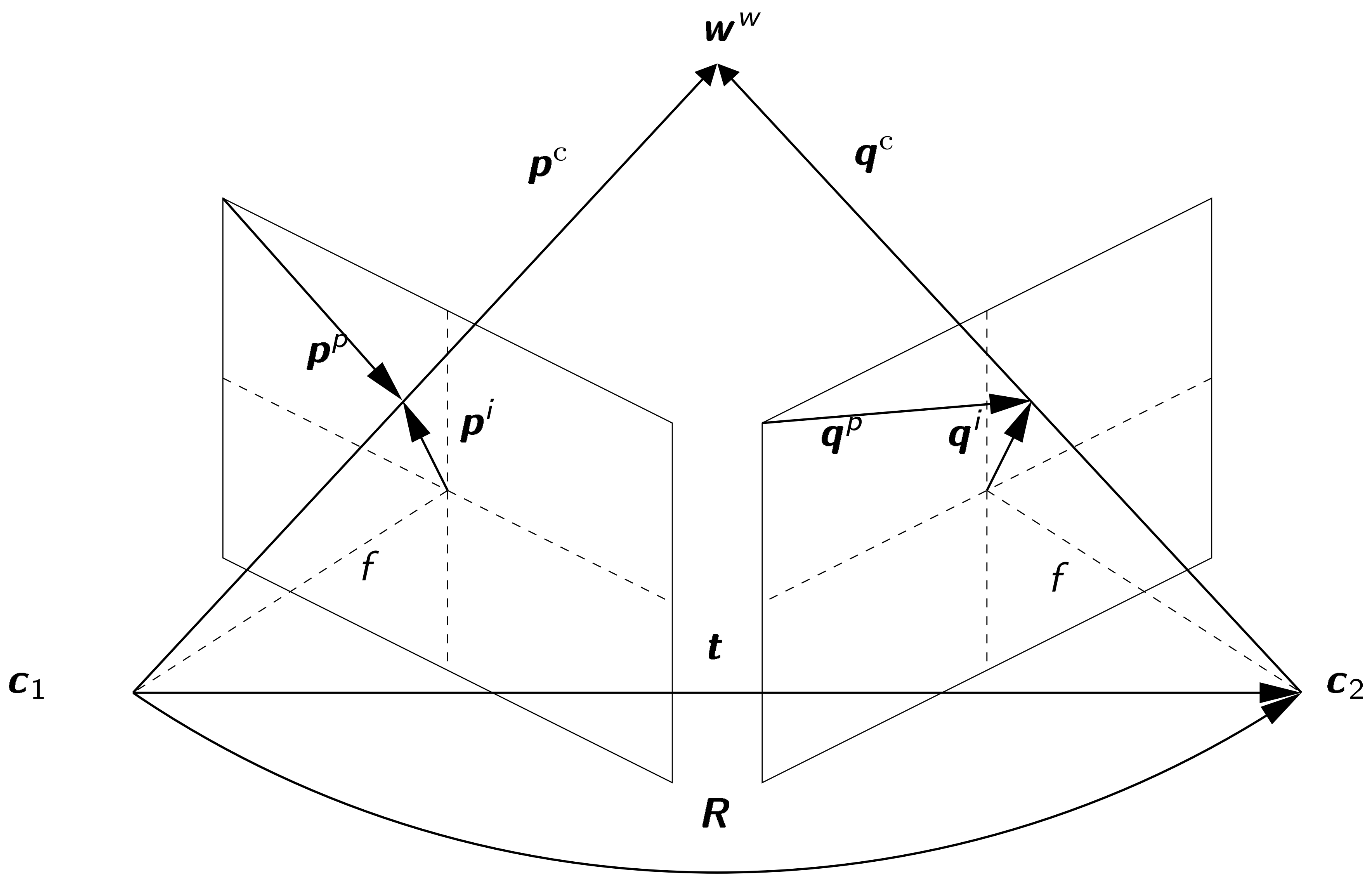

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- observed 2-D points in the image planes can be expressed in terms of image coordinates, where the *origin* of the coordinate system *coincides with the central projection ray*: $\boldsymbol{p}^i, \boldsymbol{q}^i \in \mathbb{R}^2$.

Epipolar Constraint

We observe the following relationships:

- $\boldsymbol{q}^{\mathrm{c}} = \boldsymbol{R}\left(\boldsymbol{p}^{\mathrm{c}} - \boldsymbol{t}\right) \quad \Rightarrow \quad \boldsymbol{R}^{\top}\boldsymbol{q}^{\mathrm{c}} = \boldsymbol{p}^{\mathrm{c}} - \boldsymbol{t}$

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- $\boldsymbol{p}^{\mathrm{c}}$, $\boldsymbol{p}^{\mathrm{c}} - \boldsymbol{t}$ and $\boldsymbol{t}$ lie in one plane:

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- The line through both optical centers intersects the image planes in the so-called epipoles $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

Skew Matrix and Projections

- The cross product is linear in the components of each vector and can be rewritten in matrix notation:

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$$[\boldsymbol{t}]_\times = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}$$

(please check as an exercise).

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- The projection $\tilde{\boldsymbol{q}}^\mathrm{i}$ of $\boldsymbol{q}^\mathrm{c}$ to the image plane is computed by

$$\tilde{\boldsymbol{q}}^\mathrm{i} = \boldsymbol{P}_\mathrm{per} \tilde{\boldsymbol{q}}^\mathrm{c} \quad ,$$

  where $\boldsymbol{P}_\mathrm{per}$ denotes the 3×4 projection matrix and $\tilde{\boldsymbol{q}}^\mathrm{c}$ is the homogeneous point corresponding to $\boldsymbol{q}^\mathrm{c}$.

Epipolar Constraint and Essential Matrix

This is the so-called ***epipolar constraint*** (Longuet-Higgins, 1982):

$$(\tilde{\boldsymbol{q}}^{\mathrm{i}})^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \tilde{\boldsymbol{p}}^{\mathrm{i}} = 0,$$

which is valid for normalized coordinates, i. e., those resulting from $f = 1$.

The matrix $\boldsymbol{E}$ is called ***essential matrix***.

Topics

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Further Readings

Take Home Messages

- We started considering scenes in 3-D space.
- In epipolar geometry, we look at the mapping from a point on one image plane to its corresponding point on another image plane.
- The essential matrix as part of the epipolar constraint depends on the relative position and orientation of both cameras. We will learn more about it in the next unit.

Further Readings

Epipolar geometry is nicely introduced in:

Emanuele Trucco and Alessandro Verri. *Introductory Techniques for 3-D Computer Vision*. Upper Saddle River, NJ, USA: Prentice Hall, 1998

All the math regarding epipolar geometry can be found in:

Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685

Magnetic navigation:

Michelle P. Armacost et al. "Accurate and Reproducible Target Navigation with the Stereotaxis Niobe® Magnetic Navigation System". In: *Journal of Cardiovascular Electrophysiology* 18 (Jan. 2007), S26–S31. DOI: 10.1111/j.1540-8167.2007.00708.x

Medical Image Processing for Interventional Applications

Epipolar Constraint and Essential Matrix

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TECHNISCHE FAKULTÄT

Topics

Epipolar Constraint and Essential Matrix

- Properties of the Essential Matrix
- System of Linear Equations
- The Seaman’s Algorithm

Summary

- Take Home Messages
- Further Readings

Epipolar Constraint and Essential Matrix

In the last unit we have introduced the ***epipolar constraint***:

$$(\tilde{\boldsymbol{q}}^{\mathrm{i}})^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \tilde{\boldsymbol{p}}^{\mathrm{i}} = 0,$$

for normalized coordinates ($f = 1$) and with the essential matrix $\boldsymbol{E}$.

Now we want to discuss some of its properties.

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- If we inspect the matrix

$$\boldsymbol{E}^{\mathsf{T}}\boldsymbol{E} = \left(\boldsymbol{R}[\boldsymbol{t}]_\times\right)^{\mathsf{T}}\boldsymbol{R}[\boldsymbol{t}]_\times = [\boldsymbol{t}]_\times{}^{\mathsf{T}}[\boldsymbol{t}]_\times,$$

  we recognize that $\boldsymbol{E}^{\mathsf{T}}\boldsymbol{E}$ is independent from $\boldsymbol{R}$.

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- $\boldsymbol{E}$ has 5 degrees of freedom (DOF).
- Two nonzero singular values are identical.
- It is $\boldsymbol{l}_2 \boldsymbol{E} \tilde{\boldsymbol{p}}^i = 0$ and $(\tilde{\boldsymbol{q}}^i)^T \boldsymbol{E} \boldsymbol{l}_1^T = 0$

System of Linear Equations

For all points $(\tilde{\boldsymbol{q}}^{\mathrm{i}}_k, \tilde{\boldsymbol{p}}^{\mathrm{i}}_k)$, $k = 1, 2, \ldots, N$, we have

$$\left(\tilde{\boldsymbol{q}}^{\mathrm{i}}_k\right)^{\mathrm{T}} \cdot \boldsymbol{E} \cdot \tilde{\boldsymbol{p}}^{\mathrm{i}}_k = 0.$$

These equations are **linear** in the unknowns, i. e., the components of $\boldsymbol{E}$:

$$\left(\tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,1}, \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,2}, \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,3}\right) \begin{pmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,1} \\ \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,2} \\ \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,3} \end{pmatrix} = 0.$$

For the k-th equation we get:

$$\left(\tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,1} e_{1,1} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,2} e_{2,1} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,3} e_{3,1}\right) \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,1} + \left(\tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,1} e_{1,2} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,2} e_{2,2} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,3} e_{3,2}\right) \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,2} + \left(\tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,1} e_{1,3} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,2} e_{2,3} + \tilde{\boldsymbol{q}}^{\mathrm{i}}_{k,3} e_{3,3}\right) \tilde{\boldsymbol{p}}^{\mathrm{i}}_{k,3} = 0.$$

Measurement Matrix

$$\boldsymbol{Me} = \begin{pmatrix} \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,3} \\ \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,3} \\ \vdots & & \ddots & & & \vdots \\ \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,3} \end{pmatrix} \begin{pmatrix} e_{1,1} \\ e_{1,2} \\ e_{1,3} \\ e_{2,1} \\ e_{2,2} \\ e_{2,3} \\ e_{3,1} \\ e_{3,2} \\ e_{3,3} \end{pmatrix}$$

The Seaman's Algorithm

read point correspondences $\{(\tilde{\boldsymbol{p}}_i^{\mathrm{i}}, \tilde{\boldsymbol{q}}_i^{\mathrm{i}}), i = 1, \ldots, N\}$
rewrite the epipolar constraints in terms of linear equations: $\boldsymbol{Me} = 0$
compute the SVD of $\boldsymbol{M} = \boldsymbol{U\Sigma V}^{\mathrm{T}}$
return last column of $\boldsymbol{V}$

Figure 1: Seaman's algorithm for $\boldsymbol{E}$

Topics

Take Home Messages

- We can compute the essential matrix by solving a linear system of equations.
- One possibility to implement this is by using Seaman's algorithm.

Further Readings

Epipolar geometry is nicely introduced in:

Emanuele Trucco and Alessandro Verri. *Introductory Techniques for 3-D Computer Vision*. Upper Saddle River, NJ, USA: Prentice Hall, 1998

All the math regarding epipolar geometry can be found in:

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Medical Image Processing for Interventional Applications

Eight Point Algorithm

Online Course – Unit 32
Andreas Maier, Joachim Hornegger, Frank Schebesch
Pattern Recognition Lab (CS 5)

Topics

The Eight Point Algorithm

read point correspondences $\{(\tilde{\boldsymbol{p}}_k^{\mathrm{i}}, \tilde{\boldsymbol{q}}_k^{\mathrm{i}}), k = 1, \dots, N\}$

rewrite the epipolar constraints in terms of linear equations: $\boldsymbol{Me} = 0$

compute the SVD of $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$

$\sigma_9 > \varepsilon$ ($\boldsymbol{M}$ has full rank)

- `true`: stop (epipolar constraint not fulfilled)
  - $\varnothing$
- `false`:
  - nullspace of $\boldsymbol{M}$ defines the components of the essential matrix $\boldsymbol{E}$
  - compute the SVD of $\boldsymbol{E} = \boldsymbol{U}'\boldsymbol{\Sigma}'\boldsymbol{V}'^{\mathrm{T}}$
  - $\sigma_3 > \varepsilon$ ($\boldsymbol{E}$ has rank 3)
    - `true`: stop (essential matrix violates rank criterion)
      - $\varnothing$
    - `false`: $\sigma_1 - \sigma_2 > \varepsilon$
      - `true`: stop (essential matrix has different singular values σ_1, σ_2)
      - `false`: return $\boldsymbol{E} = \boldsymbol{U}' \begin{pmatrix} \frac{\sigma_1+\sigma_2}{2} & 0 & 0 \\ 0 & \frac{\sigma_1+\sigma_2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{V}'^{\mathrm{T}}$

Figure 1: Eight point algorithm for $\boldsymbol{E}$

Input Data

For now we ...

... have two images of a patient with one camera,

... use point features,

... assume the correspondence problem to be already solved:

$$\{(\tilde{\boldsymbol{p}}_k^{\mathrm{i}}, \tilde{\boldsymbol{q}}_k^{\mathrm{i}}), k = 1, \ldots, N\},$$

i. e., point $\tilde{\boldsymbol{p}}_k^{\mathrm{i}}$ in image 1 corresponds to point $\tilde{\boldsymbol{q}}_k^{\mathrm{i}}$ in image 2 [the tilde indicates homogeneous coordinates],

... have the points given as normalized homogeneous image coordinates, i. e., the third component is set to 1,

... use perspective projection (pinhole camera).

Intrinsic Camera Parameters

- Intrinsic parameters (summarized by the matrix $\boldsymbol{K} \in \mathbb{R}^{3\times3}$) are known.
- Intrinsic parameters do not change when camera moves.
- Origin of image coordinate system does not coincide with the intersection of optical axis and image plane in general.
- Axes of the camera's CCD-Chip are not orthogonal.
- Pixels are non-quadratic.

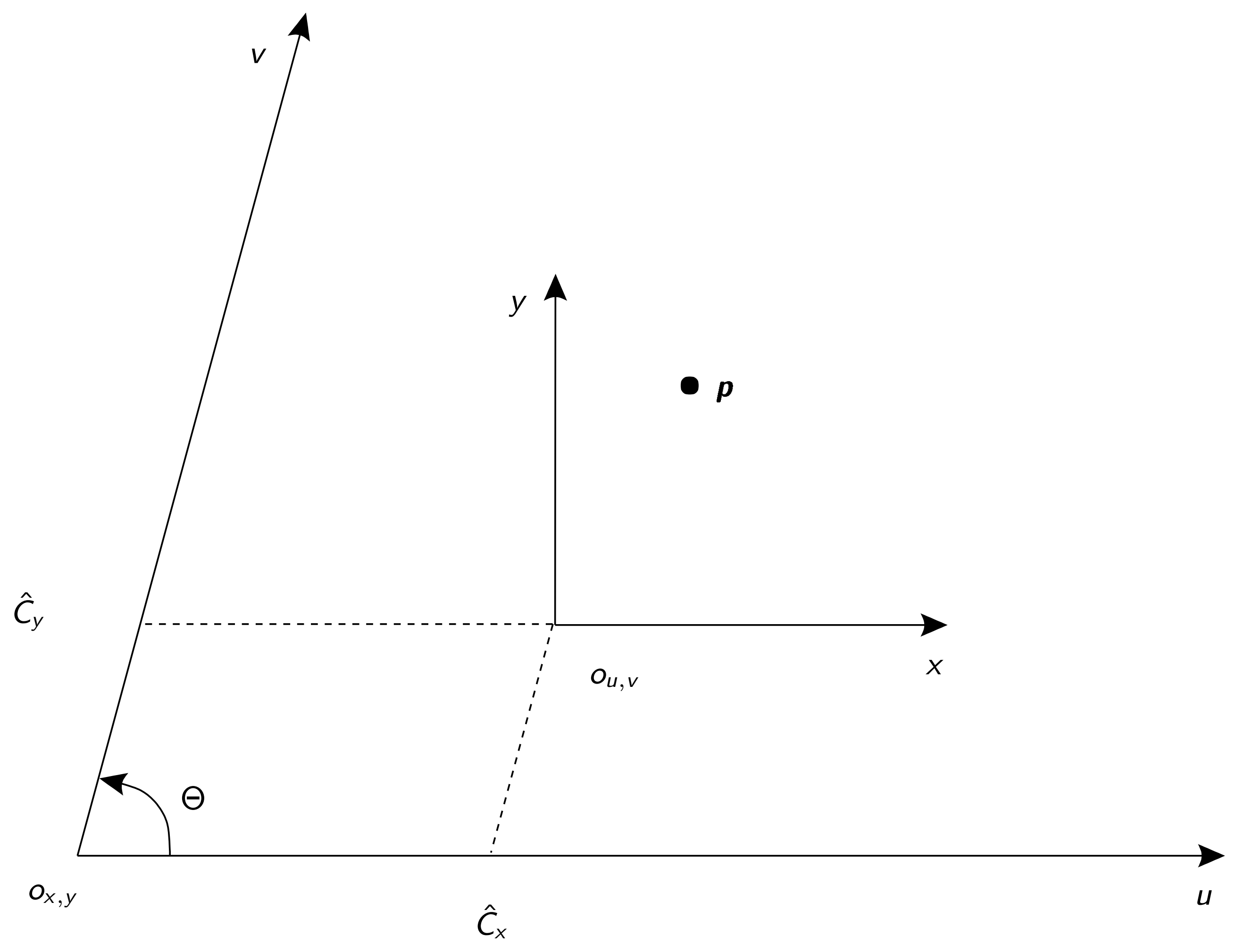

Figure 2: Pixel coordinate system

Coordinate System

(x, y) - coordinate system:

- ideal coordinate system used so far (image coordinate system with origin o^i)

(u, v) - coordinate system:

- real system, in which pixels are addressed, (pixel coordinate system, origin o^p)
- Θ: angle between axes, skew $s = -k_x \tan\Theta$
- k_x, k_y: units of u and v axis, with respect to units in x/y system

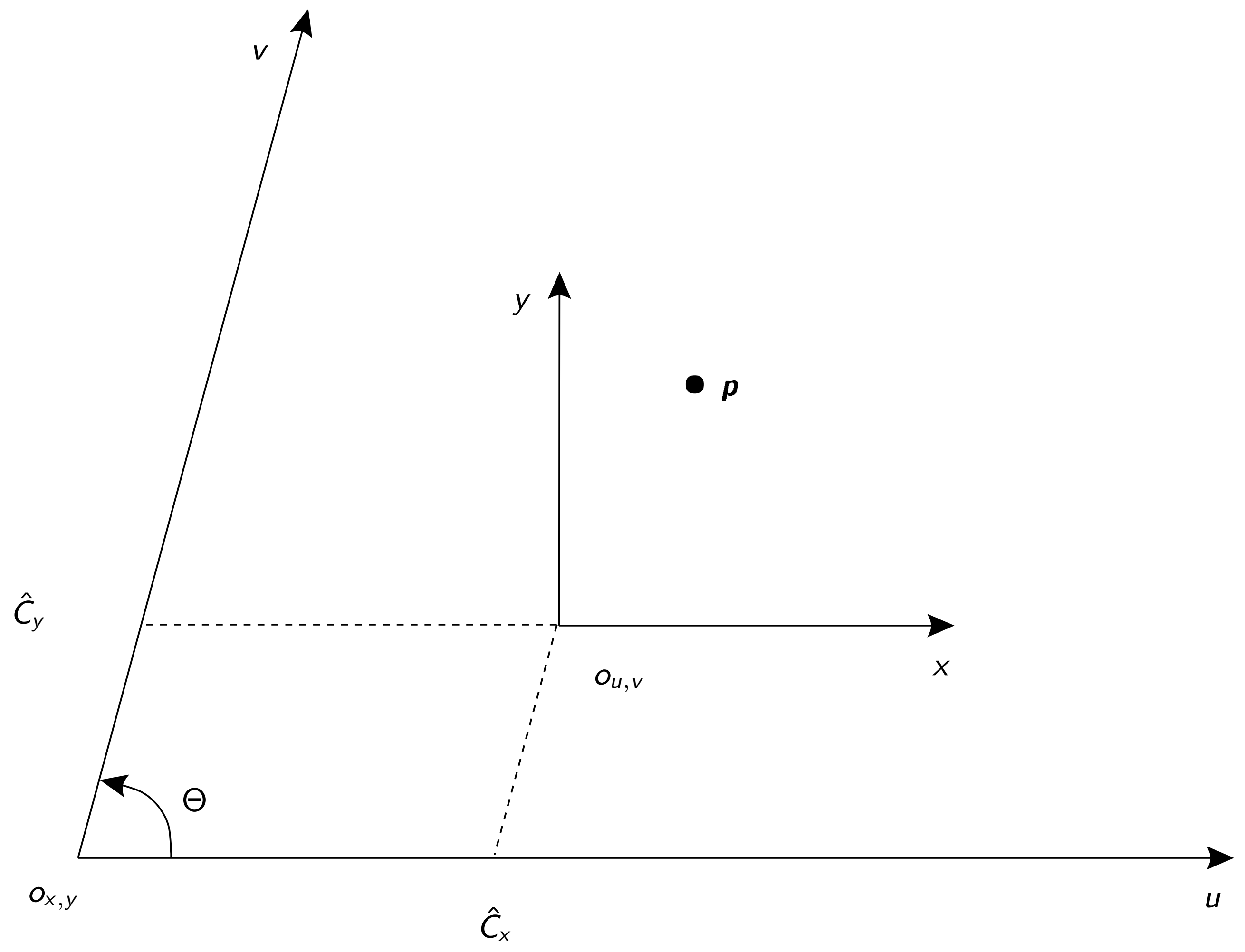

Figure 3: Pixel coordinate system

Fundamental Matrix

If pixel coordinates are used instead of ideal image coordinates

$$\tilde{\boldsymbol{p}}^{\mathrm{p}} = \boldsymbol{K}\tilde{\boldsymbol{p}}^{\mathrm{i}},$$

we substitute

$$\tilde{\boldsymbol{p}}^{\mathrm{i}} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{p}}^{\mathrm{p}}, \qquad \tilde{\boldsymbol{q}}^{\mathrm{i}} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{q}}^{\mathrm{p}},$$

and get:

$$(\tilde{\boldsymbol{q}}^{\mathrm{p}})^{\mathsf{T}} \cdot \underbrace{\left(\left(\boldsymbol{K}^{-1}\right)^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \boldsymbol{K}^{-1}\right)}_{\boldsymbol{F}} \cdot \tilde{\boldsymbol{p}}^{\mathrm{p}} = 0.$$

F is called the ***fundamental matrix***.

Properties of Fundamental Matrix

- $\boldsymbol{F}$ has rank 2.
- $\boldsymbol{F}$ encodes intrinsic and extrinsic parameters.
- $\boldsymbol{F}$ maps a point $\tilde{\boldsymbol{p}}^{\mathrm{p}}$ to its epipolar line $\boldsymbol{l}$ in pixel coordinates by $\boldsymbol{l}^{\mathsf{T}} = \boldsymbol{F} \cdot \tilde{\boldsymbol{p}}^{\mathrm{p}}$:

$$\boldsymbol{l}_2{}^{\mathsf{T}} = \boldsymbol{F}\tilde{\boldsymbol{p}}^{\mathrm{p}}, \qquad \boldsymbol{l}_1{}^{\mathsf{T}} = \boldsymbol{F}^{\mathsf{T}}\tilde{\boldsymbol{q}}^{\mathrm{p}}.$$

- All epipolar lines intersect in the epipole (left epipole $\tilde{\boldsymbol{e}}_l^p$, right epipole $\tilde{\boldsymbol{e}}_r^p$):
  - computation of the left null space:

$$(\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}}\boldsymbol{l}_1{}^{\mathsf{T}} = (\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}}\boldsymbol{F}\tilde{\boldsymbol{p}}^{\mathrm{p}} = 0$$

    implies $(\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}}\boldsymbol{F} = 0$,
  - computation of the right null space:

$$\boldsymbol{l}_2\tilde{\boldsymbol{e}}_r^p = \left(\tilde{\boldsymbol{e}}_r^p\boldsymbol{F}\right)^{\mathsf{T}}\tilde{\boldsymbol{q}}^{\mathrm{p}} = 0$$

    implies $\boldsymbol{F}^{\mathsf{T}}\tilde{\boldsymbol{e}}_l^p = 0$.

