Medical Image Processing for Interventional Applications

Ultrasound

Online Course – Unit 37

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Topics

Ultrasound

- Historical Remarks
- Facts on Ultrasound

Summary

- Take Home Messages
- Further Readings

Historical Remarks

1942: Discovery of medical ultrasound by Theodore Dussik

1984: First 3-D ultrasound system reported by Kazunori Baba

Figure 1: First applications of sound (echometry due to Aristoteles)

Generation of Ultrasound Waves

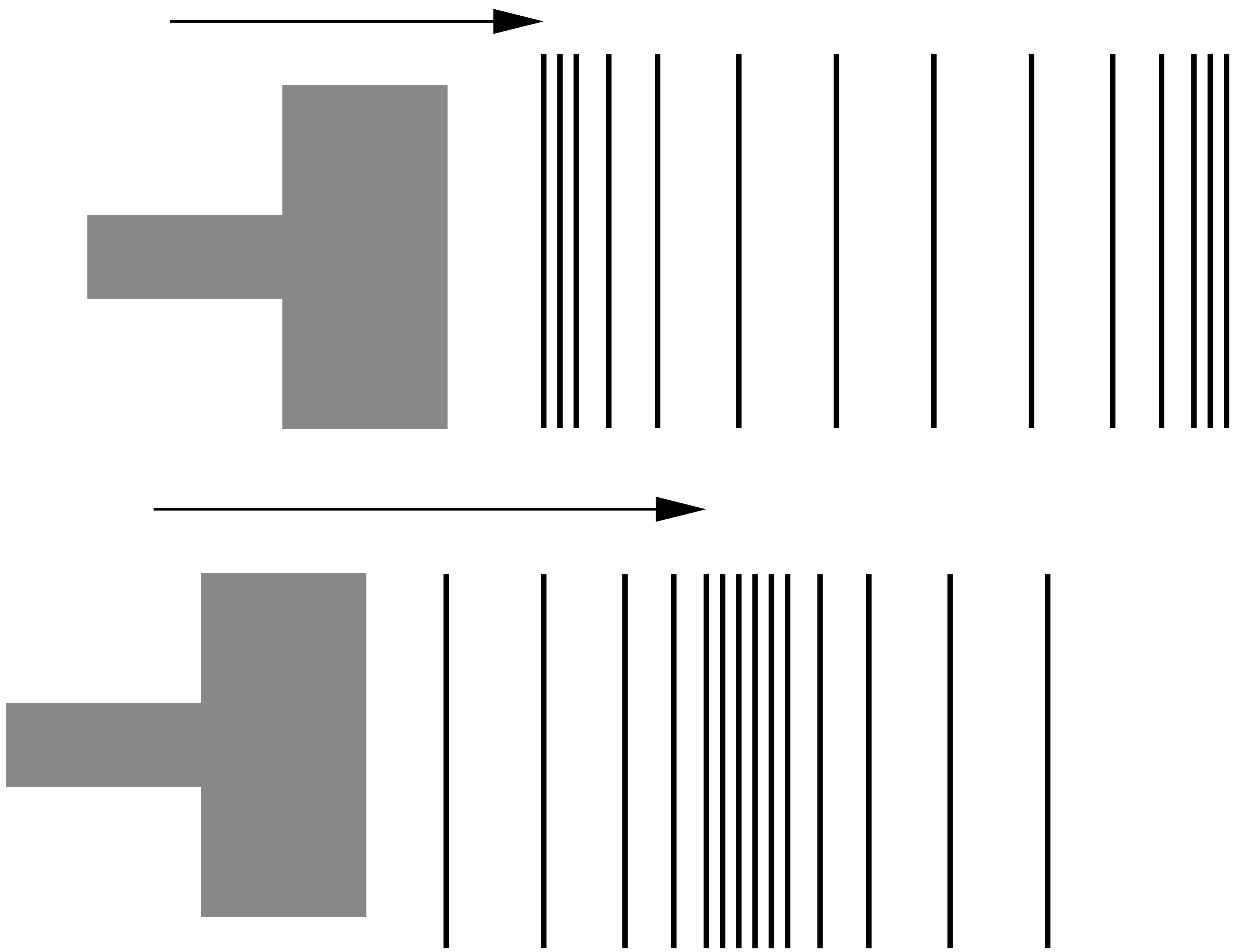

Figure 2: Pressure waves generated by periodic motion

Properties of Waves

- **Reflection:** At the boundary of two media waves are not transmitted, but reflected.
- **Refraction:** At the boundary of two media waves are bended.
- **Absorption:** Conversion of acoustic energy to heat causes attenuation of waves.

Figure 3: Siemens-ACUSON Aspen Echo System (left), Siemens-ACUSON US CV 70 (right)

2-D Ultrasound Images

Figure 4: Portable ultrasound system (Siemens Healthcare)

2-D Ultrasound Images

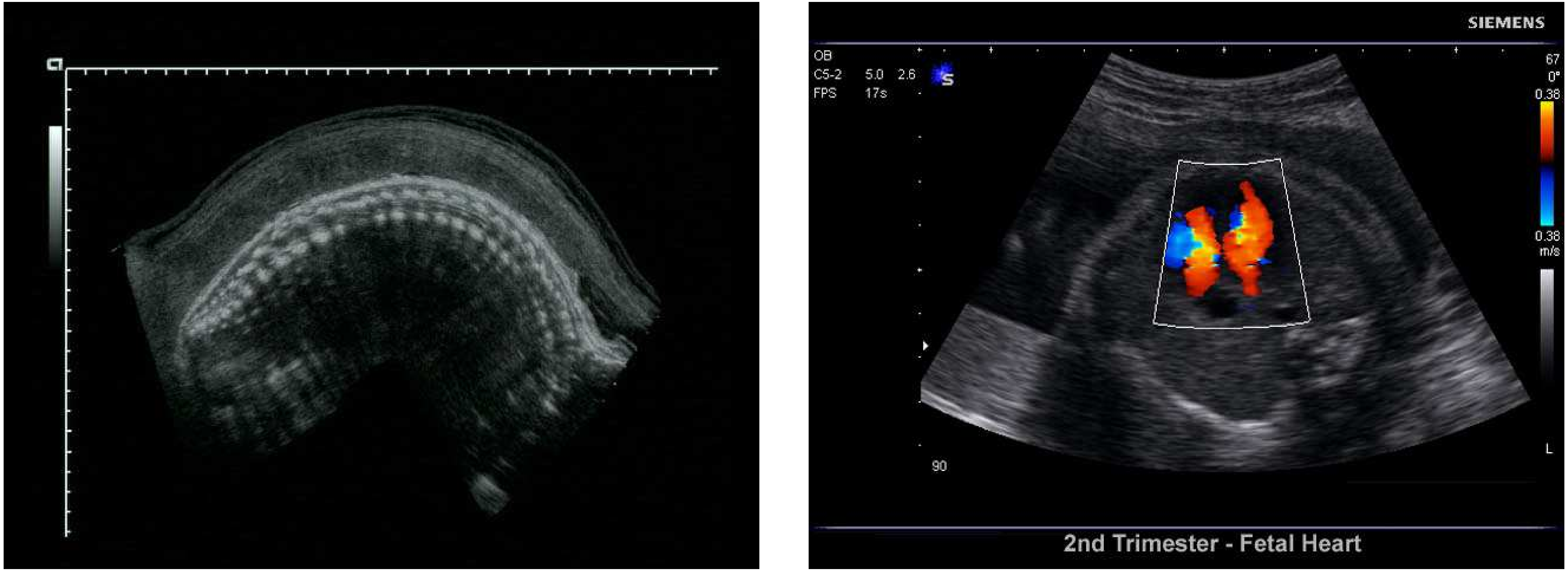

Figure 5: 2-D ultrasound showing a fetal spine and heart (images courtesy of Siemens Healthcare)

