Medical Image Processing for Diagnostic Applications

Singular Value Decomposition

Online Course – Unit 4

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Topics

Singular Value Decomposition (SVD) – Part 1

- General Remarks
- On the Geometry of Linear Mappings
- Normal Form of Matrices: SVD

Summary

- Take Home Messages
- Further Readings

Singular Value Decomposition

- Powerful normal form for matrices that allows for a simple solution of most linear problems in imaging and image processing.
- SVD is a method from linear algebra ...
  - ... invented in the 19th century.
  - ... rediscovered and pushed for practical applications by Gene Golub.
  - ... established in computer vision by Carlo Tomasi's famous factorization algorithm to compute structure and camera motion from image sequences.
  - ... which is extremely robust and simple to use.

Singular Value Decomposition

SVD is a perfect tool, e. g., for

- the computation of singular values,
- the computation of the null space,
- the computation of the (pseudo-) inverse,
- the solution of overdetermined linear equations,
- the computation of condition numbers,
- enforcing a rank criterion (numerical rank),
- and other applications of matrices.

On the Geometry of Linear Mappings

From linear algebra, we know that a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ maps the unit vectors $\boldsymbol{e}_i \in \mathbb{R}^n$ of the standard base to the corresponding column vectors $\boldsymbol{a}_i \in \mathbb{R}^m$ of the matrix $\boldsymbol{A}$, $i = 1, \ldots, n$.

Example

$$\boldsymbol{A} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_6) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{a}_4$$

On the Geometry of Linear Mappings

In the example we have made use of the following notation:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{pmatrix} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)\,.$$

We can write:

$$\boldsymbol{Ax} = \boldsymbol{a}_1 x_1 + \boldsymbol{a}_2 x_2 + \ldots + \boldsymbol{a}_n x_n$$

and for the first two unit vectors $\boldsymbol{e}_1 = (1, 0, 0, ..., 0)^T$, $\boldsymbol{e}_2 = (0, 1, 0, ..., 0)^T$ find:

$$\boldsymbol{Ae}_1 = \boldsymbol{a}_1\,, \quad \boldsymbol{Ae}_2 = \boldsymbol{a}_2\,.$$

On the Geometry of Linear Mappings

Example

Compute the orthogonal matrix $\boldsymbol{R}$, i. e., $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathsf{T}}$, that rotates points in the 2-D image plane by the angle θ.

Solution:
The base vectors are mapped as follows:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

and thus the 2-D rotation matrix is:

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

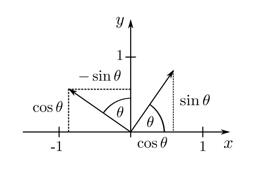

Figure 1: Rotation of 2-D unit vectors

On the Geometry of Linear Mappings

If $\boldsymbol{A}$ is a real $m \times n-$matrix of rank r, then $\boldsymbol{A}$ maps the unit hyper-sphere in the n-dimensional space to an r-dimensional hyperellipsoid in the m-dimensional space.

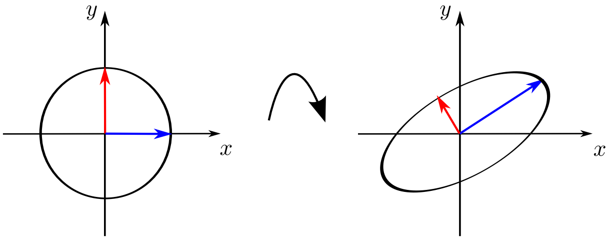

Figure 2: A rank 2-matrix $\boldsymbol{A}$ maps the 2-D unit sphere to a 2-D ellipse.

Normal Form of Matrices: SVD

Theorem

If $\boldsymbol{A}$ *is a real* $m \times n-$*matrix, then there exist orthogonal matrices* $\boldsymbol{U} \in \mathbb{R}^{m \times m}$ *and* $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ *such that*

$$\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathsf{T}},$$

where

$$\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \in \mathbb{R}^{m \times n}$$

with $p = \min\{m, n\}$. *The diagonal elements* σ_i *are the* **singular values** *that fulfill*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0.$$

Topics

Take Home Messages

- SVD is a useful tool to solve a multitude of problems.
- We studied the effect of a matrix on unit vectors and the unit sphere.
- An arbitrary real matrix $\boldsymbol{A}$ can be decomposed by $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathsf{T}}$.

Further Readings

Read the original:

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 3rd ed. Johns Hopkins Studies in the Mathematical Sciences. Baltimore: The Johns Hopkins University Press, Oct. 1996

A very detailed and easy to follow introduction of the SVD can be found in:

Carlo Tomasi's class notes, chapter 3 (a **must-read**).

The theory is described in an easy to read format in:

Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Philadelphia: SIAM, 1997

For details about the numerical computation of SVD see:

William H. Press et al. *Numerical Recipes – The Art of Scientific Computing*. 3rd ed. Cambridge University Press, 2007. Get at `http://numerical.recipes/` (August 2016).

Finally, have a look at:

Kaare Brandt Petersen and Michael Syskind Pedersen. *The Matrix Cookbook*. Online. Accessed: 25. April 2017. Technical University of Denmark, Nov. 2012. URL: `http://www2.imm.dtu.dk/pubdb/p.php?3274`