Eight Point Algorithm for $\boldsymbol{F}$

Computation of $\boldsymbol{F}$:

- We get N equations of the form $(\tilde{\boldsymbol{q}}_i^{\mathrm{p}})^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \tilde{\boldsymbol{p}}_i^{\mathrm{p}} = 0$.
- This system of equations is **linear** in the components of $\boldsymbol{F} = [f_{ij}]_{i,j \in \{1,2,3\}}$:

$$\boldsymbol{M} \cdot \boldsymbol{f} = 0, \quad \boldsymbol{f} = \begin{pmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{33} \end{pmatrix}, \quad \boldsymbol{M} \in \mathbb{R}^{N \times 9},$$

  where $\mathrm{rank}(\boldsymbol{M}) = 8$.
- Solve this system using singular value decomposition.
- Make sure that $\mathrm{rank}(\boldsymbol{F}) = 2$.

Eight Point Algorithm

Starting point:

- Over-determined system of equations $\boldsymbol{M} \cdot \boldsymbol{f} = 0$
- $\boldsymbol{f}$ lies in the null space of $\boldsymbol{M}$. The null space is non-trivial, since $\boldsymbol{M} \in \mathbb{R}^{N \times 9}$ and $\mathrm{rank}(\boldsymbol{M}) = 8$.

Eight Point Algorithm

Starting point:

- Over-determined system of equations $\boldsymbol{M} \cdot \boldsymbol{f} = 0$
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1. SVD of $\boldsymbol{M} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathsf{T}}$:
   - $\sigma_9 \approx 0 \quad \Rightarrow \quad \boldsymbol{f} = \lambda \cdot \boldsymbol{v}_9, \quad$ and since $\|\boldsymbol{F}\|_F = \|\boldsymbol{f}\|_2 = 1 \quad \Rightarrow \quad \boldsymbol{f} = \boldsymbol{v}_9$
   - If $\sigma_9 > \varepsilon \rightarrow$ error

Eight Point Algorithm

Starting point:

- Over-determined system of equations $\boldsymbol{M} \cdot \boldsymbol{f} = 0$
- $\boldsymbol{f}$ lies in the null space of $\boldsymbol{M}$. The null space is non-trivial, since $\boldsymbol{M} \in \mathbb{R}^{N \times 9}$ and $\mathrm{rank}(\boldsymbol{M}) = 8$.

1. SVD of $\boldsymbol{M} = \boldsymbol{U}\Sigma \boldsymbol{V}^{\mathsf{T}}$:
   - $\sigma_9 \approx 0 \quad \Rightarrow \quad \boldsymbol{f} = \lambda \cdot \boldsymbol{v}_9, \quad$ and since $\|\boldsymbol{F}\|_F = \|\boldsymbol{f}\|_2 = 1 \quad \Rightarrow \quad \boldsymbol{f} = \boldsymbol{v}_9$
   - If $\sigma_9 > \varepsilon \rightarrow$ error
2. Enforce $\mathrm{rank}(\boldsymbol{F}) = 2$ using SVD of $\boldsymbol{F} = U_{\boldsymbol{F}}\Sigma_{\boldsymbol{F}} V_{\boldsymbol{F}}^{\mathsf{T}}$:
   - For the fundamental matrix it is: $\sigma_1 \geq \sigma_2 > 0$, $\sigma_3 = 0$.
   - If $\sigma_3 > \varepsilon \rightarrow$ error
   - Set $\sigma_3 = 0$, and compute $\boldsymbol{F}$ using Σ_F' anew:

$$\boldsymbol{F} = U_{\boldsymbol{F}} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_{\boldsymbol{F}}^{\mathsf{T}}.$$

Numerical Instabilities

- Image coordinates are usually defined with respect to the top left corner of the image.
- Thus coordinates vary from 0 to a few hundred.
- The third (homogeneous) coordinate is usually set to 1.

Numerical Instabilities

Normalize the coordinates $\tilde{\boldsymbol{p}}_i^{\mathrm{p}} = (\boldsymbol{p}_{1,i}, \boldsymbol{p}_{2,i}, 1)^\mathsf{T}$ and $\tilde{\boldsymbol{q}}_i^{\mathrm{p}} = (\boldsymbol{q}_{1,i}, \boldsymbol{q}_{2,i}, 1)^\mathsf{T}$ such that the entries of $\boldsymbol{M}$ are of comparable size:

- Translate the origin of the image coordinate system to the centroid of the feature points, that is:
  - to $\left(\frac{1}{N}\sum_i^N \boldsymbol{p}_{1,i}, \frac{1}{N}\sum_i^N \boldsymbol{p}_{2,i}, 1\right)^\mathsf{T}$ for the left side,
  - and $\left(\frac{1}{N}\sum_i^N \boldsymbol{q}_{1,i}, \frac{1}{N}\sum_i^N \boldsymbol{q}_{2,i}, 1\right)^\mathsf{T}$ for the right image.
- Scale the feature points such that the mean homogeneous point looks like $\frac{1}{\sqrt{2}}(1, 1, 1)^\mathsf{T}$, i. e., the mean norm of a 2-D point is $\sqrt{2}$.

This is called ***balancing***.

Topics

The Eight Point Algorithm

Algorithm

Definitions

Fundamental Matrix

Data Balancing

Summary

Take Home Messages

Further Readings

Take Home Messages

- We have gone over the eight point algorithm using the fundamental matrix.
- The fundamental matrix maps a point to its epipolar line.
- The epipolar constraint is linear in components of $\boldsymbol{E}$ and $\boldsymbol{F}$.
- Balancing can be used to make an estimation of an essential matrix numerically robust.

Further Readings

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Medical Image Processing for Interventional Applications

Camera Rotation and Translation

Online Course – Unit 33

Andreas Maier, Joachim Hornegger, Frank Schebesch
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TECHNISCHE FAKULTÄT

Topics

Camera Rotation and Translation

Summary

Take Home Messages

Further Readings

Motivation

What needs to be computed:

- Translation $\rightarrow$ null space of the essential matrix
- Rotation $\rightarrow$ linear estimator based on quaternions
- Coordinates of 3-D points $\rightarrow$ triangulation

Computation of Camera Rotation and Translation

For computing rotation and translation, compute ***E*** from ***F***:

$$\boldsymbol{F} = \left(\boldsymbol{K}^{-1}\right)^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \boldsymbol{K}^{-1}.$$

Since intrinsic parameters are known

$$\boldsymbol{E} = \boldsymbol{K}^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{K}.$$

$\rightarrow$ ***E*** depends on the extrinsic camera parameters only.

Computation of Camera Rotation and Translation

Known:

- Essential matrix $\boldsymbol{E} = \boldsymbol{R} \cdot [t]_\times$
- Translation vector $\boldsymbol{t}$ (spans nullspace of essential matrix)

Computation of rotation matrix $\boldsymbol{R}$

Compute by solving the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{E} - \boldsymbol{R} \cdot [\boldsymbol{t}]_\times\|_2^2, \\ \text{subject to} \quad & \det(\boldsymbol{R}) = 1, \\ & \boldsymbol{R} \text{ is an orthogonal matrix, i. e., } \boldsymbol{R}^{-1} = \boldsymbol{R}^\mathsf{T}. \end{aligned}$$

This is a **nonlinear** and **difficult** optimization problem.

But it can be converted into a **linear** problem by using quaternions.

Essential Matrix and Quaternions

First we rewrite the objective function:

$$\|\boldsymbol{E} - \boldsymbol{R} \cdot [\boldsymbol{t}]_\times\|_2^2 = \|\boldsymbol{R}^\mathsf{T}\boldsymbol{E} - [\boldsymbol{t}]_\times\|_2^2,$$

where $\det \boldsymbol{R} = 1$.

Writing $\boldsymbol{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, where $\boldsymbol{e}_i \in \mathbb{R}^3$, we therefore minimize:

$$\min_{\boldsymbol{R}} \|\boldsymbol{R}^\mathsf{T}\boldsymbol{E} - [\boldsymbol{t}]_\times\|_2^2 = \min_{\boldsymbol{R}} \sum_{i=1}^{3} \|\boldsymbol{R}^\mathsf{T}\boldsymbol{e}_i - \left([\boldsymbol{t}]_\times\right)_i\|_2^2.$$

Essential Matrix and Quaternions

In quaternion notation:

- $\boldsymbol{R}^\mathsf{T}$ defines a quaternion $\boldsymbol{r}$,
- $\boldsymbol{p}_i = (0, \boldsymbol{e}_i{}^\mathsf{T})$,
- $\boldsymbol{p}'_i = (0, ([\boldsymbol{t}]_\times)_i{}^\mathsf{T})$.

Thus we get

$$\min_{\boldsymbol{r}} \sum_{i=1}^{3} \|\boldsymbol{r} \cdot \boldsymbol{p}_i \cdot \bar{\boldsymbol{r}} - \boldsymbol{p}'_i\|^2,$$

where $\|\boldsymbol{r}\| = 1$.

Essential Matrix and Quaternions

Multiplication by $\boldsymbol{r}$ from the right results in the objective function:

$$\widehat{\boldsymbol{r}} = \min_{\boldsymbol{r}} \sum_{i=1}^{3} \|\boldsymbol{r} \cdot \boldsymbol{p}_i - \boldsymbol{p}_i' \cdot \boldsymbol{r}\|^2.$$

Since this expression is linear in $\boldsymbol{r}$, there exist matrices $\boldsymbol{M}_i$ with:

$$\widehat{\boldsymbol{r}} = \min_{\boldsymbol{r}} \sum_{i=1}^{3} \left\|\boldsymbol{M}_i \cdot \boldsymbol{r}^{\mathrm{T}}\right\|^2.$$

Topics

Camera Rotation and Translation

Summary

Take Home Messages

Further Readings

Take Home Messages

- Both camera translation and rotation can be computed using the essential matrix.
- If we use quaternions, we can formulate the optimization problem for the rotation linearly.

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Medical Image Processing for Interventional Applications

Data Consistency

Online Course – Unit 34

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Topics

Motion Compensation

Data Consistency Conditions

Summary

Take Home Messages

Further Readings

Motion Compensation

- Result in reconstruction artifacts in the final images:

  motion blur

- Degraded image quality lowers **diagnostic confidence.**

- **Effects increase** with acquisition time.

Figure 1: Motion-free

Figure 2: Motion-corrupted

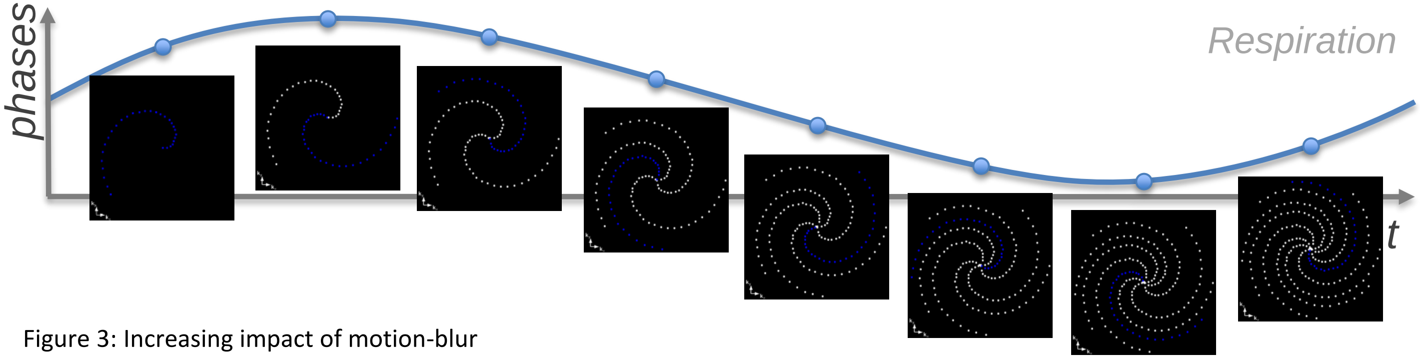
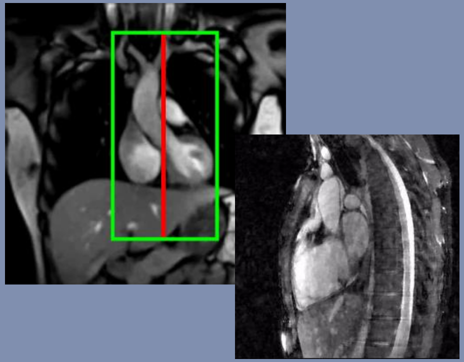
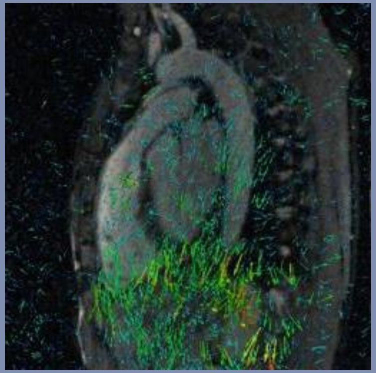


Figure 3: Increasing impact of motion-blur

Motion Compensated Reconstruction



Motion-Detection



Motion-Compensation

Reconstruction

- **Initial reconstruction** of individual respiratory phases with **weighted compressed sensing** reconstruction.

- **Image registration** to estimate the **displacement** due to respiratory motion.
- Apply **displacement field** on acquired data.

- Reconstruction of the resulting **motion-compensated** data.

Motion Compensated Reconstruction: Resulting Images

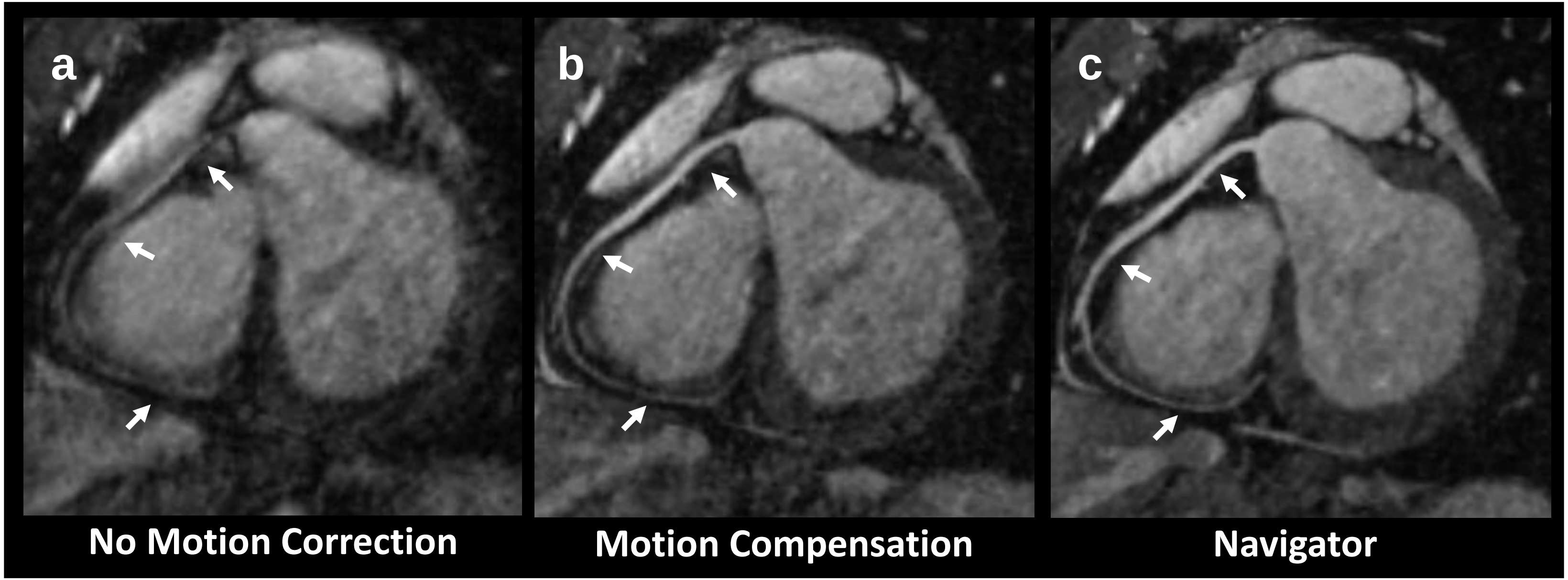


Figure 7: Reformatted images showing the right coronary artery (RCA) reconstructed (a) without any correction and (b) using motion compensation. (c) An additional navigator-gated scan is performed for reference.

Motion Compensation

Standard FDK reconstruction | ECG-gated reconstruction | PICCS + iTV reference reconstruction | Result of CMHPR algorithm

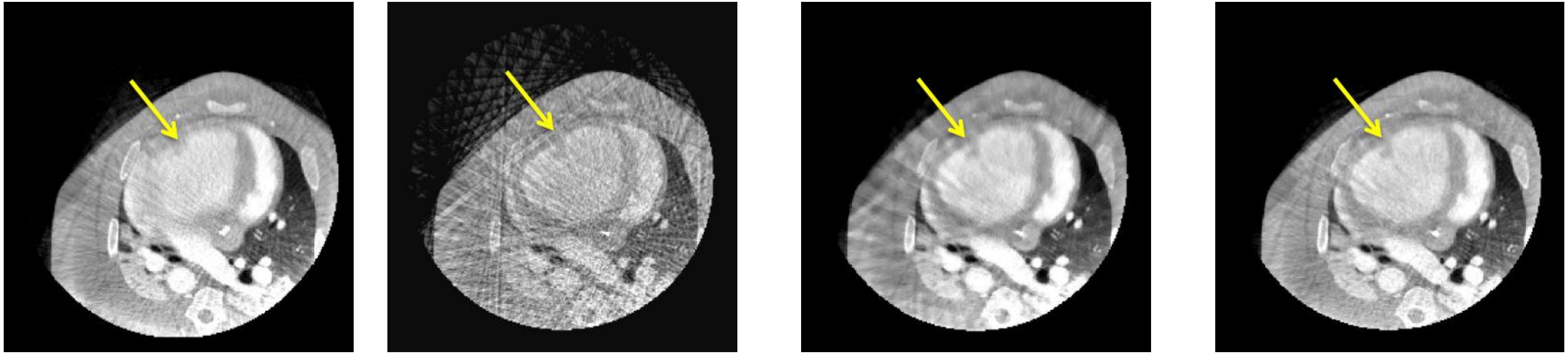


Figure 8: Experimental results in porcine model p1 of the central slice and a relative heart phase of 80 %. (W 1630 HU, C 50 HU, slice thickness 1.0 mm). The ECG-gated reconstruction was windowed to be visually comparable (images courtesy of Kerstin Müller, Pattern Recognition Lab, FAU).

Bulk Rigid Motion / Online Calibration

- Scanner-related:
  - Imperfect system geometry (C-Arm wobble)
  - Outdated calibration
- Patient-related:
  - Patients suffering from stroke (rigid head motion)
  - Slight tremor while standing (rigid case)

→ Motion model with few parameters

→ Interventional application requires fast computation.

Data Consistency Metrics

Goal:

- Describe consistency that is inherent to a CT acquisition.
- Allow the correction of sources of “inconsistency”: motion, scatter, truncation.

Data Consistency Metrics

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- Describe consistency that is inherent to a CT acquisition.
- Allow the correction of sources of “inconsistency”: motion, scatter, truncation.

Pros:

- Without the use of markers
  → No change in patient preparation
- Purely image-based
  → No additional sensors
- Before/without reconstruction
  → Fast

Data Consistency Metrics

Goal:

- Describe consistency that is inherent to a CT acquisition.
- Allow the correction of sources of "inconsistency": motion, scatter, truncation.

Pros:

- Without the use of markers
  → No change in patient preparation
- Purely image-based
  → No additional sensors
- Before/without reconstruction
  → Fast

Cons:

- Works only for certain applications/geometries
- Many results from simulations published
  (since over 25 years)
- Few real data results published
- Never applied in real systems

Topics

Motion Compensation

Data Consistency Conditions

Summary

Take Home Messages

Further Readings

Data Consistency Conditions

Parallel beam conditions:

- Projection moments (Helgason & Ludwig, 1985)
- Sinogram Fourier space (Natterer & Edholm, 1986)

Extensions:

- Moments:
  - Fan beam (Clackdoyle)
  - Cone beam (Clackdoyle & Desbat)
- Fourier Space:
  - Fan beam (Mazin & Pelc)
  - Cone beam (Brokish & Bresler)

Moments (Parallel)

$$\int_0^{2\pi} \left(\int_{-\infty}^{\infty} s^m p(s,\Theta) ds \right) e^{-ik\Theta} d\Theta = 0$$

$$\forall k, m \in \mathbb{N} \ \wedge \ k > m \geq 0$$

Moments (Parallel)

m-th order moment of the projection

$$\int_0^{2\pi} \left(\boxed{\int_{-\infty}^{\infty} s^m p(s,\Theta) ds} \right) e^{-ik\Theta} d\Theta = 0$$

$$\forall k, m \in \mathbb{N} \quad \wedge \quad k > m \geq 0$$

Moments (Parallel)

m-th order moment of the projection

$$\boxed{\int_0^{2\pi}\left(\boxed{\int_{-\infty}^{\infty} s^m p(s,\Theta)ds}\right)e^{-ik\Theta}d\Theta} = 0$$

$$\forall k, m \in \mathbb{N} \wedge \quad k > m \geq 0$$

k-th order Fourier expansion

Moments (Parallel)

$$\int_0^{2\pi} \left(\int_{-\infty}^{\infty} s^m p(s,\Theta) ds \right) e^{-ik\Theta} d\Theta = 0$$

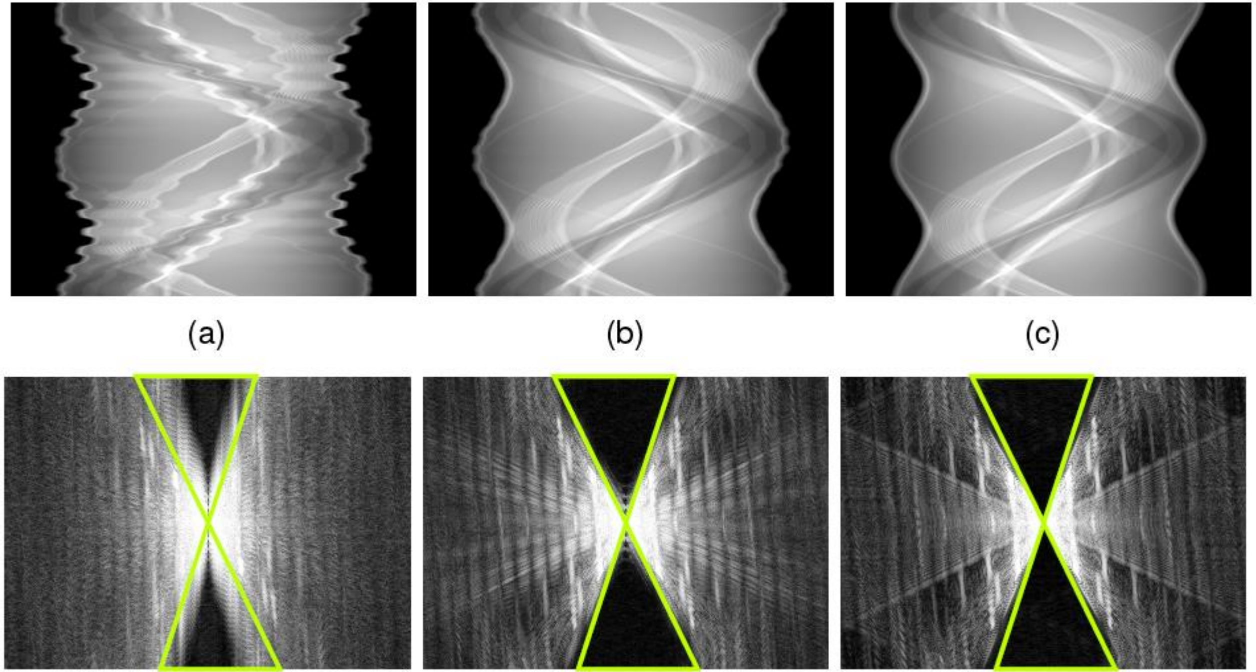
$$\forall k, m \in \mathbb{N} \quad \wedge \quad k > m \geq 0$$

Example (*m=0*)

- Inner integral becomes the total mass of the projection.
- Outer integral constrains all frequencies (*k>m*) to 0.

