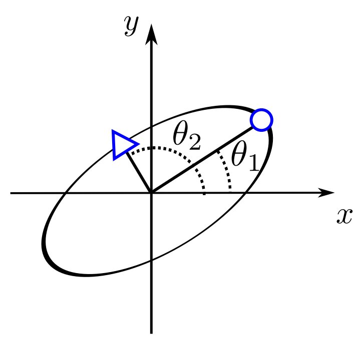
Diagnostic Medical Image Processing
Prof. Dr.-Ing. Andreas Maier
Exercises (DMIP-E)
WS 2016/17

Singular Value Decomposition (SVD)

Frank Schebesch (frank.schebesch@fau.de) Exercise Sheet 1

1 Understanding SVD

(i) Take a look at the ellipse on the right. Let $\theta_1 = 30°$, $\theta_2 = 120°$, the coordinates of the circle and triangle in the shown axes are $(3\sqrt{3}, 3)$ and $(-1, \sqrt{3})$.
Use your knowledge about the SVD to find a matrix, that maps the ellipse to the unit sphere. Is that a unique mapping?
Can you also find a transformation that preserves the direction from the origin to both the circle and the triangle, respectively?
(You can solve this exercise analytically, but a correct numerical solution using SVD is also accepted.)



$\boxed{2}$

(ii) Which of the following are common applications of the SVD?

- ☐ computation of condition numbers
- ☐ ranking matrices
- ☐ low-rank approximations of images
- ☐ solving linear systems
- ☐ computation of multiple values
- ☐ computation of the null space

$\boxed{1}$

2 Condition of a matrix

In the lecture we have seen the matrix

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix}.$$

(i) Compute the condition number of $\mathbf{A}$ with respect to the 2-norm and compare your result with the lecture (*hint:* class `DecompositionSVD`).

(ii) Recall that the numerical rank of a matrix $\mathbf{M}$ is defined by the number

$$\text{rank}_\epsilon(\mathbf{M}) = \#\{\sigma_i > \epsilon,\, \sigma_i \text{ singular value of } \mathbf{M}\}.$$

By setting $\epsilon = 10^{-3}$, we get a rank deficiency in $\mathbf{A}$. Can you directly tell nullspace and range from your SVD computations in (i)? What are those?

(iii) Given the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1.001 \\ 0.999 \\ 1.001 \end{pmatrix}$, show that a variation of the elements of $\mathbf{b}$ by 0.1 % implies a change in $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ by at least 240 %.

(iv) Compute the condition number (w.r.t. 2-norm) for the matrix $\mathbf{B} \in \mathbb{R}^{20\times 20}$ defined by

$$\mathbf{B} = \mathbf{U}\,\text{diag}\,(a_1, ..., a_{20})\,\mathbf{V}^T, \quad a_n = \frac{1}{(n-5)^2+4}, \quad n = 1, ..., 20.$$

2+1+2+1

3 Optimization Problems

(i) Implement and verify optimization problem 1 from the lecture.

(ii) Optimization problem 2: Four 2-D vectors were given on the lecture slides. Implement the optimization problem for the general case, e.g. 5, 6, 20 or N vectors.

(iii) Implement the third optimization problem using the image `mr_head_angio.jpg`.
How many approximations do we have? Which rank-k-approximations are sufficient?
Plot the RMSE for k=1,...,150 (*hint:* class `NumericGridOperator`).

(iv) Compute the regression line through the following set of 2-D points:

$$\{(-3,7),\ (-2,8),\ (-1,9),\ (0,3.3),\ (1.5,2),\ (2,-3),\ (3.1,4),\ (5.9,-0.1),\ (7.3,-0.5)\}.$$

2+1+3+1

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Preprocessing/Image Undistortion

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Exercise Sheet 2

4 Preprocessing

(i) Shortly explain the term "preprocessing".

(ii) Artifacts are often encountered in medical imaging, and might obscure or simulate pathologies. There are many types of image artifacts, name at least three types of image artifacts **and** their origins.

(iii) Explain interpolation and regression and the difference of both in your own words.

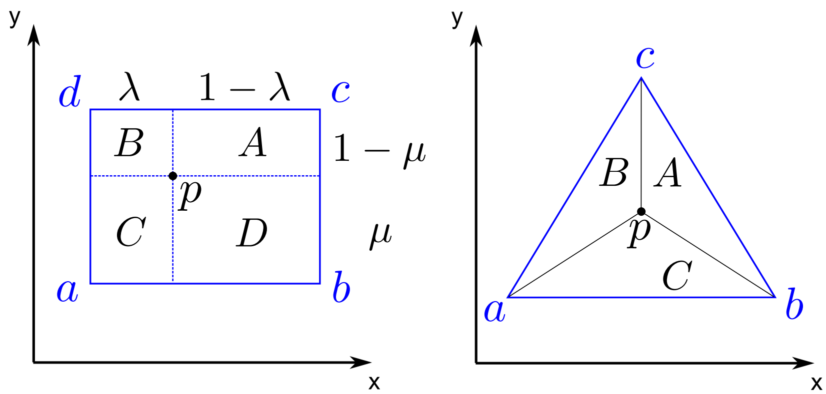
(iv) An undistorted image $\mathbf{I}_1$ has to be generated from a distorted image $\mathbf{I}_0$. Describe the interpolation problems associated with the mapping from the distorted image to the undistorted image. What are the disadvantages of using the points of $\mathbf{I}_0$ for sampling to generate $\mathbf{I}_1$? Outline a good solution and a solution fixing the associated problems directly.

$\boxed{\mathbf{1+1.5+1.5+2}}$

5 Polynomial Mappings

A parametric mapping for the distortion correction of a distorted image can be estimated by bivariate polynomials, i.e. $X(x', y')$ for the mapped x-coordinate of a point, in which (x', y') is a point in the undistorted image:

$$x = X(x', y') = \sum_{i=0}^{d} \sum_{j=0}^{d-i} u_{i,j} x'^i y'^j.$$



(i) How many multiplications does a naive evaluation of this polynomial take?

