Projection Models and Homogeneous Coordinates

Projection Models

Refresher Course
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Topics

Motivation

For 3-D imaging by means of X-ray projections we require detailed knowledge about projection rays.

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- How can we characterize different projection geometries?
- What is the mechanical setup for the calibration of projection parameters?
- How can we estimate the camera parameters?
- How can we compute the path of X-rays?
- How reliable are the estimates?

Projections

X-ray projection geometry is best modeled by a perspective projection.

$\longrightarrow$ All X-ray beams intersect at the focal point of the X-ray tube.

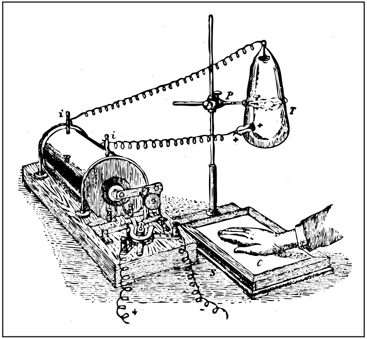

Figure 1: Conventional Röntgen scheme using photographic paper (Fölsing 1995, [2])

Projections

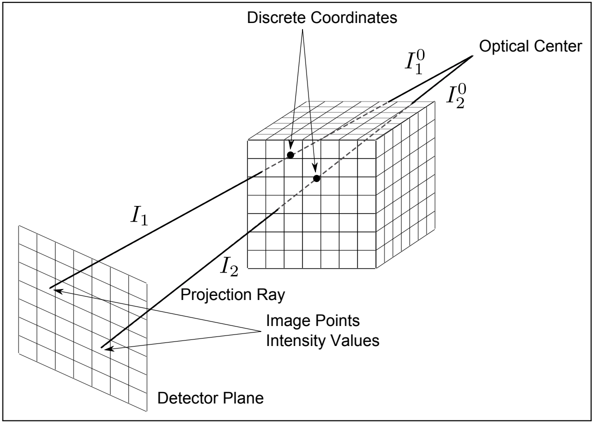

Figure 2: Projection from 3-D to 2-D

Projection Geometries

In the following discussion we assume that the image plane is in a fixed position in 3-D space:

- The 2-D image plane is embedded parallel to the (x, y)–plane of the 3-D coordinate system.
- The distance of the image plane to the origin of the 3-D coordinate system is denoted by f, that is the image plane's z-coordinate is constant ($z = f$).

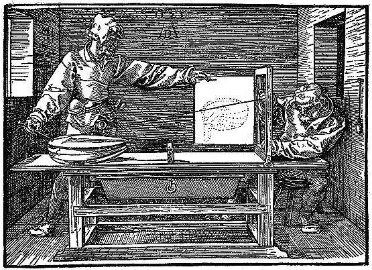

Figure 3: Illustration of the perspective projection (Dürer 1525, [1])

Projection Models

In computer vision and graphics several projection models are used:

1. orthographic projection,
2. weak perspective projection,
3. para-perspective projection,
4. perspective projection.

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Projection Models

1. **Orthographic projection:** The projected point results from the cancelation of the z components:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}.$$

Obviously, this is a linear mapping and can be written in homogeneous coordinates:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Projection Models

2. **Weak perspective projection** is a scaled orthographic projection, i. e., the coordinates (x, y) are scaled by a scaling factor k:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} k \cdot x \\ k \cdot y \end{pmatrix}.$$

This is again a linear mapping and can be written in homogeneous coordinates:

$$\begin{pmatrix} k \cdot x \\ k \cdot y \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Projection Models

3. **Perspective projection:** The projected point is the intersection of the line connecting point and optical center (focal point) with the image plane.
   This **nonlinear mapping** of points is given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f \cdot x/z \\ f \cdot y/z \end{pmatrix}$$

   where f is the distance of the image plane to the origin.

   Observation: The projection model of X-ray systems can be approximated by perspective projection.