3-D Ultrasound Images

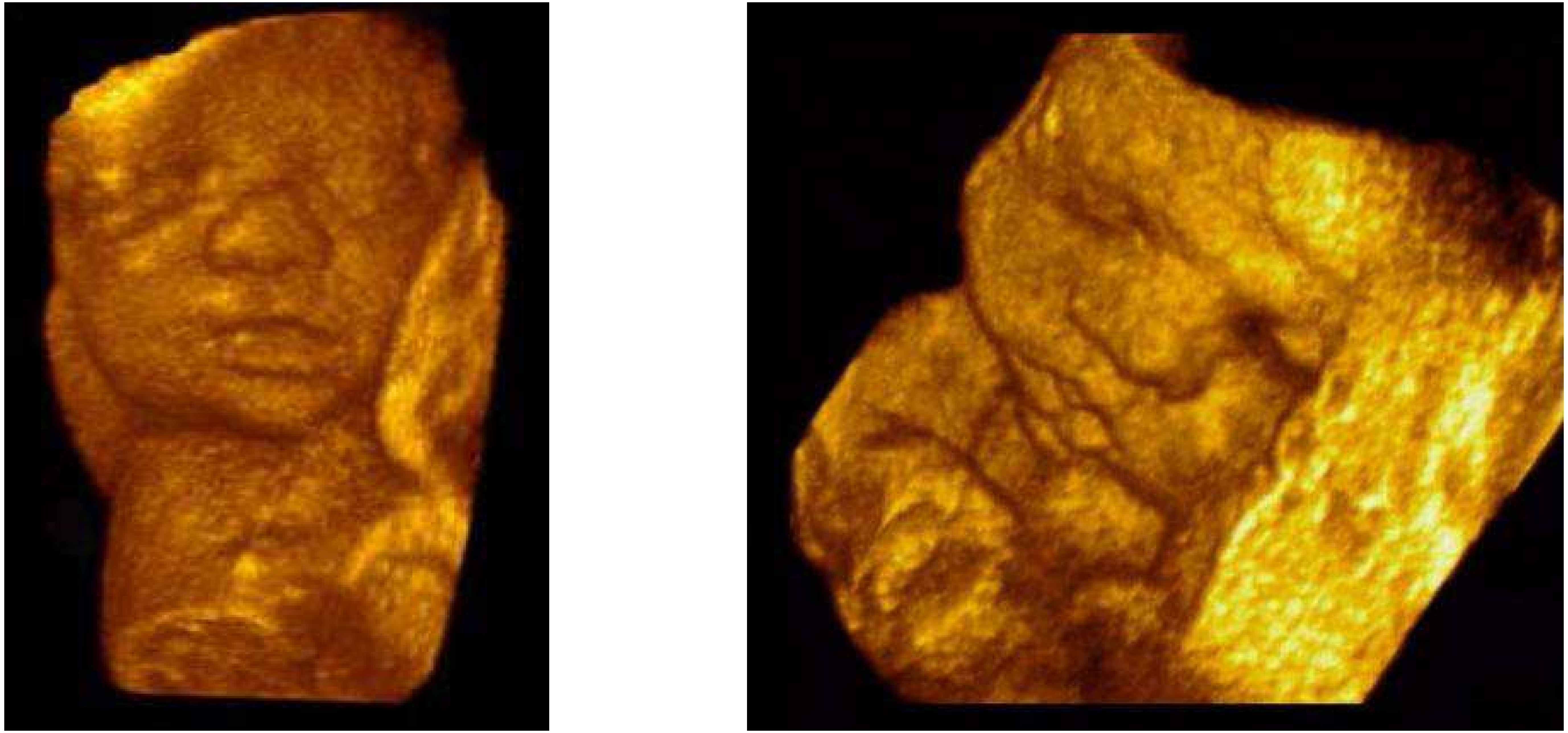

Figure 6: 3-D ultrasound images of a fetus (images courtesy of Siemens Healthcare)

Carotid Artery

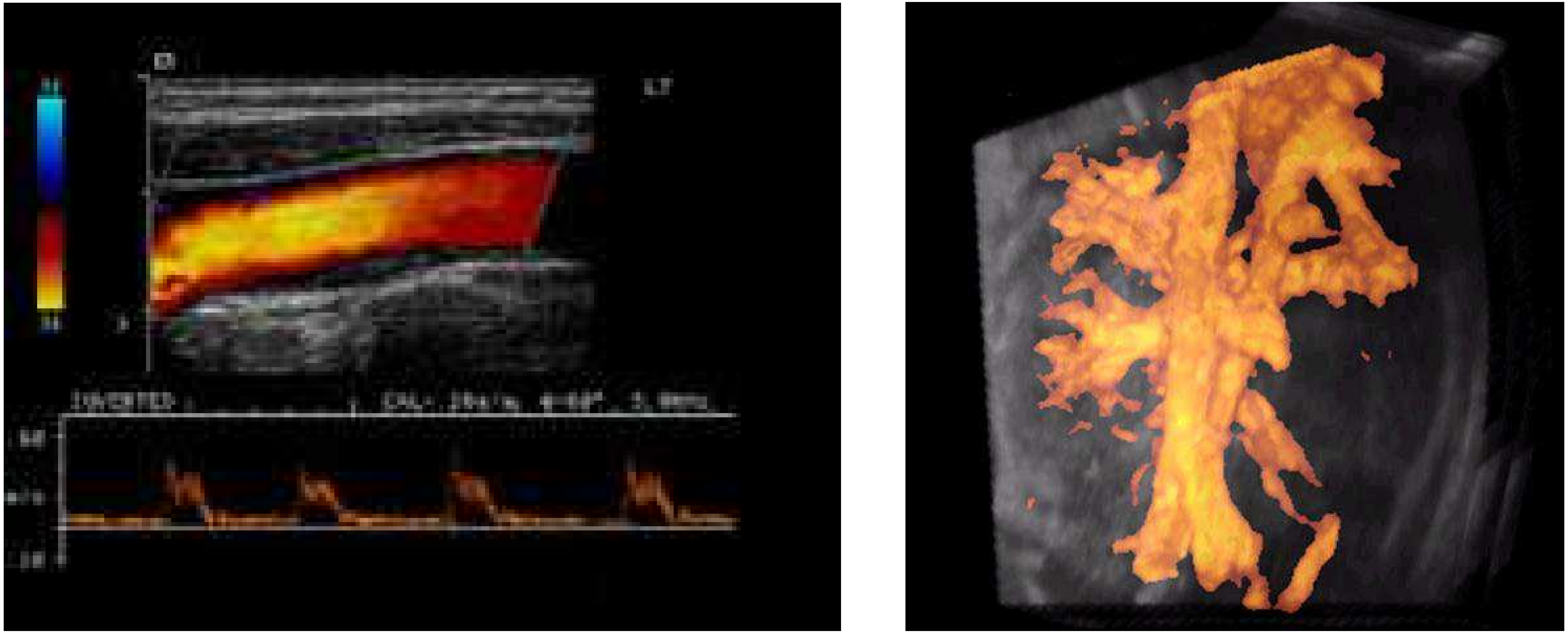

Figure 7: Carotid artery and 3-D ultrasound image of vessels (images courtesy of Siemens Healthcare)

Basics from Physics

- Physical medium can vibrate and produce sound.
- Sound waves are due to tissue vibrations.
- Sine waves: peak represents the maximum, nadir represents the minimum pressure.
- Characteristics of sound waves: period, frequency, speed, amplitude, power, intensity, wavelength
- Propagation speed in human tissue: $\sim 1500\,\mathrm{m\,s^{-1}}$
- Hearable sound: 20–20000 Hz
- Clinical ultrasound: 1–10 MHz

Biological Media

medium	$c\,[\mathrm{m\,s^{-1}}]$	$Z\,[\mathrm{g\,cm^{-2}\,s^{-1}}]$	$\rho\,[\mathrm{g\,cm^{-3}}]$
air	331	430	0.013
grease	1470	1.42×10^5	0.97
water	1492	1.48×10^5	0.9982
brain tissue	1530	1.56×10^5	1.02
muscles	1568	1.63×10^5	1.04
bones	3600	6.12×10^5	1.7

Table 1: Data of different biological media (speed of sound in the medium, acoustic impedance, density)

Basics from Physics

Important observations:

- Medium determines the speed of sound.
- Sound of different frequencies propagates at the same speed in the same medium.

Basics from Physics

Important observations:

- Medium determines the speed of sound.
- Sound of different frequencies propagates at the same speed in the same medium.

Definition

The ***acoustic impedance*** Z is:

$$Z = \rho \cdot c\,,$$

where ρ is the density of the medium, and c the speed of sound waves in the medium.

Basics from Physics

Reflection at the boundary of two different tissue classes can be described by:

$$I_R = I_I \frac{1 - \frac{Z_2}{Z_1}}{1 + \frac{Z_2}{Z_1}},$$

where

- I_I: intensity of incoming wave,
- I_R: intensity of reflected wave,
- Z_1: impedance of tissue class 1,
- Z_2: impedance of tissue class 2.

Basics from Physics

The **relationship of speed** c**, frequency** f **and wavelength** λ is:

$$c = f \cdot \lambda \,.$$

- The denser a medium, the higher the speed of sound through the medium.
  $\rightarrow$ Sound propagates faster through bones than liquids.
- The higher the frequency, the lower the wavelength.
  $\rightarrow$ Echocardiographic imaging: Higher image resolution due to smaller wavelength; deeper penetration results from larger wavelength.