(ii) How can the number of multiplications for the evaluation be efficiently reduced and how many multiplications are necessary to evaluate the polynomial after the reduction?

(iii) Use the Horner scheme to evaluate the polynomial for $d = 2$.

(iv) Given six calibration points from the undistorted image, i.e., (x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4), (x'_5, y'_5), (x'_6, y'_6), and $d = 2$, write the measurement matrix $\mathbf{A}$. How would you call this matrix in 1D?

(v) Have a look at the diagrams. What is the value at $p = (3, 5)$ when the corner nodes of the rectangle in the left diagram are $(2, 4)$, $(6, 4)$, $(6, 7)$, $(2, 7)$, and $a = 2$, $b = 5$,$c = 3$,$d = -1$?

(vi) Looking at the process of how you computed the last value, can you also compute the value at $p = (3, 2.464)$ in the right diagram with the corner nodes $(1, 1)$, $(5, 1)$, $(3, 4.464)$, and $a = 100$, $b = 250$, and $c = 250$.

$\boxed{\mathbf{1{+}1{+}1{+}1{+}1{+}1}}$

6 Image Undistortion – Programming Exercise

Have a look at the code file *exercise2.java*. First, search for the main method and see what is happening there. Go through the code of `generateDistortedImage()` to understand how the distorted test images for this exercise are artificially created.
Your task is to undo this distortion by implementing the method `doImageUndistortion()` using the polynomial undistortion approach which you have seen in the lecture.

1. In real world we would know the relation between the undistorted and the distorted image by point correspondences of a calibration pattern. However, in this example we use the artificial distortion field.
   Therefore, we assume lattice points distributed over the whole image domain, e.g. a 8×8 lattice. For these 64 positions we know the relation between the undistorted (Xu,Yu) and the distorted image (Xd,Yd).

Set the number of lattice points and sample the undistorted and distorted image at those positions.
Attention: We want to **sample the distorted image** to get the corrected image. When creating the distortion field the ideal image was sampled to get the distorted one. Check the direction of the distortion regarding the undistorted and distorted image!

2. The distortion correction is described in the slide set for unit 11. d is the polynomial's degree. What is the maximum number for d w.r.t. the given number of sampling points?

3. Build the matrix $\mathbf{A}$ where the number of rows is the number of correspondences, and the number of columns is the number of coefficients (see "Gaußsche Summenformel").

4. Compute the pseudoinverse $\mathbf{A}^\dagger$ and use it to estimate the coefficients $u_{i,j}$ and $v_{i,j}$.

5. Compute the sampling grid for an undistorted image and resample the distorted image to obtain the undistorted image using bilinear interpolation. Show the result.

$\boxed{\mathbf{2+1+0.5+1.5+1}}$

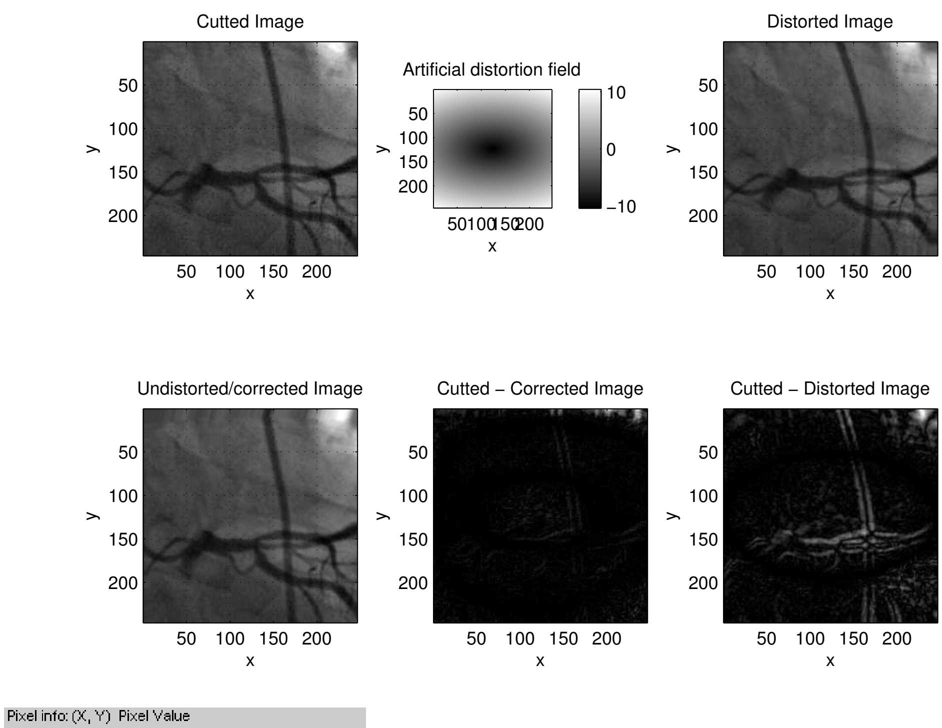


Figure 1: Original image, artificial distortion field and image undistortion result.

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Projection Models

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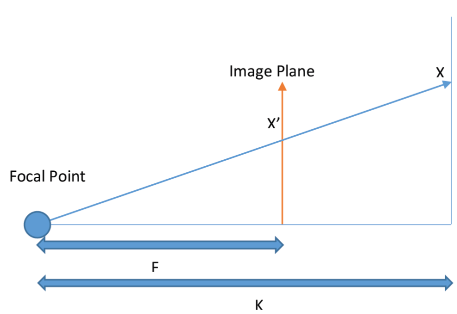
Exercise Sheet 3

7 Projections

(i) We want to find the projection of the point X on the image plane as shown in the figure.

