

Medical Image Processing for Diagnostic Applications

SVD in Optimization - Part 2

Online Course – Unit 7

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Pattern Recognition Lab (CS 5)

Topics

Optimization Problem III

Optimization Problem IV

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Optimization Problem III

Another quite important optimization problem in image processing and pattern recognition is the following:

Problem: Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Compute the matrix $\hat{\mathbf{B}} \in \mathbb{R}^{n \times n}$ of rank $k < n$ that minimizes:

$$\hat{\mathbf{B}} = \arg \min_{\mathbf{B}} \|\mathbf{A} - \mathbf{B}\|_2, \quad \text{subject to} \quad \text{rank}(\mathbf{B}) = k.$$

Solution: Using SVD, the solution can be computed easily by:

$$\hat{\mathbf{B}} = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$

Optimization Problem III

Example

The SVD can be used to compute the image matrix of rank 1 that approximates an image best w. r. t. $\|\cdot\|_2$.

Figure 1 shows an image I and its rank 1-approximation $I' = \sigma_1 u_1 v_1^T$.

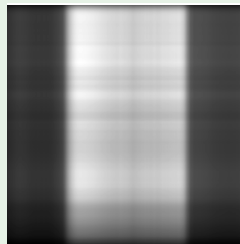
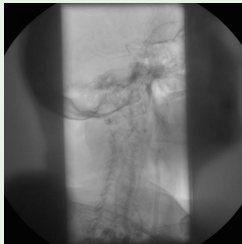


Figure 1: Original X-ray image (left) and its rank 1-approximation (right)

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Optimization Problem IV

Problem: The *Moore–Penrose pseudoinverse* is required to find the solution to the following optimization problem:

$$\| \mathbf{Ax} - \mathbf{b} \|_2 \rightarrow \min.$$

Solution: The least squares solution of this optimization problem is given by

$$\mathbf{x} = \mathbf{A}^\dagger \mathbf{b},$$

where we get $\mathbf{A}^\dagger \in \mathbb{R}^{n \times m}$ based on the SVD of $\mathbf{A} \in \mathbb{R}^{m \times n}$ by:

$$\mathbf{A}^\dagger = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top = \mathbf{V} \mathbf{\Sigma}^\dagger \mathbf{U}^\top.$$

Optimization Problem IV

Proof: We start with the optimization problem:

$$\frac{1}{2} \|\mathbf{Ax} - \mathbf{b}\|_2^2 \rightarrow \min,$$

which can be solved analytically by derivation of this functional:

$$\mathbf{A}^T (\mathbf{Ax} - \mathbf{b}) = 0$$

$$\Leftrightarrow \mathbf{A}^T \mathbf{Ax} - \mathbf{A}^T \mathbf{b} = 0$$

$$\Leftrightarrow \mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.$$

Optimization Problem IV

The diagonal matrix Σ^\dagger in the SVD of the pseudo-inverse of \mathbf{A} is given by:

$$\Sigma^\dagger = \begin{pmatrix} \frac{1}{\sigma_1} & & & & 0 & \dots & 0 \\ & \ddots & & & & & \\ & & \frac{1}{\sigma_r} & & \vdots & & \vdots \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n \times m},$$

where $\sigma_r > 0$ is the smallest nonzero singular value of \mathbf{A} .

Optimization Problem IV

Example

Compute the regression line through the following 2-D points:

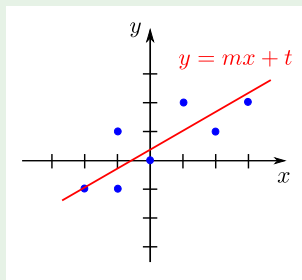
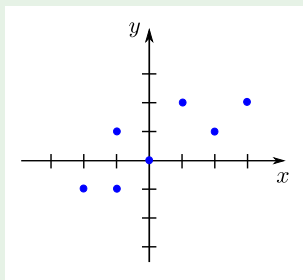


Figure 2: Regression line through a set of 2-D points

Optimization Problem IV

All points (x_i, y_i) , $i = 1, \dots, 7$, have to fulfill the line equation:

$$y_i = mx_i + t, \quad \text{for } i = 1, \dots, 7.$$

Thus we get the following system of linear equations:

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} m \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Optimization Problem IV

The Moore-Penrose pseudo-inverse for this particular problem is:

$$\mathbf{A}^{\dagger} = \begin{pmatrix} 0.14 & 0.09 & 0.04 & -0.01 & -0.07 & -0.07 & -0.12 \\ 0.11 & 0.12 & 0.13 & 0.15 & 0.16 & 0.16 & 0.18 \end{pmatrix}.$$

Therefore, for the regression line we get the equation:

$$y = 0.56x + 0.41.$$

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Remarks on SVD Computation

- SVD can be computed for every matrix.
- SVD is numerically robust.
- In most practical situations we have more rows than columns:

$$m \gg n.$$

- The time complexity to decompose $\mathbf{A} \in \mathbb{R}^{m \times n}$ is:

$$4m^2n + 8mn^2 + 9n^3.$$

- For us, the SVD is a black box. We do not consider algorithms to compute the SVD numerically.

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- We have studied two additional applications (see also previous unit):
 - low-rank approximation,
 - fitting of a regression line.
- SVD is *the* tool for linear equations – it cannot fail (but in many special cases there may exist better solutions).
- SVD is provided by all standard libraries.

Further Readings

Read the original:

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 3rd ed. Johns Hopkins Studies in the Mathematical Sciences. Baltimore: The Johns Hopkins University Press, Oct. 1996

A very detailed and easy to follow introduction of the SVD can be found in:

Carlo Tomasi's class notes, chapter 3 (a **must-read**).

The theory is described in an easy to read format in:

Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Philadelphia: SIAM, 1997

For details about the numerical computation of SVD see:

William H. Press et al. *Numerical Recipes – The Art of Scientific Computing*. 3rd ed. Cambridge University Press, 2007. Get at <http://numerical.recipes/> (August 2016).

Finally, have a look at:

Kaare Brandt Petersen and Michael Syskind Pedersen. *The Matrix Cookbook*. Online. Accessed: 25. April 2017. Technical University of Denmark, Nov. 2012. URL: <http://www2.imm.dtu.dk/pubdb/p.php?3274>