Projection Models

4. **Para-perspective projection:**
   (i) If multiple points are projected, an auxiliary plane through the points' centroid and parallel to the image plane is defined.
   (ii) Then a parallel projection is applied to map all points onto this auxiliary plane, where the projection direction is parallel to the vector that defines the centroid.
   (iii) The points on the auxiliary plane are mapped by perspective projection into the image plane, i. e., we perform a scaled orthographic projection.

   Note: The para-perspective projection is an affine mapping.

Illustration of the Different Projection Models

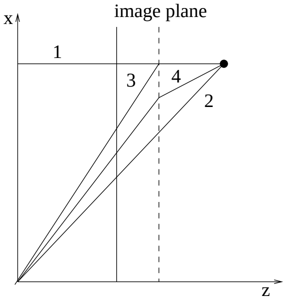

Figure 4: Projection models in computer vision and graphics: orthographic (1), perspective (2), weak perspective (3), para-perspective (4)

Illustration of the Different Projection Models

In summary we find:

- the projection models (1) and (3) can be expressed in terms of a linear mapping in 3-D,
- projection model (4) is defined by an affine mapping, and
- the perspective projection (2) is non-linear.

Too bad: The perspective projection model is the model we are required to deal with.

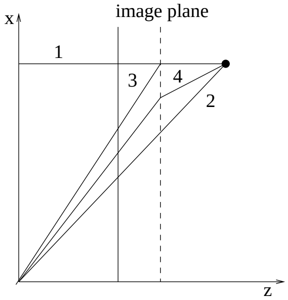

Figure 4: Projection models in computer vision and graphics: orthographic (1), perspective (2), weak perspective (3), para-perspective (4)

Topics

Take Home Messages

- 3-D X-ray imaging requires profound knowledge of appropriate projection models.
- We have learned about four different projection models:
  - orthographic projection,
  - weak perspective projection,
  - para-perspective projection,
  - perspective projection.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993

References:

Albrecht Dürer. *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*. Nüremberg: [Hieronymus Andreae], 1525.

Albrecht Fölsing. *Wilhelm Conrad Röntgen: Aufbruch ins Innere der Materie*. München Wien: Carl Hanser Verlag, 1995.

Projection Models and Homogeneous Coordinates

Homogeneous Coordinates

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Topics

Homogeneous Coordinates

Using a simple trick, we can extend 2-D or 3-D vectors by an additional component that allows us to write

- affine mappings as linear mappings, and
- the perspective projection as a linear mapping.

Let us first consider the 2-D case.

Homogeneous Coordinates

We extend $\mathbb{R}^2$ by a third coordinate to create the projective space $\mathbb{P}^2$:

Definition

A two-dimensional point in Cartesian coordinates $\boldsymbol{p} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$ is represented by $\widetilde{\boldsymbol{p}} = (wx, wy, w)^{\mathrm{T}} \in \mathbb{P}^2$ in ***homogeneous coordinates***, where $w \in \mathbb{R} \setminus \{0\}$ is an arbitrary real value.

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Note: A vector $(\widetilde{x}, \widetilde{y}, \widetilde{z})^{\mathrm{T}}$ in homogeneous coordinates can be transformed into a 2-D vector by dividing the first two components $\widetilde{x}$ and $\widetilde{y}$ with the third component $\widetilde{z} \neq 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \widetilde{x}/\widetilde{z} \\ \widetilde{y}/\widetilde{z} \end{pmatrix}.$$

Homogeneous Coordinates

- A 2-D point $(x, y)^{\mathrm{T}}$ in Cartesian coordinates corresponds to a line in 3-D:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left\{ w \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \;\middle|\; w \in \mathbb{R} \right\}.$$

- There exists an infinite number of homogeneous points that correspond to one and the same 2-D point.
- The representation in homogeneous coordinates has a singularity for $w \to 0$.

Homogeneous Coordinates

We now define an equivalence relation:

Definition

We call two homogeneous points $\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}$ equivalent, if $\widetilde{\boldsymbol{p}} = \lambda \widetilde{\boldsymbol{q}}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. This equivalence is denoted by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$.