⇒ Total mass is identical in all projections.

Sinogram Fourier Space



(a) (b) (c)

(d) (e) (f)

Topics

Motion Compensation

Data Consistency Conditions

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- In image processing algorithms that depend on multiple acquisitions like image reconstruction, motion creates artifacts in uncompensated images.
- Motion compensation is a broad research topic, but approaches using data consistency are rather scarce and mostly deal with simulated data only.
- In the following units we look deeper into data consistency and epipolar geometry.

Further Readings

André Aichert et al. "Epipolar Consistency in Transmission Imaging". In: *IEEE Transactions on Medical Imaging* 34.11 (Nov. 2015), pp. 2205–2219. DOI: 10.1109/TMI.2015.2426417

Acknowledgements:

MEDICAL VALLEY
Europäische Metropolregion Nürnberg
SIEMENS
Universitätsklinikum Erlangen

Medical Image Processing for Interventional Applications

Epipolar Geometry – Part 2

Online Course – Unit 35

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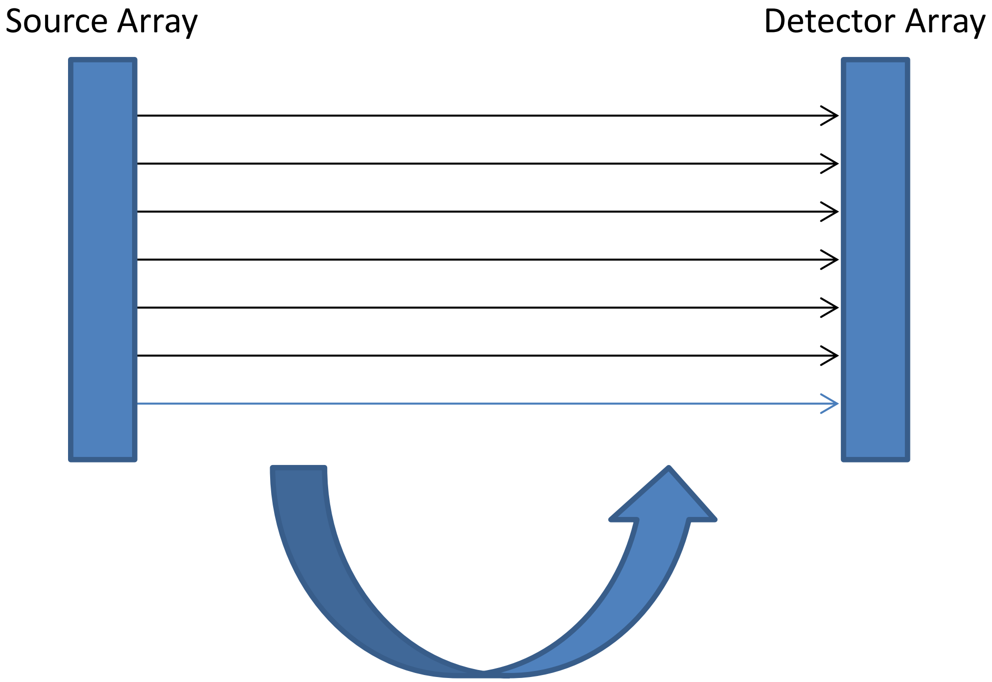
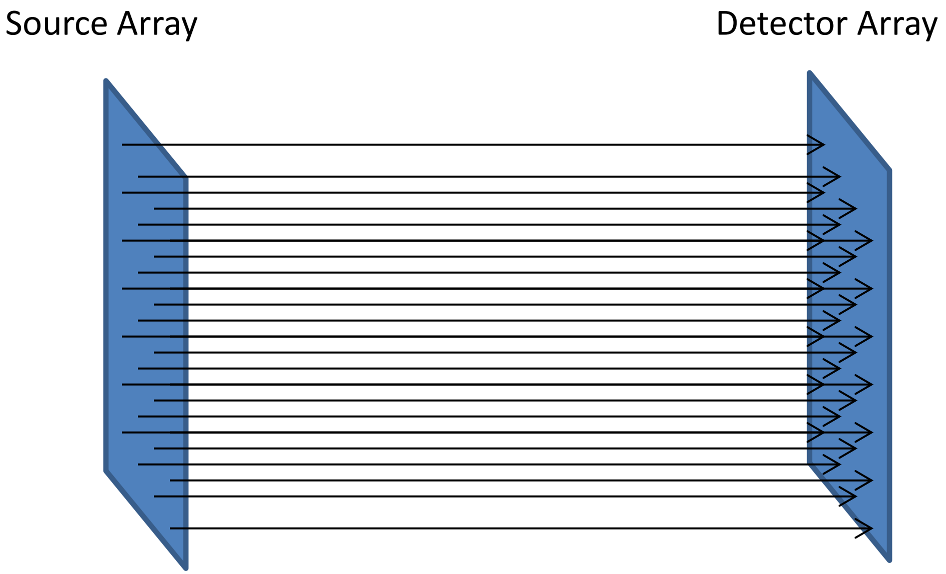
Topics

Radon Transform: 2-D Case



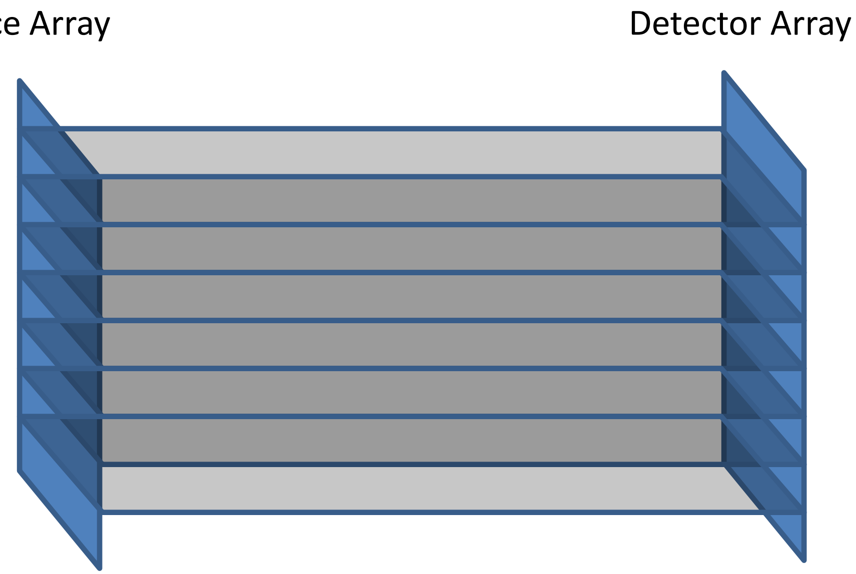
2-D Radon Transform　　Sinogram　　X-Ray Transform

Radon Transform: 3-D Case



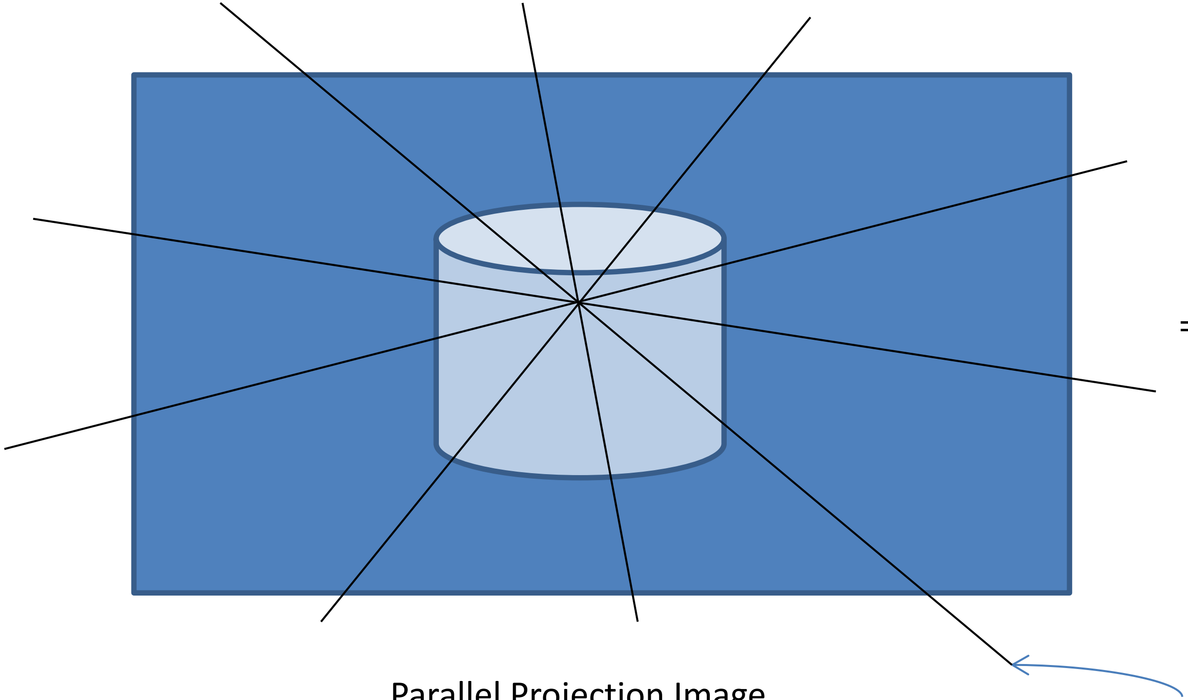
3-D X-Ray Transform !

Radon Transform: 3-D Case



3-D X-Ray Transform !

X-Ray Transform and Radon Transform



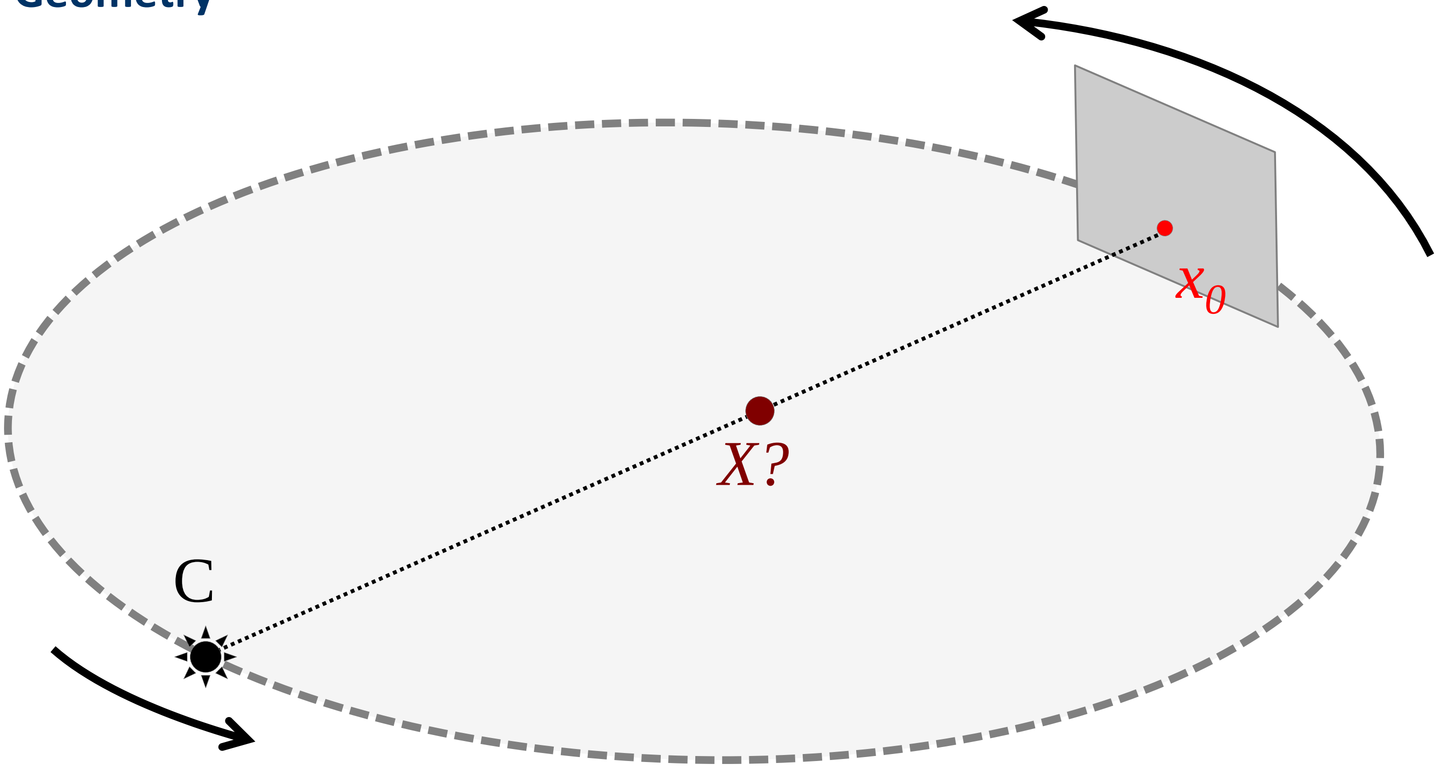
Parallel Projection Image

⇒ 2-D Radon Transfrom

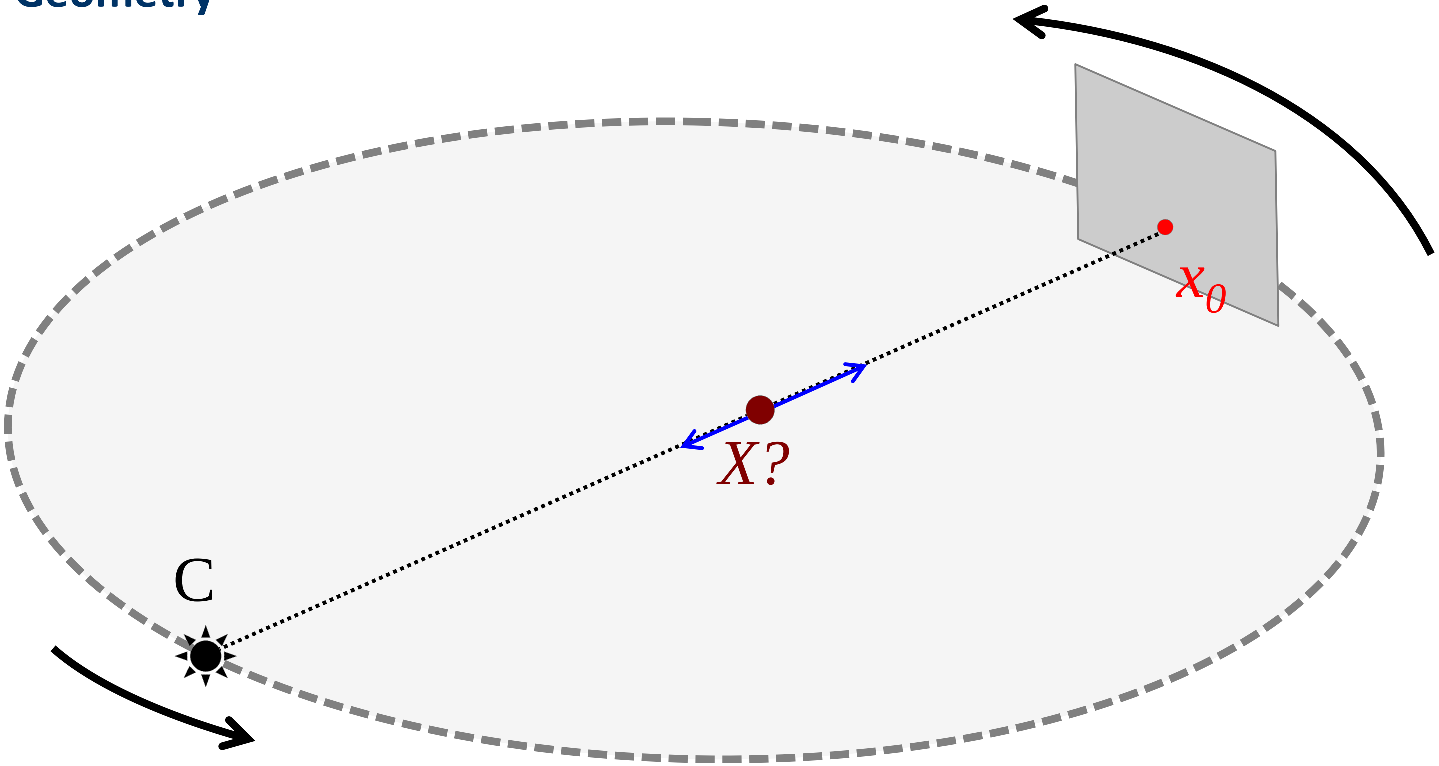
Arbitrary Line

Topics

Epipolar Geometry



Epipolar Geometry



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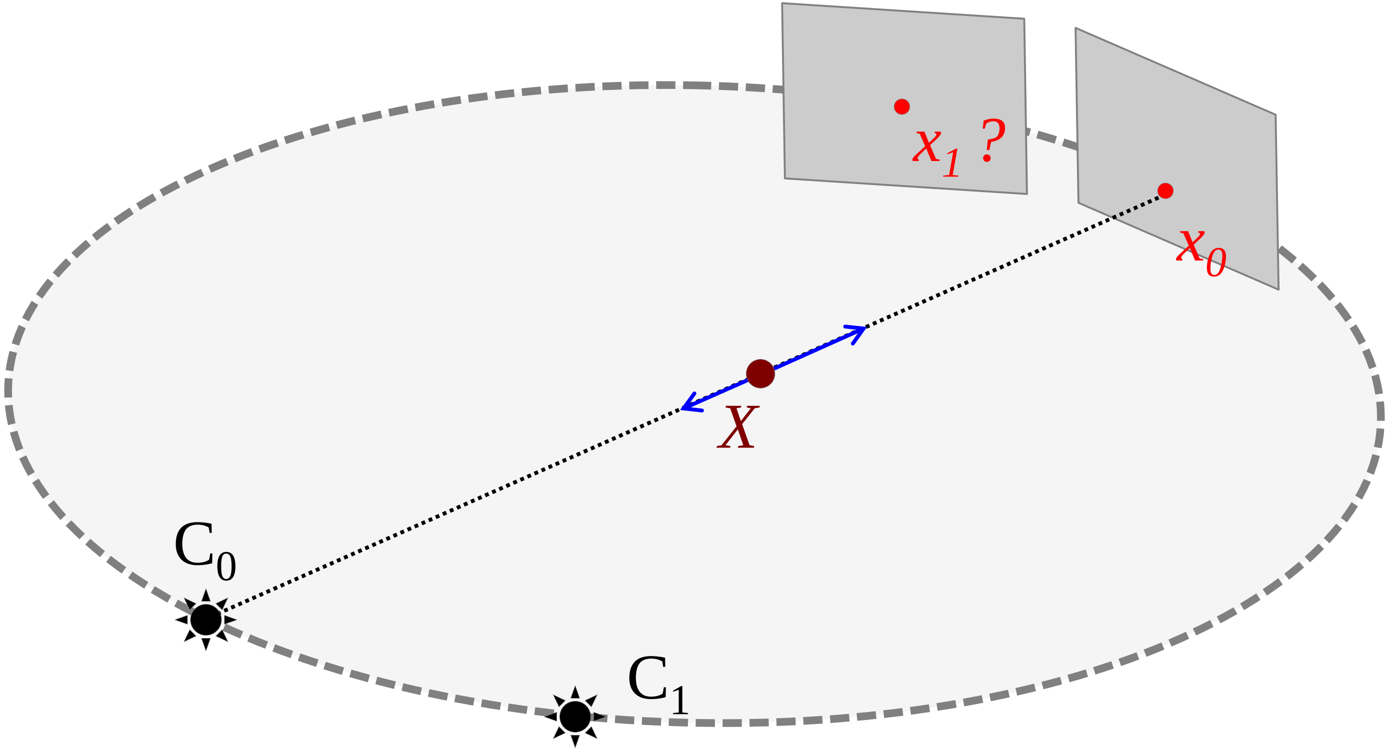
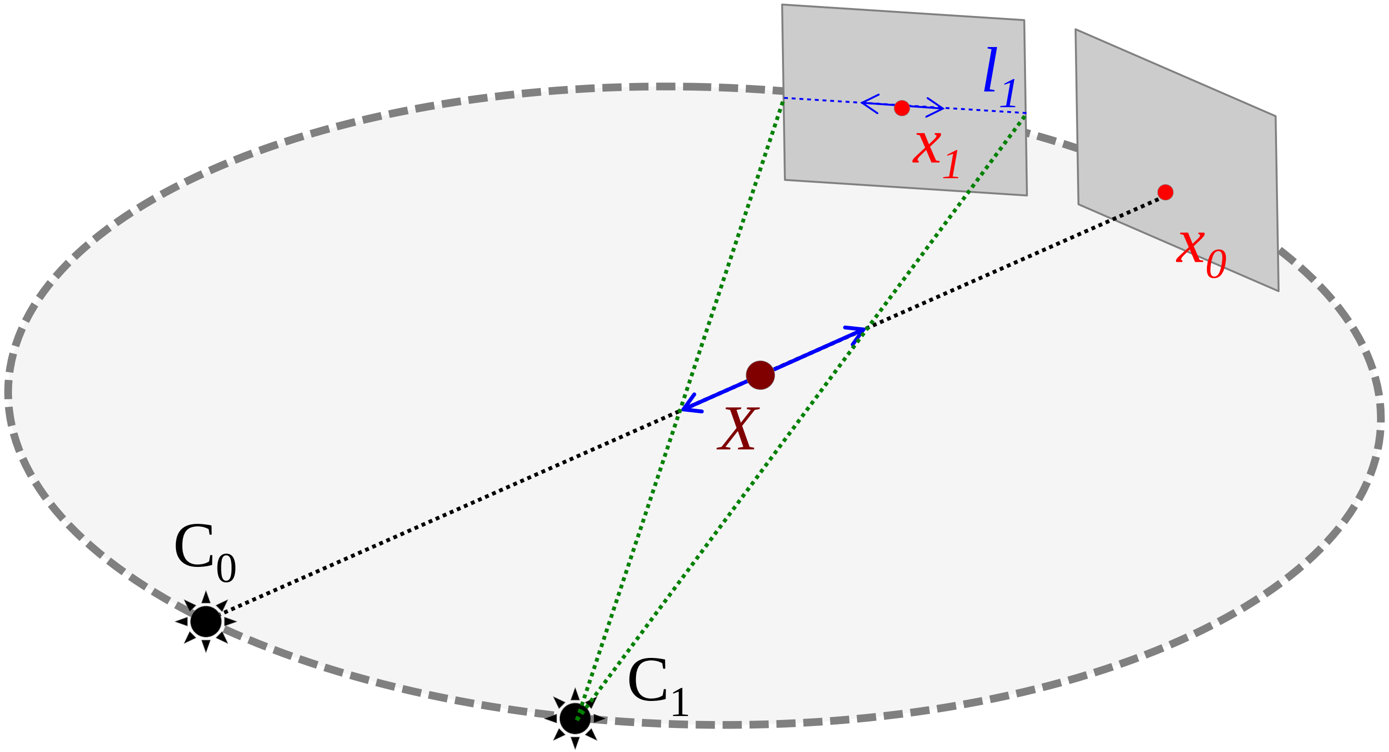
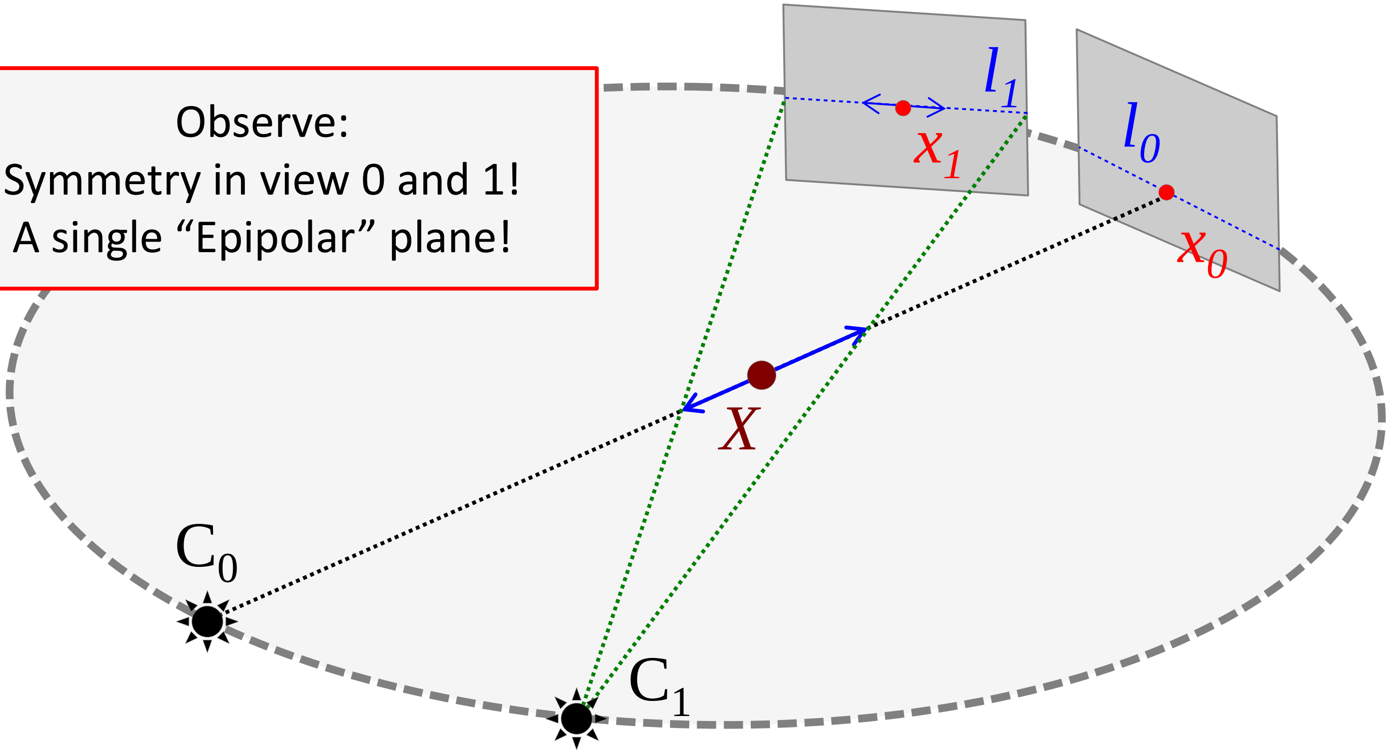