Basics from Physics

The **distance between ultrasound source and boundary** can be computed as

$$d = \frac{1}{2}ct\,,$$

where

- d: distance between source and tissue boundary,
- t: runtime of signal,
- c: speed of sound.

Note: Factor 0.5 results from the fact that the signal moves from the source to the tissue boundary and back.

Topics

Take Home Messages

- Ultrasound is using sound waves to generate images. This is possible due to different acoustic characteristics of the tissue materials.
- There are various medical applications and, in contrast to X-ray imaging, US does not depend on possibly harmful radiation.

Further Readings

- Carlo Tomasi and Takeo Kanade. "Shape and Motion from Image Streams Under Orthography: A Factorization Method". In: *International Journal of Computer Vision* 9.2 (Nov. 1992), pp. 137–154. DOI: 10.1007/BF00129684
- C. J. Poelman and T. Kanade. "A Paraperspective Factorization Method for Shape and Motion Recovery". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 19.3 (Mar. 1997), pp. 206–218. DOI: 10.1109/34.584098
- Mei Han and Takeo Kanade. "A Perspective Factorization Method for Euclidean Reconstruction with Uncalibrated Cameras". In: *The Journal of Visualization and Computer Animation* 13.4 (2002), pp. 211–223. DOI: 10.1002/vis.290
- Peter Sturm and Bill Triggs. "A Factorization Based Algorithm for Multi-Image Projective Structure and Motion". In: *Computer Vision — ECCV '96: 4th European Conference on Computer Vision Cambridge, UK, April 15–18, 1996 Proceedings Volume II*. ed. by Bernard Buxton and Roberto Cipolla. Vol. 1065. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 709–720. DOI: 10.1007/3-540-61123-1_183

Medical Image Processing for Interventional Applications

3-D Ultrasound

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Topics

3-D Ultrasound

Interpolation Techniques

Summary
- Take Home Messages
- Further Readings

3-D Reconstruction

- 3-D reconstruction requires computation of relative positions and orientations of acquired 2-D ultrasound images.
- 3-D volume results from interpolation of voxel values.
- Adjustment of 2-D images:
  - Feature-based reconstruction
  - Intensity-based reconstruction

3-D Ultrasound: Principle

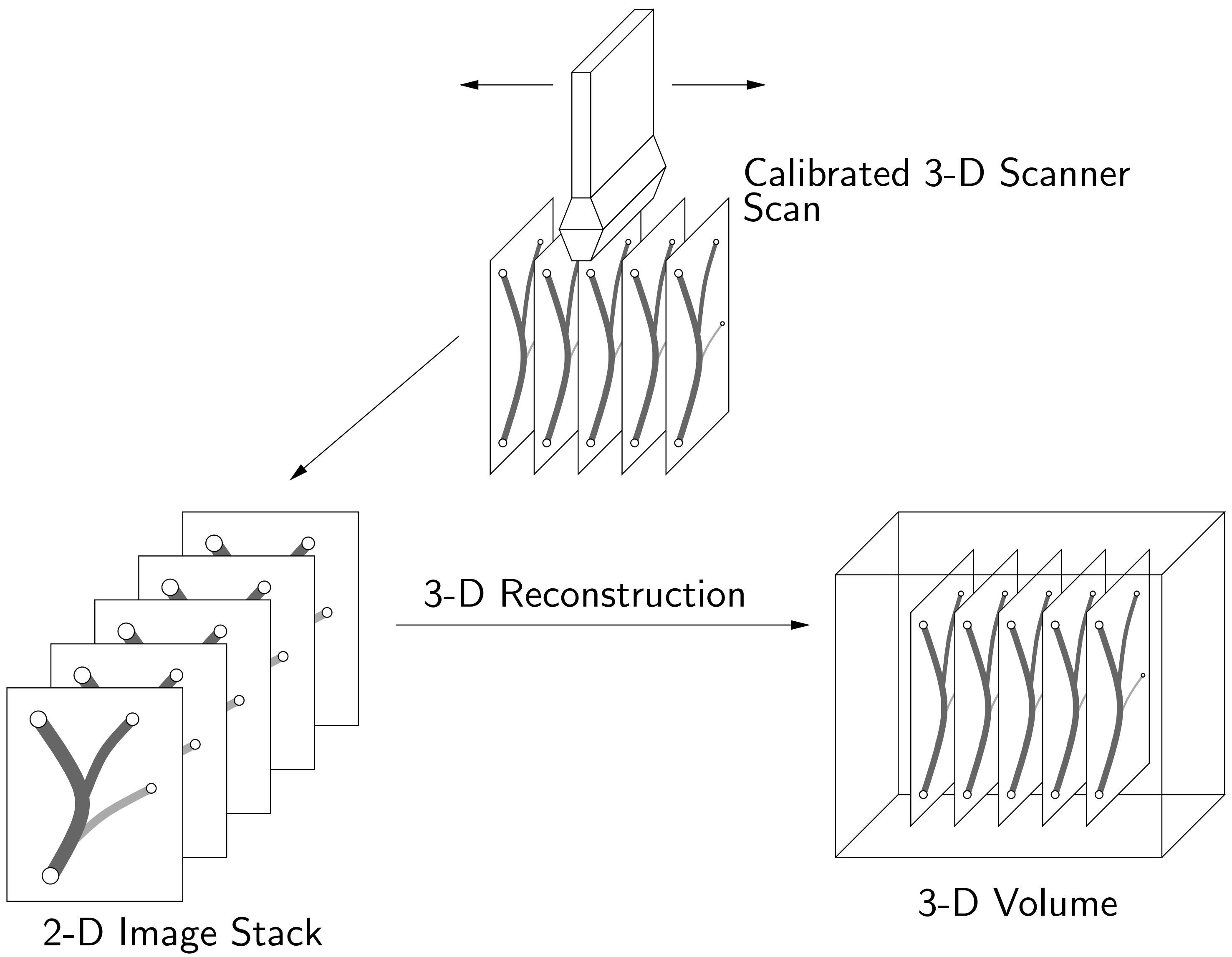

Figure 1: Ideal 3-D ultrasound acquisition

3-D Ultrasound: Principle

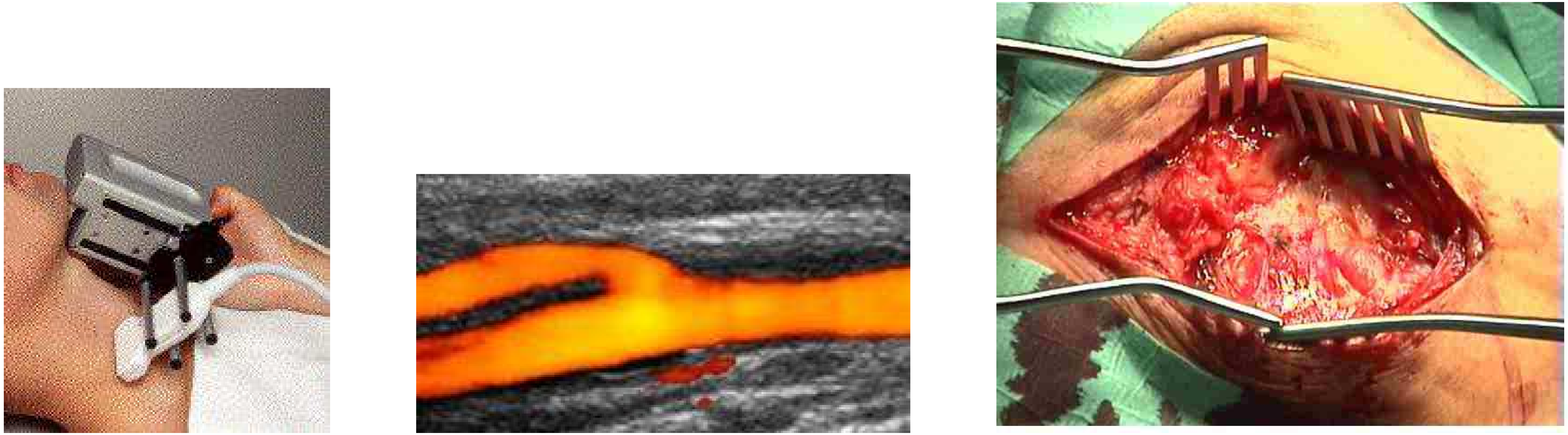

Figure 2: Special device for 3-D ultrasound acquisition (left) (image courtesy of A. Fenster & D. B. Downey), ultrasound image of carotis bifurcation (middle), surgery of carotis (right) (image courtesy of Schön Klinik Vogtareuth)

