Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Linear Equations

Online Course – Unit 55

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Topics

Linear Equations

- Example
- Kaczmarz Method

Summary

- Take Home Messages
- Further Readings

Example: Backprojection

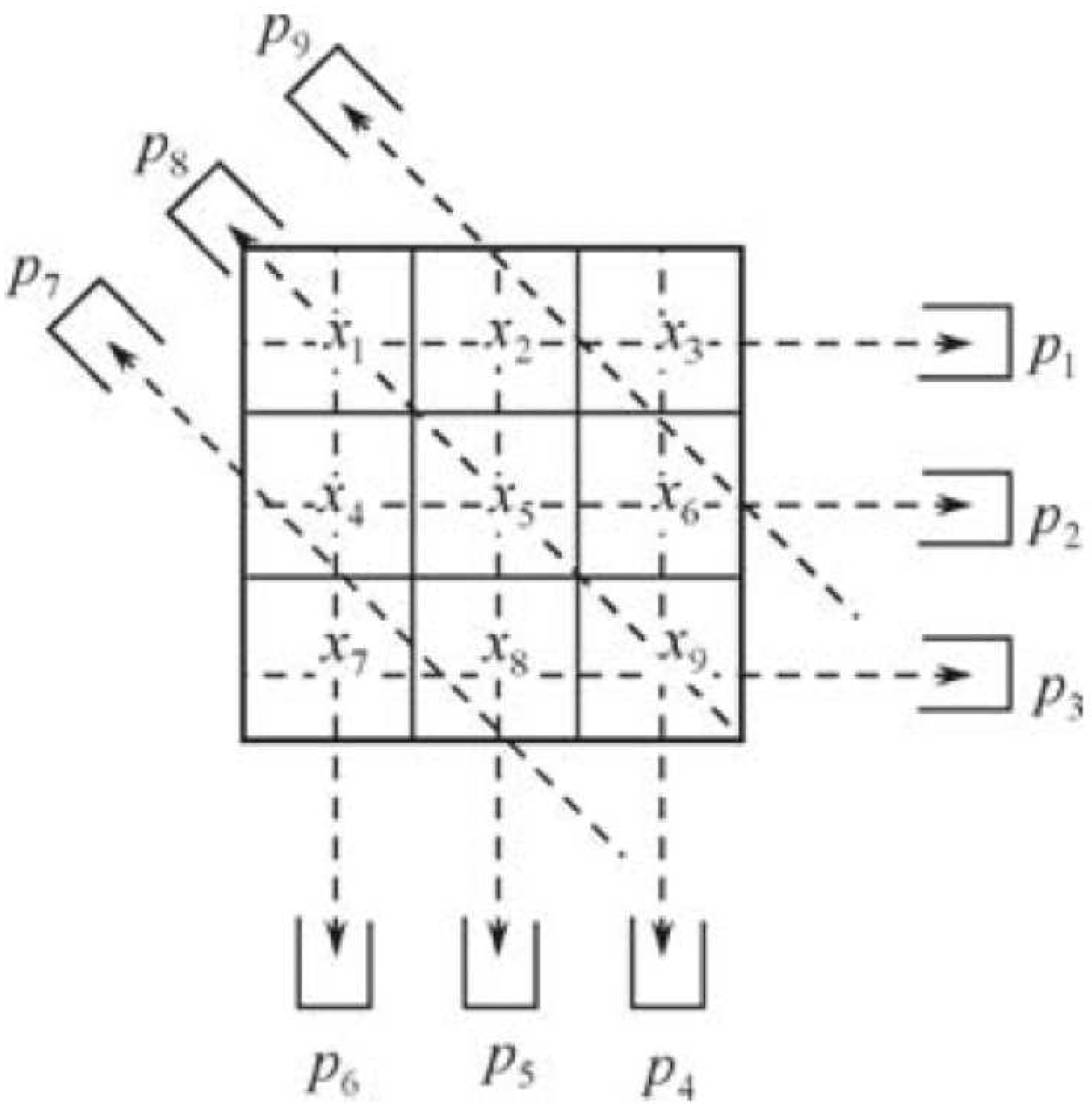

Figure 1: A 3×3 volume is backprojected from 3 projections with 3 detector pixels (Zeng, 2009).

$$
\begin{aligned}
x_1 + x_2 + x_3 &= p_1 \\
x_4 + x_5 + x_6 &= p_2 \\
x_7 + x_8 + x_9 &= p_3 \\
x_3 + x_6 + x_9 &= p_4 \\
x_2 + x_5 + x_8 &= p_5 \\
x_1 + x_4 + x_7 &= p_6 \\
2(\sqrt{2}-1)x_4 + (2-\sqrt{2})x_7 + 2(\sqrt{2}-1)x_8 &= p_7 \\
\sqrt{2}x_1 + \sqrt{2}x_5 + \sqrt{2}x_9 &= p_8 \\
2(\sqrt{2}-1)x_2 + (2-\sqrt{2})x_3 + 2(\sqrt{2}-1)x_6 &= p_9
\end{aligned}
$$

Example: Linear Equation

Rewrite the single equations to

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{P}$$

with

$$\boldsymbol{X} = (x_1, x_2, \ldots, x_9)^{\mathrm{T}} \in \mathbb{R}^n, \quad \boldsymbol{P} = (p_1, p_2, \ldots, p_9)^{\mathrm{T}} \in \mathbb{R}^m.$$

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the system matrix with elements a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$.
- The a_{ij} describe the contribution of each voxel to each ray.

Example: Solution?

The linear system of equations

$$\boldsymbol{AX} = \boldsymbol{P}$$

can mathematically be solved by using

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{P},$$

or

$$\boldsymbol{X} = \left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{P},$$

or

$$\boldsymbol{X} = \boldsymbol{A}^{\mathsf{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathsf{T}}\right)^{-1}\boldsymbol{P},$$

but practically these methods are infeasible (Gauss-Seidel, SVD, etc.).

$\rightarrow$ A solution that does not require the inversion of $\boldsymbol{A}$, or a matrix product is desirable.

Kaczmarz Method

- Each pixel can be interpreted as a linear equation.
- This equation forms a line (2-D) or a hyperplane (higher dimensions) in the solution space.
- The point of intersection forms the correct solution.
- Projection onto the respective hyperplane forms a solution that fulfills the respective equation.
- Repetition yields an improved solution.

Example with Two Voxels

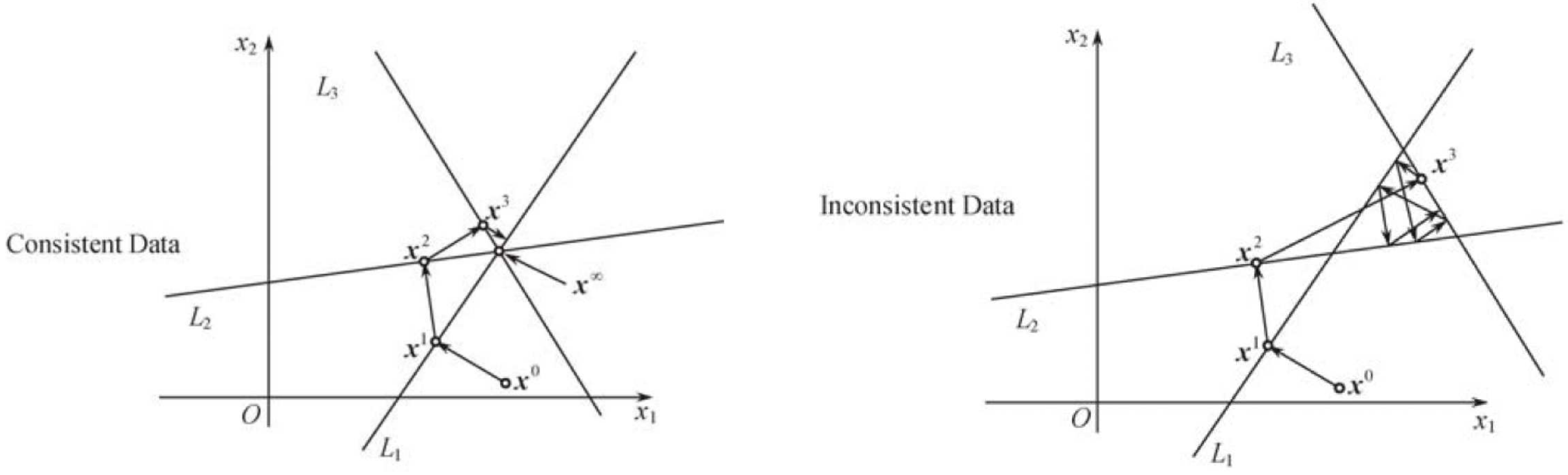

Figure 2: In practice, it is unlikely to get consistent data (left), it usually is inconsistent (right) (Zeng, 2009).

Projection onto a Hyperplane

- Consider a point $\boldsymbol{x} \in \mathbb{R}^n$ and a hyperplane
$$\{\boldsymbol{c} \in \mathbb{R}^n \,|\, \boldsymbol{n}^{\mathsf{T}}\boldsymbol{c} = d\}$$
for $d \in \mathbb{R}$ and a given $\boldsymbol{n} \in \mathbb{R}^n$.
- The projection $\boldsymbol{x}'$ of $\boldsymbol{x}$ must be in direction of the normal vector $\boldsymbol{n}$:
$$\boldsymbol{x}' = \boldsymbol{x} + \lambda \boldsymbol{n}.$$
- $\boldsymbol{x}'$ is on the hyperplane:
$$\begin{aligned} \boldsymbol{n}^{\mathsf{T}}\boldsymbol{x}' &= d, \\ \boldsymbol{n}^{\mathsf{T}}(\boldsymbol{x} + \lambda \boldsymbol{n}) &= d, \\ \boldsymbol{n}^{\mathsf{T}}\boldsymbol{x} + \lambda \boldsymbol{n}^{\mathsf{T}}\boldsymbol{n} &= d, \\ \lambda &= \frac{d - \boldsymbol{n}^{\mathsf{T}}\boldsymbol{x}}{\boldsymbol{n}^{\mathsf{T}}\boldsymbol{n}}, \\ \Rightarrow \qquad \boldsymbol{x}' &= \boldsymbol{x} + \frac{d - \boldsymbol{n}^{\mathsf{T}}\boldsymbol{x}}{\boldsymbol{n}^{\mathsf{T}}\boldsymbol{n}}\boldsymbol{n}. \end{aligned}$$

Topics

Linear Equations
Example
Kaczmarz Method

Summary
Take Home Messages
Further Readings

Take Home Messages

- The projection process can be formulated as a system of linear equations.
- Using Kaczmarz method, we iteratively project approximate solutions to different hyperplanes.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. "Adaptive Iterative Reconstruction". In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953

Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Algebraic Reconstruction Technique

Online Course – Unit 56
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Topics

Algebraic Reconstruction Technique

Summary

Take Home Messages

Further Readings

Algebraic Reconstruction Technique (ART)

Idea: Find an iterative solution of

$$\boldsymbol{AX} = \boldsymbol{P},$$

using the Kaczmarz method:

1. For each pixel p_i and each row $\boldsymbol{A}_i$ of $\boldsymbol{A}$ perform the following update:

$$\boldsymbol{X}^{k+1} = \boldsymbol{X}^k + \frac{p_i - \boldsymbol{A_i X}^k}{\boldsymbol{A_i A_i}^\mathsf{T}} \boldsymbol{A_i}^\mathsf{T}.$$

2. Repeat until convergence.

Algebraic Reconstruction Technique: Remarks

- Tanabe has shown in 1971 that the iterative scheme converges to the solution if there exists a unique solution.
- The angle between hyperplanes influences the rate of convergence to the solution.
- If hyperplanes are orthogonal to each other, it is obvious that the method converges rapidly (consider the 2-D case for plausibility).
- Orthogonalization methods applied in advance to iterations will improve convergence.
  - **Cons:** This is computationally prohibitive, and orthogonalization amplifies noise in measurements.
  - An alternative to orthogonalization is careful selection of the sequence of projections.
- Overdetermined systems and noise often have no unique solution and suffer from oscillations.

