Medical Image Processing for Interventional Applications

Variational Calculus

Online Course – Unit 46

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Topics

Variational Calculus

- Variational Image Smoothing
- Calculus of Variations
- Euler-Lagrange Equation

Summary

- Take Home Messages
- Further Readings

Variational Methods for Image Enhancement

Optimization problem:
Compute a smoothing image transform according to a selected optimality criterion.

Model assumptions:

- The filtered image g shall be as similar to the original image f as possible.
- The filtered image g shall be smooth.

Variational Methods for Image Enhancement

Objective function for image smoothing:

Given an acquired image f, we compute the enhanced image g by minimization of the objective function:

$$D(f,g) = \frac{1}{2}\int_{\Omega} (f-g)^2 + \mu \,\|\nabla g\|_p^2 \,\mathrm{d}x\,\mathrm{d}y\,,$$

where

- the first term is the *similarity measure*,
- the second term is the *smoothness measure* or *regularizer* w. r. t. the p-norm, and
- the parameter μ denotes the *regularization parameter*.

Variational Methods for Image Enhancement

Problem:

How can we minimize the functional $D(f, g)$ with respect to g?

Solution:

$\rightarrow$ Use calculus of variations

Calculus of Variations

Basics of optimization theory:

- The *regular derivative* of a function $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$, is:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}.$$

- The *directional derivative* for a function $f : \mathbb{R}^d \to \mathbb{R}, \boldsymbol{x} \mapsto f(\boldsymbol{x})$, in the direction $\boldsymbol{u} \in \mathbb{R}^d$ is:

$$\nabla_{\boldsymbol{u}} f(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \frac{f(\boldsymbol{x}+\varepsilon\boldsymbol{u}) - f(\boldsymbol{x})}{\varepsilon} = \frac{\partial}{\partial \varepsilon} f(\boldsymbol{x}+\varepsilon\boldsymbol{u}) \Big|_{\varepsilon=0} .$$

Calculus of Variations

Basics of optimization theory:

- Let us consider real-valued functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$.
- If $\boldsymbol{x}^*$ is the minimum of f, then $\boldsymbol{x}^*$ fulfills:

$$\nabla f(\boldsymbol{x}^*) = 0.$$

- $\boldsymbol{x}^*$ is a unique minimum if f is strictly convex.

Calculus of Variations

Variational calculus:

- Consider a real-valued *functional* $I(f)$ that maps *functions* $f \in \mathbb{C}^2\left(\mathbb{R}^d\right)$ to real numbers.
- If f_0 is a minimum of $I(f)$, then f_0 necessarily satisfies the corresponding *Euler-Lagrange equation*, a differential equation in f.
- f_0 is a unique minimum if $I(f)$ is strictly convex.

Euler-Lagrange Equation

Optimization problem:

Let $f \in \mathcal{C}^2([x_1, x_2])$. We are interested in the following problem:

$$\begin{aligned} \text{minimize} \quad & I(f) = \int_{x_1}^{x_2} F(x, f, f')\,\mathrm{d}x\,, \\ \text{subject to:} \quad & f(x_1) = f_1\,, \\ & f(x_2) = f_2\,. \end{aligned}$$

Here we slightly abuse notation since the symbols f and f' are used for two different meanings:

- In $F(x, f, f')$ the symbols f and f' denote the second and third variable of the function F.
- Usually, f and f' are used as symbols for the function f and its derivative f' on $[x_1, x_2]$.

Euler-Lagrange Equation

Euler-Lagrange equation:
A necessary condition for the minimizing functional is:

$$F_f - \frac{\mathrm{d}}{\mathrm{d}x} F_{f'} = 0\,,$$

where we use:

$$F_f = \frac{\partial}{\partial f} F(x, f, f') \quad \text{and} \quad F_{f'} = \frac{\partial}{\partial f'} F(x, f, f').$$

Topics

Variational Calculus

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- Euler-Lagrange Equation

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Take Home Messages

- Variational calculus can be used to formulate a lot of problems in image processing as variational problems, e. g., image enhancement or image registration problems.
- The Euler-Lagrange equation is a differential equation originally used in classical mechanics and an important tool for solving variational problems.
- We derive the equation in the next unit.

