Medical Image Processing for Diagnostic Applications

3-D Rotations – Euler Angles and Rodrigues Formula

Online Course – Unit 65

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Topics

Representations of 3-D Rotations

- Overview
- Euler Angles
- Axis-Angle Representation

Summary

- Take Home Messages
- Further Readings

Rotations in 3-D

Various representations for rotations:

- **Euler angles**
- **Axis-angle representation**
- Quaternions

Euler Angle Representation

- A 3-D rotation can be expressed by a 3×3 rotation matrix.
- An arbitrary rotation can be composed of 3 rotations around the axes of the coordinate system using the angles φ_x (roll), φ_y (pitch), φ_z (yaw).

$$\boldsymbol{R} = \boldsymbol{R}_x \boldsymbol{R}_y \boldsymbol{R}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_x & -\sin\varphi_x \\ 0 & \sin\varphi_x & \cos\varphi_x \end{pmatrix} \begin{pmatrix} \cos\varphi_y & 0 & \sin\varphi_y \\ 0 & 1 & 0 \\ -\sin\varphi_y & 0 & \cos\varphi_y \end{pmatrix} \begin{pmatrix} \cos\varphi_z & -\sin\varphi_z & 0 \\ \sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi_y \cos\varphi_z & -\cos\varphi_y \sin\varphi_z & \sin\varphi_y \\ \sin\varphi_x \sin\varphi_y \cos\varphi_z + \cos\varphi_x \sin\varphi_z & -\sin\varphi_x \sin\varphi_y \sin\varphi_z + \cos\varphi_x \cos\varphi_z & -\sin\varphi_x \cos\varphi_y \\ -\cos\varphi_x \sin\varphi_y \cos\varphi_z + \sin\varphi_x \sin\varphi_z & \cos\varphi_x \sin\varphi_y \sin\varphi_z + \sin\varphi_x \cos\varphi_z & \cos\varphi_x \cos\varphi_y \end{pmatrix}$$

Euler Angle Representation

Remark: The order is essential for the resulting rotation matrix!

- Matrix multiplication is not commutative:

$$\boldsymbol{R}_x \boldsymbol{R}_y \boldsymbol{R}_z \neq \boldsymbol{R}_y \boldsymbol{R}_x \boldsymbol{R}_z,$$

- only for small rotation angles commutativity is approximately true.

Gimbal Lock (Shoemaker):

When object points are first rotated around the x-axis by $-\frac{\pi}{2}$, then the y- and the z-axis are aligned and the rotations around the y- and z-axis, respectively, can no longer be distinguished.

- Conversion between angles and matrices is computationally not very robust.
- This representation of rotations is not unique and there exist singularities.

Axis-Angle Representation

Before we introduce the commonly used axis-angle representation of rotations, we briefly consider the linearity of the cross-product.

For 3-D vectors we have:

$$\boldsymbol{u} \times \boldsymbol{v} = [\boldsymbol{u}]_\times \boldsymbol{v},$$

where

$$[\boldsymbol{u}]_\times = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

The matrix $[\boldsymbol{u}]_\times$ is called the ***skew matrix*** of $\boldsymbol{u}$.

Axis-Angle Representation

Alternatively to the Euler representation, an arbitrary rotation $\boldsymbol{R}$ can be represented as a single rotation by the angle Θ with respect to a single axis defined by a unit vector $\boldsymbol{u}$.

Given: rotation axis $\boldsymbol{u} = (u_1, u_2, u_3)^{\mathsf{T}}$ and angle Θ

Compute: rotation matrix $\boldsymbol{R}$

Solution:

$$\boldsymbol{R} = f(\boldsymbol{u}, \Theta) = \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}} + \left(\boldsymbol{I}_3 - \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}}\right) \cdot \cos\Theta + [\boldsymbol{u}]_{\times} \sin\Theta,$$

or in components:

$$\boldsymbol{R} = \begin{pmatrix} u_1^2 + (1 - u_1^2)\cos\Theta & u_1 u_2(1 - \cos\Theta) - u_3 \sin\Theta & u_1 u_3(1 - \cos\Theta) + u_2 \sin\Theta \\ u_1 u_2(1 - \cos\Theta) + u_3 \sin\Theta & u_2^2 + (1 - u_2^2)\cos\Theta & u_2 u_3(1 - \cos\Theta) - u_1 \sin\Theta \\ u_1 u_3(1 - \cos\Theta) - u_2 \sin\Theta & u_2 u_3(1 - \cos\Theta) + u_1 \sin\Theta & u_3^2 + (1 - u_3^2)\cos\Theta \end{pmatrix}$$