Algebraic Reconstruction Technique: Case Studies

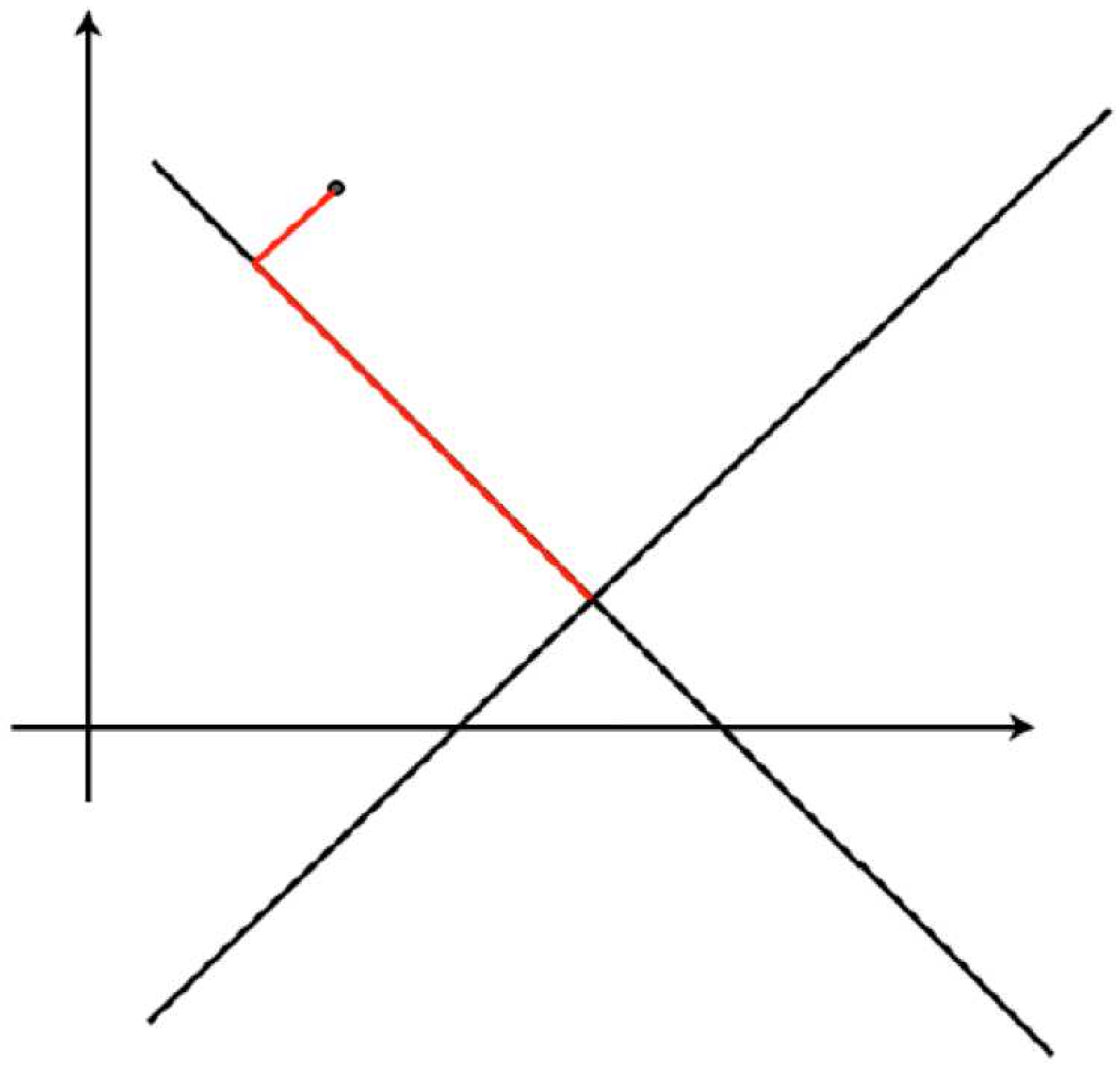

Figure 1: Two orthogonal projections

Algebraic Reconstruction Technique: Case Studies

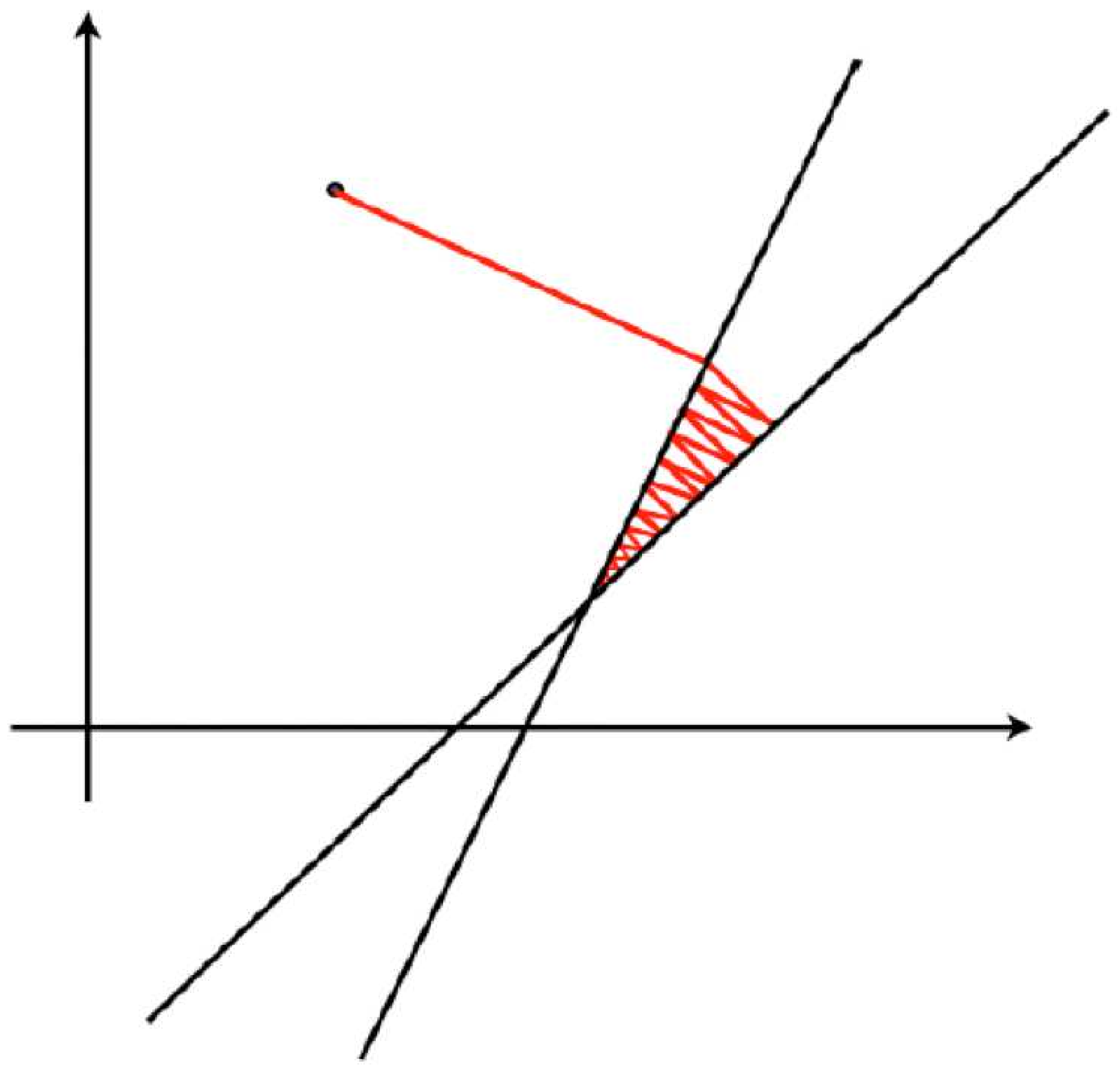

Figure 2: Oscillations

Algebraic Reconstruction Technique: Case Studies

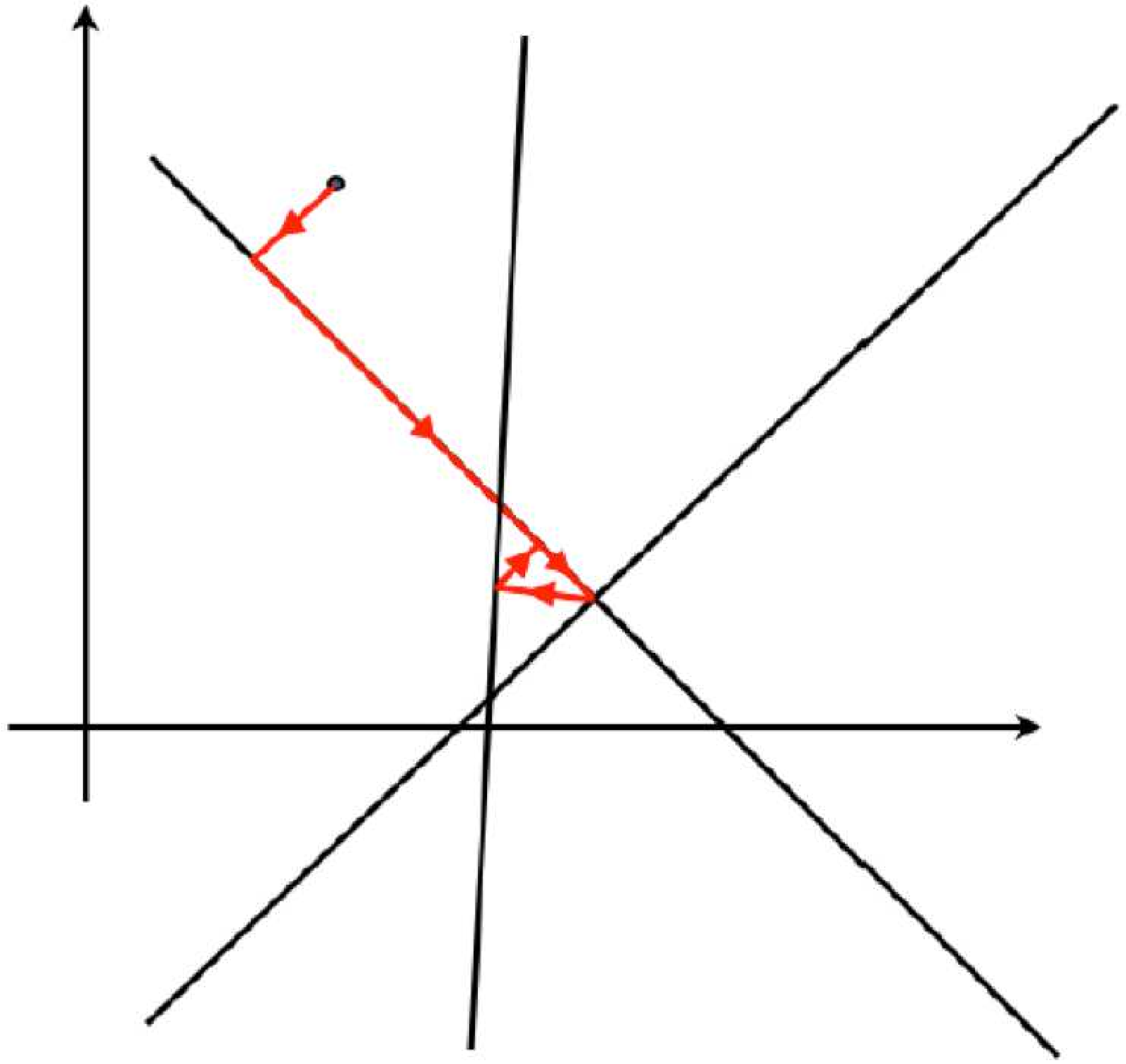

Figure 3: Four iterations

Algebraic Reconstruction Technique: Case Studies

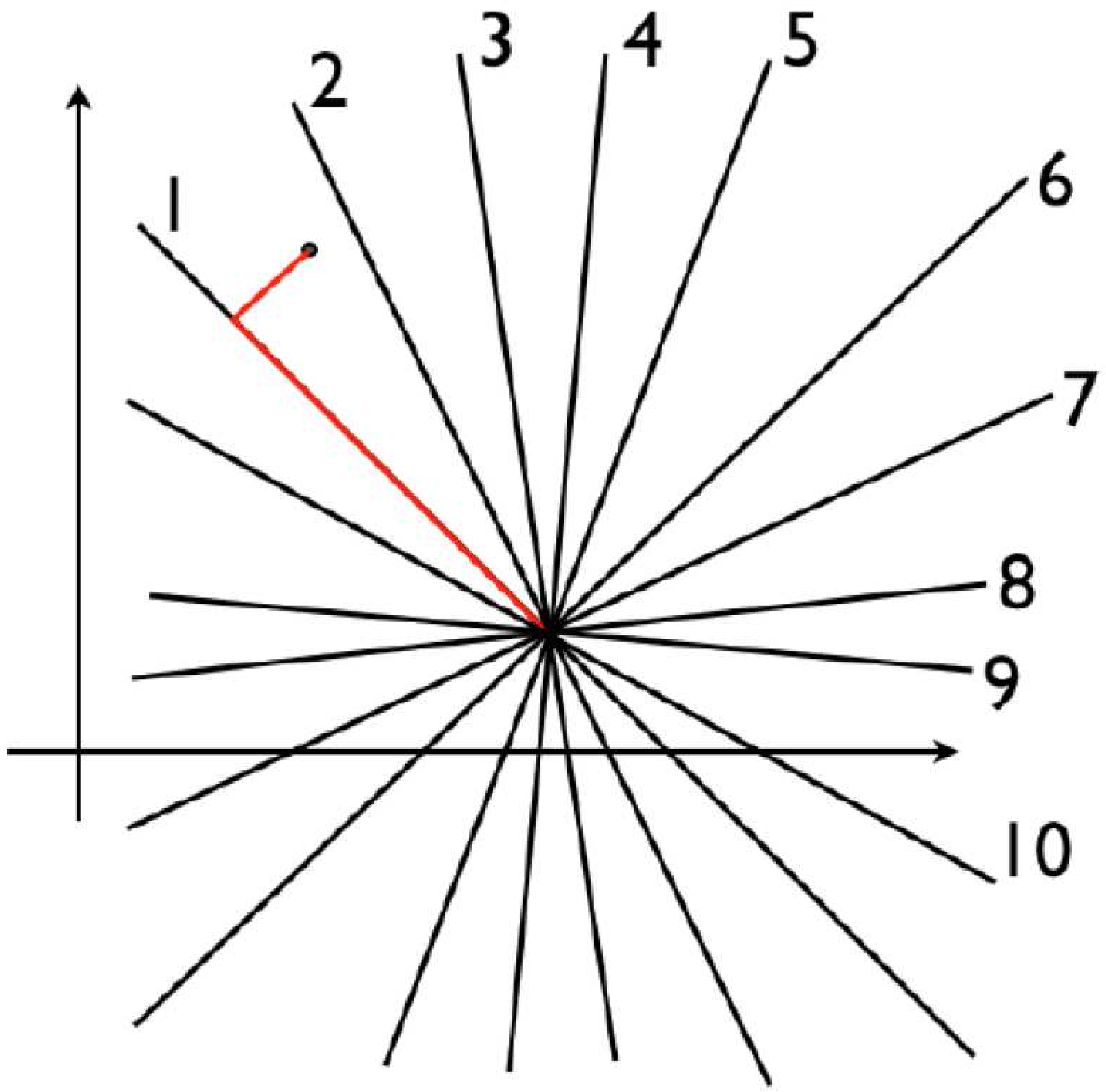

Figure 4: Consideration on convergence

Algebraic Reconstruction Technique: Case Studies

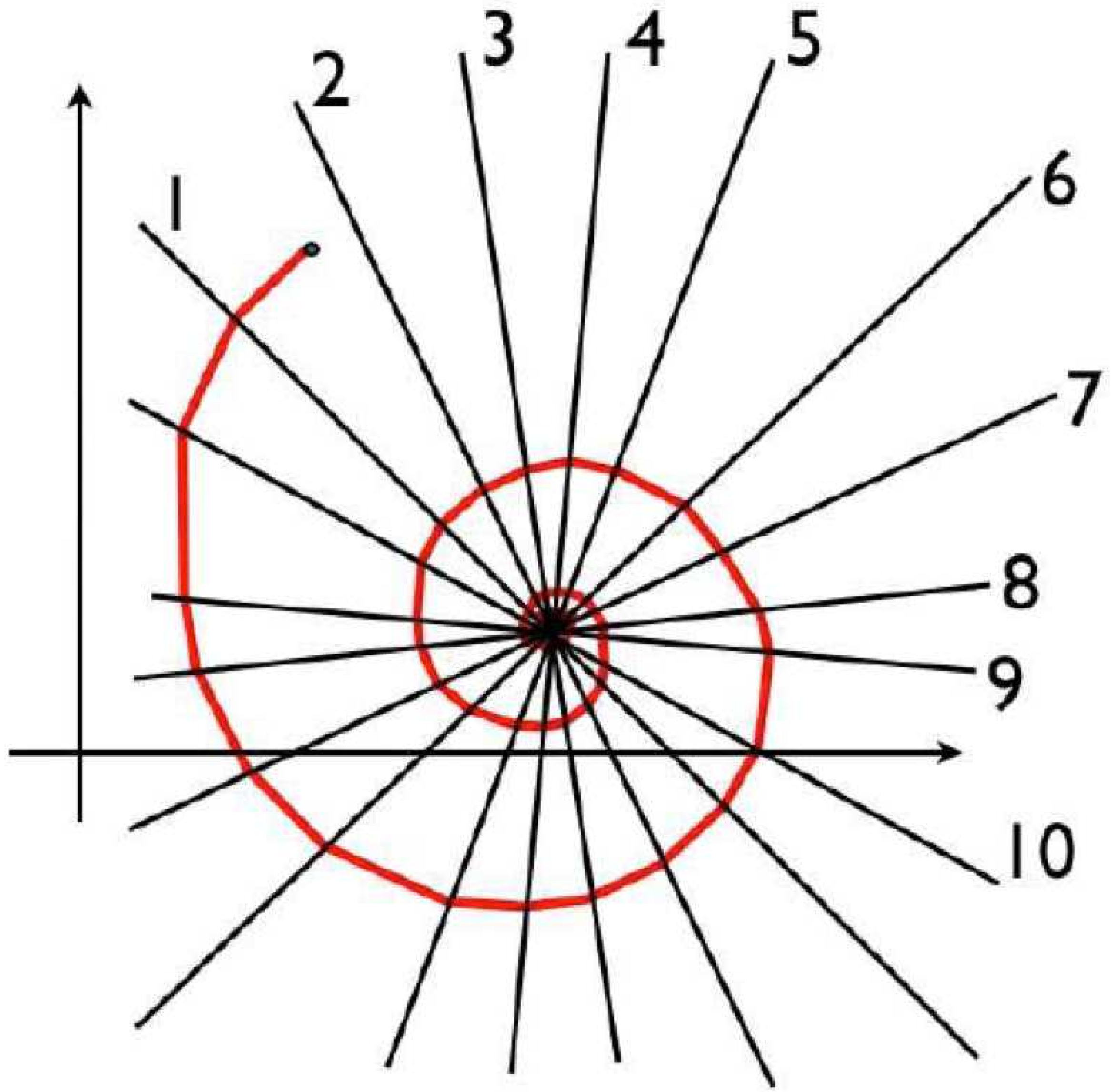

Figure 5: Consideration on convergence

Topics

Algebraic Reconstruction Technique

Summary

Take Home Messages

Further Readings

Take Home Messages

- ART exploits the Kaczmarz method to iteratively compute a solution for a reconstruction problem.
- Iterative reconstruction usually demands significant higher computation times.
- Note that for this approach we do not have to make requirements on data completeness or acquisition geometry.
- The naive implementation has its problems, so we discuss some extensions in the next unit.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

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Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – ART Extensions

Online Course – Unit 57

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Topics

Extensions of ART

SART

SIRT

Ordered Subsets

Towards More Realism

Summary

Take Home Messages

Further Readings

Extensions of ART

- Slow convergence is the main drawback of ART.
  $\rightarrow$ Extensions aim at improving convergence speed.
- Compute update using more than one projected pixel:
  - Simultaneous ART (SART): multiple updates at the same time and combination of the result,
  - Simultaneous Iterative Reconstruction Technique (SIRT): compute update once per iteration.
- Use intelligent methods to select the order of the update equations (ordered subsets).
- Use more realistic models within the system matrix.

SART

1. Estimate an initial solution of the system of linear equations.
2. Compute the orthogonal projections of the current estimate to all hyperplanes.
3. Compute the centroid of all projected points.
4. Use this centroid for the next iteration.

SART

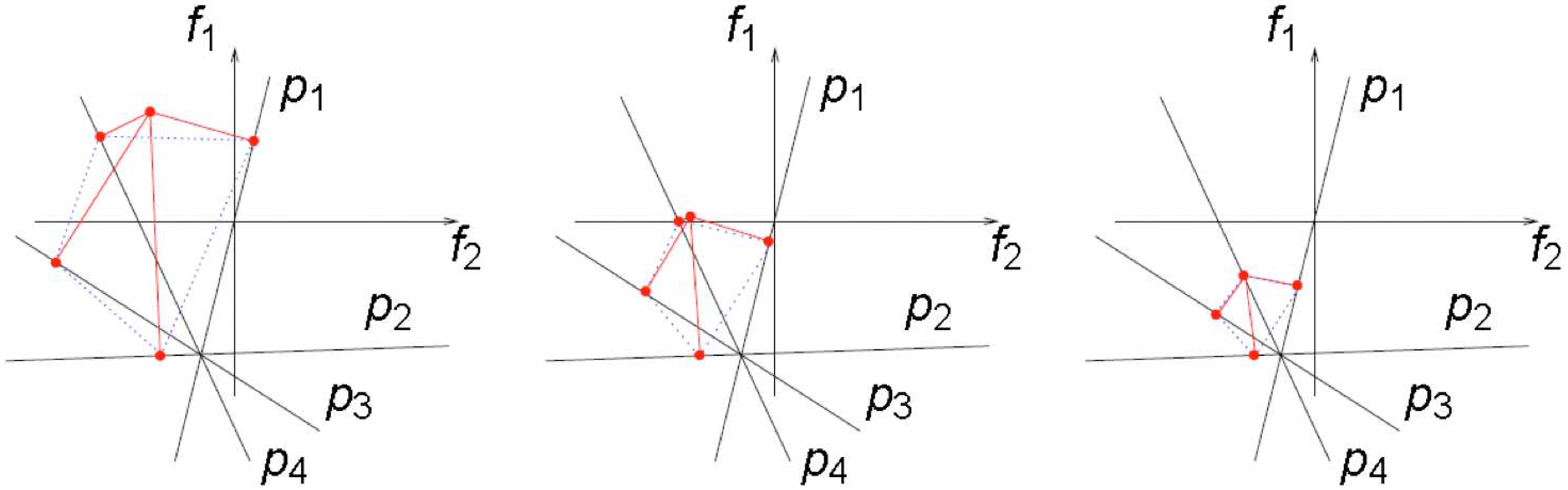

Figure 1: Scheme of the SART algorithm

SART

All pixels p_i are considered in the update simultaneously with the following update rule:

$$\boldsymbol{X}^{k+1} = \boldsymbol{X}^k + \lambda_k \sum_i u_{k,i} \frac{p_i - \boldsymbol{A_i}\boldsymbol{X}^k}{\boldsymbol{A_i}\boldsymbol{A_i}^\top} \boldsymbol{A_i}^\top,$$

where

$$\sum_i u_{k,i} = 1, \quad k = 0, 1, 2, ...,$$

and the parameter λ_k is used to control the step size in each iteration.

SIRT

- All voxels of the volume are updated simultaneously using the **same** projection $\boldsymbol{P_j}$ in the following update rule:

$$\boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} + \frac{\boldsymbol{P_j} - \boldsymbol{A_j}\boldsymbol{X}^{k}}{||\boldsymbol{A_j}||}\boldsymbol{A_j}^{\mathsf{T}}$$

- $\boldsymbol{A_j}$ is the part of the system matrix that projects all the voxels of $\boldsymbol{X}$ onto the pixels of $\boldsymbol{P_j}$.
- $\boldsymbol{A_j}\boldsymbol{X}$ is a forward projection of $\boldsymbol{X}$.
- $\boldsymbol{P_j}\boldsymbol{A_j}^{\mathsf{T}}$ is a backprojection of $\boldsymbol{P_j}$.

Ordered Subsets ...

