Medical Image Processing for Diagnostic Applications

Iterative Reconstruction – Linear Equations

Online Course – Unit 55

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

virtuelle hochschule bayern

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Linear Equations
Example
Kaczmarz Method

Summary
Take Home Messages
Further Readings

Example: Backprojection

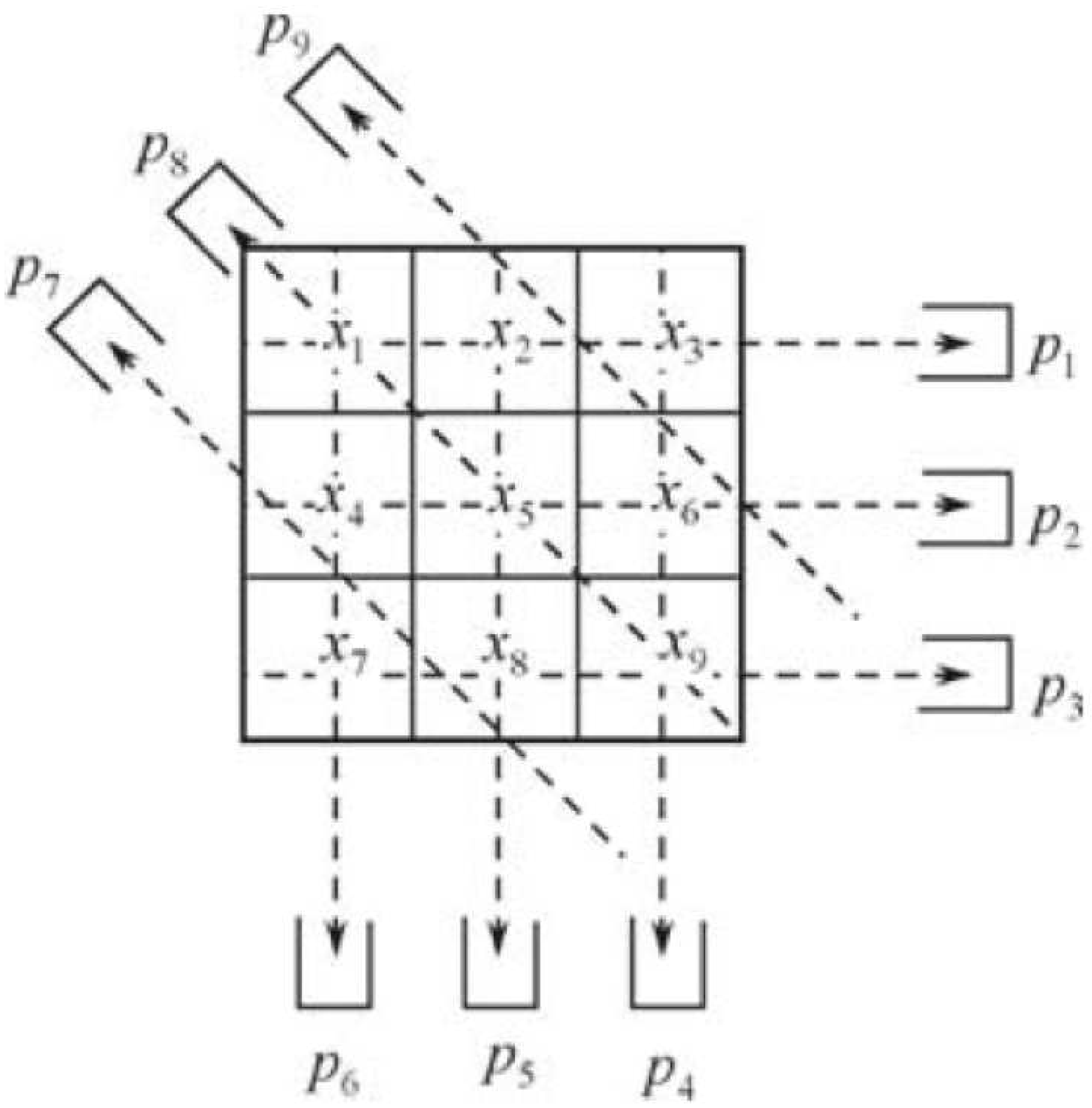

Figure 1: A 3×3 volume is backprojected from 3 projections with 3 detector pixels (Zeng, 2009).