Axis-Angle Representation

We construct three pairwise orthogonal vectors:

$$\boldsymbol{u} \times \boldsymbol{v}, \quad (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u} \quad \text{and} \quad \boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}.$$

We will subsequently use these vectors as basis.

$\rightarrow$ The rotated vector $\boldsymbol{Rv}$ can be written as a linear combination of $\boldsymbol{u} \times \boldsymbol{v}$, $(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}$ and $\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}$.

Axis-Angle Representation

Formula of Rodrigues:

$$\boldsymbol{R} = f\left(\boldsymbol{u}, \Theta\right) = \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}} + \left(\boldsymbol{I}_3 - \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}}\right) \cdot \cos\Theta + [\boldsymbol{u}]_{\times} \sin\Theta,$$

i. e., if axis and angle are known, the computation of $\boldsymbol{R}$ is possible.
We require that

$$\boldsymbol{u} = \left(u_1, u_2, u_3\right)^{\mathsf{T}} \quad \text{with} \quad \|\boldsymbol{u}\|_2 = 1$$

Note: This description still has three degrees of freedom, two for the direction of the rotation axis and one for the angle.

Axis-Angle Representation

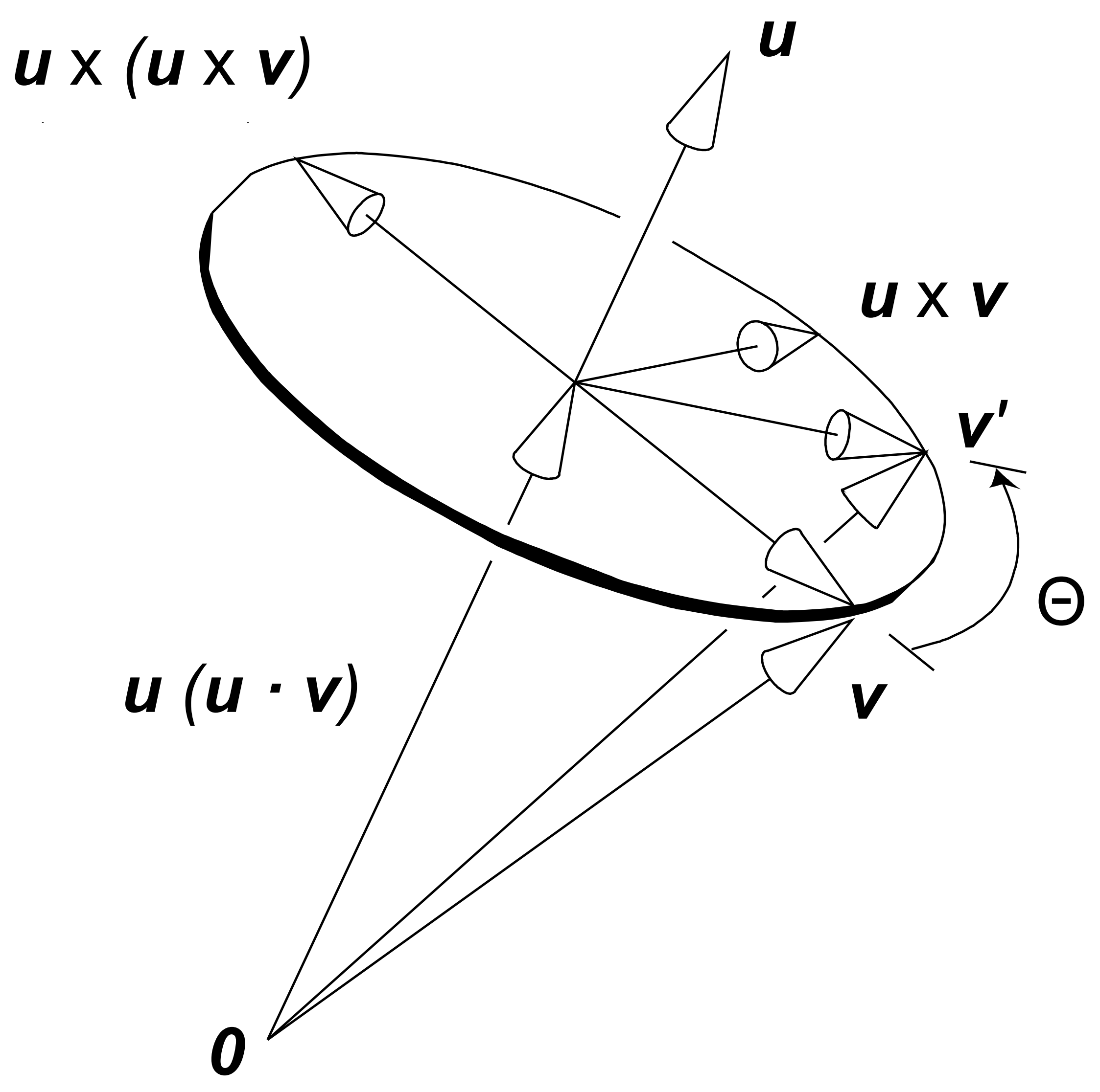

Figure 1: Formula of Rodrigues: schematic of the particular base vectors

Axis-Angle Representation

Question: How can we get Θ and $\boldsymbol{u}$ from $\boldsymbol{R}$?

Use the eigenvalues and eigenvectors of $\boldsymbol{R}$:

- the eigenvalues of $\boldsymbol{R}$ are
  - all equal to 1 (if $\boldsymbol{R} = \boldsymbol{I}_3$), or
  - 1, $\cos\Theta + i\sin\Theta$, $\cos\Theta - i\sin\Theta$,
- the eigenvector for eigenvalue 1 of $\boldsymbol{R}$ is collinear with $\boldsymbol{u}$.
- Θ can also be obtained via $\text{trace}(\boldsymbol{R}) = 1 + 2\cos(\Theta)$.