- What kind of projection is that and how is X mapped to its projection point X' on the image plane?
- Which difficulty is connected with this mapping?
- Which kind of mapping could we use instead if we wanted to approximate the projection from above?
- For both projection models, write down the mapping for 3-D to 2-D *cartesian* coordinates.



(ii) Of the following projection models, for which ones do all projected points pass through the origin of the camera coordinate system?

- ☐ respective projection
- ☐ orthographic projection
- ☐ weak-perspective projection
- ☐ paraperspective projection
- ☐ perspective projection
- ☐ no-perspective projection

2+1

8 Homogeneous Coordinates

(i) Find the intersection point of the following two lines:

$$l_1 : 3x + 4y = 6,$$
$$l_2 : \quad x + y = 2.$$

(ii) Compute the intersection of the parallel 2-D lines [a,b,c] and [a,b,c'] (notation from the lecture).
(*Hint*: Check out which 2-D point you get if you calculate the cross product of the coordinate axes.)

(iii) Can you find an inhomogeneous 2-D representation of the intersection point? Otherwise describe why you cannot.

(iv) Do these results in 2-D match the intuition that parallel lines "meet at infinity"?

1+1+1+1

9 Camera Parameters

(i) What are extrinsic camera parameters and intrinsic camera parameters?

(ii) With your new digital camera (focal length f=0.3 cm, zero pinhole offset, perfect square pixels, i.e., $k_x = k_y$, camera skew is exactly 90°) you want to study the effects of perspective distortion. Therefore you take a picture of the rails at an inoperative side track near your local train station. The coordinates of the camera's optical center with respect to the world coordinate system are $C = (0, 0, h)^T$, where $h = 60$ cm. The camera's principal axis is parallel to the rails. Write the expression for the camera's full projection matrix P.

(iii) An object is observed and its center is located at $\mathbf{x}_0 = (1.2, 3.6, 2.0)^T \in \mathbb{R}^3$. The object is rotated around the x-axis by $\theta_x = 30°$ and by $\theta_y = 90°$ around the y-axis. Furthermore the camera is translated by $\mathbf{t} = (1.3, 2.2, 2.0)^T$. Finally the object is projected perspectively to the image plane with focal length $f = 4$.

- Does it make a difference which rotation to perform first? Why?
- State the translation, rotation and projection mapping in a single transformation matrix T and apply T appropriately to find the homogeneous point $\mathbf{x}'_1 \in \mathbb{P}^3$.
- Calculate the projected center $\mathbf{x}_1 \in \mathbb{R}^3$.

0.5+1+1.5

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MR Intensity Inhomogeneities

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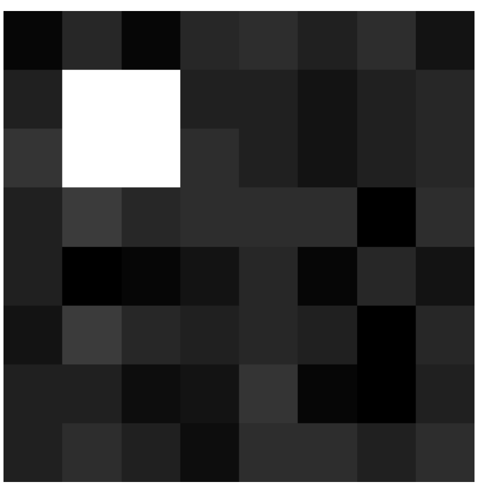
Exercise Sheet 4

11 Histograms and KL Divergence

(i) Originally, the pixels in the following image were distributed by a nice and well-known distribution. Unfortunately, the image got somehow corrupted on the disk and the white area now shows the wrong pixel value (see matrix on the right). Can you guess what value it has originally shown? Can you also support your guess by an argument using histograms?
[The image data can also be found in the file `whatpixelvalue.csv`.]



$$\begin{bmatrix} 2 & 7 & 2 & 7 & 8 & 6 & 8 & 4 \\ 6 & 40 & 40 & 6 & 6 & 4 & 6 & 7 \\ 9 & 40 & 40 & 8 & 6 & 4 & 6 & 7 \\ 6 & 10 & 7 & 8 & 8 & 8 & 1 & 8 \\ 6 & 1 & 2 & 4 & 7 & 2 & 7 & 4 \\ 4 & 10 & 7 & 6 & 7 & 6 & 1 & 7 \\ 6 & 6 & 3 & 4 & 9 & 2 & 1 & 6 \\ 6 & 8 & 6 & 3 & 8 & 8 & 6 & 8 \end{bmatrix}$$

(ii) Write down the definition of the Kullback-Leibler (KL) divergence between two discrete probability density functions p and q. Show its relation to entropy.

(iii) Suppose we have four events a, b, c, d which are distributed as follows:

$$p(a) = \frac{3}{5},\ p(b) = \frac{1}{5},\ p(c) = \frac{1}{5},\ p(d) = 0.$$

From noisy measurements we determined the following relative frequencies:

$$q(a) = \frac{5}{9},\ q(b) = \frac{3}{9},\ \text{and } q(d) = \frac{1}{9}.$$

We want to use the KL divergence $\mathrm{KL}(p, q)$ to decide if the measurement approximates the actual distribution. Can you compute it? State the reason if not.
Hint:

$$\lim_{p\to 0} p \log p = 0, \quad \lim_{q\to 0} p \log \frac{p}{q} = \infty,\ p \neq 0$$

(iv) We now apply a smoothing trick for both distributions, i.e., we add a probability of $\epsilon = 10^{-3}$ to those events with zero probability/frequency and distribute the error made to equal parts on the events with nonzero probability/frequency:

$$p_\epsilon(a) = \frac{3}{5} - \frac{\epsilon}{3},\ p_\epsilon(b) = \frac{1}{5} - \frac{\epsilon}{3},\ p_\epsilon(c) = \frac{1}{5} - \frac{\epsilon}{3},\ p_\epsilon(d) = \epsilon,$$
$$q_\epsilon(a) = \frac{5}{9} - \frac{\epsilon}{3},\ q_\epsilon(b) = \frac{3}{9} - \frac{\epsilon}{3},\ q_\epsilon(c) = \epsilon,\ q_\epsilon(d) = \frac{1}{5} - \frac{\epsilon}{3}.$$

Compute the KL divergence $\mathrm{KL}(p_\epsilon, q_\epsilon)$. What is your conclusion?