Epipolar Geometry

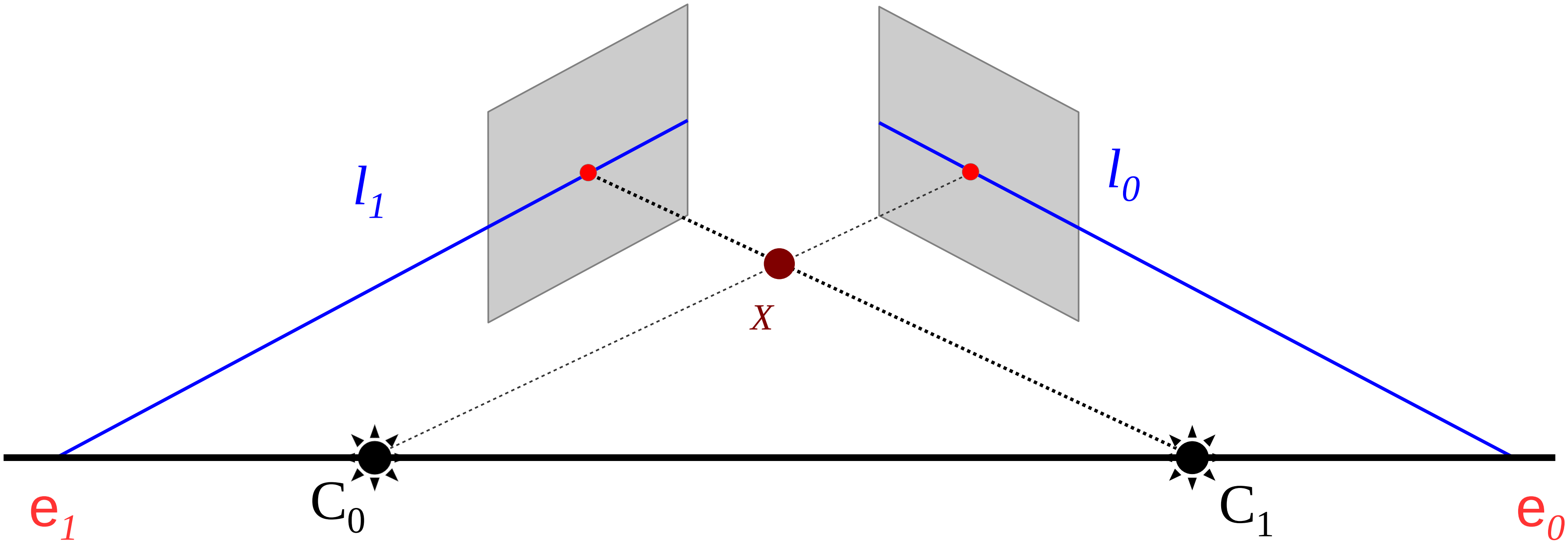


Epipolar Geometry

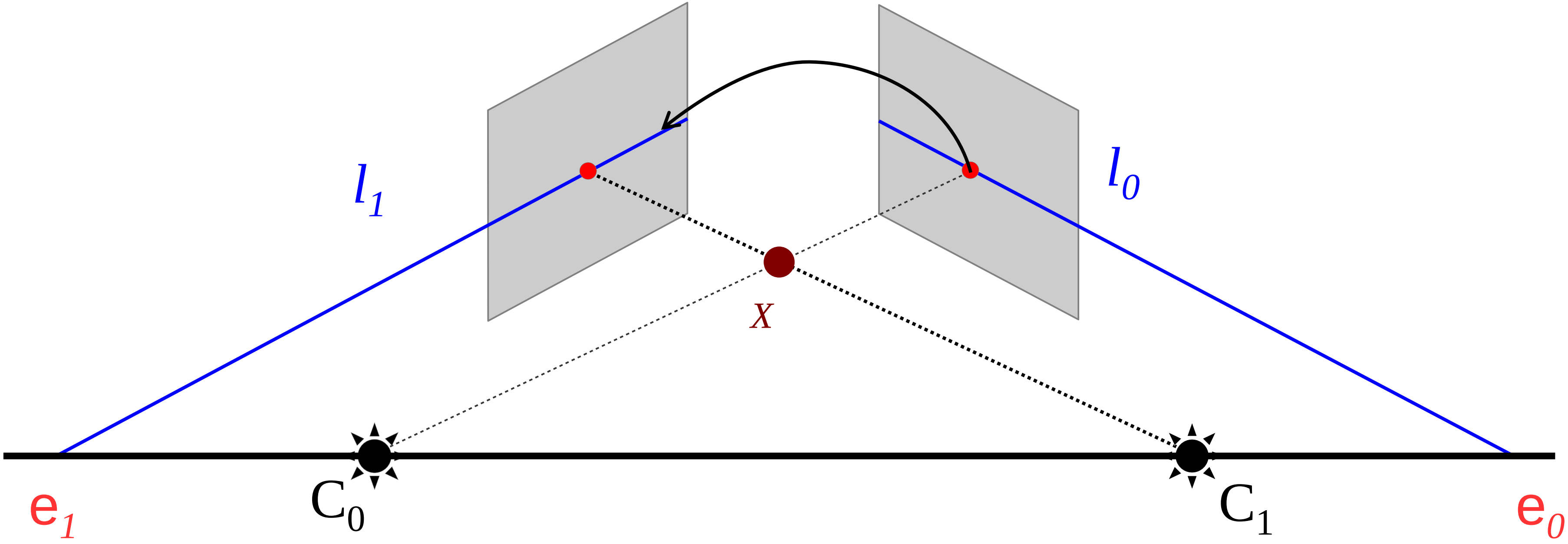
Observe:
Symmetry in view 0 and 1!
A single "Epipolar" plane!



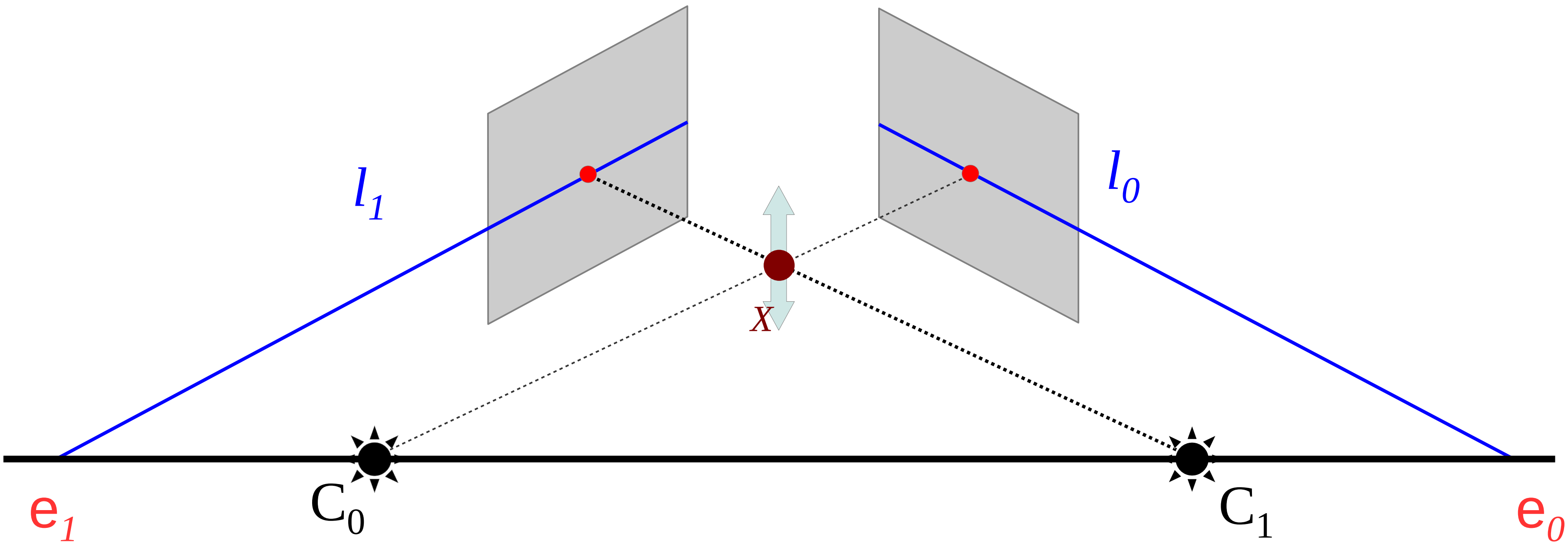
Epipolar Geometry



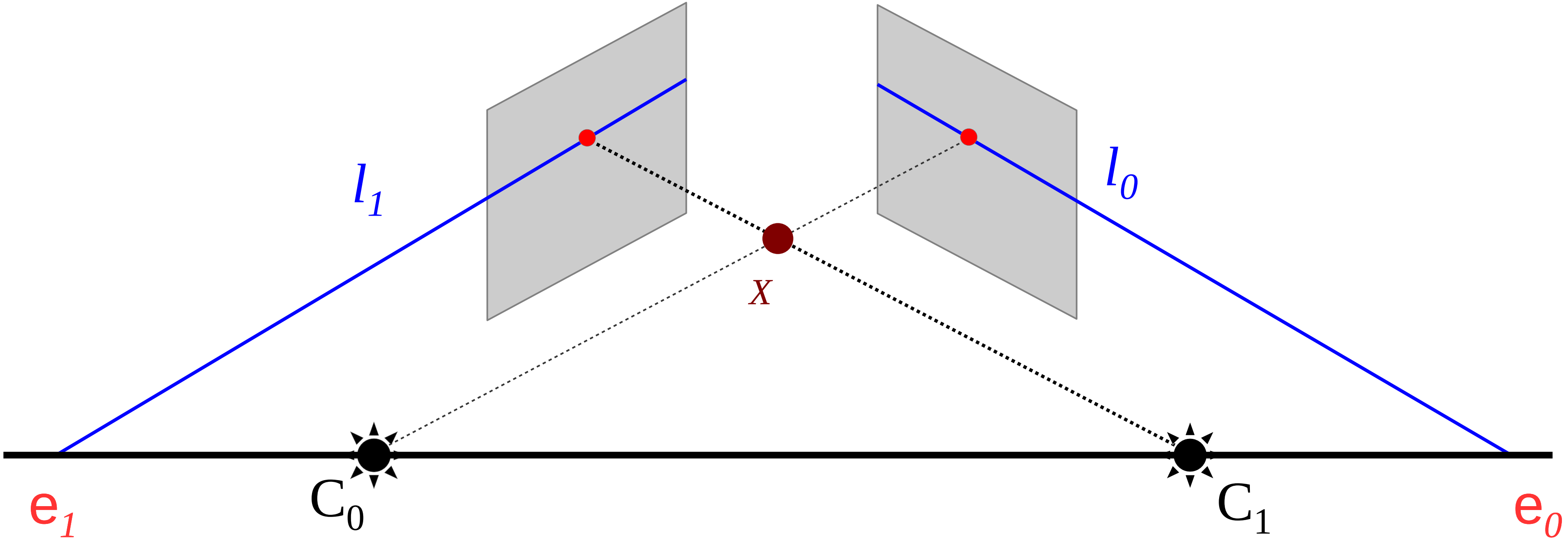
Epipolar Geometry



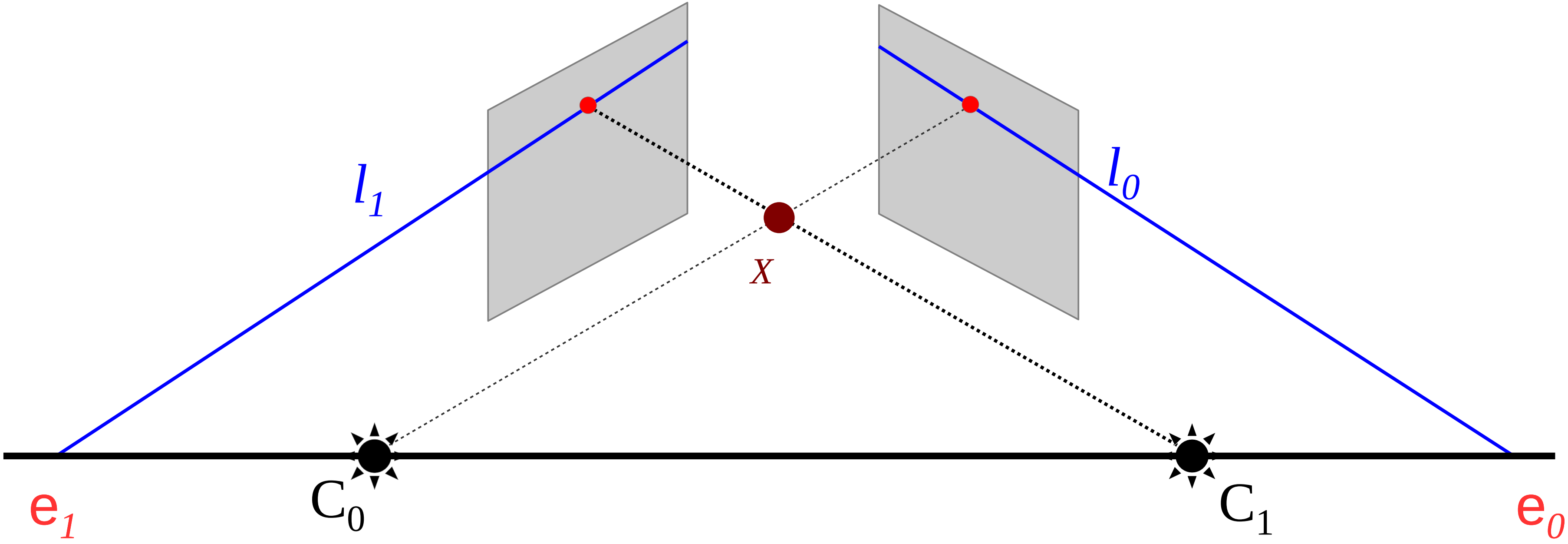
Epipolar Geometry



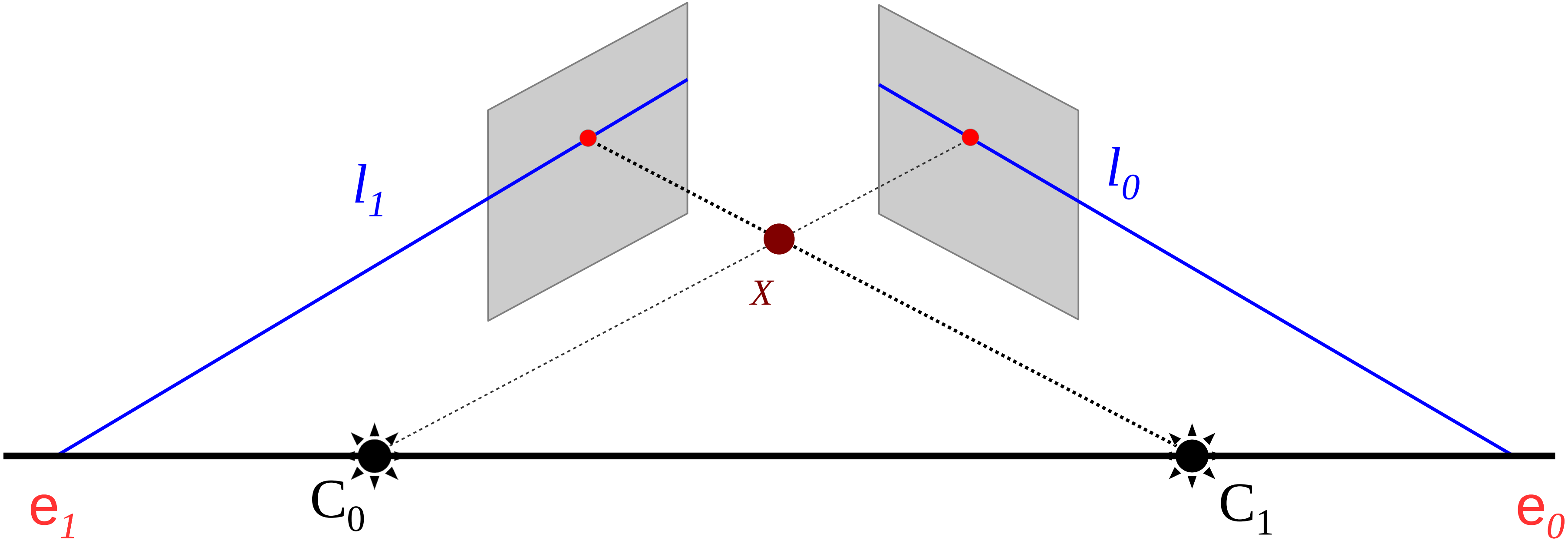
Epipolar Geometry



Epipolar Geometry



Epipolar Geometry



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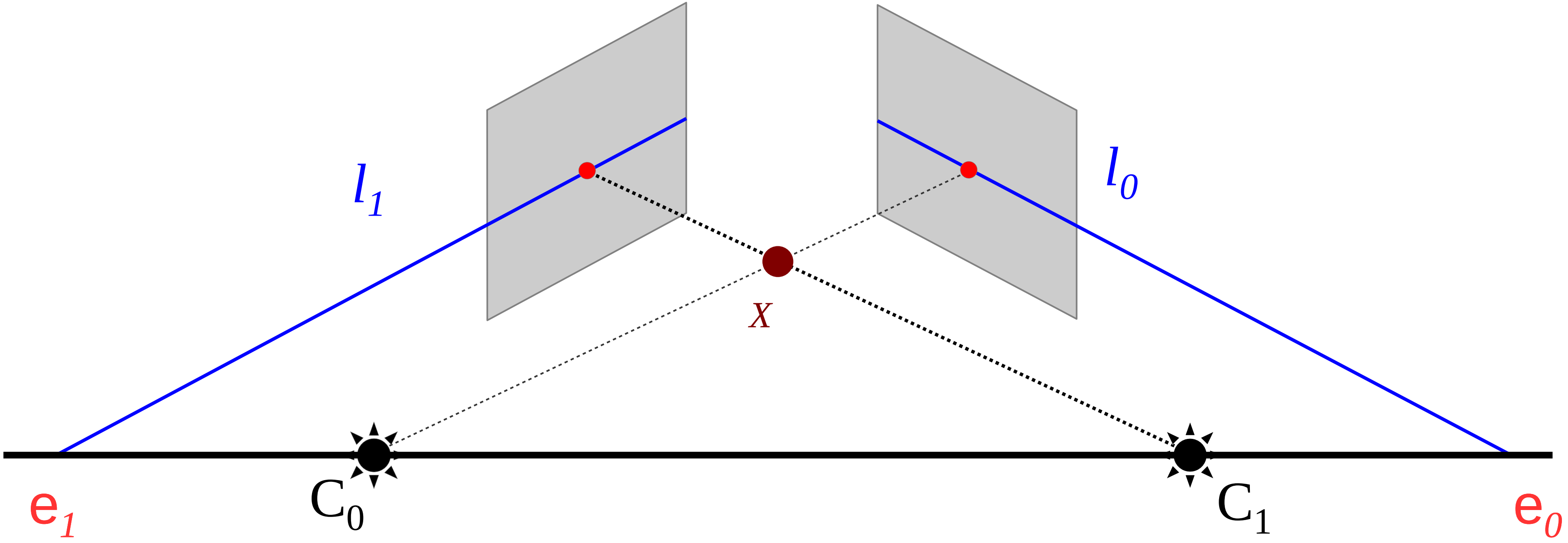

