You have 60 minutes for the exam. It contains 4 sections, each is worth 15 points. Write your answers on a separate piece of paper.

MIPIA Test Exam

1 Gaussian Filtering

Question 1. 15 P.

A Gaussian filter with zero mean and the standard deviation σ is given as

$$\mathrm{g}_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

and its Fourier transform as

$$\mathrm{G}_\sigma(f) = \exp\left(-\frac{(\sigma 2\pi f)^2}{2}\right) .$$

Show that the chaining of two Gaussian filters, using the standard deviations σ_1 and σ_2 respectively, is equivalent to one Gaussian filter using the standard deviation $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$

$$\left(\mathrm{g}_{\sigma_1} * \mathrm{g}_{\sigma_2}\right)(x) = \mathrm{g}_{\sigma_3}(x) .$$

Solution 1.

$$\begin{aligned}
\left(\mathrm{g}_{\sigma_1} * \mathrm{g}_{\sigma_2}\right)(x) &= \mathrm{F}^{-1}\left(\mathrm{G}_{\sigma_1}(f) \cdot \mathrm{G}_{\sigma_2}(f)\right) && (1)\\
&= \mathrm{F}^{-1}\left(\exp\left(-\frac{\sigma_1^2(2\pi f)^2}{2}\right) \cdot \exp\left(-\frac{\sigma_2^2(2\pi f)^2}{2}\right)\right) && (2)\\
&= \mathrm{F}^{-1}\left(\exp\left(-\frac{\sigma_1^2(2\pi f)^2}{2} - \frac{\sigma_2^2(2\pi f)^2}{2}\right)\right) && (3)\\
&= \mathrm{F}^{-1}\left(\exp\left(-\frac{\sigma_1^2(2\pi f)^2 + \sigma_2^2(2\pi f)^2}{2}\right)\right) && (4)\\
&= \mathrm{F}^{-1}\left(\exp\left(-\frac{\left(\sigma_1^2 + \sigma_2^2\right)(2\pi f)^2}{2}\right)\right) && (5)\\
&= \mathrm{F}^{-1}\left(\exp\left(-\frac{\left(\sqrt{\sigma_1^2 + \sigma_2^2}\right)^2 (2\pi f)^2}{2}\right)\right) && (6)\\
&= \mathrm{F}^{-1}\left(\mathrm{G}_{\sigma_3}(f)\right) && (7)\\
&= \mathrm{g}_{\sigma_3}(x) && (8)
\end{aligned}$$

2 Structure Tensor

Applying the tensor product to the gradients of an image f yields the structure tensor

$$\mathbf{J} = \nabla f (\nabla f)^T = \begin{pmatrix} f_x \\ f_y \end{pmatrix} (f_x, f_y) = \begin{pmatrix} f_x^2 & f_x f_y \\ f_x f_y & f_y \end{pmatrix}$$

with f_x and f_y being the derivatives of f in x and y direction respectively.

Spatial averaging of the individual components of $\mathbf{J}$ with the Gaussian kernel K_ϱ will result in the structure tensor

$$\mathbf{J}_{\varrho,\sigma} = K_\varrho * (\nabla f_\sigma \otimes \nabla f_\sigma)$$

with

$$\nabla f_\sigma = (\nabla K_\sigma) * f \ .$$

In this context, the standard deviations ϱ and σ act as regularization parameters.

Question 2. 7.5 P.

$\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \geq \lambda_2$ are the eigenvalues and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are the eigenvectors of the structure tensor $\mathbf{J}_{\varrho,\sigma} \in \mathbb{R}^{2\times 2}$ of an image f. Which conditions apply to the eigenvalues λ_1 and λ_2 when $\mathbf{J}_{\varrho,\sigma}$ denotes

1. a flat area?
2. a straight edge?
3. a corner?

Solution 2.

For the eigenvalues λ_1, λ_2 of a structure tensor $\mathbf{J}_{\varrho,\sigma}$ the following applies in the case of a

1. flat area: $\lambda_1 \approx \lambda_2 \approx 0$
2. straight edge: $\lambda_1 \gg \lambda_2 \approx 0$
3. corner: $\lambda_1 \geq \lambda_2 \gg 0$

Question 3. 7.5 P.

Figure 1-a shows an image of a fingerprint and figures b), c) and d) are showing the direction of the eigenvectors of the structure tensors of figure 1-a with varying parameters. The mathematical formulation of the structure tensor is given in the equations above.

1. What change of parameters causes the differences between figures 1-b, 1-c and 1-d?
2. What can you say about the changed parameters between figures 1-b, 1-c and 1-d, where are they increased and where are they decreased?

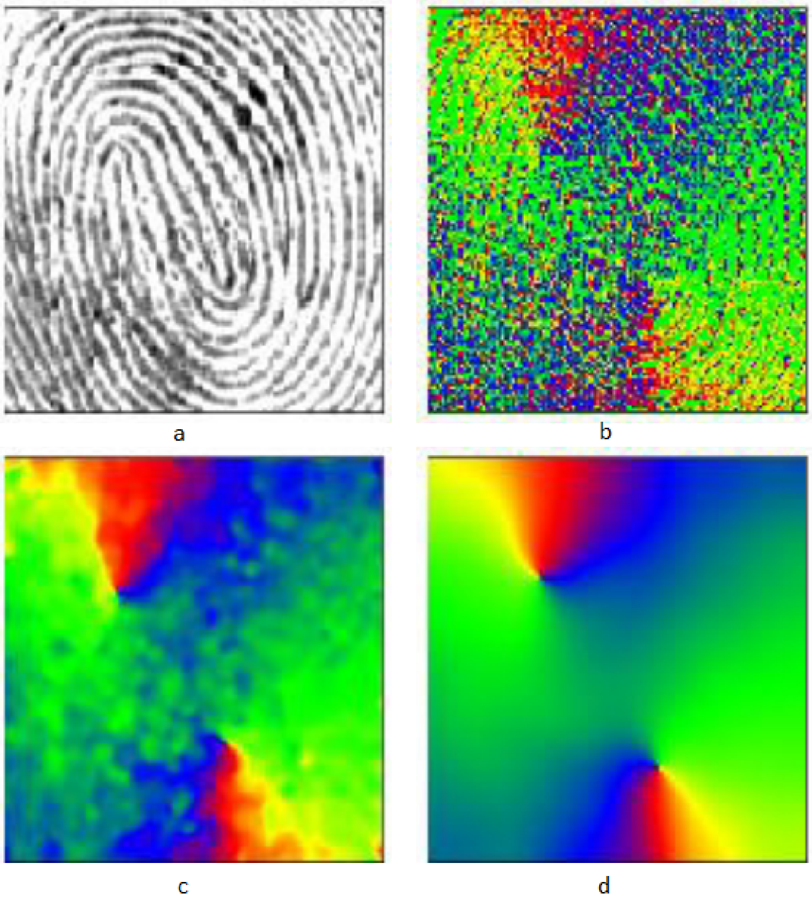

Figure 1: Subfigure a shows an image of a finger print and figures b, c and d are the computed structure tensor of image a with different parameter(s) (image: Joachim Weickert).

Solution 3.
The standard deviations ϱ and σ for the structure tensor $\mathbf{J}_{\varrho,\sigma}$ are the regularization parameters. Therefore, change in these parameters can result in differently localized structure tensors in figure 3-b, 3-c and 3-d.

For the sake of comparison, consider the σ to be the same in 3-b, 3-c and 3-d, we will have $\varrho_b < \varrho_c < \varrho_d$.

3 Epipolar Geometry

Question 4. 5 P.
Label the *epipole(s)*, *epipolar line(s)*, *epipolar plane(s)* and *baseline(s)* in figure 2.

Solution 4.
Epipole(s), epipolar line(s), epipolar plane(s) and baseline(s) are shown in figure 3.

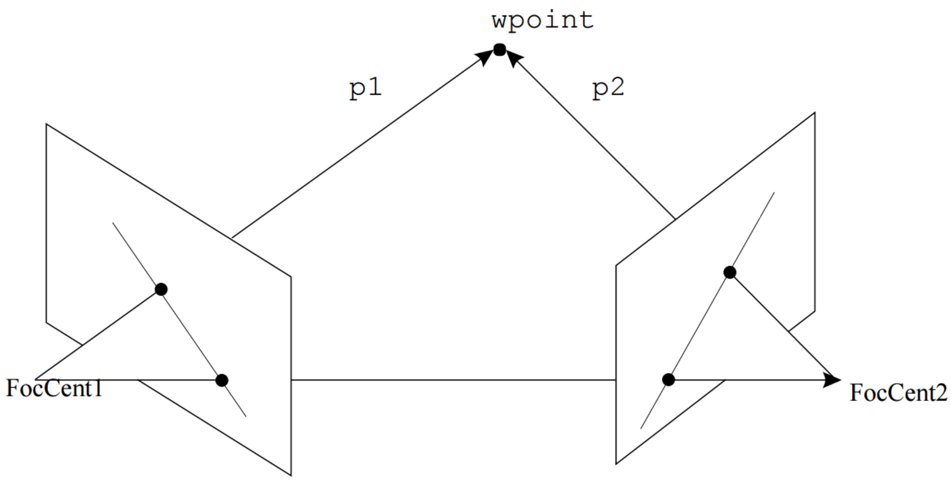

Figure 2: Epipolar geometry.

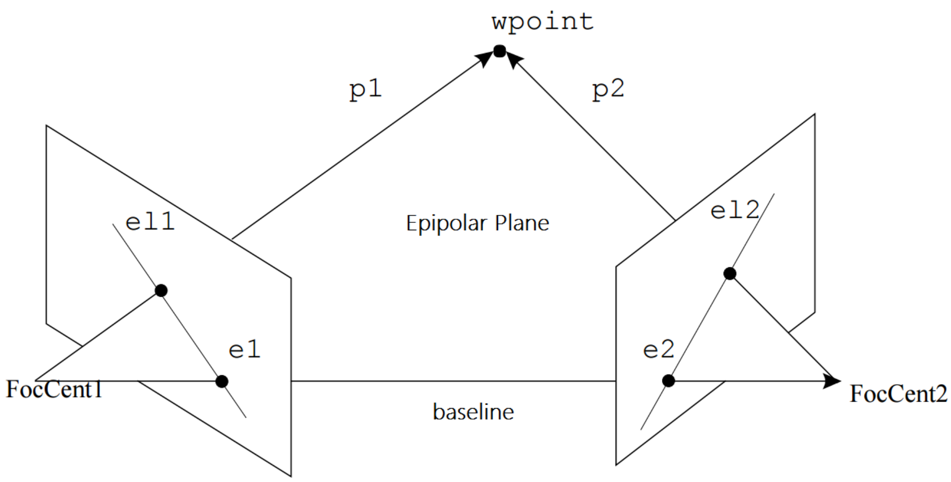

Figure 3: Epipolar geometry.

Question 5. 10 P.

1. Describe the *epipolar constraint*.

2. Consider a 3D world point $\mathbf{w} \in R^3$. $\mathbf{w}$ is mapped to $\mathbf{p} \in \mathbb{R}^3$ in the left image and to $\mathbf{q} \in \mathbb{R}^3$ in the right image. $\mathbf{p}^c$ and $\mathbf{q}^c$ denote the points corresponding to the world point $\mathbf{w}$ in 3-D camera coordinates. $\mathbf{t} \in \mathbb{R}^3$ denotes the translation vector and $\mathbf{R} \in \mathbb{R}^{3\times 3}$ denotes the rotation matrix.

   Derive the epipolar constraint, i.e. the essential matrix $\mathbf{E}$ mathematically. You can use the the equations below as a starting point.

   - $\mathbf{q}^c = \mathbf{R}(\mathbf{p}^c - \mathbf{t})$.
   - $\mathbf{p}^c$, $\mathbf{p}^c - t$ and $\mathbf{t}$ lie on the same plane, i.e. $(\mathbf{p}^c - \mathbf{t})^T(\mathbf{t} \times \mathbf{p}^c) = 0$.

Solution 5.

1. Projection of a 3D world point $\mathbf{w}$ on the left/right image plane lies on the epipolar line on the left/right image plane.

2.
$$\mathbf{q}^c = \mathbf{R}(\mathbf{p}^c - \mathbf{t}) \Rightarrow \mathbf{R}^T\mathbf{q}^c = \mathbf{p}^c - \mathbf{t} \tag{9}$$

   $\mathbf{p}^c$, $\mathbf{p}^c - \mathbf{t}$ and $\mathbf{t}$ lie on the same plane, i.e.

$$(\mathbf{p}^c - \mathbf{t})^T (\mathbf{t} \times \mathbf{p}^c) = 0 \ . \tag{10}$$

   We then get

$$(\mathbf{R}^T\mathbf{q}^c)^T(\mathbf{t} \times \mathbf{p}^c) = \mathbf{q^c}^T R(\mathbf{t} \times \mathbf{p}^c) = \mathbf{q}^{cT} \underbrace{\mathbf{R}[t]_\times}_{\mathbf{E}} \mathbf{p}^c = \mathbf{q}^{cT}\mathbf{E}\mathbf{p}^c \tag{11}$$

   with $\mathbf{E}$ being the essential matrix.

3. Same solution as the second part.

4 Variational Calculus

Question 6. 15 P.

The Euler-Lagrange equation

$$\frac{\delta}{\delta \mathrm{f}}\mathrm{F}\left(x, \mathrm{f}(x), \mathrm{f}'(x)\right) - \frac{d}{dx}\frac{\delta}{\delta \mathrm{f}'}\mathrm{F}\left(x, \mathrm{f}(x), \mathrm{f}'(x)\right) = 0$$

is satisfied for the functional

$$\mathrm{I}\left(\mathrm{f}\right) = \int_{x_1}^{x_2} \mathrm{F}\left(x, \mathrm{f}(x), \mathrm{f}'(x)\right) dx$$

with $x_1, x_2 \in \mathbb{R}^2$ and f' being the first degree derivative of f, when $\mathrm{f}(x)$ is a minimum for $\mathrm{I}(\mathrm{f})$.

The integral

$$\mathrm{L}\left(\mathrm{c}\right) = \int_a^b \sqrt{1 + \left(\mathrm{c}'(x)\right)^2} dx$$

gives the length for the curve described by the function $\mathrm{c}(x) \in ?$ between the points $a, b \in \mathbb{R}^2$.

1. Minimize L with respect to c to find the minimal function c_0.

2. How can c_0 be interpreted?

Solution 6.

We try to find the minimum c_0 for the functional L so we apply the Euler-Lagrange equation to L. The integrand from L

$$\mathrm{L}(\mathrm{c}) = \int_a^b \underbrace{\sqrt{1 + (\mathrm{c}'(x))^2}}_{\text{integrand}} \, dx \tag{12}$$

is now

$$\mathrm{F}(x, \mathrm{c}(x), \mathrm{c}'(x)) = \sqrt{1 + (\mathrm{c}'(x))^2} \, . \tag{13}$$

The derivatives of F with respect to $\mathrm{c}(x)$ and $\mathrm{c}'(x)$ are

$$\frac{\delta}{\delta \mathrm{c}} \mathrm{F}(x, \mathrm{c}(x), \mathrm{c}'(x)) = 0 \tag{14}$$

$$\frac{\delta}{\delta \mathrm{c}'} \mathrm{F}(x, \mathrm{c}(x), \mathrm{c}'(x)) = \frac{\mathrm{c}'(x)}{\sqrt{1 + (\mathrm{c}'(x))^2}} \, . \tag{15}$$

These are inserted into the Euler-Lagrange equation

$$\frac{\delta}{\delta \mathrm{c}} \mathrm{F}(x, \mathrm{c}(x), \mathrm{c}'(x)) - \frac{d}{dx} \frac{\delta}{\delta \mathrm{c}'} \mathrm{F}(x, \mathrm{c}(x), \mathrm{c}'(x)) = 0 \tag{16}$$

$$0 - \frac{d}{dx} \frac{\mathrm{c}'(x)}{\sqrt{1 + (\mathrm{c}'(x))^2}} = 0 \, . \tag{17}$$

Since the derivative above is 0, its parent function must be a constant

$$k = \frac{\mathrm{c}'(x)}{\sqrt{1 + (\mathrm{c}'(x))^2}} \, . \tag{18}$$

This can be solved for $\mathrm{c}'(x)$ by

$$\mathrm{c}'(x) = k \sqrt{1 + (\mathrm{c}'(x))^2} \tag{19}$$

$$(\mathrm{c}'(x))^2 = k^2 \left(1 + (\mathrm{c}'(x))^2\right) = k^2 + k^2 (\mathrm{c}'(x))^2 \tag{20}$$

$$(\mathrm{c}'(x))^2 - k^2 (\mathrm{c}'(x))^2 = k^2 \tag{21}$$

$$(\mathrm{c}'(x))^2 \left(1 - k^2\right) = k^2 \tag{22}$$

$$(\mathrm{c}'(x))^2 = \frac{k^2}{(1 - k^2)} \tag{23}$$

$$\mathrm{c}'(x) = \frac{k}{\sqrt{1 - k^2}} \, . \tag{24}$$

$\mathrm{c}'(x)$ is a constant which leads to its parent function

$$\mathrm{c}_0(x) = \frac{k}{\sqrt{1 - k^2}} x + t = mx + t \tag{25}$$

with $m, t \in \mathbb{R}$ being constants. The optimum $\mathrm{c}_0(x)$ describes a line, which is the shortest connection between the points a and b.