# Medical Image Processing for Diagnostic Applications

## Iterative Closest Point Algorithm – Theory

Online Course – Unit 70

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# Topics

Iterative Closest Point (ICP)

- Theory
- Point-to-Point Error Metric
- Point-to-Plane Error Metric

Summary

- Take Home Messages
- Further Readings

# Algorithm Outline

**Given:** Two point sets $P = \{\mathbf{p}_i\}$, $Q = \{\mathbf{q}_i\}$, $i = 1, \ldots, N$, where $\mathbf{p}_i$, $\mathbf{q}_i$ are $3 \times 1$ column vectors

**Wanted:** Best transformation $\mathbf{T}$ between these two point sets, consisting of

- rotation matrix $\mathbf{R}$,
- translation $\mathbf{t}$

**Algorithm 1:** Iterative closest point [5]

**Input :** Two point clouds: $P$, $Q$

**Output:** Transformation $\mathbf{T}$, which aligns $P$ and $Q$

$\mathbf{T} \leftarrow \mathbf{T}_0$;<br>
**while** *not converged* **do**<br>
    **for** $i \leftarrow 1$ **to** $N$ **do**<br>
        $\mathbf{c}_i \leftarrow \texttt{GetClosestPointInQ}(\mathbf{T} \cdot \mathbf{p}_i)$;<br>
        **if** $\|\mathbf{T} \cdot \mathbf{p}_i - \mathbf{c}_i\| \leq \theta_{max}$ **then**<br>
            $\omega_i \leftarrow 1$;<br>
        **else**<br>
            $\omega_i \leftarrow 0$;<br>
        **end**<br>
    **end**<br>
    $\mathbf{T} \leftarrow \arg\min_{\mathbf{T}} \sum_{i=1}^{N} \omega_i \|\mathbf{T} \cdot \mathbf{p}_i - \mathbf{c}_i\|^2$;<br>
**end**

# Point-to-Point Error Metric

$\rightarrow$ minimizes the Euclidean distance between selected point pairs.

The optimization problem can be solved by:

- a singular value decomposition (SVD) based method [1],
- a quaternion method [3],
- orthonormal matrices [4], or
- calculation based on dual quaternions [6].

# Point-to-Point Error Metric (SVD) [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2,$$

where

$$\mathbf{q}_i = \underset{\mathbf{q}_x \in Q}{\arg\min}\ \mathrm{d}(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i)$$

Decouple translation and rotation:
$\rightarrow$ computation of the center points $(\bar{\mathbf{p}}, \bar{\mathbf{q}})$ of both point sets for translation

$$\left.\begin{array}{l} P' : \mathbf{p}_i' = \mathbf{p}_i - \bar{\mathbf{p}}, \\ Q' : \mathbf{q}_i' = \mathbf{q}_i - \bar{\mathbf{q}} \end{array}\right\} \quad \Rightarrow \quad \varepsilon = \sum_{i=1}^{N} \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i'\|^2$$

| | |
|---|---|
| $\mathbf{R}$ | rotation matrix |
| $\mathbf{t}$ | translation vector |
| $\mathrm{d}(x,y)$ | Euclidean distance between two points $x$ and $y$ |
| $Q = \{\mathbf{q}_i\}$ | 3-D point sets |
| $P = \{\mathbf{p}_i\}$ | 3-D point sets |
| $i = 1, ..., N$ | number of points |

# Point-to-Point Error Metric (SVD) [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \left\| \mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right\|^2$$

Two simplifications:

- Index $i$ matches the closest points of both point clouds.
- Both point clouds contain the same number of points.