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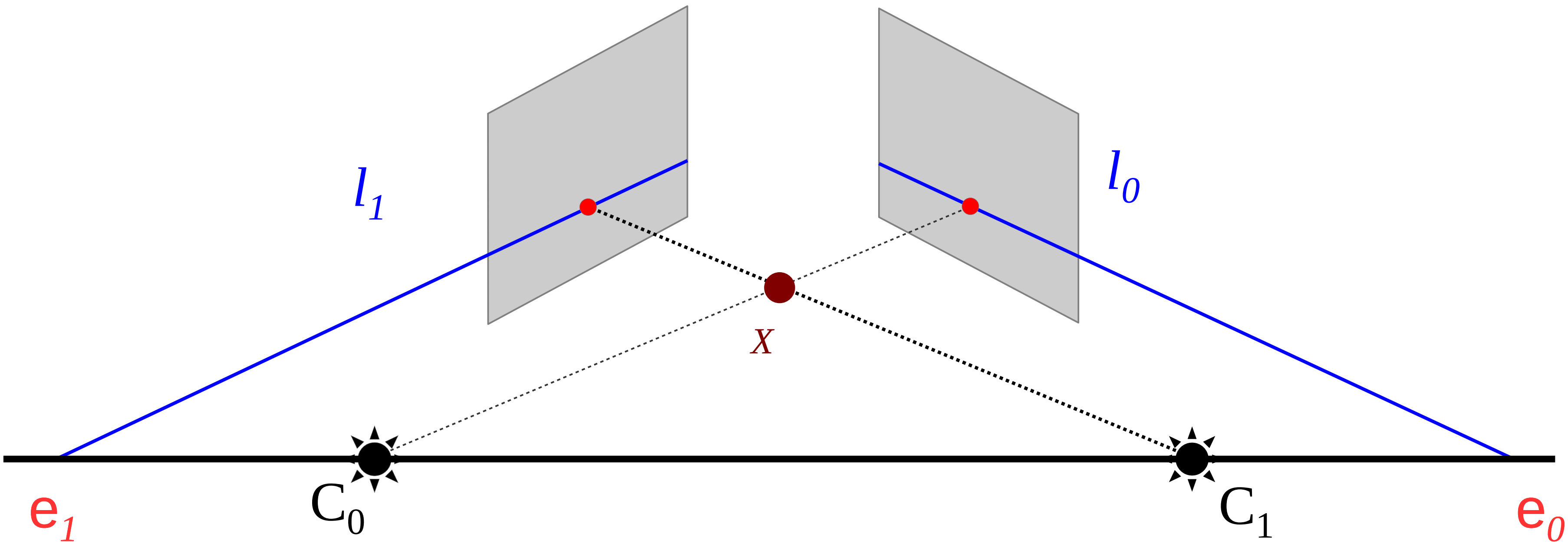
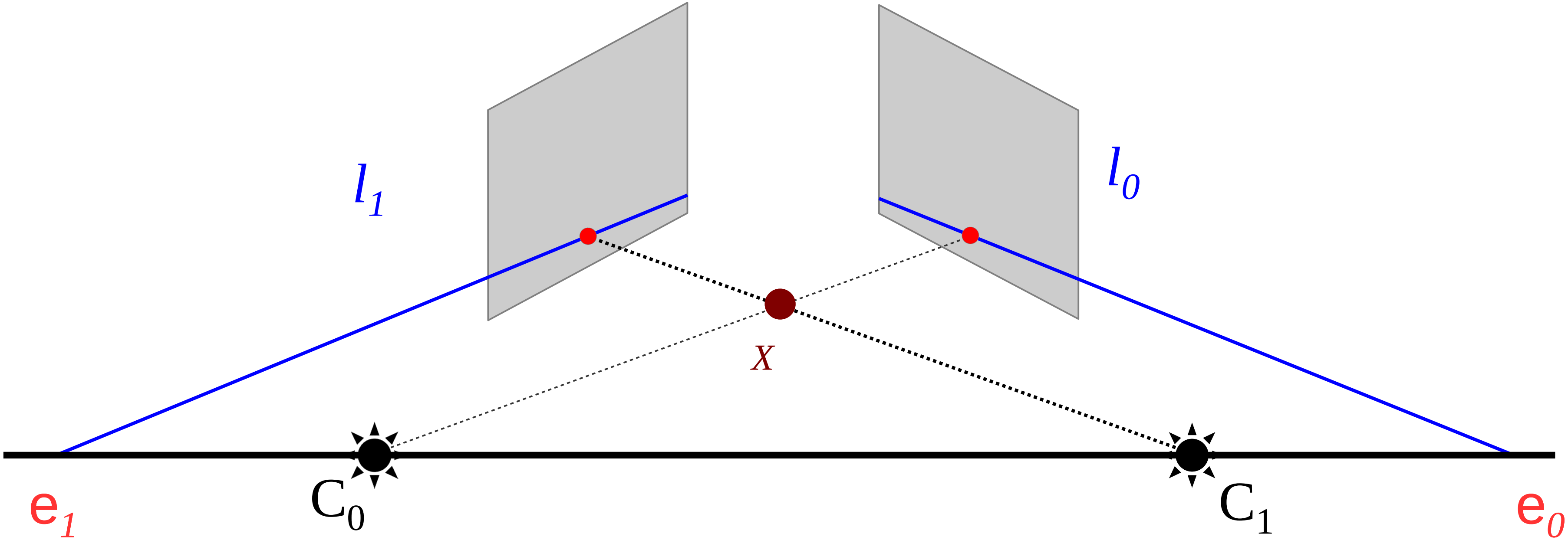


Epipolar Geometry



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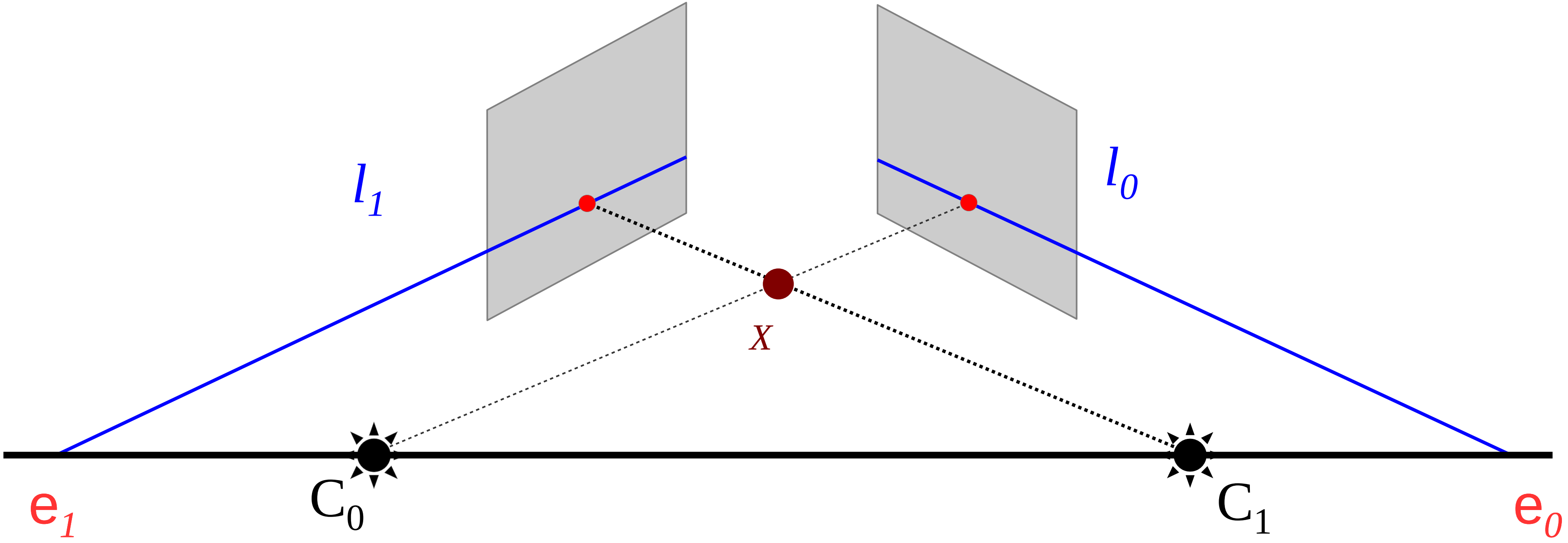

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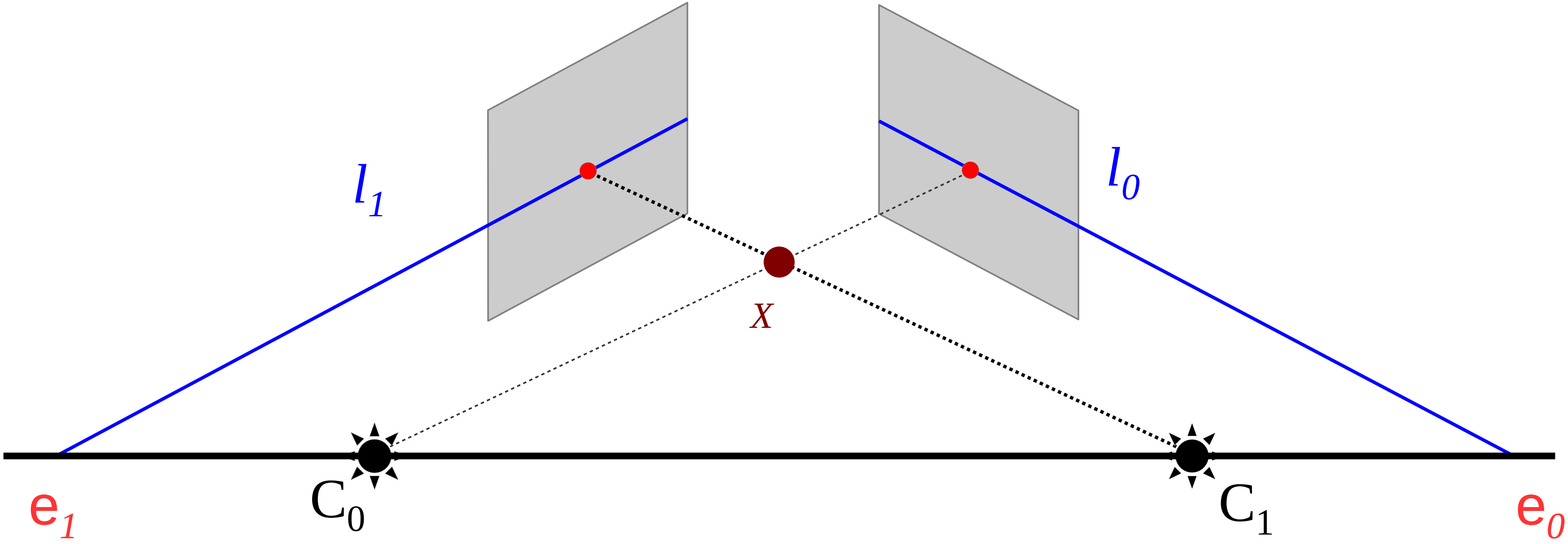
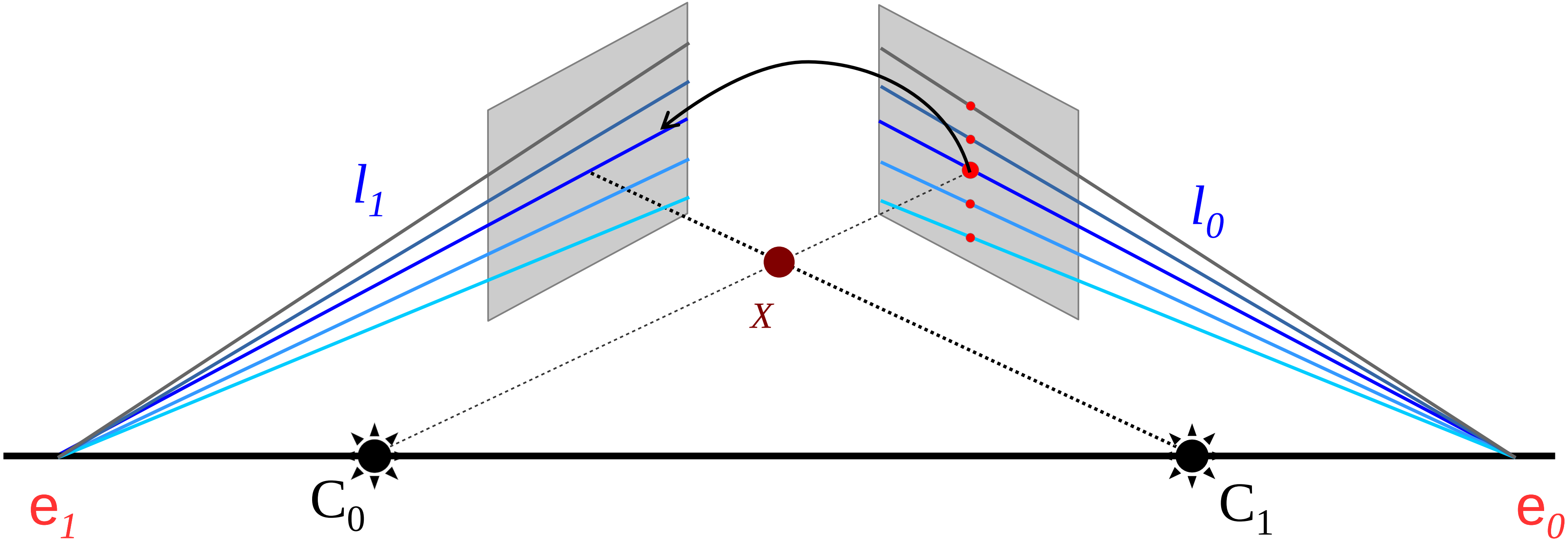
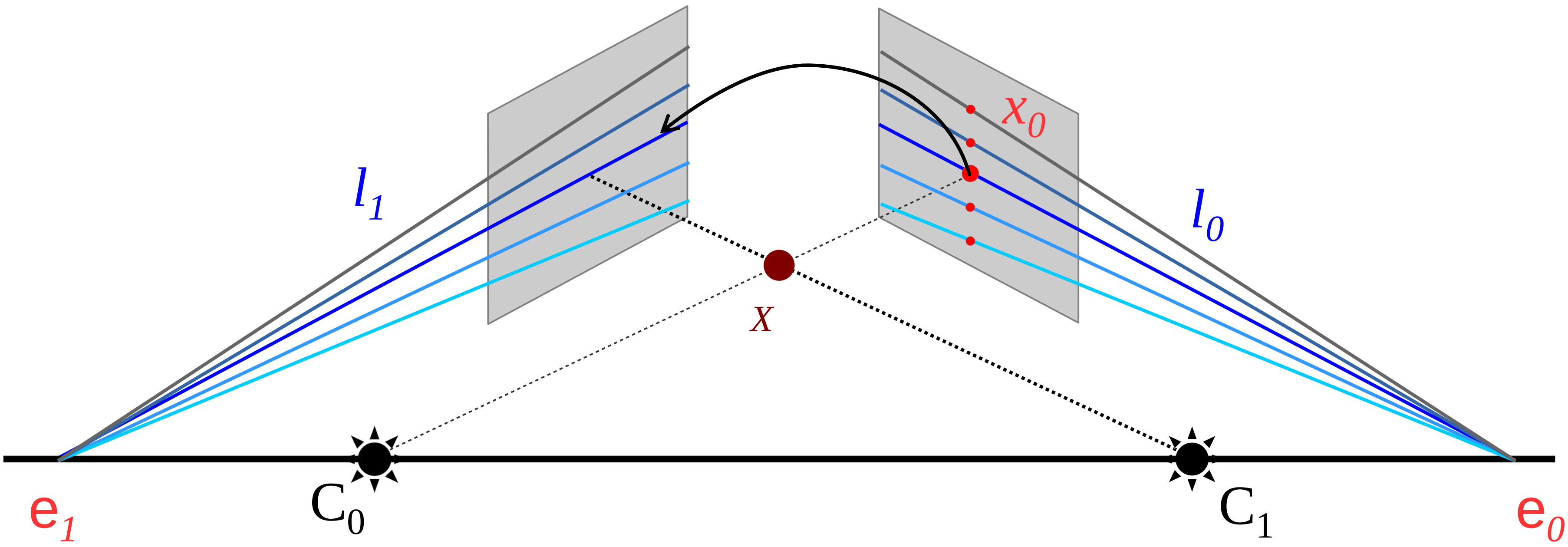


Epipolar Geometry



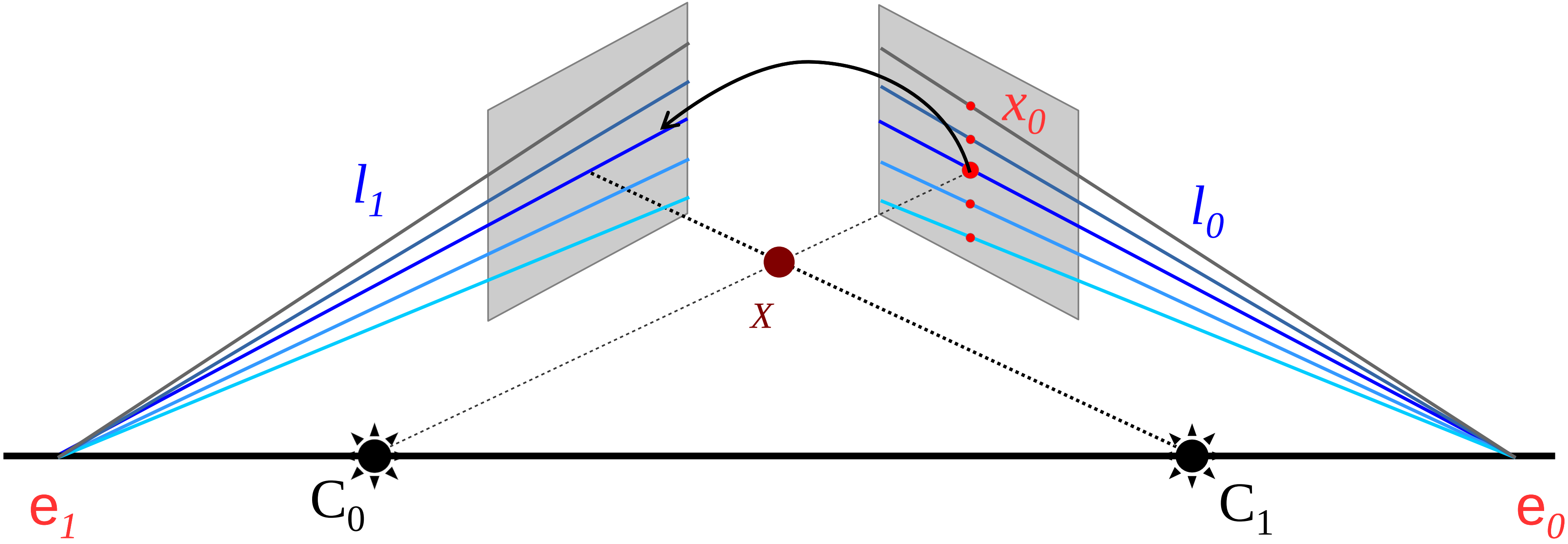
Two line bundles with 1-1 correspondences!

Epipolar Geometry



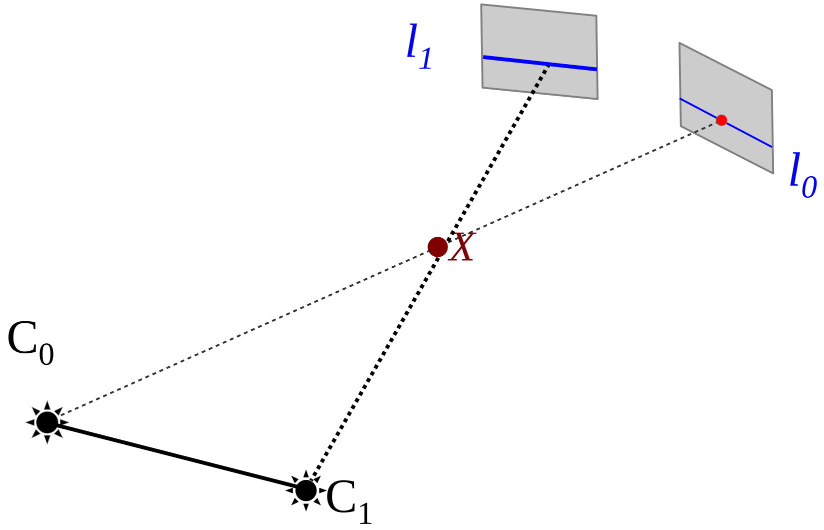
A single 3x3 matrix ***F*** encodes the relative geometry!

Epipolar Geometry

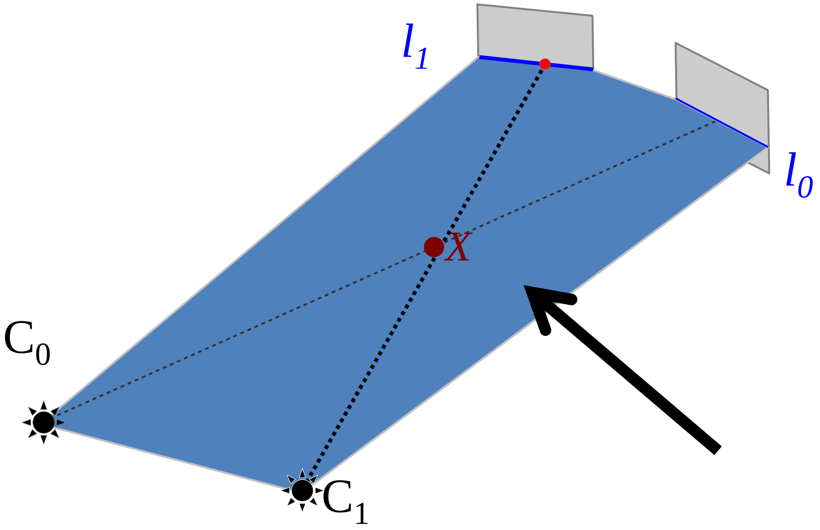


All epipolar lines intersect in the epipole!