Further Readings

There are many standard books on variational calculus. We recommend to check out the following:

- Gilles Aubert and Pierre Kornprobst. *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. Ed. by J. E. Antman S. S.and Marsden and L. Sirovic. 2nd ed. Vol. 147. Applied Mathematical Sciences. Springer New York, 2006. DOI: 10.1007/978-0-387-44588-5
- David G. Costa. *An Invitation to Variational Methods in Differential Equations*. Birkhäuser Boston, 2007. DOI: 10.1007/978-0-8176-4536-6

Medical Image Processing for Interventional Applications

Derivation of the Euler-Lagrange Equation

Online Course – Unit 47

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Euler-Lagrange Equation

In the last unit we have seen the Euler-Lagrange equation:

$$F_f - \frac{\mathrm{d}}{\mathrm{d}x} F_{f'} = 0\,,$$

where

$$F_f = \frac{\partial}{\partial f} F(x, f, f') \quad \text{and} \quad F_{f'} = \frac{\partial}{\partial f'} F(x, f, f').$$

We now want to see how we can derive this formula.

Derivation of the Euler-Lagrange Equation

Assumption:
Let the function $f : [x_1, x_2] \rightarrow \mathbb{R}$ be a minimum of the functional $I(f)$.

Idea:
- We add a perturbation function
$$\eta \in \mathcal{C}^2([x_1, x_2])$$
with
$$\eta(x_1) = \eta(x_2) = 0.$$
- It is added with a scaled amplitude ε to the original function $f(x)$.
- This small variation $\varepsilon\eta(x)$ should **not change** the value of the functional **too much**.

Derivation of the Euler-Lagrange Equation

Variation of the original function f:

We define:

$$g(x) := f(x) + \varepsilon\eta(x)$$

with the derivative:

$$g'(x) = f'(x) + \varepsilon\eta'(x).$$

The boundary constraints:

$$g(x_1) = f(x_1) = f_1 \quad \text{and} \quad g(x_2) = f(x_2) = f_2$$

are also fulfilled for the function g due to $\eta(x_1) = 0$ and $\eta(x_2) = 0$.

Derivation of the Euler-Lagrange Equation

Necessary optimality condition:

$$\forall \eta : \quad \lim_{\varepsilon \to 0} \frac{I(f(x)+\varepsilon\eta(x)) - I(f(x))}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(g)\bigg|_{\varepsilon=0} = 0$$

Derivation of the Euler-Lagrange Equation

Exchange differentiation and integration and apply the chain rule to compute the total derivative of $F(x, g, g')$ with respect to ε:

$$
\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(g)\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_{x_1}^{x_2} F(x, g, g')\,\mathrm{d}x \bigg|_{\varepsilon=0} \\
&= \int_{x_1}^{x_2} \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(x, g, g') \right) \mathrm{d}x \bigg|_{\varepsilon=0} \\
&= \int_{x_1}^{x_2} \left(F_g(x, g, g')\eta(x) + F_{g'}(x, g, g')\eta'(x) \right) \mathrm{d}x \bigg|_{\varepsilon=0} \\
&= \int_{x_1}^{x_2} F_f(x, f, f')\eta(x) + F_{f'}(x, f, f')\eta'(x)\,\mathrm{d}x\,.
\end{aligned}
$$

Derivation of the Euler-Lagrange Equation

Partial integration of the second term using:

$$\int_a^b u \cdot v' \, \mathrm{d}x = [u \cdot v]_a^b - \int_a^b u' \cdot v \, \mathrm{d}x,$$

and using $\eta(x_1) = \eta(x_2) = 0$ results in:

$$\int_{x_1}^{x_2} F_{f'}(x,f,f')\eta'(x) \, \mathrm{d}x = \underbrace{[F_{f'}(x,f,f')\eta(x)]_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2} \left(\frac{\mathrm{d}}{\mathrm{d}x} F_{f'}(x,f,f') \right) \eta(x) \, \mathrm{d}x.$$

Inserting into the necessary optimality condition yields:

$$\int_{x_1}^{x_2} \left(F_f(x,f,f') - \frac{\mathrm{d}}{\mathrm{d}x} F_{f'}(x,f,f') \right) \eta(x) \, \mathrm{d}x = 0.$$

Derivation of the Euler-Lagrange Equation

Lemma (Fundamental Lemma of Variational Calculus)

If

$$\int_a^b g(x)h(x)\,dx = 0$$

for all $h \in \mathcal{C}^2([a,b])$ *with* $h(a) = h(b) = 0$, *then*

$$g(x) = 0.$$

Applying this lemma on the derivation above yields the Euler-Lagrange equation:

$$F_f(x,f,f') - \frac{\mathrm{d}}{\mathrm{d}x} F_{f'}(x,f,f') = 0.$$

Topics

Derivation of the Euler-Lagrange Equation

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Further Readings

Take Home Messages

- We derived the Euler-Lagrange equation by making use of an essential lemma of variational calculus.
- This “trick” should be understood as variation of the function over all possible approximations.

