Diagnostic Medical Image Processing
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Exercises (DMIP-E)
WS 2016/17

LEHRSTUHL FÜR MUSTER-ERKENNUNG

Singular Value Decomposition (SVD)

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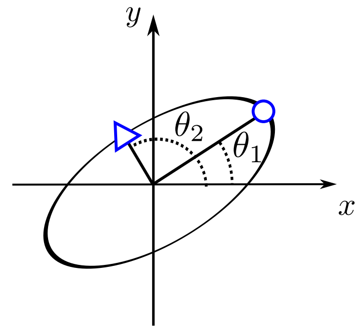
1 Understanding SVD

(i) Take a look at the ellipse on the right. Let $\theta_1 = 30°$, $\theta_2 = 120°$, the coordinates of the circle and triangle in the shown axes are $\left(3\sqrt{3},\, 3\right)$ and $\left(-1,\, \sqrt{3}\right)$.
Use your knowledge about the SVD to find a matrix, that maps the ellipse to the unit sphere. Is that a unique mapping?
Can you also find a transformation that preserves the direction from the origin to both the circle and the triangle, respectively?
(You can solve this exercise analytically, but a correct numerical solution using SVD is also accepted.)



$\boxed{\mathbf{2}}$

From the lecture, we know that there is a matrix $\mathbf{A} = \mathbf{U\Sigma V}^T$ which represents the required transformation. In order to map the ellipse to the unit sphere, $\mathbf{\Sigma V}^T$ is sufficient, which resembles the intuitive way of rotating the semiaxes to the coordinate system axes and then scaling. If we need to preserve the directions of the semiaxes, then we need to rotate back.

A correct mapping to the unit sphere is for instance:

$$\widetilde{\mathbf{V}} = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\widetilde{\mathbf{\Sigma}} = \begin{pmatrix} \frac{1}{\left\|\left(3\sqrt{3},3\right)^T\right\|_2} & 0 \\ 0 & \frac{1}{\left\|\left(-1,\sqrt{3}\right)^T\right\|_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

$\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^T$ maps the semiaxes to the canonical unit vectors:

$$\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^T \begin{pmatrix} 3\sqrt{3} \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^T \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any orthogonal transformation of $\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^T$ maps the ellipse to the unit sphere, so this is not unique. To restore the directions, we need to rotate back by

$$\widetilde{\mathbf{U}} = \left(\widetilde{\mathbf{V}}^T\right)^{-1} = \widetilde{\mathbf{V}}.$$

The SVD is defined uniquely by sorting the singular values which in the end gives

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \frac{1}{12}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{pmatrix}.$$

(ii) Which of the following are common applications of the SVD?

- ■ computation of condition numbers
- ☐ ranking matrices
- ■ low-rank approximations of images
- ■ solving linear systems
- ☐ computation of multiple values
- ■ computation of the null space

2 Condition of a matrix – Programming Exercise

In the lecture we have seen the matrix

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix}.$$

(i) Compute the condition number of $\mathbf{A}$ with respect to the 2-norm and compare your result with the lecture (*hint:* class `DecompositionSVD`).

(ii) Recall that the numerical rank of a matrix $\mathbf{M}$ is defined by the number

$$\text{rank}_\epsilon\left(\mathbf{M}\right) = \#\left\{\sigma_i > \epsilon,\, \sigma_i \text{ singular value of } \mathbf{M}\right\}.$$

By setting $\epsilon = 10^{-3}$, we get a rank deficiency in $\mathbf{A}$. Can you directly tell nullspace and range from your SVD computations in (i)? What are those?

(iii) Given the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1.001 \\ 0.999 \\ 1.001 \end{pmatrix}$, show that a variation of the elements of $\mathbf{b}$ by 0.1 % implies a change in $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ by at least 240 %.

(iv) Compute the condition number (w.r.t. 2-norm) for the matrix $\mathbf{B} \in \mathbb{R}^{20\times 20}$ defined by

$$\mathbf{B} = \mathbf{U}\,\mathrm{diag}\,(a_1, ..., a_{20})\,\mathbf{V}^T, \quad a_n = \frac{1}{(n-5)^2+4}, \quad n = 1, ..., 20.$$

2+1+2+1

(i) code implementation

(ii) code implementation

(iii) code implementation

(iv) correct answer is

$$\kappa\,(\mathbf{B}) = \frac{a_5}{a_{20}} = \frac{15^2+4}{0^2+4} = \frac{229}{4} = 57.25$$

3 Optimization Problems – Programming Exercise

(i) Implement and verify optimization problem 1 from the lecture.

(ii) Optimization problem 2: Four 2-D vectors were given on the lecture slides. Implement the optimization problem for the general case, e.g. 5, 6, 20 or N vectors.

(iii) Implement the third optimization problem using the image `mr_head_angio.jpg`. How many approximations do we have? Which rank-k-approximations are sufficient? Plot the RMSE for k=1,...,150 (*hint:* class `NumericGridOperator`).

(iv) Compute the regression line through the following set of 2-D points:

$$\{(-3,7), (-2,8), (-1,9), (0,3.3), (1.5,2), (2,-3), (3.1,4), (5.9,-0.1), (7.3,-0.5)\}.$$

2+1+3+1

(i) code implementation

(ii) code implementation

(iii) code implementation

(iv) code implementation

Total: 16