Redundancies on Epipolar Lines

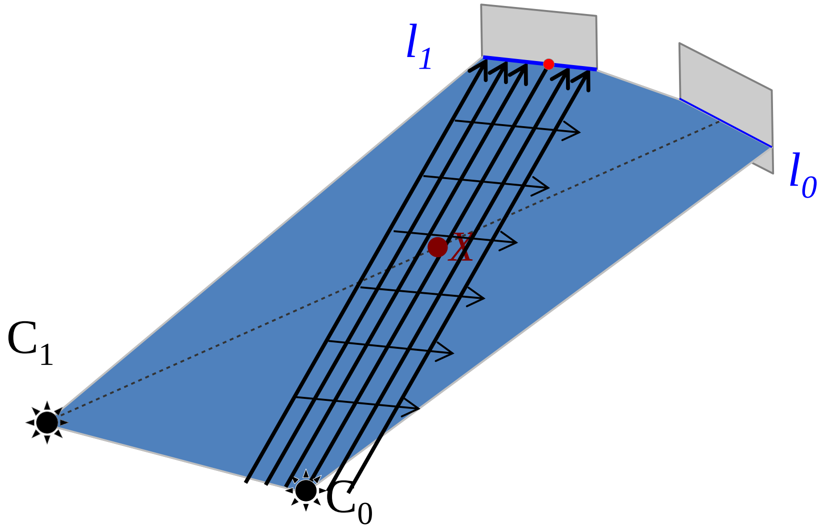


Redundancies on Epipolar Lines

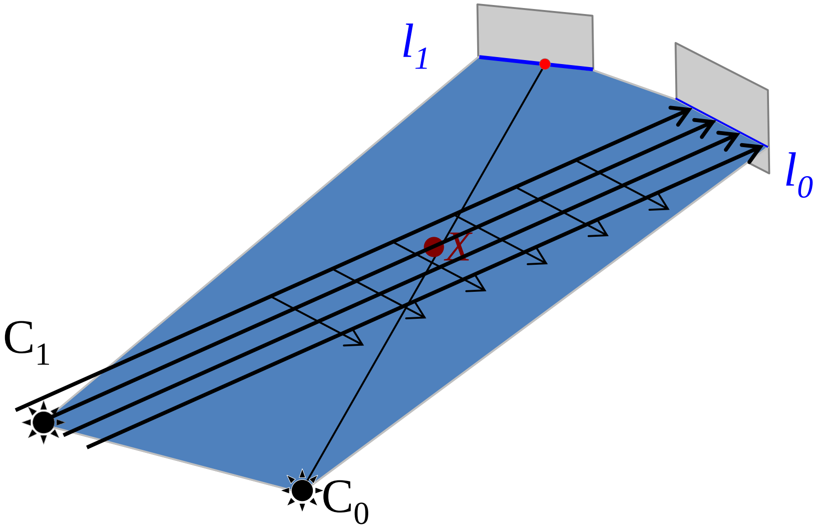


Epipolar Plane

Redundancies on Epipolar Lines



Redundancies on Epipolar Lines



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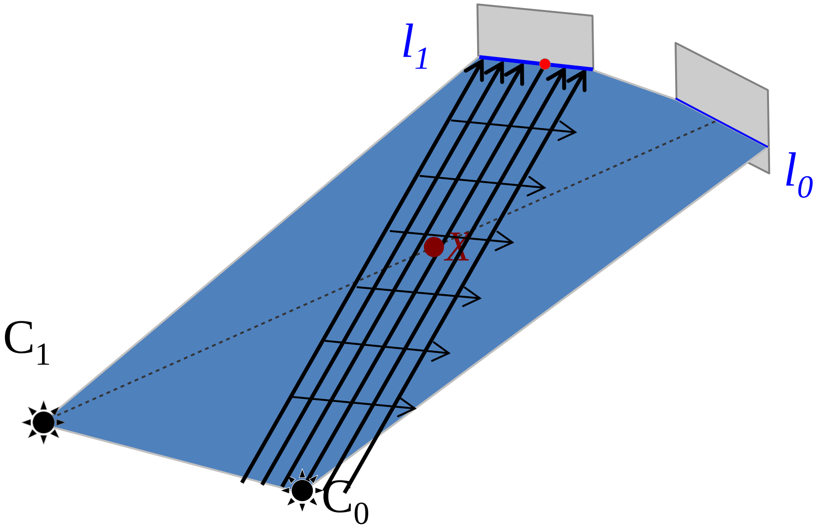

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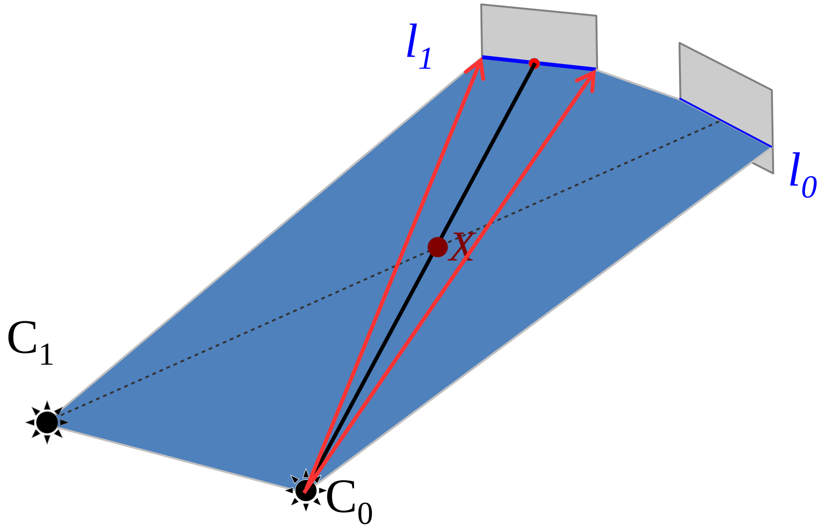


Not true for fan-beam geometry

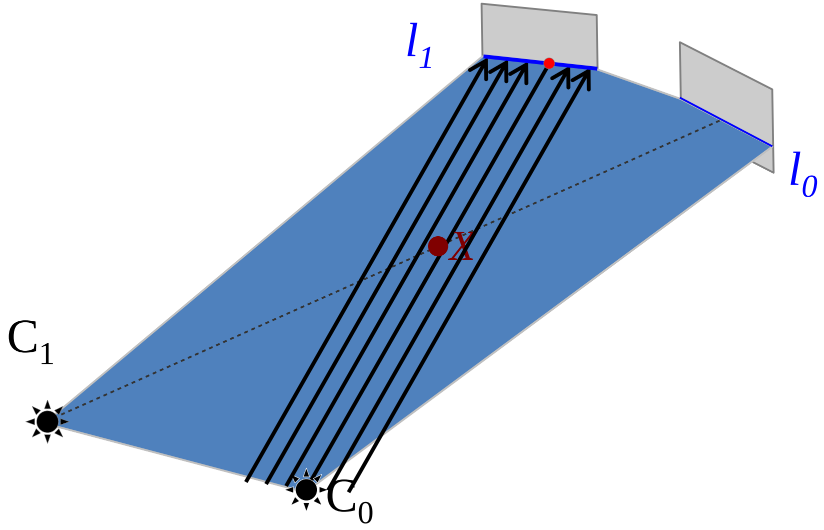
Suppose C were far away.

-> Rays would be parallel.

-> Then: **plane integral = line integral!**

Grangeat's Theorem

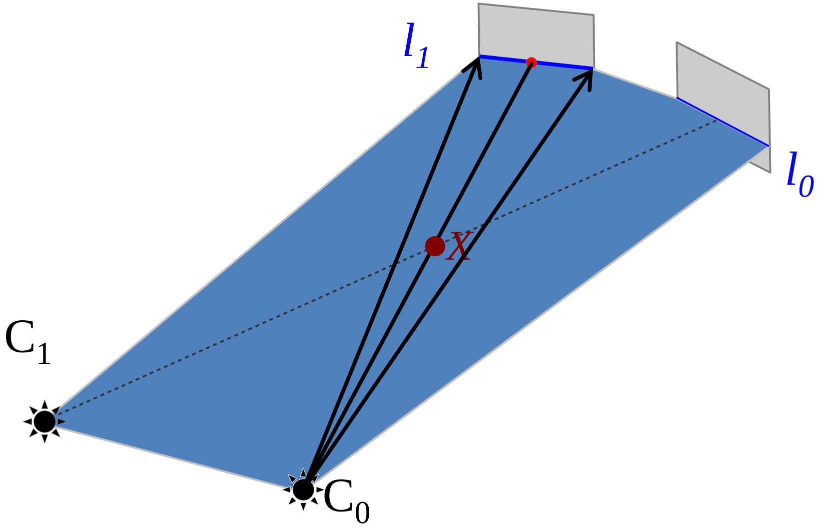
Grangeat's Theorem



Integral of Epipolar Plane

$$\rho_f(\mathbf{E}) = \iint f(x, y, 0)\,dx\,dy$$

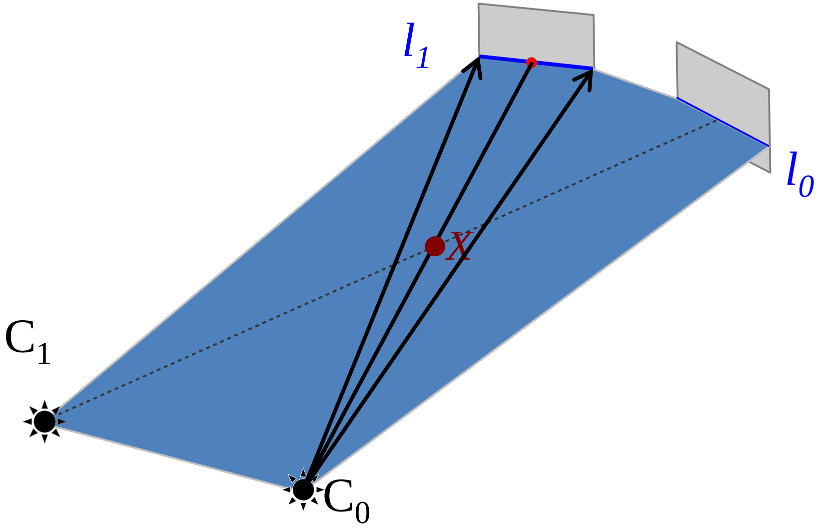
Grangeat's Theorem



Integral of Epipolar Plane

$$\rho_f(\mathbf{E}) = \iint f(x,y,0)dxdy$$

Grangeat's Theorem



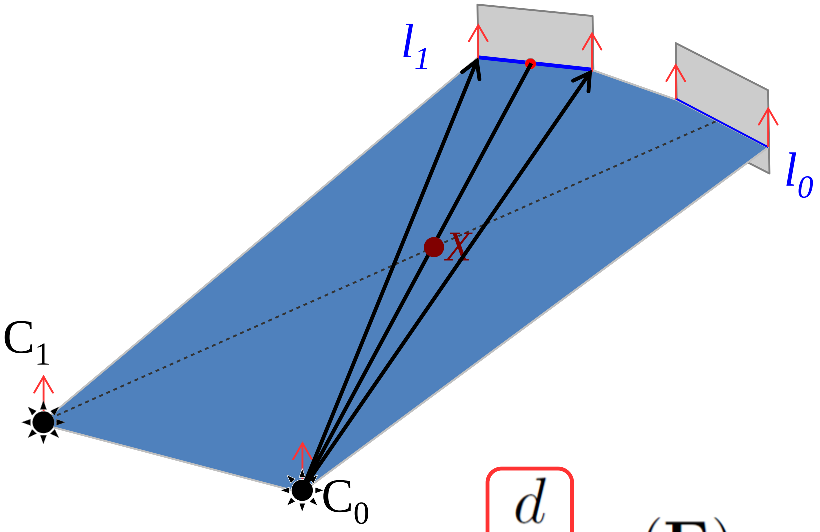
Integral of Epipolar Plane

$$\rho_f(\mathbf{E}) = \iint f(x,y,0)dxdy$$

$$= \iint f(\Phi(\varphi,r))\boxed{det(J_\Phi)}drd\varphi$$

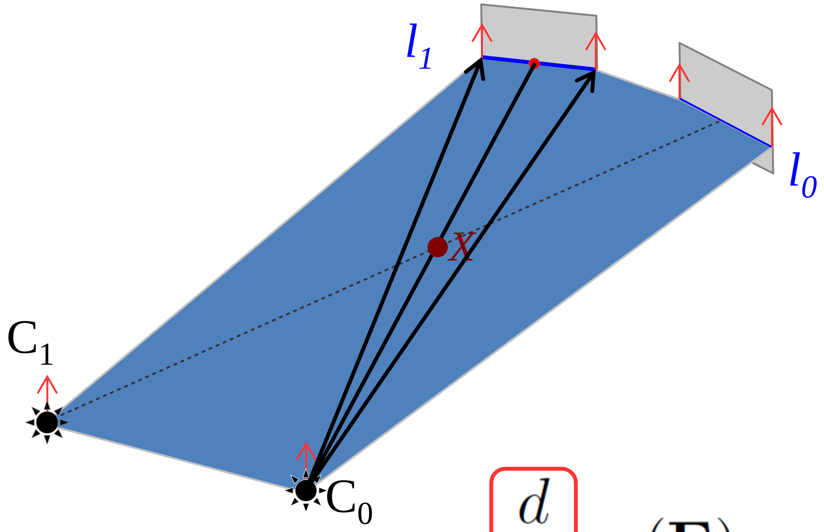
Weighted line integral on detector

Grangeat's Theorem



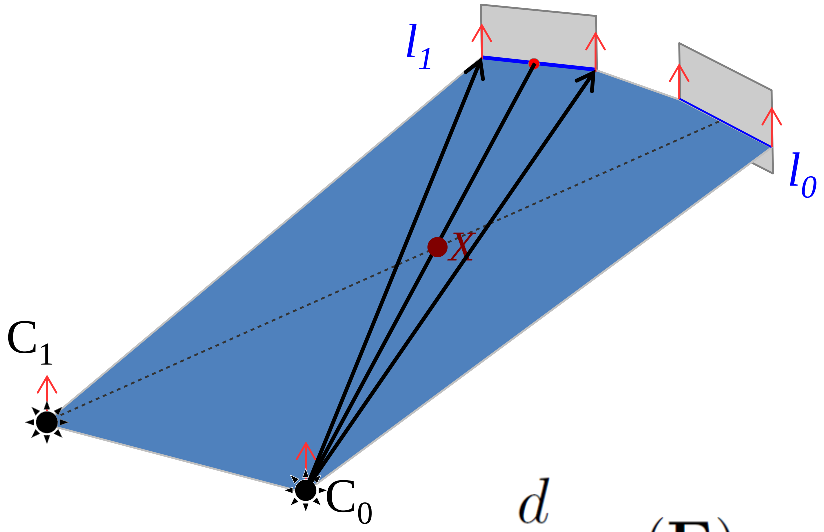
$$\boxed{\frac{d}{dn}}\rho_f(\mathbf{E})$$

Grangeat's Theorem



$$\boxed{\frac{d}{dn}}\rho_f(\mathbf{E}) \approx \frac{d}{d\kappa}\iint f(\Phi(\varphi, r))dr d\varphi \approx \boxed{\frac{d}{dt}}\rho_I(\mathbf{l})$$

Grangeat's Theorem



The derivative of the 3-D Radon transform in normal direction is approximately the derivative of the 2-D Radon transform in intercept direction.

$$\frac{d}{dn}\rho_f(\mathbf{E}) \approx \frac{d}{d\kappa}\iint f(\Phi(\varphi, r))dr d\varphi \approx \frac{d}{dt}\rho_I(\mathbf{l})$$

Topics

Radon Transform (Refresher)

Epipolar Geometry

- In Diagrams
- Redundancies on Epipolar Lines
- Grangeat's Theorem

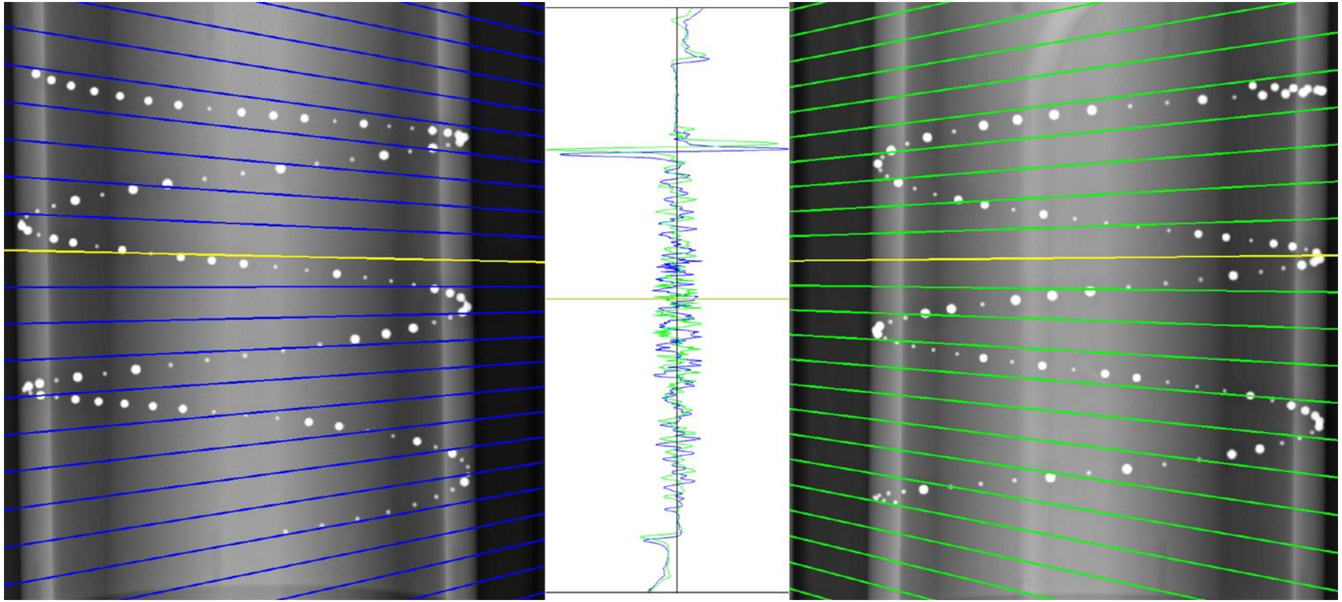
Applied Example

Summary

- Take Home Messages
- Further Readings

Metric for Geometric Consistency

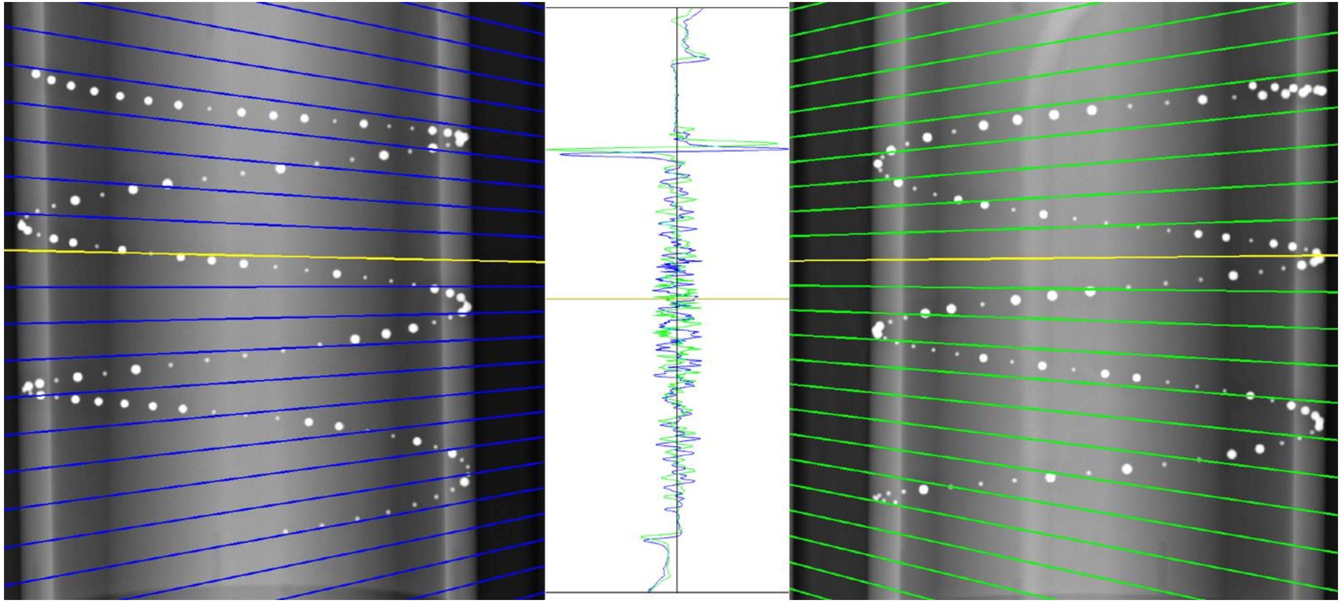
Derivative of line integrals



Plane angle (around baseline)

Metric for Geometric Consistency

Derivative of line integrals

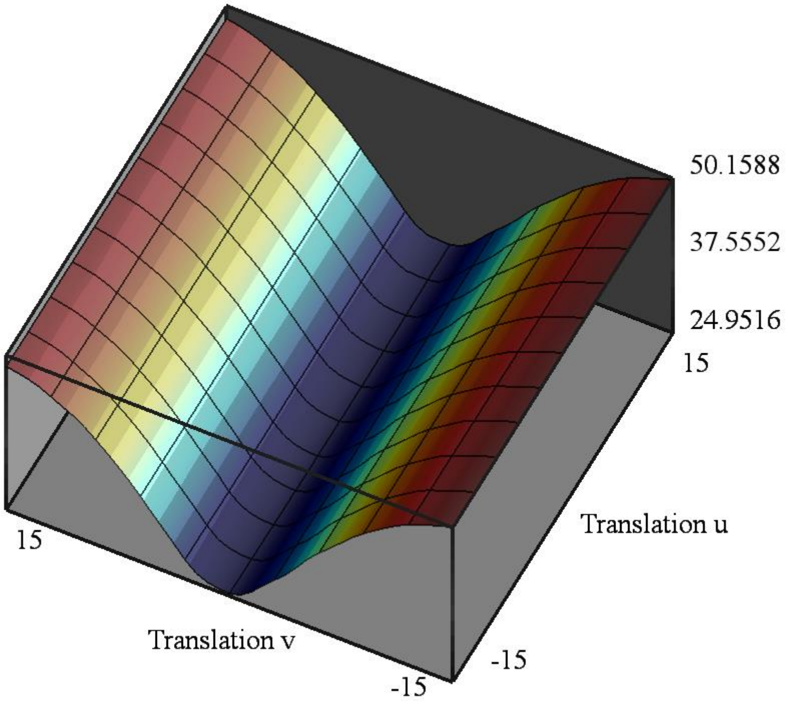


Plane angle (around baseline)

Metric defined as difference between blue and green curves!

A Plot for Detector Shifts and Just Two Views

- For a “close” image pair
- Range: 15 pixels
- Epipolar lines almost parallel to u-axis



Topics

Take Home Messages

- In this unit you got visual insight into the intricacies of epipolar geometry.
- We also connected line integrals on the detector with integrals of the epipolar plane.
- In the next unit we will learn how to use this to develop an epipolar consistency metric.