Further Readings

There are many standard books on variational calculus. We recommend to check out the following:

- Gilles Aubert and Pierre Kornprobst. *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. Ed. by J. E. Antman S. S.and Marsden and L. Sirovic. 2nd ed. Vol. 147. Applied Mathematical Sciences. Springer New York, 2006. DOI: 10.1007/978-0-387-44588-5
- David G. Costa. *An Invitation to Variational Methods in Differential Equations*. Birkhäuser Boston, 2007. DOI: 10.1007/978-0-8176-4536-6

Medical Image Processing for Interventional Applications

Euler-Lagrange – Examples and Extensions

Online Course – Unit 48

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Topics

Examples

- Computational Example
- Curve of Minimal Length

Extensions

Summary

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- Further Readings

Computational Example

Consider the functional:

$$F(x, f, f') = x\ f(x)^5 + f'(x)^2.$$

Obviously we get:

$$\frac{\partial}{\partial f} F(x, f, f') = 5x\ f(x)^4, \quad \text{and} \quad \frac{\partial}{\partial f'} F(x, f, f') = 2f'(x).$$

Example: Curve of Minimal Length

Problem:
Find the function f of shortest length connecting two points (x_1, y_1) and (x_2, y_2).

Curve length of $f(x)$ between $x = x_1$ and $x = x_2$:
Pythagoras tells us:

$$\delta s = \sqrt{\delta x^2 + \delta y^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

Example: Curve of Minimal Length

By reducing δx and δy infinitesimally to the limit, we get the integral that is the length of the curve:

$$I(f) = \int_{x_1}^{x_2} \sqrt{1 + (f')^2}\,\mathrm{d}x\,.$$

Partial derivatives of the integrand:

$$F(x, f, f') = \sqrt{1 + (f')^2}$$

are:

$$F_f = 0 \quad \text{and} \quad F_{f'} = \frac{f'}{\sqrt{1 + (f')^2}}\,.$$

Example: Curve of Minimal Length

Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f'(x)}{\sqrt{1+(f'(x))^2}}=0$$

The derivative vanishes if the original function is a constant $c \in \mathbb{R}$:

$$\frac{f'(x)}{\sqrt{1+(f'(x))^2}}=c.$$

This is an equation in $f'(x)$.

Example: Curve of Minimal Length

Solving for $f'(x)$ yields:

$$f'(x) = \frac{c}{\sqrt{1-c^2}}.$$

Thus, the derivative of $f(x)$ is also a constant and $f(x)$ is required to be an affine function in x:

$$f(x) = \frac{c}{\sqrt{1-c^2}} x + t,$$

which is of the form $mx + t$.

Result:
$f(x)$ is a straight line, values of m and t obviously result from the boundary conditions $f(x_1) = f_1$, $f(x_2) = f_2$.

Topics

Explicit Form

The operator $\frac{\mathrm{d}}{\mathrm{d}x}$ defines the total derivative of the functional $F_{f'}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{f'} = \frac{\partial}{\partial x}F_{f'}(x,f,f') + \frac{\partial}{\partial f}F_{f'}(x,f,f')f' + \frac{\partial}{\partial f'}F_{f'}(x,f,f')f'' = F_{f',x} + F_{f',f}f' + F_{f',f'}f''\,.$$

Euler-Lagrange equation in **explicit form**:

$$0 = F_f - \frac{\mathrm{d}}{\mathrm{d}x}F_{f'} = F_f - F_{f',x} - F_{f',f}f' - F_{f',f'}f''$$

Generalization to Higher Order Derivatives

Integrand with **higher order derivatives:**

$$I(f) = \int_{x_1}^{x_2} F(x, f, f', f'', \ldots)\,\mathrm{d}x$$

The Euler-Lagrange equation scales up as follows:

$$F_f - \frac{\mathrm{d}}{\mathrm{d}x} F_{f'} + \frac{d^2}{dx^2} F_{f''} \pm \ldots = 0\,.$$

Note: The alternating sign comes from iterated partial integration.