$\boxed{\mathbf{1.5+1+1+1}}$

12 Bias Field Correction

(i) What are the three major causes for intensity inhomogeneities in MR imaging?

(ii) MRI Inhomogenities are often modeled by a pixelwise gain field $b_{i,j}$. For b_{ij} different mathematical models can be used, where $g_{i,j}$ are the observed intensities and n_{ij} is additive Gaussian noise. Which of the following models are <u>not</u> commonly used to model MRI inhomogeneities?

☐ M$_1$: $g_{i,j} = f_{i,j} \cdot b_{i,j} + n_{i,j}$

☐ M$_2$: $\log(g_{i,j}) = \log(f_{i,j}) \cdot \log(b_{i,j})$

☐ M$_3$: $g_{i,j} = f_{i,j} + b_{i,j}$

☐ M$_4$: $\log(g_{i,j}) = \log(f_{i,j} \cdot b_{i,j} + n_{i,j})$

☐ M$_5$: $g_{i,j} = f_{i,j} + n_{i,j}$

☐ M$_6$: $\log(g_{i,j}) = \log(f_{i,j} \cdot b_{i,j})$

(iii) Assume Fig. 1 shows data points of a 1-D MR-image. By a simple polynomial fitting the gain field is estimated by the *blue line*. What numerical problem can arise if the correction of the gain field is applied? How can we account for it, without changing the polynomial fitting?

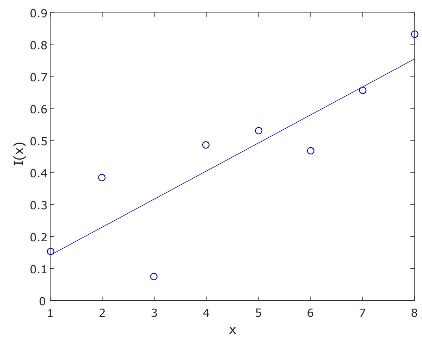


Figure 1: MRI-measurements with polynomial fitting for gain-field correction.

(iv) Suppose we have the following minimalistic MRI image $\mathbf{g} = (g_{i,j})$ inherently with bias $\mathbf{b} = (b_{i,j})$ and we know the bias field:

$$\mathbf{g} = \begin{pmatrix} 4 & 6 & 9 \\ 9 & 24 & 12 \\ 7 & 15 & 22 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 8 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

If we assume the multiplicative model for the bias field, and the noise to be zero, correct the image matrix by removing the bias field effect.

$\boxed{\mathbf{0.5{+}1{+}1{+}1}}$

13 Fuzzy C-means Clustering

Complete the following partition matrix related to fuzzy c-means:

0.25	1	0.1	0.6	0.75
⬚	⬚	0.85	0.2	⬚
0.25	⬚	⬚	0.1	0.1
0.5	⬚	0	⬚	0.1

How many clusters does it have? And how many data points?

$\boxed{\mathbf{2}}$

14 Random Sample Consensus – Programming Exercise

Model estimations on noisy data are usually error-prone. Even a small number of outliers will influence the estimation such that the resulting model can produce large errors. The goal is to identify models that minimize the error and ignore outliers.

In the RANSAC algorithm we assume that a model built with a minimum number of data points does not contain outliers. If we imagine the minimum number of points for a line, the generated line will exactly fit through those two points. To consider every point, the model error is evaluated on the whole data set. In a scenario where we expect the majority of the

data points to be in a valid range we can use this strategy to find a model fitting the inliers and ignoring the outliers.
We want to complete the gaps in `exercise41.java`. It implements a RANSAC algorithm to fit a line through a point plot. Sample points are assumed to be obtained on a line where outliers occur due to measurement errors.

(i) Compute the correct number of iterations needed to satisfy a given probability that only inliers are picked.

(ii) An error function considering the whole dataset should be implemented. The error should measure how many points lie within a certain range of the estimated line. Implement this function in the following method:
`lineError(SimpleVector line_params, SimpleMatrix points)`.

RANSAC algorithm

1. Determine the minimum number N of data points required to build the model.
2. FOR n iterations DO
   i. Choose randomly N points out of your data to estimate the model.
   ii. Determine the error of the current model using all data points.
3. Choose the model with the lowest error.

1+4

15 Unsharp Masking – Programming Exercise

In the lecture several approaches to correct MR-images for the bias field are introduced. In this exercise we have a look at both methods which are referred to as "unsharp masking". Basically, in these methods you want to subtract a low frequency field from the (hopefully) high frequency content that has actual diagnostic value.
Therefore, have a look at the java code in `exercise42.java`.

(i) First, we implement homomorphic unsharp masking. In this method, we attempt to reweight the image by the quotient of the global mean μ and the local means $\mu_{i,j}$, assuming the bias field is approximated by $\frac{\mu_{i,j}}{\mu}$.

(ii) Second, the low frequencies are reduced or cut-off in frequency domain directly. Use the internal Conrad methods to transform the image and try both, the filter given in the lecture and a hard cut-off. For the later, just review the code and find the fitting method.

Use the windowing tool of ImageJ to analyze the corrected images and discuss your results.