3-D Ultrasound: Principle

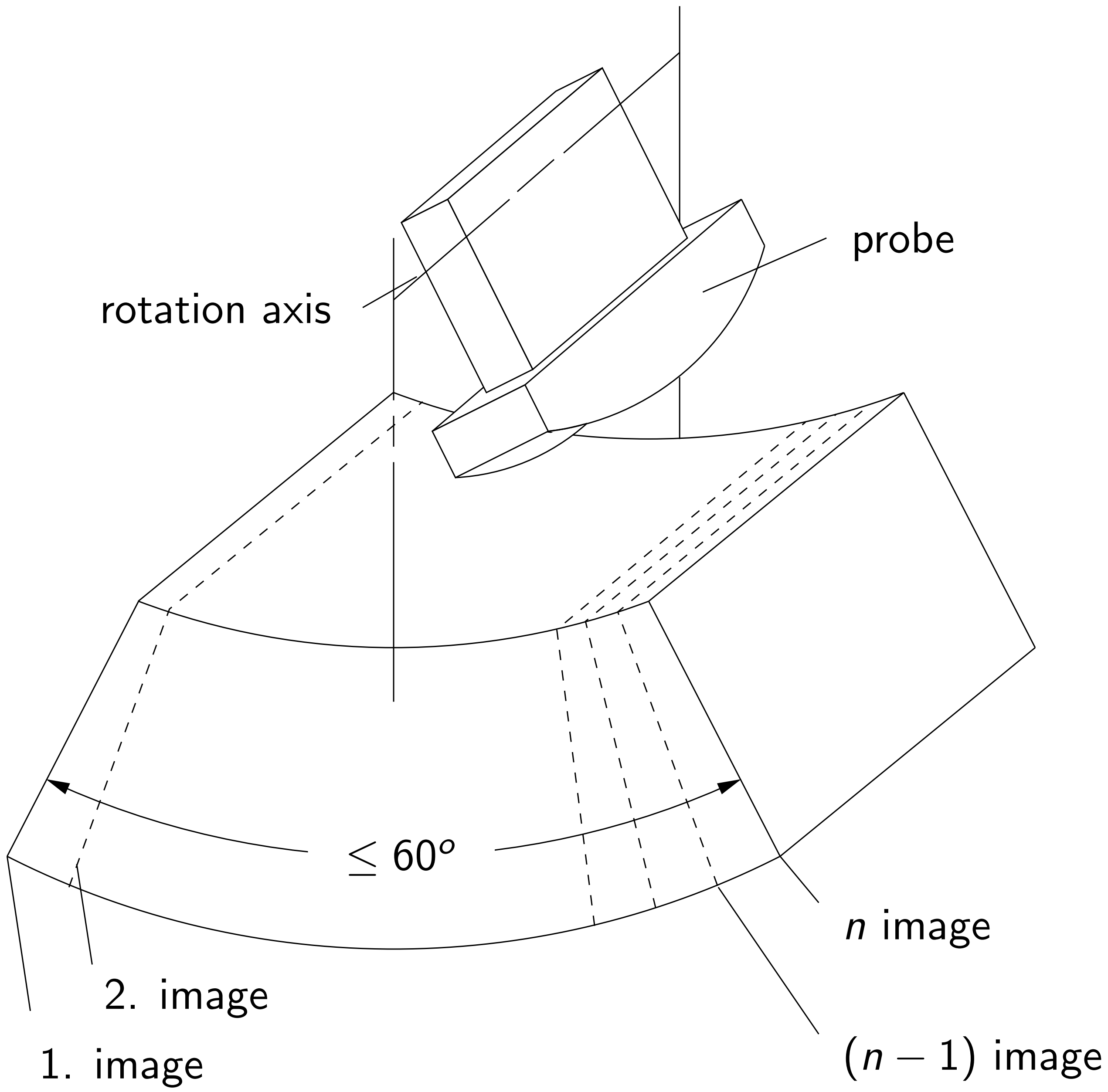

Figure 3: 3-D ultrasound acquisition

3-D Ultrasound: Principle

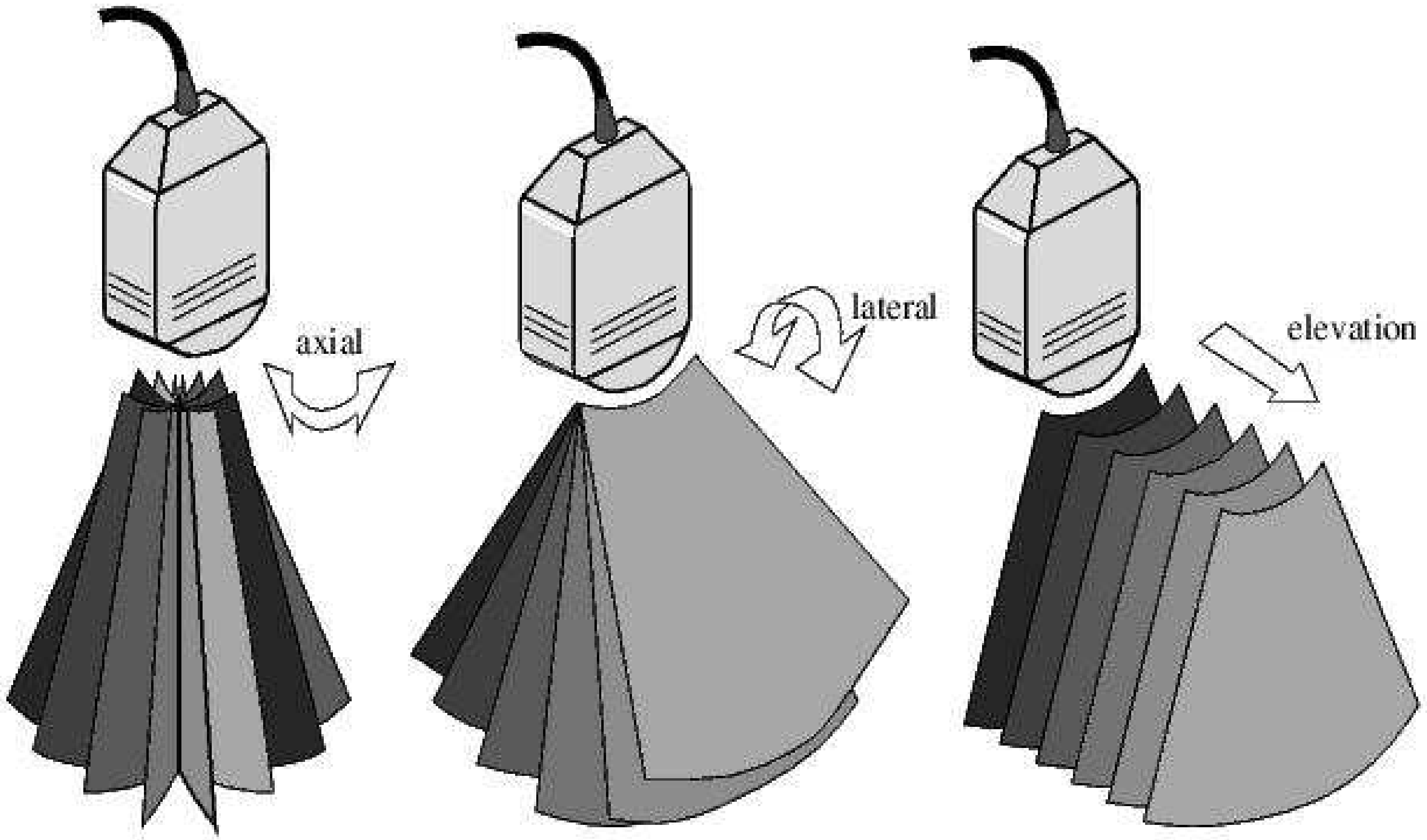

Figure 4: Mechanical sweep motion for 3-D US acquisition

3-D Ultrasound: Principle

Figure 5: Mechanical motion of the probe by a rotation device (left), a sweep device (middle), or a pullback device (right) (TOMTEC, Munich)

