Projection Models and Homogeneous Coordinates

Projection Models

Refresher Course

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

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- What is the mechanical setup for the calibration of projection parameters?
- How can we estimate the camera parameters?
- How can we compute the path of X-rays?
- How reliable are the estimates?

Projections

X-ray projection geometry is best modeled by a perspective projection.

$\longrightarrow$ All X-ray beams intersect at the focal point of the X-ray tube.

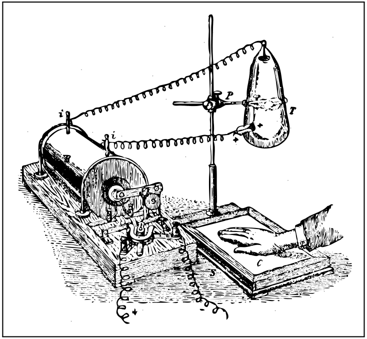

Figure 1: Conventional Röntgen scheme using photographic paper (Fölsing 1995, [2])

Projections

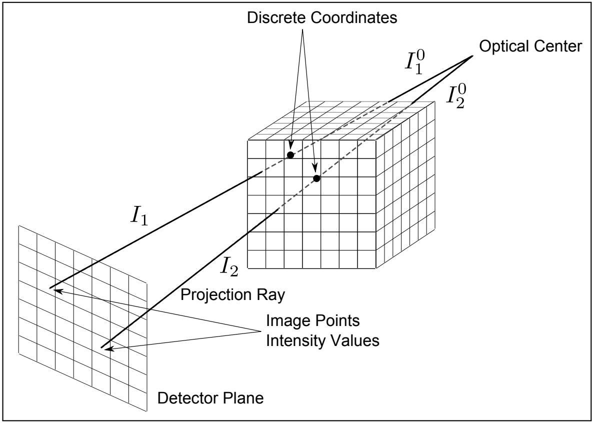

Figure 2: Projection from 3-D to 2-D

Projection Geometries

In the following discussion we assume that the image plane is in a fixed position in 3-D space:

- The 2-D image plane is embedded parallel to the (x, y) – plane of the 3-D coordinate system.
- The distance of the image plane to the origin of the 3-D coordinate system is denoted by f, that is the image plane's z-coordinate is constant $(z = f)$.

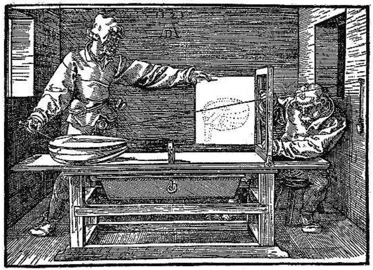

Figure 3: Illustration of the perspective projection (Dürer 1525, [1])

Projection Models

In computer vision and graphics several projection models are used:

1. orthographic projection,
2. weak perspective projection,
3. para-perspective projection,
4. perspective projection.

Projection Models

1. **Orthographic projection:** The projected point results from the cancelation of the z components:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}.$$

Obviously, this is a linear mapping and can be written in homogeneous coordinates:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Projection Models

2. **Weak perspective projection** is a scaled orthographic projection, i. e., the coordinates (x, y) are scaled by a scaling factor k:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} k \cdot x \\ k \cdot y \end{pmatrix}.$$

This is again a linear mapping and can be written in homogeneous coordinates:

$$\begin{pmatrix} k \cdot x \\ k \cdot y \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Projection Models

3. **Perspective projection:** The projected point is the intersection of the line connecting point and optical center (focal point) with the image plane.
   This **nonlinear mapping** of points is given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f \cdot x/z \\ f \cdot y/z \end{pmatrix}$$

   where f is the distance of the image plane to the origin.

   Observation: The projection model of X-ray systems can be approximated by perspective projection.

Projection Models

4. **Para-perspective projection:**
   (i) If multiple points are projected, an auxiliary plane through the points' centroid and parallel to the image plane is defined.
   (ii) Then a parallel projection is applied to map all points onto this auxiliary plane, where the projection direction is parallel to the vector that defines the centroid.
   (iii) The points on the auxiliary plane are mapped by perspective projection into the image plane, i. e., we perform a scaled orthographic projection.

Note: The para-perspective projection is an affine mapping.

Illustration of the Different Projection Models

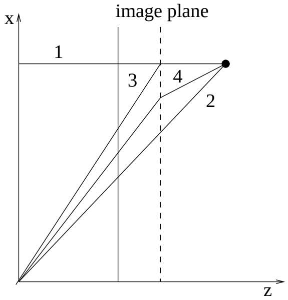

Figure 4: Projection models in computer vision and graphics: orthographic (1), perspective (2), weak perspective (3), para-perspective (4)

Illustration of the Different Projection Models

In summary we find:

- the projection models (1) and (3) can be expressed in terms of a linear mapping in 3-D,
- projection model (4) is defined by an affine mapping, and
- the perspective projection (2) is non-linear.

Too bad: The perspective projection model is the model we are required to deal with.

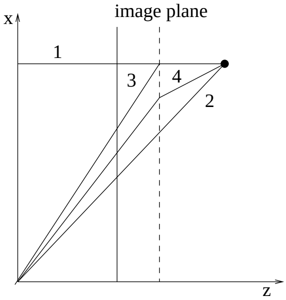

Figure 4: Projection models in computer vision and graphics: orthographic (1), perspective (2), weak perspective (3), para-perspective (4)

Topics

Take Home Messages

- 3-D X-ray imaging requires profound knowledge of appropriate projection models.
- We have learned about four different projection models:
  - orthographic projection,
  - weak perspective projection,
  - para-perspective projection,
  - perspective projection.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993

References:

Albrecht Dürer. *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*. Nüremberg: [Hieronymus Andreae], 1525.

Albrecht Fölsing. *Wilhelm Conrad Röntgen: Aufbruch ins Innere der Materie*. München Wien: Carl Hanser Verlag, 1995.