Dependence on Several Functions

Integrand with **dependence on the functions** $f_1, f_2, \dots$:

$$I(f_1, f_2, \dots) = \int_{x_1}^{x_2} F(x, f_1, f_2, \dots, f_1', f_2', \dots) \, \mathrm{d}x$$

The Euler-Lagrange equation scales up as follows:

$$\begin{aligned} F_{f_1} - \frac{\mathrm{d}}{\mathrm{d}x} F_{f_1'} &= 0\,, \\ F_{f_2} - \frac{\mathrm{d}}{\mathrm{d}x} F_{f_2'} &= 0\,, \\ &\vdots \end{aligned}$$

We derive as many equations as we have functional dependencies.

Two-Dimensional Variational Calculus

Functional is an **integral in two dimensions:**

$$I(f) = \int_{\Omega} F(x, y, f, f_x, f_y)\,\mathrm{d}x\,\mathrm{d}y$$

with partial derivatives $f_x := \frac{\partial f}{\partial x}$, $f_y := \frac{\partial f}{\partial y}$.

Boundary constraints: The values of $f(x, y)$ are given on the boundary $\partial\Omega$ of the region Ω.

Euler-Lagrange equation for the **2-D case**:

$$F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0$$

This can be derived similarly to the 1-D case, based on small variations $\varepsilon\eta(x, y)$.

Topics

Examples
- Computational Example
- Curve of Minimal Length

Extensions

Summary
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Take Home Messages

- One possible application of the Euler-Lagrange formalism is the computation of minimal curves.
- Besides the common way of writing the Euler-Lagrange equation, it can also be written in explicit form.
- It can also be extended to higher dimensional functions and dependencies on more than a single function.

Further Readings

There are many standard books on variational calculus. We recommend to check out the following:

- Gilles Aubert and Pierre Kornprobst. *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. Ed. by J. E. Antman S. S.and Marsden and L. Sirovic. 2nd ed. Vol. 147. Applied Mathematical Sciences. Springer New York, 2006. DOI: 10.1007/978-0-387-44588-5
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Variational Methods for Image Processing

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Topics

Variational Methods for Image Enhancement

Variational Calculus and Nonlinear Diffusion

Summary

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Variational Methods for Image Enhancement

Problem:

Find a smooth image function g that minimizes the cost:

$$D(f,g) = \frac{1}{2}\int_{\Omega} \Big(\underbrace{(f-g)^2}_{\text{similarity}} + \mu \underbrace{\|\nabla g\|_2^2}_{\text{smoothness}} \Big)\, \mathrm{d}x\, \mathrm{d}y$$

$$= \int_{\Omega} \left(\frac{1}{2}(f-g)^2 + \frac{\mu}{2}(g_x^2 + g_y^2) \right) \mathrm{d}x\, \mathrm{d}y$$

$$= \int_{\Omega} F(x, y, g, g_x, g_y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

Variational Methods for Image Enhancement

The partial derivatives of the integrand are:

$$F_g = g - f, \quad F_{g_x} = \mu g_x, \quad F_{g_y} = \mu g_y .$$

Euler-Lagrange equation in this case:

$$\begin{aligned}
0 &= F_g - \frac{\partial}{\partial x} F_{g_x} - \frac{\partial}{\partial y} F_{g_y} \\
&= g(x,y) - f(x,y) - \frac{\partial}{\partial x}(\mu g_x(x,y)) - \frac{\partial}{\partial y}(\mu g_y(x,y)) \\
&= g(x,y) - f(x,y) - \mu \underbrace{(g_{xx}(x,y) + g_{yy}(x,y))}_{\Delta g} \\
&= g(x,y) - f(x,y) - \mu \Delta g(x,y)
\end{aligned}$$

Variational Methods for Image Enhancement

- The Euler-Lagrange equation includes partial derivatives of the unknown function $g(x, y)$, thus we have to solve a **partial differential equation (PDE)**.
- PDEs are usually solved numerically.
- Discretization using finite difference approximation results in a linear system of equations.

Boundary Conditions

Natural boundary conditions $\boldsymbol{n}^{\mathsf{T}} \begin{pmatrix} F_{g_x} \\ F_{g_y} \end{pmatrix} = 0$ on the image boundary $\partial\Omega$, where $\boldsymbol{n}$ is the normal vector to the boundary, give:

$$0 = \boldsymbol{n}^{\mathsf{T}} \nabla f = \partial_{\boldsymbol{n}} f \,,$$

where $\partial_{\boldsymbol{n}} f$ denotes the directional derivative of f in the direction of n.