- ... is a clever technique to speed up convergence.
- ... employs knowledge on the acquisition sequence.
- ... selects subsets of equations (projections) that are most likely orthogonal to each other.

Iterations are performed in a nested way:

```
For all subsets
{
    Update using all items in the subset
}
```

Ordered Subsets

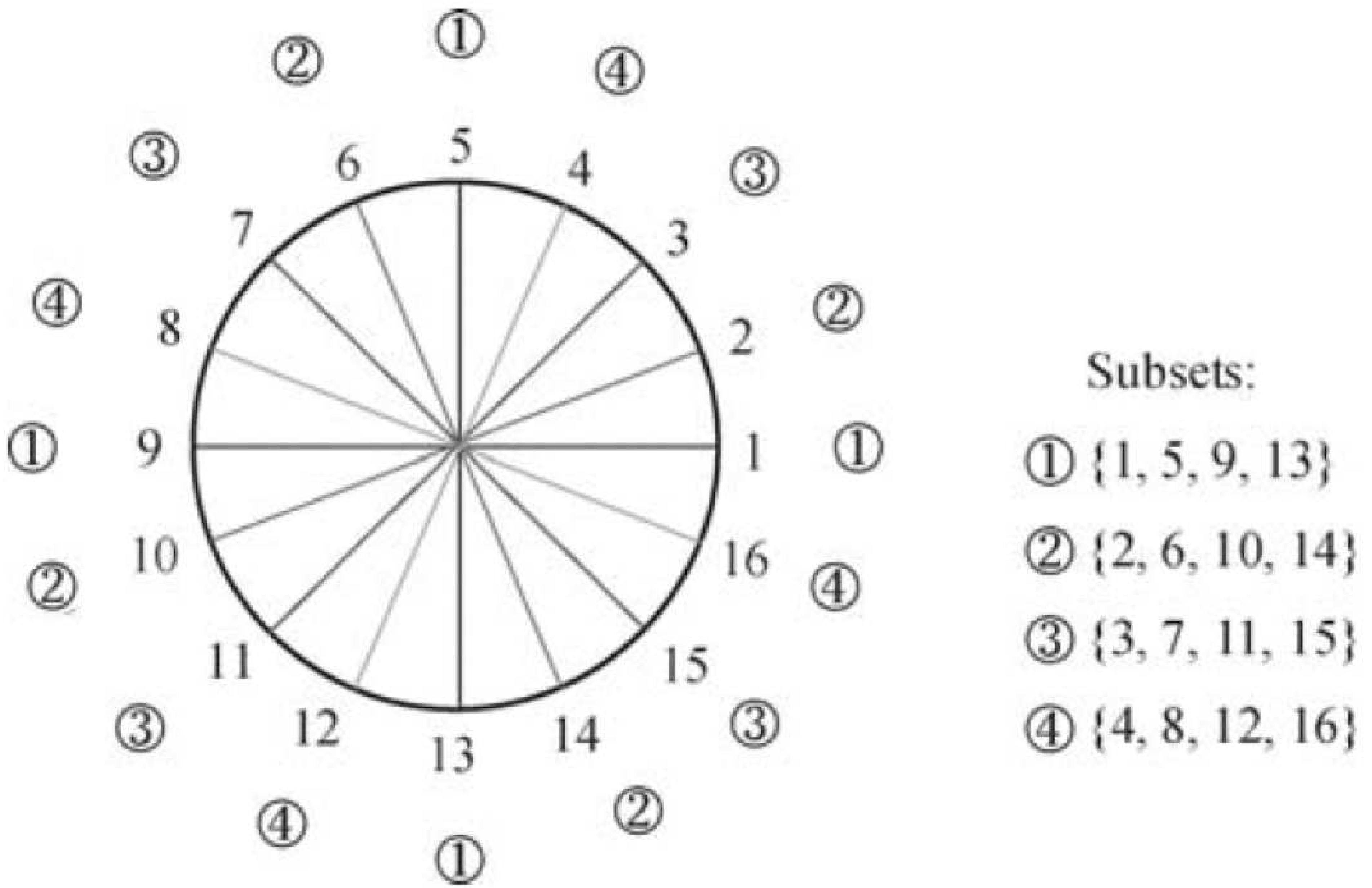

Figure 2: Illustration of the ordered subset scheme (Zeng, 2009)

Towards More Realism: Ray Intersection Model

The path length of the ray through a voxel is not the only way of modeling the imaging grid.

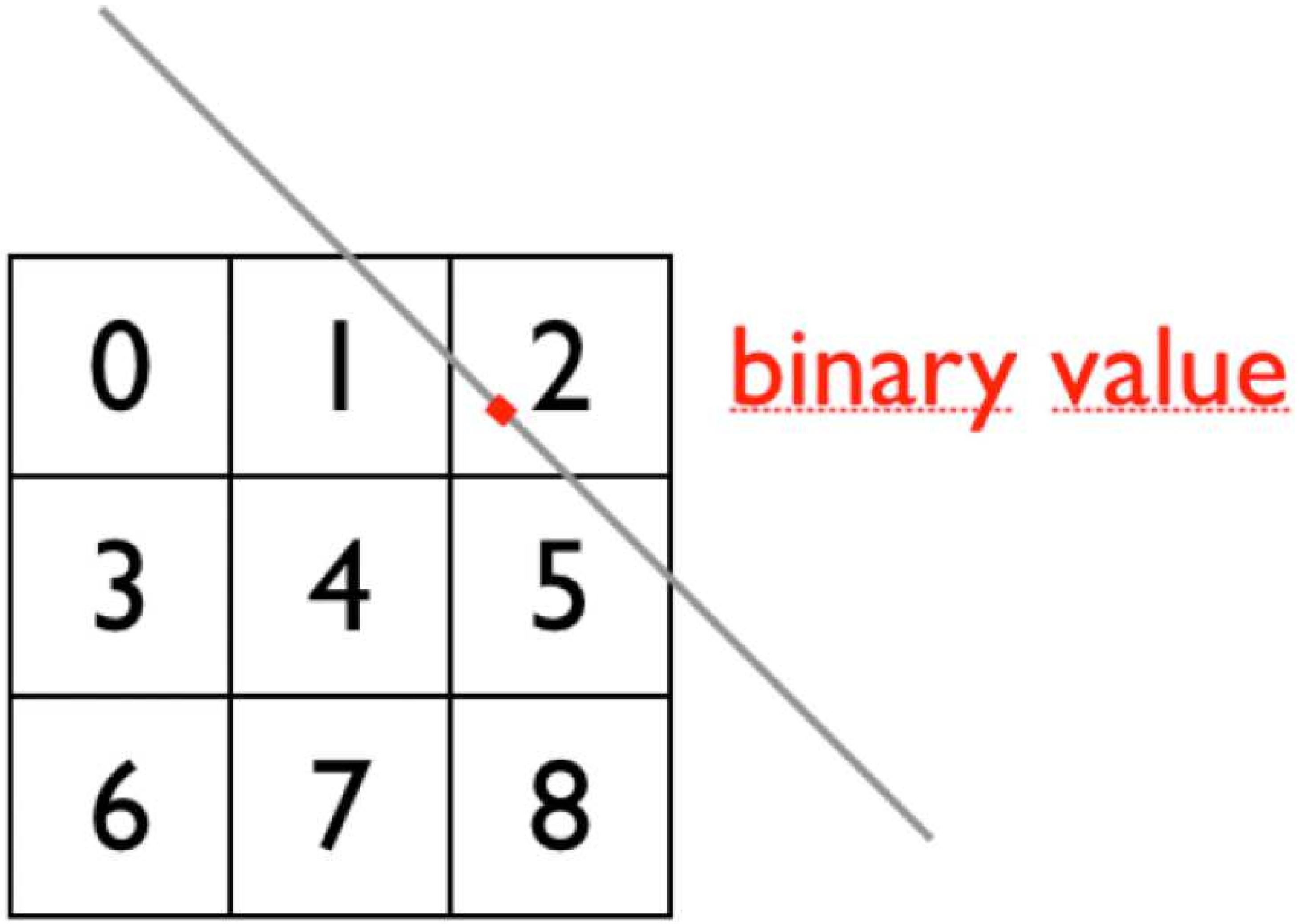

Figure 3: The projection value is evaluated in a single point.

Towards More Realism: Ray Intersection Model

The path length of the ray through a voxel is not the only way of modeling the imaging grid.

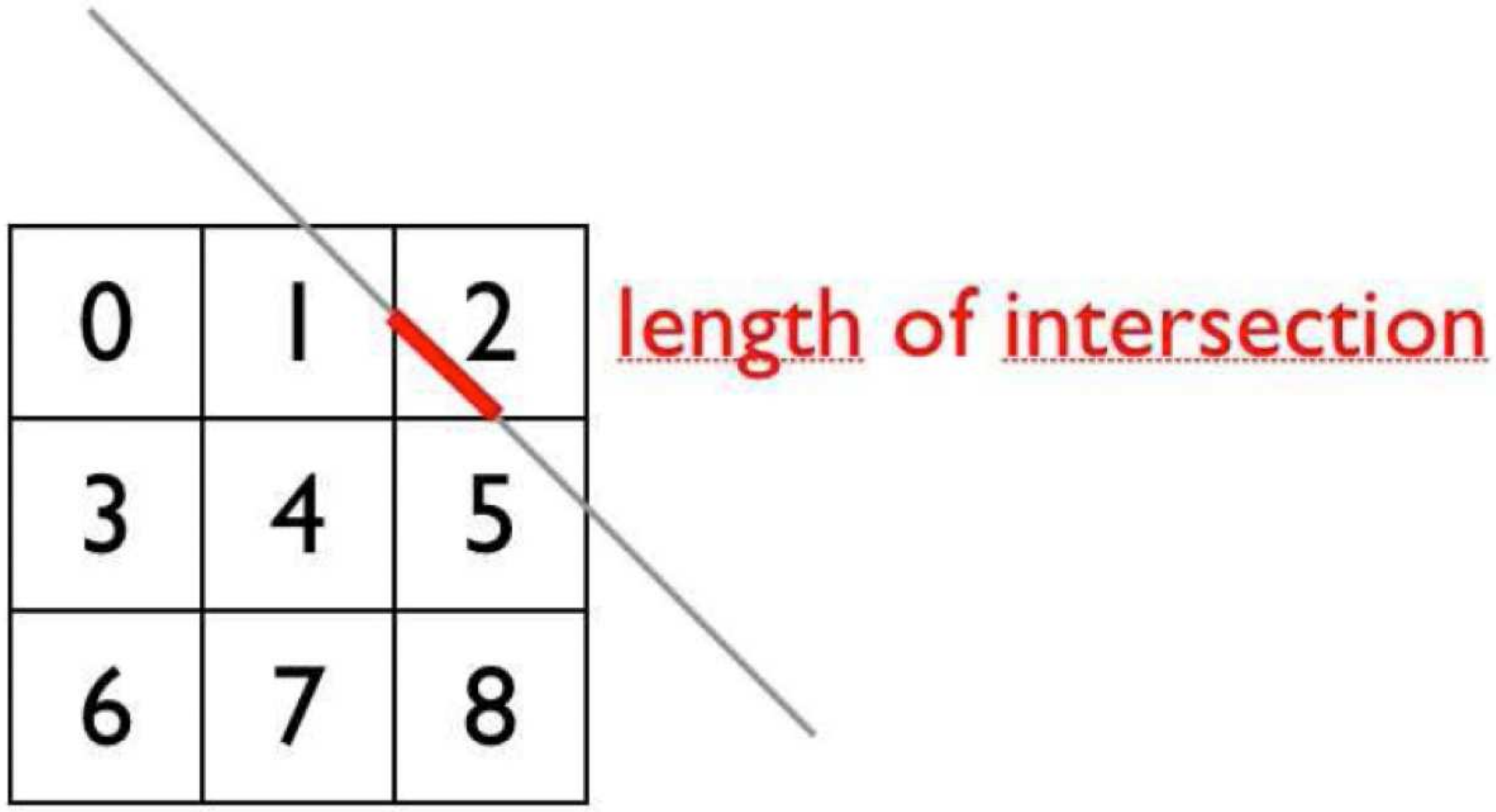

Figure 4: The length of a ray passing through a voxel is taken into account.

Towards More Realism: Ray Intersection Model

The path length of the ray through a voxel is not the only way of modeling the imaging grid.

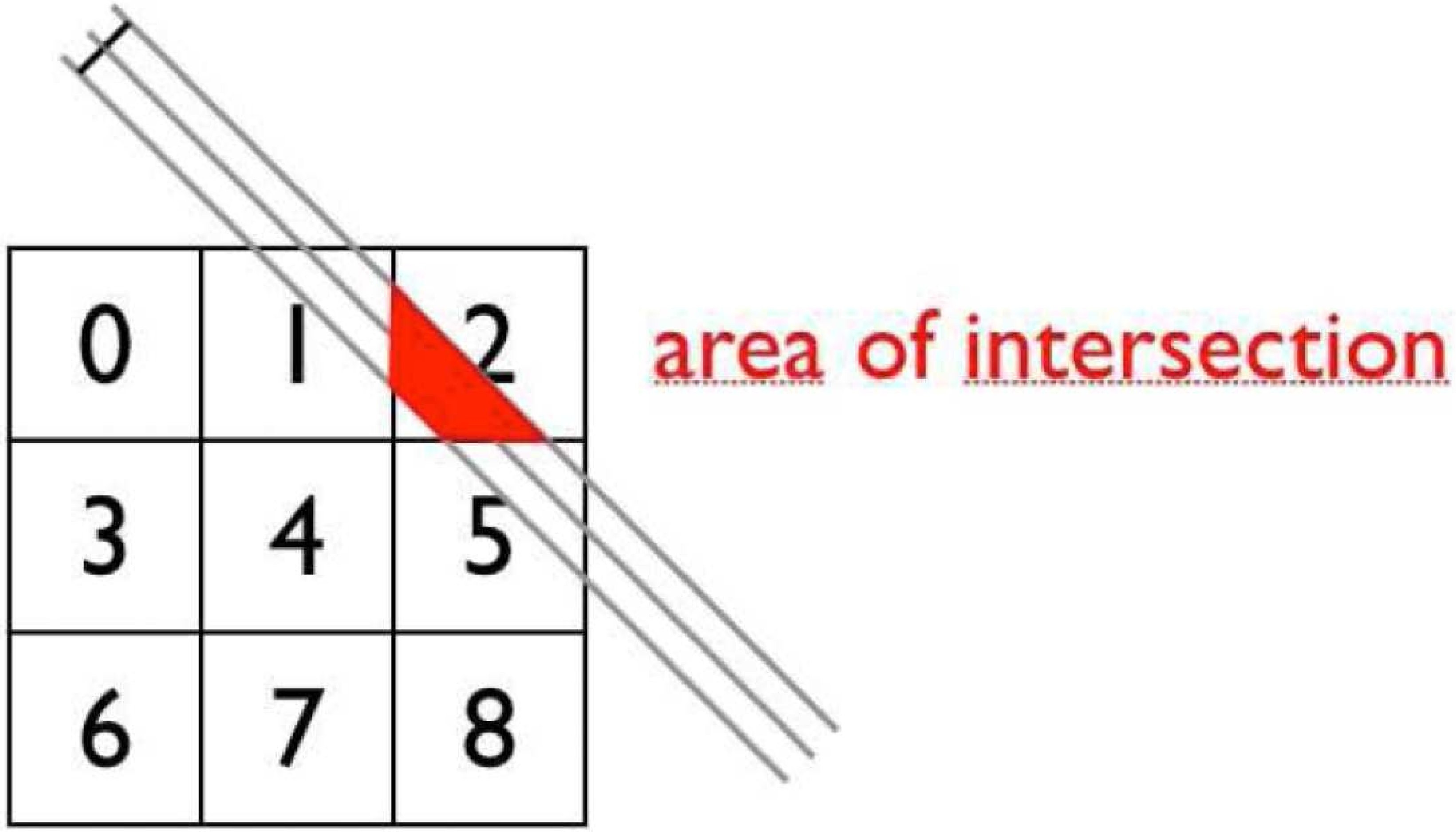

Figure 5: The ray model includes a second dimension such that the projection value scales with the area of intersection.

Towards more Realism: Alternative Voxels

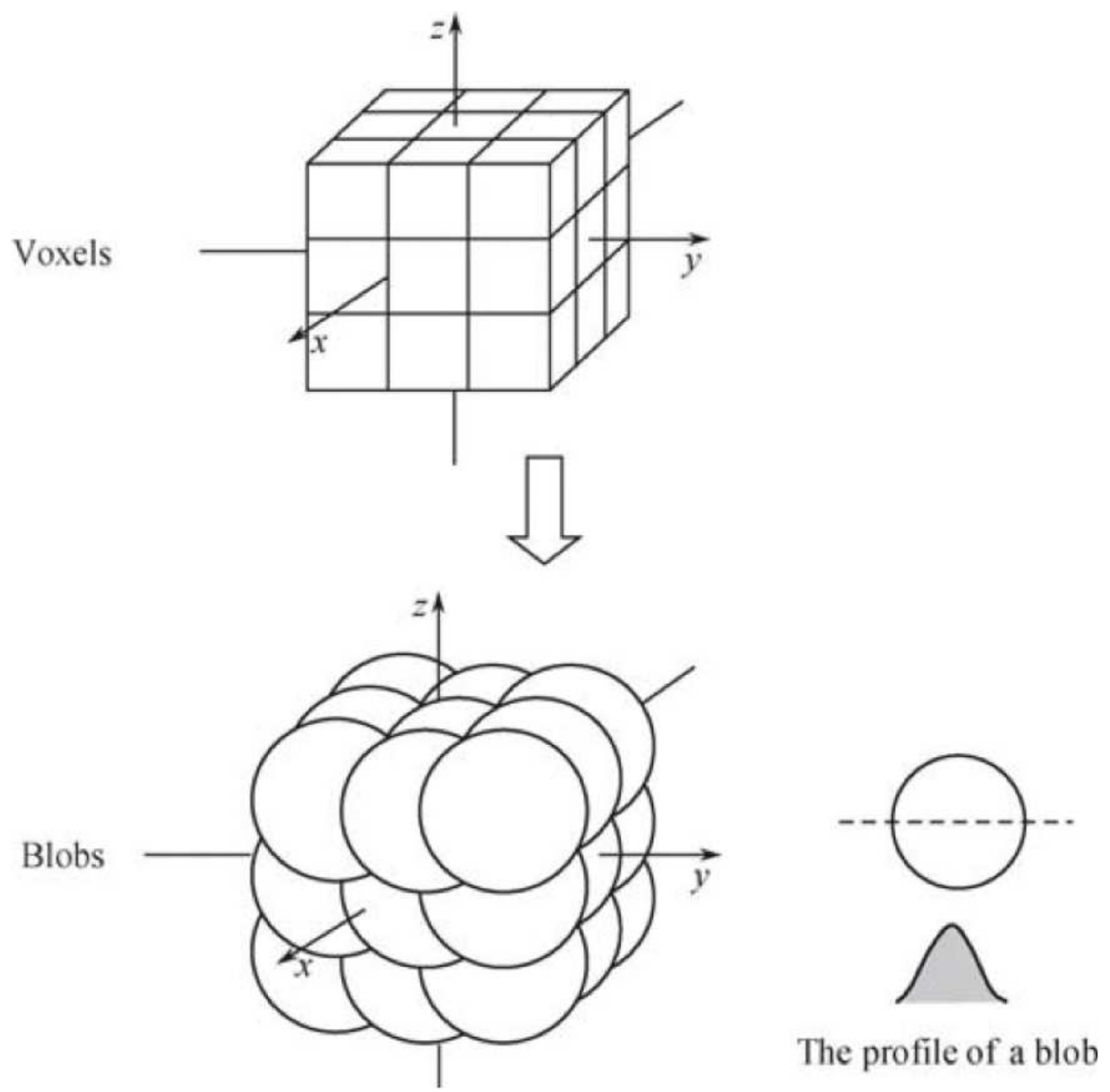

Figure 6: Use blobs as elements of the reconstruction volume (Zeng, 2009).

