Medical Image Processing for Diagnostic Applications

3-D Rotations – Quaternions

Online Course – Unit 66

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

virtuelle hochschule bayern

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Rotations in 3-D

Various representations for rotations:

- Euler angles
- Axis-angle representation
- **Quaternions**

Quaternion representation

- Rotations in $\mathbb{R}^3$ can be elegantly described by so-called *quaternions*.
- Quaternions can be understood as an extension of complex numbers:
  - Three different numbers that are all square roots of -1:

$$i * i = -1, \qquad j * j = -1, \qquad k * k = -1.$$

  - The products between these numbers are defined as:

$$i * j = -j * i = k, \qquad j * k = -k * j = i, \qquad k * i = -i * k = j.$$

Quaternions

Definition

A ***quaternion*** is a linear combination $\mathbf{r} = w + xi + yj + zk$ where $w, x, y, z \in \mathbb{R}$.

Definition

Similar to complex numbers we define the conjugate $\bar{\mathbf{r}}$ and the magnitude $|\mathbf{r}|$ of a quaternion $\mathbf{r} = w + xi + yj + zk$ as

$$\bar{\mathbf{r}} = w - xi - yj - zk,$$
$$|\mathbf{r}| = \sqrt{\mathbf{r} * \bar{\mathbf{r}}} = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Properties of Quaternions

Definition

A quaternion $\mathbf{r}$ which has length 1 is called a ***unit quaternion***.

A few important properties of quaternions:

- Multiplication and summation are associative.
- Multiplication is *not* commutative, i. e., $\mathbf{r}_1 * \mathbf{r}_2 \neq \mathbf{r}_2 * \mathbf{r}_1$.

$\rightarrow$ Quaternions are no algebraic field, they form a division ring.

- For unit quaternions the inverse is determined as follows:

$$|\mathbf{r}| = 1 \quad \Rightarrow \quad \mathbf{r}^{-1} = \bar{\mathbf{r}}.$$

Multiplication of Quaternions

Definition

We represent quaternions by a row vector $(w, x, y, z) = (w, \boldsymbol{v})$ where $\boldsymbol{v}^\mathsf{T} = (x, y, z)$.

In this notation the product of two quaternions $\mathbf{r}_1 = (w_1, \mathbf{v}_1)$ and $\mathbf{r}_2 = (w_2, \mathbf{v}_2)$ is given by

$$\mathbf{r}_1 * \mathbf{r}_2 = \left(w_1 w_2 - \mathbf{v}_1{}^\mathsf{T}\mathbf{v}_2\,,\; w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2\right).$$

Multiplication of Quaternions

Using the other notation

$$\mathbf{r}_1 = w_1 + x_1 i + y_1 j + z_1 k\,, \quad \mathbf{r}_2 = w_2 + x_2 i + y_2 j + z_2 k\,,$$

we get:

$$\begin{aligned}\mathbf{r}_1 * \mathbf{r}_2 = &\left(w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2\right)\\ &+\left(w_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2\right) i\\ &+\left(w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2\right) j\\ &+\left(w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2\right) k\,.\end{aligned}$$

Multiplication of Quaternions

This quaternion product can be rewritten in matrix notation. For that purpose define the matrix ${}_*[\mathbf{r}_2]$ such that

$$\mathbf{r}_1\, {}_*[\mathbf{r}_2] = \mathbf{r}_1 * \mathbf{r}_2.$$

We find

$${}_*[\mathbf{r}_2] = \begin{pmatrix} w_2 & x_2 & y_2 & z_2 \\ -x_2 & w_2 & -z_2 & y_2 \\ -y_2 & z_2 & w_2 & -x_2 \\ -z_2 & -y_2 & x_2 & w_2 \end{pmatrix}.$$

Note: This matrix shows similarities to the skew matrix that is used to express the cross product of vectors by matrix multiplication, where $\boldsymbol{x} \times \boldsymbol{y} = [\boldsymbol{x}]_\times \boldsymbol{y}$.

Rotation Quaternion

Let

- $\boldsymbol{p} \in \mathbb{R}^3$ be a 3-D point to be rotated,
- $\boldsymbol{u} \in \mathbb{R}^3$ be the axis of rotation with $\|\boldsymbol{u}\| = 1$,
- $\Theta \in \mathbb{R}$ be the angle of rotation.

Definition

- The ***rotation quaternion***, according to a rotation given in axis-angle representation, is defined by:

$$\mathbf{r} = \left(\cos\frac{\Theta}{2}, \sin\frac{\Theta}{2} \cdot \boldsymbol{u} \right).$$

- The quaternion associated with a 3-D point $\boldsymbol{p}$ is defined by $\mathbf{p}' = (0, \boldsymbol{p})$.