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Example

The homogeneous points $\widetilde{\boldsymbol{p}} = (2,3,1)^{\mathrm{T}}$ and $\widetilde{\boldsymbol{q}} = (4,6,2)^{\mathrm{T}}$ are equivalent by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$ as both project to the same point which is $(2,3)^{\mathrm{T}} \in \mathbb{R}^2$.
They are not equal considered as vectors in $\mathbb{R}^3$, i. e., $\widetilde{\boldsymbol{p}} \neq \widetilde{\boldsymbol{q}}$.
Note: It is $\widetilde{\boldsymbol{p}} \ncong (4,6,1)^{\mathrm{T}}$.

Homogeneous Coordinates

Let us now consider lines in 2-D.

- A line in $\mathbb{R}^2$ is fully determined by the equation

$$ax + by + c = 0, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$

  This equation can be multiplied by an arbitrary factor $w \in \mathbb{R} \setminus \{0\}$, and it still represents the same line.

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- Each vector $(a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$ represents a line, and

$$ax + by + c = (w \cdot a)x + (w \cdot b)y + (w \cdot c) = 0$$

  holds for each non-zero w.

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  $$ax + by + c = (w \cdot a)x + (w \cdot b)y + (w \cdot c) = 0$$

  holds for each non-zero w.
- In terms of homogeneous coordinates we can state that each 2-D line can be represented by a corresponding vector $\boldsymbol{l} = (a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$.

Homogeneous Coordinates

- A point $\widetilde{\boldsymbol{p}}$ (represented in homogeneous coordinates) lies on the line $\boldsymbol{l}$ if

$$\boldsymbol{l}^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0.$$

- Intersection of lines: Two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ intersect in point $\widetilde{\boldsymbol{p}}$ if

$$\boldsymbol{l}_1^{\mathrm{T}}\widetilde{\boldsymbol{p}} = \boldsymbol{l}_2^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0,$$

so we find

$$\widetilde{\boldsymbol{p}} = \boldsymbol{l}_1 \times \boldsymbol{l}_2.$$

Homogeneous Coordinates

Definition

The set of ***ideal points*** lies on the line at infinity $\boldsymbol{l}_\infty = (0,0,1)^\mathrm{T}$:

$$(0,0,1)^\mathrm{T}(x,y,0) = 0.$$

Note: The tupel $(0,0,0)^\mathrm{T}$ describes no valid coordinate in $\mathbb{P}^2$.

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Exercise: Do parallel lines intersect in $\mathbb{P}^2$? Where?

The concept of homogeneous coordinates can be transferred to higher dimensional spaces. We will not continue to look into the details of this theory. Interested students are referred to the literature on perspective geometry (see for instance Hartley's book).

Orthographic Projection

We will now formulate projections from 3-D to 2-D using homogeneous coordinates:

The *orthographic projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (x, y, 1)^{\mathrm{T}}.$$

This mapping from $\mathbb{P}^3 \to \mathbb{P}^2$ can be simply written in matrix form as

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

Weak Perspective Projection

The *weak perspective projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \;\mapsto\; \widetilde{\boldsymbol{p}}' = (kx, ky, 1)^{\mathrm{T}},$$

where $k \in \mathbb{R}$ is a scaling factor.
This mapping from $\mathbb{P}^3 \to \mathbb{P}^2$ can be simply written in matrix form as:

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/k \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

Perspective Projection

Using homogeneous coordinates, the *perspective projection* becomes a *linear* mapping:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \;\mapsto\; \widetilde{\boldsymbol{p}}' = (fx, fy, z)^{\mathrm{T}} \cong (fx/z, fy/z, 1)^{\mathrm{T}}.$$

We get the following linear mapping from $\mathbb{P}^3 \to \mathbb{P}^2$:

$$\widetilde{\boldsymbol{p}}' = \underbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\boldsymbol{P}} \widetilde{\boldsymbol{p}}.$$

Topics

Homogeneous Coordinates
- Definition
- Lines in $\mathbb{R}^2$ and Points in $\mathbb{P}^2$
- Projections in Homogeneous Coordinates

Summary
- Take Home Messages
- Further Readings

Take Home Messages

- Points on a line through the origin in real vector space correspond to a single point in the projective plane.
- The nonlinear projective mapping in $\mathbb{R}^3$ can be written as a linear mapping using homogeneous coordinates.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993

Projection Models and Homogeneous Coordinates

Extrinsic and Intrinsic Camera Parameters

Refresher Course

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Topics

Extrinsic Camera Parameters

Intrinsic Camera Parameters

Complete Projection

Summary
- Take Home Messages
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Extrinsic Camera Parameters

So far we have described the projection of a 3-D point into the image plane.

