# Medical Image Processing for Diagnostic Applications

## Parallel Beam – Ram-Lak Filter

Online Course – Unit 33

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# Topics

## How to Implement a Parallel Beam Algorithm - Part 1

### Example Projection

### Implementation Scheme

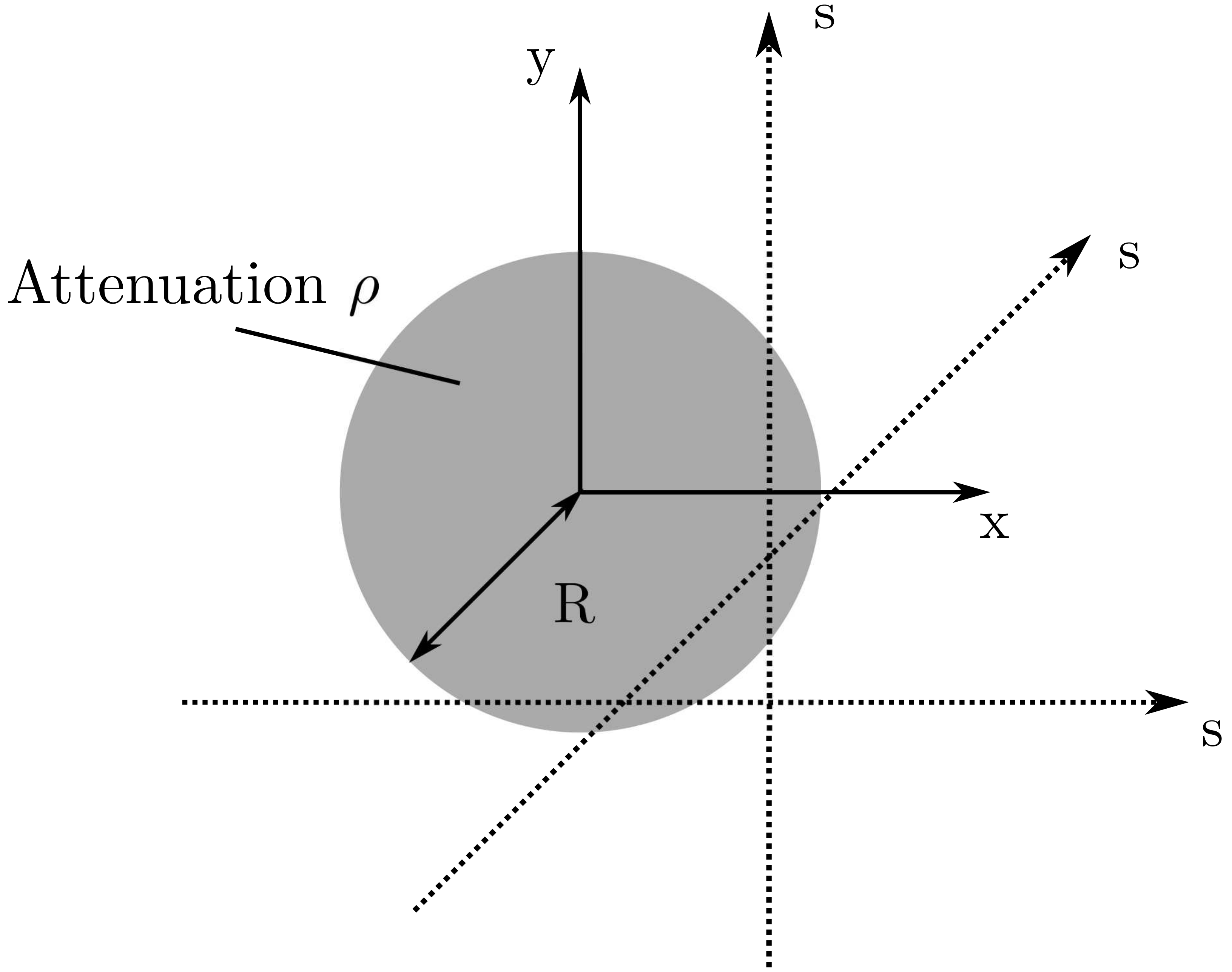
### Discrete Spatial Form of the Ramp Filter

## Summary

### Take Home Messages

### Further Readings

# Example: Homogeneous Cylinder (My First Phantom)




Disc of radius $R$ is in the coordinate center $\rightarrow$ projection is the same in all views:

$$p(s) = \begin{cases} 2\rho\sqrt{R^2 - s^2} & s \leq R, \\ 0 & s > R. \end{cases}$$

(The dotted lines indicate rays from different projection angles.)

