Projection Models and Homogeneous Coordinates

Calibration – Part 2

Refresher Course

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Topics

Camera Calibration

Let

- $X = \{\boldsymbol{x}_i = (x_{i,0}, x_{i,1}, x_{i,2})^{\mathrm{T}} \mid i = 1, \ldots, N\}$ be the set of *3-D points of the calibration pattern*, and
- $Y = \{\boldsymbol{y}_i = (y_{i,0}, y_{i,1})^{\mathrm{T}} \mid i = 1, \ldots, N\}$ be the set of *2-D observations*.

Looking at the set of all corresponding points $\{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid i = 1, \ldots, N\}$, we get N homogeneous equations:

$$\boldsymbol{P}\widetilde{\boldsymbol{x}}_i = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \end{pmatrix} \begin{pmatrix} x_{i,0} \\ x_{i,1} \\ x_{i,2} \\ 1 \end{pmatrix} \cong \begin{pmatrix} y_{i,0} \\ y_{i,1} \\ 1 \end{pmatrix},$$

where $i = 1, \ldots, N$.

The unknowns are the components of the projection matrix $\boldsymbol{P}$.

Camera Calibration

Using the definition of homogeneous coordinates, we get $2N$ equations

$$\frac{p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} = y_{i,0}, \tag{1}$$

$$\frac{p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} = y_{i,1}, \tag{2}$$

for $i = 1, \ldots, N$, which are nonlinear in the components of $\boldsymbol{P}$.

Note: The points in the image plane are computed by applying segmentation methods on real images. Segmentation errors and noise will be present, and the equations will not be fulfilled exactly.

Camera Calibration

We apply the idea of least squares estimation, and estimate the projection matrix according to:

$$\widehat{\boldsymbol{P}} = \underset{\boldsymbol{P}}{\arg\min} \sum_{i=1}^{N} \left(\frac{p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} - y_{i,0} \right)^2 + \sum_{i=1}^{N} \left(\frac{p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} - y_{i,1} \right)^2 .$$

This nonlinear optimization problem is hard to solve. Therefore, numerial optimization usually requires a *good initialization*.

Camera Calibration

A linear method to estimate the projection matrix results from multiplication of the equations (1), (2) by the respective denominators:

$$p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,0},$$
$$p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,1}.$$

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$$p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,1}.$$

Observations:

- These equations are linear in the components of the projection matrix $\boldsymbol{P}$.
- They can be rewritten in matrix form, where a so-called measurement matrix $\boldsymbol{M}$ will include the information on the 3-D calibration points and the measured 2-D points accordingly.

Camera Calibration

Camera calibration thus reduces to the nullspace computation of the measurement matrix $\boldsymbol{M}$:

$$\boldsymbol{M}\begin{pmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{3,3} \\ p_{3,4} \end{pmatrix} = 0, \qquad \text{where}$$

$$\boldsymbol{M} = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 1 & 0 & 0 & 0 & 0 & -x_{1,0}y_{1,0} & -x_{1,1}y_{1,0} & -x_{1,2}y_{1,0} & -y_{1,0} \\ \vdots & & & & & \ddots & & & & & & \vdots \\ x_{N,0} & x_{N,1} & x_{N,2} & 1 & 0 & 0 & 0 & 0 & -x_{N,0}y_{N,0} & -x_{N,1}y_{N,0} & -x_{N,2}y_{N,0} & -y_{N,0} \\ 0 & 0 & 0 & 0 & x_{1,0} & x_{1,1} & x_{1,2} & 1 & -x_{1,0}y_{1,1} & -x_{1,1}y_{1,1} & -x_{1,2}y_{1,1} & -y_{1,1} \\ \vdots & & & & & \ddots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & x_{N,0} & x_{N,1} & x_{N,2} & 1 & -x_{N,0}y_{N,1} & -x_{N,1}y_{N,1} & -x_{N,2}y_{N,1} & -y_{N,1} \end{pmatrix}.$$

Camera Calibration

Observations:

- The calibration problem is reduced to the computation of the nullspace of the measurement matrix $\boldsymbol{M}$.
- We know how to compute the nullspace of $\boldsymbol{M}$ using SVD.
- The rank of $\boldsymbol{M}$ is 11.

Camera Calibration

The estimation problem can also be reduced to an eigenvalue/eigenvector problem:

$$\|\boldsymbol{Mp}\|^2 \rightarrow \min, \quad \text{subject to} \quad \|\boldsymbol{p}\|^2 = 1.$$

Camera Calibration

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Applying the Lagrange multiplier method this becomes:

$$\boldsymbol{p}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{p} - \lambda(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} - 1) \rightarrow \min.$$

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Now we find the zero crossings of the gradient w. r. t. the components of $\boldsymbol{P}$:

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and so we obtain:

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{p} = \lambda\boldsymbol{p}.$$

Camera Calibration

Conclusions:

- The components of the projection matrix $\boldsymbol{P}$ result from the eigenvector belonging to the smallest eigenvalue.
- The linear estimate of $\boldsymbol{P}$ is an excellent initialization for the nonlinear least squares estimate of the projection matrix.

Topics

Take Home Messages

- Computation of projection matrices originally is a nonlinear problem.
- We have studied how a linear estimate can be computed and in doing so build the measurement matrix $\boldsymbol{M}$.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993