- The normal derivative has to vanish at the image boundarys.
- Numerically, this can be established by extending the image by mirroring the boundary pixels.

Variational Methods for Image Enhancement

There is a direct connection to **linear diffusion**. The Euler-Lagrange equation:

$$g_{xx} + g_{yy} + \frac{g - f}{\mu} = 0$$

can be interpreted as steady-state $(t \to \infty)$ of linear diffusion equipped with an additional bias term:

$$g_t = g_{xx} + g_{yy} + \frac{g - f}{\mu}.$$

Conclusion: Discretization of the biased linear diffusion process gives a gradient descent method for minimizing $D(f, g)$.

Topics

Variational Calculus and Nonlinear Diffusion

- Nonlinear diffusion reduces blurring of edges.
- Replace smoothing term $\|\nabla f\|_2^2$ by potential function $\Psi(\|\nabla f\|_2^2)$ which penalizes large gradients less severely.

Example

The **Perona-Malik potential** is defined as:

$$\Psi(\|\nabla f\|_2^2) = \frac{\lambda^2}{2} \log\left(1 + \frac{\|\nabla f\|_2^2}{\lambda^2}\right).$$

If we *ignore the similarity term*, the functional for minimization with Perona-Malik potential yields:

$$I(f) = \int_\Omega \Psi\left(\|\nabla f\|_2^2\right) \mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{\lambda^2}{2} \log\left(1 + \frac{\|\nabla f\|_2^2}{\lambda^2}\right) \mathrm{d}x\,\mathrm{d}y.$$

Variational Calculus and Nonlinear Diffusion

Example (cont.)

Computation of the required partial derivatives of $\Psi\left(\|\nabla f\|_2^2\right)$:

$$\Psi_f = 0, \quad \Psi_{f_x} = \frac{f_x}{1 + \|\nabla f\|_2^2/\lambda^2}, \quad \Psi_{f_y} = \frac{f_y}{1 + \|\nabla f\|_2^2/\lambda^2}$$

Euler-Lagrange equation:

$$0 = \frac{\partial}{\partial x}\Psi_{f_x} + \frac{\partial}{\partial y}\Psi_{f_y} - \Psi_f = \operatorname{div}\left(\frac{1}{1 + \|\nabla f\|_2^2/\lambda^2}\nabla f\right) \approx f_t$$

Conclusion: The diffusion process defines a gradient descent method for minimizing $I(f)$.

Topics

Variational Methods for Image Enhancement

Variational Calculus and Nonlinear Diffusion

Summary

Take Home Messages

Further Readings

Take Home Messages

- Variational calculus and the Euler-Lagrange equation in particular can be used for image enhancements like optimizing for smooth approximations.
- For more than one image dimension, the Euler-Lagrange equation is a PDE that we have to solve.
- Potentials for nonlinear diffusion can be used to enhance solutions further.

Further Readings

There are many standard books on variational calculus. We recommend to check out the following:

- Gilles Aubert and Pierre Kornprobst. *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. Ed. by J. E. Antman S. S.and Marsden and L. Sirovic. 2nd ed. Vol. 147. Applied Mathematical Sciences. Springer New York, 2006. DOI: 10.1007/978-0-387-44588-5
- David G. Costa. *An Invitation to Variational Methods in Differential Equations*. Birkhäuser Boston, 2007. DOI: 10.1007/978-0-8176-4536-6

Medical Image Processing for Interventional Applications

Non-Rigid Image Registration

Online Course – Unit 50
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Topics

Introduction

Non-rigid image registration methods are required:

- Rigid registration is a limitation for many medical applications.
- CT scans: hands up $\leftrightarrow$ SPECT scans: hands down
- Patients are repositioned on different acquisition devices.
- Patient change (e. g., filled bladder)
- Motion compensation (e. g., for the reconstruction of the beating heart)

Introduction

Figure 1: Hybrid SPECT/CT-scanner (Image courtesy of Nuclear Medicine, FAU)

Categorization

Image registration problem:
Given a pair of images, a source image S and a target image T, estimate a (non-rigid) transform ϕ that maps S to T under constraints. After registration S and T share the same coordinate system.