2+3

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LEHRSTUHL FÜR MUSTER-ERKENNUNG

Parallel Beam Reconstruction

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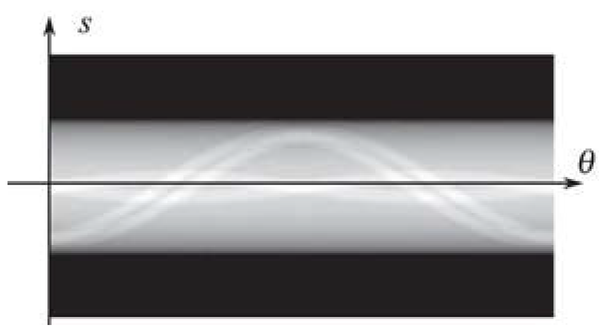
Yixing Huang, Asmaa Khdeir, Houman Mirzaalian

Exercise Sheet 5

16 Parallel Beam Projection & Backprojection

(i) Look at the following figure:



What is this type of image called? Explain what information the image contains and how it can be obtained.

(ii) Consider the following 2×2 image:

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

Given four projections:

$$\begin{aligned}
p_0 &: x+z=9, w+y=11,\\
p_{\frac{\pi}{4}} &: w+z=7,\\
p_{\frac{\pi}{2}} &: w+x=12, y+z=8,\\
p_{\frac{3\pi}{4}} &: y+x=13,
\end{aligned}$$

build the system matrix and compute the 2×2 image values. Check your result.

(iii) Now we acquire four projections of the image:

$$\begin{bmatrix} 1 & 3 & 2 \\ 6 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$$

at angles 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$. Obtain the sinogram. When computing the diagonal projections, assume, for simplicity, that the image intensities contribute to equal parts to the values of the detector pixels $1, 2, 3$ according to the following scheme (analoguously for the other diagonal angle):

2	1	1
3	2	1
3	3	2

(iv) In the same fashion, backproject the data of the sinogram. Is this a perfect reconstruction, and why?

(v) Which angular range is necessary for a complete parallel beam backprojection: $[0, \pi]$ or $[0, \pi)$? Explain graphically.

$\boxed{\mathbf{1+1+1+1+1}}$

17 Filtered Backprojection

In this exercise we want to show the filtered backprojection algorithm in the Fourier domain. From the lecture, we know that simple backprojection of parallel beam data is not sufficient to obtain the original object. Filtering has to be applied which is usually implemented as a multiplication in Fourier domain. Here we derive the necessary steps:

(i) Start with the inverse Fourier transform of $f(x, y)$:

$$f(x,y) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F(u,v)e^{2\pi i(xu+yv)}\,\mathrm{d}u\,\mathrm{d}v$$

and transform the integration variables from cartesian (detector) coordinates (u, v) to polar coordinates (ω, θ). The transformation of $F(u, v)$ is denoted by $F_{\text{polar}}(\omega, \theta)$.

(ii) Since we do not know F_{polar} directly, we want to connect this formula with something we can compute, for instance the 1-D Fourier transform of the sinogram $p(s, \theta)$:

$$P(\omega, \theta) = \int_{-\infty}^{\infty} p(s, \theta) e^{-2\pi i \omega s} \, \mathrm{d}s.$$

Show that $F_{\text{polar}}(\omega, \theta) = P(\omega, \theta)$ for all $\omega \geq 0$, $\theta \in [0, 2\pi)$.

(iii) Finally, use what you have shown so far to state the filtered backprojection algorithm in Fourier domain. What steps do you have to compute?

(iv) What is the name of the filter you have found? Why do we need it in the algorithm? Please describe the reason from the aspect of sampling.

1.5+1.5+1+1

18 Ram-Lak Filter – Programming Exercise

In this exercise we have a look at the filtered backprojection of a Shepp-Logan phantom. Your task is to implement the appropriate filter methods. Therefore, please fill in the missing parts in `exercise5.java`. You can set the type of filtering in the `main()`-method.

(i) First, try backprojecting without filtering. Then implement the Ram-Lak filter and apply it in Fourier domain.

(ii) Compare your result with another filter. For that purpose implement the Shepp-Logan filter as well.

4.5+1.5

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LEHRSTUHL FÜR MUSTER-ERKENNUNG

Fan Beam Reconstruction

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Exercise Sheet 6

19 Fan Beam Reconstruction

(i) Please explain graphically how the point spread function (PSF) for fan beam looks like and show its relation to the parallel beam PSF.
Using this insight, what trick have you learned in the lecture to derive an analytical reconstruction algorithm for fan beam reconstruction? State its name and a very short description of the general idea.

(ii) The parallel beam sinogram is denoted by $p(s, \theta)$ and the equal angle fan beam sinogram is denoted by $g(\gamma, \beta)$. The distance from the X-ray source to the isocenter is denoted by D. The filtered backprojection (FBP) algorithm for parallel beam reconstruction is given as follows:

$$f(x, y) = \frac{1}{2} \int_0^{2\pi} \int_{-\infty}^{\infty} p(s, \theta) h(x \cos\theta + y \sin\theta - s) \mathrm{d}s \mathrm{d}\theta.$$

The FBP algorithm for fan beam reconstruction is derived from the parallel beam reconstruction. Before we apply the fan beam filter, we apply the cosine weighting $\cos\gamma$ first. Why do we need the cosine weighting?
Hint: We used a transformation to polar coordinates for the derivation of the algorithm.

(iii) In order to write the final fan beam algorithm in form of a convolution, we used a specific property of the ramp filter h:

$$h(D' \sin\gamma) = \left(\frac{\gamma}{D' \sin\gamma}\right)^2 h(\gamma).$$

Show that this relation is correct.