The rotation matrix **R** is computed via help of the matrix **H**:

$$\mathbf{H} = \sum_{i=1}^{N} \mathbf{p}'_i \mathbf{q}'_i = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{T}}, \qquad \mathbf{R} = \mathbf{V}\mathbf{U}^{\mathsf{T}}.$$

The decoupled translation vector is recovered by

$$\mathbf{t} = \bar{\mathbf{q}} - \mathbf{R}\bar{\mathbf{p}}.$$

# Point-to-Point Error Metric (SVD) [1]

Derivation for **H**:

1. Expanding right-hand side of $\varepsilon$:

$$\varepsilon = \sum_{i=1}^{N} \left\| \mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right\|^2 = \sum_{i=1}^{N} \left(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\right)^{\mathsf{T}} \left(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\right) = \sum_{i=1}^{N} \left( \mathbf{p}'^{\mathsf{T}}_i \mathbf{R}^{\mathsf{T}} \mathbf{R}\mathbf{p}'_i + \mathbf{q}'^{\mathsf{T}}_i \mathbf{q}'_i - 2\mathbf{q}'^{\mathsf{T}}_i \mathbf{R}\mathbf{p}'_i \right)$$

2. Minimizing $\varepsilon$ is equivalent to maximizing:

$$F = \sum_{i=1}^{N} \mathbf{q}'^{\mathsf{T}}_i \mathbf{R}\mathbf{p}'_i = \operatorname{tr}\left( \sum_{i=1}^{N} \mathbf{R}\mathbf{p}'_i \mathbf{q}'^{\mathsf{T}}_i \right) = \operatorname{tr}\left(\mathbf{R}\mathbf{H}\right)$$

# Point-to-Point Error Metric (SVD) [1]

Depending on the properties of the point clouds, the solution may not be unique:

- If $P'$ is collinear
  $\rightarrow$ infinitely many rotations and reflections.
- If $P'$ is coplanar
  $\rightarrow$ two unique solutions, the desired rotation matrix and its reflection:
  - the reflection is given if $\det \mathbf{R} = -1$,
  - the correct rotation matrix is given by $\mathbf{R} = \mathbf{V}'\mathbf{U}^\mathsf{T}$,
  - $\mathbf{V}'$ is constructed by flipping the sign of the $i$-th column of $\mathbf{V}$, where index $i$ identifies the zero singular value.
- If $P'$ is not coplanar
  $\rightarrow$ one unique solution for $\mathbf{R} = \mathbf{V}\mathbf{U}^\mathsf{T}$.

# Point-to-Point Error Metric (Quaternions) [2, 3]

Optimization function of the ICP:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} \left\|(\mathbf{R}(\mathbf{q}_R)\mathbf{p}_i + \mathbf{q}_T) - \mathbf{q}_i\right\|^2,$$

where

$$\mathbf{q}_i = \underset{\mathbf{q}_x \in Q}{\arg\min}\ \mathrm{d}\left(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i\right)$$

| | |
|---|---|
| $\mathbf{R} = \mathbf{R}(\mathbf{q}_R)$ | rotation matrix generated by a unit quaternion |
| $\mathbf{q}_R = [q_0 q_1 q_2 q_3]^{\mathrm{T}}$ | unit quaternion (rotation) |
| $\mathbf{q}_T = [q_4 q_5 q_6]^{\mathrm{T}}$ | unit quaternion (translation) |
| $\mathbf{q} = [\mathbf{q}_R \vert \mathbf{q}_T]^{\mathrm{T}}$ | registration state vector |
| $\mathrm{d}(x, y)$ | Euclidean distance between two points $x$ and $y$ |
| $Q = \{\mathbf{q}_i\}$ | 3-D point sets |
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Rotation matrix generated by a unit quaternion:

$$\mathbf{R} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{pmatrix}$$

# Point-to-Point Error Metric (Quaternions) [2, 3]

Optimization function of the ICP:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} \left\|(\mathbf{R}(\mathbf{q}_R)\mathbf{p}_i + \mathbf{q}_T) - \mathbf{q}_i\right\|^2,$$

where

$$\mathbf{q}_i = \underset{\mathbf{q}_x \in Q}{\arg\min}\ \mathrm{d}\left(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i\right)$$