3-D Ultrasound: Principle

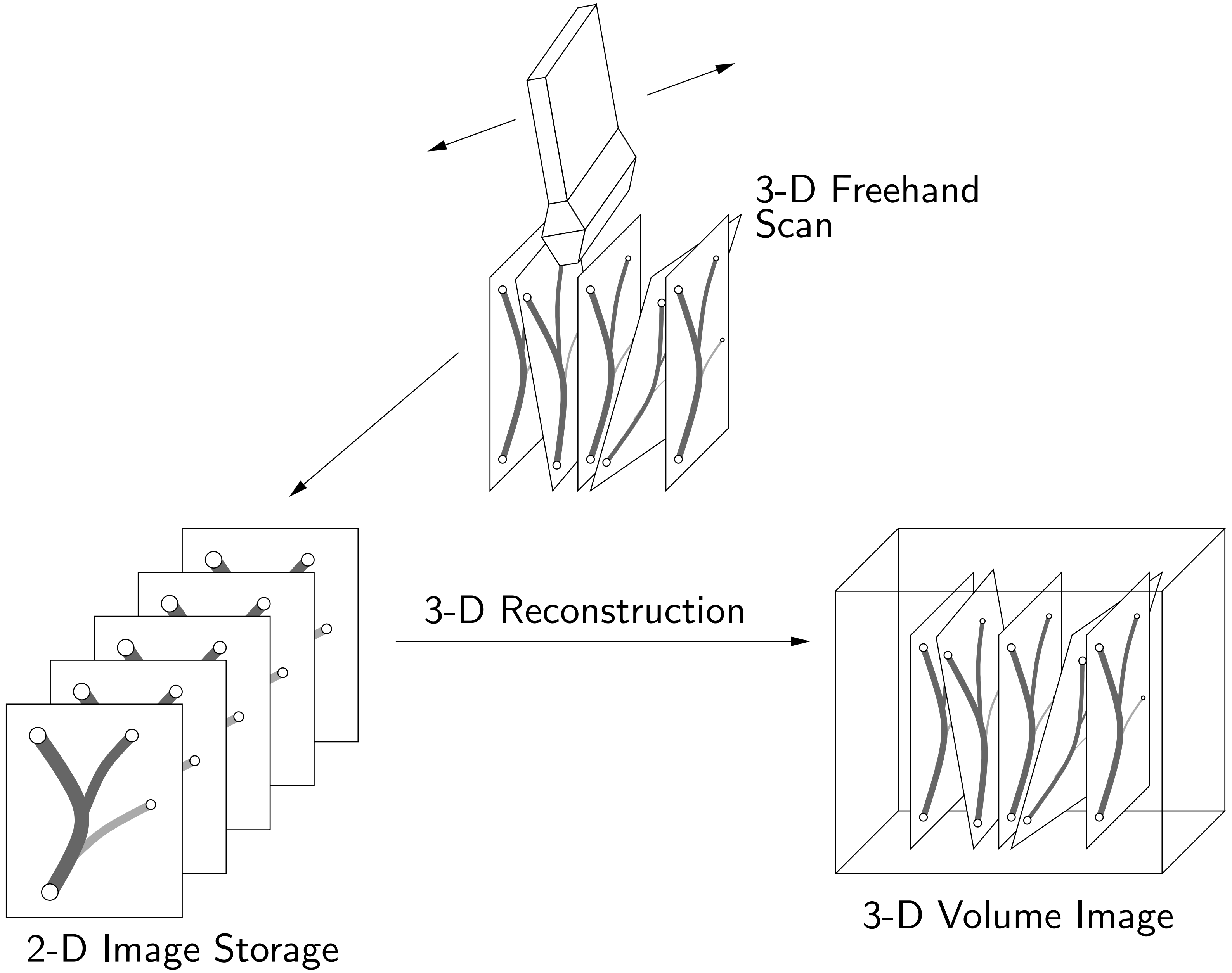

Figure 6: Freehand ultrasound 3-D acquisition

3-D Ultrasound: Principle

Figure 7: Markers for automatic pose estimation of the probe (left), stereo camera system (right)

Exercise: Describe an algorithm that computes the rotation and translation parameters between two different positions of the probe (use fundamental and essential matrix).

ECG Triggered US Reconstruction

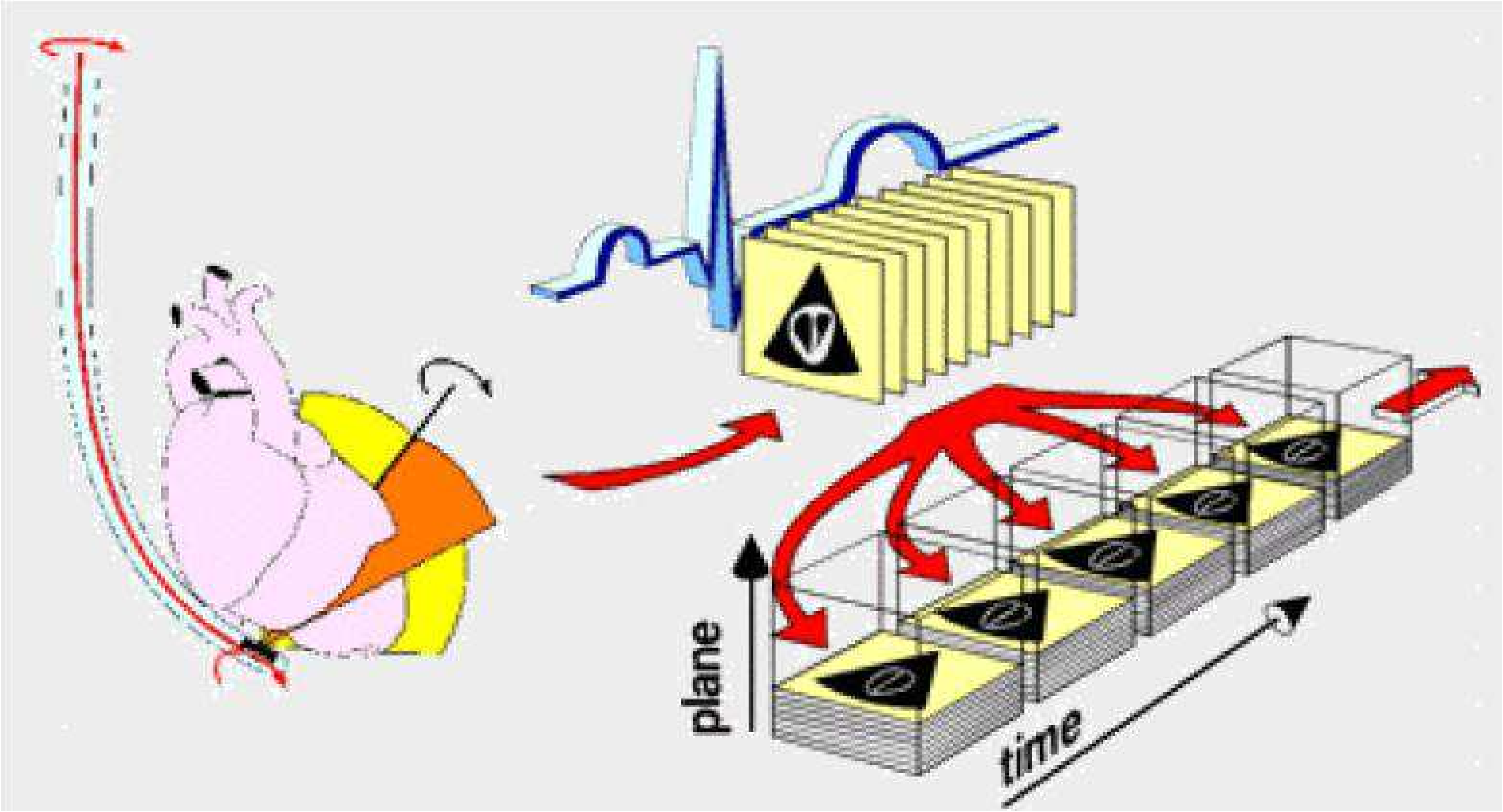

Figure 8: ECG triggered US reconstruction (TOMTEC, Munich)