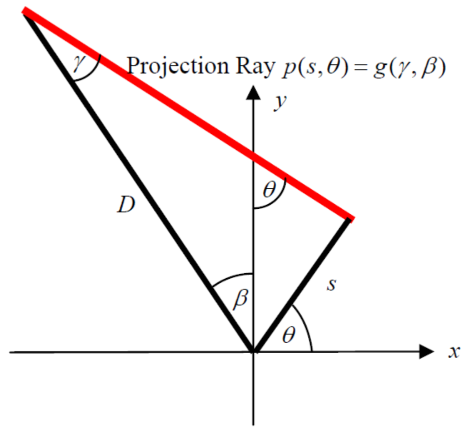

Figure 1: Geometry and notation for the fan beam system

(iv) Assume there is a point object at position (x_0, y_0) and we use the fan beam system as shown in **Fig.** 1. When the X-ray source and the curved detector rotate over 360°, the projection of the point will be a curve in the sinogram. Please give the function of the sinogram curve, i.e., the relation of γ and β.

$\boxed{4 \times 0.5\mathbf{+1+1+1}}$

20 Short Scan

(i) A CT acquisition of a Christmas tree (see **Fig.** 2) is reconstructed by a fan beam reconstruction with $\delta = 11°$. It is performed by a short-scan. Projections are taken with $\Delta\gamma = 2°$. How many projections have to be acquired to reconstruct the image?

(ii) In **Fig.** 3, you find a short-scan and three other trajectories. The object is assumed to occupy the circular shaded region of radius R_m, and the trajectory lies concentrically on a circle of radius R_0. Assume that the detector is always large enough to image the full object, thereby accommodating a maximum fan angle of $\gamma_m (= \delta)$. Explain for each figure if we can get a whole object from the reconstruction process?

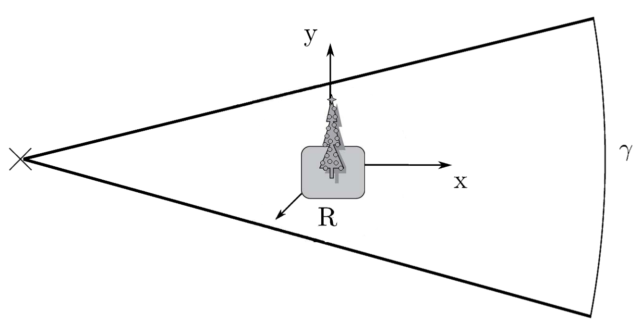

Figure 2: CT image for Christmas

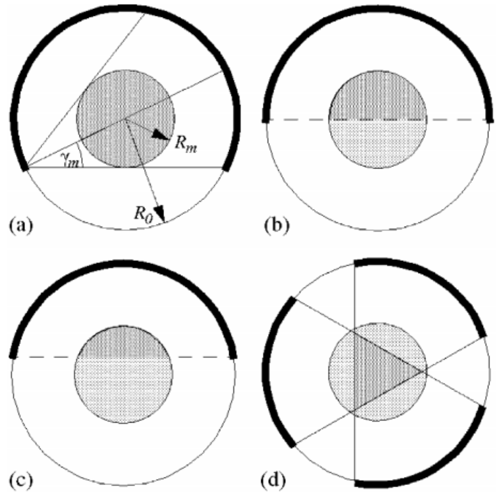

Figure 3: Different scan trajectories, how much of the object do they cover completely? (Source: `http://iopscience.iop.org/article/10.1088/0031-9155/47/14/311/`)

(iii) What problem arises with short scans and how can it be fixed?

$\boxed{\mathbf{1+2+1}}$

21 FBP for a Fan Beam Short Scan – Programming Exercise

The goal for this exercise is to reconstruct short scan data using the fan beam FBP algorithm with Parker weights.

Fan Beam FBP Algorithm

Given the fan beam opening angle δ and the flat panel 'fanogram' data $g(s, \beta)$, we use the following reconstruction steps from the lecture (see **Fig.** 4(a) for the notation):

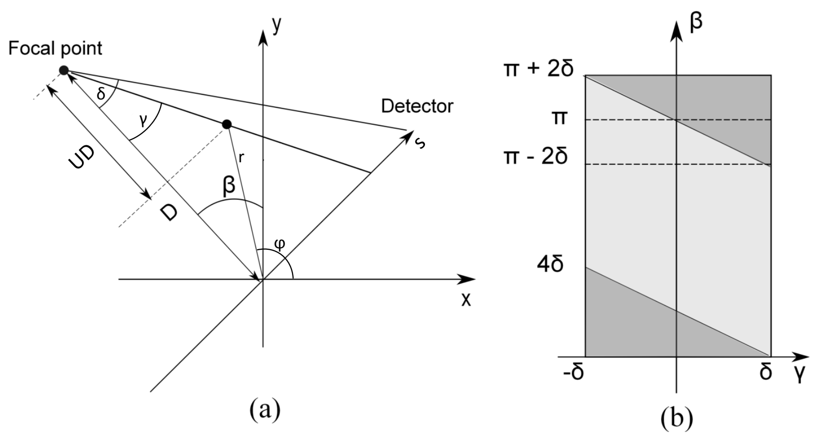

Figure 4: (a) Fan beam imaging geometry, (b) fan beam sinogram and data redundancy.

1. Cosine weighting of projection data to obtain $g_1(s, \beta)$:
$$g_1(s, \beta) = \frac{D}{\sqrt{D^2 + s^2}} g(s, \beta) \ ,$$
2. Perform fan beam filtering:
$$g_F(s, \beta) = \int_{-\infty}^{\infty} h_R(s - s') \, g_1(s', \beta) \, ds' \ ,$$
where $h_R(s)$ is the filter kernel.
3. Backprojection with a weighting function of object to focal point distance U:
$$f(r, \varphi) = \int_0^{2\pi} \frac{1}{U^2} g_F(s, \beta) \, d\beta, \qquad U = \frac{D + r\sin(\beta - \varphi)}{D}$$

Parker Weights

A short scan measures some redundant rays at the beginning and at the end of data acquisition (see the two dark triangles in **Fig.** 4(b)). Corresponding redundant rays determined by the relation $g(\gamma, \beta) = g(-\gamma, \beta + \pi + 2\gamma)$ are commonly by the Parker weighting function:

$$\omega(\gamma, \beta) = \begin{cases} \sin^2\left(\frac{\pi}{4} \frac{\beta}{\delta - \gamma}\right), & 0 \leq \beta \leq 2\delta - 2\gamma \\ 1, & 2\delta - 2\gamma \leq \beta \leq \pi - 2\gamma \\ \sin^2\left(\frac{\pi}{4} \frac{\pi + 2\delta - \beta}{\delta + \gamma}\right), & \pi - 2\gamma \leq \beta \leq \pi + 2\delta \end{cases}$$

Projection rays measured twice are normalized to unity while guaranteeing smooth transitions between non-redundant and redundant data.