Decouple translation and rotation:
$\rightarrow$ computation of the center points $(\bar{\mathbf{p}}, \bar{\mathbf{q}})$ of both point sets for translation

$$\left.\begin{array}{l} P' : \mathbf{p}'_i = \mathbf{p}_i - \bar{\mathbf{p}}, \\ Q' : \mathbf{q}'_i = \mathbf{q}_i - \bar{\mathbf{q}} \end{array}\right\} \quad \Rightarrow \quad \Sigma_{pq} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{p}'_i \mathbf{q}'^{\mathrm{T}}_i$$

| | |
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| $\mathbf{R} = \mathbf{R}(\mathbf{q}_R)$ | rotation matrix generated by a unit quaternion |
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# Point-to-Point Error Metric (Quaternions) [2, 3]

- Cross-covariance matrix:

$$\Sigma_{pq} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{p}_i - \bar{\mathbf{p}})(\mathbf{q}_i - \bar{\mathbf{q}})^{\mathsf{T}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{p}_i'\mathbf{q}_i'^{\mathsf{T}},$$

$\rightarrow$ Build anti-symmetric matrix $\mathbf{A}$ with $A_{ij} = \left(\Sigma_{pq} - \Sigma_{pq}{}^{\mathsf{T}}\right)_{ij}$

$\rightarrow$ Form column vector $\Delta = (A_{23}, A_{31}, A_{12})^{\mathsf{T}}$

$\rightarrow$ Form symmetric $4 \times 4$ matrix $\mathbf{Q}(\Sigma_{pq})$:

$$\mathbf{Q}(\Sigma_{pq}) = \begin{pmatrix} \mathrm{tr}\,(\Sigma_{pq}) & \Delta^{\mathsf{T}} \\ \Delta & \Sigma_{pq} + \Sigma_{pq}{}^{\mathsf{T}} - \mathrm{tr}\,(\Sigma_{pq})\,\boldsymbol{I}_3 \end{pmatrix}, \quad \boldsymbol{I}_3 \text{ is the } 3 \times 3 \text{ identity matrix}$$

- Optimal rotation $\mathbf{q}_R$: unit eigenvector corresponding to the maximum eigenvalue of matrix $\mathbf{Q}(\Sigma_{pq})$
- Optimal translation vector $\mathbf{q}_T = \bar{\mathbf{q}} - \mathbf{R}(\mathbf{q}_R)\bar{\mathbf{p}}$

# Point-to-Plane Error Metric [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \left\| \left( (\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}'_j \right) \mathbf{n}_i \right\|^2 ,$$

where

$$\mathbf{q}'_j = \left\{ \mathbf{q} \mid \arg\min_{\mathbf{q} \in s_j} \left\| \mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q} \right\| \right\}$$