Further Readings

André Aichert et al. "Epipolar Consistency in Transmission Imaging". In: *IEEE Transactions on Medical Imaging* 34.11 (Nov. 2015), pp. 2205–2219. DOI: 10.1109/TMI.2015.2426417

Acknowledgements:

MEDICAL VALLEY
Europäische Metropolregion Nürnberg

SIEMENS

Universitätsklinikum
Erlangen

Medical Image Processing for Interventional Applications

Epipolar Consistency Metric

Online Course – Unit 36
Andreas Maier, André Aichert, Frank Schebesch
Pattern Recognition Lab (CS 5)

virtuelle hochschule bayern

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Epipolar Consistency Metric

- Idea
- Formalism

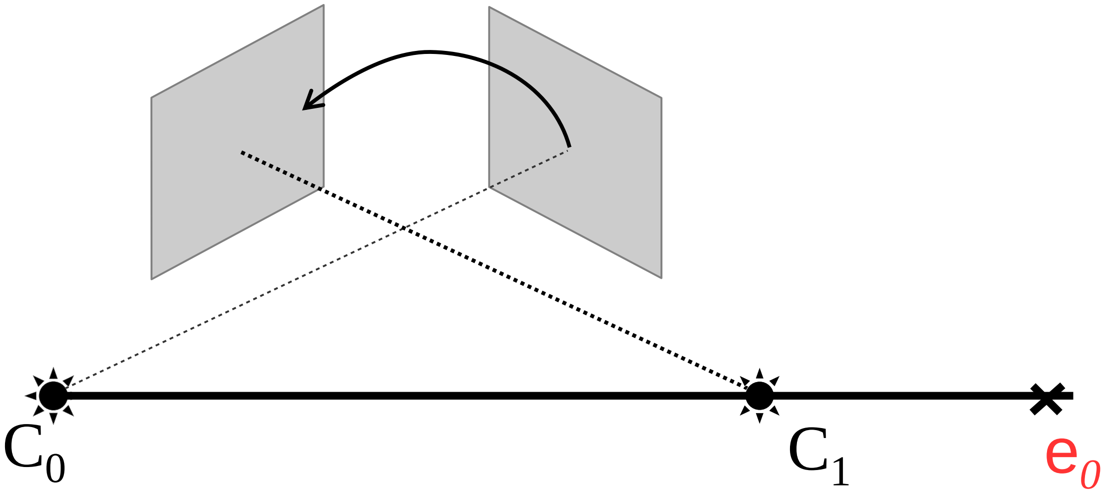
Special Case – Circular Trajectory

Summary

- Take Home Messages
- Further Readings

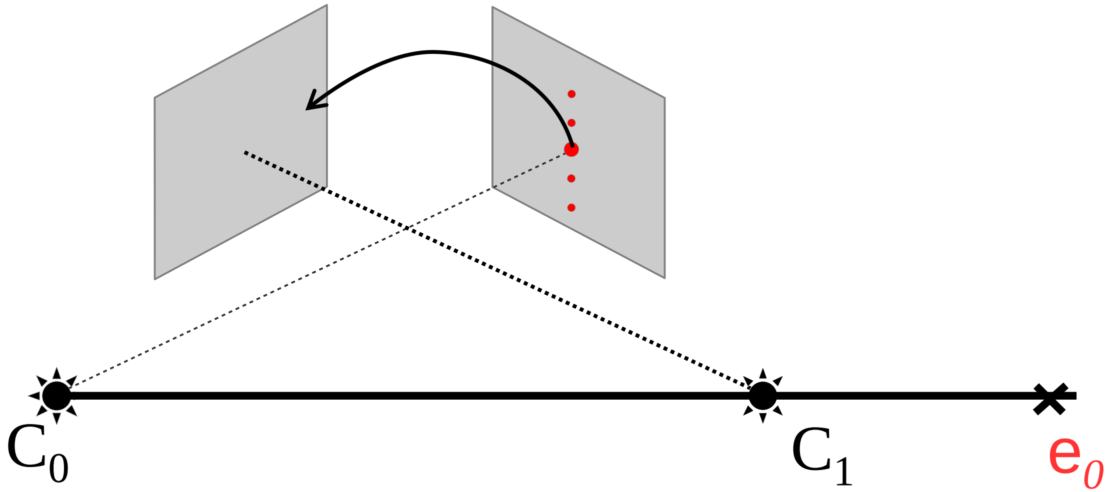
Idea

1. Compute fundamental matrix and epipoles



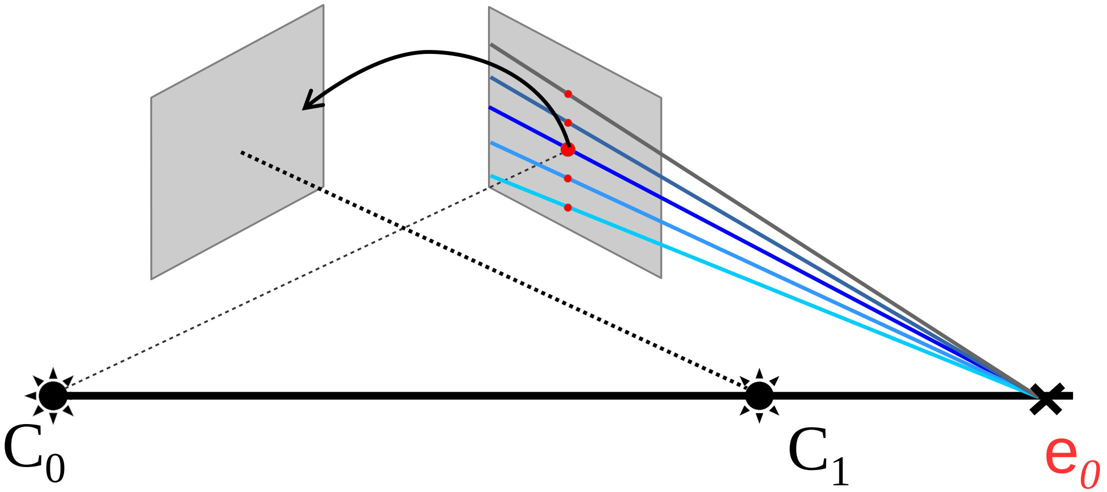
Idea

1. Compute fundamental matrix and epipoles
2. Select points in reference image



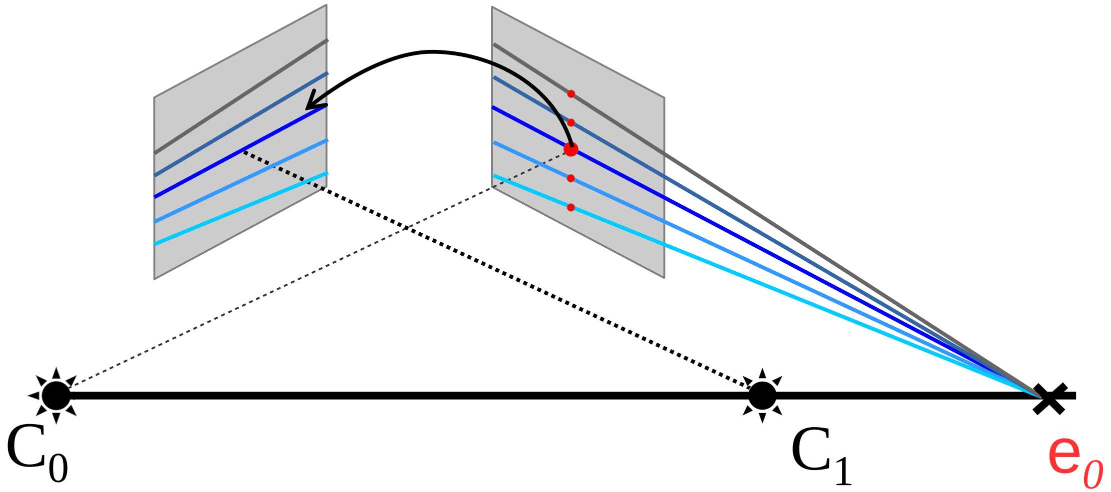
Idea

1. Compute fundamental matrix and epipoles
2. Select points in reference image
3. Compute lines in reference image as the join with the epipole



Idea

1. Compute fundamental matrix and epipoles
2. Select points in reference image
3. Compute lines in reference image as the join with the epipole
4. Use fundamental matrix to project to lines in the other image



Formalism

$$M_0^i = \frac{1}{|\mathcal{X}_0^i|} \sum_{x_0 \in \mathcal{X}_0^i} \left(\frac{d}{dt} \rho_{I_0}(x_0 \times e_0) - \frac{d}{dt} \rho_{I_i}(F_0^i x_0) \right)^2$$

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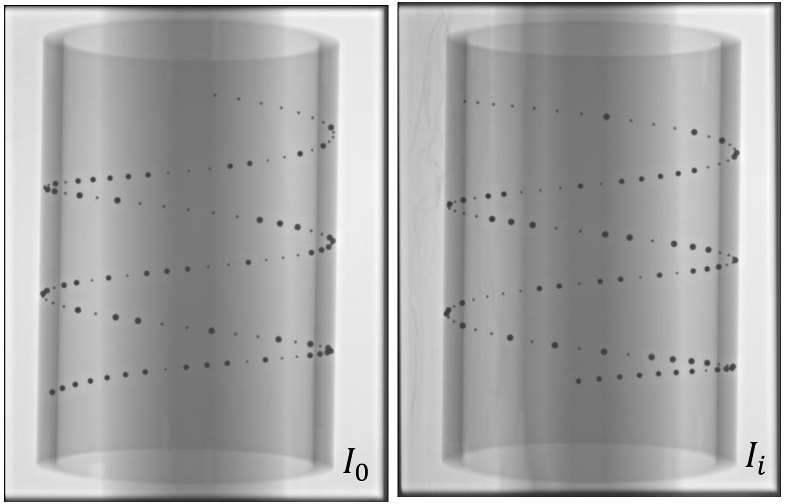

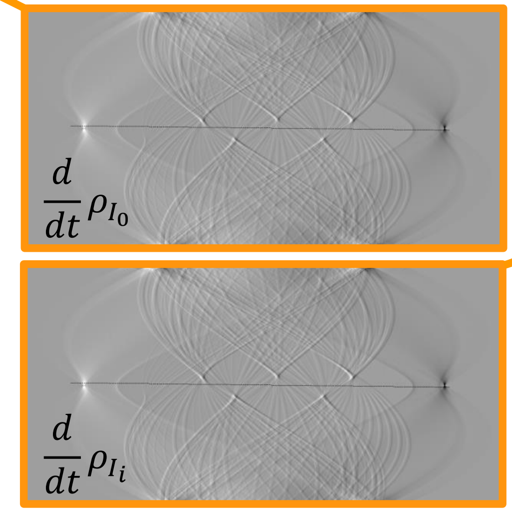

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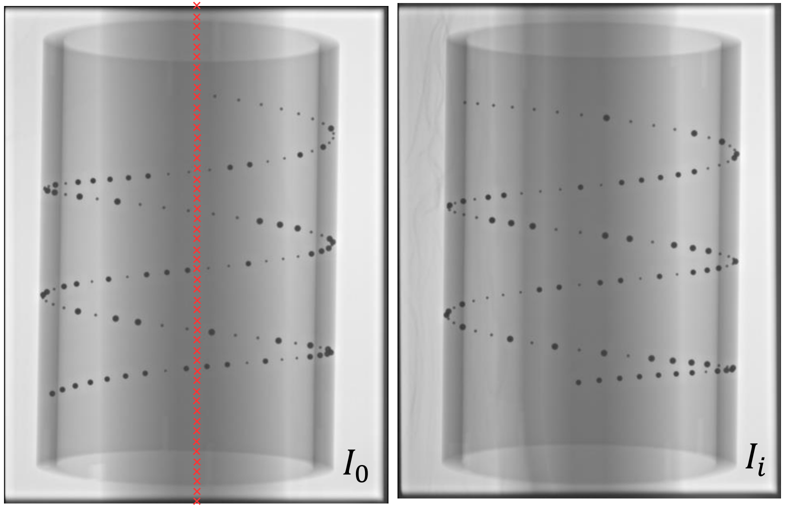

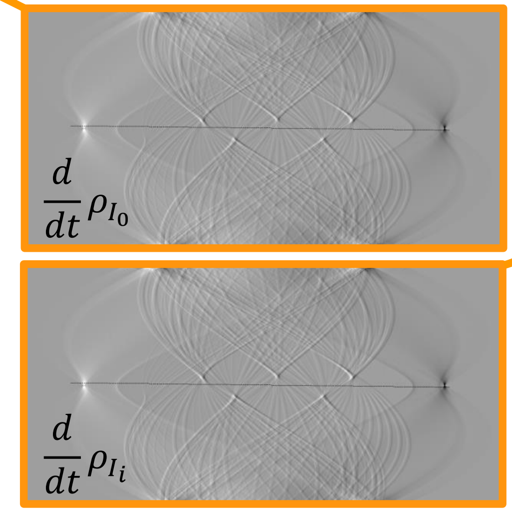

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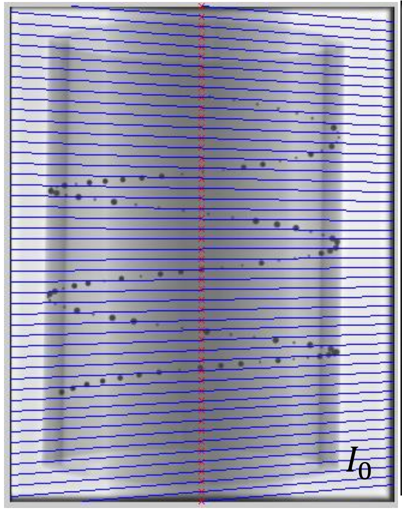

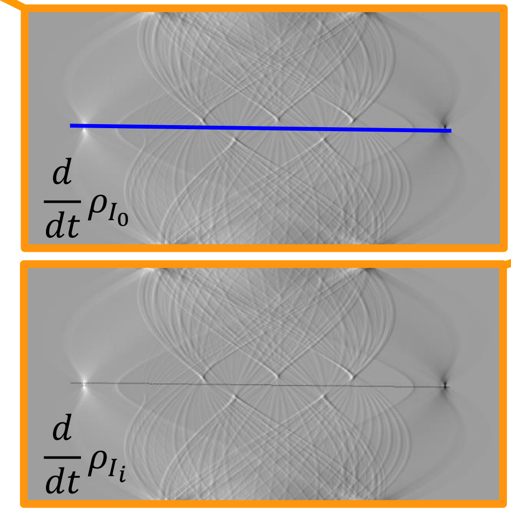

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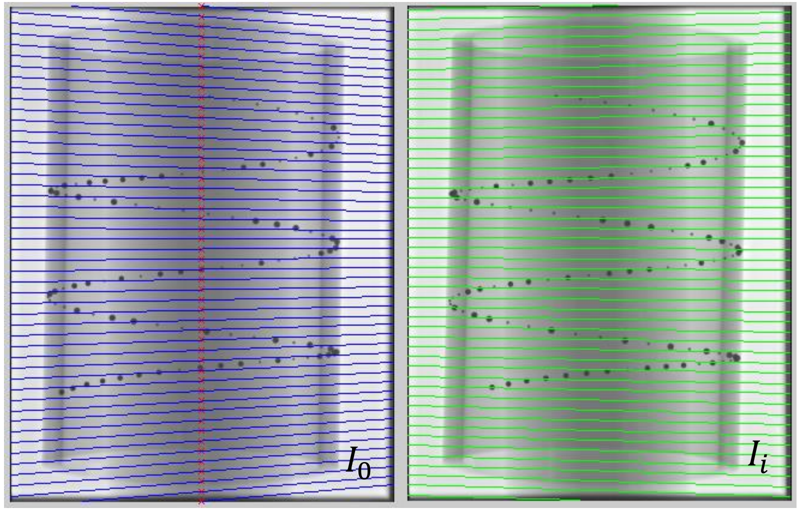

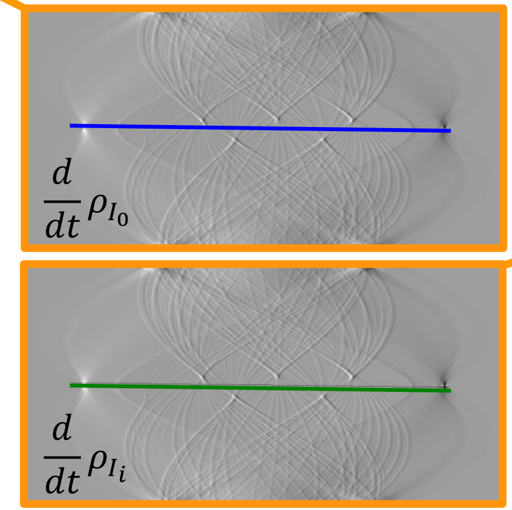

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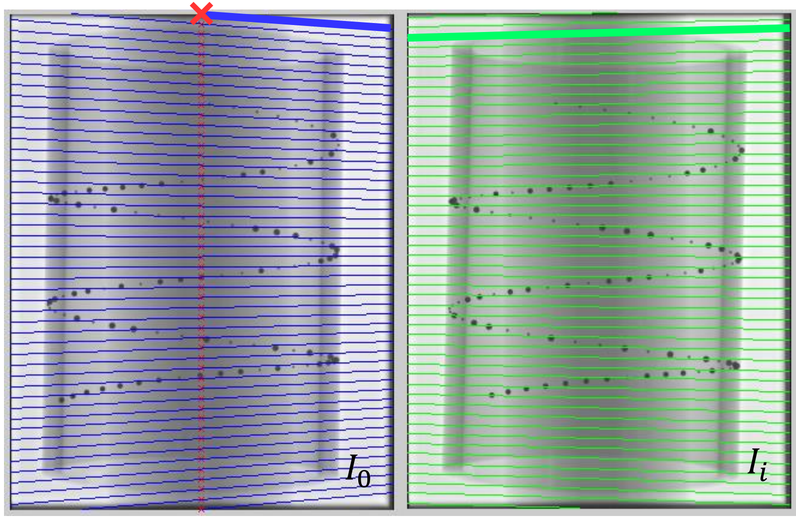

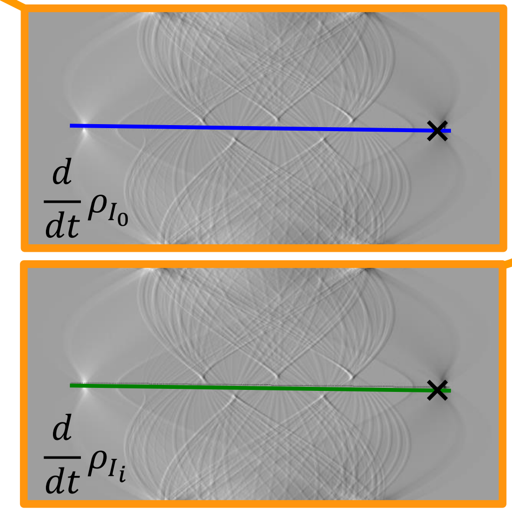

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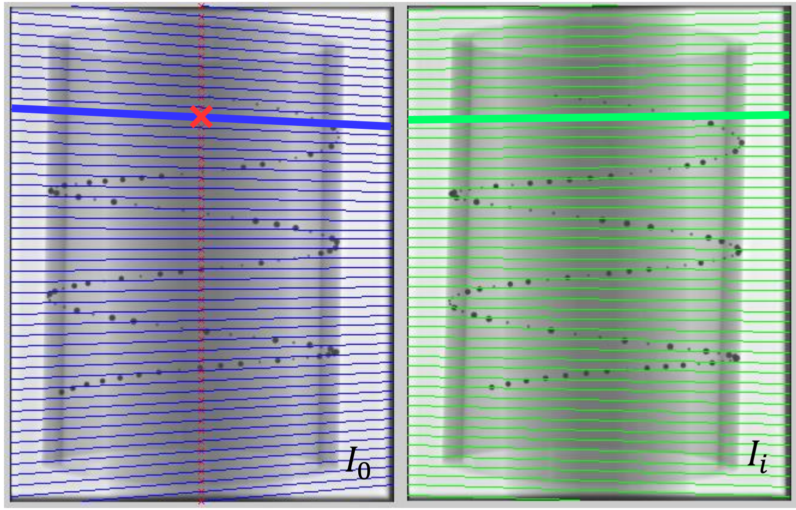

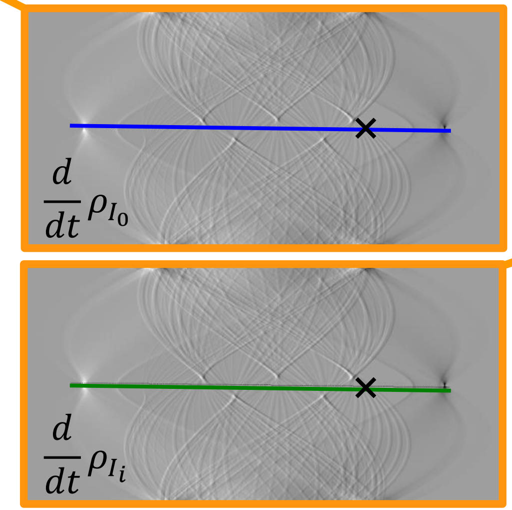

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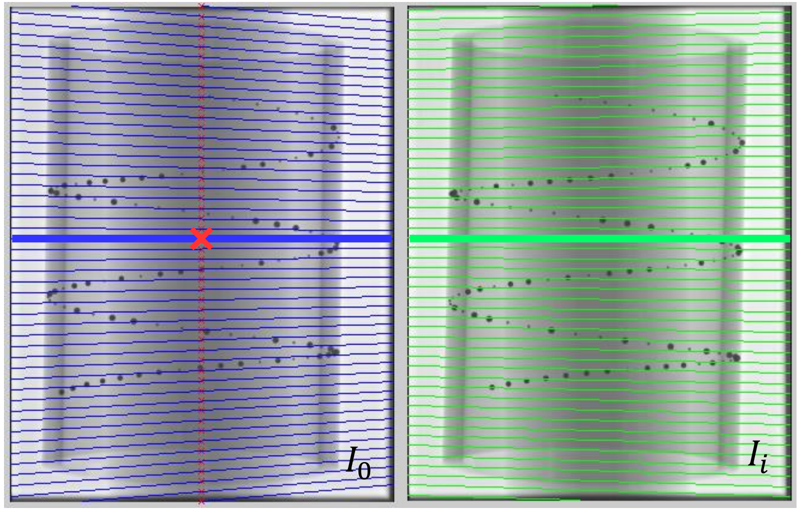

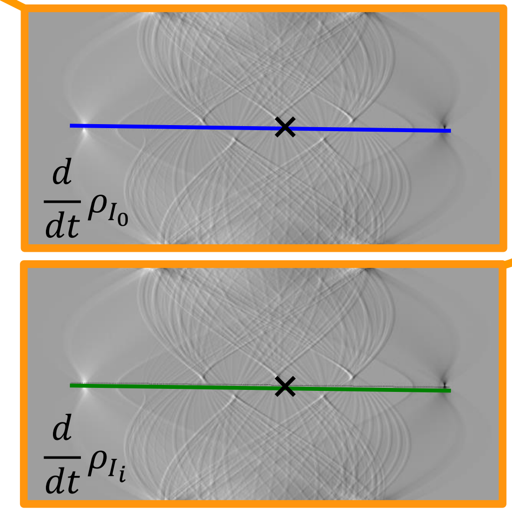

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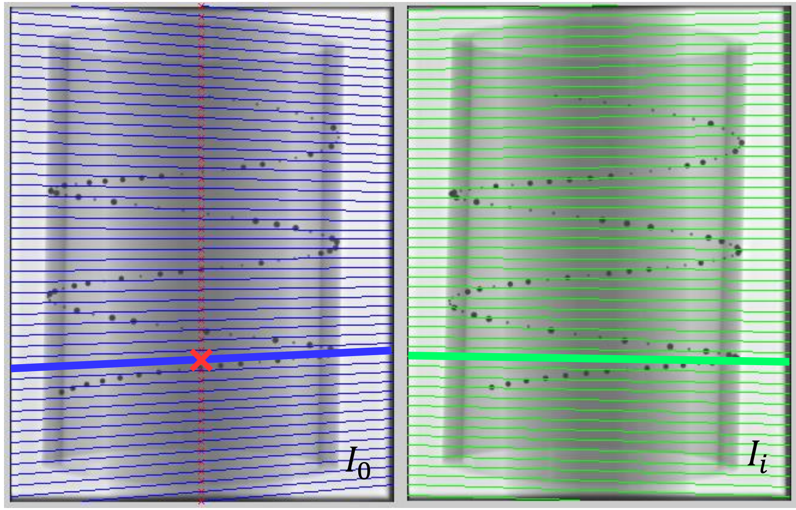