We have not considered the motion of the position and orientation of the acquisition device yet:

- an X-ray source can be translated in 3-D,
- an X-ray source can be rotated in 3-D.

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- an X-ray source can be rotated in 3-D.

Definition

Extrinsic parameters characterize the *pose*, i. e., position and orientation of the camera with respect to a world coordinate system. The position is defined by a 3-D translation vector, the orientation by three rotation angles.

Extrinsic Camera Parameters

Figure 1: C-arm device in different positions and orientations that can be characterized by the extrinsic parameters of the acquisition device (image courtesy of Siemens Healthcare)

Extrinsic Camera Parameters

Mathematical characterization:
Rotation and translation of a 3-D point can be expressed by:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \boldsymbol{t},$$

where

- $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ denotes a rotation matrix (with its known properties), and
- $\boldsymbol{t} \in \mathbb{R}^{3}$ represents a translation in Euclidean space.

This is an affine mapping.

Extrinsic Camera Parameters

Using homogeneous coordinates we can rewrite the affine as a linear mapping:

$$\begin{pmatrix} wx' \\ wy' \\ wz' \\ w \end{pmatrix} = \boldsymbol{D} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & \boldsymbol{R} & & \boldsymbol{t} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} .$$

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Problem: How does the rotation matrix look like?

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Problem: How does the rotation matrix look like?

Solution: As we already know, the columns of the linear mapping are the images of the base vectors of the original coordinate system.

Topics

Intrinsic Camera Parameters

Besides the position and orientation of the acquisition device, in a real imaging system we have to take another set of parameters into account. There is a mapping of projected points in the ideal image plane to the used detector.

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Definition

Intrinsic parameters define the mapping of 2-D coordinates from the ideal image plane to the 2-D detector coordinates.

Intrinsic Camera Parameters

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- The coordinate axes of the detector are not necessarily orthogonal, but intersect with a skew angle Θ.
- The pixels in the detector coordinate system are not necessarily square pixels, but scaled by k_x and k_y.
- There might exist a radial distortion due to the camera optics (not considered here).

Intrinsic Camera Parameters

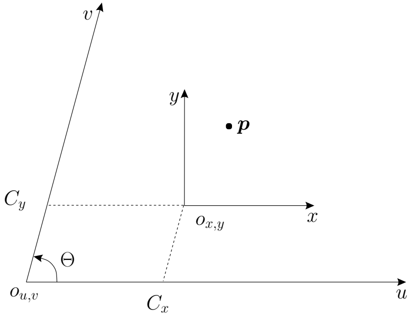

Figure 2: Detector and ideal image coordinate system

(x, y) – ideal image coordinate system:

- used in all formulas so far
- $o_{x,y}$: origin

(u, v) – detector coordinate system:

- real image matrix of measurements
- Θ: skew angle between axes
- k_x, k_y: scaling of u and v axis with respect to units in (x, y)-system
- (C_x, C_y): offset of origins of both coordinate systems

Intrinsic Camera Parameters

Transformation between (u, v)- and (x, y)-coordinate system

At first, we consider the images of base vectors of the detector coordinate system in the image coordinate system:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{k_x} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{k_y}\cos\Theta \\ \frac{1}{k_y}\sin\Theta \end{pmatrix}.$$

The required transform from the (x, y)- to the (u, v)-coordinate system is given by the inverse of the mapping above:

$$\boldsymbol{T} = \begin{pmatrix} \frac{1}{k_x} & \frac{1}{k_y}\cos\Theta \\ 0 & \frac{1}{k_y}\sin\Theta \end{pmatrix}^{-1} = \begin{pmatrix} k_x & -k_x\frac{\cos\Theta}{\sin\Theta} \\ 0 & \frac{k_y}{\sin\Theta} \end{pmatrix}.$$