# Example: Homogeneous Cylinder

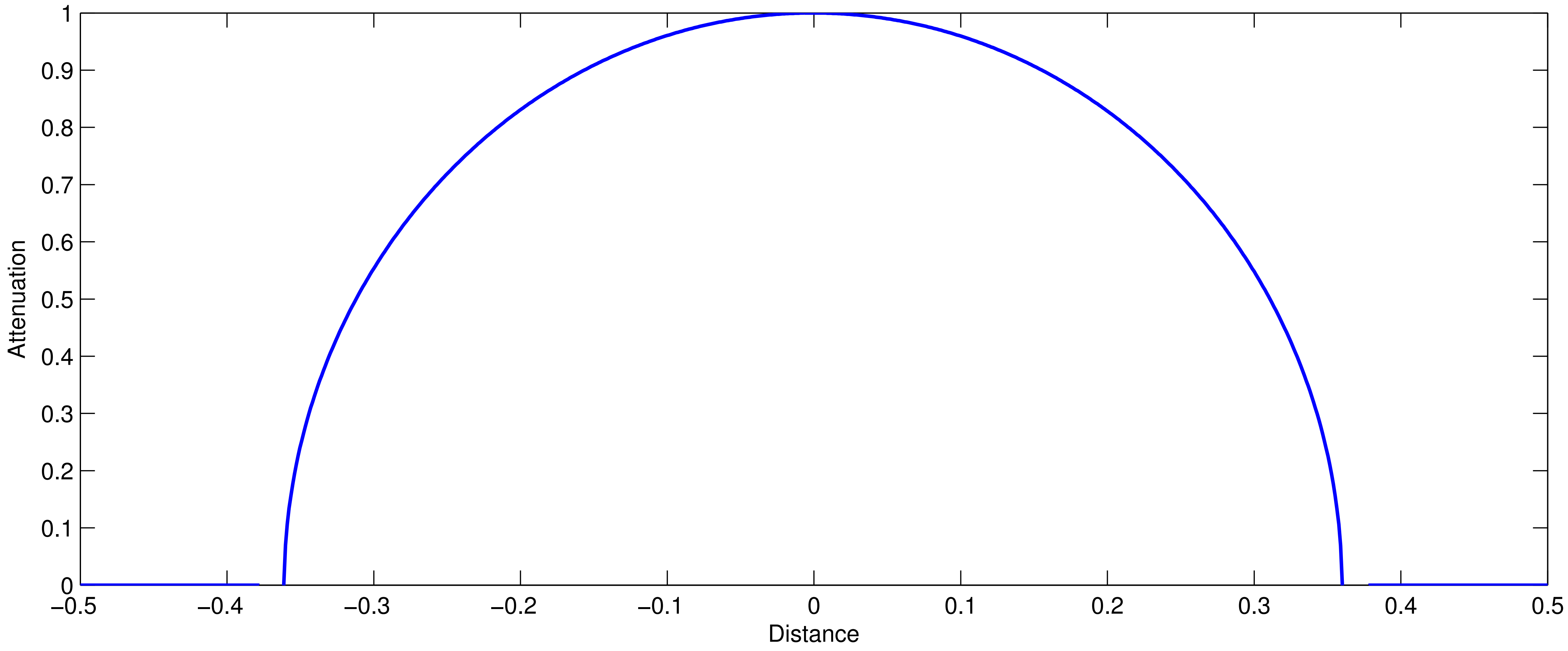



Figure 1: 1-D projection profile of the cylinder object in 2-D

# Filtered Backprojection: Implementation Scheme

- Apply filter on the detector row:

$$q(s,\theta) = h(s) * p(s,\theta),$$

$$h(s) = \int_{-\infty}^{\infty} |\omega| e^{2\pi i \omega s} \mathrm{d}\omega.$$

- Backproject $q(s,\theta)$:

$$f(x,y) = \int_{0}^{\pi} q(s,\theta)\big|_{s = x\cos\theta + y\sin\theta} \mathrm{d}\theta.$$

# Discrete Spatial Form of the Ramp Filter

**Problem:** Find the inverse Fourier transform of $|\omega|$.

Given a detector spacing $\tau$, we know from the Nyquist-Shannon sampling theorem the maximum frequency that can be represented by the DFT:

$$2B = \frac{1}{\tau}.$$

Therefore, we set the cut-off frequency of the ramp filter at $\omega = B$.