Topics

3-D Ultrasound

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Interpolation Techniques

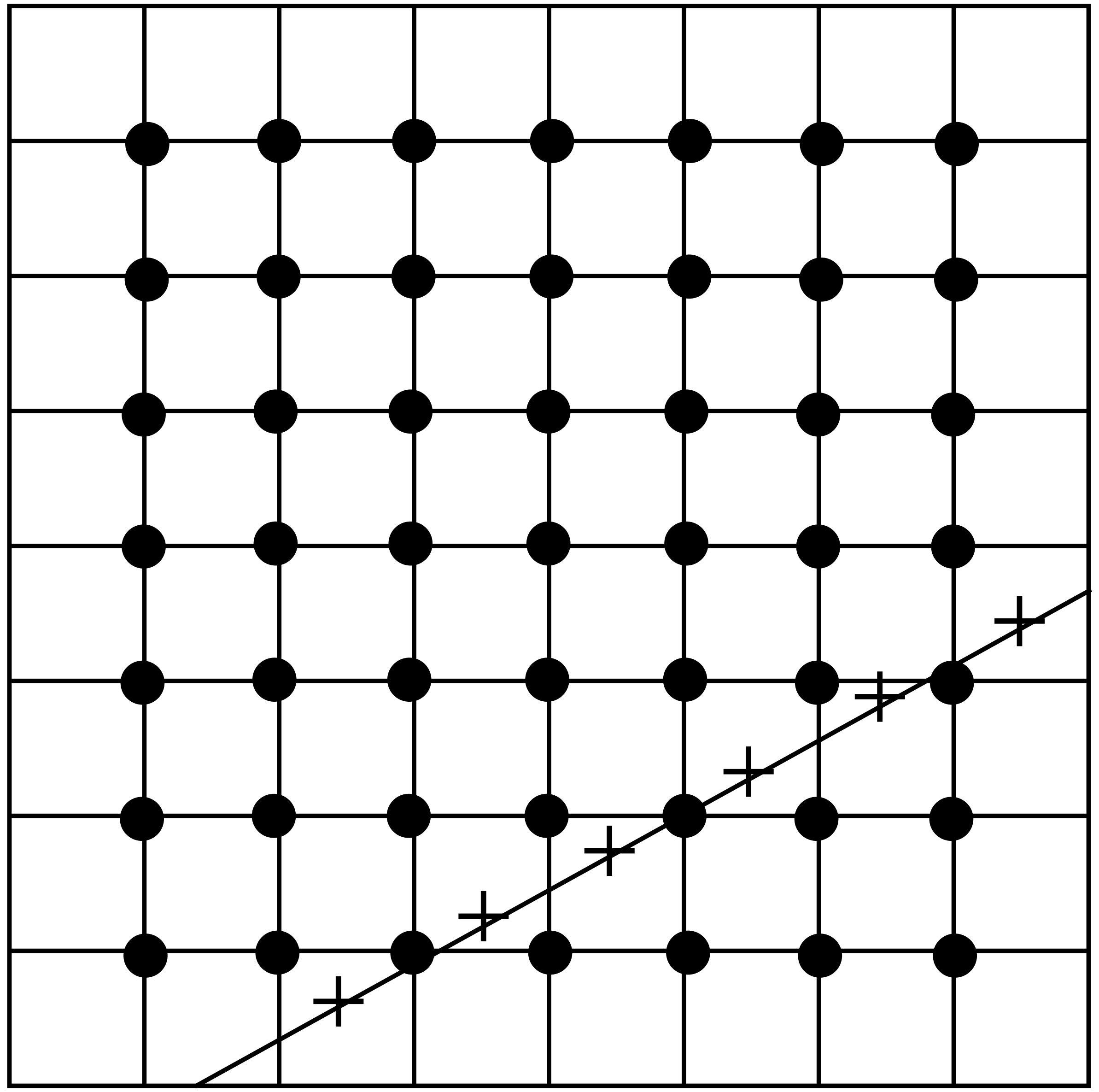

Figure 9: Interpolation

Interpolation Techniques

Voxel nearest neighbor:

- Assign voxel to nearest pixel.
- No parameterization is required.
- Avoid gaps in the reconstruction.

Interpolation Techniques

Nearest neighbor interpolation:

- Run through all pixels.
- Fill the nearest voxel with the intensity value.
- Multiple contributions are averaged or use maximum value alternatively.
- Gap filling is done in a second step.

Interpolation Techniques

Distance weighted interpolation:

- Sampling is done voxel by voxel.
- Voxel values are assigned the averaged sum of nearby voxels.
- For example, consider a spherical neighborhood centered around each voxel.

Topics

3-D Ultrasound

Interpolation Techniques

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Take Home Messages

- There are several ways to acquire US data for 3-D reconstruction, some of them even involve freehand acquisition.
- Implementing 3-D ultrasound reconstruction requires interpolation methods, where there is also a variety of methods to choose from (differing in runtime and accuracy).

Further Readings

- Carlo Tomasi and Takeo Kanade. "Shape and Motion from Image Streams Under Orthography: A Factorization Method". In: *International Journal of Computer Vision* 9.2 (Nov. 1992), pp. 137–154. DOI: 10.1007/BF00129684
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Factorization for Orthographic Projections

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Topics

Factorization Methods

Preliminaries:

- Orthogonal projection model
- Number of frames $N_\mathrm{F} \geq 3$
- Each world point $\tilde{\boldsymbol{p}}_j^w$ is visible in **all** frames.
- The world points are **not** all coplanar.
- $(x_{ij}, y_{ij})^\mathrm{T} \in \mathbb{R}^2$ is the j-th image point in the i-th frame.

Factorization Methods

Idea:

- Put all image points together in one matrix $\boldsymbol{M}$,
- then **factorize** $\boldsymbol{M}$ into a product of two matrices, a projection-matrix $\boldsymbol{R}$, and a matrix $\boldsymbol{S}$ (world-points):

$$\boldsymbol{M} = \boldsymbol{R}\boldsymbol{S}.$$

In general we have:

- $\boldsymbol{R}$ is a $3N_\text{F} \times 4$ matrix containing all projection matrices,
- $\boldsymbol{S}$ is a $4 \times N_\text{p}$ matrix containing all world points (N_p = number of all points).

In the case of orthogonal projections, the homogeneous form is not necessary, thus:

- $\boldsymbol{R}$ is $2N_\text{F} \times 3$,
- $\boldsymbol{S}$ is $3 \times N_\text{p}$.

Measurement Matrix

Form the so-called ***measurement matrix*** $\boldsymbol{M}$ of size $2N_\mathrm{F} \times N_\mathrm{p}$ from the image points:

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix},$$

where

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1N_\mathrm{p}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N_\mathrm{F}1} & x_{N_\mathrm{F}2} & \dots & x_{N_\mathrm{F}N_\mathrm{p}} \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1N_\mathrm{p}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{N_\mathrm{F}1} & y_{N_\mathrm{F}2} & \dots & y_{N_\mathrm{F}N_\mathrm{p}} \end{pmatrix}.$$

Registered Measurement Matrix

For factorization we need the **registered** measurement matrix $\widehat{\boldsymbol{M}}$ containing all 2-D points $(x_{ij}, y_{ij})^\mathsf{T}$ shifted so that their mean is 0, i. e.,

$$\widehat{\boldsymbol{M}} = \begin{pmatrix} \widehat{\boldsymbol{X}} \\ \widehat{\boldsymbol{Y}} \end{pmatrix},$$

where the entries of $\widehat{\boldsymbol{X}}$, $\widehat{\boldsymbol{Y}}$ are:

$$\widehat{x}_{ij} = x_{ij} - \bar{x}_i, \qquad \widehat{y}_{ij} = y_{ij} - \bar{y}_i,$$

with

$$\bar{x}_i = \frac{1}{N_\mathrm{p}} \sum_{j=1}^{N_\mathrm{p}} x_{ij}, \qquad \bar{y}_i = \frac{1}{N_\mathrm{p}} \sum_{j=1}^{N_\mathrm{p}} y_{ij}.$$

Representation of 2-D Image Points

Now consider the following representation of image points:

$$x_{ij} = \boldsymbol{u}_i^{\mathrm{T}} \left(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i\right), \quad y_{ij} = \boldsymbol{v}_i^{\mathrm{T}} \left(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i\right),$$

where

- $\boldsymbol{u}_i$, $\boldsymbol{v}_i$ are unit vectors of image reference frame i (3-D vectors),
- $\boldsymbol{t}_i$ is the translation vector from world-origin to frame origin,
- $\tilde{\boldsymbol{p}}_j^w$ is a 3-D world point,
- the world coordinate system is object-centered:

$$\frac{1}{N_\mathrm{p}} \sum_{j=1}^{N_\mathrm{p}} \tilde{\boldsymbol{p}}_j^w = 0.$$