Towards more Realism: Alternative Voxels

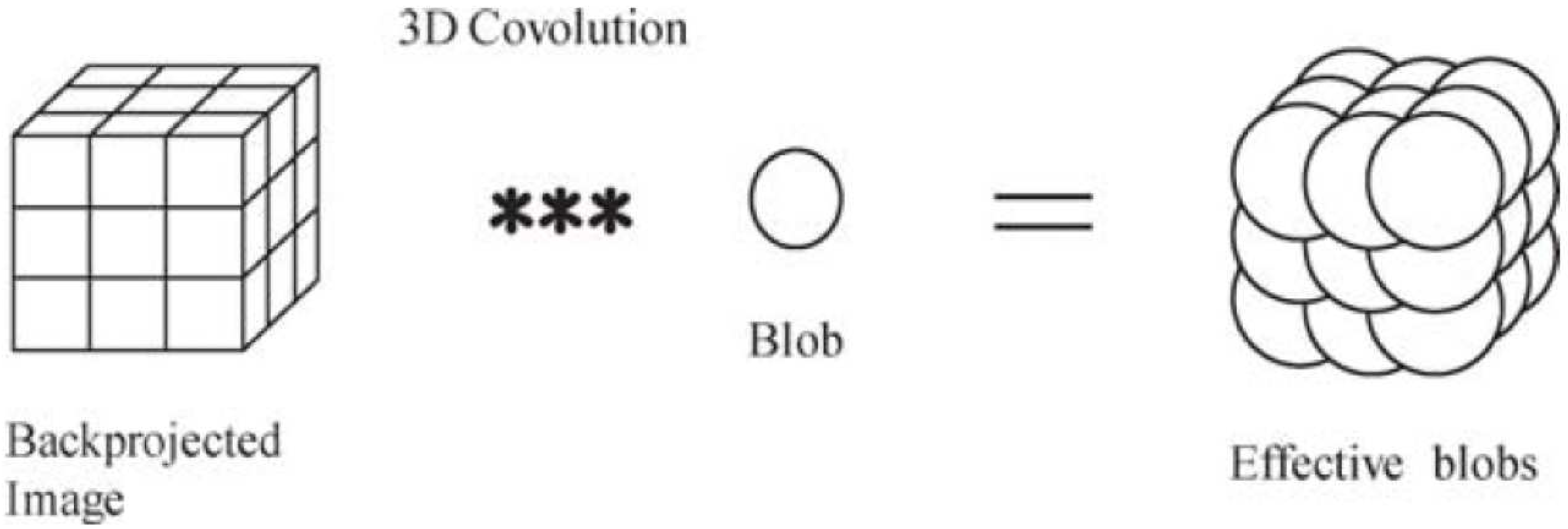

Figure 7: Illustration of the transfer from voxels to blobs by convolution (Zeng, 2009)

Towards more Realism: Final Remarks

Further extensions can be performed to include more realism:

- modeling of scattered photons,
- modeling of non-linear absorption effects,
- explicit noise modeling,
- modeling of the PSF of the imaging system.

Topics

Extensions of ART
- SART
- SIRT
- Ordered Subsets
- Towards More Realism

Summary
- Take Home Messages
- Further Readings

Take Home Messages

- Several extensions for ART are available (SART, SIRT, ...).
- Note that the projection and reconstruction model can be modified to achieve possibly better results (usually together with higher computational cost).

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

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Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Gradient Descent Algorithms

Online Course – Unit 58

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Topics

Gradient Descent Algorithms

Summary

Take Home Messages

Further Readings

Gradient Descent Algorithms

Idea:

- Formulate the reconstruction problem as an optimization problem.
- Find the optimum via a peak condition.

This enables the use of various methods that are common in optimization like:

- fast descent using **conjugate gradients**,
- or **regularization**.

Gradient Descent Algorithms: Example

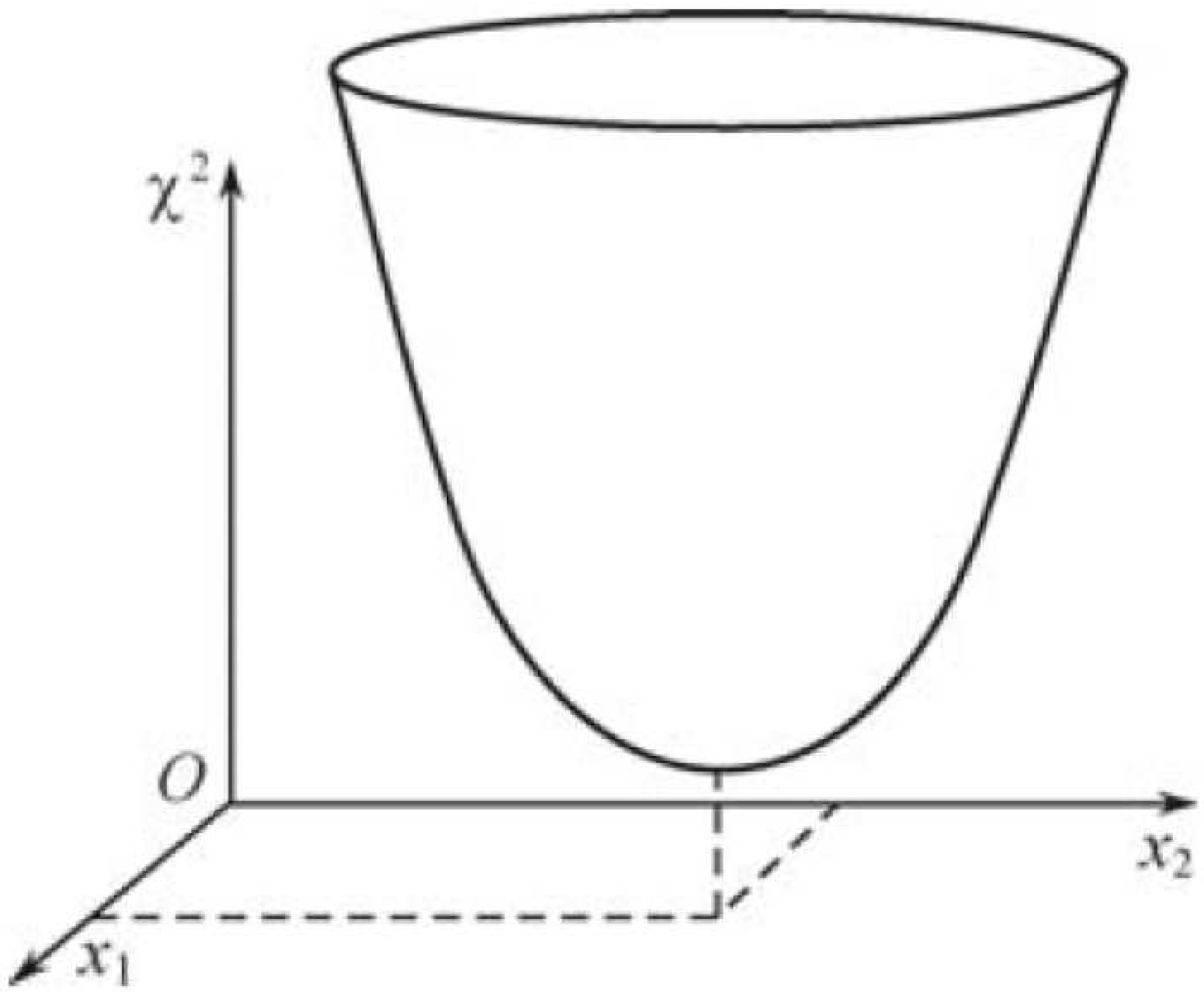

Figure 1: Finding the optimal value as an extreme point of the objective function (Zeng, 2009)

Gradient Descent Algorithms: Iterative Scheme

Let the objective function be:

$$\chi(\boldsymbol{X}) = \|\boldsymbol{AX} - \boldsymbol{P}\| = (\boldsymbol{AX} - \boldsymbol{P})^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}).$$

Then the gradient is found as:

$$\nabla\chi(\boldsymbol{X}) = 2\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}).$$

Using the peak condition $\nabla\chi(\boldsymbol{X}) = 0$ immediately yields:

$$\begin{aligned} \boldsymbol{A}^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}) &= 0, \\ \boldsymbol{A}^{\mathsf{T}}\boldsymbol{AX} &= \boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}, \\ \boldsymbol{X} &= (\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}. \end{aligned}$$

Gradient Descent Algorithms: Iterative Scheme

Instead of the analytic solution, we can formulate an iterative procedure:

$$\boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} + \lambda \Delta,$$

with an update Δ and a step scale λ.

In each step we want to go one step towards the minimum, i. e., the update is chosen as the opposite gradient direction.

Therefore, we set $\Delta = -\nabla \chi(\boldsymbol{X})$:

$$\begin{aligned} & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} - \lambda \left(2\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{X} - \boldsymbol{P})\right), \\ \Leftrightarrow \quad & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} - \lambda \left(\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{X} - \boldsymbol{P})\right), \\ \Leftrightarrow \quad & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} + \lambda \left(\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{P} - \boldsymbol{A}\boldsymbol{X})\right). \end{aligned}$$

Gradient Descent Algorithms: Iterative Scheme

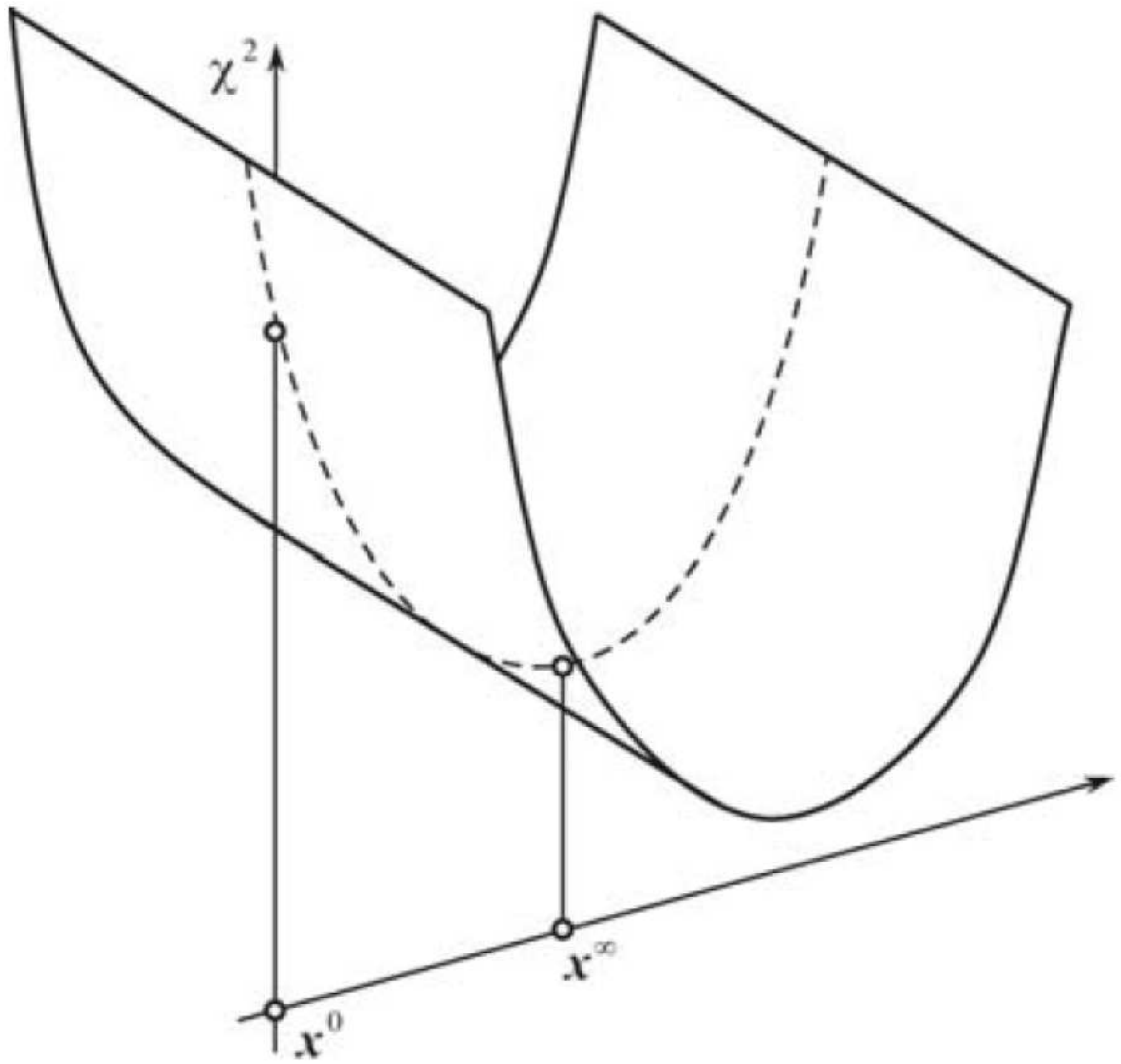

Figure 2: The solution $\boldsymbol{X}^\infty$ can depend on the initialization $\boldsymbol{X}^0$ (Zeng, 2009).

Topics

Gradient Descent Algorithms

Summary

Take Home Messages

Further Readings

Take Home Messages

- Gradient descent algorithms are iterative methods for which the iteration update is dependent on the gradient of the objective function.
- If the objective function is not convex, the algorithm might not find the global minimum/maximum.
- Even if the objective function is convex, but not strictly convex, the found minimum/maximum depends on the initialization.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

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Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – ML-EM Algorithms

Online Course – Unit 59

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Topics

Maximum-Likelihood Expectation-Maximization Methods

Summary

Take Home Messages

Further Readings

Maximum Likelihood (ML) Methods

Idea:

- Formulate the objective function as a likelihood function.
- Find the optimum of the likelihood function, i. e., the most likely solution.

Commonly solved with the “**Expectation Maximization Algorithm**” (EM).

ML-EM Reconstruction: Poisson Distribution

- In the following we consider an *emission tomography* problem.
- The probability density function for the random variable r, that describes the emission of a certain amount of energy, follows the Poisson distribution:

$$P(r|\lambda) = e^{-\lambda}\frac{\lambda^r}{r!}.$$

- The expected value of this random variable is λ.
- The observed value p_i at each detector bin i is

$$p_i = \sum_j c_{ij},$$

where each c_{ij} is a random variable distributed by the Poisson distribution.
- Therefore, we have $\lambda_{ij} = E(c_{ij}) = a_{ij}x_j$.

ML-EM Reconstruction: Objective Function

We set the following likelihood function:

$$L = \prod_{i,j} P(c_{ij}|\lambda_{ij}) = \prod_{i,j} e^{-\lambda_{ij}} \frac{{\lambda_{ij}}^{c_{ij}}}{c_{ij}!} = \prod_{i,j} e^{-a_{ij}x_j} \frac{(a_{ij}x_j)^{c_{ij}}}{c_{ij}!}.$$

Taking the logarithm yields:

$$\ln(L) = \sum_{i,j} (c_{ij} \ln(a_{ij}x_j) - a_{ij}x_j) - \sum_{i,j} \ln(c_{ij}!).$$

Note that $\sum_{i,j} \ln(c_{ij}!)$ is independent of x_j. Hence, it is valid to optimize with

$$\ln(L) = \sum_{i,j} c_{ij} \ln(a_{ij}x_j) - a_{ij}x_j.$$

ML-EM Reconstruction: Expectation Maximization

Compute the expected value of c_{ij}:

$$E(c_{ij}|p_i, \boldsymbol{X}^k) = \frac{a_{ij}x_j^k}{\sum_l a_{il}x_l^k} p_i.$$

Set c_{ij} to its expected value (E-step):

$$E(L|p_i, \boldsymbol{X}^k) = \sum_{i,j} \left(\frac{a_{ij}x_j^k}{\sum_l a_{il}x_l^k} p_i \ln(a_{ij}x_j) - a_{ij}x_j \right).$$

Maximize the expected value of the objective function (M-step):

$$\frac{\partial E(L|p_i, \boldsymbol{X}^k)}{\partial x_j} = 0.$$

ML-EM Reconstruction: Expectation Maximization

Compute the derivative of $E(L|p_i, \boldsymbol{X}^k)$:

$$\frac{\partial E(L|p_i, \boldsymbol{X}^k)}{\partial x_j} = \sum_i \left(\frac{a_{ij} x_j^k}{\sum_l a_{il} x_l^k} p_i \frac{a_{ij}}{a_{ij} x_j} - a_{ij} \right) = \frac{1}{x_j} \sum_i \frac{a_{ij} x_j^k}{\sum_l a_{il} x_l^k} p_i - \sum_i a_{ij} \stackrel{!}{=} 0.$$

Solving for x_j yields the ML-EM update rule:

$$x_j^{k+1} = \frac{x_j^k}{\sum_i a_{ij}} \sum_i a_{ij} \frac{p_i}{\sum_l a_{il} x_l^k}.$$

This can be interpreted as follows:

$$x_j^{k+1} = \frac{x_j^k}{\text{backproject}(1)} \text{backproject} \left(\frac{p_i}{\text{project}(x_l^k)} \right).$$

Topics

Maximum-Likelihood Expectation-Maximization Methods

Summary
- Take Home Messages
- Further Readings

Take Home Messages

- The ML-EM algorithm is based on likelihoods and the maximization of the expected value of the objective function.
- We have learned how emission events are modeled by random variables.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. “Angenäherte Auflösung von Systemen linearer Gleichungen”. In: *Bulletin International de l’Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. “Adaptive Iterative Reconstruction”. In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953

Medical Image Processing for Diagnostic Applications

Regularized Reconstruction – L_p-Norms

Online Course – Unit 60
Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
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TECHNISCHE FAKULTÄT

Topics

Regularized Reconstruction

L_p-Norms

Summary

Take Home Messages

Further Readings

Regularized Reconstruction

- Introduction of additional information into the reconstruction process helps to enforce a certain solution.
- This is especially advantageous if the problem is underdetermined.
- Additional weighting terms are also used to suppress noise or artifacts.

