Projection Models and Homogeneous Coordinates

Homogeneous Coordinates

Refresher Course

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Homogeneous Coordinates
- Definition
- Lines in $\mathbb{R}^2$ and Points in $\mathbb{P}^2$
- Projections in Homogeneous Coordinates

Summary
- Take Home Messages
- Further Readings

Homogeneous Coordinates

Using a simple trick, we can extend 2-D or 3-D vectors by an additional component that allows us to write

- affine mappings as linear mappings, and
- the perspective projection as a linear mapping.

Let us first consider the 2-D case.

Homogeneous Coordinates

We extend $\mathbb{R}^2$ by a third coordinate to create the projective space $\mathbb{P}^2$:

Definition

A two-dimensional point in Cartesian coordinates $\boldsymbol{p} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$ is represented by $\widetilde{\boldsymbol{p}} = (wx, wy, w)^{\mathrm{T}} \in \mathbb{P}^2$ in ***homogeneous coordinates***, where $w \in \mathbb{R} \setminus \{0\}$ is an arbitrary real value.

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Note: A vector $(\widetilde{x}, \widetilde{y}, \widetilde{z})^{\mathrm{T}}$ in homogeneous coordinates can be transformed into a 2-D vector by dividing the first two components $\widetilde{x}$ and $\widetilde{y}$ with the third component $\widetilde{z} \neq 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \widetilde{x}/\widetilde{z} \\ \widetilde{y}/\widetilde{z} \end{pmatrix}.$$

Homogeneous Coordinates

- A 2-D point $(x, y)^{\mathrm{T}}$ in Cartesian coordinates corresponds to a line in 3-D:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left\{ w \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \,\middle|\, w \in \mathbb{R} \right\}.$$

- There exists an infinite number of homogeneous points that correspond to one and the same 2-D point.
- The representation in homogeneous coordinates has a singularity for $w \to 0$.

Homogeneous Coordinates

We now define an equivalence relation:

Definition

We call two homogeneous points $\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}$ equivalent, if $\widetilde{\boldsymbol{p}} = \lambda \widetilde{\boldsymbol{q}}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. This equivalence is denoted by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$.

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Example

The homogeneous points $\widetilde{\boldsymbol{p}} = (2,3,1)^{\mathrm{T}}$ and $\widetilde{\boldsymbol{q}} = (4,6,2)^{\mathrm{T}}$ are equivalent by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$ as both project to the same point which is $(2,3)^{\mathrm{T}} \in \mathbb{R}^2$.
They are not equal considered as vectors in $\mathbb{R}^3$, i. e., $\widetilde{\boldsymbol{p}} \neq \widetilde{\boldsymbol{q}}$.
Note: It is $\widetilde{\boldsymbol{p}} \ncong (4,6,1)^{\mathrm{T}}$.

Homogeneous Coordinates

Let us now consider lines in 2-D.

- A line in $\mathbb{R}^2$ is fully determined by the equation

$$ax + by + c = 0, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$

  This equation can be multiplied by an arbitrary factor $w \in \mathbb{R} \setminus \{0\}$, and it still represents the same line.

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- Each vector $(a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$ represents a line, and

$$ax + by + c = (w \cdot a)x + (w \cdot b)y + (w \cdot c) = 0$$

  holds for each non-zero w.

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- In terms of homogeneous coordinates we can state that each 2-D line can be represented by a corresponding vector $\boldsymbol{l} = (a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$.

Homogeneous Coordinates

- A point $\widetilde{\boldsymbol{p}}$ (represented in homogeneous coordinates) lies on the line $\boldsymbol{l}$ if

$$\boldsymbol{l}^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0.$$

- Intersection of lines: Two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ intersect in point $\widetilde{\boldsymbol{p}}$ if

$$\boldsymbol{l}_1^{\mathrm{T}}\widetilde{\boldsymbol{p}} = \boldsymbol{l}_2^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0,$$

  so we find

$$\widetilde{\boldsymbol{p}} = \boldsymbol{l}_1 \times \boldsymbol{l}_2.$$

Homogeneous Coordinates

Definition

The set of ***ideal points*** lies on the line at infinity $\boldsymbol{l}_\infty = (0,0,1)^{\mathrm{T}}$:

$$(0,0,1)^{\mathrm{T}}(x,y,0) = 0.$$

Note: The tupel $(0,0,0)^{\mathrm{T}}$ describes no valid coordinate in $\mathbb{P}^2$.

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Exercise: Do parallel lines intersect in $\mathbb{P}^2$? Where?

The concept of homogeneous coordinates can be transferred to higher dimensional spaces. We will not continue to look into the details of this theory. Interested students are referred to the literature on perspective geometry (see for instance Hartley's book).

Orthographic Projection

We will now formulate projections from 3-D to 2-D using homogeneous coordinates:

The *orthographic projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \;\mapsto\; \widetilde{\boldsymbol{p}}' = (x, y, 1)^{\mathrm{T}}.$$

This mapping from $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ can be simply written in matrix form as

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

Weak Perspective Projection

The *weak perspective projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (kx, ky, 1)^{\mathrm{T}},$$

where $k \in \mathbb{R}$ is a scaling factor.
This mapping from $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ can be simply written in matrix form as:

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/k \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

Perspective Projection

Using homogeneous coordinates, the *perspective projection* becomes a *linear* mapping:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (fx, fy, z)^{\mathrm{T}} \cong (fx/z, fy/z, 1)^{\mathrm{T}}.$$

We get the following linear mapping from $\mathbb{P}^3 \to \mathbb{P}^2$:

$$\widetilde{\boldsymbol{p}}' = \underbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\boldsymbol{P}} \widetilde{\boldsymbol{p}}.$$

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Take Home Messages

- Points on a line through the origin in real vector space correspond to a single point in the projective plane.
- The nonlinear projective mapping in $\mathbb{R}^3$ can be written as a linear mapping using homogeneous coordinates.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993