Diagnostic Medical Image Processing
Prof. Dr.-Ing. Andreas Maier
Exercises (DMIP-E)
WS 2016/17

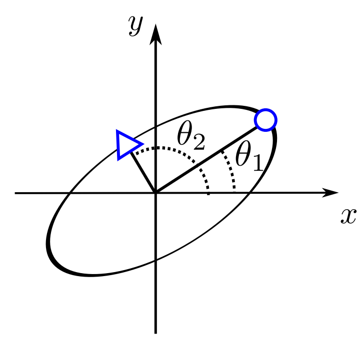
LEHRSTUHL FÜR MUSTER-ERKENNUNG

Singular Value Decomposition (SVD)

Frank Schebesch (frank.schebesch@fau.de) Exercise Sheet 1

1 Understanding SVD

(i) Take a look at the ellipse on the right. Let $\theta_1 = 30°$, $\theta_2 = 120°$, the coordinates of the circle and triangle in the shown axes are $\left(3\sqrt{3},\, 3\right)$ and $\left(-1,\, \sqrt{3}\right)$.
Use your knowledge about the SVD to find a matrix, that maps the ellipse to the unit sphere. Is that a unique mapping?
Can you also find a transformation that preserves the direction from the origin to both the circle and the triangle, respectively?
(You can solve this exercise analytically, but a correct numerical solution using SVD is also accepted.)



$\boxed{2}$

(ii) Which of the following are common applications of the SVD?

- ☐ computation of condition numbers
- ☐ ranking matrices
- ☐ low-rank approximations of images
- ☐ solving linear systems
- ☐ computation of multiple values
- ☐ computation of the null space

$\boxed{1}$

2 Condition of a matrix

In the lecture we have seen the matrix

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix}.$$

(i) Compute the condition number of $\mathbf{A}$ with respect to the 2-norm and compare your result with the lecture (*hint:* class `DecompositionSVD`).

(ii) Recall that the numerical rank of a matrix $\mathbf{M}$ is defined by the number

$$\mathrm{rank}_\epsilon(\mathbf{M}) = \#\left\{\sigma_i > \epsilon,\ \sigma_i \text{ singular value of } \mathbf{M}\right\}.$$

By setting $\epsilon = 10^{-3}$, we get a rank deficiency in $\mathbf{A}$. Can you directly tell nullspace and range from your SVD computations in (i)? What are those?

(iii) Given the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1.001 \\ 0.999 \\ 1.001 \end{pmatrix}$, show that a variation of the elements of $\mathbf{b}$ by 0.1 % implies a change in $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ by at least 240 %.

(iv) Compute the condition number (w.r.t. 2-norm) for the matrix $\mathbf{B} \in \mathbb{R}^{20\times 20}$ defined by

$$\mathbf{B} = \mathbf{U}\,\mathrm{diag}\,(a_1, ..., a_{20})\,\mathbf{V}^T, \quad a_n = \frac{1}{(n-5)^2 + 4}, \quad n = 1, ..., 20.$$

2+1+2+1

3 Optimization Problems

(i) Implement and verify optimization problem 1 from the lecture.

(ii) Optimization problem 2: Four 2-D vectors were given on the lecture slides. Implement the optimization problem for the general case, e.g. 5, 6, 20 or N vectors.

(iii) Implement the third optimization problem using the image `mr_head_angio.jpg`.
How many approximations do we have? Which rank-k-approximations are sufficient? Plot the RMSE for k=1,...,150 (*hint:* class `NumericGridOperator`).

(iv) Compute the regression line through the following set of 2-D points:

$$\{(-3, 7),\ (-2, 8),\ (-1, 9),\ (0, 3.3),\ (1.5, 2),\ (2, -3),\ (3.1, 4),\ (5.9, -0.1),\ (7.3, -0.5)\}.$$

2+1+3+1

Latest submission date: 11/4/2016 **Total: 16**