Intrinsic Camera Parameters

The complete mapping of (x, y)- to (u, v)-coordinates in Euclidean space is thus given by:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k_x & -k_x \frac{\cos\Theta}{\sin\Theta} \\ 0 & \frac{k_y}{\sin\Theta} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} C_x \\ C_y \end{pmatrix}.$$

Using homogeneous coordinates we get the matrix including the described intrinsic parameters that maps the ideal image coordinates to the detector coordinates:

$$\boldsymbol{K} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{T}$}} & -C_x \\ & & -C_y \\ \hline 0 & 0 & 1 \end{array}\right).$$

Topics

Complete Projection

The total perspective transformation is:

$$\boldsymbol{P}\widetilde{\boldsymbol{p}} = \boldsymbol{K}\boldsymbol{P}_{\text{proj}}\boldsymbol{D}\widetilde{\boldsymbol{p}}.$$

- $\boldsymbol{D}$: extrinsic camera parameters
  $\rightarrow$ position and orientation of camera w. r. t. the world coordinate system
- $\boldsymbol{P}_{\text{proj}}$: projection model matrix, ideal perspective projection
- $\boldsymbol{K}$: intrinsic camera parameters
  - optical and geometric characteristics of the camera
  - do *not* change with camera movement

Topics

Extrinsic Camera Parameters

Intrinsic Camera Parameters

Complete Projection

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- For projections with a real detector extrinsic and intrinsic camera parameters have to be considered.
- Extrinsic parameters describe the source/camera movement, and can be written as a linear mapping in homogeneous coordinates.
- Intrinsic parameters describe the (usually constant) deviations of the detector from an ideal image plane, and can be written as a linear mapping in homogeneous coordinates as well.

Further Readings

For further details on geometric aspects of imaging see:

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Projection Models and Homogeneous Coordinates

Calibration – Part 1

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Topics

Calibration of X-ray Systems

Definition

The estimation of the projection parameters is called ***calibration***.

Let us first consider a *simplified case* for calibration, where we consider the following problems in detail:

- Design a calibration pattern whose 3-D geometry is known exactly.
- Capture an X-ray image of the calibration pattern.
- Compute the correspondence of 2-D to 3-D points.
- Compute the focal length f using a least square estimator.

Calibration Patterns

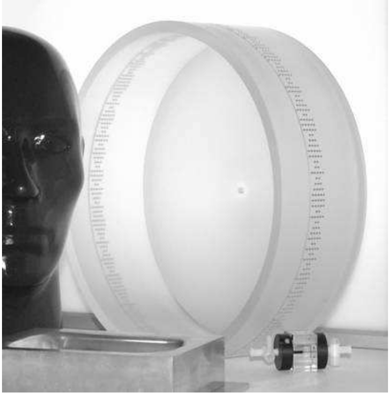

Figure 1: Calibration pattern used for C-arm calibration (image courtesy of Siemens Healthcare)

Figure 2: Observed 2-D point features (image courtesy of Siemens Healthcare)

PDS2 Calibration Phantom

Example

Let us assume we have a so-called calibration pattern that is manufactured precisely, and N centroids of the 3-D metal spheres are known in 3-D world coordinates (x_0, x_1, x_2):

$$X = \{\boldsymbol{x}_i = (x_{i,0}, x_{i,1}, x_{i,2})^\mathrm{T} \mid i = 1, \ldots, N\}.$$

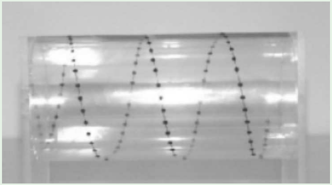

Figure 3: Calibration pattern used for the computation of the projection geometry of a C-arm

PDS2 Calibration Phantom

Example

Properties of the PDS2 calibration pattern used for today's C-arm systems:

- 108 steel spheres,
- large spheres represent a logical one,
- small spheres represent a logical zero,
- 8 Bit binary encoding,
- position of spheres is known in the world coordinate system,
- the world coordinate system is attached to the phantom.