Regularization of the Reconstruction Problem

- The objective functions altered:

$$\chi(\boldsymbol{X}) = |\boldsymbol{\Psi X}|_p, \qquad \text{subject to } \boldsymbol{AX} = \boldsymbol{P}.$$

- $\boldsymbol{\Psi}$ is a transformation that transforms the problem into a different domain.
- $|\cdot|_p$ is a L_p norm:

$$|\boldsymbol{x}|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}.$$

The Transformation Ψ

- Ψ is a transformation that transforms the problem into a different domain.
- The selection of Ψ is problem dependent.

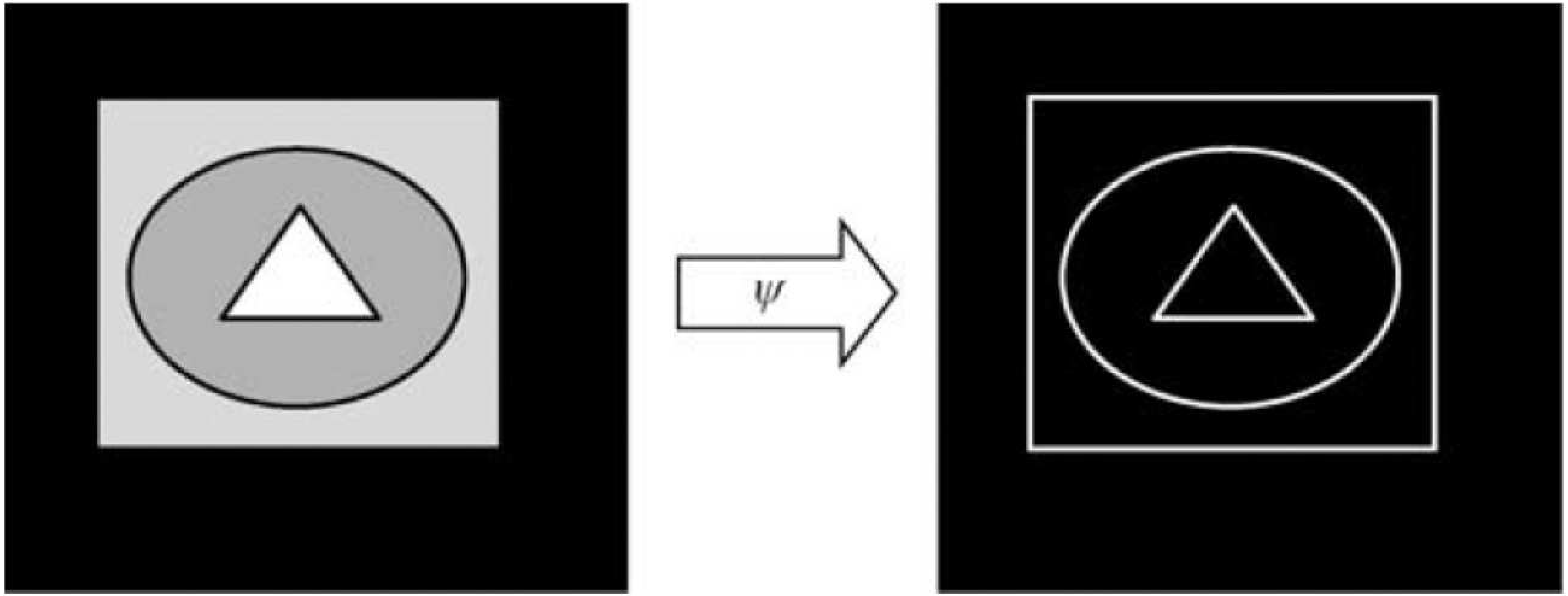

Figure 1: Example for a sparsifying transformation (Zeng, 2009)

The Transformation Ψ

Many transforms have already been investigated as sparsifying transform:

- gradient image,
- wavelet transform,
- Fourier transform,
- discrete cosine transform,
- and many more.

Topics

Regularized Reconstruction

L_p-Norms

Summary

Take Home Messages

Further Readings

L_p-Norms

Definition

We denote a L_p-norm as $|\cdot|_p$ which is defined by

$$|\boldsymbol{x}|_p = \left(\sum_i x_i^p\right)^{\frac{1}{p}}.$$

This definition yields a valid vector space norm for $p \in [1, \infty) \cup \{\infty\}$.

- For $p = 2$ we get:

$$|\boldsymbol{x}|_2 = \|\boldsymbol{x}\| = \sqrt{\sum_i x_i^2}.$$

- For $p = 1$ we get:

$$|\boldsymbol{x}|_1 = \sum_i |x_i|.$$

L_p-Norms: Special Cases

- For $p = 0$ we get:
$$|\boldsymbol{x}|_0 = \sum_i |x_i|^0,$$
where we define $0^0 = 0$ for this purpose. This is strictly speaking not a norm in the mathematical sense, but it is a useful tool.
- For $p = \infty$ we get:
$$|\boldsymbol{x}|_\infty = \max_i(|x_i|),$$
which is a mathematical norm.

Examples

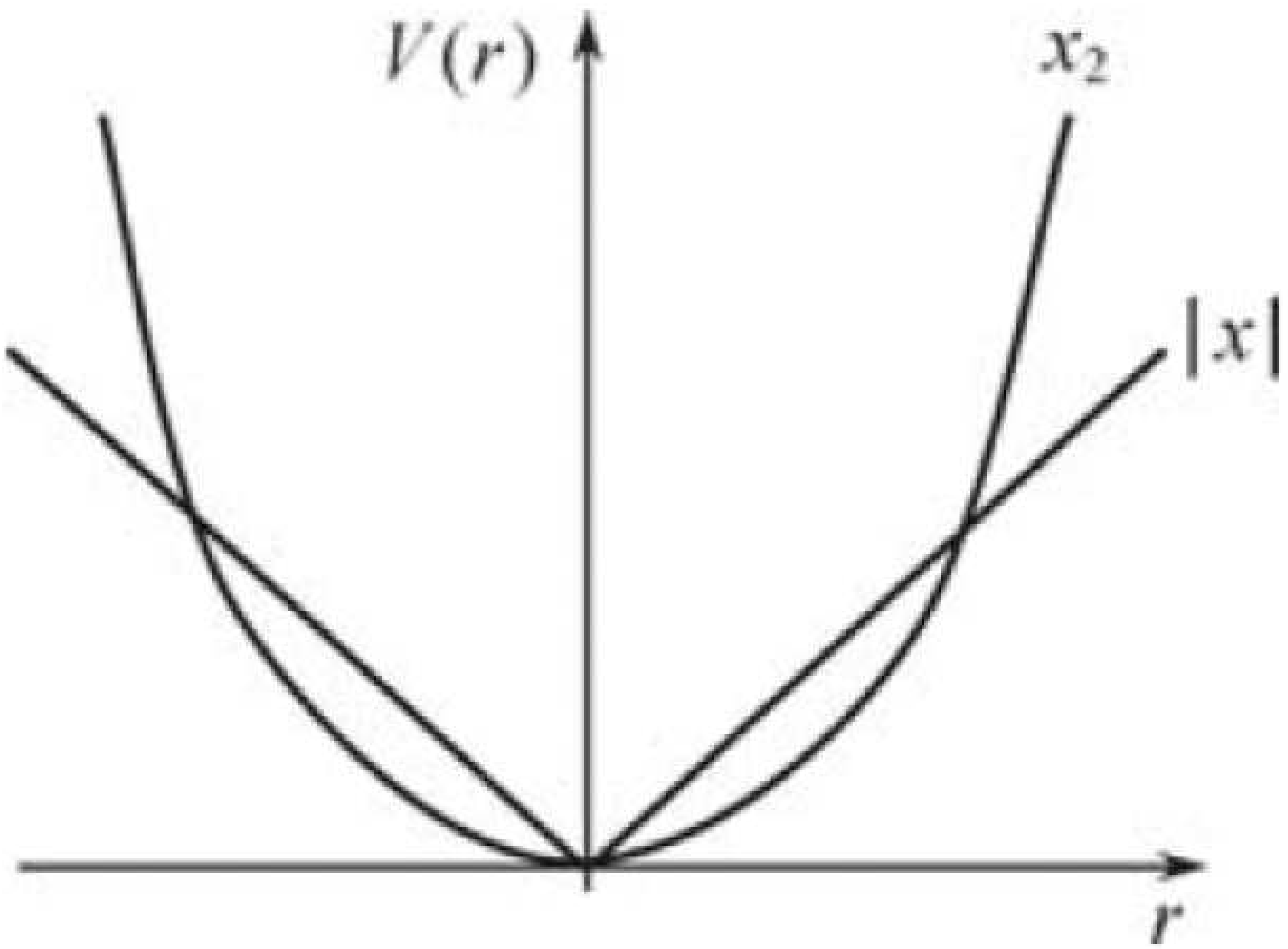

Figure 2: Regularization with L_1-norm (Zeng, 2009)

Examples

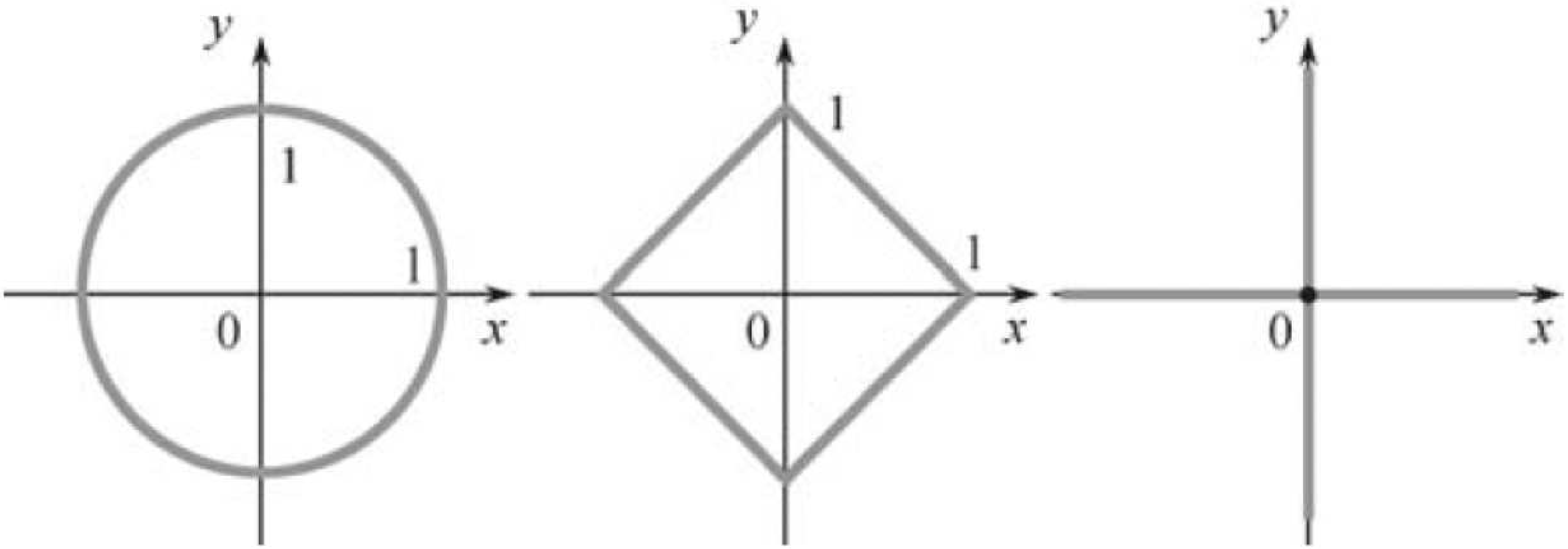

Figure 3: The unit sphere for L_2, L_1, and L_0 (Zeng, 2009)

Topics

Regularized Reconstruction

L_p-Norms

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- Regularization of a reconstruction problem is done via a specific transformation and optimization with respect to a specific objective function.
- This objective function is defined in almost every case by use of an L_p-norm.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. "Adaptive Iterative Reconstruction". In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953

Medical Image Processing for Diagnostic Applications

Regularized Reconstruction Methods

Online Course – Unit 61

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Topics

Regularized Reconstruction Methods

- TV-Norm
- Gradient Methods
- ML Methods

Summary

- Take Home Messages
- Further Readings

Regularized Reconstruction: TV-Norm

- The **total variation norm (TV-norm)** is a popular combination of a sparsifying transform and the L_1-norm.
- The sparsifying transform is the gradient image:

$$\Psi(f(x,y)) = \sqrt{\left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2}.$$

- The norm is the L_1-norm:

$$|\boldsymbol{X}|_1 = \sum_{i,j} |x|.$$

- The TV-norm for the discretized image $\boldsymbol{f}$ then is computed by:

$$|\Psi(\boldsymbol{f})|_1 = \sum_{i,j} \left| \sqrt{\left(\frac{\partial f}{\partial x}(i,j)\right)^2 + \left(\frac{\partial f}{\partial y}(i,j)\right)^2} \right| = \sum_{i,j} \sqrt{\left(\frac{\partial f}{\partial x}(i,j)\right)^2 + \left(\frac{\partial f}{\partial y}(i,j)\right)^2}.$$

Regularized Reconstruction: TV-Norm

The TV-Norm ...

- ... promotes sparsity in the gradient domain.
- ... suppresses noise.
- ... preserves edges.
- ... promotes images with large uniform areas and sharp boundaries.

$\rightarrow$ These properties make it ideal for phantoms.

$\rightarrow$ Images are often described as “patchy”.

Regularized Reconstruction: TV-Norm

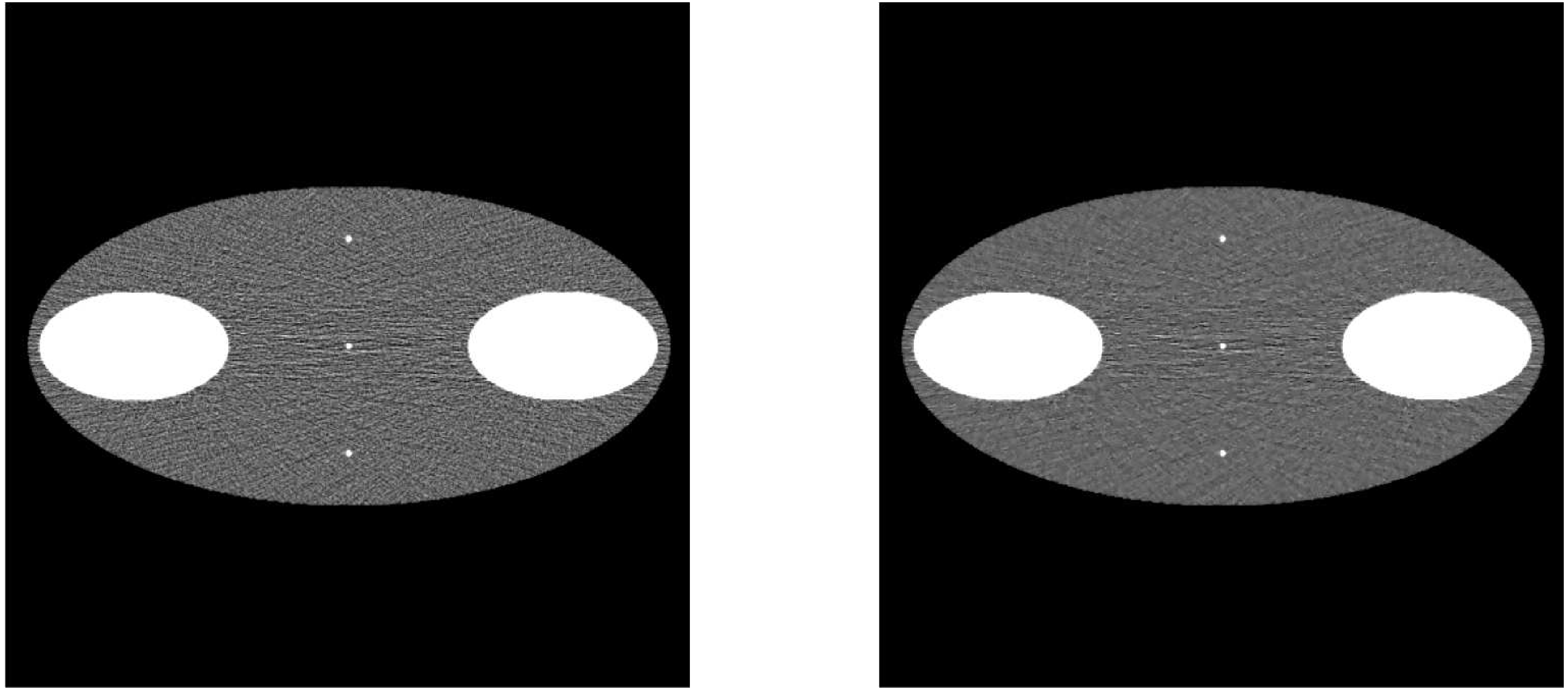

Figure 1: Streak artifacts in a phantom reconstruction without (left), and with TV regularization (right)

Regularized Reconstruction: TV-Norm

- Bruder et al. have shown that reconstruction with TV regularization is equivalent to FBP reconstruction with subsequent nonlinear edge preserving filtering.
- Non-linear filtering makes image quality assessment difficult.