Implementation Tasks

Complete the gaps in the provided CONRAD class that are marked with "**TODO**"

(i) Initialize the parameters. Compute the fan angle and the short scan range.

(ii) Compute direction and position of the detector border at a rotation angle β.

(iii) Transform the pixel coordinates to world coordinates.

(iv) Compute the intersection point of a ray with the detector.

(v) Compute the distance weights for this point at the given rotation angle β.

(vi) Complete the cosine weights computation.

(vii) Complete the Parker weights implementation.

(viii) Test all available sinograms `Sinogram0.tif`, `Sinogram1.tif`, and `Sinogram2.tif`.

1+0.5+0.5+1+1+0.5+1+0.5

Your tutors wish you merry Christmas and some relaxing holidays!

Latest submission date: 01/10/2017

Total: 15

Diagnostic Medical Image Processing
Prof. Dr.-Ing. Andreas Maier
Exercises (DMIP-E)
WS 2016/17

LEHRSTUHL FÜR MUSTER-ERKENNUNG

Reconstruction in 3-D

Frank Schebesch, Tobias Würfl, Matthias Utzschneider,
Yixing Huang, Asmaa Khdeir, Houman Mirzaalian

Exercise Sheet 7

22 Analytic and Iterative Reconstruction

(i) The X-ray transform models the process of how CT raw data is acquired. In 3-D, is there a difference between this transform and the Radon transform? Include the respective integrals in your answer.
Hint: How and in which geometry are the integrals computed?

(ii) The algebraic reconstruction technique (ART) is a practical application of the Kaczmarc method. To get an idea how it works, assume that the following two equations describe two lines in the solution space of a synthetic CT projection:

$$3y - x = 5,$$
$$11x + 4y = 19.$$

Compute two iteration steps using ART to find an approximate solution $\mathbf{X}^{(2)} \in \mathbb{R}^2$. Initialize your algorithm with $\mathbf{X}^{(0)} = (0, 0)^\mathsf{T}$.
How good is this estimate? Compute the exact solution of the linear system and compare. Comment on the convergence rate for this specific example.

(iii) What is the main drawback of the elementary ART? Name and explain in a few words three different techniques of how we can tackle this problem.

$\boxed{\mathbf{2+2+2}}$

23 Data Completeness

(i) What is Orlov's condition? What is Tuy's condition? Show for each condition a trajectory that meets its criterion.

(ii) Use Tuy's condition to explain under which condition the FDK algorithm performs an exact reconstruction.

(iii) In the lecture, we have taken a closer look at one specific algorithm that can be used for a helical trajectory. Which reconstruction algorithm was that? This algorithm used a concept called π-lines. Explain them in your own words and discuss the treatment of redundant data.

$\boxed{\mathbf{2+1+1}}$

Submit to your tutor until the last exercise session on 01/18/2017 or 01/19/2017.

$\boxed{\textbf{Total: 10}}$

Diagnostic Medical Image Processing
Prof. Dr.-Ing. Andreas Maier
Exercises (DMIP-E)
WS 2016/17

Rigid Registration

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Exercise Sheet 8

Rigid registration only allows rotations and translations which means that the objects maintain their shape and size in the process. In 2-D rigid transformations between image points $\mathbf{p}_k$ and $\mathbf{q}_k$ can be described as

$$\mathbf{p}_k = \mathbf{R}\mathbf{q}_k + \mathbf{t}, \tag{1}$$

where $\mathbf{R}$ is a rotation matrix for the angle $\varphi \in [0, 2\pi)$

$$\mathbf{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and $\mathbf{t}$ is a translation vector $\mathbf{t} = (t_1, t_2)$, $t_1, t_2 \in \mathbb{R}$.

Solving the Correspondence Problem

Given N point correspondences $(\mathbf{p}_k, \mathbf{q}_k) \in \mathbb{R}^2$ from one of two images to the other, the associated optimization problem is

$$\left(\hat{\mathbf{R}}, \hat{\mathbf{t}}\right) = \underset{\varphi, t_1, t_2}{\operatorname{argmin}} \sum_{k=1}^{N} \|\mathbf{p}_k - \mathbf{R}\mathbf{q}_k - \mathbf{t}\|^2 .$$

To avoid solving this nonlinear problem, one can make use of complex numbers. The image points $\mathbf{p}_k = (p_{k,1}, p_{k,2})$ and $\mathbf{q}_k = (q_{k,1}, q_{k,2})$ are thereby identified with the complex numbers $p_{k,1}+ip_{k,2}$ and $q_{k,1}+iq_{k,2}$. The rigid transformation is then described by two complex numbers r_1+ir_2 and t_1+it_2 for rotation and translation, respectively, as follows:

$$p_{k,1}+ip_{k,2} = (r_1+ir_2)(q_{k,1}+iq_{k,2}) + t_1+it_2, \quad \text{for } k = 1, 2, \ldots, N. \tag{2}$$

Splitting real and imaginary part we get two equations

$$p_{k,1} = r_1 q_{k,1} - r_2 q_{k,2} + t_1 = (q_{k,1}, -q_{k,2}, 1, 0) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix},$$

$$p_{k,2} = r_1 q_{k,2} + r_2 q_{k,1} + t_2 = (q_{k,2}, q_{k,1}, 0, 1) \begin{pmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{pmatrix}.$$

A sufficient amount of correspondences provided, these equations can be used to determine the optimal rigid transform for these correspondences.

