Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Gradient Descent Algorithms

Online Course – Unit 58

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

virtuelle hochschule bayern

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Gradient Descent Algorithms

Summary

Take Home Messages

Further Readings

Gradient Descent Algorithms

Idea:

- Formulate the reconstruction problem as an optimization problem.
- Find the optimum via a peak condition.

This enables the use of various methods that are common in optimization like:

- fast descent using **conjugate gradients**,
- or **regularization**.

Gradient Descent Algorithms: Example

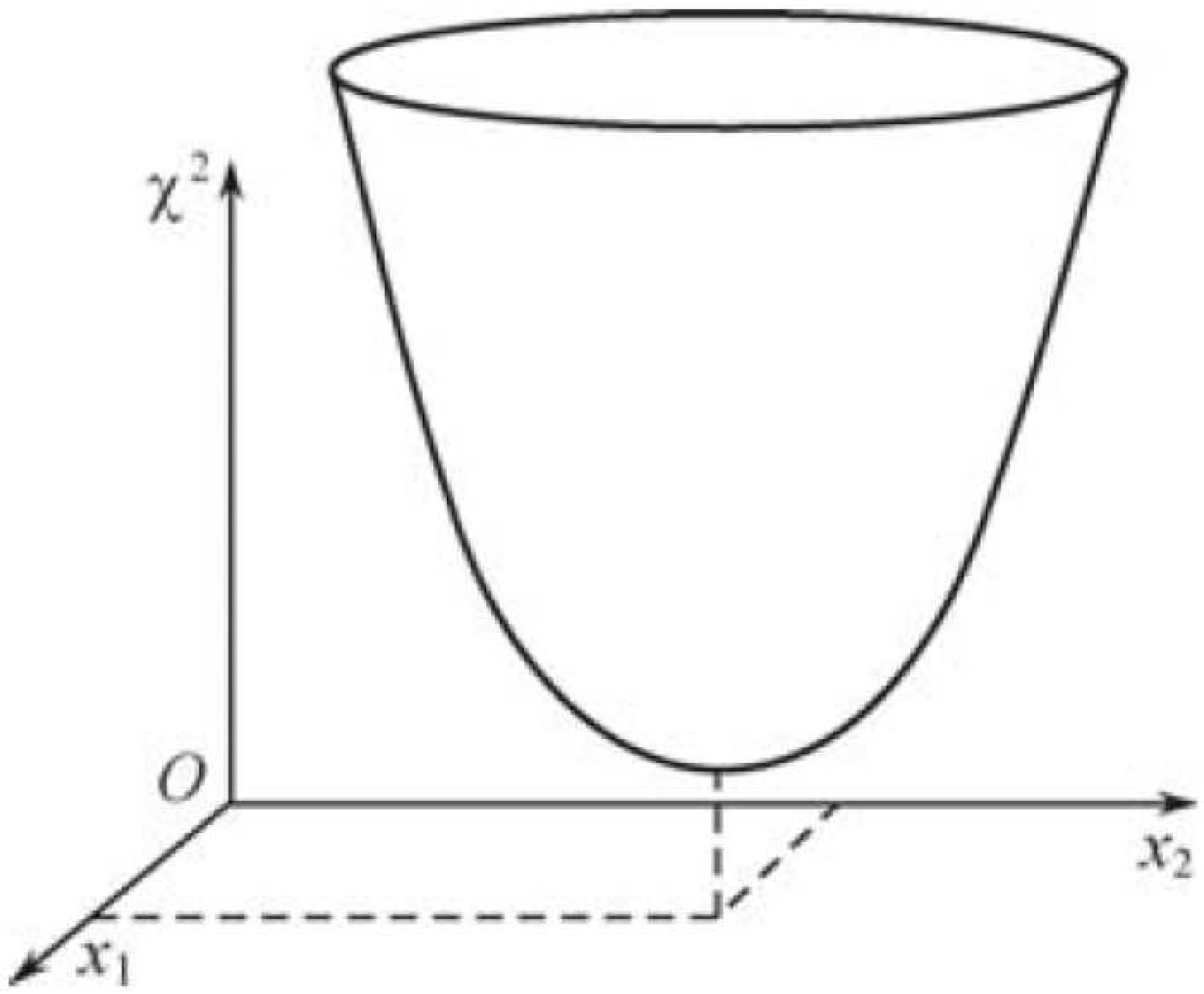

Figure 1: Finding the optimal value as an extreme point of the objective function (Zeng, 2009)

Gradient Descent Algorithms: Iterative Scheme

Let the objective function be:

$$\chi(\boldsymbol{X}) = \|\boldsymbol{AX} - \boldsymbol{P}\| = (\boldsymbol{AX} - \boldsymbol{P})^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}).$$

Then the gradient is found as:

$$\nabla\chi(\boldsymbol{X}) = 2\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}).$$

Using the peak condition $\nabla\chi(\boldsymbol{X}) = 0$ immediately yields:

$$\begin{aligned} \boldsymbol{A}^{\mathsf{T}}(\boldsymbol{AX} - \boldsymbol{P}) &= 0, \\ \boldsymbol{A}^{\mathsf{T}}\boldsymbol{AX} &= \boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}, \\ \boldsymbol{X} &= (\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}. \end{aligned}$$

Gradient Descent Algorithms: Iterative Scheme

Instead of the analytic solution, we can formulate an iterative procedure:

$$\boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} + \lambda \Delta,$$

with an update Δ and a step scale λ.

