

Signal Processing

Fourier Transformation

Refresher Course
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Topics

The Fourier Transformation and Its Properties

- Fourier Transform

- Sampling Theorem

- Convolution Theorem

- Symmetry Property

- Other Properties

Further Readings



Fourier Transform

Definition

The discrete **Fourier transform** of a 1-D signal $f : \{0, \dots, N-1\} \rightarrow \mathbb{C}$ is defined by:

$$F(\xi) = \sum_{n=0}^{N-1} f(n) e^{\frac{-2\pi i n \xi}{N}}.$$



Sampling Theorem

Theorem

Given a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a limited bandwidth $B \in \mathbb{R}$, i. e., the Fourier transform F of f satisfies

$$F(\xi) = 0 \quad \text{for} \quad \xi > B.$$

The interpolation using discrete sampling points allows the exact reconstruction of f if the function is sampled with a sampling rate that is larger than twice the bandwidth B , i. e.,

$$f_{\text{sampling}} > 2B.$$



Convolution Theorem

Theorem

Convolution theorem: *The convolution of two signals f, h in the time domain corresponds to a multiplication of the respective Fourier transforms F, H in the frequency domain:*

$$g(n) = \sum_{k=0}^{N-1} f(k)h(n-k) = (f * h)(n),$$

$$G(\xi) = F(\xi)H(\xi),$$

where $G = \text{FT}(g)$.

Convolution Theorem

$$\begin{aligned} FT(f * h)(\xi) &= \sum_{n=0}^{N-1} (f * h)(n) e^{-\frac{2\pi i n \xi}{N}} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} f(k) h(n-k) e^{-\frac{2\pi i n \xi}{N}} \\ &= \sum_{k=0}^{N-1} f(k) \sum_{n=0}^{N-1} h(n-k) e^{-\frac{2\pi i n \xi}{N}} \\ &= \sum_{k=0}^{N-1} f(k) \sum_{n'=0}^{N-1} h(n') e^{-\frac{2\pi i (n'+k) \xi}{N}} \quad (n' \rightarrow n-k, h \text{ periodic}) \\ &= \sum_{k=0}^{N-1} f(k) e^{-\frac{2\pi i k \xi}{N}} \sum_{n'=0}^{N-1} h(n') e^{-\frac{2\pi i n' \xi}{N}} \\ &= F(\xi) H(\xi) = G(\xi) \end{aligned}$$



Symmetry Property for Real Valued Signals

Theorem

If $f(n)$ is a real valued discrete signal of length N , its discrete Fourier transform $F(\xi)$ fulfills the following symmetry property:

$$F(\xi) = \overline{F(N - \xi)},$$

where \overline{F} denotes the complex conjugate of F .

Symmetry Property for Real Valued Signals

Using the property that for real numbers $a \in \mathbb{R} \subset \mathbb{C}$ they are identical with their complex conjugate $a = \bar{a}$, we have $f(n) = \bar{f}(n)$ for all sampling points n , and thus:

$$\bar{F}(N - \xi) = \overline{\sum_{k=0}^{N-1} f(k) e^{\frac{-2\pi i k(N-\xi)}{N}}} = \sum_{k=0}^{N-1} \bar{f}(k) e^{\frac{2\pi i k(N-\xi)}{N}} = \sum_{k=0}^{N-1} f(k) e^{\frac{-2\pi i k(\xi-N)}{N}}.$$

The N -th unit roots further fulfill

$$e^{\frac{2\pi i k \xi}{N}} = e^{\frac{2\pi i k(\xi + mN)}{N}},$$

where $m \in \mathbb{Z}$, so we finally get:

$$\bar{F}(N - \xi) = F(\xi).$$

Some Useful Properties of the Fourier Transform

	spatial domain	frequency domain
scaling	$f(at)$	$\frac{1}{ a } F\left(\frac{\xi}{a}\right)$
shifting	$f(t - t_0)$	$e^{-i\xi t_0} F(\xi)$
symmetry	$-\frac{1}{2\pi} F(t)$	$f(-\xi)$
differentiation	$\frac{d^n}{dt^n} f(t)$	$(i\xi)^n F(\xi)$

Table 1: Summary of important properties of the Fourier transform



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Further Readings

This topic is so important and well-known that further information can easily be found searching on the internet, or in certain calculus textbooks.

A concise mathematical overview is given in the rather old book:

R. D. Stuart. *An Introduction to Fourier Analysis*. Methuen's
Monographs on Physical Subjects. London: Methuen & Co Ltd., 1961

It can be found online [here](#) (November 2016).