Mutual Information

Distance measures (or similarity measures) are used to calculate similarity between objects. Let F be a reference image and M be a moving image. Mutual information is defined as

$$\mathcal{D}_{MI} = H(F) + H(M) - H(F, M),$$

where $H(F)$ and $H(M)$ are the entropies of the images F and M, and $H(F, G)$ is the joint entropy. $\mathcal{D}_{MI}$ evaluates how much information is shared in both pictures. The mutual entropy should be maximized.

24 Rigid Registration in 2-D – Programming Exercise

Complete the gaps in the provided CONRAD classes that are marked with "**TODO**".

(i) Solving the correspondence problem:
Two point clouds are given. Calculate the translation and rotation using the information at the beginning of this sheet and what you have learned about quaternions in the lecture. Build the linear system resulting from the linear optimization problem and solve it by using SVD. Plot the result and discuss with your tutor.

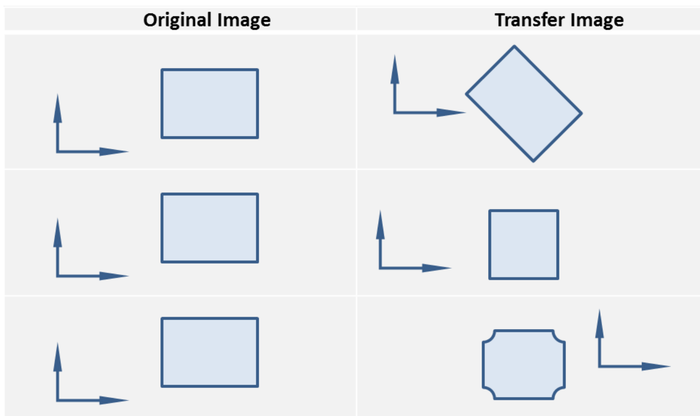
(ii) Rigid registration with mutual information:
Load the images `Proton.png` and `T1.png`. The given code produces a blurred version of both images.

- (a) Calculate the joint histogram and a histogram for each image.
- (b) Calculate the joint and the marginal entropies.
- (c) Calculate the mutual information.
- (d) Show your result.

$\boxed{\mathbf{4+4}}$

25 Rigid Transformations

(i) Assume the images below are shown in 2-D. Which of them show rigid transformations? Now imagine you are just looking from the top view at a 3-D object (an ashlar for the originals). What is your answer now, which of the cases could represent a rigid transformation? (The arrows mark origin and direction of the coordinate system, in 3-D the third axis is orthogonal to the given ones.)



(ii) Compute the complex number $r = r_1 + ir_2$ (where $r_1^2 + r_2^2 = 1$) that corresponds to the rotation matrix $\mathbf{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.

(iii) What is the complex number for a rotation by 45°? What complex number do you get if you rotate $2+2i$ by 45°? And finally, consider rotating $3+3i$ about $2+2i$ by 45°, what do you get?

$\boxed{\mathbf{1+0.5+1.5}}$

26 Quaternions

(i) Which geometric degrees of freedom are typically described using quaternions?

(ii) Is there a difference between a four dimensional vector space and quaternions? Describe this difference.

(iii) Let the quaternion $\mathbf{r}$ be defined as $\mathbf{r} = w + ix + jy + kz = (w, \mathbf{q})$, where 1, i, j, and k denote the imaginary units and $\mathbf{q} = (x, y, z)^\top$. Regarding $\mathbf{r} = (w, \mathbf{q})$ as a rotation, please give its rotation axis $\mathbf{u}$ and rotation angle θ in the axis-angle representation of rotations.

(iv) Given $\mathbf{r}_1 = i + 2j + 3k$ and $\mathbf{r}_2 = 1 + 2i + 3j + 4k$, calculate the product of $\mathbf{r}_1$ and $\mathbf{r}_2$.

(v) Compute the inverse of $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e., $\mathbf{r}_1^{-1}$ and $\mathbf{r}_2^{-1}$.

$\boxed{\mathbf{0.5+0.5+0.5+0.5+1}}$

27 Iterative Closest Point (ICP)

Here is a simple example to show the idea of the ICP algorithm. There are two curves P and Q. On the first curve P, four points are observed: $\mathbf{p}_1 = (0, 1)$, $\mathbf{p}_2 = (2, 2)$, $\mathbf{p}_3 = (4, 1)$, and $\mathbf{p}_4 = (6, 2)$. On the second curve Q, eight points are observed: $\mathbf{q}_1 = (-1.5, -1)$, $\mathbf{q}_2 = (0, 0)$, $\mathbf{q}_3 = (0.5, 0.5)$, $\mathbf{q}_4 = (2, 1)$, $\mathbf{q}_5 = (3, 0.5)$, $\mathbf{q}_6 = (4, 0)$, $\mathbf{q}_7 = (5, 0.5)$, and $\mathbf{q}_8 = (6, 1)$.

(i) Determine the four closest of the given points on curve Q for the four given points on curve P w.r.t. Euclidean distance (graphical solution accepted).

(ii) With these point correspondences, compute the translation $\mathbf{t} = (t_x, t_y)$ of curve Q that makes the average square distance of the corresponding points minimal when no additional rotation is used.

(iii) Describe a procedure to compute an estimate of the translation between two point clouds which you want to register.

$\boxed{\mathbf{0.5{+}1{+}0.5}}$

Submit until 01/27/2017 either directly to your tutor or leave it in Frank's office.

$\boxed{\textbf{Total: 16}}$