Categorization of registration problems:

- 2-D/2-D, 3-D/3-D, 2-D/3-D
- Mono- or multi-modal
- Intensity- or feature-based
- Rigid or non-rigid
- Parametric vs. non-parametric

Categorization

Dependent on the properties of the transform, we have two major classes for image registration:

- The term **rigid registration** subsumes the process of computing a rigid transform for registration.
- The term **non-rigid registration** includes all the methods of deforming the different images such that they can be represented in one common coordinate system.

Basic Principle

- Define type of transform $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where n is the image dimension.
- Define objective function (image similarity dependent on images and the transform) and the constraints for the mapping.
- Efficient solution of the constrained optimization problem
- Experimental (clinical) evaluation of the results

Mathematical Formalization

Notation:

- Source image S
- Target image T
- Similarity measure $\mathcal{D}(\cdot,\cdot)$
- Regularization term $\mathcal{R}(\cdot)$
- Weight $\alpha \in \mathbb{R}$
- Displacement vector field u
- Objective function for estimation of the displacement vector field:

$$\mathcal{J}(u) = \mathcal{D}(S, T, u) + \alpha \cdot \mathcal{R}(u)$$

- Optimization problem to compute the optimal displacement vector field:

$$\hat{u} = \arg\min_{u} \mathcal{J}(u)$$

Mathematical Formalization

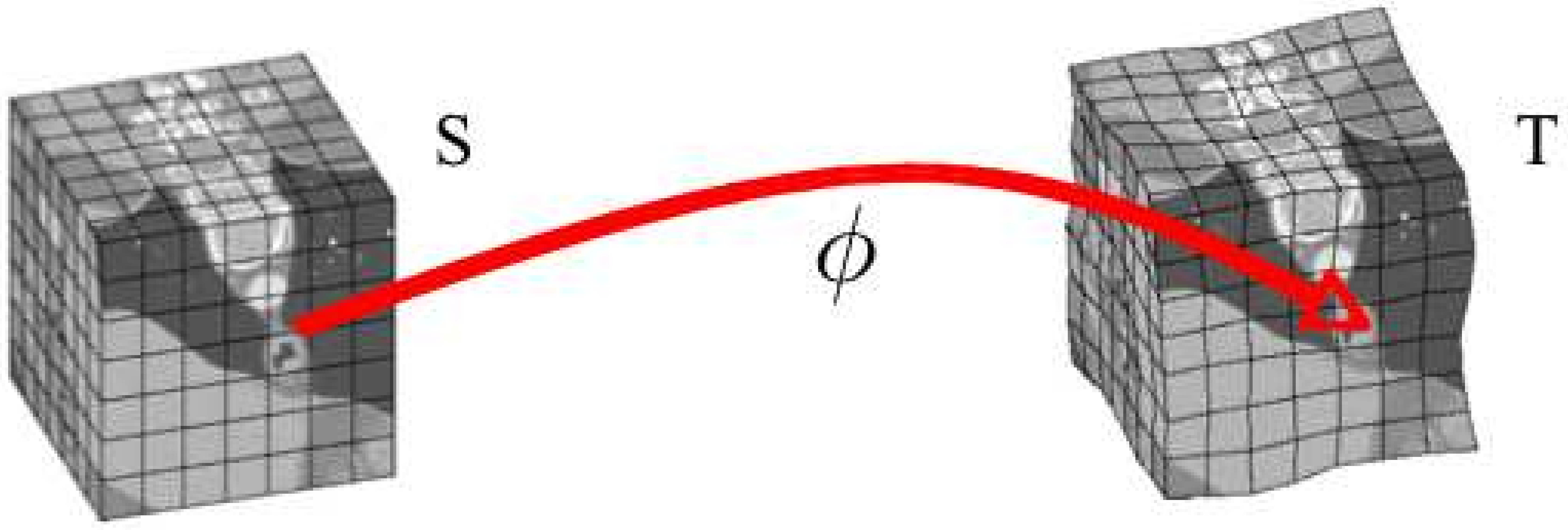

Figure 2: Principle of image registration

Similarity Measures

Common similarity measures $\mathcal{D}(\cdot,\cdot)$:

- *Sum of squared differences* (SSD):

$$\mathcal{D}^{\mathrm{SSD}}(S,T,u) = \frac{1}{2}\int_{\Omega} \left(S(x) - T(\boldsymbol{u}(x))\right)^2 \, \mathrm{d}x$$

- *Mutual information* (MI):

$$\mathcal{D}^{\mathrm{MI}}(S,T,u) = \int\int p_u(s,t)\, \log \frac{p_u(s,t)}{p(s)p_u(t)} \, \mathrm{d}s\mathrm{d}t$$

  where

  - $p(s)$ is the probability density function (p. d. f.) of intensity values of the source image,
  - $p_u(t)$ the p. d. f. of the transformed target image,
  - and $p_u(s,t)$ the joint intensity p. d. f. of the pair of registered images.