Calibration of X-ray Systems

Let $(x_{i,0}, x_{i,1}, x_{i,2})^{\mathrm{T}}$, $i = 1, \ldots, N$, be the set of 3-D points of the calibration pattern, and let $(y_{i,0}, y_{i,1})^{\mathrm{T}}$ be the corresponding set of 2-D points.

The projection is defined by:

$$\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{i,0} \\ x_{i,1} \\ x_{i,2} \\ 1 \end{pmatrix} \cong \begin{pmatrix} y_{i,0} \\ y_{i,1} \\ 1 \end{pmatrix}.$$

Calibration of X-ray Systems

For each pair of points 2-D/3-D points we get the pair of equations:

$$\frac{fx_{i,0}}{x_{i,2}} = y_{i,0}, \tag{1}$$

$$\frac{fx_{i,1}}{x_{i,2}} = y_{i,1}. \tag{2}$$

Due to image noise and segmentation errors, the equations have to be replaced by the following minimization problem:

$$\hat{f} = \arg\min_{f} \sum_{i=1}^{N} \sum_{k=0}^{1} \left(\frac{fx_{i,k}}{x_{i,2}} - y_{i,k} \right)^2. \tag{OP 1}$$

Calibration of X-ray Systems

Solution: We compute the zero crossings of the partial derivatives w. r. t. the unknown parameter f and get the following estimator for f:

$$\hat{f} = \frac{\sum_{i=1}^{N} \sum_{k=0}^{1} \frac{y_{i,k} x_{i,k}}{x_{i,2}}}{\sum_{i=1}^{N} \sum_{k=0}^{1} \left(\frac{x_{i,k}}{x_{i,2}} \right)^2}.$$

Alternative Objective Functions

We can multiply (1) and (2) by $x_{i,2}$ and get:

$$fx_{i,0} = x_{i,2} y_{i,0} ,$$
$$fx_{i,1} = x_{i,2} y_{i,1} .$$

The resulting objective function is:

$$\hat{f} = \arg\min_{f} \sum_{i=1}^{N} \sum_{k=0}^{1} \left(fx_{i,k} - x_{i,2} y_{i,k} \right)^2 . \tag{OP 2}$$

Different Objective Functions

There is yet another option to rewrite (1) and (2):

$$f = \frac{x_{i,2} y_{i,0}}{x_{i,0}},$$
$$f = \frac{x_{i,2} y_{i,1}}{x_{i,1}}.$$

Using these equations, the resulting objective function is:

$$\hat{f} = \arg\min_{f} \sum_{i=1}^{N} \sum_{k=0}^{1} \left(f - x_{i,2} \frac{y_{i,k}}{x_{i,k}} \right)^2. \tag{OP 3}$$

Different Objective Functions

Question: Which objective function is the best one? Should we optimize (OP 1), (OP 2) or (OP 3)?

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Rule of thumb: Always optimize differences in the image space (or in the respective space of observations)!

Exercise: Compute the estimators for (OP 1), (OP 2), and (OP 3) and analyze the sensitivity of estimates by adding noise to the input data.

Topics

Take Home Messages

- In a real imaging system the detector has to be calibrated.
- Calibration can be performed by use of calibration phantoms that contain several markers whose pattern is exactly known.
- We have studied how a projection parameter like the focal length can be estimated by solving an optimization problem with measured 2-D points and known 3-D positions.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993

Projection Models and Homogeneous Coordinates

Calibration – Part 2

Refresher Course

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Topics

Camera Calibration

Let

- $X = \{\boldsymbol{x}_i = (x_{i,0}, x_{i,1}, x_{i,2})^{\mathrm{T}} \mid i = 1, \ldots, N\}$ be the set of *3-D points of the calibration pattern*, and
- $Y = \{\boldsymbol{y}_i = (y_{i,0}, y_{i,1})^{\mathrm{T}} \mid i = 1, \ldots, N\}$ be the set of *2-D observations*.