Rotation Quaternion

Then the rotation of $\boldsymbol{p}$ can be computed by:

$$\mathbf{p}'_{\text{rot}} = \mathbf{r} * \mathbf{p}' * \bar{\mathbf{r}}.$$

Note:

- The quaternion $\boldsymbol{p}'_{\text{rot}}$ should be $(0, \boldsymbol{p}_{\text{rot}})$. Actually, we could put any value into the scalar part of $\mathbf{p}'$, i. e., $\mathbf{p}' = (c, \boldsymbol{p})$ and after performing the quaternion multiplication, we should get back $\boldsymbol{p}'_{\text{rot}} = (c, \boldsymbol{p}_{\text{rot}})$.
- You may want to confirm that $\mathbf{r}$ is a *unit quaternion*, since that will allow us to use the fact that the inverse of $\mathbf{r}$ is $\bar{\mathbf{r}}$ if $\mathbf{r}$ is a unit quaternion, i. e., $\|\mathbf{r}\| = 1$, $\mathbf{r}^{-1} = \bar{\mathbf{r}}$.

Estimation of 3-D Rotation

The optimization problem to estimate the 3-D rotation is:

$$\widehat{\boldsymbol{R}} = \operatorname*{arg\,min}_{\boldsymbol{R}} \sum_{i=1}^{N} \left\| \boldsymbol{p}_{\text{rot},i} - \boldsymbol{R} \cdot \boldsymbol{p}_i \right\|^2 .$$

Using quaternions for representing rotations we get the following relationship between original and rotated points:

$$(0, \boldsymbol{p}_{\text{rot},i}) = \boldsymbol{q}(0, \boldsymbol{p}_i)\bar{\boldsymbol{q}} \quad \Leftrightarrow \quad (0, \boldsymbol{p}_{rot,i})\boldsymbol{q} = \boldsymbol{q}(0, \boldsymbol{p}_i),$$

and thus we get the optimization problem:

$$\operatorname*{arg\,min}_{\boldsymbol{q}} \sum_{i=1}^{N} \left\| (0, \boldsymbol{p}_{\text{rot},i})\boldsymbol{q} - \boldsymbol{q}(0, \boldsymbol{p}_i) \right\|^2 .$$

Conclusion: The objective function is linear in the rotation quaternion. The rotation can be estimated by solving a system of linear equations.

Topics

Take Home Messages

- Quaternions can be regarded as an expansion of the idea of complex numbers. They allow a useful representation of rotation operations.
- Using quaternions, we have found a linear method to estimate 3-D rotation.
- The translation has to be known.

Further Readings – Part 1

Survey papers on medical image registration:

- Derek L. G. Hill et al. “Medical Image Registration”. In: *Physics in Medicine and Biology* 46.3 (2001), R1–R45
- J. B.Antoine Maintz and Max A. Viergever. “A Survey of Medical Image Registration”. In: *Medical Image Analysis* 2.1 (1998), pp. 1–36. DOI: 10.1016/S1361-8415(01)80026-8
- L. G. Brown. “A Survey of Image Registration Techniques”. In: *ACM Computing Surveys* 24.4 (Dec. 1992), pp. 325–376. DOI: 10.1145/146370.146374
- Josien P. W. Pluim, J. B. Antoine Maintz, and Max A. Viergever. “Mutual-Information-Based Registration of Medical Images: A Survey”. In: *IEEE Transactions on Medical Imaging* 22.8 (Aug. 2003), pp. 986–1004. DOI: 10.1109/TMI.2003.815867

A paper that inspired all the sections on complex numbers, quaternions, and dual quaternions:

Konstantinos Daniilidis. “Hand-Eye Calibration Using Dual Quaternions”. In: *The International Journal of Robotics Research* 18.3 (Mar. 1999), pp. 286–298. DOI: 10.1177/02783649922066213

Further Readings – Part 2

Non-parametric mappings for image registration:

- Nonlinear registration methods applied to DSA can be found in Erik Meijering's papers.
- Jan Modersitzki. *Numerical Methods for Image Registration*. Numerical Mathematics and Scientific Computations. Oxford Scholarship Online, 2007. Oxford: Oxford University Press, 2003. DOI: 10.1093/acprof:oso/9780198528418.001.0001
- Many of Jan Modersitzki's and Bernd Fischer's papers on image registration can be found in the publication list of the Institute of Mathematics and Image Computing (Lübeck).
- The group of Martin Rumpf also published on non-parametric image registration. Details on their work can be found on the institute's webpage.