Regularizers

Common regularizers $\mathcal{R}(\cdot)$:

- *Diffusion* regularizer:

$$\mathcal{R}(u) = \frac{1}{2}\int_\Omega \|\nabla u(x)\|_2^2 \, \mathrm{d}x$$

- Linear *curvature-based* regularizer:

$$\mathcal{R}(u) = \frac{1}{2}\int_\Omega \sum_i (\Delta u_i(x))^2 \, \mathrm{d}x$$

Incorporation of *prior knowledge* in terms of point correspondences $(\boldsymbol{p}_i, \boldsymbol{q}_i)$, $i = 1, 2, \ldots, N$:

$$\mathcal{R}(u, \{(\boldsymbol{p}_i, \boldsymbol{q}_i), i = 1, 2, \ldots, N\}) = \sum_{i=1}^{N} \|u(\boldsymbol{p}_i) - \boldsymbol{q}_i\|_2^2$$

Topics

Take Home Messages

- Non-rigid registration is an important field of research for medial imaging due to all kinds of motion from one image acquisition set to another.
- We can choose different similarity measures, like SSD or MI.
- There are also different regularizers available, e. g., regularization w. r. t. diffusion or curvature.

Further Readings

- Gerardo Hermosillo, Christophe Chefd'Hotel, and Olivier Faugeras. "Variational Methods for Multimodal Image Matching". In: *International Journal of Computer Vision* 50.3 (Dec. 2002), pp. 329–343. DOI: 10.1023/A:1020830525823
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001

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Non-Rigid Registration – Variational Formulations

Online Course – Unit 51
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Pattern Recognition Lab (CS 5)

Topics

Euler-Lagrange Equation for Diffusion Registration

For diffusion registration we have to solve the following functional equation:

$$
\begin{aligned}
\hat{u} &= \arg\min_{u} \; \mathcal{J}(u) \\
&= \arg\min_{u} \; \mathcal{D}^{\text{SSD}}(S, T, u) + \alpha \cdot \mathcal{R}(u) \\
&= \arg\min_{u} \; \frac{1}{2} \int_{\Omega} \Big(\big(S(x) - T(u(x))\big)^2 + \alpha \|\nabla u(x)\|_2^2 \Big) \, \mathrm{d}x \\
&= \arg\min_{u} \; \int_{\Omega} F(x, u, u') \, \mathrm{d}x \, .
\end{aligned}
$$

Euler-Lagrange Equation for Diffusion Registration

The Euler-Lagrange equation for this particular case is:

$$0 = F_u(x,u,u') - \frac{\mathrm{d}}{\mathrm{d}x} F_{u'}(x,u,u') = -(S(x) - T(u(x))) \cdot \frac{\mathrm{d}T}{\mathrm{d}u} - \alpha \Delta u(x)\,,$$

where Δ is the Laplace operator.

Finally we get the following PDE:

$$-(S(x) - T(u(x))) \cdot \frac{\mathrm{d}T}{\mathrm{d}u} = \alpha \Delta u(x)\,.$$

Topics

Euler-Lagrange Equation for Diffusion Registration

Euler-Lagrange Equation for Curvature Registration

First Variation of Mutual Information

Summary
- Take Home Messages
- Further Readings

Euler-Lagrange Equation for Curvature Registration

For curvature registration we have to solve the following functional equation:

$$
\begin{aligned}
\hat{u} &= \arg\min_{u} \mathcal{J}(u) \\
&= \arg\min_{u} \mathcal{D}^{\mathrm{SSD}}(S, T, u) + \alpha \cdot \mathcal{R}(u) \\
&= \arg\min_{u} \frac{1}{2} \int_{\Omega} \left((S(x) - T(u(x)))^2 + \sum_{i} (\Delta u_i(x))^2 \right) \mathrm{d}x \\
&= \arg\min_{u} \int_{\Omega} F(x, u, u') \,\mathrm{d}x \,.
\end{aligned}
$$

Euler-Lagrange Equation for Curvature Registration

The Euler-Lagrange equation for this particular case is:

$$0 = F_u(x, u, u') - \frac{\mathrm{d}}{\mathrm{d}x} F_{u'}(x, u, u') = -(S(x) - T(u(x))) \cdot \frac{\mathrm{d}T}{\mathrm{d}u} - \alpha \Delta^2 u(x)\,,$$

where Δ is the Laplace operator.