$\rightarrow$ FBP reconstruction is much faster.

$\rightarrow$ The same “patch” image characteristic can be obtained if too much filtering is applied.

Regularized Reconstruction: Gradient Methods

Recall: The objective function is determined by:

$$\chi(\boldsymbol{X}) = |\boldsymbol{\Psi X}|_p, \qquad \text{subject to } \boldsymbol{AX} = \boldsymbol{P}.$$

We can formulate a single new objective function:

$$\chi(\boldsymbol{X}) = |\boldsymbol{\Psi X}|_p + \boldsymbol{\Lambda}(\boldsymbol{AX} - \boldsymbol{P})$$

with Lagrange multipliers

$$\boldsymbol{\Lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

→ This method is often referred to as "**Compressed Sensing**".

Regularized Reconstruction: Gradient Methods

Similarly, we can add further constraints:

$$\chi(\boldsymbol{X}) = |\boldsymbol{\Psi}\boldsymbol{X}|_p + \boldsymbol{\Lambda}(\boldsymbol{A}\boldsymbol{X} - \boldsymbol{P}) + k(|\boldsymbol{X} - \boldsymbol{X}_{\text{prior}}|_p)$$

with weighting factor k and prior knowledge $\boldsymbol{X}_{\text{prior}}$.

$\rightarrow$ This method is known as “**Prior Image Constrained Compressed Sensing**”.

Regularized Reconstruction: Gradient Methods

A solution is based on the following principles:

- compute the gradient & set to 0,
- follow the opposite gradient direction,
- perform projection onto convex sets:
  - enforce constraint A,
  - enforce constraint B,
  - repeat until convergence.

Regularized Reconstruction: ML Methods

- Introduction of new constraints is done using Bayes' rule:

$$P(X|p) = \frac{P(p|X)P(X)}{P(p)}.$$

- Taking the logarithm yields:

$$\ln P(X|p) = \ln P(p|X) + \ln P(X) - \ln P(p).$$

- The optimization is independent of $\ln P(p)$.
- This yields the following objective function:

$$L(X) = \ln P(p|X) + \ln P(X),$$

which could be interpreted as:

$$\text{Posterior Function} = \text{Likelihood Function} + \beta(\text{Prior Knowledge}).$$

Regularized Reconstruction: ML Methods

Recall: The unconstrained ML-EM update formula is:

$$x_j^{k+1} = \frac{x_j^k}{\sum_i a_{ij}} \sum_i a_{ij} \frac{p_i}{\sum_l a_{il} x_l^k}.$$

Adding the constraint $P(\boldsymbol{X})$ yields:

$$x_j^{k+1} = \frac{x_j^k}{\sum_i a_{ij} + \beta \frac{\partial \ln P(\boldsymbol{X}^k)}{\partial x_j^k}} \sum_i a_{ij} \frac{p_i}{\sum_l a_{il} x_l^k}.$$

$\rightarrow$ This method is also known as “Green’s One-Step-Late Method”.

$\rightarrow$ Methods using this Bayesian scheme are also known as **maximum-a-posteriori (MAP) methods**.

Regularized Reconstruction: Remarks

- Regularization leads to a new reconstruction problem that does not necessarily lead to the solution of the original problem.
- Sophisticated cost functions often also lead to increased computational effort.
- Regularization can lead to results that look beautiful.
- The result may not be the true image!

Topics

Regularized Reconstruction Methods

- TV-Norm
- Gradient Methods
- ML Methods

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- You have learned about several approaches to regularized reconstruction: TV-norm, gradient based and ML-EM methods.
- Regularization may change the image content towards, but also away from clinical realism, so these methods need to be tuned for the problem at hand.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

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H. Bruder et al. "Adaptive Iterative Reconstruction". In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953

Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Resolution and Noise

Online Course – Unit 62
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Topics

Iterative Reconstruction Methods
Resolution
Noise

Summary
Take Home Messages
Further Readings

Iterative Reconstruction Methods: Resolution

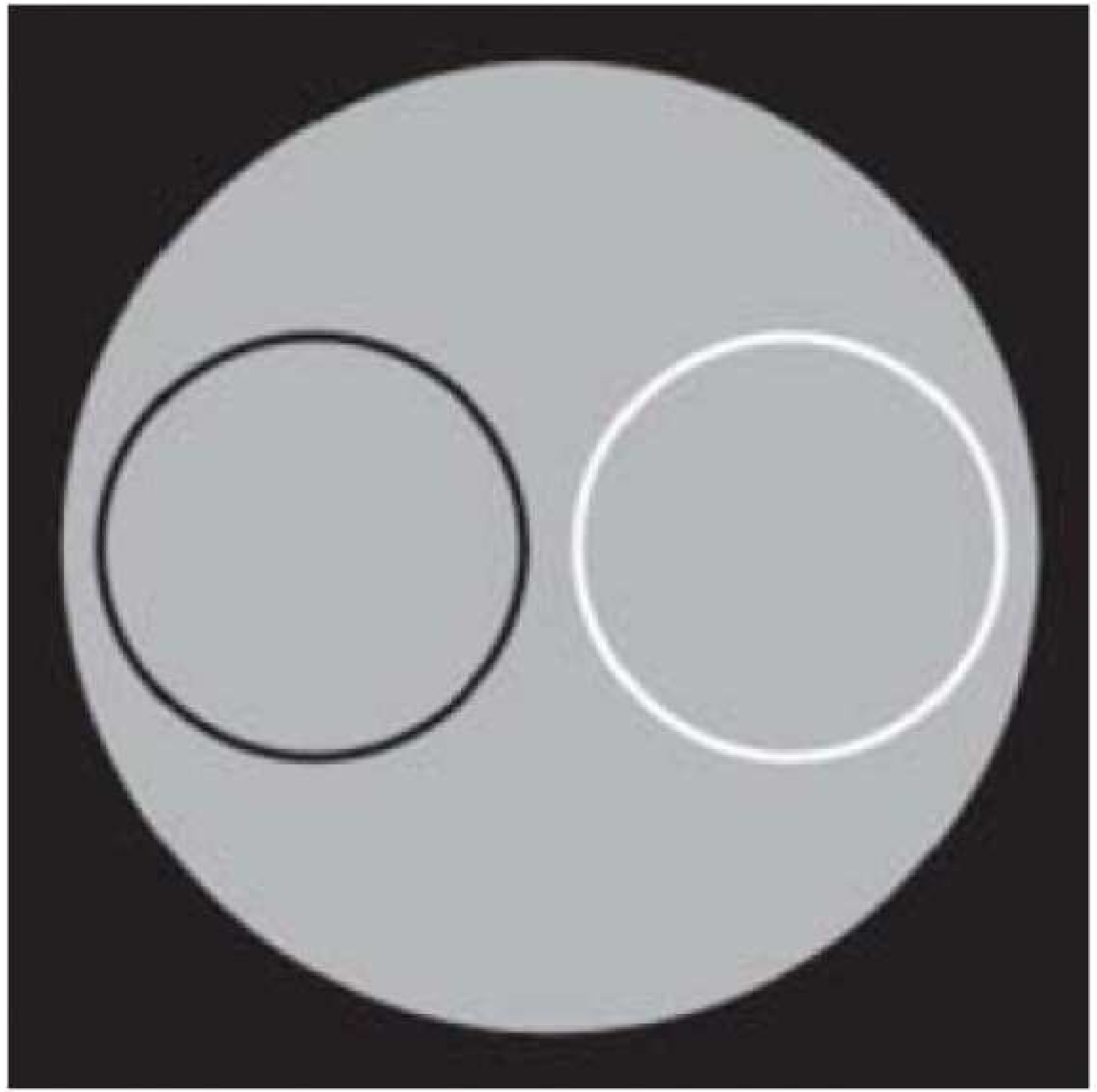

Figure 1: Phantom (Zeng, 2009)

Iterative Reconstruction Methods: Resolution

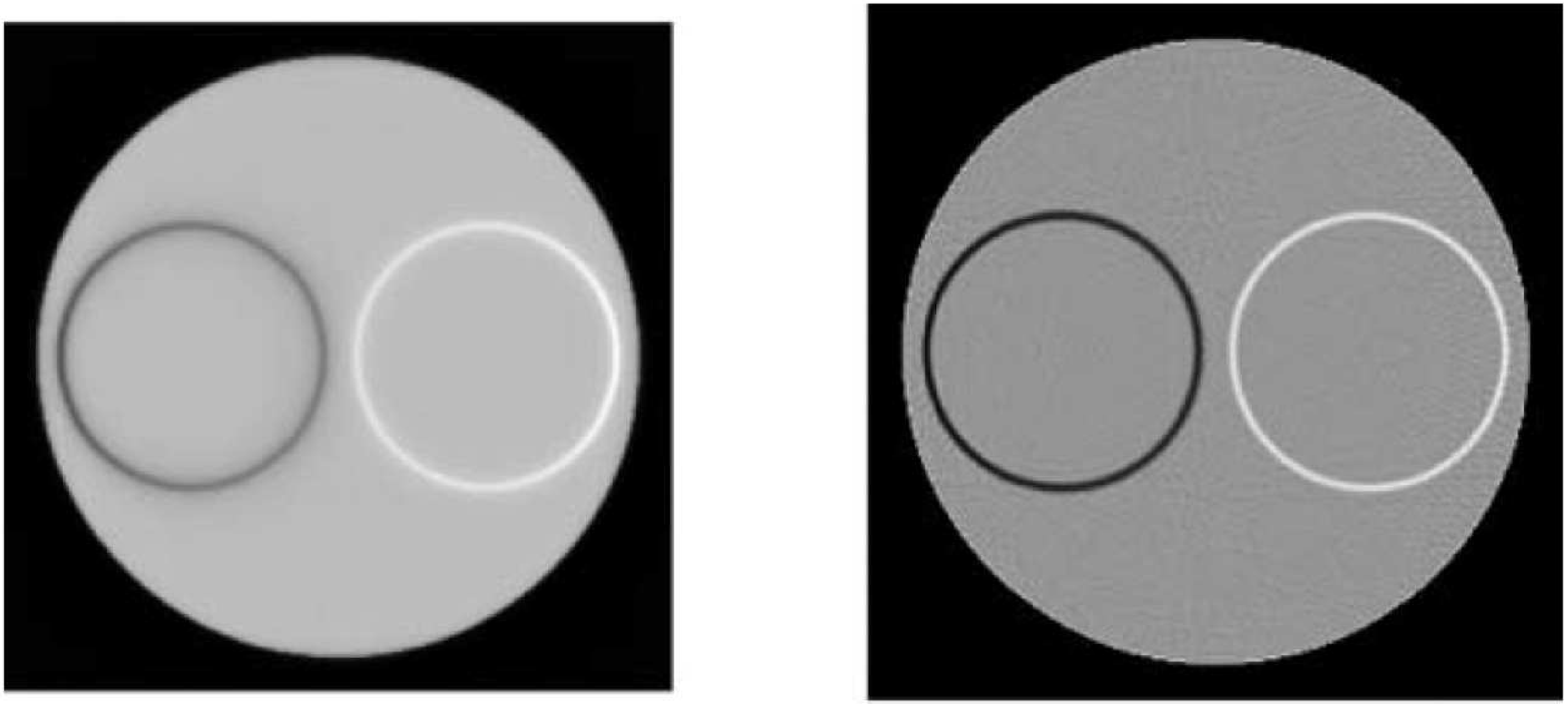

Figure 2: 25 iterations (left), 250 iterations (right) (Zeng, 2009)

Iterative Reconstruction Methods: Resolution

- Iterative methods do not necessarily have a uniform image resolution.
- Resolution increases with the number of iterations.
- This makes image quality assessment difficult.

Iterative Reconstruction Methods: Noise

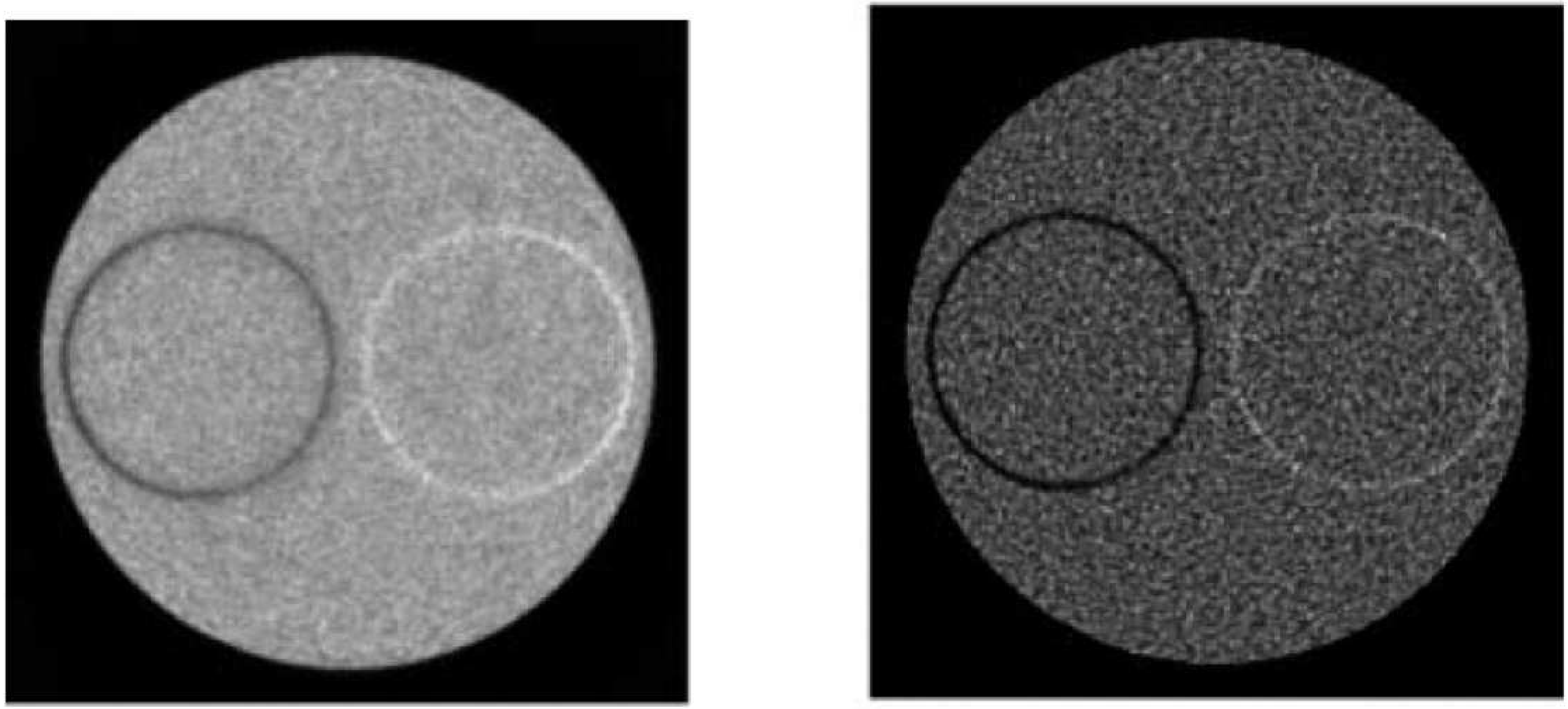

Figure 3: 25 iterations (left), 250 iterations (right) (Zeng, 2009)

Iterative Reconstruction Methods: Noise

- Image noise is dependent on the number of iterations.
- The more iterations, the more noise is in the image.
- A common means to control noise is to adjust the number of iterations.
- This is yet another reason why it is difficult to measure image quality in iterative methods.

Iterative Reconstruction Methods: Noise

Overiteration plus low-pass filtering can help to make the image resolution more uniform:

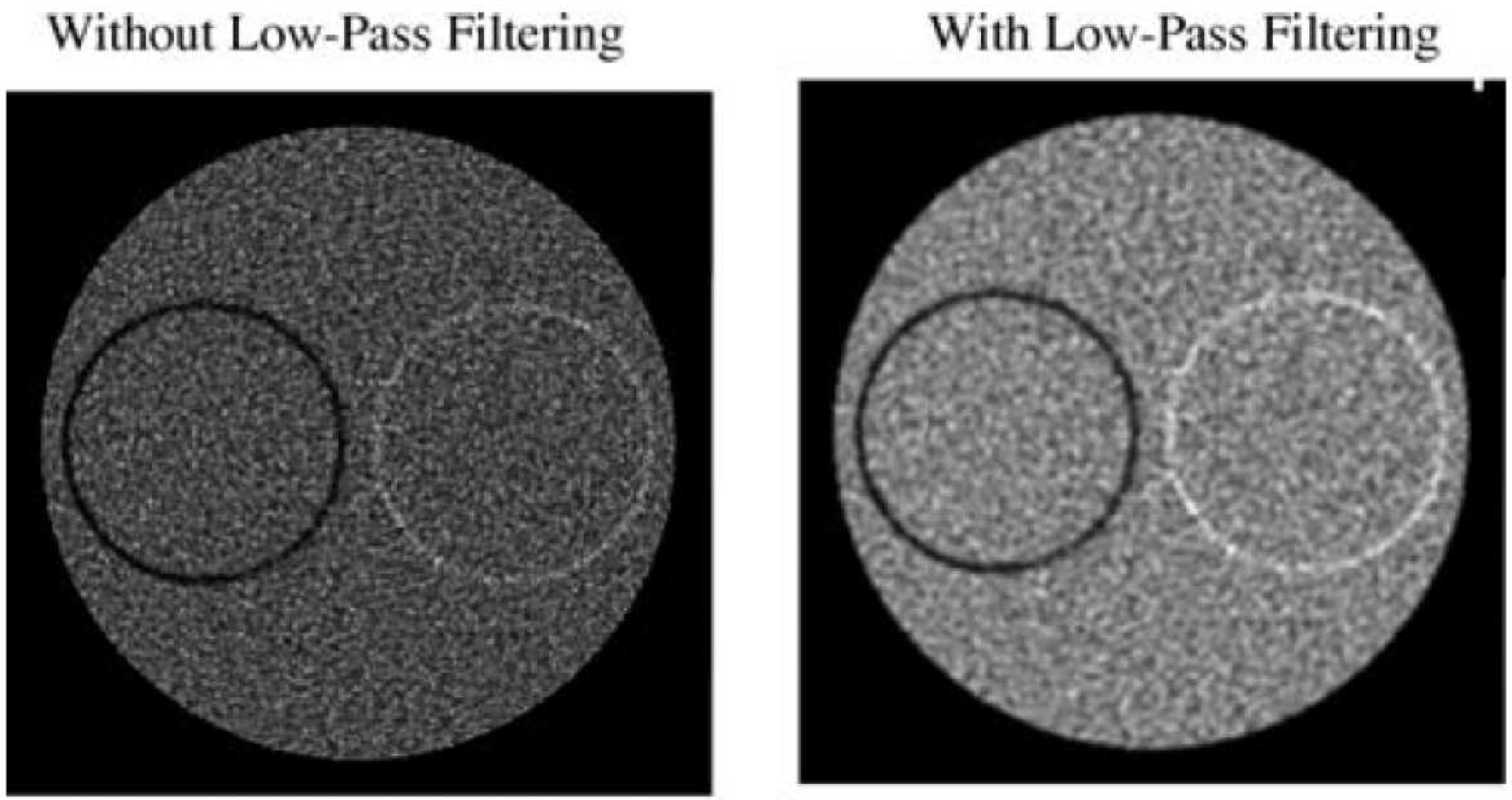

Figure 4: Low-pass filtering effect on reconstruction result (Zeng, 2009)

Topics

Iterative Reconstruction Methods

- Resolution
- Noise

Summary

- Take Home Messages
- Further Readings

Take Home Messages

[Unit:]

- Resolution and noise are important aspects in iterative reconstruction and both are dependent on the number of iterations.
- Image quality assessment for iterative reconstruction remains challenging.