In each step we want to go one step towards the minimum, i. e., the update is chosen as the opposite gradient direction.

Therefore, we set $\Delta = -\nabla \chi(\boldsymbol{X})$:

$$\begin{aligned} & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} - \lambda \left(2\boldsymbol{A}^{\mathsf{T}} (\boldsymbol{A}\boldsymbol{X} - \boldsymbol{P}) \right), \\ \Leftrightarrow \quad & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} - \lambda \left(\boldsymbol{A}^{\mathsf{T}} (\boldsymbol{A}\boldsymbol{X} - \boldsymbol{P}) \right), \\ \Leftrightarrow \quad & \boldsymbol{X}^{k+1} = \boldsymbol{X}^{k} + \lambda \left(\boldsymbol{A}^{\mathsf{T}} (\boldsymbol{P} - \boldsymbol{A}\boldsymbol{X}) \right). \end{aligned}$$

Gradient Descent Algorithms: Iterative Scheme

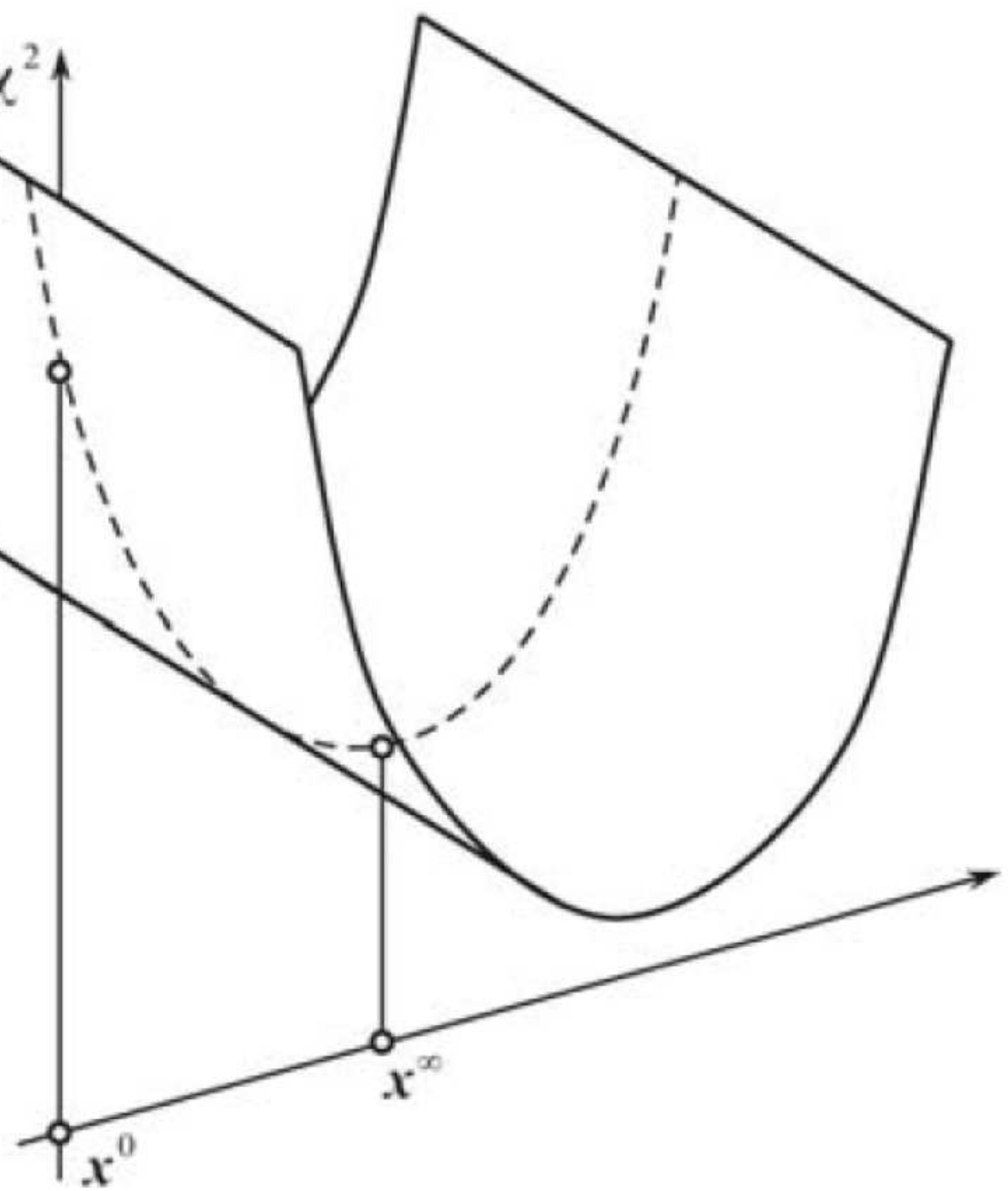

Figure 2: The solution $\boldsymbol{X}^\infty$ can depend on the initialization $\boldsymbol{X}^0$ (Zeng, 2009).

Topics

Gradient Descent Algorithms

Summary

Take Home Messages

Further Readings

Take Home Messages

- Gradient descent algorithms are iterative methods for which the iteration update is dependent on the gradient of the objective function.
- If the objective function is not convex, the algorithm might not find the global minimum/maximum.
- Even if the objective function is convex, but not strictly convex, the found minimum/maximum depends on the initialization.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. "Adaptive Iterative Reconstruction". In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953