Finally we get the following PDE:

$$-(S(x) - T(u(x))) \cdot \frac{\mathrm{d}T}{\mathrm{d}u} = \alpha \Delta^2 u(x)\,.$$

Topics

First Variation of Mutual Information

$$\frac{\partial \mathcal{D}^{\mathrm{MI}}(S,T,u+\varepsilon\eta)}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\int\int p_{u+\varepsilon\eta}(s,t)\,\log\frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\,\mathrm{d}s\mathrm{d}t$$

First Variation of Mutual Information

$$
\begin{aligned}
\frac{\partial \mathcal{D}^{\mathrm{MI}}(S, T, u+\varepsilon\eta)}{\partial \varepsilon} &= \frac{\partial}{\partial \varepsilon} \int\int p_{u+\varepsilon\eta}(s,t)\, \log \frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\, \mathrm{d}s\mathrm{d}t \\
&= \int\int \left(1 + \log \frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\right) \frac{\partial p_{u+\varepsilon\eta}(s,t)}{\partial \varepsilon}\, \mathrm{d}s\mathrm{d}t + \underbrace{\int\int \frac{p_{u+\varepsilon\eta}(s,t)}{p_{u+\varepsilon\eta}(t)} \frac{\partial p_{u+\varepsilon\eta}(t)}{\partial \varepsilon}\, \mathrm{d}s\mathrm{d}t}_{=0}
\end{aligned}
$$

First Variation of Mutual Information

$$\frac{\partial \mathcal{D}^{\mathrm{MI}}(S,T,u+\varepsilon\eta)}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\int\int p_{u+\varepsilon\eta}(s,t)\,\log\frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\,\mathrm{d}s\mathrm{d}t$$

$$= \int\int\left(1+\log\frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\right)\frac{\partial p_{u+\varepsilon\eta}(s,t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t + \underbrace{\int\int\frac{p_{u+\varepsilon\eta}(s,t)}{p_{u+\varepsilon\eta}(t)}\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t}_{=0}$$

$$\int\int\frac{p_{u+\varepsilon\eta}(s,t)}{p_{u+\varepsilon\eta}(t)}\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t = \int\int p_{u+\varepsilon\eta}(s|t)\,\mathrm{d}s\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}t$$

First Variation of Mutual Information

$$\frac{\partial \mathcal{D}^{\mathrm{MI}}(S,T,u+\varepsilon\eta)}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\int\int p_{u+\varepsilon\eta}(s,t)\,\log\frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\,\mathrm{d}s\mathrm{d}t$$
$$= \int\int\left(1+\log\frac{p_{u+\varepsilon\eta}(s,t)}{p(s)p_{u+\varepsilon\eta}(t)}\right)\frac{\partial p_{u+\varepsilon\eta}(s,t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t + \underbrace{\int\int\frac{p_{u+\varepsilon\eta}(s,t)}{p_{u+\varepsilon\eta}(t)}\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t}_{=0}$$

$$\int\int\frac{p_{u+\varepsilon\eta}(s,t)}{p_{u+\varepsilon\eta}(t)}\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}s\mathrm{d}t = \int\int p_{u+\varepsilon\eta}(s|t)\,\mathrm{d}s\frac{\partial p_{u+\varepsilon\eta}(t)}{\partial\varepsilon}\,\mathrm{d}t = \frac{\partial}{\partial\varepsilon}\int p_{u+\varepsilon\eta}(t)\,\mathrm{d}t = 0$$

Topics

Take Home Messages

- Different regularizers yield different PDEs from the Euler-Lagrange equation.
- From functionals as those presented, we derive solutions to problems for non-rigid image registration problems. Practically, this is most often done numerically.

Further Readings

- Gerardo Hermosillo, Christophe Chefd'Hotel, and Olivier Faugeras. "Variational Methods for Multimodal Image Matching". In: *International Journal of Computer Vision* 50.3 (Dec. 2002), pp. 329–343. DOI: 10.1023/A:1020830525823
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001