Looking at the set of all corresponding points $\{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid i = 1, \ldots, N\}$, we get N homogeneous equations:

$$\boldsymbol{P}\widetilde{\boldsymbol{x}}_i = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \end{pmatrix} \begin{pmatrix} x_{i,0} \\ x_{i,1} \\ x_{i,2} \\ 1 \end{pmatrix} \cong \begin{pmatrix} y_{i,0} \\ y_{i,1} \\ 1 \end{pmatrix},$$

where $i = 1, \ldots, N$.

The unknowns are the components of the projection matrix $\boldsymbol{P}$.

Camera Calibration

Using the definition of homogeneous coordinates, we get $2N$ equations

$$\frac{p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} = y_{i,0}, \tag{1}$$

$$\frac{p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} = y_{i,1}, \tag{2}$$

for $i = 1, \ldots, N$, which are nonlinear in the components of $\boldsymbol{P}$.

Note: The points in the image plane are computed by applying segmentation methods on real images. Segmentation errors and noise will be present, and the equations will not be fulfilled exactly.

Camera Calibration

We apply the idea of least squares estimation, and estimate the projection matrix according to:

$$\widehat{\boldsymbol{P}} = \arg\min_{\boldsymbol{P}} \sum_{i=1}^{N} \left(\frac{p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} - y_{i,0} \right)^2 + \sum_{i=1}^{N} \left(\frac{p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4}}{p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4}} - y_{i,1} \right)^2 .$$

This nonlinear optimization problem is hard to solve. Therefore, numerial optimization usually requires a *good initialization*.

Camera Calibration

A linear method to estimate the projection matrix results from multiplication of the equations (1), (2) by the respective denominators:

$$p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,0},$$
$$p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,1}.$$

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$$p_{1,1}x_{i,0} + p_{1,2}x_{i,1} + p_{1,3}x_{i,2} + p_{1,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,0},$$
$$p_{2,1}x_{i,0} + p_{2,2}x_{i,1} + p_{2,3}x_{i,2} + p_{2,4} = (p_{3,1}x_{i,0} + p_{3,2}x_{i,1} + p_{3,3}x_{i,2} + p_{3,4})y_{i,1}.$$

Observations:

- These equations are linear in the components of the projection matrix $\boldsymbol{P}$.
- They can be rewritten in matrix form, where a so-called measurement matrix $\boldsymbol{M}$ will include the information on the 3-D calibration points and the measured 2-D points accordingly.

Camera Calibration

Camera calibration thus reduces to the nullspace computation of the measurement matrix $\boldsymbol{M}$:

$$\boldsymbol{M}\begin{pmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{3,3} \\ p_{3,4} \end{pmatrix} = 0, \qquad \text{where}$$

$$\boldsymbol{M} = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 1 & 0 & 0 & 0 & 0 & -x_{1,0}y_{1,0} & -x_{1,1}y_{1,0} & -x_{1,2}y_{1,0} & -y_{1,0} \\ \vdots & & & & & \ddots & & & & & & \vdots \\ x_{N,0} & x_{N,1} & x_{N,2} & 1 & 0 & 0 & 0 & 0 & -x_{N,0}y_{N,0} & -x_{N,1}y_{N,0} & -x_{N,2}y_{N,0} & -y_{N,0} \\ 0 & 0 & 0 & 0 & x_{1,0} & x_{1,1} & x_{1,2} & 1 & -x_{1,0}y_{1,1} & -x_{1,1}y_{1,1} & -x_{1,2}y_{1,1} & -y_{1,1} \\ \vdots & & & & & \ddots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & x_{N,0} & x_{N,1} & x_{N,2} & 1 & -x_{N,0}y_{N,1} & -x_{N,1}y_{N,1} & -x_{N,2}y_{N,1} & -y_{N,1} \end{pmatrix}.$$

Camera Calibration

Observations:

- The calibration problem is reduced to the computation of the nullspace of the measurement matrix $\boldsymbol{M}$.
- We know how to compute the nullspace of $\boldsymbol{M}$ using SVD.
- The rank of $\boldsymbol{M}$ is 11.