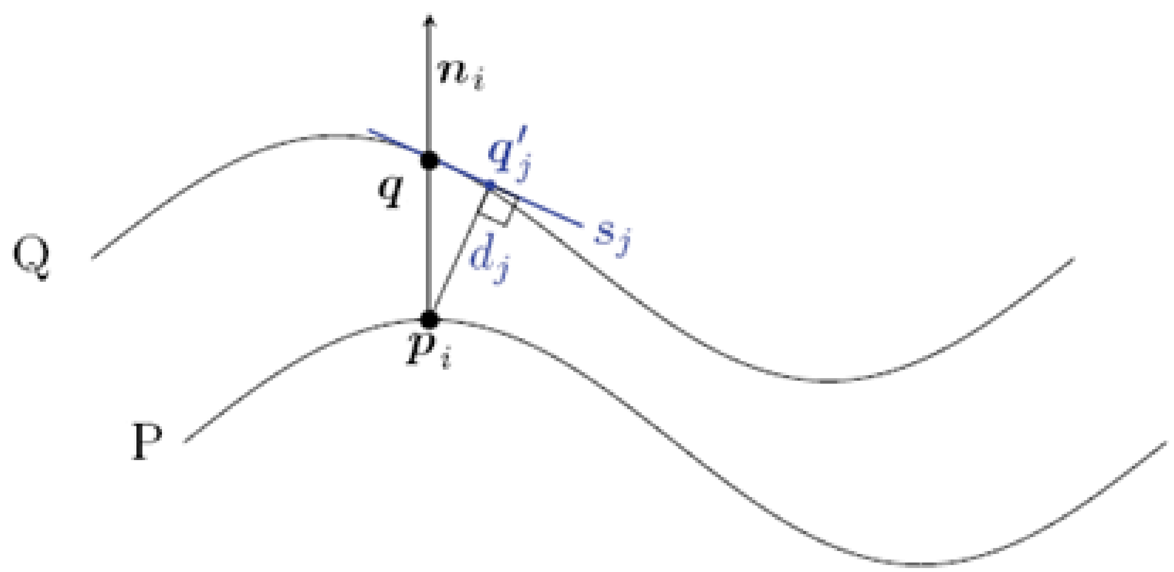


Figure 1: Image courtesy of Konrad Sickel [5]

| | |
|---|---|
| $\mathbf{R}$ | rotation matrix |
| $\mathbf{t}$ | translation vector |
| $d_j$ | distance between the tangent plane $s_j$ and the point $\mathbf{p}_i$ |
| $Q = \{\mathbf{q}_i\}$ | 3-D point sets |
| $P = \{\mathbf{p}_i\}$ | 3-D point sets |
| $s_j$ | tangent plane of $Q$ at $\mathbf{q}$ |
| $n_i$ | surface normal |
| $i = 1, \ldots, N$ | number of points |

# Point-to-Plane Error Metric [1]

- More robust, accurate, and converges faster than the point-to-point error metric
- Utilizes surface normal as an additional input and allows that smooth or planar areas of the meshes slide over each other easily

# Topics

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- Point-to-Plane Error Metric

Summary

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# Take Home Messages

- Several ways to solve the optimization problem of the ICP algorithm are known, two of which we have seen: using SVD and quaternions.
- Several error metrics can be used, we learned about point-to-point error metrics and point-to-plane metrics.

# Further Readings


[1] K. S. Arun, T. S. Huang, and S. D. Blostein. "Least-Squares Fitting of Two 3-D Point Sets". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* PAMI-9.5 (Sept. 1987), pp. 698–700. DOI: 10.1109/TPAMI.1987.4767965.

[2] Paul J. Besl and Neil D. McKay. "A Method for Registration of 3-D Shapes". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 14.2 (Feb. 1992), pp. 239–256. DOI: 10.1109/34.121791.

[3] Berthold K. P. Horn. "Closed-form Solution of Absolute Orientation Using Unit Quaternions". In: *Journal of the Optical Society of America A* 4.4 (Apr. 1987), pp. 629–642. DOI: 10.1364/JOSAA.4.000629.

[4] Berthold K. P. Horn, Hugh M. Hilden, and Shahriar Negahdaripour. "Closed-form Solution of Absolute Orientation Using Orthonormal Matrices". In: *Journal of the Optical Society of America A* 5.7 (July 1988), pp. 1127–1135. DOI: 10.1364/JOSAA.5.001127.

[5] Konrad Sickel. "Computerized Automatic Modeling of Medical Prostheses". PhD Thesis. Erlangen: Pattern Recognition Lab, Friedrich-Alexander-Universität Erlangen-Nürnberg, Apr. 2013.

[6] Michael W. Walker, Lejun Shao, and Richard A. Volz. "Estimating 3-D Location Parameters Using Dual Number Quaternions". In: *CVGIP: image understanding* 54.3 (Nov. 1991), pp. 358–367. DOI: 10.1016/1049-9660(91)90036-O.