Topics

Take Home Messages

- Rotation can be represented by using Euler angles, however this approach is not very robust.
- Another representation is describing an arbitrary rotation by rotation around a single axis and a certain angle, which is the essence of the Rodrigues formula.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. "Medical Image Registration". In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
- J. B.Antoine Maintz and Max A. Viergever. "A Survey of Medical Image Registration". In: *Medical Image Analysis* 2.1 (1998), pp. 1–36. DOI: 10.1016/S1361-8415(01)80026-8
- L. G. Brown. "A Survey of Image Registration Techniques". In: *ACM Computing Surveys* 24.4 (Dec. 1992), pp. 325–376. DOI: 10.1145/146370.146374
- Josien P. W. Pluim, J. B. Antoine Maintz, and Max A. Viergever. "Mutual-Information-Based Registration of Medical Images: A Survey". In: *IEEE Transactions on Medical Imaging* 22.8 (Aug. 2003), pp. 986–1004. DOI: 10.1109/TMI.2003.815867

A paper that inspired all the sections on complex numbers, quaternions, and dual quaternions:

Konstantinos Daniilidis. "Hand-Eye Calibration Using Dual Quaternions". In: *The International Journal of Robotics Research* 18.3 (Mar. 1999), pp. 286–298. DOI: 10.1177/02783649922066213

Further Readings – Part 2

Non-parametric mappings for image registration:

- Nonlinear registration methods applied to DSA can be found in Erik Meijering's papers.
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001
- Many of Jan Modersitzki's and Bernd Fischer's papers on image registration can be found in the publication list of the Institute of Mathematics and Image Computing (Lübeck).
- The group of Martin Rumpf also published on non-parametric image registration. Details on their work can be found on the institute's webpage.