$$
\begin{aligned}
x_1 + x_2 + x_3 &= p_1 \\
x_4 + x_5 + x_6 &= p_2 \\
x_7 + x_8 + x_9 &= p_3 \\
x_3 + x_6 + x_9 &= p_4 \\
x_2 + x_5 + x_8 &= p_5 \\
x_1 + x_4 + x_7 &= p_6 \\
2(\sqrt{2}-1)x_4 + (2-\sqrt{2})x_7 + 2(\sqrt{2}-1)x_8 &= p_7 \\
\sqrt{2}x_1 + \sqrt{2}x_5 + \sqrt{2}x_9 &= p_8 \\
2(\sqrt{2}-1)x_2 + (2-\sqrt{2})x_3 + 2(\sqrt{2}-1)x_6 &= p_9
\end{aligned}
$$

Example: Linear Equation

Rewrite the single equations to

$$\boldsymbol{AX} = \boldsymbol{P}$$

with

$$\boldsymbol{X} = (x_1, x_2, \dots, x_9)^\mathsf{T} \in \mathbb{R}^n, \quad \boldsymbol{P} = (p_1, p_2, \dots, p_9)^\mathsf{T} \in \mathbb{R}^m.$$

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the system matrix with elements a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$.
- The a_{ij} describe the contribution of each voxel to each ray.

Example: Solution?

The linear system of equations

$$\boldsymbol{AX} = \boldsymbol{P}$$

can mathematically be solved by using

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{P},$$

or

$$\boldsymbol{X} = \left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{P},$$

or

$$\boldsymbol{X} = \boldsymbol{A}^{\mathsf{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathsf{T}}\right)^{-1}\boldsymbol{P},$$

but practically these methods are infeasible (Gauss-Seidel, SVD, etc.).

$\rightarrow$ A solution that does not require the inversion of $\boldsymbol{A}$, or a matrix product is desirable.

Kaczmarz Method

- Each pixel can be interpreted as a linear equation.
- This equation forms a line (2-D) or a hyperplane (higher dimensions) in the solution space.
- The point of intersection forms the correct solution.
- Projection onto the respective hyperplane forms a solution that fulfills the respective equation.
- Repetition yields an improved solution.