[Chapter:]

- Iterative reconstruction allows the definition of very flexible objective functions and therewith very flexible modeling of the imaging system.
- Analytic reconstruction methods are still often used, not least because superior image quality has not yet been shown for many applications.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

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Medical Image Processing for Diagnostic Applications

Bilateral Filtering

Online Course – Unit 63

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Topics

Edge Preserving Filtering

- Motivation
- Low-Pass Domain Filter
- Low-Pass Range Filter
- The Combination to the Bilateral Filter

Examples

Summary

- Take Home Messages
- Further Readings

Motivation

Problem: How can we prevent averaging across edges while still averaging within a smooth region?

Definition

Bilateral filtering is a method for edge preserving noise reduction. Instead of simply averaging image values in dependence on their geometric closeness, also the photometric similarity of nearby pixels is considered.

Low-Pass Domain Filter

Let us consider spatial domain filtering of continuous 2-D image functions defined as follows:

$$h_{\text{domain}}(x,y) = k_d^{-1}(x,y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\mu,\nu)c(x,y,\mu,\nu)\,\mathrm{d}\mu\,\mathrm{d}\nu,$$

where

- the filtering kernel is usually restricted to a local neighborhood,
- $f(x,y)$ denotes the observed image,
- $c(x,y,\mu,\nu)$ measures the *geometric closeness* between the image point (x,y) and the point (μ,ν), and
- the normalization function k_d is defined by

$$k_d(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} c(x,y,\mu,\nu)\,\mathrm{d}\mu\,\mathrm{d}\nu.$$

Low-Pass Domain Filter

We can make use of a standard definition from system theory:

Definition

The filter is called ***shift-invariant*** if $c(x, y, \mu, \nu)$ is bivariate in (μ, ν) and $k_d(x, y)$ is constant.

Low-Pass Range Filter

Now we apply a similar idea to the intensity values:

$$h_{\text{range}}(x,y) = k_r^{-1}(x,y) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\mu,\nu)s(f(x,y),f(\mu,\nu))\,\mathrm{d}\mu\,\mathrm{d}\nu,$$

where

- again the filtering kernel is usually restricted to a local neighborhood,
- $f(x,y)$ denotes the observed image,
- $s(f(x,y),f(\mu,\nu))$ measures the *photometric similarity* between the intensity value at (x,y) and a neighboring position (μ,ν), and
- the normalization function k_r is defined by

$$k_r(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(f(x,y),f(\mu,\nu))\,\mathrm{d}\mu\,\mathrm{d}\nu.$$

Low-Pass Range Filter

Definition

The similarity function $s(f(x,y), f(\mu,\nu))$ is ***unbiased*** if it depends only on the difference $f(x,y) - f(\mu,\nu)$.

Bilateral Filter

Now we combine domain and range filtering in a proper manner:

$$h_{\text{bilateral}}(x,y) = k^{-1}(x,y) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\mu,\nu)c(x,y,\mu,\nu)s(f(x,y),f(\mu,\nu))\,\mathrm{d}\mu\,\mathrm{d}\nu$$

where

- $f(x,y)$ denotes the observed image,
- $c(x,y,\mu,\nu)$ measures the *geometric closeness* between the image point (x,y) and the point (μ,ν),
- $s(f(x,y),f(\mu,\nu))$ measures the *photometric similarity* between the intensity value at (x,y) and a neighboring position (μ,ν), and
- the normalization function k is defined by

$$k(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c(x,y,\mu,\nu)s(f(x,y),f(\mu,\nu))\,\mathrm{d}\mu\,\mathrm{d}\nu.$$

Geometric Closeness and Photometric Similarity: Examples

Example

Closeness function

$$c(x, y, \mu, \nu) = \exp\left(-\frac{1}{2}\frac{\left\|\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}\mu\\\nu\end{pmatrix}\right\|^2}{\sigma_d^2}\right)$$

Example

Similarity function

$$s(f_1, f_2) = \exp\left(-\frac{1}{2}\frac{\|f_1 - f_2\|^2}{\sigma_r^2}\right)$$

Topics

Edge Preserving Filtering
- Motivation
- Low-Pass Domain Filter
- Low-Pass Range Filter
- The Combination to the Bilateral Filter

Examples

Summary
- Take Home Messages
- Further Readings

Edge Preserving Smoothing

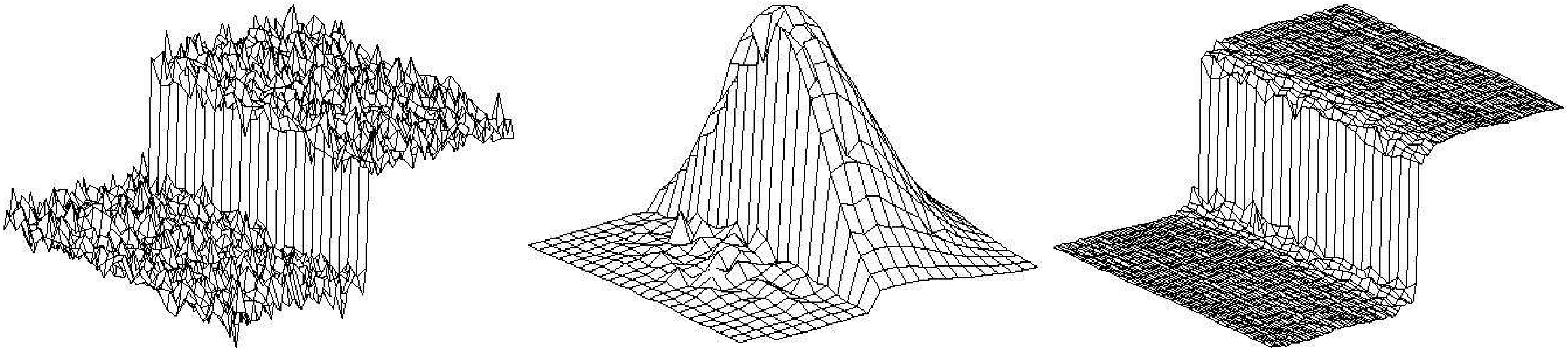


Figure 1: A 100-gray level step perturbed by Gaussian noise with $\sigma = 10$ gray levels (left). Combined similarity weights $c(x,y,\mu,\nu)s(f(x,y),f(\mu,\nu))$ for a 23×23 neighborhood centered two pixels to the right of the step (middle). Result of bilateral filtering with $\sigma_r = 50$ gray levels and $\sigma_d = 5$ pixels (right) (Carlo Tomasi)

Image Example: CT Slice

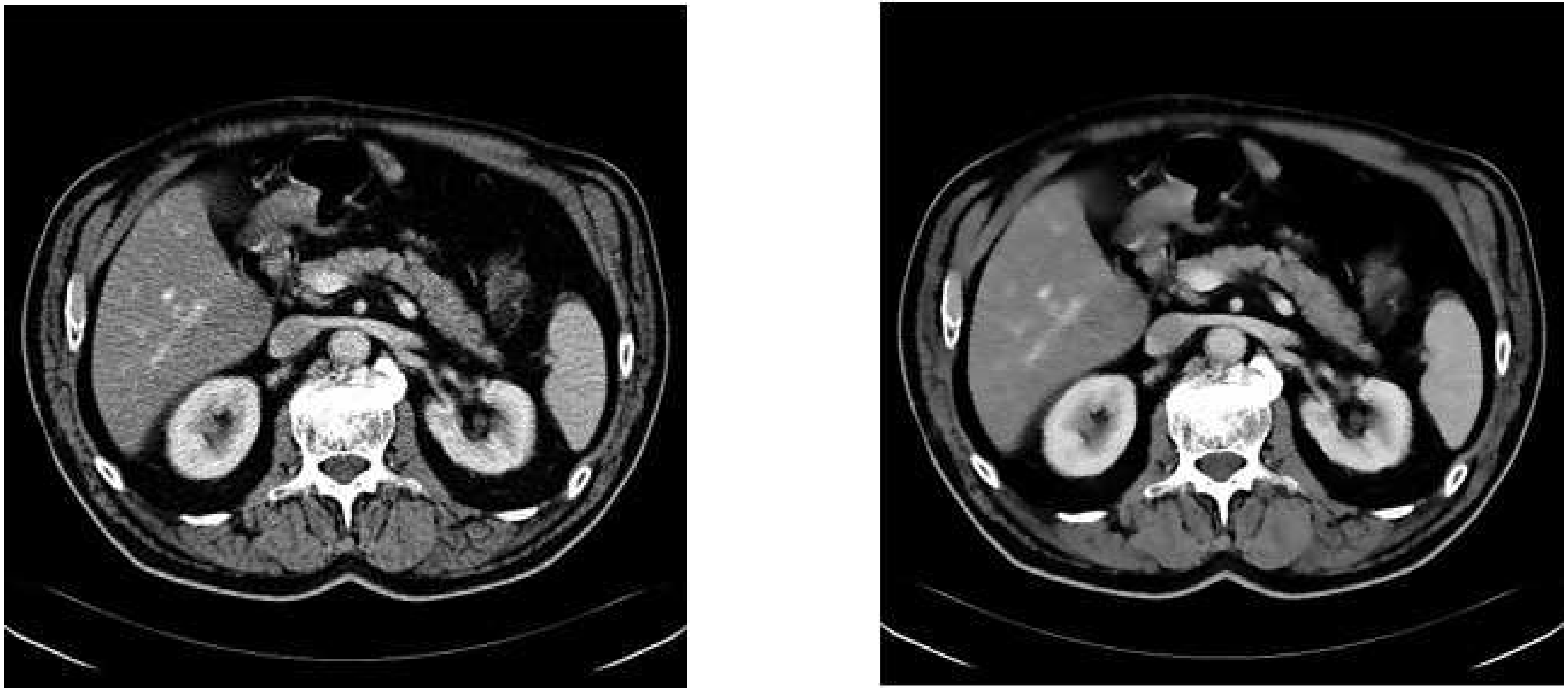


Figure 2: (Anja Borsdorf, Pattern Recognition Lab, FAU)

Example: Limited Angle Reconstruction (180 Degrees Only)

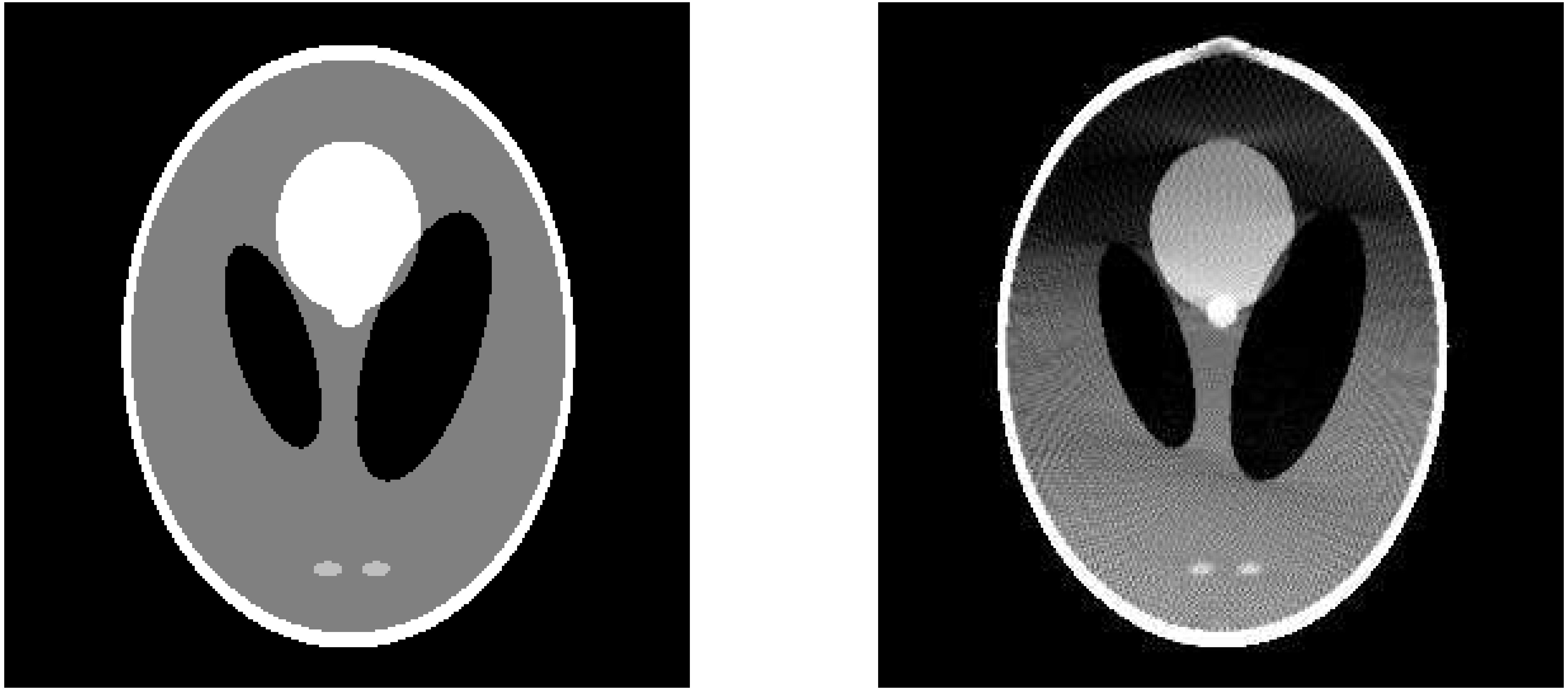


Figure 3: Shepp-Logan phantom (left), FBP with Parker weights (right)

Example: Limited Angle Reconstruction (Line Profiles)

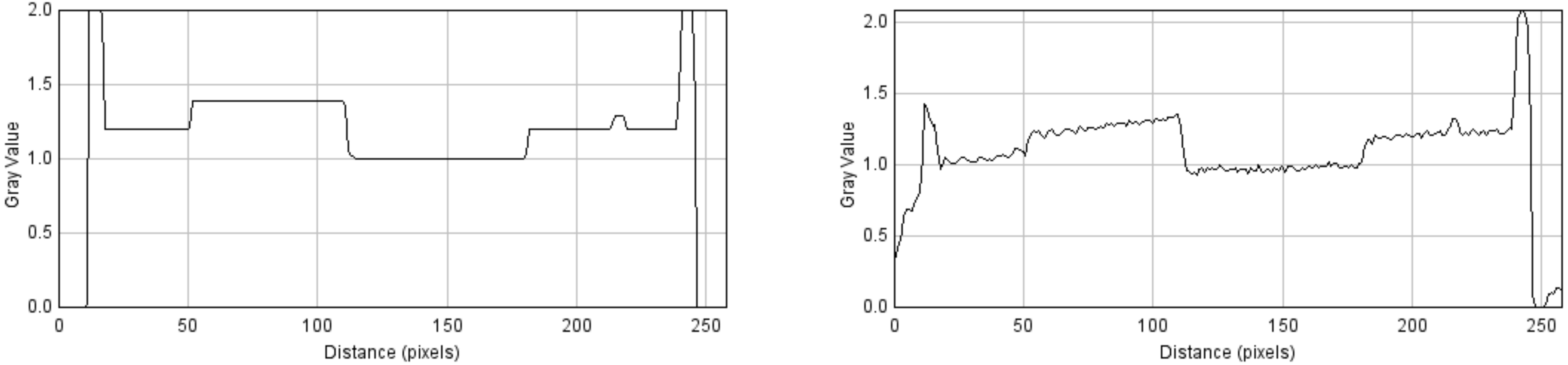


Figure 4: Shepp-Logan phantom (left), FBP with Parker weights (right)