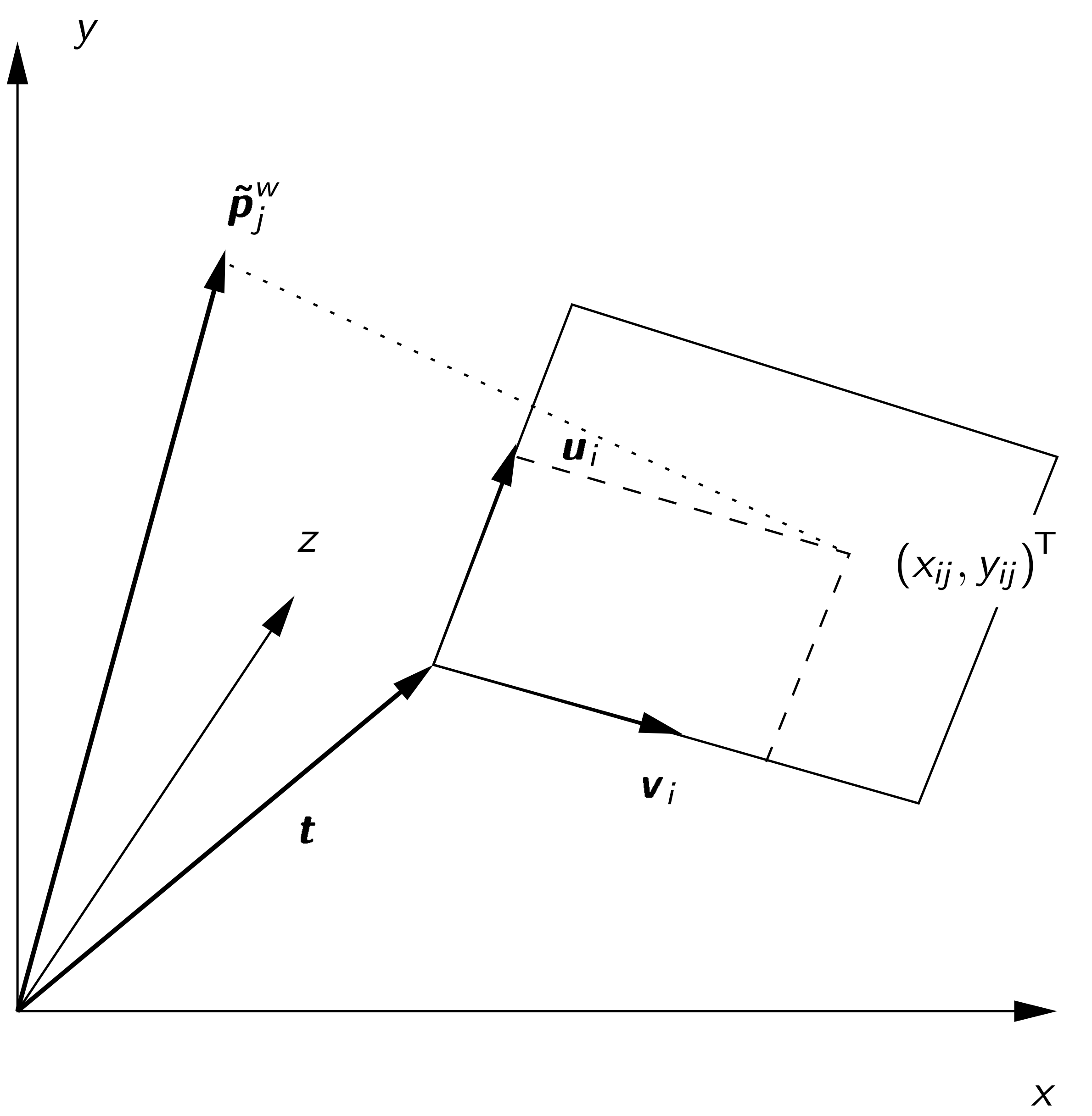

Figure 1: Image planes in 3-D

Representation of 2-D Image Points

Thus we get:

$$
\begin{aligned}
\widehat{x}_{ij} = x_{ij} - \bar{x}_i &= \boldsymbol{u}_i^\mathrm{T}(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i) - \frac{1}{N_\mathrm{p}} \sum_{m=1}^{N_\mathrm{p}} \left(\boldsymbol{u}_i^\mathrm{T}(\tilde{\boldsymbol{p}}_m^w - \boldsymbol{t}_i)\right) \\
&= \boldsymbol{u}_i^\mathrm{T}\tilde{\boldsymbol{p}}_j^w - \boldsymbol{u}_i^\mathrm{T}\boldsymbol{t}_i - \boldsymbol{u}_i^\mathrm{T}\left(\left(\frac{1}{N_\mathrm{p}} \sum_{m=1}^{N_\mathrm{p}} \tilde{\boldsymbol{p}}_m^w\right) - \boldsymbol{t}_i\right) \\
&= \boldsymbol{u}_i^\mathrm{T}\tilde{\boldsymbol{p}}_j^w \, .
\end{aligned}
$$

Computation of Registered Image Points

With $\widehat{x}_{ij} = \boldsymbol{u}_i^\mathsf{T} \tilde{\boldsymbol{p}}_j^w$ and $\widehat{y}_{ij} = \boldsymbol{v}_i^\mathsf{T} \tilde{\boldsymbol{p}}_j^w$ the registered measurement matrix looks as follows:

$$\widehat{\boldsymbol{M}} = \begin{pmatrix} \boldsymbol{u}_1^\mathsf{T}\tilde{\boldsymbol{p}}_1^w & \boldsymbol{u}_1^\mathsf{T}\tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{u}_1^\mathsf{T}\tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{u}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_1^w & \boldsymbol{u}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{u}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ \boldsymbol{v}_1^\mathsf{T}\tilde{\boldsymbol{p}}_1^w & \boldsymbol{v}_1^\mathsf{T}\tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{v}_1^\mathsf{T}\tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_1^w & \boldsymbol{v}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{v}_{N_\mathrm{F}}^\mathsf{T}\tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \end{pmatrix} = \overset{R}{\begin{pmatrix} \boldsymbol{u}_1^\mathsf{T} \\ \vdots \\ \boldsymbol{u}_{N_\mathrm{F}}^\mathsf{T} \\ \boldsymbol{v}_1^\mathsf{T} \\ \vdots \\ \boldsymbol{v}_{N_\mathrm{F}}^\mathsf{T} \end{pmatrix}} \overset{S}{\begin{pmatrix} \tilde{\boldsymbol{p}}_1^w & \tilde{\boldsymbol{p}}_2^w & \dots & \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \end{pmatrix}} .$$

Notes

- $\widehat{\boldsymbol{M}}$ can be factorized into:
  - a $2N_{\mathrm{F}} \times 3$ matrix $\boldsymbol{R}$ containing camera movement,
  - and a $3 \times N_{\mathrm{p}}$ matrix $\boldsymbol{S}$ containing 3-D points.
- $\widehat{\boldsymbol{M}}$ is always of rank 3, since
  - $\boldsymbol{u}_i$, $\boldsymbol{v}_i$, $\tilde{\boldsymbol{p}}_j^w$ are 3-vectors,
  - and the world points are **not** all coplanar.
- Factorization can be done using the SVD.
- The factorization is not unique.

Rank theorem: $\widehat{\boldsymbol{M}}$ has rank 3.

Factorization of the Measurement Matrix

If the factorization is $\widehat{\boldsymbol{M}} = \boldsymbol{RS}$, then

$$\widehat{\boldsymbol{M}} = (\boldsymbol{RQ})(\boldsymbol{Q}^{-1}\boldsymbol{S})$$

is also a valid factorization. The matrix $\boldsymbol{Q}$ is an invertible 3×3 matrix.

The following constraints are useful:

- $\boldsymbol{u}_i$, $\boldsymbol{v}_i$ are orthogonal,
- $|\boldsymbol{u}_i| = |\boldsymbol{v}_i| = 1$.

Tomasi's Factorization Algorithm

1. Track points.

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2. Compute SVD of $\widehat{\boldsymbol{M}}$:

$$\widehat{\boldsymbol{M}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\top}.$$

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$$\widehat{\boldsymbol{M}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathsf{T}}.$$
3. Set all σ_k for $k \geq 4$ to zero, since $\operatorname{rank}\left(\widehat{\boldsymbol{M}}\right) = 3$.