Example with Two Voxels

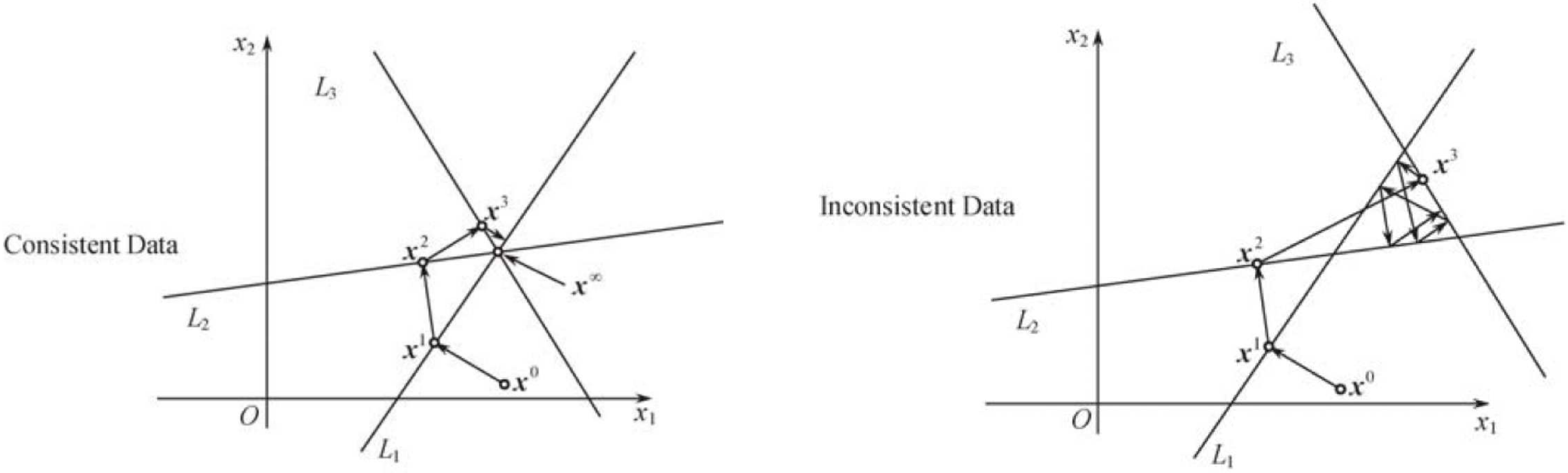

Figure 2: In practice, it is unlikely to get consistent data (left), it usually is inconsistent (right) (Zeng, 2009).

Projection onto a Hyperplane

- Consider a point $\boldsymbol{x} \in \mathbb{R}^n$ and a hyperplane
$$\{\boldsymbol{c} \in \mathbb{R}^n \mid \boldsymbol{n}^\top \boldsymbol{c} = d\}$$
for $d \in \mathbb{R}$ and a given $\boldsymbol{n} \in \mathbb{R}^n$.
- The projection $\boldsymbol{x}'$ of $\boldsymbol{x}$ must be in direction of the normal vector $\boldsymbol{n}$:
$$\boldsymbol{x}' = \boldsymbol{x} + \lambda \boldsymbol{n}.$$
- $\boldsymbol{x}'$ is on the hyperplane:
$$\begin{aligned} \boldsymbol{n}^\top \boldsymbol{x}' &= d, \\ \boldsymbol{n}^\top (\boldsymbol{x} + \lambda \boldsymbol{n}) &= d, \\ \boldsymbol{n}^\top \boldsymbol{x} + \lambda \boldsymbol{n}^\top \boldsymbol{n} &= d, \\ \lambda &= \frac{d - \boldsymbol{n}^\top \boldsymbol{x}}{\boldsymbol{n}^\top \boldsymbol{n}}, \\ \Rightarrow \qquad \boldsymbol{x}' &= \boldsymbol{x} + \frac{d - \boldsymbol{n}^\top \boldsymbol{x}}{\boldsymbol{n}^\top \boldsymbol{n}} \boldsymbol{n}. \end{aligned}$$

Topics

Linear Equations

Example

Kaczmarz Method

Summary

Take Home Messages

Further Readings

Take Home Messages

- The projection process can be formulated as a system of linear equations.
- Using Kaczmarz method, we iteratively project approximate solutions to different hyperplanes.

Further Readings

References and related books for the discussed topics in iterative reconstruction:

Gengsheng Lawrence Zeng. *Medical Image Reconstruction – A Conceptual Tutorial*. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-05368-9

Stefan Kaczmarz. "Angenäherte Auflösung von Systemen linearer Gleichungen". In: *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques* 35 (1937), pp. 355–357 For this article you can find an English translation here (December 2016).

Avinash C. Kak and Malcolm Slaney. *Principles of Computerized Tomographic Imaging*. Classics in Applied Mathematics. Accessed: 21. November 2016. Society of Industrial and Applied Mathematics, 2001. DOI: 10.1137/1.9780898719277. URL: http://www.slaney.org/pct/

H. Bruder et al. "Adaptive Iterative Reconstruction". In: *Medical Imaging 2011: Physics of Medical Imaging*. Ed. by Norbert J. Pelc, Ehsan Samei, and Robert M. Nishikawa. Vol. 7961. Proc. SPIE 79610J. Feb. 2011, pp. 1–12. DOI: 10.1117/12.877953