

Medical Image Processing for Diagnostic Applications

Properties of the SVD

Online Course – Unit 5

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

Topics

Singular Value Decomposition (SVD) – Part 2

Properties of the SVD

Ill-conditioned Matrix

Summary

Take Home Messages

Further Readings

Properties of the SVD: Rank and Norm

The singular value decomposition shows many extremely useful properties that are listed here without proof:

- **rank** of matrix \mathbf{A} :

$$\text{rank}(\mathbf{A}) = \#\{\sigma_i > 0\},$$

- **numerical ε -rank** of matrix \mathbf{A} :

$$\text{rank}_\varepsilon(\mathbf{A}) = \#\{\sigma_i > \varepsilon\},$$

- the **Frobenius norm** of the matrix $\mathbf{A} = (a_{i,j})_{i,j}$ is given by

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{i,j}^2} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}.$$

Properties of the SVD: Eigenvectors

The singular value decomposition shows many extremely useful properties that are listed here without proof:

- decomposition into rank 1 – matrices:

$$\mathbf{A} = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \quad r = \text{rank}(\mathbf{A}),$$

- $\mathbf{A} \mathbf{v}_i = \sigma_i \mathbf{u}_i$ and $\mathbf{A}^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$,
- the column vectors of \mathbf{U} are the eigenvectors of $\mathbf{A} \mathbf{A}^T$:

$$\mathbf{A} \mathbf{A}^T \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i,$$

- the column vectors of \mathbf{V} are the eigenvectors of $\mathbf{A}^T \mathbf{A}$:

$$\mathbf{A}^T \mathbf{A} \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i.$$

Mapping Effect of a Matrix

We want to find the directional vector \mathbf{n} which \mathbf{A} maps to a vector of maximal length compared to other vectors of the unit sphere:

$$\|\mathbf{A}\mathbf{n}\|_2^2 \longrightarrow \max.$$

A Lagrange multiplier is used to add the constraint $\|\mathbf{n}\|_2^2 = 1$:

$$\mathcal{L}(\mathbf{n}) = \|\mathbf{A}\mathbf{n}\|_2^2 - \lambda \left(\|\mathbf{n}\|_2^2 - 1 \right) = \mathbf{n}^\top \mathbf{A}^\top \mathbf{A} \mathbf{n} - \lambda \mathbf{n}^\top \mathbf{n} - \lambda,$$

which can be solved by setting $\frac{d\mathcal{L}(\mathbf{n})}{d\mathbf{n}} = 0$:

$$2\mathbf{A}^\top \mathbf{A} \mathbf{n} - 2\lambda \mathbf{n} = 0 \quad \Leftrightarrow \quad \mathbf{A}^\top \mathbf{A} \mathbf{n} = \lambda \mathbf{n}.$$

Thus, the solution is an eigenvector of $\mathbf{A}^\top \mathbf{A}$.

Properties of the SVD

- The SVD yields orthonormal bases for the kernel (null-space) and the range of a matrix **A**:
 - The **kernel** of matrix **A** is spanned by the column vectors \mathbf{v}_i of **V**, where the corresponding singular values fulfill $\sigma_i = 0$.
 - The **range** of matrix **A** is spanned by the column vectors \mathbf{u}_i of **U**, where σ_i are the corresponding non-zero singular values.
- For the 2-norm of matrix **A** we get:

$$\|\mathbf{A}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} = \sigma_1^2,$$

and if **A** is regular, we even have:

$$\|\mathbf{A}^{-1}\|_2^2 = \frac{1}{\sigma_p^2}.$$

Example

$$\mathbf{A} = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

where

$$\mathbf{U} = \begin{pmatrix} 0.1285 & 0.8375 & 0.5311 \\ -0.2396 & 0.5459 & -0.8028 \\ -0.9623 & -0.0241 & 0.2708 \end{pmatrix},$$

$$\mathbf{\Sigma} = \begin{pmatrix} 71.3967 & 0 & 0 \\ 0 & 21.7831 & 0 \\ 0 & 0 & 0.0006 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} -0.2092 & 0.7082 & -0.6743 \\ -0.1941 & 0.6458 & 0.7384 \\ 0.9584 & 0.2854 & 0.0024 \end{pmatrix}.$$