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4. Let $\boldsymbol{U}'$ be the $2N_{\mathrm{F}} \times 3$ submatrix of $\boldsymbol{U}$, and $\boldsymbol{V}'$ the $3 \times N_{\mathrm{p}}$ submatrix of $\boldsymbol{V}$ corresponding to σ_1, σ_2, and σ_3. Let $\boldsymbol{\Sigma}' = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, then compute:
$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^{\mathsf{T}}.$$

Tomasi's Factorization Algorithm

1. Track points.
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$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^{\mathsf{T}}.$$
5. Solve the following (nonlinear) equations for $\boldsymbol{Q}$:
$$\begin{aligned} \hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{u}}_i &= 1, \\ \hat{\boldsymbol{v}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i &= 1, \\ \hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i &= 0. \end{aligned}$$

Tomasi's Factorization Algorithm

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3. Set all σ_k for $k \geq 4$ to zero, since $\operatorname{rank}\left(\widehat{\boldsymbol{M}}\right) = 3$.
4. Let $\boldsymbol{U}'$ be the $2N_{\mathrm{F}} \times 3$ submatrix of $\boldsymbol{U}$, and $\boldsymbol{V}'$ the $3 \times N_{\mathrm{p}}$ submatrix of $\boldsymbol{V}$ corresponding to σ_1, σ_2, and σ_3. Let $\boldsymbol{\Sigma}' = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, then compute:
$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^{\mathsf{T}}.$$
5. Solve the following (nonlinear) equations for $\boldsymbol{Q}$:
$$\begin{aligned} \hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{u}}_i &= 1, \\ \hat{\boldsymbol{v}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i &= 1, \\ \hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i &= 0. \end{aligned}$$
6. Compute the output:
$$\boldsymbol{R} = \hat{\boldsymbol{R}}\boldsymbol{Q}, \qquad \boldsymbol{S} = \boldsymbol{Q}^{-1}\hat{\boldsymbol{S}}.$$

Remarks

- Nonlinear optimization for $\boldsymbol{Q}$ is not very pleasant.
- Elegant “democratic” method: All points are treated equally.
- It is mathematically simple and stable.
- The algorithm yields only the **rotation** of the world points.
- It is used in industry.
- Translation parallel to the image plane is proportional to the translation of the image centroid between two frames.
- The translational component along the optical axis cannot be computed because of the orthogonal projection model.
- Adding new frames is easy and gives a more stable reconstruction.
- *Problem:* **All** 3-D points must be visible in **all** frames.
- Check the assumption that the camera gives an orthogonal image.

Topics

Factorization Methods for Orthographic Projections

- Preliminaries
- Registered Measurement Matrix
- Factorization of the Measurement Matrix

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- If we put all image points from several ultrasound acquisitions into a single measurement matrix for 3-D reconstruction, we can perform a factorization of this matrix.
- One of the factorized matrices contains the projective information, the other contains the world points.
- We need to register a given measurement matrix towards the centroid center of the image points.
- Tomasi's algorithm can be used to compute a factorization in case of orthogonal projections.

Further Readings

- Carlo Tomasi and Takeo Kanade. "Shape and Motion from Image Streams Under Orthography: A Factorization Method". In: *International Journal of Computer Vision* 9.2 (Nov. 1992), pp. 137–154. DOI: 10.1007/BF00129684
- C. J. Poelman and T. Kanade. "A Paraperspective Factorization Method for Shape and Motion Recovery". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 19.3 (Mar. 1997), pp. 206–218. DOI: 10.1109/34.584098
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Medical Image Processing for Interventional Applications

Factorization for Perspective Projections

Online Course – Unit 40

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Perspective Factorization

Historical remarks:

- Projective factorization method introduced by Sturm and Triggs (1996)
- Algorithm very similar to orthographic factorization

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Preliminaries:

- Perspective projection model
- Projection matrix for the i-th frame is denoted by $\boldsymbol{P}_i \in \mathbb{R}^{3\times 4}$.
- Number of frames $N_\mathrm{F} \geq 3$
- j-th world point $\tilde{\boldsymbol{p}}_j^w \in \mathbb{R}^4$ is represented in homogeneous coordinates and is visible in **all** frames.
- World points are **not** all coplanar.
- $\tilde{\boldsymbol{q}}_{i,j}^{\boldsymbol{i}} = (x_{i,j}, y_{i,j}, 1)^\mathrm{T} \in \mathbb{R}^3$ is the homogeneous vector associated with the j-th image point in the i-th frame.
- $\lambda_{i,j}$ is the scaling factor (**projective depth**) of the j-th image point in the i-th frame.

Perspective Factorization

Factorization using homogeneous coordinates:

The world point is projected to the image point by perspective projection:

$$\lambda_{i,j}\tilde{\boldsymbol{q}}^{i}_{i,j} = \boldsymbol{P}_i\tilde{\boldsymbol{p}}^{w}_{j}.$$

In components:

$$\lambda_{i,j}\begin{pmatrix} x_{i,j} \\ y_{i,j} \\ 1 \end{pmatrix} = \begin{pmatrix} p_{i,1,1} & p_{i,1,2} & p_{i,1,3} & p_{i,1,4} \\ p_{i,2,1} & p_{i,2,2} & p_{i,2,3} & p_{i,2,4} \\ p_{i,3,1} & p_{i,3,2} & p_{i,3,3} & p_{i,3,4} \end{pmatrix}\begin{pmatrix} x_j \\ y_j \\ z_j \\ 1 \end{pmatrix}$$

Perspective Factorization

Considering all points simultaneously, matrix notation gives us the following factorization of the measurement matrix $\boldsymbol{M}$:

$$\begin{pmatrix} \lambda_{1,1}\tilde{\boldsymbol{q}}^i_{1,1} & \lambda_{1,2}\tilde{\boldsymbol{q}}^i_{1,2} & \cdots & \lambda_{1,N_\text{p}}\tilde{\boldsymbol{q}}^i_{1,N_\text{p}} \\ \lambda_{2,1}\tilde{\boldsymbol{q}}^i_{2,1} & \lambda_{2,2}\tilde{\boldsymbol{q}}^i_{2,2} & \cdots & \lambda_{2,N_\text{p}}\tilde{\boldsymbol{q}}^i_{2,N_\text{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N_\text{F},1}\tilde{\boldsymbol{q}}^i_{N_\text{F},1} & \lambda_{N_\text{F},2}\tilde{\boldsymbol{q}}^i_{N_\text{F},2} & \cdots & \lambda_{N_\text{F},N_\text{p}}\tilde{\boldsymbol{q}}^i_{N_\text{F},N_\text{p}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \vdots \\ \boldsymbol{P}_{N_\text{F}} \end{pmatrix} \left(\tilde{\boldsymbol{p}}^w_1, \tilde{\boldsymbol{p}}^w_2, \ldots, \tilde{\boldsymbol{p}}^w_{N_\text{p}}\right).$$

Perspective Factorization

Assumption: Projective depth $\lambda_{i,j}$ is known.

- In the perspective case the measurement matrix has rank 4, since it is a product of two matrices where the first factor has 4 columns and the second one 4 rows.
- Rank criterion can be enforced by SVD: all but the first four singular values are set to zero.

Perspective Factorization

- Use the SVD of the rank enforced measurement matrix

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}.$$

- Up to a 4×4 projective transform we get the following motion and scene parameters:
  - projection matrices result from:

$$\begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \vdots \\ \boldsymbol{P}_{N_{\mathrm{F}}} \end{pmatrix} = \boldsymbol{U}\boldsymbol{\Sigma},$$

  - 3-D scene points are:

$$\left(\tilde{\boldsymbol{p}}_1^w, \tilde{\boldsymbol{p}}_2^w, \ldots, \tilde{\boldsymbol{p}}_{N_{\mathrm{p}}}^w\right) = \boldsymbol{V}^{\mathrm{T}}.$$

Perspective Factorization

Estimating projective depths:

Apply the following iterative scheme:

1. Initialize $\lambda_{i,j} := 1$.
2. Normalize projective depths by scaling $\boldsymbol{M}$ such that column and row vectors have norm 1.
3. Enforce the rank criterion for the measurement matrix.
4. Use SVD to estimate projection matrices and 3-D structure.
5. Project estimated points anew into each frame and update all $\lambda_{i,j}$.
6. If the projective depths change significantly, go back to step 2.

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- In many ways, perspective factorization is similar to orthogonal factorization, but here we additionally need to estimate the projective depths.
- The projective depths are estimated iteratively along with the factorization algorithm.

Further Readings

- Carlo Tomasi and Takeo Kanade. "Shape and Motion from Image Streams Under Orthography: A Factorization Method". In: *International Journal of Computer Vision* 9.2 (Nov. 1992), pp. 137–154. DOI: 10.1007/BF00129684
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- Mei Han and Takeo Kanade. "A Perspective Factorization Method for Euclidean Reconstruction with Uncalibrated Cameras". In: *The Journal of Visualization and Computer Animation* 13.4 (2002), pp. 211–223. DOI: 10.1002/vis.290
- Peter Sturm and Bill Triggs. "A Factorization Based Algorithm for Multi-Image Projective Structure and Motion". In: *Computer Vision — ECCV '96: 4th European Conference on Computer Vision Cambridge, UK, April 15–18, 1996 Proceedings Volume II*. ed. by Bernard Buxton and Roberto Cipolla. Vol. 1065. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 709–720. DOI: 10.1007/3-540-61123-1_183