Medical Image Processing for Diagnostic Applications

Regularized Reconstruction – L_p-Norms

Online Course – Unit 60

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Topics

Regularized Reconstruction

L_p-Norms

Summary

- Take Home Messages
- Further Readings

Regularized Reconstruction

- Introduction of additional information into the reconstruction process helps to enforce a certain solution.
- This is especially advantageous if the problem is underdetermined.
- Additional weighting terms are also used to suppress noise or artifacts.

Regularization of the Reconstruction Problem

- The objective functions altered:

$$\chi(\boldsymbol{X}) = |\boldsymbol{\Psi}\boldsymbol{X}|_p, \qquad \text{subject to } \boldsymbol{AX} = \boldsymbol{P}.$$

- $\boldsymbol{\Psi}$ is a transformation that transforms the problem into a different domain.
- $|\cdot|_p$ is a L_p norm:

$$|\boldsymbol{x}|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}.$$

The Transformation Ψ

- Ψ is a transformation that transforms the problem into a different domain.
- The selection of Ψ is problem dependent.

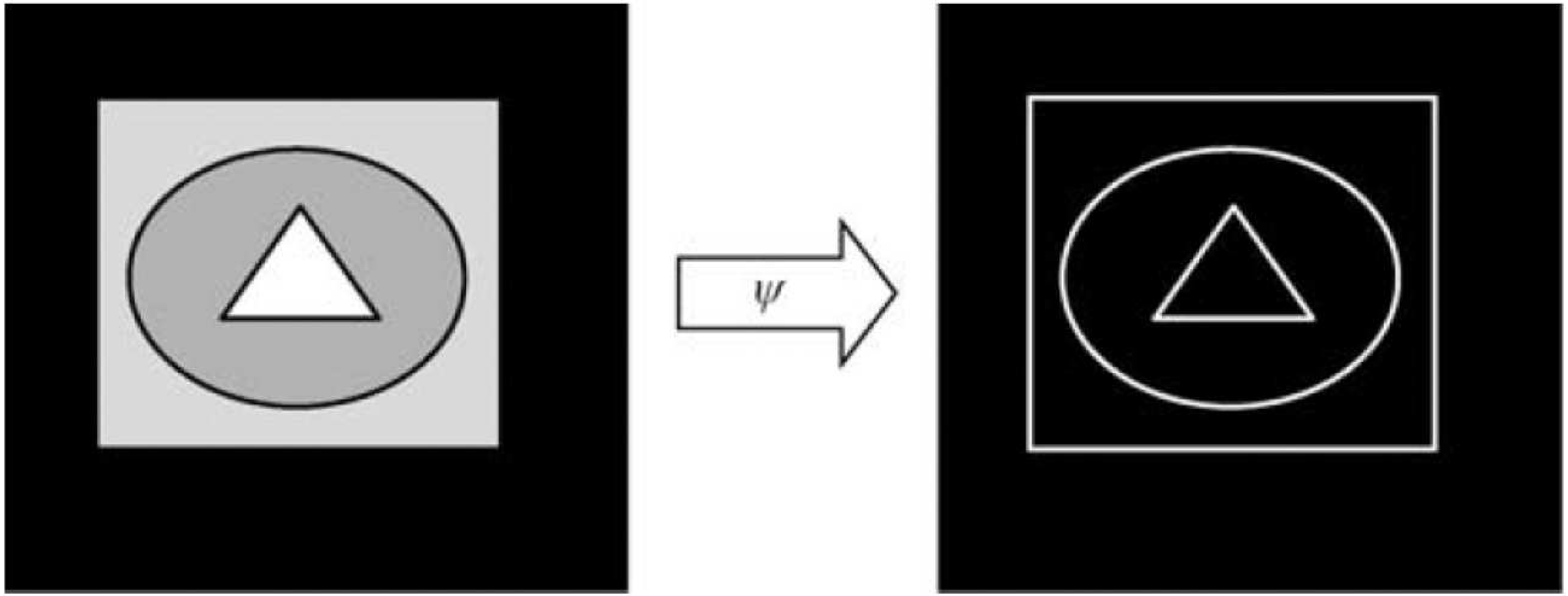

Figure 1: Example for a sparsifying transformation (Zeng, 2009)

The Transformation Ψ

Many transforms have already been investigated as sparsifying transform:

- gradient image,
- wavelet transform,
- Fourier transform,
- discrete cosine transform,
- and many more.