Example

- Obviously, matrix **A** has a rank deficiency if we select $\varepsilon = 10^{-3}$.
- The kernel of **A** is given by:

$$\ker(\mathbf{A}) = \left\{ \lambda \cdot \begin{pmatrix} -0.6743 \\ 0.7384 \\ 0.0024 \end{pmatrix} ; \lambda \in \mathbb{R} \right\}.$$

- The range (or image) of **A** is:

$$\text{im}(\mathbf{A}) = \left\{ \lambda \cdot \begin{pmatrix} 0.1285 \\ -0.2396 \\ -0.9623 \end{pmatrix} + \mu \cdot \begin{pmatrix} 0.8375 \\ 0.5459 \\ -0.0241 \end{pmatrix} ; \lambda, \mu \in \mathbb{R} \right\}.$$

Ill-conditioned Matrix

Definition

A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is called **ill-conditioned** if for a given linear system

$$\mathbf{Ax} = \mathbf{b}$$

minor changes in $\mathbf{b} \in \mathbb{R}^m$ cause major changes in $\mathbf{x} \in \mathbb{R}^n$.

Definition

The **condition number** of a regular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with respect to a matrix norm $\|\cdot\|$ is defined by

$$\kappa(\mathbf{A}) = \|\mathbf{A}^{-1}\| \cdot \|\mathbf{A}\|.$$

If \mathbf{A} is singular, $\kappa(\mathbf{A}) = +\infty$.

Ill-conditioned Matrix: Remarks

- A matrix with $\kappa(\mathbf{A})$ close to 1 is called ***well-conditioned***.
- A matrix with $\kappa(\mathbf{A})$ significantly greater than 1 is said to be ***ill-conditioned***.
- The condition number is a measure of the stability or sensitivity of a matrix.
- Using the 2-norm, the condition number of a regular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ can be computed by SVD:

$$\kappa(\mathbf{A}) = \frac{\sigma_1}{\sigma_n},$$

where σ_1 is the largest, and σ_n is the smallest singular value.

- The SVD allows for the exact computation of the condition number, but this is computationally expensive.

Ill-conditioned Matrix

Example

Consider the previous matrix

$$\mathbf{A} = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix},$$

where we have $\det(\mathbf{A}) = 1$. The singular value decomposition of \mathbf{A} results in the singular values:

$$\sigma_1 = 71.3967, \sigma_2 = 21.7831, \text{ and } \sigma_3 = 0.0006.$$

Thus the condition number is $\kappa(\mathbf{A}) = 118994.5 \gg 1$.

Exercise: Show that a variation in \mathbf{b} by 0.1% implies a change in \mathbf{x} by 240%.

Topics

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Further Readings

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- We learned about important properties of the SVD, like
 - analytical and numerical rank definition,
 - Frobenius norm and 2-norm,
 - the relation between \mathbf{U} , \mathbf{V} and the eigenvectors of $\mathbf{A}\mathbf{A}^T$, $\mathbf{A}^T\mathbf{A}$,
 - the relation between the kernel/range of \mathbf{A} and the columns of \mathbf{V} , \mathbf{U} .
- For every matrix a condition number can be computed. Ill-conditioned matrices are numerically rather instable.

Further Readings

Read the original:

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 3rd ed. Johns Hopkins Studies in the Mathematical Sciences. Baltimore: The Johns Hopkins University Press, Oct. 1996

A very detailed and easy to follow introduction of the SVD can be found in:

Carlo Tomasi's class notes, chapter 3 (a **must-read**).

The theory is described in an easy to read format in:

Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Philadelphia: SIAM, 1997

For details about the numerical computation of SVD see:

William H. Press et al. *Numerical Recipes – The Art of Scientific Computing*. 3rd ed. Cambridge University Press, 2007. Get at <http://numerical.recipes/> (August 2016).

Finally, have a look at:

Kaare Brandt Petersen and Michael Syskind Pedersen. *The Matrix Cookbook*. Online. Accessed: 25. April 2017. Technical University of Denmark, Nov. 2012. URL: <http://www2.imm.dtu.dk/pubdb/p.php?3274>