Camera Calibration

The estimation problem can also be reduced to an eigenvalue/eigenvector problem:

$$\|\boldsymbol{Mp}\|^2 \to \min, \quad \text{subject to} \quad \|\boldsymbol{p}\|^2 = 1.$$

Camera Calibration

The estimation problem can also be reduced to an eigenvalue/eigenvector problem:

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Applying the Lagrange multiplier method this becomes:

$$\boldsymbol{p}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Mp} - \lambda(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} - 1) \rightarrow \min.$$

Camera Calibration

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Now we find the zero crossings of the gradient w. r. t. the components of $\boldsymbol{P}$:

$$2\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Mp} - 2\lambda\boldsymbol{p} = 0,$$

Camera Calibration

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Now we find the zero crossings of the gradient w. r. t. the components of $\boldsymbol{P}$:

$$2\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Mp} - 2\lambda\boldsymbol{p} = 0,$$

and so we obtain:

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Mp} = \lambda\boldsymbol{p}.$$

Camera Calibration

Conclusions:

- The components of the projection matrix $\boldsymbol{P}$ result from the eigenvector belonging to the smallest eigenvalue.
- The linear estimate of $\boldsymbol{P}$ is an excellent initialization for the nonlinear least squares estimate of the projection matrix.

Topics

Take Home Messages

- Computation of projection matrices originally is a nonlinear problem.
- We have studied how a linear estimate can be computed and in doing so build the measurement matrix $\boldsymbol{M}$.

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Further Readings

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Projection Models and Homogeneous Coordinates

RANSAC Algorithm

Refresher Course
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Topics

RANSAC – RANdom SAmple Consensus

Further Readings

RANSAC – RANdom SAmple Consensus

Problem: In calibration we have to deal with inaccuracies in observations and outliers in the data.

There are two types of outliers:

- badly localized points, and
- wrong correspondences.

Outliers in Linear Regression

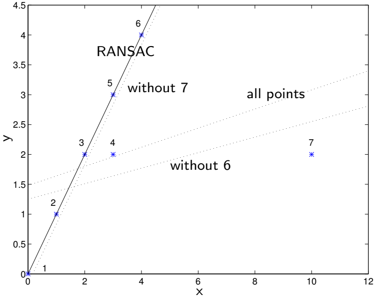

Figure 1: Example of the influence of an outlier in linear regression (least squares method)

RANSAC Algorithm

1. Draw samples uniformly and at random from the input data set.
2. Cardinality of sample set is the smallest size sufficient to estimate the model parameters.
3. Compute the model parameters for each element of the sample data.
4. Evaluate the quality of the hypothetical models on the full data set.
   - Cost function for the evaluation of the quality of the model
   - Inliers: data points which agree with the model within an error tolerance
5. The hypothesis which gets the most support from the data set is taken as the best estimate.

How Many Iterations? When Do We Need to Stop?

Problem: If not run often enough, we probably still have outliers.
Goal: Find a model that is determined only from inliers after N iterations.

- Model estimation requires K points.
- $p(x)$: probability that x is an inlier
- $p(y)$: prob. that at least one model that consists only of inliers is picked

Bernoulli trial: $1 - p(x)^K \rightarrow$ at least 1 out of K points is an outlier
After N iterations: $\left(1 - p(x)^K\right)^N \rightarrow$ prob. that all N models contain outliers

$$\Rightarrow \quad 1 - p(y) = \left(1 - p(x)^K\right)^N$$

We solve the logarithmized equation for N:

$$N = \frac{\log(1 - p(y))}{\log\left(1 - p(x)^K\right)}$$

Example

Let us consider that

- the number of model observations is 1000, and
- the number of inliers is only 100 (a worst case scenario, $p(x) = 10\%$).

Further assume:

- we have a parabolic model ($K = 3$), and
- $p(y) = 99.99999\%$.

$\Rightarrow$ N must be at least 16110.

Topics

RANSAC – RANdom SAmple Consensus

Further Readings

Further Readings

For the original work see:

Martin A. Fischler and Robert C. Bolles. "Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography". In: *CACM* 24.6 (June 1981), pp. 381–395. DOI: 10.1145/358669.358692