Idea: Apply Multiplicative Weighting for Compensation

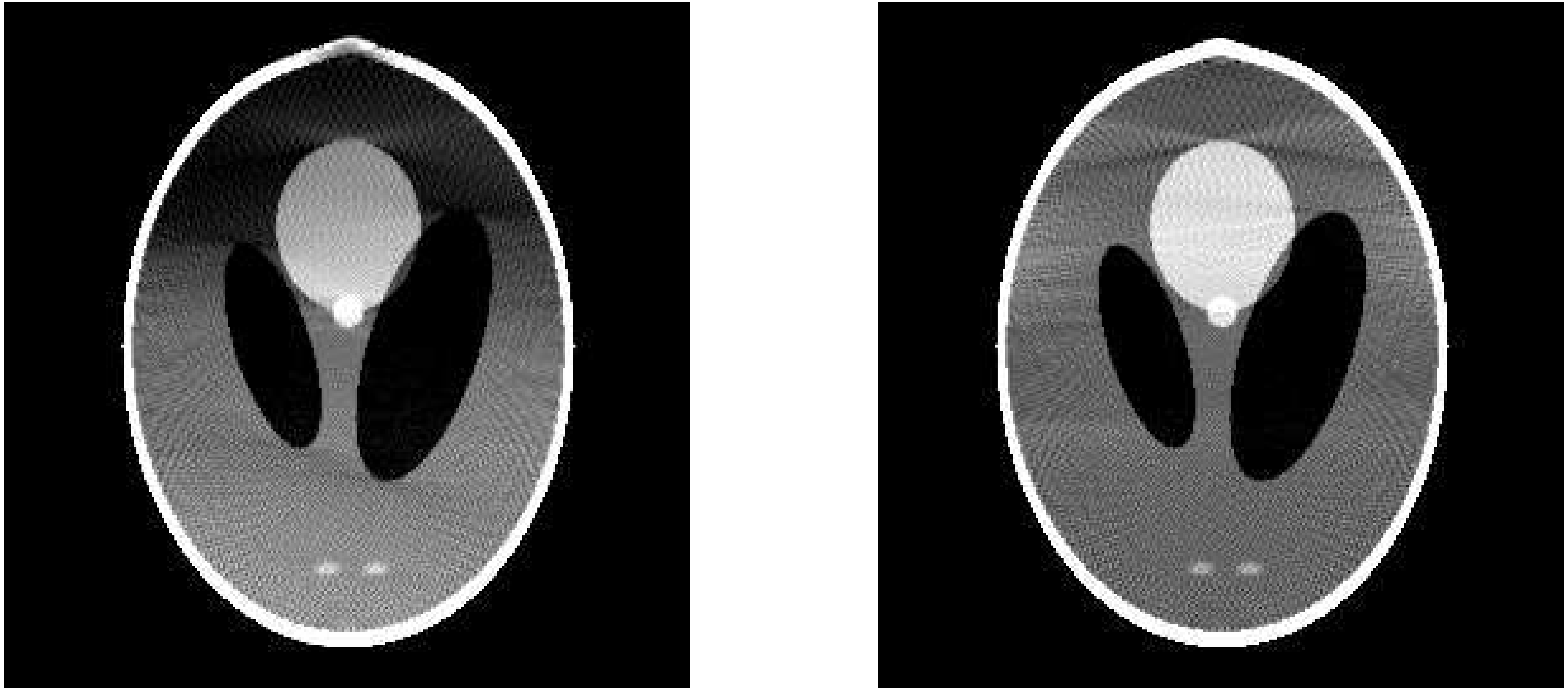
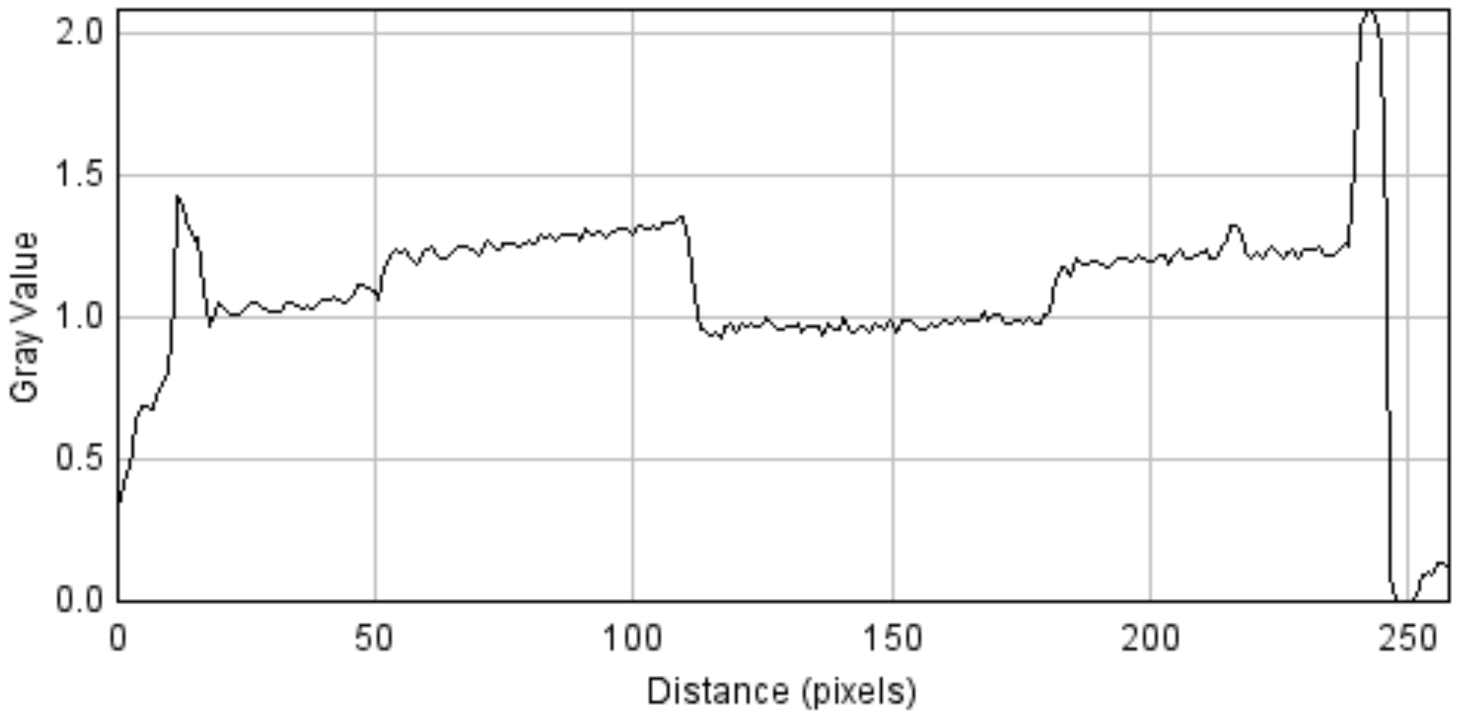


Figure 5: FBP with Parker weights (left), FBP with compensation (right)

Multiplicative Weighting (Line Profiles)



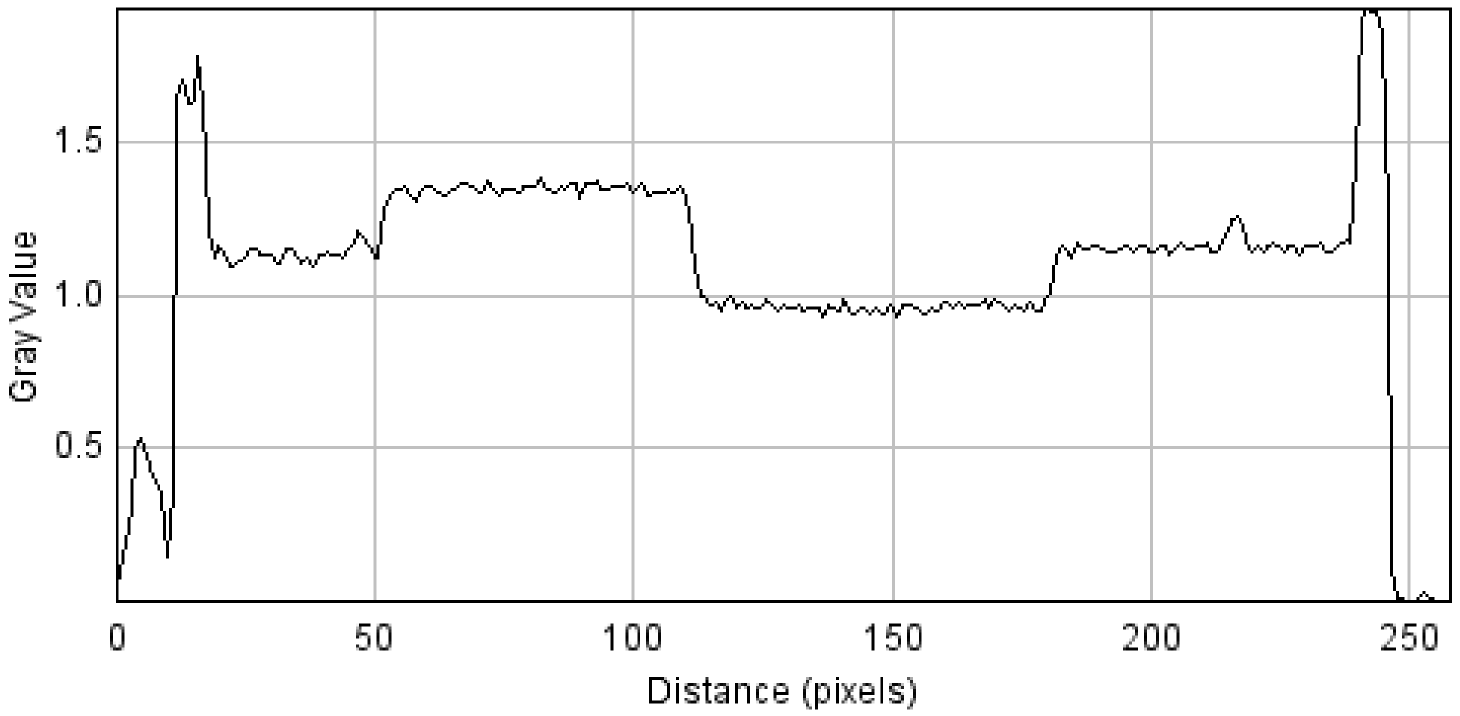


Figure 6: FBP with Parker weights (left), FBP with compensation (right)

Comparison: Regularized Iterative Reconstruction

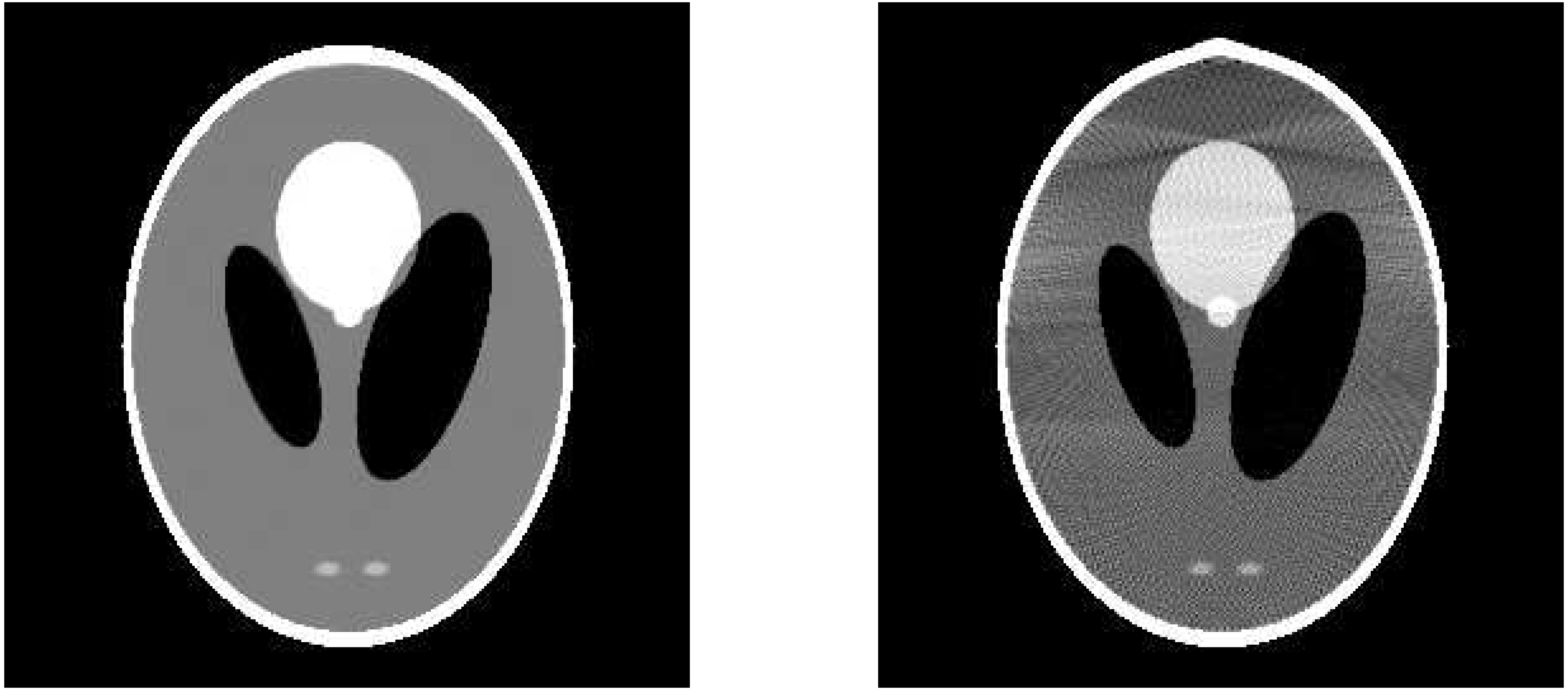


Figure 7: Iterative reconstruction with TV (left), FBP with compensation (right)

TV vs. Multiplicative Weighting (Line Profiles)

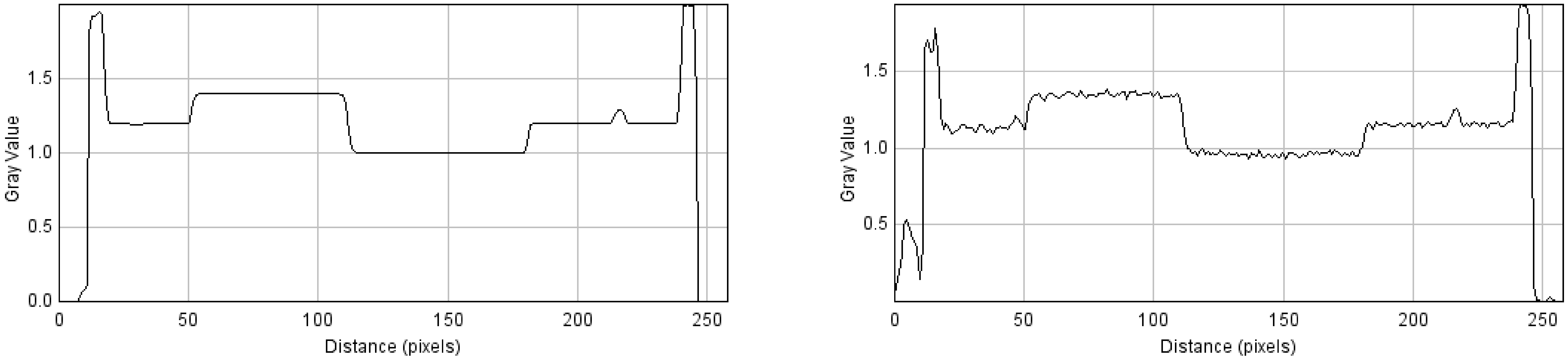


Figure 8: Iterative reconstruction with TV (left), FBP with compensation (right)

Image Space Iterative Reconstruction?

Figure 9: Iterative reconstruction with TV - 1000 iterations (left), FBP with compensation + BF, 1 iteration (right)

Line Plot Comparison

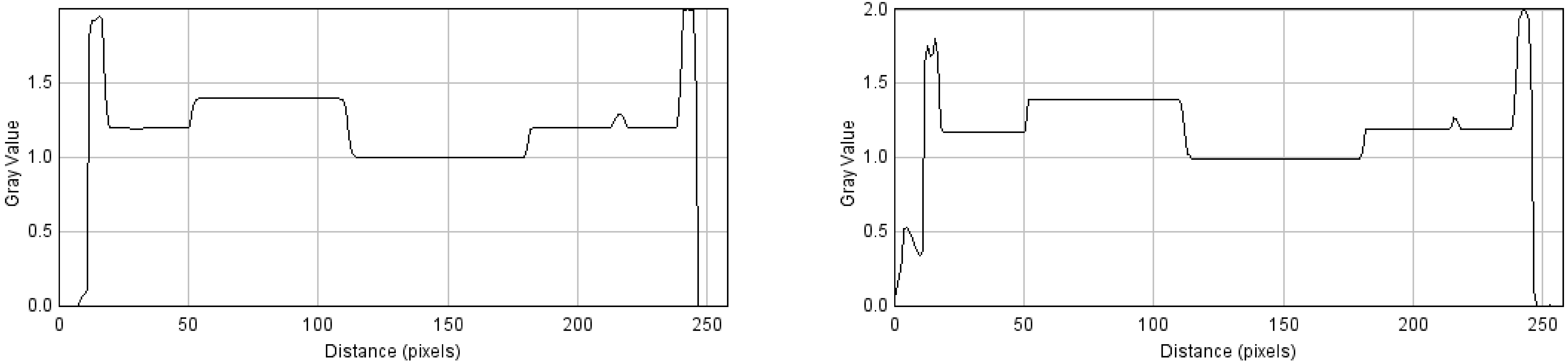


Figure 10: Iterative reconstruction with TV (left), FBP with compensation + BF (right)

TV or not TV? That is the question...

Figure 11: Iterative reconstruction with TV (left), FBP with Parker weights + BF (right)

Topics

Edge Preserving Filtering
- Motivation
- Low-Pass Domain Filter
- Low-Pass Range Filter
- The Combination to the Bilateral Filter

Examples

Summary
- Take Home Messages
- Further Readings

Take Home Messages

- Simple ideas are often the best ones!
- Bilateral filtering is computationally expensive and requires acceleration.
- Bilateral filtering is used in medical products and can be applied to any image (independent of the modality) where edge preserving smoothing is an issue.
- Edge preserving filtering can compensate for many disadvantages of expensive iterative reconstruction methods.

Further Readings

Read the original paper of the year 1998:

Carlo Tomasi and Roberto Manduchi. "Bilateral Filtering for Gray and Color Images". In: *Sixth International Conference on Computer Vision, 1998*. IEEE, Jan. 1998, pp. 839–846. DOI: 10.1109/ICCV.1998.710815

A very nice paper on ways to improve bilateral filtering can be found in:

Michael Elad. "On the Origin of the Bilateral Filter and Ways to Improve It". In: *IEEE Transactions on Image Processing* 11.10 (Oct. 2002), pp. 1141–1151. DOI: 10.1109/TIP.2002.801126

Developments of fast bilateral filtering are subject of the following webpage:

http://people.csail.mit.edu/sparis/bf/