So we want to determine

$$h(s) = \int_{-B}^{B} |\omega| e^{2\pi i \omega s} \mathrm{d}\omega = \int_{-\infty}^{\infty} |\omega| \operatorname{rect}\left(\frac{\omega}{2B}\right) e^{2\pi i \omega s} \mathrm{d}\omega,$$

where

$$\operatorname{rect}(t) = \begin{cases} 1, & \text{if } |t| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

## Discrete Spatial Form of the Ramp Filter

With the rect-function we can also rewrite $|\omega|$:

$$|\omega| = B - \mathrm{rect}\left(\frac{\omega}{B}\right) * \mathrm{rect}\left(\frac{\omega}{B}\right).$$

The convolution of both rect-functions yields a triangular shaped function with support on $[-B, B]$ and its maximum $B$ at zero.

We now have:

$$\begin{aligned} h(s) &= \mathrm{FT}^{-1}\left(\left(B - \mathrm{rect}\left(\frac{\omega}{B}\right) * \mathrm{rect}\left(\frac{\omega}{B}\right)\right)\mathrm{rect}\left(\frac{\omega}{2B}\right)\right) \\ &= \mathrm{FT}^{-1}\left(B\,\mathrm{rect}\left(\frac{\omega}{2B}\right)\right) - \mathrm{FT}^{-1}\left(\underbrace{\left(\mathrm{rect}\left(\frac{\omega}{B}\right) * \mathrm{rect}\left(\frac{\omega}{B}\right)\right)}_{\text{support on } [-B,B]}\underbrace{\mathrm{rect}\left(\frac{\omega}{2B}\right)}_{=1 \text{ on } [-B,B]}\right) \\ &= \mathrm{FT}^{-1}\left(B\,\mathrm{rect}\left(\frac{\omega}{2B}\right)\right) - \mathrm{FT}^{-1}\left(\mathrm{rect}\left(\frac{\omega}{B}\right)\right) \cdot \mathrm{FT}^{-1}\left(\mathrm{rect}\left(\frac{\omega}{B}\right)\right). \end{aligned}$$

# Discrete Spatial Form of the Ramp Filter

The Fourier transform of the $\text{rect}$-function is a $\text{sinc}$-function, and using the appropriate scaling properties of the Fourier transform, we get:

$$
\begin{aligned}
h(s) &= 2B^2 \,\text{sinc}\left(2Bs\right) - B^2 \text{sinc}^2\left(Bs\right) \\
&= \frac{1}{2\tau^2} \frac{\sin\left(\frac{\pi s}{\tau}\right)}{\frac{\pi s}{\tau}} - \frac{1}{4\tau^2} \left( \frac{\sin\left(\frac{\pi s}{2\tau}\right)}{\frac{\pi s}{2\tau}} \right)^2 .
\end{aligned}
$$

The detector is sampled by $s = n\tau$, $n \in \mathbb{Z}$, hence we find the discrete filter in the spatial domain:

$$
h(n\tau) = \begin{cases} \frac{1}{4\tau^2} & n = 0, \\ 0 & n \text{ even}, \\ -\frac{1}{\pi^2 (n\tau)^2} & n \text{ odd}, \end{cases}
$$

also known as the **“Ramachandran-Lakshminarayanan” convolver** or shortly the **“Ram-Lak” filter**.

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# Take Home Messages

- In this unit we derived the discrete spatial filter version of the ramp filter. It is also called the Ram-Lak filter.
- By design, the Ram-Lak filter “fits” optimally on the detector grid which enhances the accuracy of the reconstruction algorithm (see next unit).

# Further Readings

The original Ram-Lak article is:

G. N. Ramachandran and A. V. Lakshminarayanan. “Three-dimensional Reconstruction from Radiographs and Electron Micrographs: Application of Convolutions instead of Fourier Transforms”. In: *Proceedings of the National Academy of Sciences of the United States of America* 68.9 (Sept. 1971), pp. 2236–2240

The derivation shown in this unit is based on a document by Martin Berger.

The concise reconstruction book from ‘Larry ’Zeng:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Another mathematical examination of filtered backprojection can be found in

Thorsten Buzug. *Computed Tomography: From Photon Statistics to Modern Cone-Beam CT*. Springer Berlin Heidelberg, 2008. DOI: 10.1007/978-3-540-39408-2