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L_p-Norms

Definition

We denote a L_p-norm as $|\cdot|_p$ which is defined by

$$|\boldsymbol{x}|_p = \left(\sum_i x_i^p\right)^{\frac{1}{p}}.$$

This definition yields a valid vector space norm for $p \in [1, \infty) \cup \{\infty\}$.

- For $p = 2$ we get:

$$|\boldsymbol{x}|_2 = \|\boldsymbol{x}\| = \sqrt{\sum_i x_i^2}.$$

- For $p = 1$ we get:

$$|\boldsymbol{x}|_1 = \sum_i |x_i|.$$

L_p-Norms: Special Cases

- For $p = 0$ we get:

$$|\boldsymbol{x}|_0 = \sum_i |x_i|^0,$$

where we define $0^0 = 0$ for this purpose. This is strictly speaking not a norm in the mathematical sense, but it is a useful tool.

- For $p = \infty$ we get:

$$|\boldsymbol{x}|_\infty = \max_i \big(|x_i|\big),$$

which is a mathematical norm.

Examples

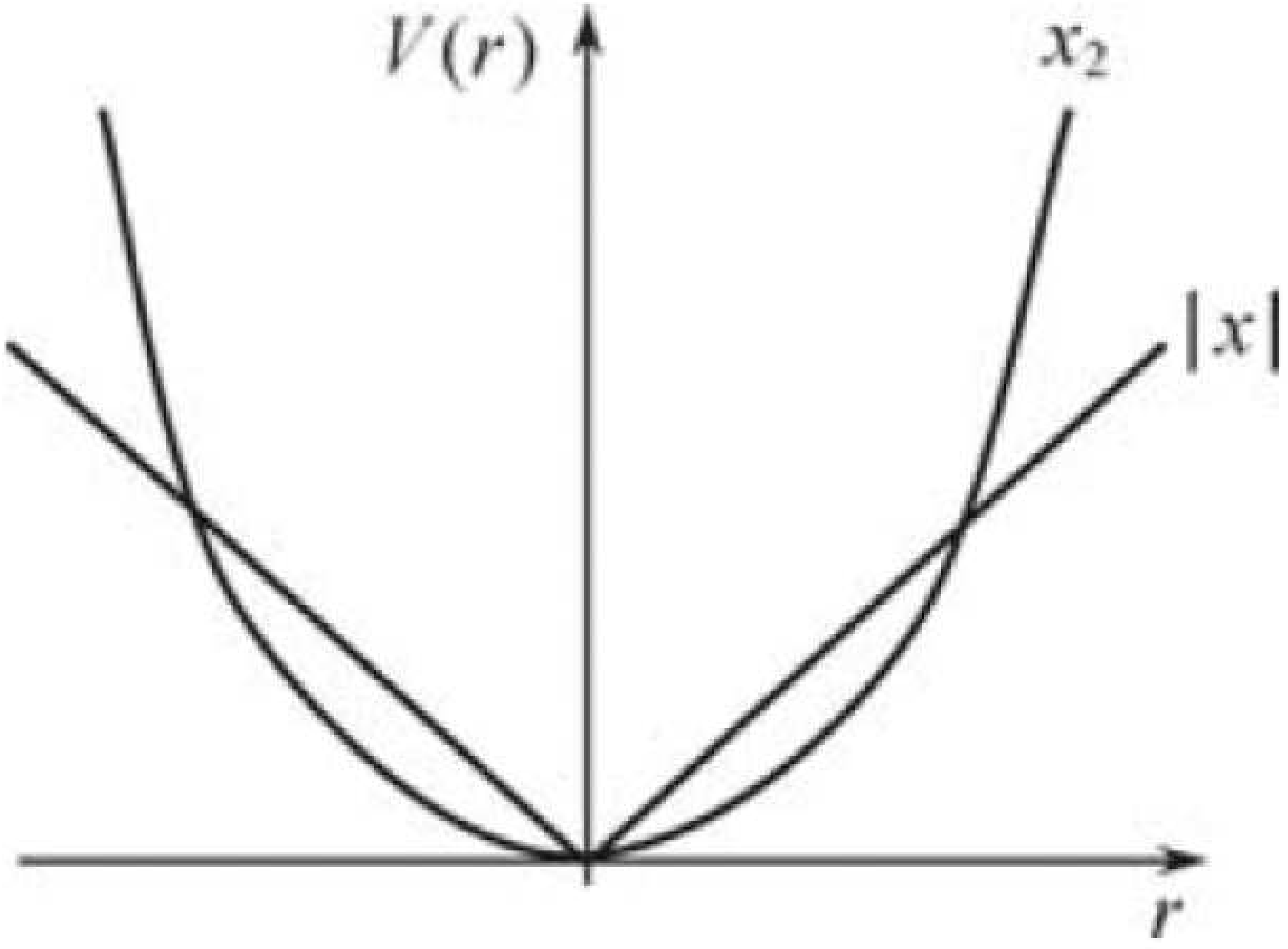

Figure 2: Regularization with L_1-norm (Zeng, 2009)

Examples

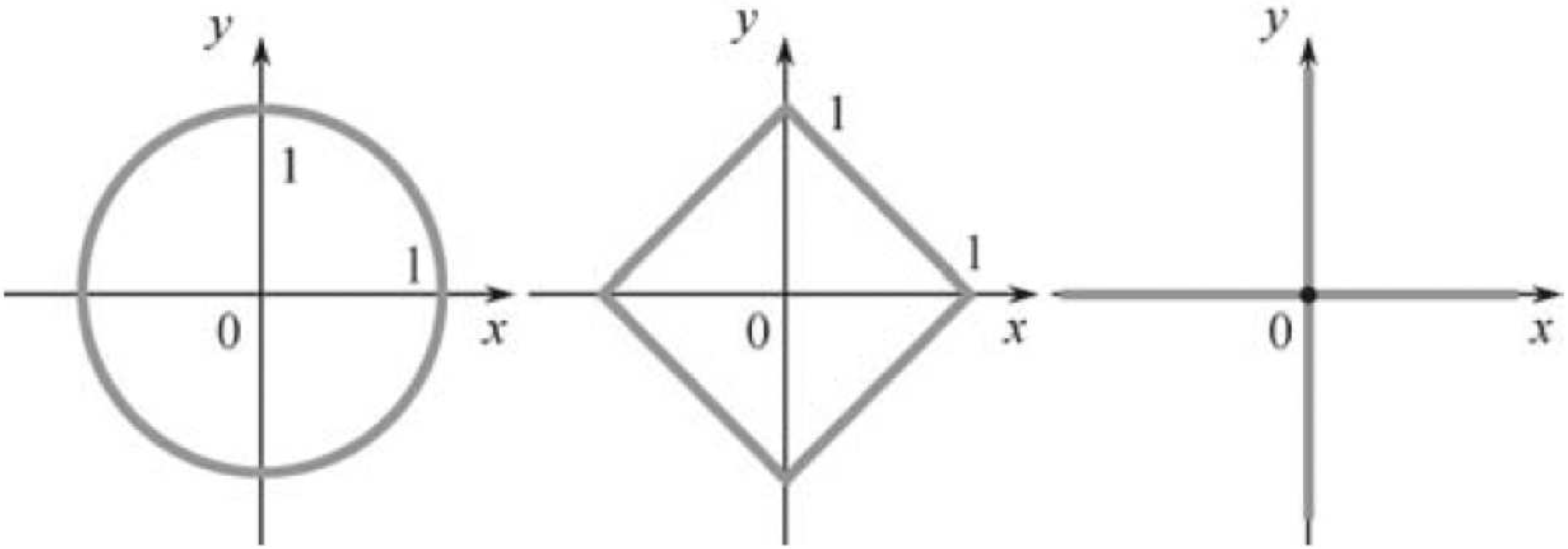

Figure 3: The unit sphere for L_2, L_1, and L_0 (Zeng, 2009)

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Take Home Messages

- Regularization of a reconstruction problem is done via a specific transformation and optimization with respect to a specific objective function.
- This objective function is defined in almost every case by use of an L_p-norm.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. “Angenäherte Auflösung von Systemen linearer Gleichungen”. In: *Bulletin International de l’Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. “Adaptive Iterative Reconstruction”. In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953