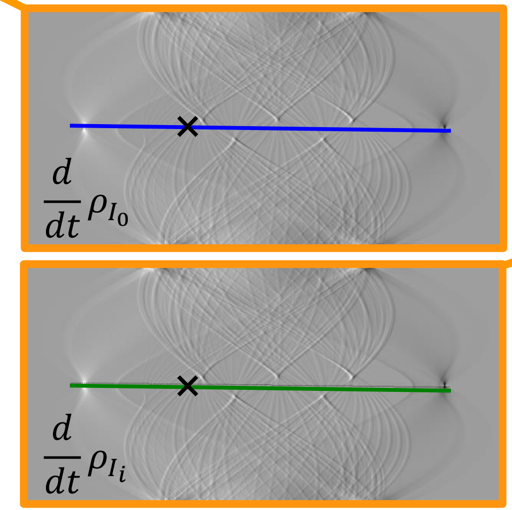

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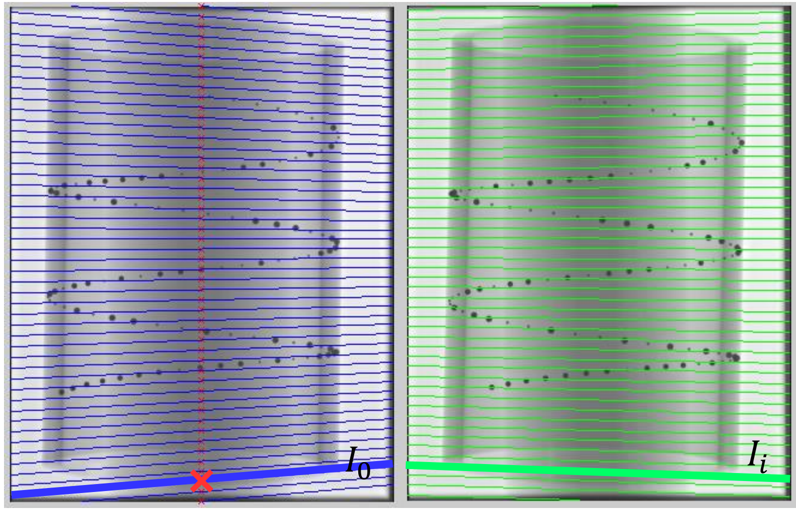
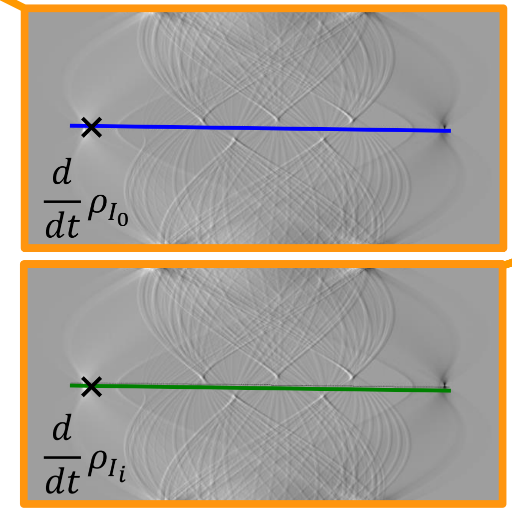




Topics

Epipolar Consistency Metric

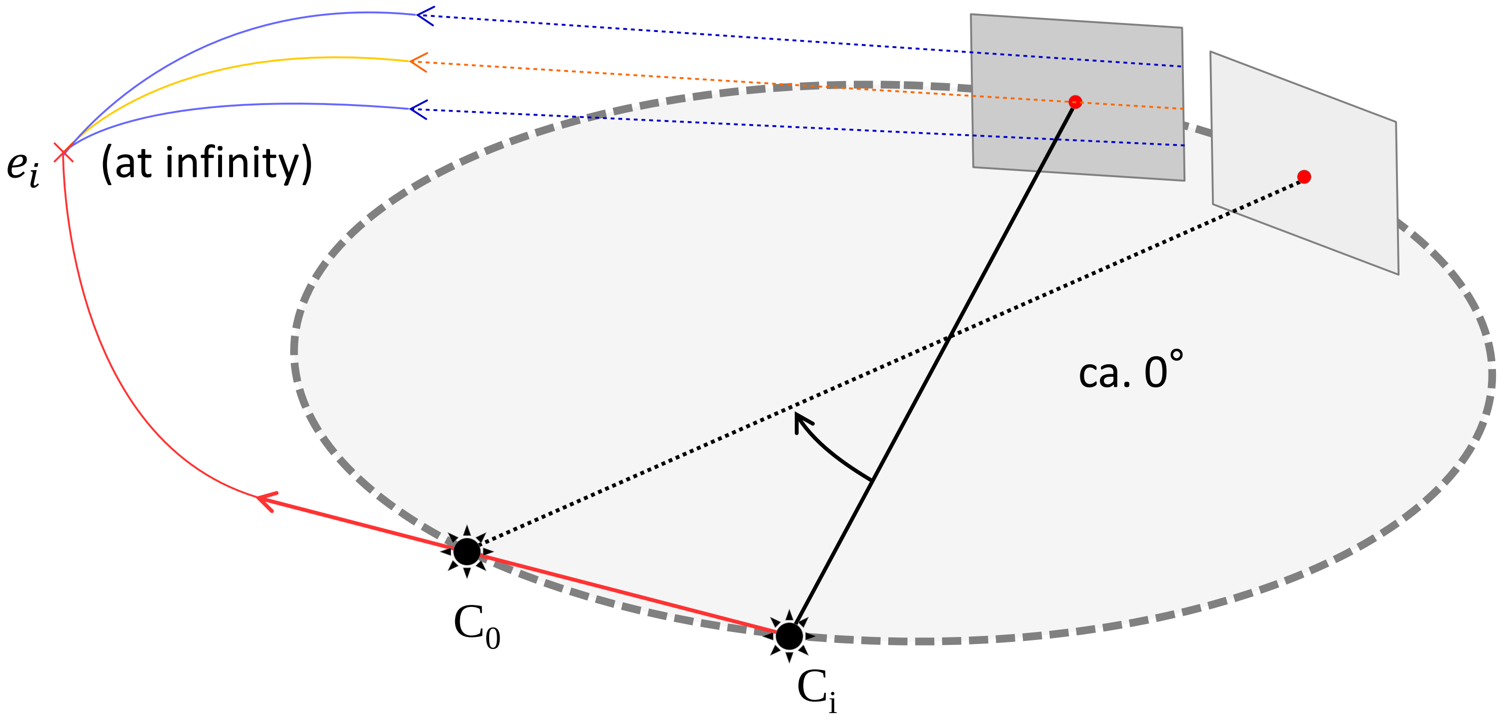
- Idea
- Formalism

Special Case – Circular Trajectory

Summary

- Take Home Messages
- Further Readings

Epipolar Geometry of a Circular Trajectory



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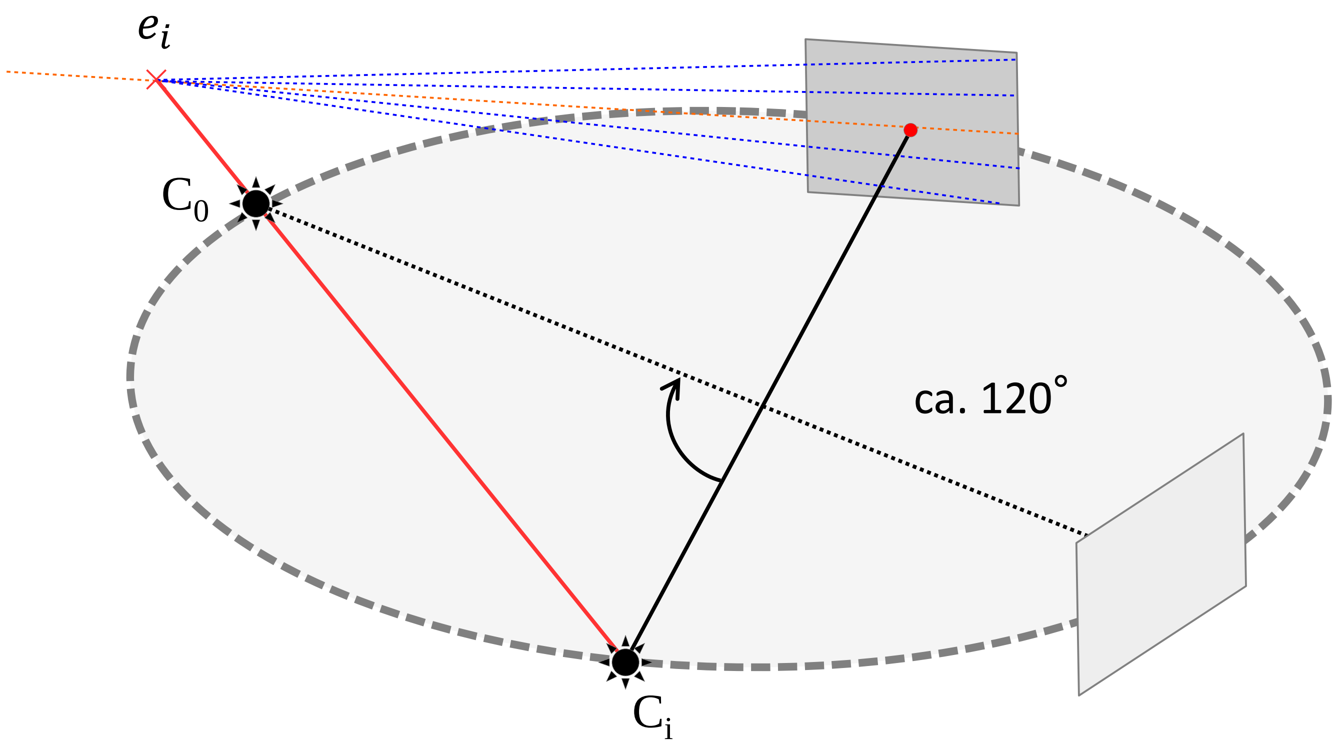

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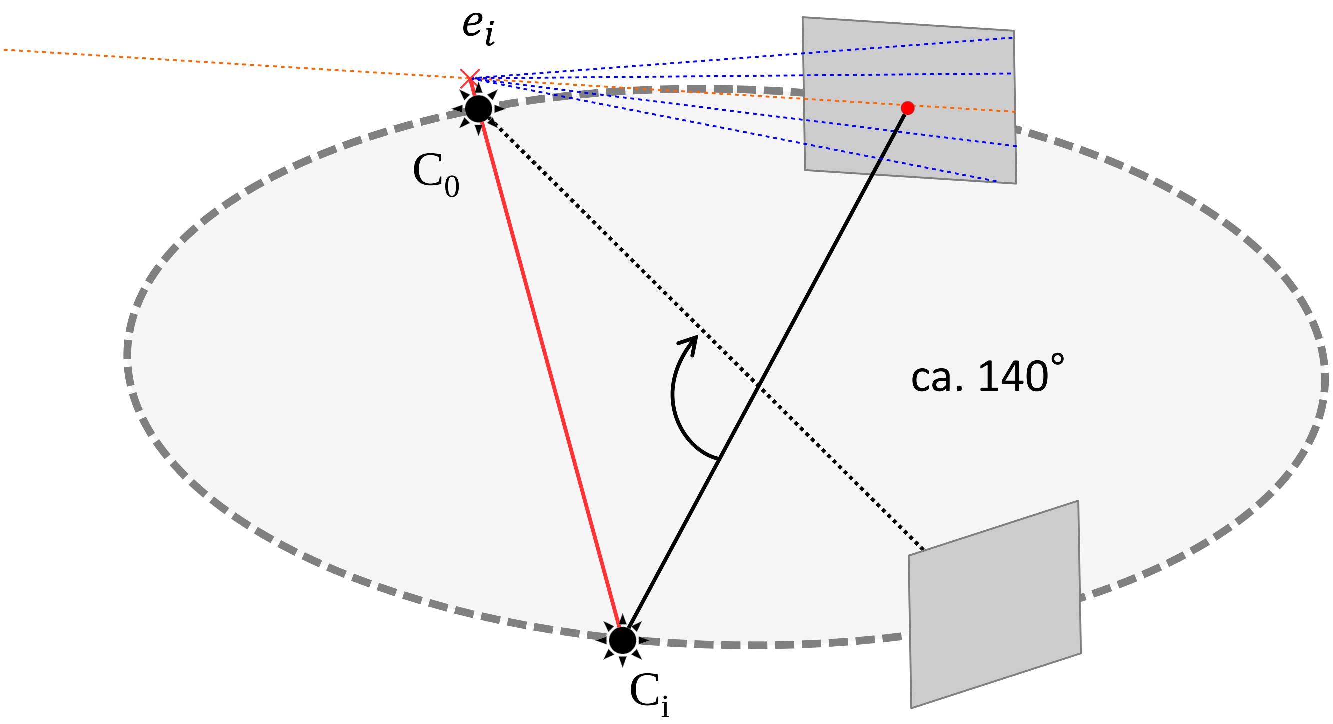
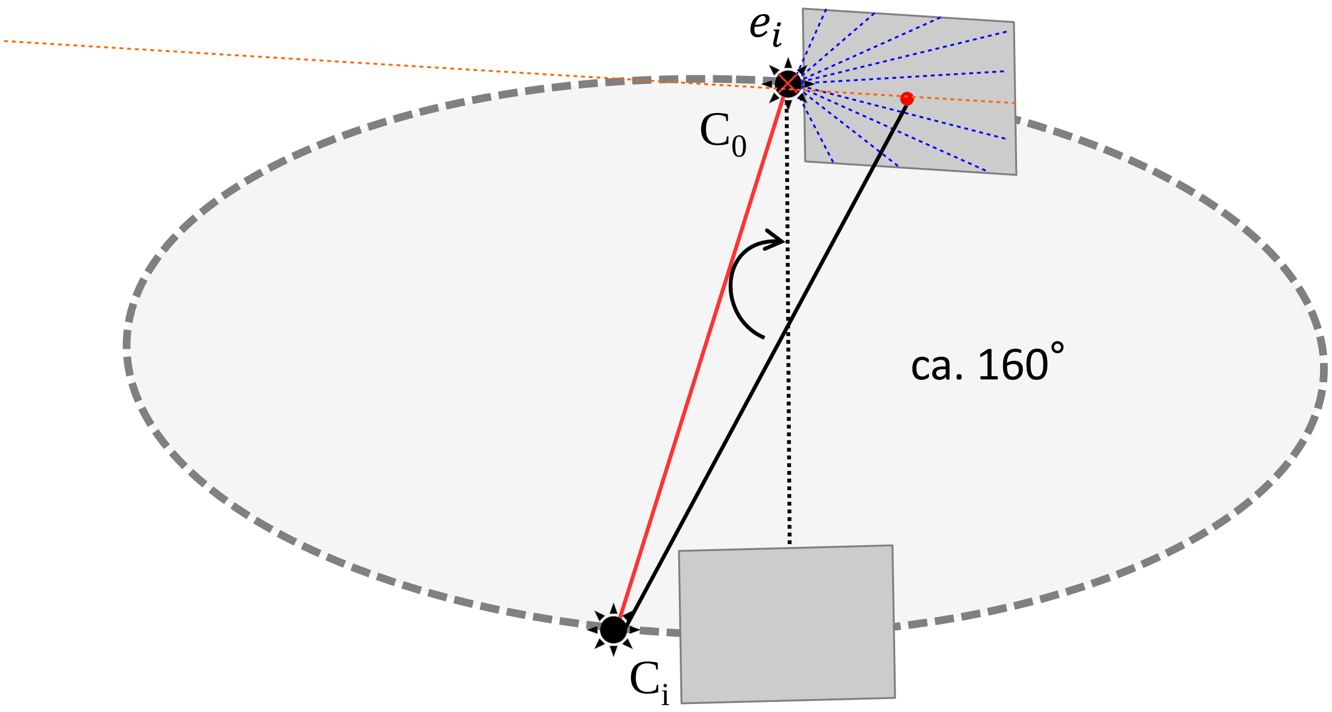
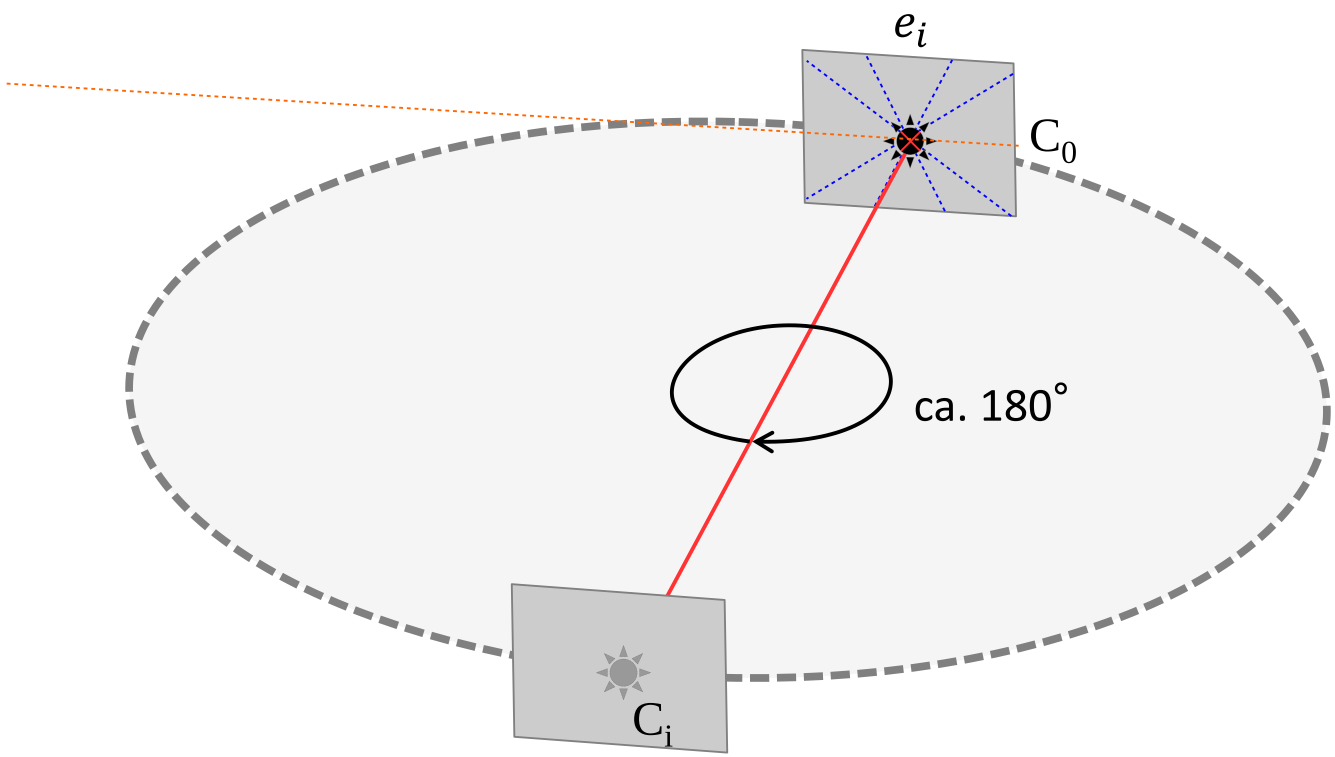


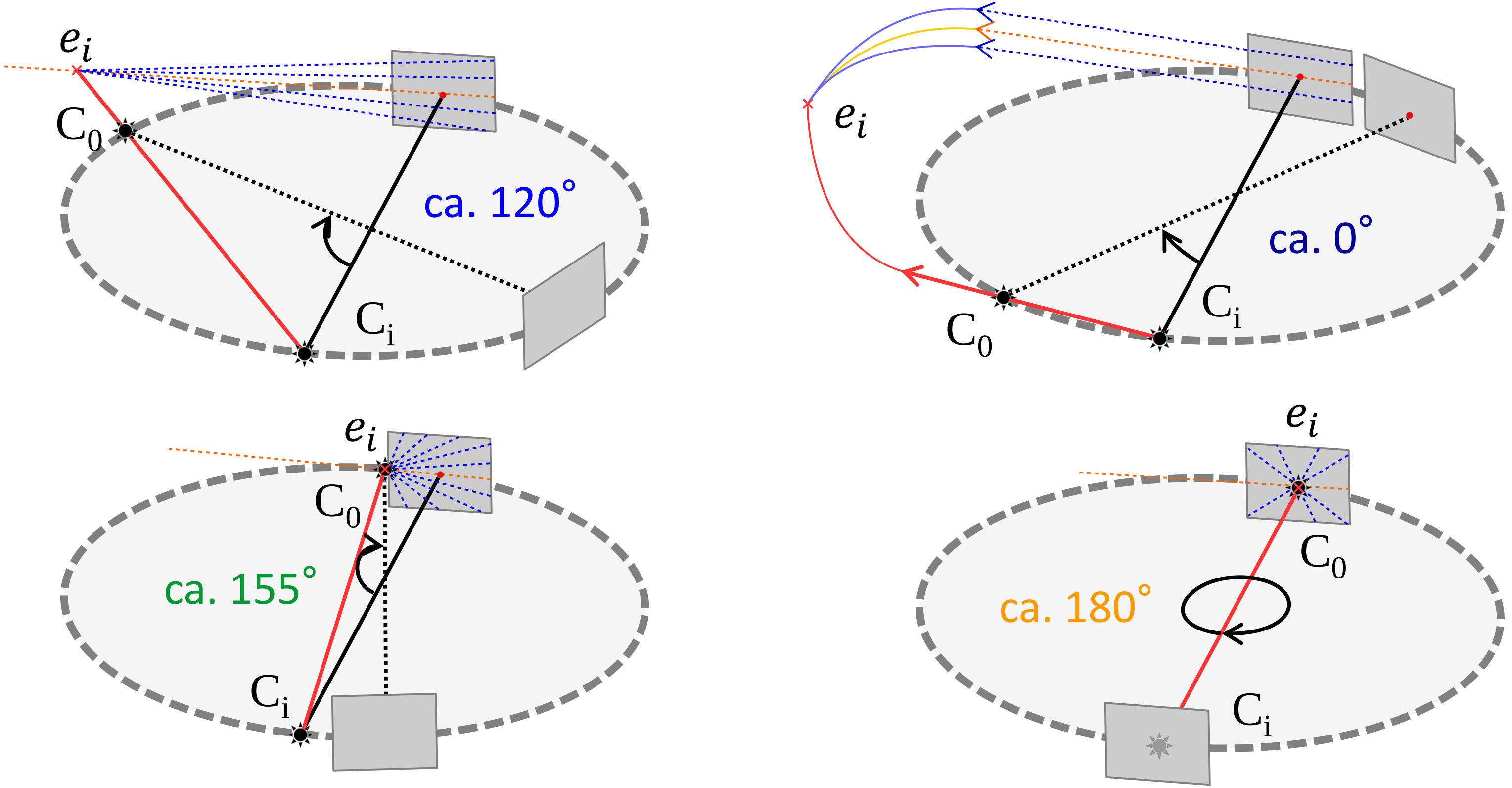
Epipolar Geometry of a Circular Trajectory



Epipolar Geometry of a Circular Trajectory



Let's Summarize ...



Optimization in “Reliable Directions“

Ideal scenario:

- Two pairs of views with orthogonal lines
- Or when epipole is within the image (opposing views)
- Ideally random positions on a sphere around the object

→ No information in directions of parallel lines (due to summation)

Topics

Epipolar Consistency Metric

Idea

Formalism

Special Case – Circular Trajectory

Summary

Take Home Messages

Further Readings

Take Home Messages

- We described the formulation of a consistency metric which works on **any** pair of transmission images.
- In case of opposing views, the epipole is within the image, or otherwise information is given only in one direction.

Further Readings

André Aichert et al. "Epipolar Consistency in Transmission Imaging". In: *IEEE Transactions on Medical Imaging* 34.11 (Nov. 2015), pp. 2205–2219. DOI: 10.1109/TMI.2015.2426417

Acknowledgements: