Medical Image Processing for Diagnostic Applications

Image Registration in 2-D

Online Course – Unit 64

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Topics

Image Registration

- Motivation
- Point Based 2-D/2-D Rigid Registration
- Complex Numbers and Rotations

Summary

- Take Home Messages
- Further Readings

Terminology

Definition

Image registration is the process of transforming two or more different images into one common coordinate system. The registration of volumes is also subsumed by the term image registration.

Dependent on the properties of the transform we have two major classes for image registration:

- The term ***rigid registration*** subsumes the process of computing a rigid transform for registration.
- The term ***non-rigid registration*** includes all the methods of deforming the different images such that they can be represented in one common coordinate system.

Fiducial Markers for Image Registration

Especially in therapeutic radiology the precise mapping of all available image information is required for the therapy of tumors.

Rigid registration methods are mostly applied to images of the skull. In this particular application physicians quite often make use of *fiducial markers* that are fixed to the patient.

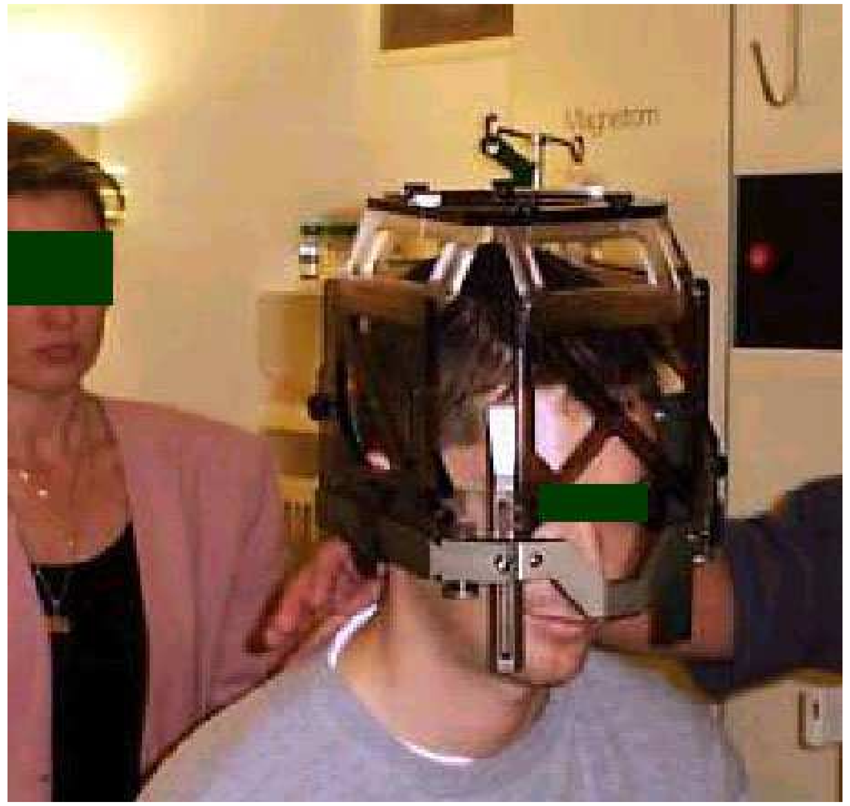

Figure 1: Fiducial markers used for Gamma Knife® treatment

Navigation System for Surgery

Further applications of markers in medical imaging:

Figure 2: Brainlab system for brain surgery

Point Based 2-D/2-D Registration

Assume a set of corresponding 2-D points in two images:

$$C = \{(\boldsymbol{p}_k, \boldsymbol{q}_k) \mid k = 1, 2, \ldots, N\},$$

where $\boldsymbol{p}_k, \boldsymbol{q}_k \in \mathbb{R}^2$ represent the k-th pair of corresponding image points.

Problem: Compute the transform that maps the $\boldsymbol{q}_k$'s to the $\boldsymbol{p}_k$'s.

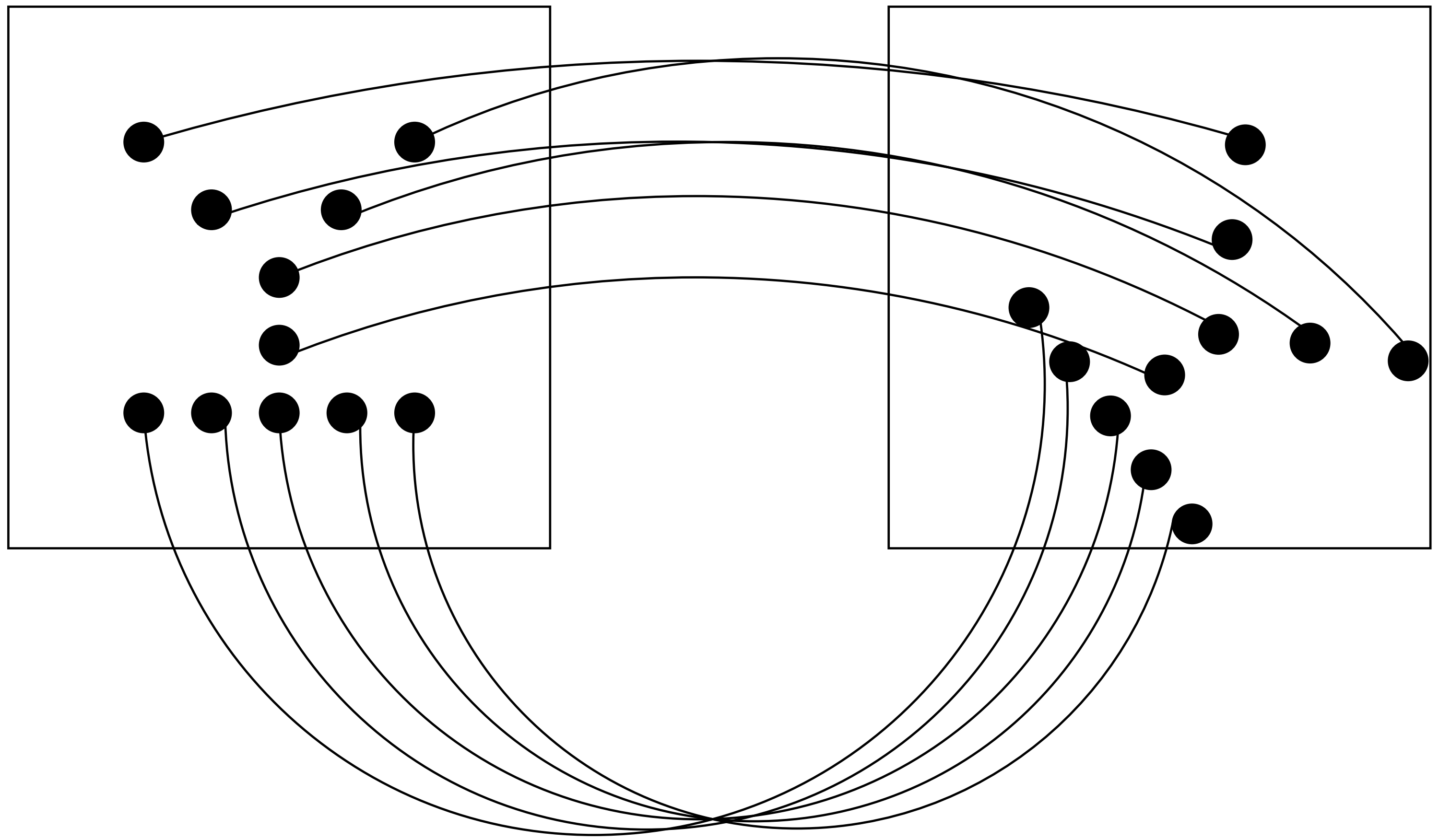

Figure 3: Corresponding 2-D point features

Optimization Problem

Using a rigid transform defined by rotation

$$\boldsymbol{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \varphi \in [0, 2\pi),$$

and translation $\boldsymbol{t} = (t_1, t_2)^\mathsf{T}$, $t_1, t_2 \in \mathbb{R}$, we want:

$$\boldsymbol{p}_k = \boldsymbol{R}\boldsymbol{q}_k + \boldsymbol{t}.$$

This leads to the optimization problem for 2-D rigid registration:

$$\underset{\varphi, t_1, t_2}{\arg\min} \sum_{k=1}^{N} \|\boldsymbol{p}_k - \boldsymbol{R}\boldsymbol{q}_k - \boldsymbol{t}\|^2 .$$

Conclusion: Rigid image registration turns out to be a nonlinear optimization problem.

Properties of Rotations

Some important properties of rotation matrices $\boldsymbol{R} \in \mathbb{R}^{n \times n}$:

- The columns of a rotation matrix are images of the base vectors of the original coordinate system (valid for all linear mappings!).
- Every rotation matrix is orthogonal:

  $$\boldsymbol{R}^{\mathsf{T}} = \boldsymbol{R}^{-1},$$

  and thus we have:

  $$\boldsymbol{R}^{\mathsf{T}}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^{\mathsf{T}} = \boldsymbol{I}_n.$$
- Rotation preserves the orientation (left/right-handedness) of the coordinate system.
- We have $\det \boldsymbol{R} = 1$.
- In 3-D, the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ defines the rotation axis.

Complex Numbers and Rotations

Complex numbers define a point in 2-D:

$$z = a + ib$$

where a is the real part and b the imaginary part of the complex number z.

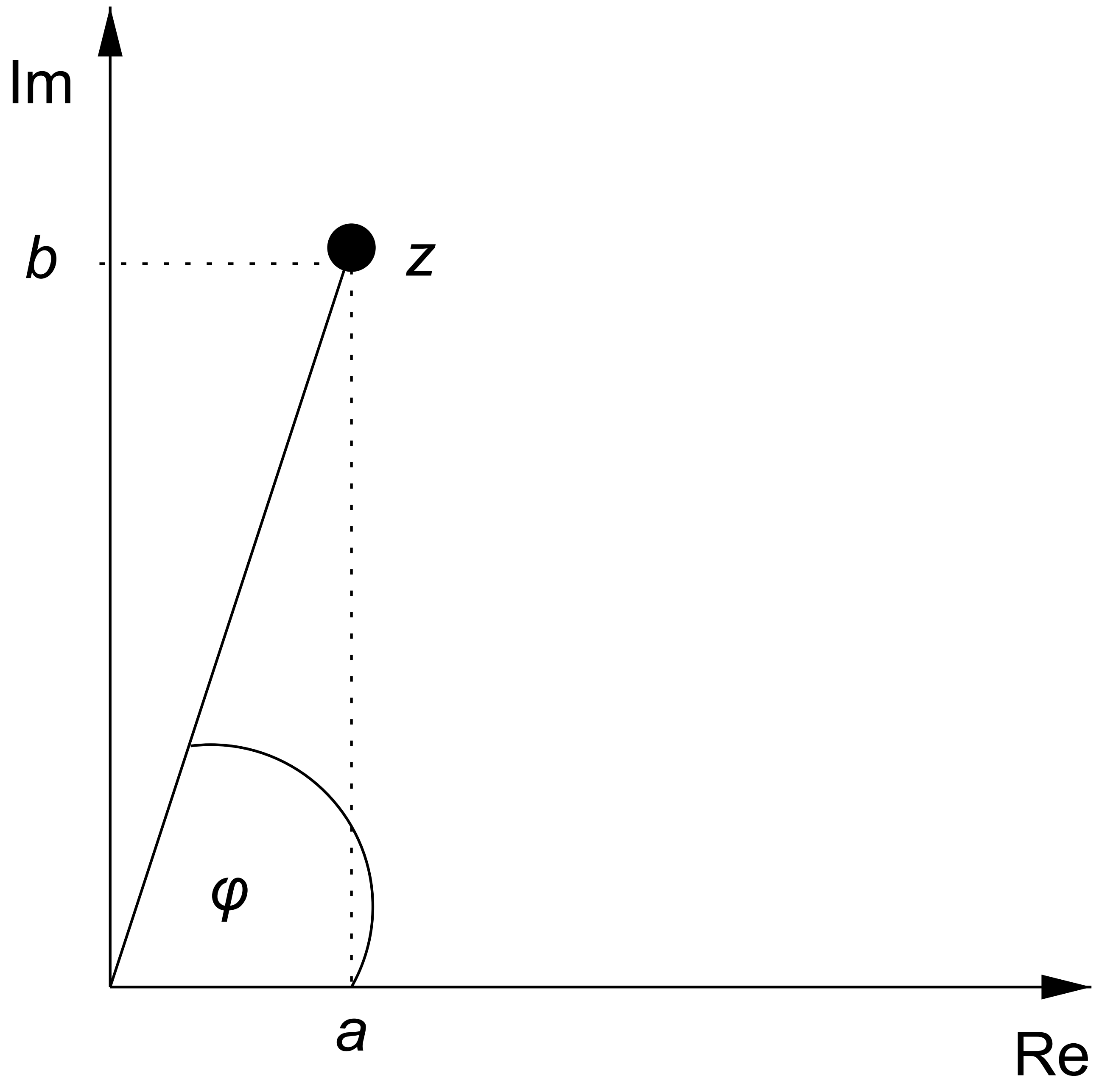

Figure 4: Geometric representation of complex numbers

Complex Numbers and Rotations

Proposition: Multiplication of complex numbers defines a 2-D scaling and rotation.

Multiplication of complex numbers is defined by:

$$z = z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$$

Complex numbers can be represented using Euler's notation:

$$z = |z| e^{i\varphi},$$

where

- $|z| = \sqrt{a^2 + b^2}$ is the length of the complex number, and
- $\varphi = \mathtt{atan2}(b, a)$ the angle w. r. t. the real axis.

Complex Numbers and Rotations

Multiplication of complex numbers using Euler notation:

$$z_1 \cdot z_2 = |z_1| e^{i\varphi_1} \cdot |z_2| e^{i\varphi_2} = |z_1||z_2| e^{i(\varphi_1+\varphi_2)}$$

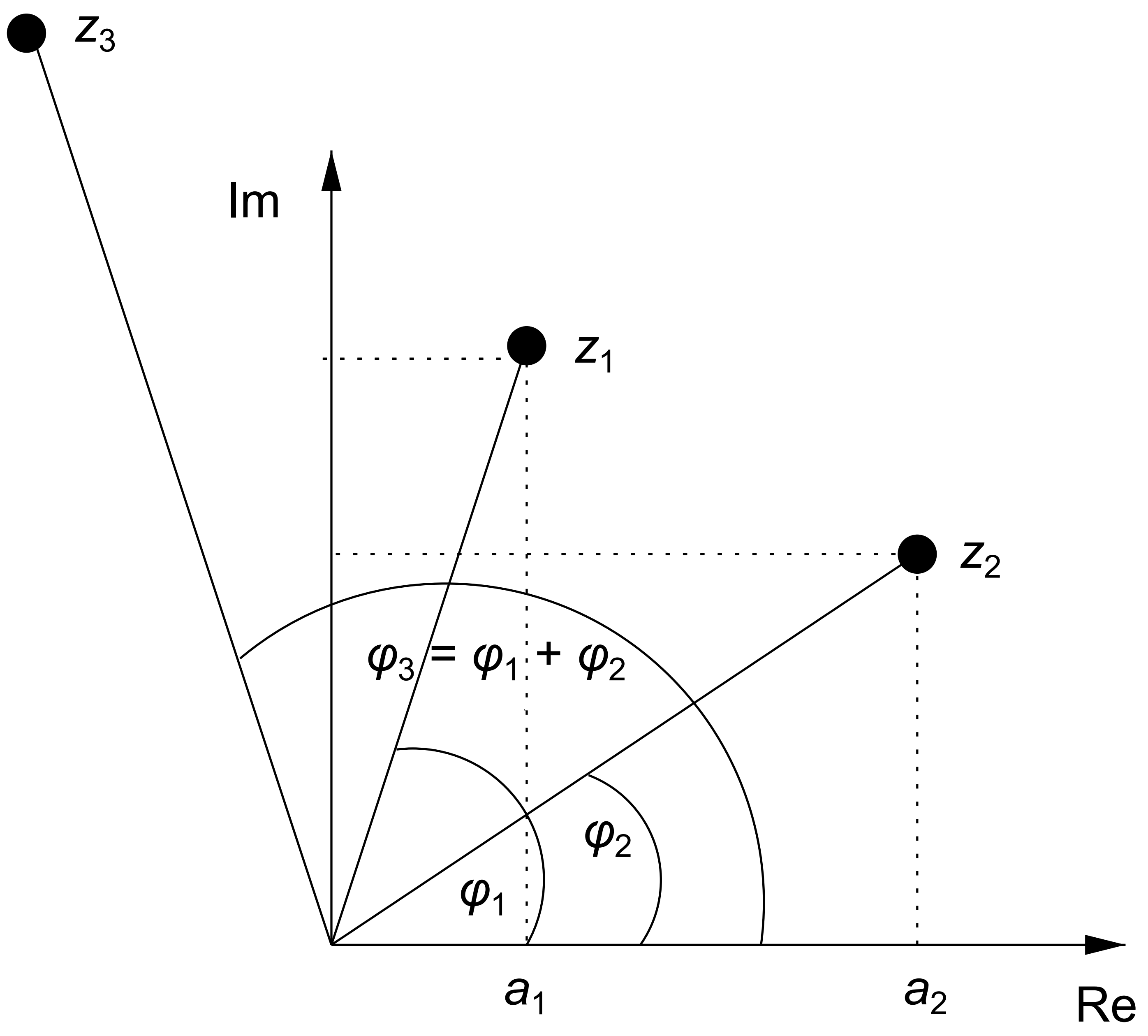

Figure 5: Geometric representation of complex numbers

Complex Numbers and Rotations

Conclusion: In terms of complex numbers we get the equations:

$$p_{k,1} + i p_{k,2} = (r_1 + i r_2)(q_{k,1} + i q_{k,2}) + t_1 + i t_2 \quad \text{for } k = 1, 2, \ldots, N.$$

This equation can be rewritten in two equations both linear in the components of the complex numbers corresponding to $\boldsymbol{R}$ and $\boldsymbol{t}$:

- for the real part we get the equation

$$p_{k,1} = r_1 q_{k,1} - r_2 q_{k,2} + t_1 = (q_{k,1}, -q_{k,2}, 1, 0) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix},$$

- the imaginary parts results in the equation

$$p_{k,2} = r_1 q_{k,2} + r_2 q_{k,1} + t_2 = (q_{k,2}, q_{k,1}, 0, 1) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

Complex Numbers and Rotations

The final system of linear equations is:

$$
\boldsymbol{Ax} = \begin{pmatrix} q_{1,1} & -q_{1,2} & 1 & 0 \\ q_{2,1} & -q_{2,2} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ q_{N,1} & -q_{N,2} & 1 & 0 \\ q_{1,2} & q_{1,1} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ q_{N,2} & q_{N,1} & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} p_{1,1} \\ p_{2,1} \\ \vdots \\ p_{N,1} \\ p_{1,2} \\ \vdots \\ p_{N,2} \end{pmatrix} = \boldsymbol{b}.
$$

Complex Numbers and Rotations

Remarks:

- In this algorithm we compute rotation and translation simultaneously.
- Using SVD we compute the pseudoinverse of $\boldsymbol{A}$ and so get both rotation and translation.
- 2-D/2-D image registration using point correspondences results in a linear problem.
- Rotation matrices imply the constraint that $r_1^2 + r_2^2 = 1$. This can be enforced by a proper scaling of the solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

Question: Can we *lift* the complex numbers to characterize 3-D rotations?

Topics

Image Registration

- Motivation
- Point Based 2-D/2-D Rigid Registration
- Complex Numbers and Rotations

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- Point based image registration describes the attempt to determine a transform that maps corresponding image points.
- In rigid registration the transform model consists of rotation and translation.
- For 2-D/2-D image registration we found a linear method to compute both rotation and translation.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. "Medical Image Registration". In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
- J. B.Antoine Maintz and Max A. Viergever. "A Survey of Medical Image Registration". In: *Medical Image Analysis* 2.1 (1998), pp. 1–36. DOI: 10.1016/S1361-8415(01)80026-8
- L. G. Brown. "A Survey of Image Registration Techniques". In: *ACM Computing Surveys* 24.4 (Dec. 1992), pp. 325–376. DOI: 10.1145/146370.146374
- Josien P. W. Pluim, J. B. Antoine Maintz, and Max A. Viergever. "Mutual-Information-Based Registration of Medical Images: A Survey". In: *IEEE Transactions on Medical Imaging* 22.8 (Aug. 2003), pp. 986–1004. DOI: 10.1109/TMI.2003.815867

A paper that inspired all the sections on complex numbers, quaternions, and dual quaternions:

Konstantinos Daniilidis. "Hand-Eye Calibration Using Dual Quaternions". In: *The International Journal of Robotics Research* 18.3 (Mar. 1999), pp. 286–298. DOI: 10.1177/02783649922066213

Further Readings – Part 2

Non-parametric mappings for image registration:

- Nonlinear registration methods applied to DSA can be found in Erik Meijering's papers.
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001
- Many of Jan Modersitzki's and Bernd Fischer's papers on image registration can be found in the publication list of the Institute of Mathematics and Image Computing (Lübeck).
- The group of Martin Rumpf also published on non-parametric image registration. Details on their work can be found on the institute's webpage.

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3-D Rotations – Euler Angles and Rodrigues Formula

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Topics

Representations of 3-D Rotations
- Overview
- Euler Angles
- Axis-Angle Representation

Summary
- Take Home Messages
- Further Readings

Rotations in 3-D

Various representations for rotations:

- **Euler angles**
- **Axis-angle representation**
- Quaternions

Euler Angle Representation

- A 3-D rotation can be expressed by a 3×3 rotation matrix.
- An arbitrary rotation can be composed of 3 rotations around the axes of the coordinate system using the angles φ_x (roll), φ_y (pitch), φ_z (yaw).

$$
\boldsymbol{R} = \boldsymbol{R}_x \boldsymbol{R}_y \boldsymbol{R}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_x & -\sin\varphi_x \\ 0 & \sin\varphi_x & \cos\varphi_x \end{pmatrix} \begin{pmatrix} \cos\varphi_y & 0 & \sin\varphi_y \\ 0 & 1 & 0 \\ -\sin\varphi_y & 0 & \cos\varphi_y \end{pmatrix} \begin{pmatrix} \cos\varphi_z & -\sin\varphi_z & 0 \\ \sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos\varphi_y \cos\varphi_z & -\cos\varphi_y \sin\varphi_z & \sin\varphi_y \\ \sin\varphi_x \sin\varphi_y \cos\varphi_z + \cos\varphi_x \sin\varphi_z & -\sin\varphi_x \sin\varphi_y \sin\varphi_z + \cos\varphi_x \cos\varphi_z & -\sin\varphi_x \cos\varphi_y \\ -\cos\varphi_x \sin\varphi_y \cos\varphi_z + \sin\varphi_x \sin\varphi_z & \cos\varphi_x \sin\varphi_y \sin\varphi_z + \sin\varphi_x \cos\varphi_z & \cos\varphi_x \cos\varphi_y \end{pmatrix}
$$

Euler Angle Representation

Remark: The order is essential for the resulting rotation matrix!

- Matrix multiplication is not commutative:

$$\boldsymbol{R}_x \boldsymbol{R}_y \boldsymbol{R}_z \neq \boldsymbol{R}_y \boldsymbol{R}_x \boldsymbol{R}_z,$$

- only for small rotation angles commutativity is approximately true.

Gimbal Lock (Shoemaker):

When object points are first rotated around the x-axis by $-\frac{\pi}{2}$, then the y- and the z-axis are aligned and the rotations around the y- and z-axis, respectively, can no longer be distinguished.

- Conversion between angles and matrices is computationally not very robust.
- This representation of rotations is not unique and there exist singularities.

Axis-Angle Representation

Before we introduce the commonly used axis-angle representation of rotations, we briefly consider the linearity of the cross-product.

For 3-D vectors we have:

$$\boldsymbol{u} \times \boldsymbol{v} = [\boldsymbol{u}]_{\times} \boldsymbol{v},$$

where

$$[\boldsymbol{u}]_{\times} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

The matrix $[\boldsymbol{u}]_{\times}$ is called the ***skew matrix*** of $\boldsymbol{u}$.

Axis-Angle Representation

Alternatively to the Euler representation, an arbitrary rotation $\boldsymbol{R}$ can be represented as a single rotation by the angle Θ with respect to a single axis defined by a unit vector $\boldsymbol{u}$.

Given: rotation axis $\boldsymbol{u} = (u_1, u_2, u_3)^\mathsf{T}$ and angle Θ

Compute: rotation matrix $\boldsymbol{R}$

Solution:

$$\boldsymbol{R} = f(\boldsymbol{u}, \Theta) = \boldsymbol{u}\boldsymbol{u}^\mathsf{T} + \left(\boldsymbol{I}_3 - \boldsymbol{u}\boldsymbol{u}^\mathsf{T}\right) \cdot \cos\Theta + [\boldsymbol{u}]_\times \sin\Theta,$$

or in components:

$$\boldsymbol{R} = \begin{pmatrix} u_1^2 + (1 - u_1^2)\cos\Theta & u_1 u_2(1 - \cos\Theta) - u_3 \sin\Theta & u_1 u_3(1 - \cos\Theta) + u_2 \sin\Theta \\ u_1 u_2(1 - \cos\Theta) + u_3 \sin\Theta & u_2^2 + (1 - u_2^2)\cos\Theta & u_2 u_3(1 - \cos\Theta) - u_1 \sin\Theta \\ u_1 u_3(1 - \cos\Theta) - u_2 \sin\Theta & u_2 u_3(1 - \cos\Theta) + u_1 \sin\Theta & u_3^2 + (1 - u_3^2)\cos\Theta \end{pmatrix}$$

Axis-Angle Representation

We construct three pairwise orthogonal vectors:

$$\boldsymbol{u} \times \boldsymbol{v}, \quad (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u} \quad \text{and} \quad \boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}.$$

We will subsequently use these vectors as basis.

$\rightarrow$ The rotated vector $\boldsymbol{Rv}$ can be written as a linear combination of $\boldsymbol{u} \times \boldsymbol{v}$, $(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}$ and $\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}$.

Axis-Angle Representation

Formula of Rodrigues:

$$\boldsymbol{R} = f(\boldsymbol{u}, \Theta) = \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}} + \left(\boldsymbol{I}_3 - \boldsymbol{u}\boldsymbol{u}^{\mathsf{T}}\right) \cdot \cos\Theta + [\boldsymbol{u}]_{\times} \sin\Theta,$$

i. e., if axis and angle are known, the computation of $\boldsymbol{R}$ is possible.
We require that

$$\boldsymbol{u} = (u_1, u_2, u_3)^{\mathsf{T}} \quad \text{with} \quad \|\boldsymbol{u}\|_2 = 1$$

Note: This description still has three degrees of freedom, two for the direction of the rotation axis and one for the angle.

Axis-Angle Representation

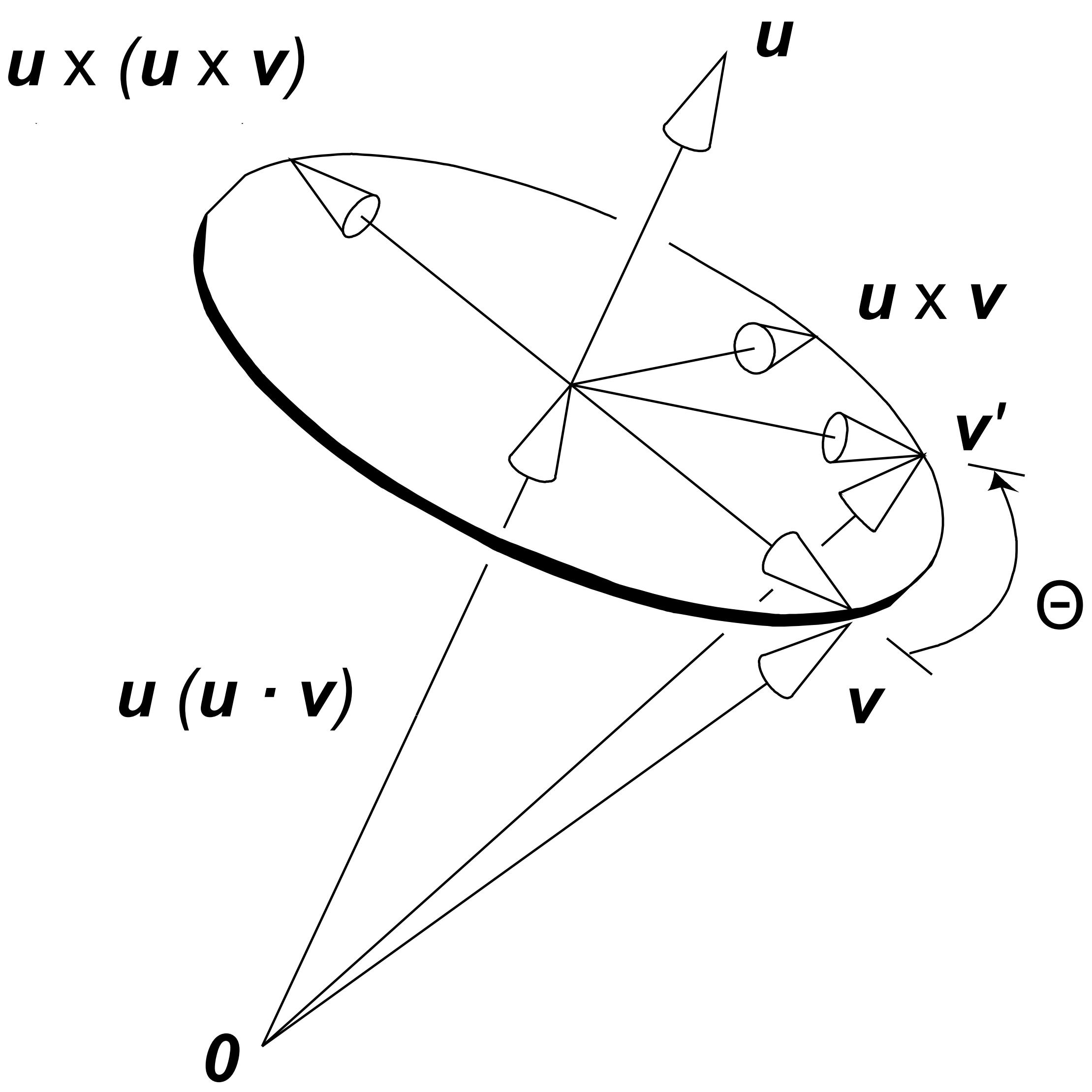

Figure 1: Formula of Rodrigues: schematic of the particular base vectors

Axis-Angle Representation

Question: How can we get Θ and $\boldsymbol{u}$ from $\boldsymbol{R}$?

Use the eigenvalues and eigenvectors of $\boldsymbol{R}$:

- the eigenvalues of $\boldsymbol{R}$ are
  - all equal to 1 (if $\boldsymbol{R} = \boldsymbol{I}_3$), or
  - $1, \cos\Theta + i\sin\Theta, \cos\Theta - i\sin\Theta,$
- the eigenvector for eigenvalue 1 of $\boldsymbol{R}$ is collinear with $\boldsymbol{u}$.
- Θ can also be obtained via $\mathrm{trace}(\boldsymbol{R}) = 1 + 2\cos(\Theta)$.

Topics

Representations of 3-D Rotations

- Overview
- Euler Angles
- Axis-Angle Representation

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- Rotation can be represented by using Euler angles, however this approach is not very robust.
- Another representation is describing an arbitrary rotation by rotation around a single axis and a certain angle, which is the essence of the Rodrigues formula.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. "Medical Image Registration". In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
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3-D Rotations – Quaternions

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Topics

Representations of 3-D Rotations

Overview

Quaternions

Multiplication of Quaternions

Rotation Quaternion

Summary

Take Home Messages

Further Readings

Rotations in 3-D

Various representations for rotations:

- Euler angles
- Axis-angle representation
- **Quaternions**

Quaternion representation

- Rotations in $\mathbb{R}^3$ can be elegantly described by so-called *quaternions*.
- Quaternions can be understood as an extension of complex numbers:
  - Three different numbers that are all square roots of -1:

$$i * i = -1, \qquad j * j = -1, \qquad k * k = -1.$$

  - The products between these numbers are defined as:

$$i * j = -j * i = k, \qquad j * k = -k * j = i, \qquad k * i = -i * k = j.$$

Quaternions

Definition

A ***quaternion*** is a linear combination $\mathbf{r} = w + xi + yj + zk$ where $w, x, y, z \in \mathbb{R}$.

Definition

Similar to complex numbers we define the conjugate $\bar{\mathbf{r}}$ and the magnitude $|\mathbf{r}|$ of a quaternion $\mathbf{r} = w + xi + yj + zk$ as

$$\bar{\mathbf{r}} = w - xi - yj - zk,$$
$$|\mathbf{r}| = \sqrt{\mathbf{r} * \bar{\mathbf{r}}} = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Properties of Quaternions

Definition

A quaternion $\mathbf{r}$ which has length 1 is called a ***unit quaternion***.

A few important properties of quaternions:

- Multiplication and summation are associative.
- Multiplication is *not* commutative, i. e., $\mathbf{r}_1 * \mathbf{r}_2 \neq \mathbf{r}_2 * \mathbf{r}_1$.

$\rightarrow$ Quaternions are no algebraic field, they form a division ring.

- For unit quaternions the inverse is determined as follows:

$$|\mathbf{r}| = 1 \quad \Rightarrow \quad \mathbf{r}^{-1} = \bar{\mathbf{r}}.$$

Multiplication of Quaternions

Definition

We represent quaternions by a row vector $(w, x, y, z) = (w, \boldsymbol{v})$ where $\boldsymbol{v}^\mathsf{T} = (x, y, z)$.

In this notation the product of two quaternions $\mathbf{r}_1 = (w_1, \mathbf{v}_1)$ and $\mathbf{r}_2 = (w_2, \mathbf{v}_2)$ is given by

$$\mathbf{r}_1 * \mathbf{r}_2 = \left(w_1 w_2 - {\mathbf{v}_1}^\mathsf{T} \mathbf{v}_2 \,,\; w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2\right) .$$

Multiplication of Quaternions

Using the other notation

$$\mathbf{r}_1 = w_1 + x_1 i + y_1 j + z_1 k\,, \quad \mathbf{r}_2 = w_2 + x_2 i + y_2 j + z_2 k\,,$$

we get:

$$\begin{aligned}\mathbf{r}_1 * \mathbf{r}_2 = & \left(w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2\right) \\ & + \left(w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2\right) i \\ & + \left(w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2\right) j \\ & + \left(w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2\right) k\,.\end{aligned}$$

Multiplication of Quaternions

This quaternion product can be rewritten in matrix notation. For that purpose define the matrix ${}_{*}[\mathbf{r}_2]$ such that

$$\mathbf{r}_1 \, {}_{*}[\mathbf{r}_2] = \mathbf{r}_1 * \mathbf{r}_2.$$

We find

$${}_{*}[\mathbf{r}_2] = \begin{pmatrix} w_2 & x_2 & y_2 & z_2 \\ -x_2 & w_2 & -z_2 & y_2 \\ -y_2 & z_2 & w_2 & -x_2 \\ -z_2 & -y_2 & x_2 & w_2 \end{pmatrix}.$$

Note: This matrix shows similarities to the skew matrix that is used to express the cross product of vectors by matrix multiplication, where $\boldsymbol{x} \times \boldsymbol{y} = [\boldsymbol{x}]_{\times} \boldsymbol{y}$.

Rotation Quaternion

Let

- $\boldsymbol{p} \in \mathbb{R}^3$ be a 3-D point to be rotated,
- $\boldsymbol{u} \in \mathbb{R}^3$ be the axis of rotation with $\|\boldsymbol{u}\| = 1$,
- $\Theta \in \mathbb{R}$ be the angle of rotation.

Definition

- The ***rotation quaternion***, according to a rotation given in axis-angle representation, is defined by:

$$\mathbf{r} = \left(\cos\frac{\Theta}{2}, \sin\frac{\Theta}{2} \cdot \boldsymbol{u}\right).$$

- The quaternion associated with a 3-D point $\boldsymbol{p}$ is defined by $\mathbf{p}' = (0, \boldsymbol{p})$.

Rotation Quaternion

Then the rotation of $\boldsymbol{p}$ can be computed by:

$$\mathbf{p}'_{\text{rot}} = \mathbf{r} * \mathbf{p}' * \bar{\mathbf{r}}.$$

Note:

- The quaternion $\boldsymbol{p}'_{\text{rot}}$ should be $(0, \boldsymbol{p}_{\text{rot}})$. Actually, we could put any value into the scalar part of $\mathbf{p}'$, i. e., $\mathbf{p}' = (c, \boldsymbol{p})$ and after performing the quaternion multiplication, we should get back $\boldsymbol{p}'_{\text{rot}} = (c, \boldsymbol{p}_{\text{rot}})$.
- You may want to confirm that $\mathbf{r}$ is a *unit quaternion*, since that will allow us to use the fact that the inverse of $\mathbf{r}$ is $\bar{\mathbf{r}}$ if $\mathbf{r}$ is a unit quaternion, i. e., $\|\mathbf{r}\| = 1$, $\mathbf{r}^{-1} = \bar{\mathbf{r}}$.

Estimation of 3-D Rotation

The optimization problem to estimate the 3-D rotation is:

$$\widehat{\boldsymbol{R}} = \operatorname*{arg\,min}_{\boldsymbol{R}} \sum_{i=1}^{N} \|\boldsymbol{p}_{\text{rot},i} - \boldsymbol{R} \cdot \boldsymbol{p}_i\|^2 .$$

Using quaternions for representing rotations we get the following relationship between original and rotated points:

$$(0, \boldsymbol{p}_{\text{rot},i}) = \boldsymbol{q}(0, \boldsymbol{p}_i)\bar{\boldsymbol{q}} \quad \Leftrightarrow \quad (0, \boldsymbol{p}_{rot,i})\boldsymbol{q} = \boldsymbol{q}(0, \boldsymbol{p}_i),$$

and thus we get the optimization problem:

$$\operatorname*{arg\,min}_{\boldsymbol{q}} \sum_{i=1}^{N} \|(0, \boldsymbol{p}_{\text{rot},i})\boldsymbol{q} - \boldsymbol{q}(0, \boldsymbol{p}_i)\|^2 .$$

Conclusion: The objective function is linear in the rotation quaternion. The rotation can be estimated by solving a system of linear equations.

Topics

Take Home Messages

- Quaternions can be regarded as an expansion of the idea of complex numbers. They allow a useful representation of rotation operations.
- Using quaternions, we have found a linear method to estimate 3-D rotation.
- The translation has to be known.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. “Medical Image Registration”. In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
- J. B.Antoine Maintz and Max A. Viergever. “A Survey of Medical Image Registration”. In: *Medical Image Analysis* 2.1 (1998), pp. 1–36. DOI: 10.1016/S1361-8415(01)80026-8
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- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001
- Many of Jan Modersitzki's and Bernd Fischer's papers on image registration can be found in the publication list of the Institute of Mathematics and Image Computing (Lübeck).
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Medical Image Processing for Diagnostic Applications

Image Registration in Practice – Part 1

Online Course – Unit 67
Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

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TECHNISCHE FAKULTÄT

Topics

Applications of Image Registration

Examples

Intramodal Registration

Summary

Take Home Messages

Further Readings

Example of 3-D/3-D Registration: Rigid Registration of the Airways

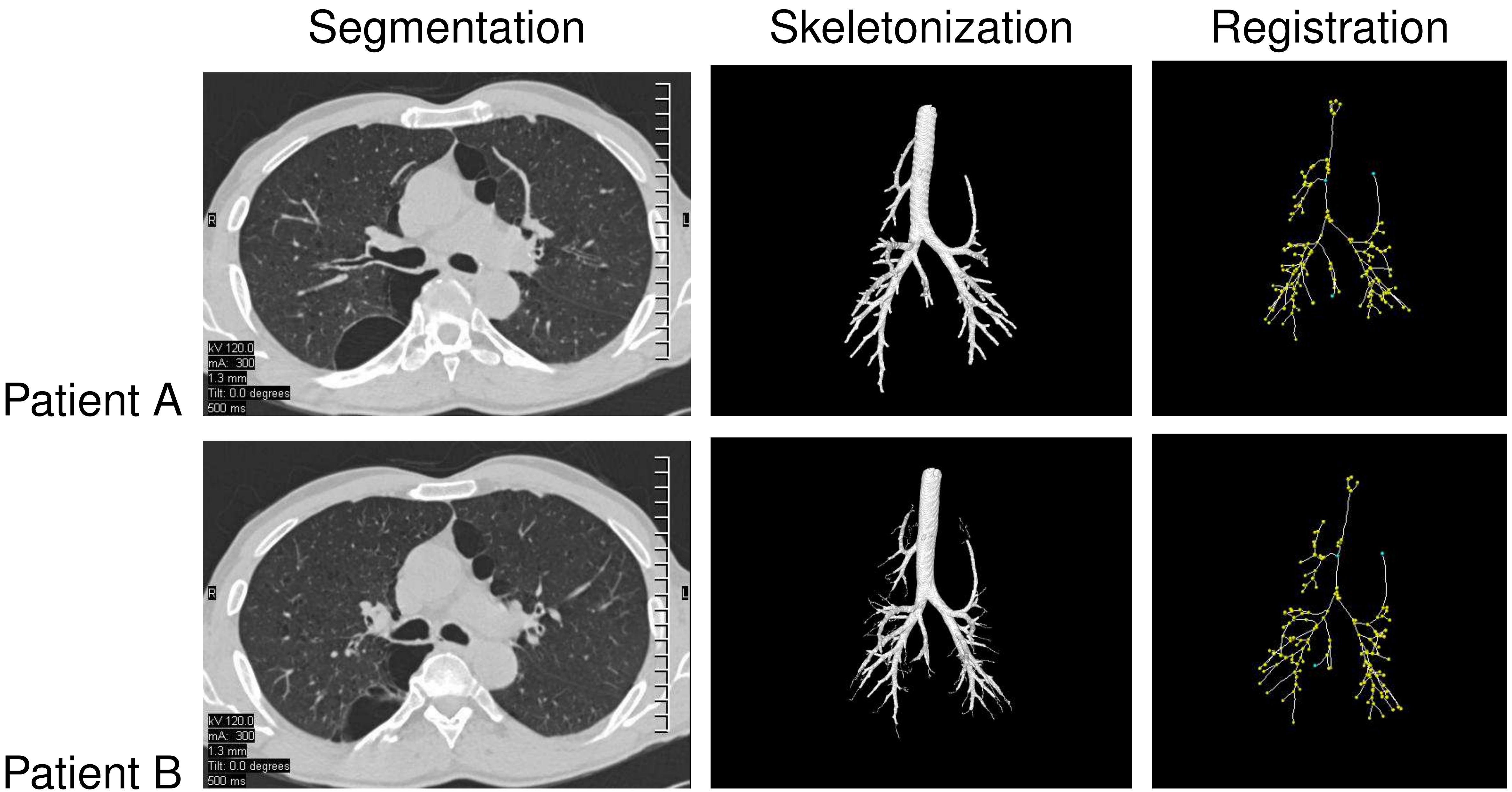


Figure 1: Rigid registration of the airways

Projections from 3-D to 2-D

Motivation: 2-D/3-D image fusion is important for applications where volume data and X-ray projections have to be registered.

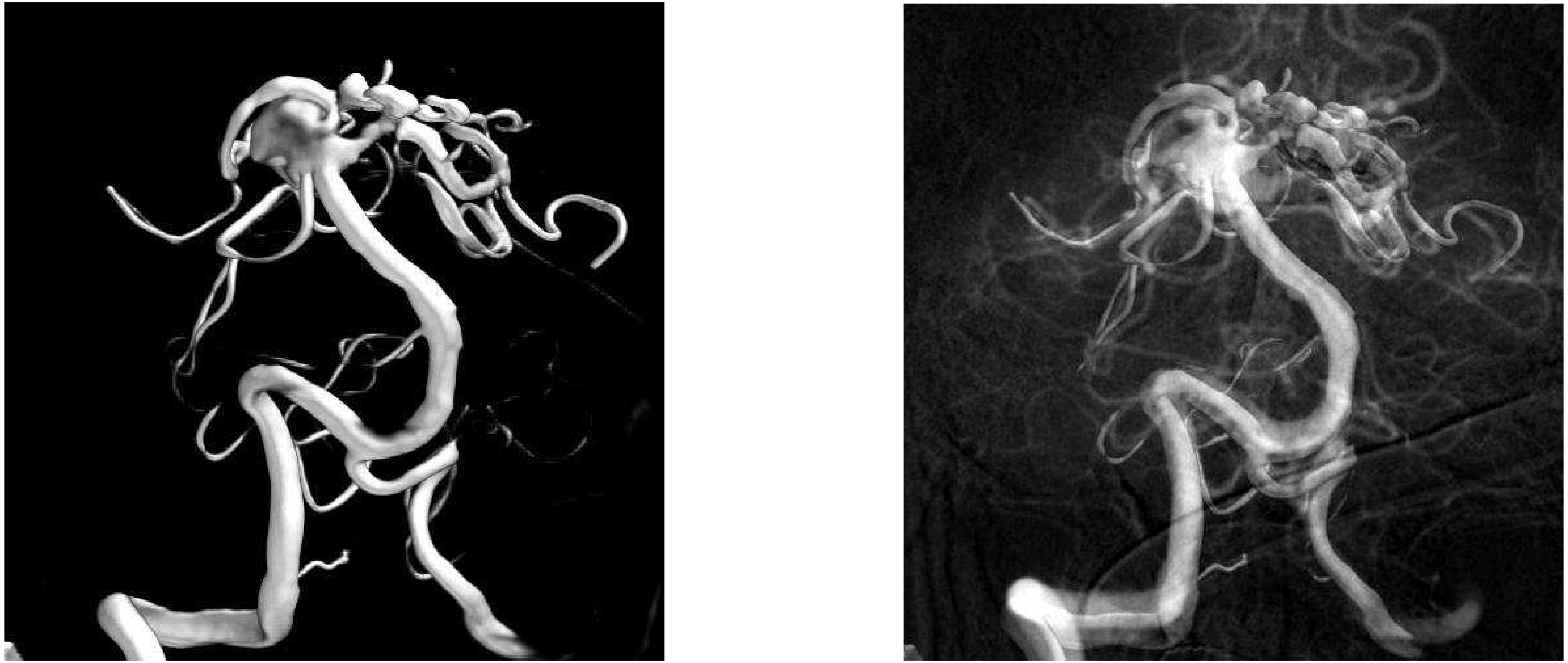


Figure 2: 2-D/3-D image fusion

Intramodal Registration

Examples for the requirement of intramodal registration are:

- digital subtraction angiography (DSA),
- dual energy X-ray and CT,
- visualization of perfusion,
- visualization of differences (therapy control),
- motion estimation (for instance, in cardiac reconstruction).

Digital Subtraction Angiography

Figure 3: Mask image (left), fill image (middle), angiogram (right)

Digital Subtraction Angiography

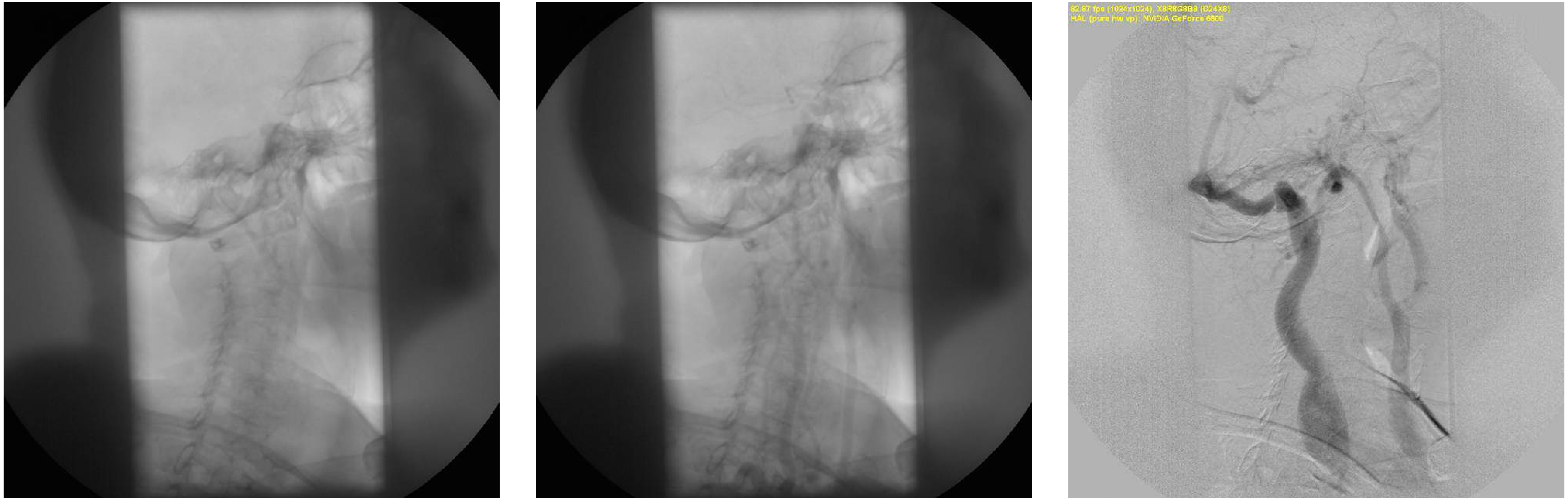
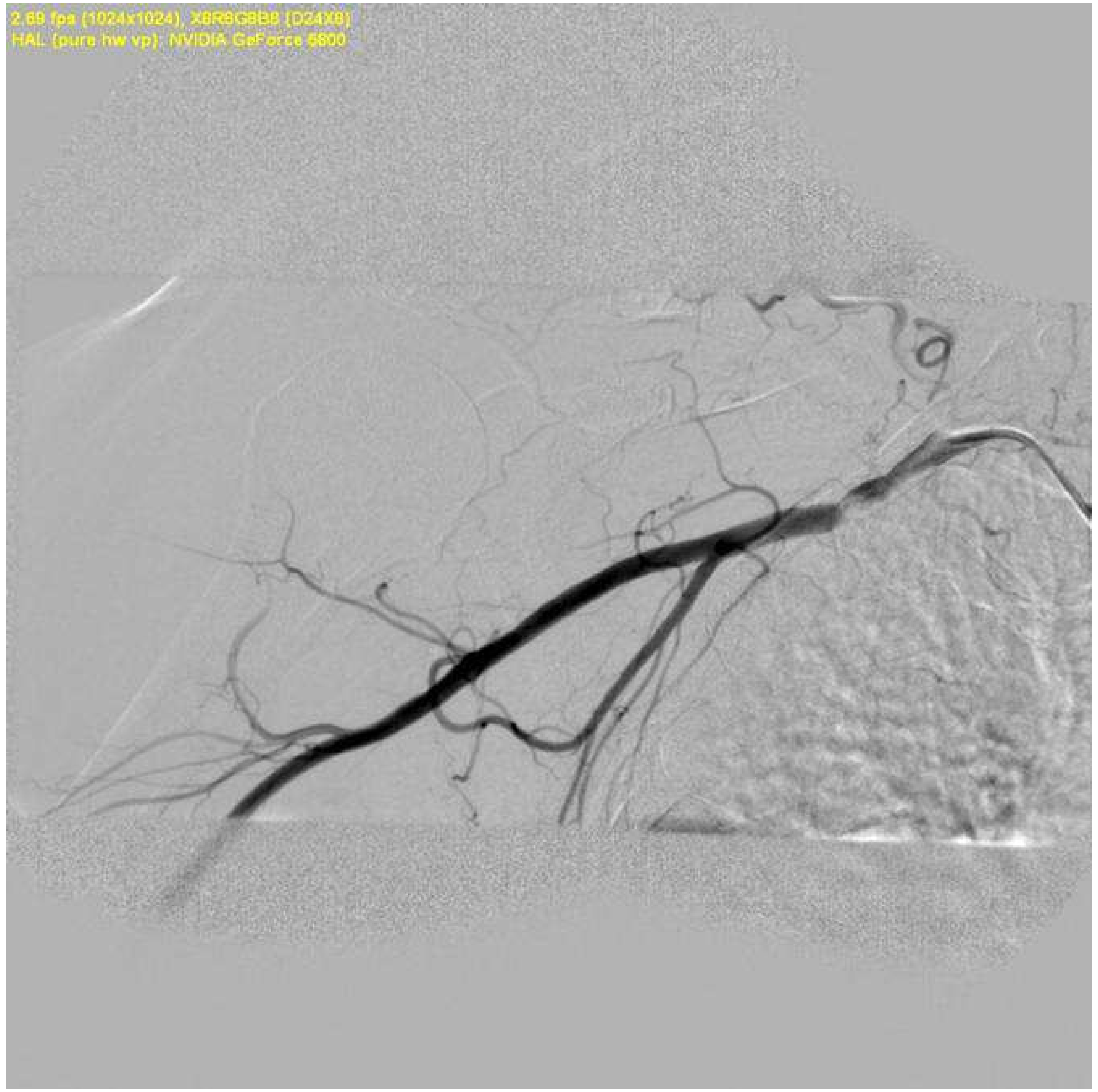


Figure 4: Mask image (left), fill image (middle), angiogram (right)

Motion Artifacts in DSA

Figure 5: Motion artifacts in DSA (Yu Deuerling-Zheng, Pattern Recognition Lab, FAU)

Motion Artifacts in DSA



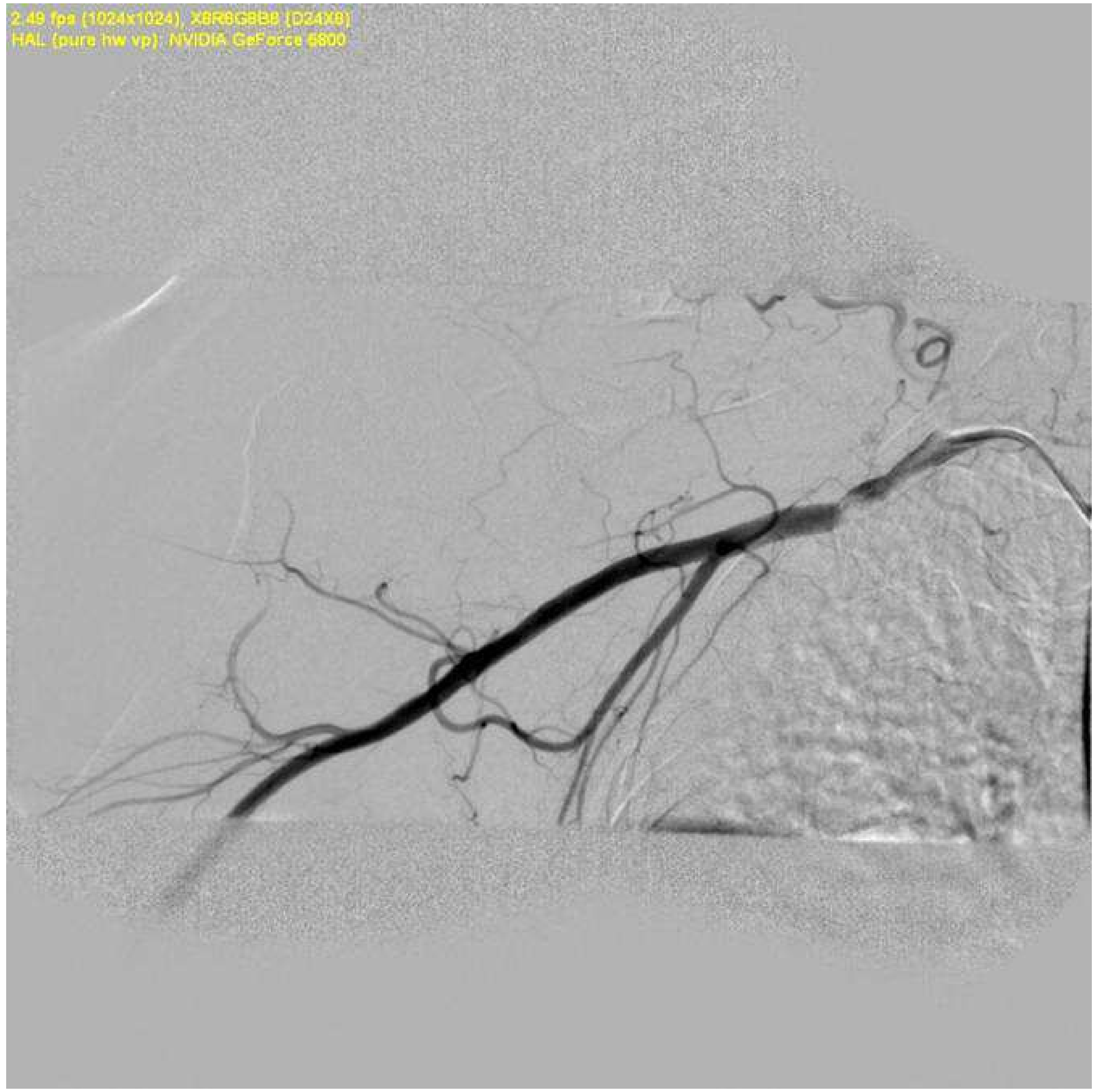


Figure 6: Motion artifacts in DSA (Yu Deuerling-Zheng, Pattern Recognition Lab, FAU)

Similarity Measures

Sum of squared differences (SSD):

$$\widehat{T} = \arg\min_{T} \sum_{i,j} \left\| f_{i,j} - T\{g_{i,j}\} \right\|^2$$

Correlation coefficient:

$$\widehat{T} = \arg\max_{T} \frac{\sum_{i,j}(f_{i,j} - \bar{f})(T\{g_{i,j}\} - \bar{g})}{\sqrt{\sum_{i,j}(f_{i,j} - \bar{f})^2 \sum_{i,j}(T\{g_{i,j}\} - \bar{g})^2}}$$

Notation:

- $[f_{i,j}]$: reference image
- $[g_{i,j}]$: image to be registered
- T: transform
- $\bar{f}$: mean intensity value of reference image
- $\bar{g}$: mean intensity value of second image

Difference Imaging in CT

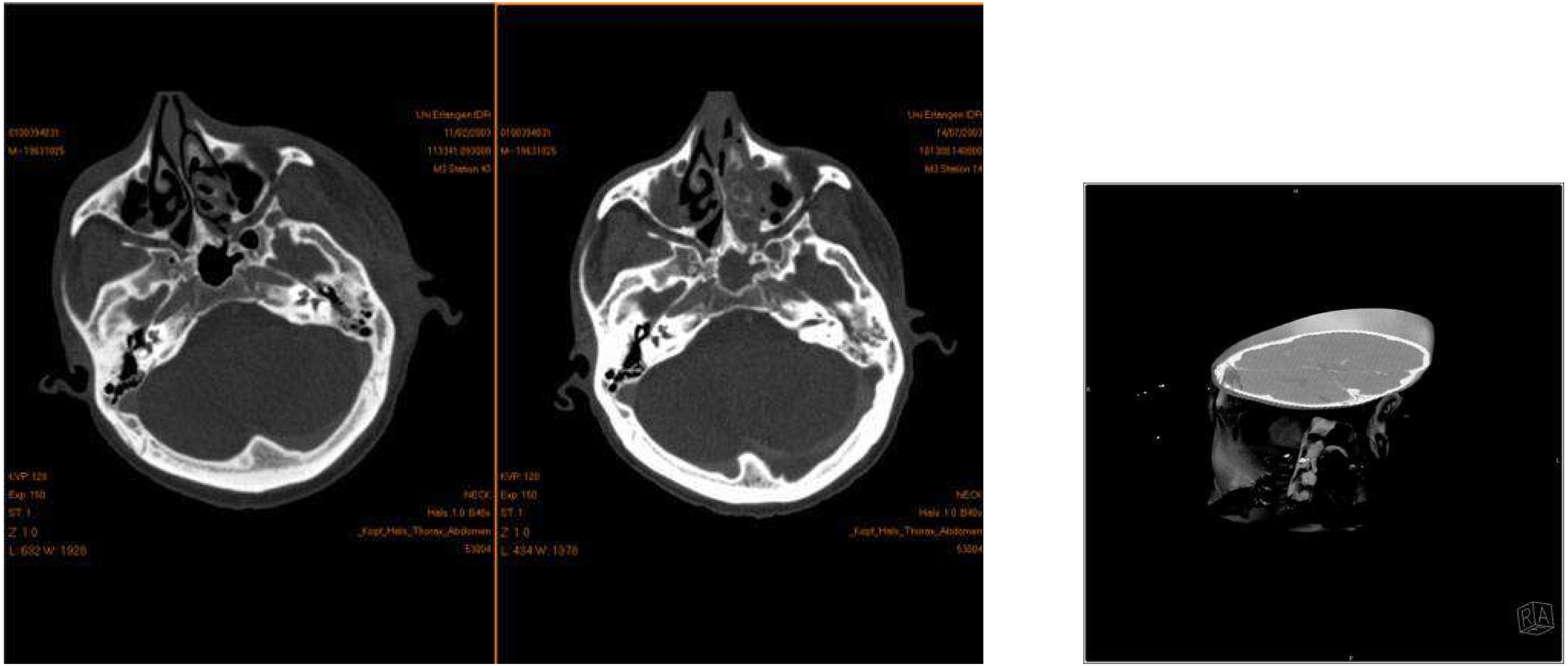


Figure 7: Difference imaging in CT (Dieter Hahn, Pattern Recognition Lab, FAU)

Registration Combined with Segmentation

Problem: Differences in images lead to a bias if all voxels are used for registration.

Solution: Apply a weighting scheme to voxels. Voxels that belong to bones are rigid and allow for a reliable estimate for the transform (high weights). Soft tissue deforms, for instance, with tumor growth and thus implies a bias (low weights).

Registration in CT using Transfer Functions

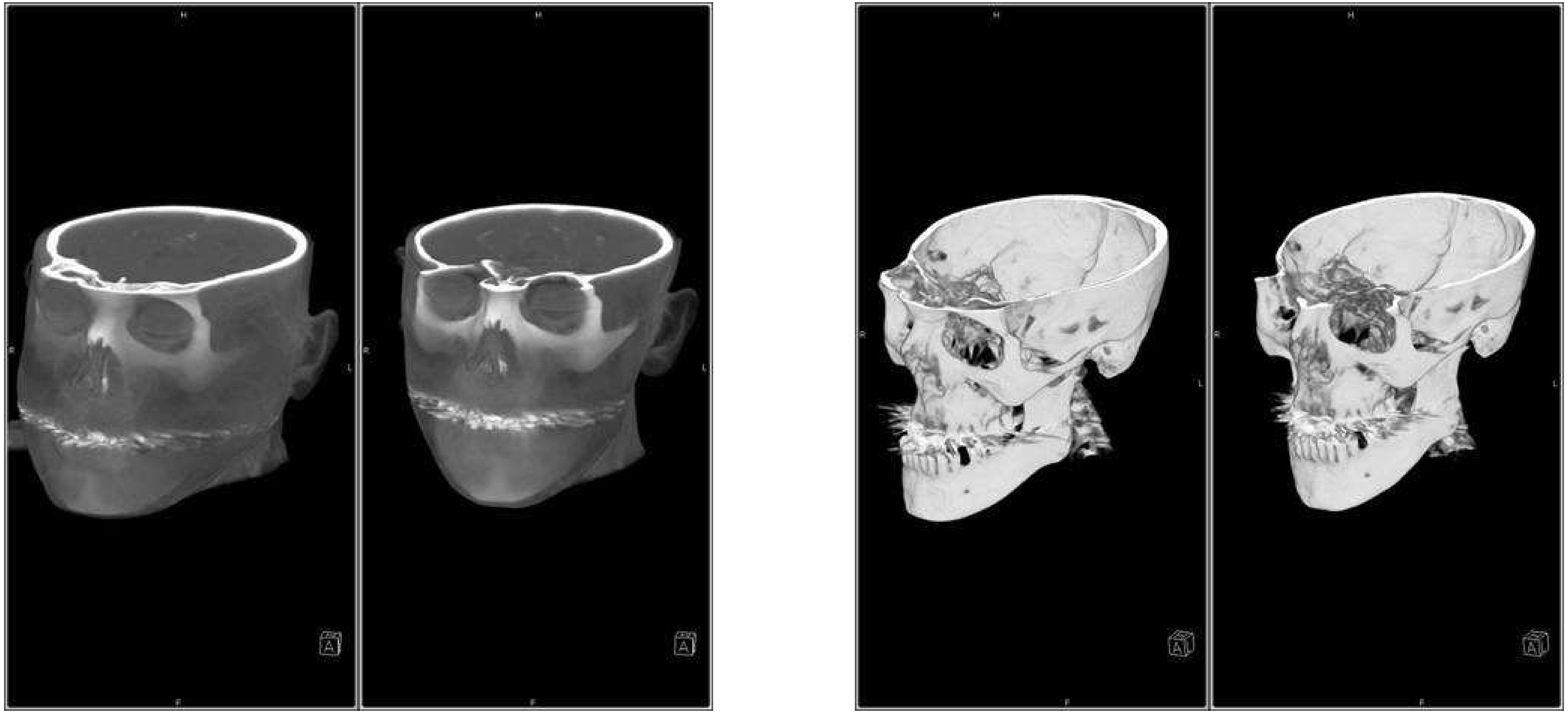


Figure 8: Difference imaging in CT: segmentation of bones (Dieter Hahn, Pattern Recognition Lab, FAU)

Registration in CT

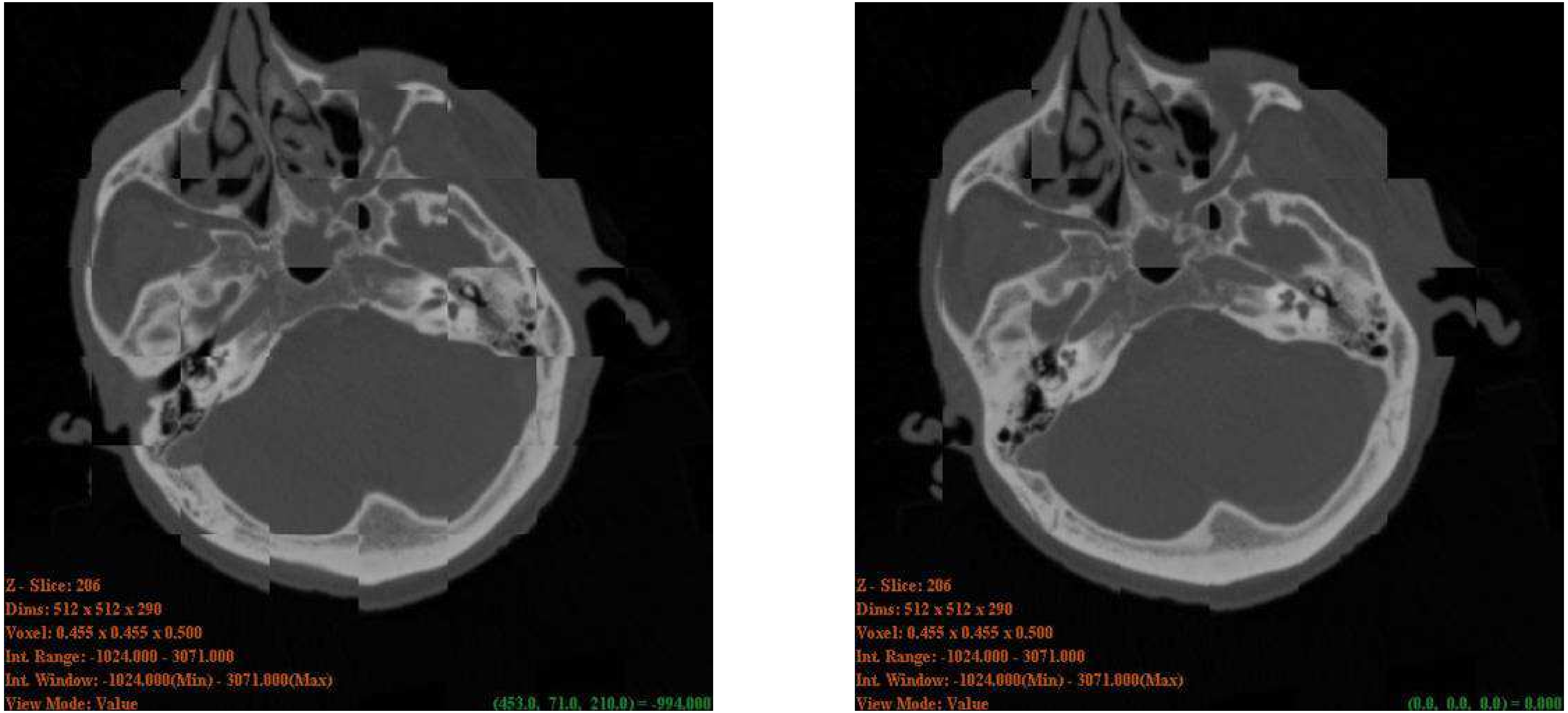


Figure 9: Checker board representation of results: no bone segmentation (left), bone segmentation (right) (Dieter Hahn, Pattern Recognition Lab, FAU)

Registration in CT

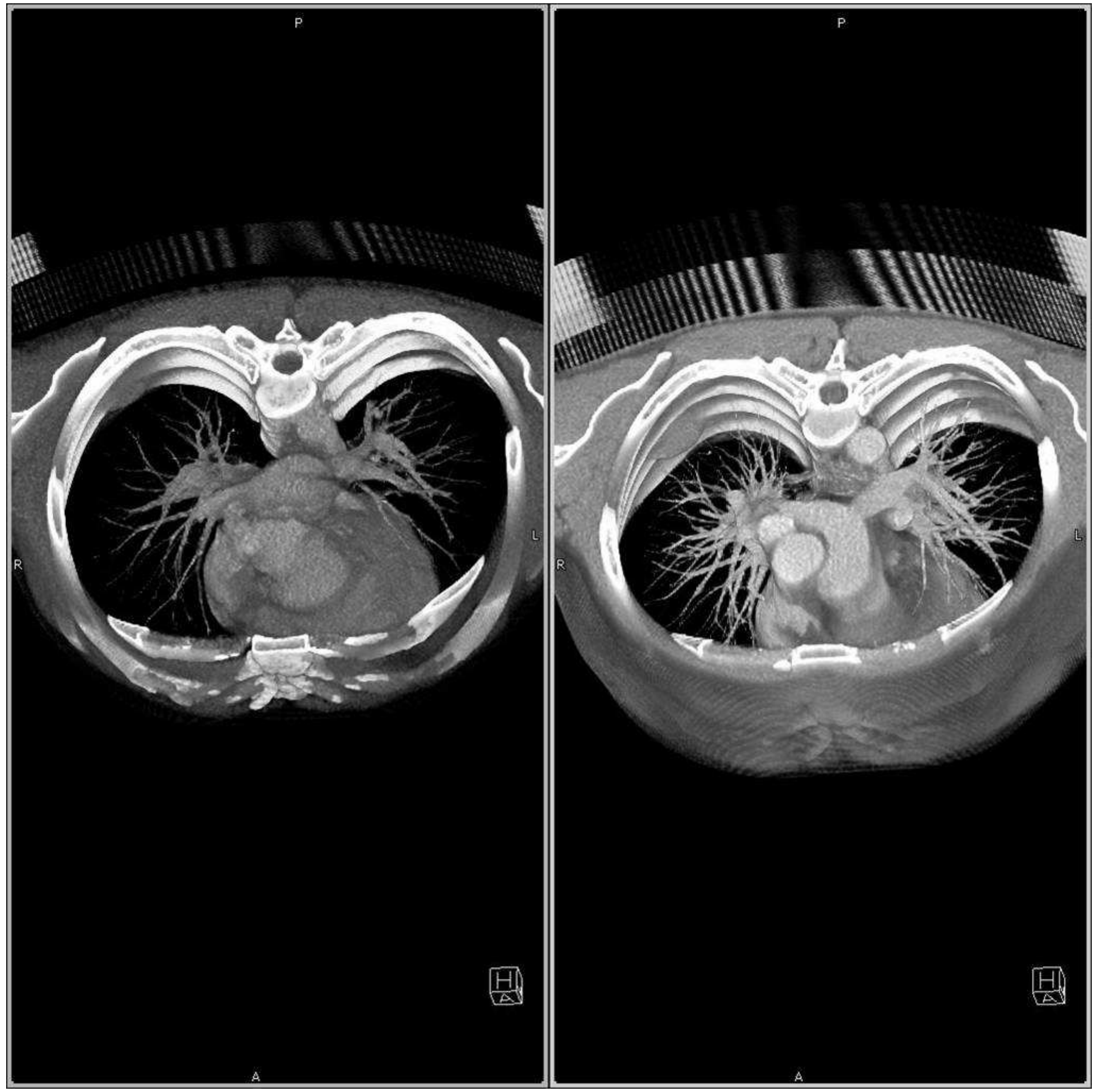


Figure 10: Thorax: tumor at two different therapy stages (Dieter Hahn, Pattern Recognition Lab, FAU)

Registration in CT

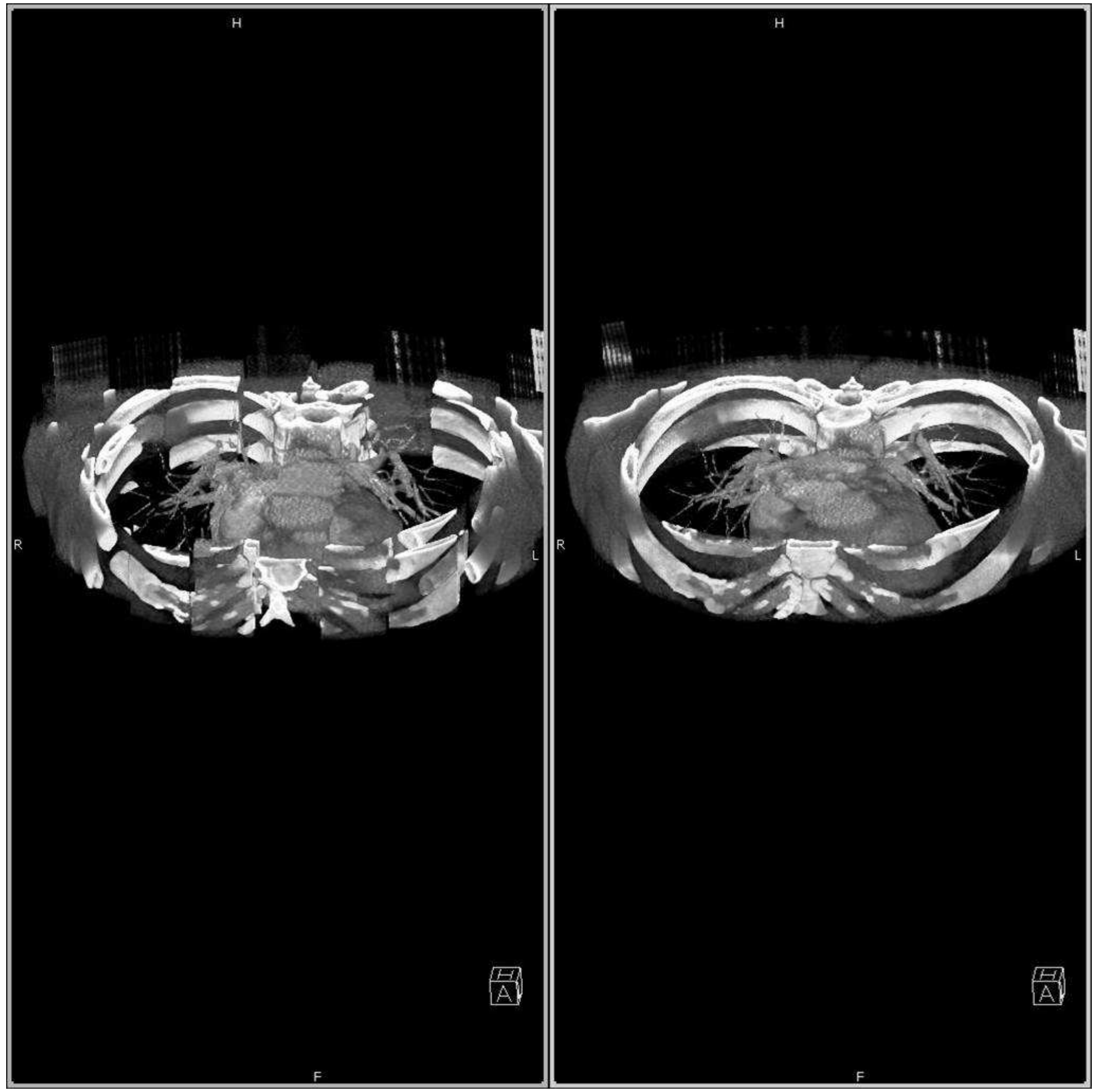


Figure 11: Checker board representation of results: no bone segmentation (left), bone segmentation (right) (Dieter Hahn, Pattern Recognition Lab, FAU)

Registration in CT

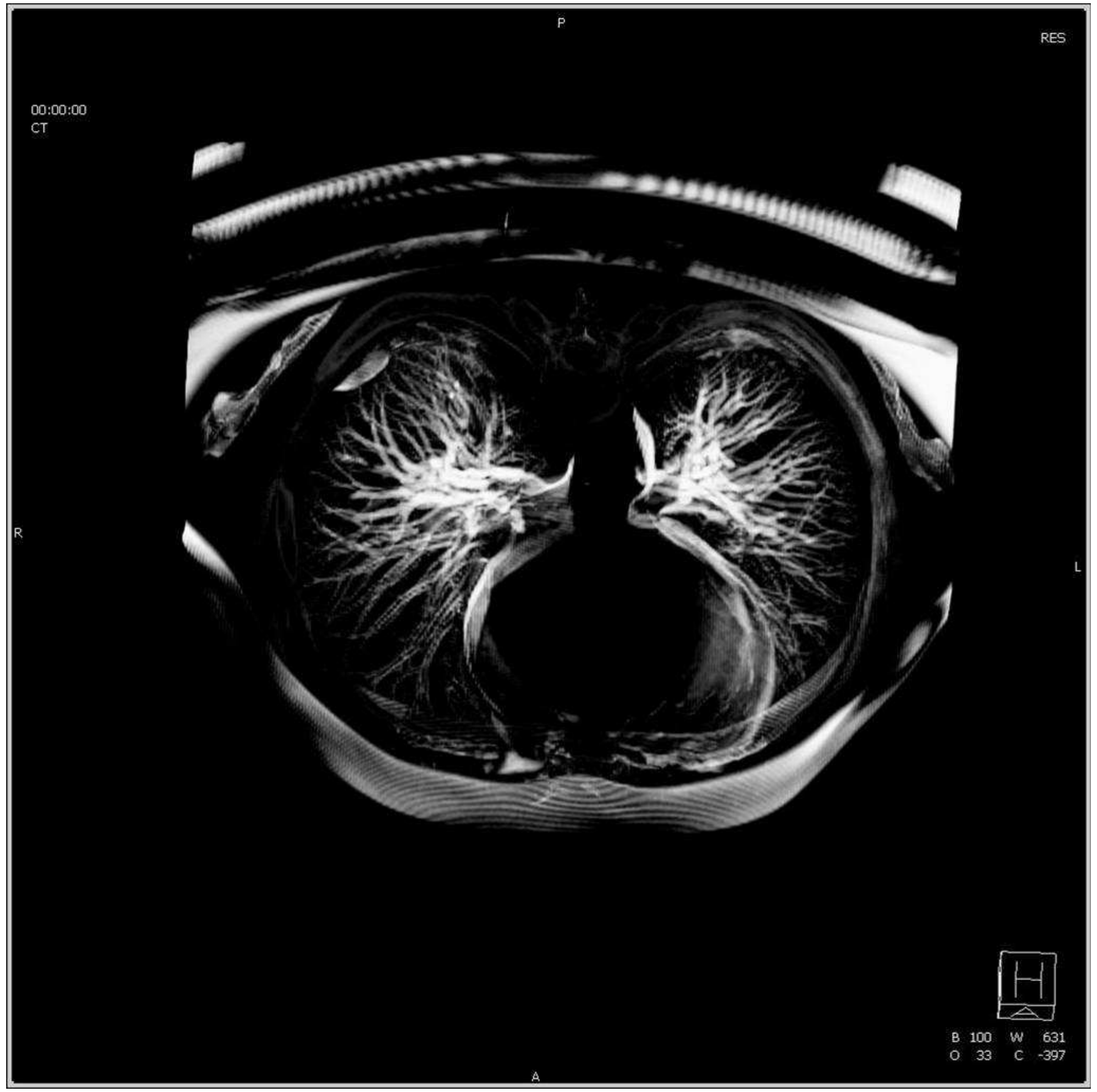


Figure 12: Difference image (Dieter Hahn, Pattern Recognition Lab, FAU)

Topics

Take Home Messages

- There is a multitude of applications of intramodal registration alone.
- It can be combined with segmentation methods, and it can be used to generate difference images.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. "Medical Image Registration". In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
- J. B.Antoine Maintz and Max A. Viergever. "A Survey of Medical Image Registration". In: *Medical Image Analysis* 2.1 (1998), pp. 1–36. DOI: 10.1016/S1361-8415(01)80026-8
- L. G. Brown. "A Survey of Image Registration Techniques". In: *ACM Computing Surveys* 24.4 (Dec. 1992), pp. 325–376. DOI: 10.1145/146370.146374
- Josien P. W. Pluim, J. B. Antoine Maintz, and Max A. Viergever. "Mutual-Information-Based Registration of Medical Images: A Survey". In: *IEEE Transactions on Medical Imaging* 22.8 (Aug. 2003), pp. 986–1004. DOI: 10.1109/TMI.2003.815867

A paper that inspired all the sections on complex numbers, quaternions, and dual quaternions:

Konstantinos Daniilidis. "Hand-Eye Calibration Using Dual Quaternions". In: *The International Journal of Robotics Research* 18.3 (Mar. 1999), pp. 286–298. DOI: 10.1177/02783649922066213

Further Readings – Part 2

Non-parametric mappings for image registration:

- Nonlinear registration methods applied to DSA can be found in Erik Meijering's papers.
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001
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Medical Image Processing for Diagnostic Applications

Image Registration in Practice – Part 2

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Topics

Applications of Image Registration

Multimodal Registration

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Multimodal Registration

Historical remarks on the application of mutual information in computer vision and image processing:

- Maximization of mutual information is applied to solve parameter estimation problems, e. g., speech recognition ($\sim$1980).
- Maximization of mutual information was first applied to image registration by P. Viola, W. Wells and by Collignon (1995).
- Maximization of mutual information was first applied to active object recognition by B. Schiele and J. Crowley (1997).

Kullback-Leibler Divergence

A proper similarity measure for density functions is the *Kullback-Leibler Divergence* (KL divergence). The KL divergence between two bivariate probability density functions $p(f, g)$ and $q(f, g)$ is defined as:

$$\mathrm{KL}(p, q) = \int\int p(f, g) \log \frac{p(f, g)}{q(f, g)} \mathrm{d}f \,\mathrm{d}g.$$

Idea for Image Registration: Images that are correctly registered imply the highest probabilistic dependency of aligned intensity values. The more the random variables depend on each other, the more the probability density functions $p(f, g)$ and $p(f) \cdot p(g)$ differ. This difference can be measured by the KL divergence:

$$\mathrm{KL}(p, q) = \int\int p(f, g) \log \frac{p(f, g)}{p(f)p(g)} \mathrm{d}f \,\mathrm{d}g.$$

Optimization Problem for Image Registration

The transform T that maps the gray level of the model image to the image point of the reference image can be estimated as follows:

$$\widehat{T} = \arg\max_T \sum_i p(f_i, g_{T(i)}) \log \frac{p(f_i, g_{T(i)})}{p(f_i)\, p(g_{T(i)})}.$$

In terms of information theory this is the maximization of **mutual information**:
We send an image $[f_i]_{i=1,\dots,N}$ and receive the image $[g_i]_{i=1,\dots,N}$, where the channel applies a transform to image coordinates and the intensity values change.

Registration of CT and SPECT Images

Figure 1: CT image (left), SPECT image (middle), result of image registration (right) (images courtesy of Dr. W. Römer, Nuclear Medicine, FAU)

Channel Model

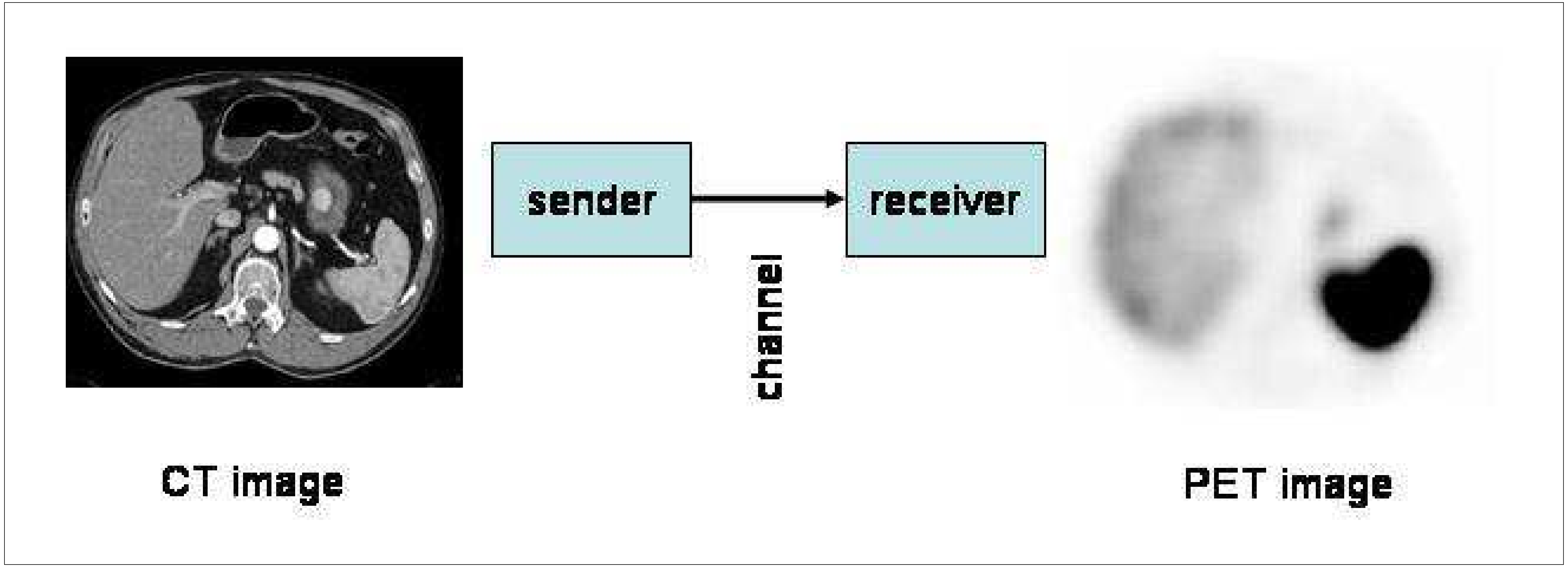

Figure 2: Channel model that defines the transition of CT to PET images (according to Wells et al., 1995).

Definition of Mutual Information

- Entropy is defined as:

$$H(F) = -\sum_{f} p(f)\ \log p(f).$$

- Entropy in the bivariate case is:

$$H(F,G) = -\sum_{f,g} p(f,g)\ \log p(f,g).$$

- Mutual information is defined as:

$$I(F,G) = H(F) + H(G) - H(F,G) = I(G,F).$$

Stochastic Maximization of Mutual Information

	draw N_A samples for i
	draw N_B samples for i
	Set $\widehat{T} = \widehat{T} + \lambda \frac{\mathrm{d}I}{\mathrm{d}T}$
UNTIL convergence	

Figure 3: Stochastic maximization

Topics

Applications of Image Registration

Multimodal Registration

Summary

Take Home Messages

Further Readings

Take Home Messages

- Multimodal registration is often performed using measures of probability theory.
- KL divergence and mutual information are important similarity measures for registration.

Further Readings – Part 1

Survey papers on medical image registration:

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Medical Image Processing for Diagnostic Applications

Iterative Closest Point Algorithm – Basics

Online Course – Unit 69

Andreas Maier, Joachim Hornegger, Eva Kollorz, Frank Schebesch
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Topics

Iterative Closest Point (ICP)

- Motivation
- Problem
- Basics

Summary

- Take Home Messages
- Further Readings

Registration of ToF and CT Data

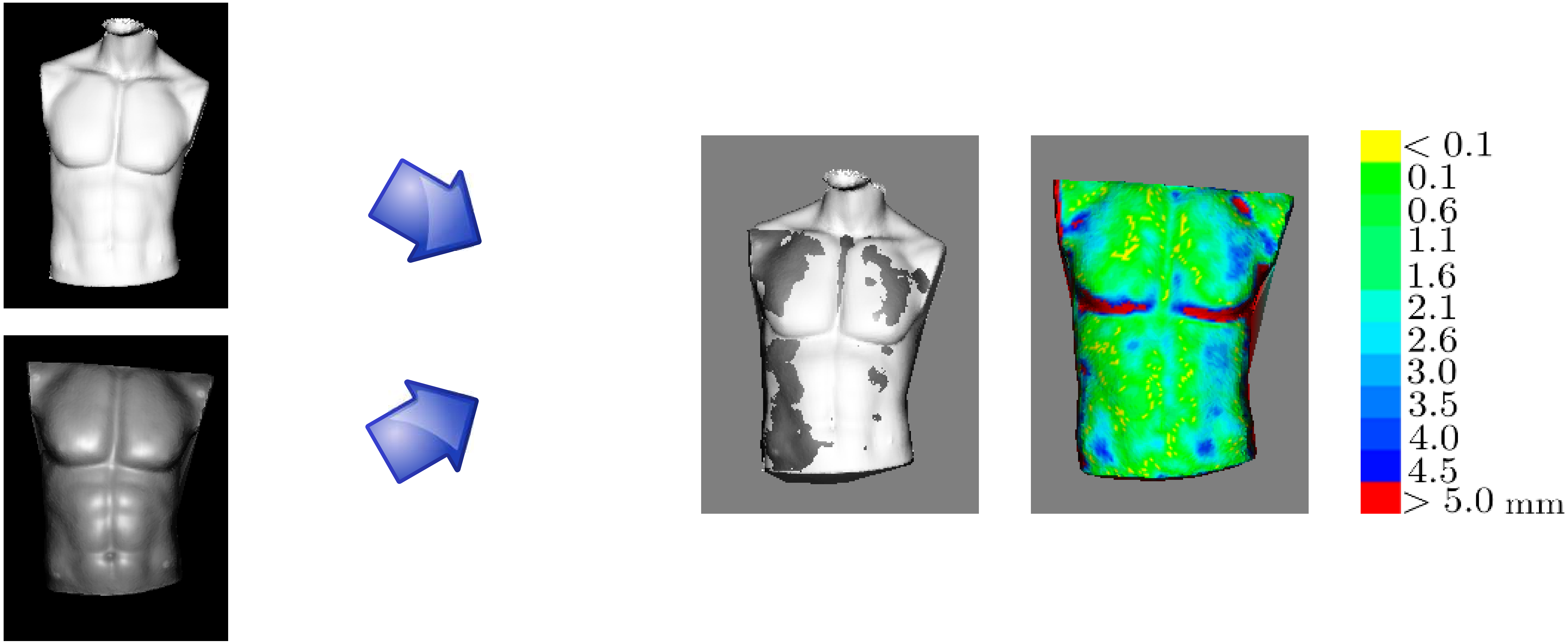


Figure 1: Images courtesy of Kerstin Müller [3]

Registration of ToF and CT Data

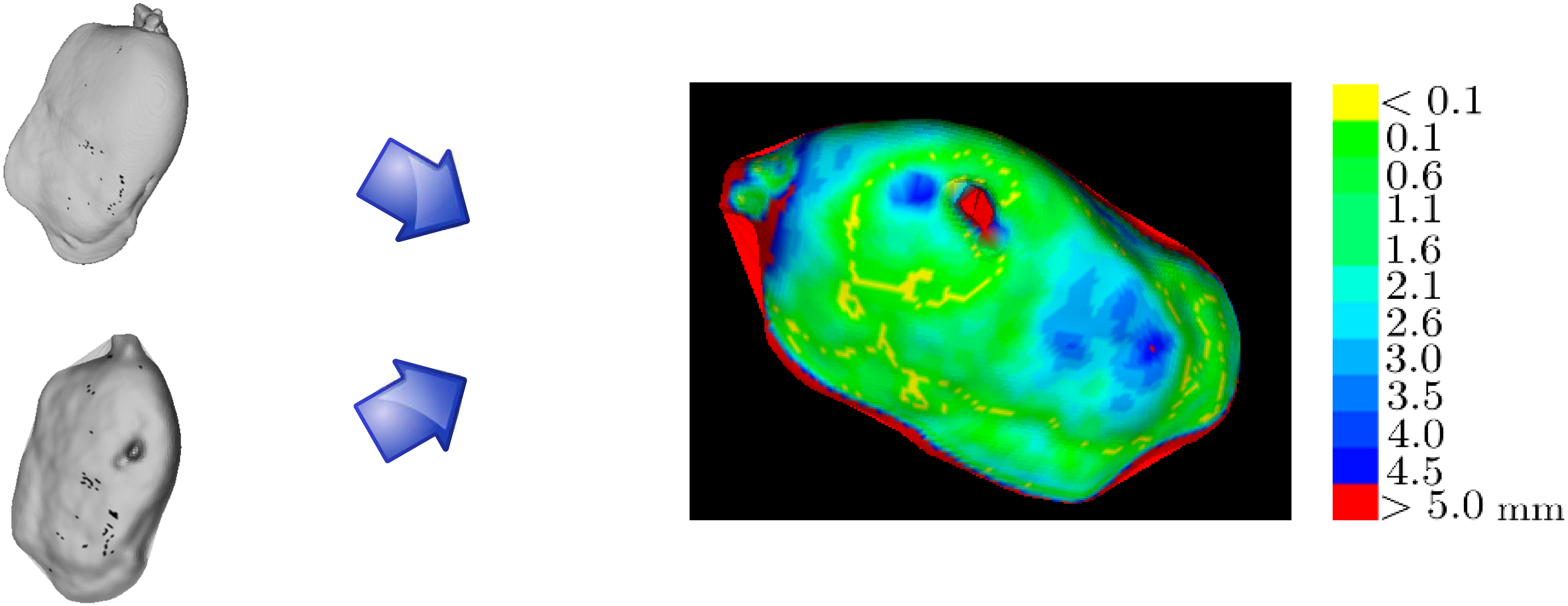


Figure 2: Images courtesy of Kerstin Müller [3]

Registration of Range Images

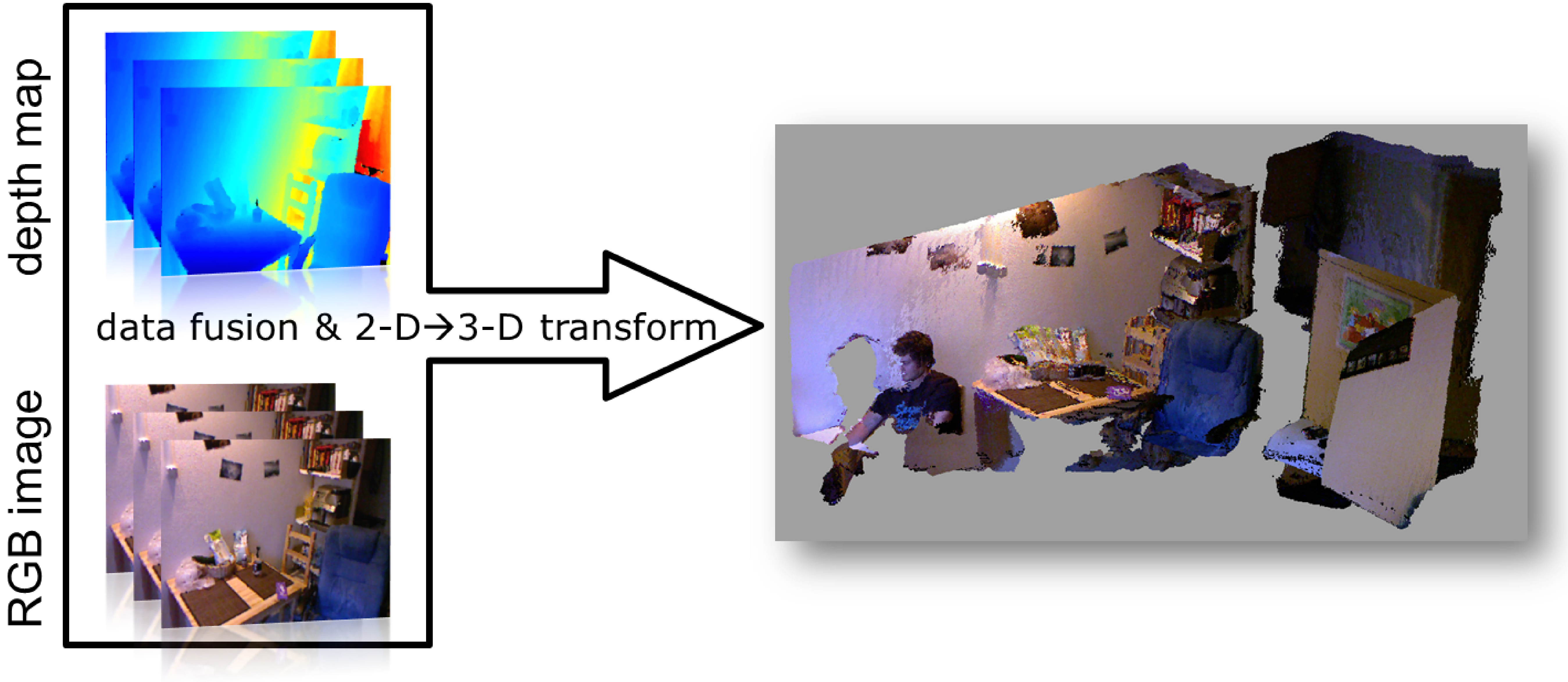


Figure 3: Images courtesy of Felix Lugauer [5]

Problem

- Input: meshes Q, P
- Output: rotation $\boldsymbol{R}$, translation $\mathbf{t}$

$$\hat{Q} = \boldsymbol{R}Q + \mathbf{t}$$
$$\min\left(\mathrm{dist}\left(\hat{Q}, P\right)\right)$$

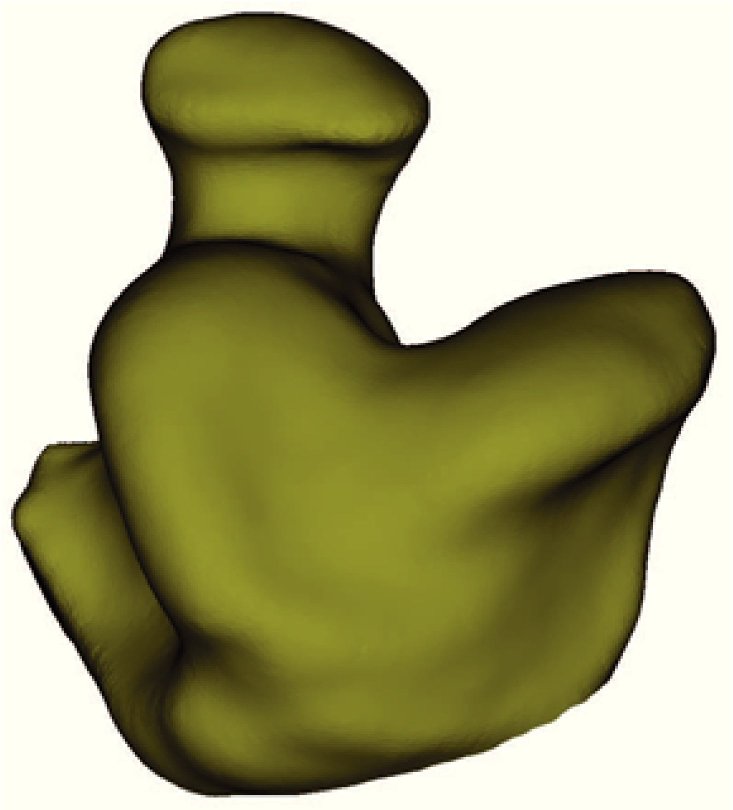


Figure 4: Images courtesy of Konrad Sickel [6]

Problem

- Input: point clouds Q, P
- Output: rotation $\boldsymbol{R}$, translation $\mathbf{t}$

$$\hat{Q} = \boldsymbol{R}Q + \mathbf{t}$$

$$\min\left(\mathrm{dist}\left(\hat{Q}, P\right)\right)$$

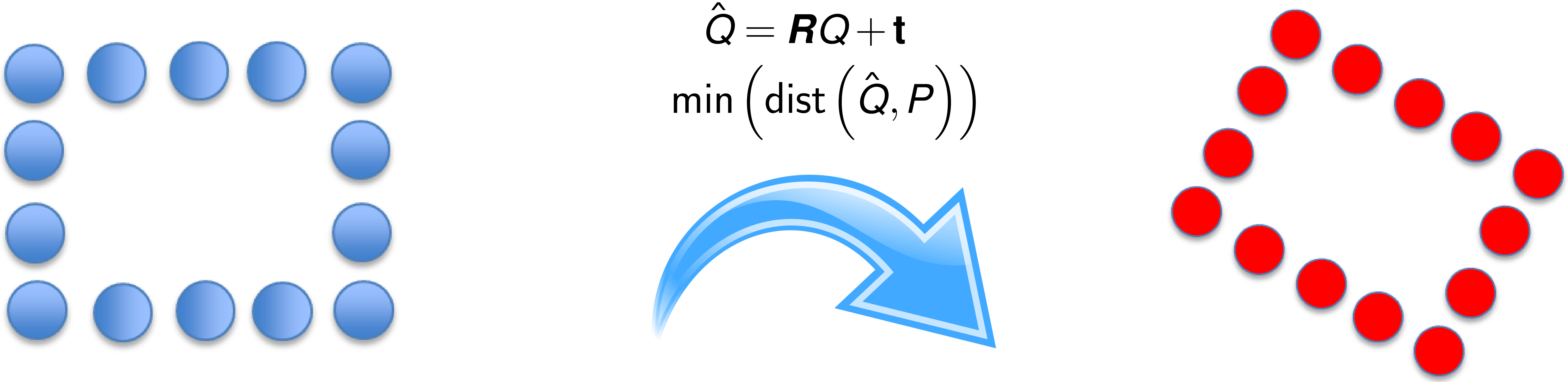


Figure 5: Scheme of a point cloud registration

Problem

- Input: point clouds Q, P
- Output: rotation $\boldsymbol{R}$, translation $\mathbf{t}$

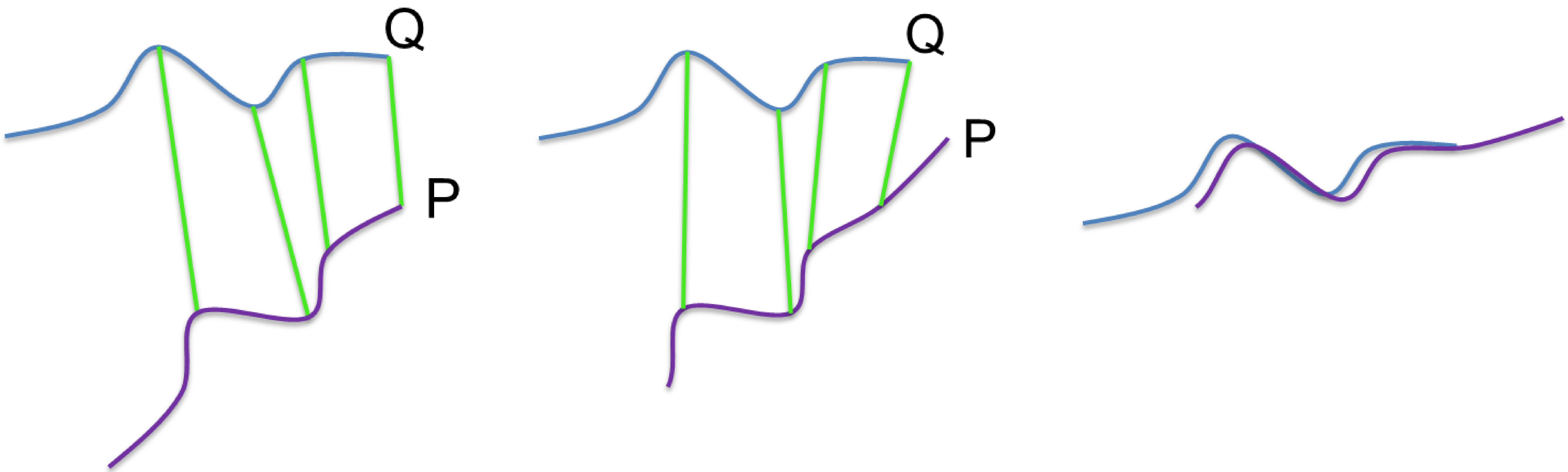
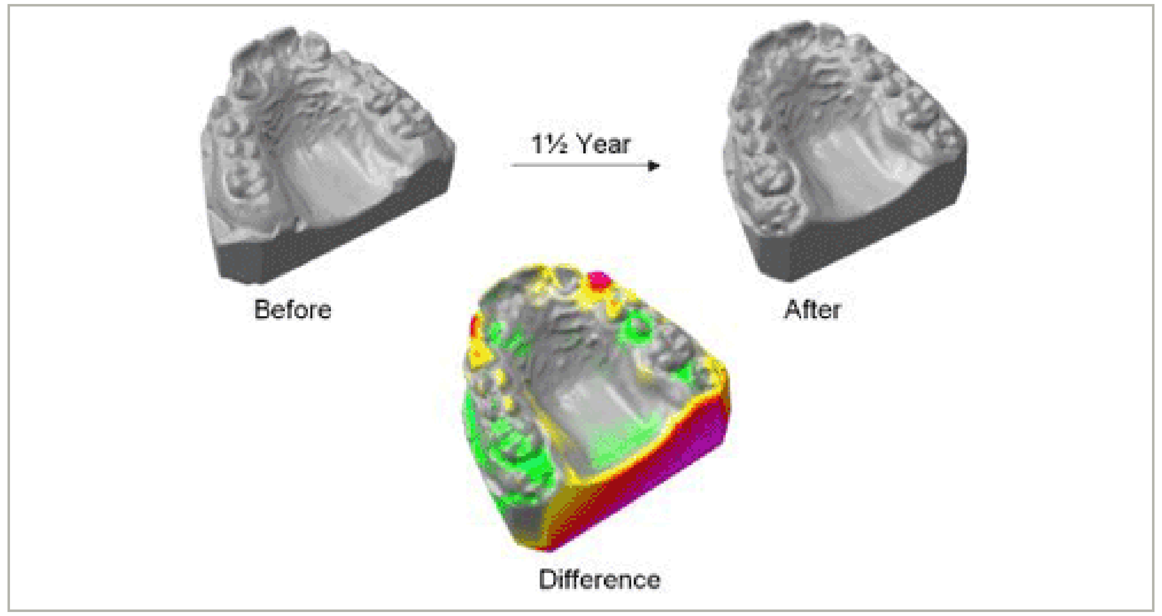


Figure 6: Curve alignment

Basics of Matching

1. Transformations:
   - rigid (rotation, translation)
   - affine (scaling)
   - projective (perspective distortion)
   - elastic (local deformation)

2. Applications of matching:
   - multi-modal (different modalities)
   - temporal (different time points)
   - viewpoint (different perspectives)



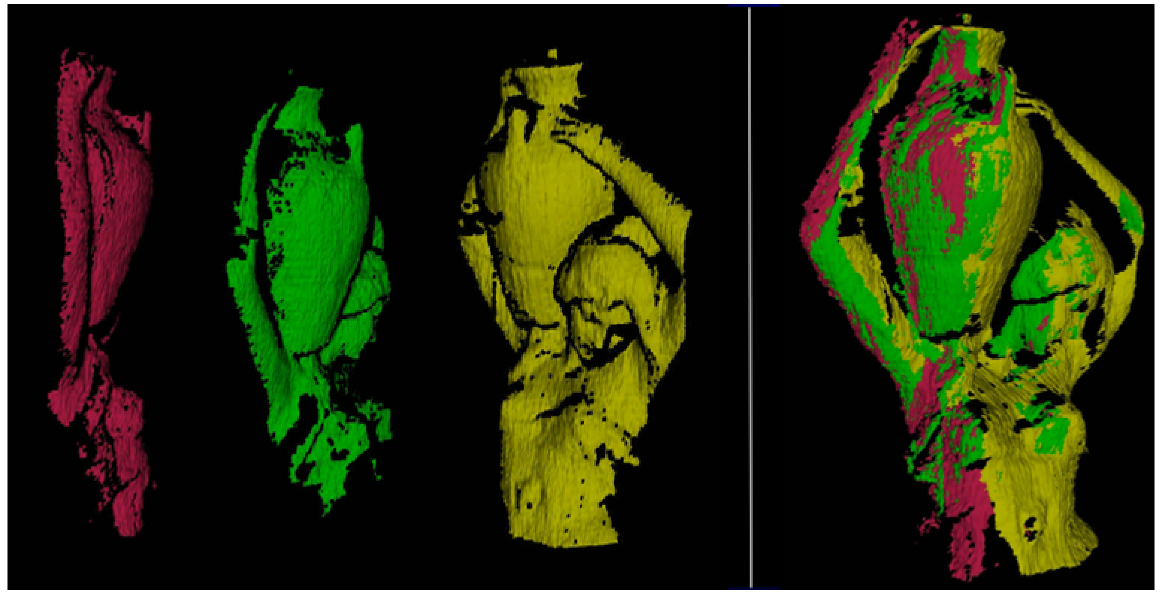


Figure 7: Images courtesy of Wilhelm Nagel [4]

Original Work

ICP was originally applied to scan-matching tasks in the early 1990s.

There were three independently published papers:

- Besl and McKay [1]:
  registration of point clouds using point-to-point error metric,
- Chen and Medioni [2]:
  working with range data for object modeling and point-to-plane error metric,
- Zhang [7]:
  robust method of outlier rejection in the selection phase of the algorithm.

Geometric Data

ICP can be used with the following representations of geometric data [1]:

- point sets,
- line segment sets (polylines),
- implicit curves,
- parametric curves,
- triangle sets (faceted surfaces),
- implicit surfaces,
- parametric surfaces.

Basic Concept

ICP computes the registration by iterating the following steps [6]:

1. computation of correspondences between two point clouds,
2. computation of a transformation which minimizes the distance between the corresponding points.

Topics

Iterative Closest Point (ICP)

- Motivation
- Problem
- Basics

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- There are a lot of applications for registration of point clouds, and simple concepts for matching are desired.
- ICP supports a lot of different geometric data.
- ICP is an iterative algorithm that is based on the minimal distance of points at each iteration step.

Further Readings

[1] Paul J. Besl and Neil D. McKay. “A Method for Registration of 3-D Shapes”. In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 14.2 (Feb. 1992), pp. 239–256. DOI: 10.1109/34.121791.

[2] Yang Chen and Gérard Medioni. “Object Modeling by Registration of Multiple Range Images”. In: *Proceedings of the 1991 IEEE International Conference on Robotics and Automation, Sacramento, California*. IEEE, Apr. 1991, pp. 2724–2729. DOI: 10.1109/ROBOT.1991.132043.

[3] Kerstin Müller. “Multi-modal Organ Surface Registration using Time-of-Flight Imaging”. Diploma Thesis. Erlangen: Pattern Recognition Lab, Friedrich-Alexander-Universität Erlangen-Nürnberg, Sept. 2010.

[4] Wilhelm Nagel. “Matchen und Mergen von 3D Punktwolken”. Seminararbeit, Universität Karlsruhe. 2002/2003.

[5] Dominik Neumann et al. “Real-time RGB-D Mapping and 3-D Modeling on the GPU using the Random Ball Cover Data Structure”. In: *2011 IEEE International Conference on Computer Vision Workshops (ICCV Workshops)*. IEEE, Nov. 2011, pp. 1161–1167. DOI: 10.1109/ICCVW.2011.6130381.

[6] Konrad Sickel. “Computerized Automatic Modeling of Medical Prostheses”. PhD Thesis. Erlangen: Pattern Recognition Lab, Friedrich-Alexander-Universität Erlangen-Nürnberg, Apr. 2013.

[7] Zhengyou Zhang. “Iterative Point Matching for Registration of Free-form Curves and Surfaces”. In: *International Journal of Computer Vision* 13.2 (Oct. 1994), pp. 119–152. DOI: 10.1007/BF01427149.

Medical Image Processing for Diagnostic Applications

Iterative Closest Point Algorithm – Theory

Online Course – Unit 70

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Topics

Iterative Closest Point (ICP)

- Theory
- Point-to-Point Error Metric
- Point-to-Plane Error Metric

Summary

- Take Home Messages
- Further Readings

Algorithm Outline

Given: Two point sets $P = \{\mathbf{p}_i\}$, $Q = \{\mathbf{q}_i\}$, $i = 1, ..., N$, where $\mathbf{p}_i$, $\mathbf{q}_i$ are 3×1 column vectors

Wanted: Best transformation $\mathbf{T}$ between these two point sets, consisting of

- rotation matrix $\mathbf{R}$,
- translation $\mathbf{t}$

Algorithm 1: Iterative closest point [5]

Input : Two point clouds: P, Q

Output: Transformation $\mathbf{T}$, which aligns P and Q

```
T ← T_0;
while not converged do
    for i ← 1 to N do
        c_i ← GetClosestPointInQ(T · p_i);
        if ||T · p_i − c_i|| ≤ θ_max then
            ω_i ← 1;
        else
            ω_i ← 0;
        end
    end
    T ← argmin_T Σ_{i=1}^{N} ω_i ||T · p_i − c_i||^2;
end
```

$$\mathbf{T} \leftarrow \arg\min_{\mathbf{T}} \sum_{i=1}^{N} \omega_i \left\|\mathbf{T}\cdot\mathbf{p}_i - \mathbf{c}_i\right\|^2$$

Point-to-Point Error Metric

$\rightarrow$ minimizes the Euclidean distance between selected point pairs.

The optimization problem can be solved by:

- a singular value decomposition (SVD) based method [1],
- a quaternion method [3],
- orthonormal matrices [4], or
- calculation based on dual quaternions [6].

Point-to-Point Error Metric (SVD) [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \left\|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right\|^2,$$

where

$$\mathbf{q}_i = \arg\min_{\mathbf{q}_x \in Q} \ \mathrm{d}\left(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i\right)$$

Decouple translation and rotation:

$\rightarrow$ computation of the center points $(\bar{\mathbf{p}}, \bar{\mathbf{q}})$ of both point sets for translation

$$\left.\begin{array}{l} P' : \ \mathbf{p}'_i = \mathbf{p}_i - \bar{\mathbf{p}}, \\ Q' : \ \mathbf{q}'_i = \mathbf{q}_i - \bar{\mathbf{q}} \end{array}\right\} \quad \Rightarrow \quad \varepsilon = \sum_{i=1}^{N} \left\|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\right\|^2$$

$\mathbf{R}$	rotation matrix
$\mathbf{t}$	translation vector
$\mathrm{d}(x, y)$	Euclidean distance between two points x and y
$Q = \{\mathbf{q}_i\}$	3-D point sets
$P = \{\mathbf{p}_i\}$	3-D point sets
$i = 1, \ldots, N$	number of points

Point-to-Point Error Metric (SVD) [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \left\| \mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' \right\|^2$$

Two simplifications:

- Index i matches the closest points of both point clouds.
- Both point clouds contain the same number of points.

The rotation matrix **R** is computed via help of the matrix **H**:

$$\mathbf{H} = \sum_{i=1}^{N} \mathbf{p}_i'\mathbf{q}_i' = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}, \qquad \mathbf{R} = \mathbf{V}\mathbf{U}^{\mathrm{T}}.$$

The decoupled translation vector is recovered by

$$\mathbf{t} = \bar{\mathbf{q}} - \mathbf{R}\bar{\mathbf{p}}.$$

Point-to-Point Error Metric (SVD) [1]

Derivation for **H**:

1. Expanding right-hand side of ε:

$$\varepsilon = \sum_{i=1}^{N} \left\| \mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right\|^2 = \sum_{i=1}^{N} \left(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right)^{\mathsf{T}} \left(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right) = \sum_{i=1}^{N} \left(\mathbf{p}'^{\mathsf{T}}_i \mathbf{R}^{\mathsf{T}} \mathbf{R}\mathbf{p}'_i + \mathbf{q}'^{\mathsf{T}}_i \mathbf{q}'_i - 2\mathbf{q}'^{\mathsf{T}}_i \mathbf{R}\mathbf{p}'_i \right)$$

2. Minimizing ε is equivalent to maximizing:

$$F = \sum_{i=1}^{N} \mathbf{q}'^{\mathsf{T}}_i \mathbf{R}\mathbf{p}'_i = \operatorname{tr}\left(\sum_{i=1}^{N} \mathbf{R}\mathbf{p}'_i \mathbf{q}'^{\mathsf{T}}_i \right) = \operatorname{tr}\left(\mathbf{R}\mathbf{H} \right)$$

Point-to-Point Error Metric (SVD) [1]

Depending on the properties of the point clouds, the solution may not be unique:

- If P' is collinear
  $\rightarrow$ infinitely many rotations and reflections.
- If P' is coplanar
  $\rightarrow$ two unique solutions, the desired rotation matrix and its reflection:
  - the reflection is given if $\det \mathbf{R} = -1$,
  - the correct rotation matrix is given by $\mathbf{R} = \mathbf{V}'\mathbf{U}^{\mathsf{T}}$,
  - $\mathbf{V}'$ is constructed by flipping the sign of the i-th column of $\mathbf{V}$, where index i identifies the zero singular value.
- If P' is not coplanar
  $\rightarrow$ one unique solution for $\mathbf{R} = \mathbf{V}\mathbf{U}^{\mathsf{T}}$.

Point-to-Point Error Metric (Quaternions) [2, 3]

Optimization function of the ICP:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} \left\| \left(\mathbf{R}(\mathbf{q}_R)\mathbf{p}_i + \mathbf{q}_T\right) - \mathbf{q}_i \right\|^2,$$

where

$$\mathbf{q}_i = \arg\min_{\mathbf{q}_x \in Q} \mathrm{d}\left(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i\right)$$

$\mathbf{R} = \mathbf{R}(\mathbf{q}_R)$	rotation matrix generated by a unit quaternion
$\mathbf{q}_R = [q_0 q_1 q_2 q_3]^\mathrm{T}$	unit quaternion (rotation)
$\mathbf{q}_T = [q_4 q_5 q_6]^\mathrm{T}$	unit quaternion (translation)
$\mathbf{q} = [\mathbf{q}_R \vert \mathbf{q}_T]^\mathrm{T}$	registration state vector
$\mathrm{d}\left(x, y\right)$	Euclidean distance between two points x and y
$Q = \{\mathbf{q}_i\}$	3-D point sets
$P = \{\mathbf{p}_i\}$	3-D point sets
$i = 1, ..., N$	number of points

Point-to-Point Error Metric (Quaternions) [2, 3]

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$Q = \{\mathbf{q}_i\}$	3-D point sets
$P = \{\mathbf{p}_i\}$	3-D point sets
$i = 1, \ldots, N$	number of points

Rotation matrix generated by a unit quaternion:

$$\mathbf{R} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{pmatrix}$$

Point-to-Point Error Metric (Quaternions) [2, 3]

Optimization function of the ICP:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} \left\|(\mathbf{R}(\mathbf{q}_R)\mathbf{p}_i + \mathbf{q}_T) - \mathbf{q}_i\right\|^2,$$

where

$$\mathbf{q}_i = \arg\min_{\mathbf{q}_x \in Q} \mathrm{d}\left(\mathbf{q}_x, \mathbf{R}\mathbf{p}_i\right)$$

Decouple translation and rotation:
$\rightarrow$ computation of the center points $(\bar{\mathbf{p}}, \bar{\mathbf{q}})$ of both point sets for translation

$$\left.\begin{array}{l} P' : \mathbf{p}'_i = \mathbf{p}_i - \bar{\mathbf{p}}, \\ Q' : \mathbf{q}'_i = \mathbf{q}_i - \bar{\mathbf{q}} \end{array}\right\} \quad \Rightarrow \quad \Sigma_{pq} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{p}'_i \mathbf{q}'^{\mathrm{T}}_i$$

$\mathbf{R} = \mathbf{R}(\mathbf{q}_R)$	rotation matrix generated by a unit quaternion
$\mathbf{q}_R = [q_0 q_1 q_2 q_3]^{\mathrm{T}}$	unit quaternion (rotation)
$\mathbf{q}_T = [q_4 q_5 q_6]^{\mathrm{T}}$	unit quaternion (translation)
$\mathbf{q} = [\mathbf{q}_R \vert \mathbf{q}_T]^{\mathrm{T}}$	registration state vector
$\mathrm{d}(x, y)$	Euclidean distance between two points x and y
$Q = \{\mathbf{q}_i\}$	3-D point sets
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$i = 1, \ldots, N$	number of points

Point-to-Point Error Metric (Quaternions) [2, 3]

- Cross-covariance matrix:

$$\Sigma_{pq} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{p}_i - \bar{\mathbf{p}})(\mathbf{q}_i - \bar{\mathbf{q}})^{\mathsf{T}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{p}_i'\mathbf{q}_i'^{\mathsf{T}},$$

$\rightarrow$ Build anti-symmetric matrix $\mathbf{A}$ with $A_{ij} = \left(\Sigma_{pq} - \Sigma_{pq}{}^{\mathsf{T}}\right)_{ij}$

$\rightarrow$ Form column vector $\Delta = (A_{23}, A_{31}, A_{12})^{\mathsf{T}}$

$\rightarrow$ Form symmetric 4×4 matrix $\mathbf{Q}(\Sigma_{pq})$:

$$\mathbf{Q}(\Sigma_{pq}) = \begin{pmatrix} \operatorname{tr}(\Sigma_{pq}) & \Delta^{\mathsf{T}} \\ \Delta & \Sigma_{pq} + \Sigma_{pq}{}^{\mathsf{T}} - \operatorname{tr}(\Sigma_{pq})\,\boldsymbol{I}_3 \end{pmatrix}, \quad \boldsymbol{I}_3 \text{ is the } 3 \times 3 \text{ identity matrix}$$

- Optimal rotation $\mathbf{q}_R$: unit eigenvector corresponding to the maximum eigenvalue of matrix $\mathbf{Q}(\Sigma_{pq})$
- Optimal translation vector $\mathbf{q}_T = \bar{\mathbf{q}} - \mathbf{R}(\mathbf{q}_R)\bar{\mathbf{p}}$

Point-to-Plane Error Metric [1]

Optimization function of the ICP:

$$\varepsilon = \sum_{i=1}^{N} \left\| \left((\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}'_j \right) \mathbf{n}_i \right\|^2,$$

where

$$\mathbf{q}'_j = \left\{ \mathbf{q} \mid \underset{\mathbf{q} \in s_j}{\arg\min} \; \| \mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q} \| \right\}$$

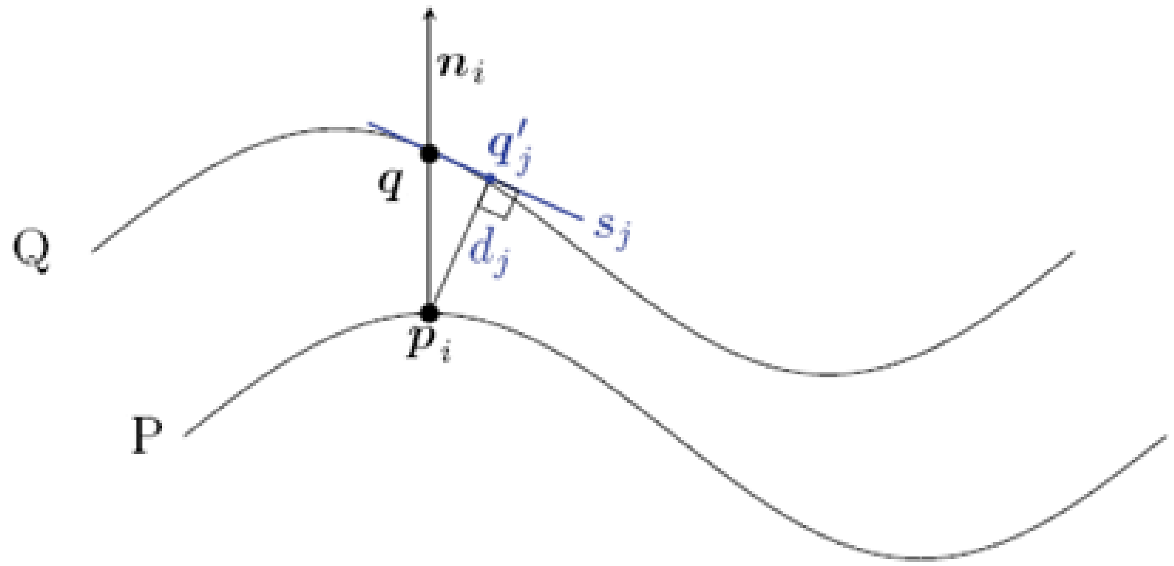

Figure 1: Image courtesy of Konrad Sickel [5]

$\mathbf{R}$	rotation matrix
$\mathbf{t}$	translation vector
d_j	distance between the tangent plane s_j and the point $\mathbf{p}_i$
$Q = \{\mathbf{q}_i\}$	3-D point sets
$P = \{\mathbf{p}_i\}$	3-D point sets
s_j	tangent plane of Q at $\mathbf{q}$
n_i	surface normal
$i = 1, ..., N$	number of points

Point-to-Plane Error Metric [1]

- More robust, accurate, and converges faster than the point-to-point error metric
- Utilizes surface normal as an additional input and allows that smooth or planar areas of the meshes slide over each other easily

Topics

Iterative Closest Point (ICP)

- Theory
- Point-to-Point Error Metric
- Point-to-Plane Error Metric

Summary

- Take Home Messages
- Further Readings

Take Home Messages

- Several ways to solve the optimization problem of the ICP algorithm are known, two of which we have seen: using SVD and quaternions.
- Several error metrics can be used, we learned about point-to-point error metrics and point-to-plane metrics.

Further Readings

[1] K. S. Arun, T. S. Huang, and S. D. Blostein. "Least-Squares Fitting of Two 3-D Point Sets". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* PAMI-9.5 (Sept. 1987), pp. 698–700. DOI: 10.1109/TPAMI.1987.4767965.

[2] Paul J. Besl and Neil D. McKay. "A Method for Registration of 3-D Shapes". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 14.2 (Feb. 1992), pp. 239–256. DOI: 10.1109/34.121791.

[3] Berthold K. P. Horn. "Closed-form Solution of Absolute Orientation Using Unit Quaternions". In: *Journal of the Optical Society of America A* 4.4 (Apr. 1987), pp. 629–642. DOI: 10.1364/JOSAA.4.000629.

[4] Berthold K. P. Horn, Hugh M. Hilden, and Shahriar Negahdaripour. "Closed-form Solution of Absolute Orientation Using Orthonormal Matrices". In: *Journal of the Optical Society of America A* 5.7 (July 1988), pp. 1127–1135. DOI: 10.1364/JOSAA.5.001127.

[5] Konrad Sickel. "Computerized Automatic Modeling of Medical Prostheses". PhD Thesis. Erlangen: Pattern Recognition Lab, Friedrich-Alexander-Universität Erlangen-Nürnberg, Apr. 2013.

[6] Michael W. Walker, Lejun Shao, and Richard A. Volz. "Estimating 3-D Location Parameters Using Dual Number Quaternions". In: *CVGIP: image understanding* 54.3 (Nov. 1991), pp. 358–367. DOI: 10.1016/1049-9660(91)90036-O.

Medical Image Processing for Diagnostic Applications

Iterative Closest Point Algorithm – Variants

Online Course – Unit 71

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Topics

Efficient Variants of the ICP Algorithm

Summary

Take Home Messages

Further Readings

Efficient Variants of the ICP Algorithm [1]

Variants grouped by affecting one of the following six stages of the algorithm:

1. Selection of some points in one or both meshes
2. **Matching** these points to samples in the other mesh
3. Weighting the corresponding pairs appropriately
4. Rejecting certain pairs based on looking at each pair individually or considering the entire set of pairs
5. Assigning an **error metric** based on the point pairs
6. **Minimizing** the error metric

(1) Selection of Points

- Always using all available points
- Uniform subsampling of the available points
- Random sampling (with a different sample of points at each iteration)
- Selection of points with high intensity gradient, in variants that use per-sample color or intensity to aid in alignment
- Each of the preceding schemes may select points on only one mesh, or select source points from both meshes
- Using distribution of normals among the selected points

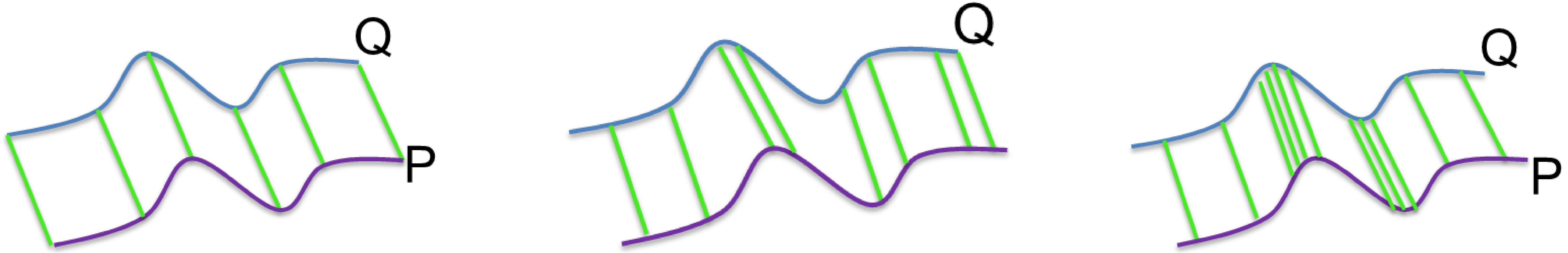

Figure 1: Possible selection strategies

(3) Weighting of Pairs

- Constant weight
- Assigning lower weights to pairs with greater point-to-point distances
- Weighting based on compatibility of normals
- Weighting based on the expected effect of scanner noise on the uncertainty in the error metric

Algorithm 1: Iterative closest point

Input : Two point clouds: P, Q
Output: Transformation $\boldsymbol{T}$, which aligns P and Q

$\boldsymbol{T} \leftarrow \boldsymbol{T}_0$;
while *not converged* **do**
  for $i \leftarrow 1$ **to** N **do**
    $\boldsymbol{c}_i \leftarrow$ `GetClosestPointInQ(`$\boldsymbol{T} \cdot \boldsymbol{p}_i$`)`;
    if $\|\boldsymbol{T} \cdot \boldsymbol{p}_i - \boldsymbol{c}_i\| \leq \theta_{max}$ **then**
      $\omega_i \leftarrow 1$;
    else
      $\omega_i \leftarrow 0$;
    end
  end
  $\boldsymbol{T} \leftarrow \arg\min_{\boldsymbol{T}} \sum_i^N \omega_i \|\boldsymbol{T} \cdot \boldsymbol{p}_i - \boldsymbol{c}_i\|^2$;
end

(4) Rejecting Pairs

- Rejection of corresponding points more than a given distance apart
- Rejection of the worst n% of pairs based on some metric
- Rejection of pairs whose point-to-point distance is larger than some multiple of the standard deviation of distances
- Rejection of pairs that are not consistent with neighboring pairs
- Rejection of pairs containing points on mesh boundaries

Figure 2: Possible rejection strategies

Pros and Cons

+ Simplicity
+ Relatively quick performance (implemented with kd-trees for closest-point look up)

- Implicit assumption of full overlap of the shapes (maximum distance threshold)
- Theoretical requirement: points are taken from a known surface (different discretizations)

Topics

Efficient Variants of the ICP Algorithm

Summary

- Take Home Messages
- Further Readings

Take Home Messages

Summary of the last three units:

- ICP = Iterative Closest Point
- Introduced early 1990s
- Goal: Find transformation between two point clouds via minimization of the difference
- Different data types
- Point-to-Point Metric
  - SVD
  - Quaternions
- Point-to-Plane Metric
- Variants of the ICP

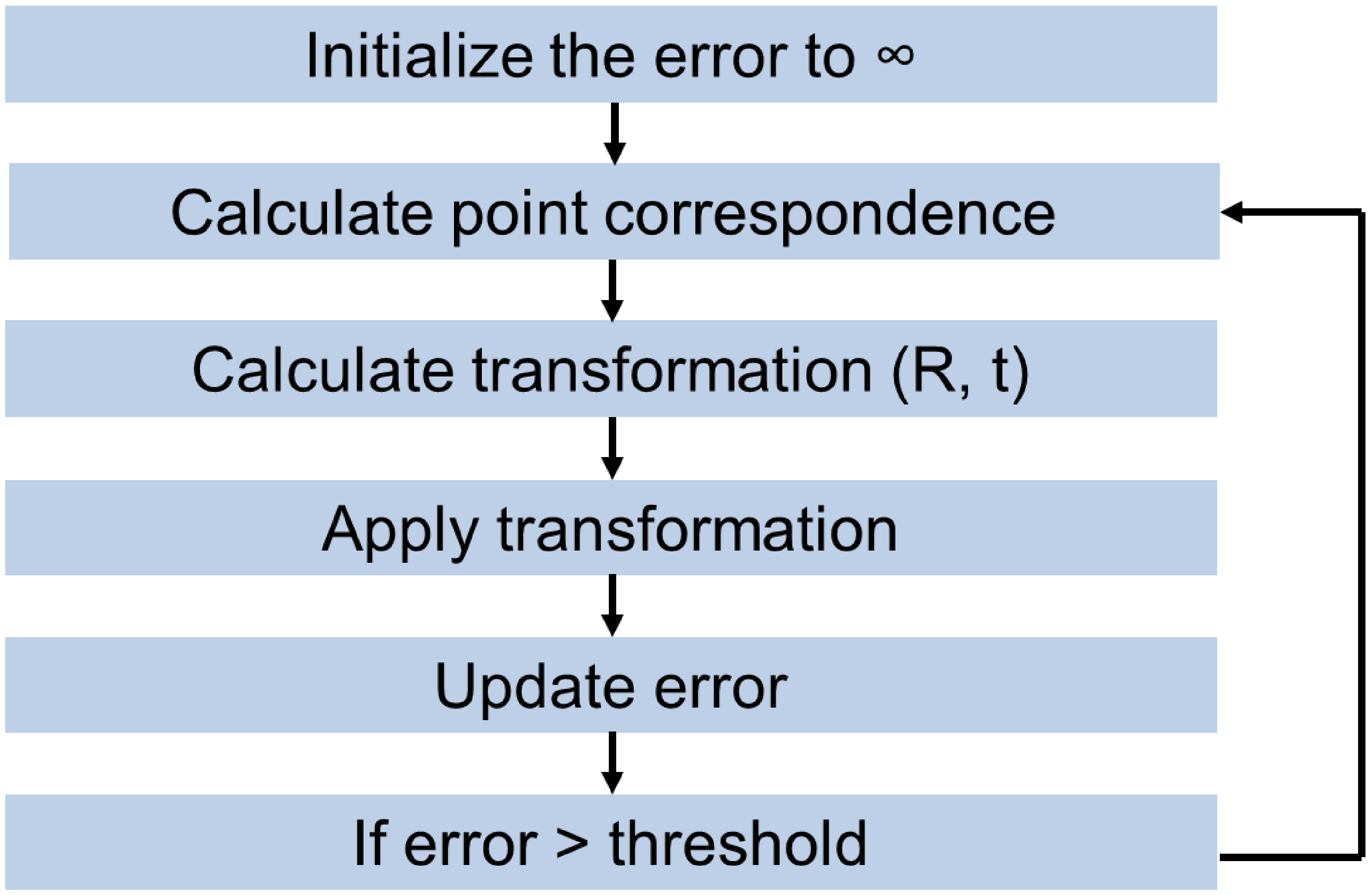

Figure 3: Scheme of the ICP algorithm

Further Readings

[1] Szymon Rusinkiewicz and Marc Levoy. "Efficient Variants of the ICP Algorithm". In: *Third International Conference on 3-D Digital Imaging and Modeling, 28 May – 1 June, Quebec City, Canada. Proceedings*. IEEE, 2001, pp. 145–152. DOI: 10.1109/IM.2001.924423.

[2] Aleksandr V. Segal, Dirk Haehnel, and Sebastian Thrun. "Generalized-ICP". In: *Robotics: Science and Systems V, Seattle, USA, June 28 – July 1, 2009*. MIT Press, 2009. DOI: 10.15607/RSS.2009.V.021.