# Projection Models and Homogeneous Coordinates

## Extrinsic and Intrinsic Camera Parameters

Refresher Course

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# Topics

# Extrinsic Camera Parameters

So far we have described the projection of a 3-D point into the image plane.

We have not considered the motion of the position and orientation of the acquisition device yet:

- an X-ray source can be translated in 3-D,
- an X-ray source can be rotated in 3-D.

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## Definition

***Extrinsic parameters*** characterize the *pose*, i. e., position and orientation of the camera with respect to a world coordinate system. The position is defined by a 3-D translation vector, the orientation by three rotation angles.

# Extrinsic Camera Parameters

Figure 1: C-arm device in different positions and orientations that can be characterized by the extrinsic parameters of the acquisition device (image courtesy of Siemens Healthcare)

# Extrinsic Camera Parameters

**Mathematical characterization:**
Rotation and translation of a 3-D point can be expressed by:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \boldsymbol{t},$$

where

- $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ denotes a rotation matrix (with its known properties), and
- $\boldsymbol{t} \in \mathbb{R}^3$ represents a translation in Euclidean space.

This is an affine mapping.

# Extrinsic Camera Parameters

Using homogeneous coordinates we can rewrite the affine as a linear mapping:

$$\begin{pmatrix} wx' \\ wy' \\ wz' \\ w \end{pmatrix} = \boldsymbol{D} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \left( \begin{array}{ccc|c} & \boldsymbol{R} & & \boldsymbol{t} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} .$$

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**Problem:** How does the rotation matrix look like?

**Solution:** As we already know, the columns of the linear mapping are the images of the base vectors of the original coordinate system.

# Topics

# Intrinsic Camera Parameters

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## Definition

***Intrinsic parameters*** define the mapping of 2-D coordinates from the ideal image plane to the 2-D detector coordinates.

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- The pixels in the detector coordinate system are not necessarily square pixels, but scaled by $k_x$ and $k_y$.
- There might exist a radial distortion due to the camera optics (not considered here).

# Intrinsic Camera Parameters

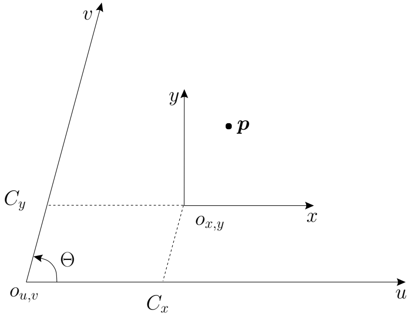


Figure 2: Detector and ideal image coordinate system

$(x, y)$ – ideal image coordinate system:

- used in all formulas so far
- $o_{x,y}$: origin

$(u, v)$ – detector coordinate system:

- real image matrix of measurements
- $\Theta$: skew angle between axes
- $k_x$, $k_y$: scaling of $u$ and $v$ axis with respect to units in $(x, y)$-system
- $(C_x, C_y)$: offset of origins of both coordinate systems

# Intrinsic Camera Parameters

**Transformation between $(u, v)$- and $(x, y)$-coordinate system**

At first, we consider the images of base vectors of the detector coordinate system in the image coordinate system:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{k_x} \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{k_y}\cos\Theta \\ \frac{1}{k_y}\sin\Theta \end{pmatrix}.$$

The required transform from the $(x, y)$- to the $(u, v)$-coordinate system is given by the inverse of the mapping above:

$$\boldsymbol{T} = \begin{pmatrix} \frac{1}{k_x} & \frac{1}{k_y}\cos\Theta \\ 0 & \frac{1}{k_y}\sin\Theta \end{pmatrix}^{-1} = \begin{pmatrix} k_x & -k_x\frac{\cos\Theta}{\sin\Theta} \\ 0 & \frac{k_y}{\sin\Theta} \end{pmatrix}.$$

# Intrinsic Camera Parameters

The complete mapping of $(x, y)$- to $(u, v)$-coordinates in Euclidean space is thus given by:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k_x & -k_x \frac{\cos\Theta}{\sin\Theta} \\ 0 & \frac{k_y}{\sin\Theta} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} C_x \\ C_y \end{pmatrix}.$$

Using homogeneous coordinates we get the matrix including the described intrinsic parameters that maps the ideal image coordinates to the detector coordinates:

$$\boldsymbol{K} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{T}$}} & -C_x \\ & & -C_y \\ \hline 0 & 0 & 1 \end{array}\right).$$

# Topics

Extrinsic Camera Parameters

Intrinsic Camera Parameters

Complete Projection

Summary
- Take Home Messages
- Further Readings

# Complete Projection

The total perspective transformation is:

$$\boldsymbol{P}\widetilde{\boldsymbol{p}} = \boldsymbol{K}\boldsymbol{P}_{\text{proj}}\boldsymbol{D}\widetilde{\boldsymbol{p}}.$$

- $\boldsymbol{D}$: extrinsic camera parameters
  $\rightarrow$ position and orientation of camera w. r. t. the world coordinate system
- $\boldsymbol{P}_{\text{proj}}$: projection model matrix, ideal perspective projection
- $\boldsymbol{K}$: intrinsic camera parameters
  - optical and geometric characteristics of the camera
  - do *not* change with camera movement

# Topics

Extrinsic Camera Parameters

Intrinsic Camera Parameters

Complete Projection

Summary

- Take Home Messages
- Further Readings

# Take Home Messages

- For projections with a real detector extrinsic and intrinsic camera parameters have to be considered.
- Extrinsic parameters describe the source/camera movement, and can be written as a linear mapping in homogeneous coordinates.
- Intrinsic parameters describe the (usually constant) deviations of the detector from an ideal image plane, and can be written as a linear mapping in homogeneous coordinates as well.

# Further Readings

For further details on geometric aspects of imaging see:


1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993