Medical Image Processing for Diagnostic Applications

Introduction

Morphologic imaging

What is morphological imaging?

Morphologic imaging is about the imaging of the **physical appearance oder PHYSICAL APPEARANCE oder Physical Appearance** of the inner human body like shape, structure or density.

Connection of modalities with physics

Assign each modality to the corresponding physical phenomenon!

sound	passt zu	US	
light	passt zu	endoscopy	
magnetic field	passt zu	MRI	
man-made radiation	passt zu	CT	
radioactive decay	passt zu	SPECT	
light	passt zu	microscopy	
man-made radiation	passt zu	X-ray imaging	
light	passt zu	OCT	
light	passt zu	retina imaging	
radioactive decay	passt zu	PET	

Imaging Types

Please name all four imaging types that we have discussed in this module.

1. Morphologic Imaging
2. Molecular Imaging
3. Diagnostic Imaging
4. Interventional Imaging

Image acquisition workflow

A medical image can be obtained from an imaging device using the following workflow:

- Choice of modality
- Acquisition of image signal
- Preprocessing

- Image manipulation (e.g. reconstruction)
- Postprocessing

Molecular imaging

What is molecular imaging?

Molecular imaging is about the imaging and visualization of **processes oder PROCESSES oder Processes** and **changes oder CHANGES oder Changes** in the organism at the molecular level.

Hybrid scanners

Which of the following is not a commercially available hybrid scanner?

○ 2-D/3-D endoscopy

○ SPECT/CT scanner

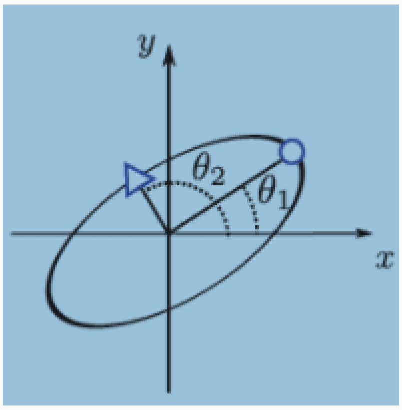
◉ PET/TFT scanner

○ PET/MR scanner

Mathematical Tools

Understanding SVD

Take a look at the ellipse on the figure. Let $\theta_1 = 30^{\circ}$; $\theta_2 = 120^{\circ}$, the coordinates of the circle and triangle in the shown axes are $(3\sqrt{3}, 3)$and $(-1, \sqrt{3})$



We consider a mapping that transforms the ellipse into a unit circle.

Because this mapping is linear it can be represented by a 2x2 matrix.
Consider the effect of the components of the SVD of this matrix. Can you complete the following statement for such a mapping?

This mapping **is not** unique, because **any orthogonal transformation** of ∑V^T maps the ellipse to the unit sphere

Image rank k - approximations

Given the definition from the third optimization problem, how many different rank k-approximations can you compute for an image of size 32x64 (at most)?

32

Test your knowledge about SVD

this unit, you will learn about certain properties of the SVD, such as rank determination, the relationship to eigenvectors of $A^\top A$ and $AA^\top$ as well as the condition of matrices.

By **decomposing** a matrix with SVD, we can easily determine the **rank** of a matrix. It is equal to the number of **singular** values greater than zero . The eigenvalues of $A^\top A$ are σ^2 . A high condition **number is bad** for computations, because the ratio of largest and smallest singular value is large .

Applications of SVD

The singular value decomposition is an important theoretical and also practical tool for a lot of applications. Every linear system can be written as a matrix-vector equation $Ax = b$.
In medical imaging we often have to deal with this type of algebraic equation and knowing about the SVD is essential in this context. In this unit we find out, how this normal form of matrices works.

- Computation of condition numbers
- Low-rank approximations of images
- Solving linear systems
- Computation of the null space

Inversion of a linear system

In this unit, we will learn more about optimization using SVD. In the first problem of this unit, we discuss rank k-approximations of matrices. The last problem of this chapter shows how the pseudoinverse optimally approximates the inverse for a overdetermined linear system (in a least square sense). This is applied to compute a regression line for a given set of points.

A linear system of type $Ax = b$ can always be solved using the **pseudoinverse oder Moore-Penrose pseudoinverse** . If we need to compute it, we could use **SVD oder singular value decomposition oder svd**

Compute condition number

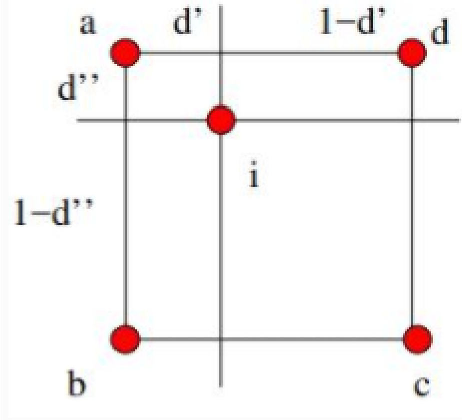
Determine the value of the condition number for the following matrices.

The matrix is defined as $B = U\ diag(a1, \dots, a20)V^\top$ with elements $a_n = \frac{1}{(n-5)2+4}, n = 1, \dots, 20$. (The matrix is defined like that, it is not necessarily a proper SVD according to the definition in the lecture.) Computing the condition number of this matrix we get the value $K_{(B)}$ = **57.25** .

Preprocessing

Bilinear interpolation (rand)

When the intensities a, b, c, d and the distances d', d'' are known, bilinear interpolation can be used to estimate the intensity of i. Use the given parameters to do that.



$a = 9, b = 16, c = 4, d = 4$

$d' = 0.8, d'' = 0.3$

$i =$ 5.42 wurde wie folgt interpretiert: 5.42

Exactly! The formula for bilinear interpolation is:

$x = a \cdot (1 - d') + d \cdot d'$
$y = b \cdot (1 - d') + c \cdot d'$
$i = x \cdot (1 - d'') + y \cdot d''$

Measurement matrix (rand)

Consider polynomial undistortion: Given are three calibration points from the undistorted image: $(x1, y1) = (3, 10), (x2, y2) = (3, 5), (x3, y3) = (9, 8)$. Let $d = 2$, and the parameter vector $u = (u0, 0, u0, 1, u0, 2, u1, 0, u1, 1, u2, 0)^T$.

Write the measurement matrix A.

$$\begin{matrix} 1 & 10 & 100 & 3 & 30 & 9 \\ 1 & 5 & 25 & 3 & 15 & 9 \\ 1 & 8 & 64 & 9 & 72 & 81 \end{matrix}$$

Exactly!

The rows have the following pattern:

$1, y, y2, x, xy, x2$

(due to the sorting of the parameter vector u)

For each row, we use one of the point correspondences, so the first row is indexed with 1 and the last with 3.

Least square estimation

Least square estimation as we use it for distortion correction is a numerical procedure that fits a parametric or non-parametric **curve oder Curve oder CURVE** to **data points oder Data Points oder DATA POINTS oder Data points oder data Points** by minimization of the sum of **squared distances oder Squared Distances oder SQUARED DISTANCES oder Squared distances oder squared Distances** of data points from the curve.

Estimates of the coefficients of the mapping are computed by using the **pseudoinverse oder Pseudoinverse oder PSEUDOINVERSE** of the measurement matrix.

Match the following statements

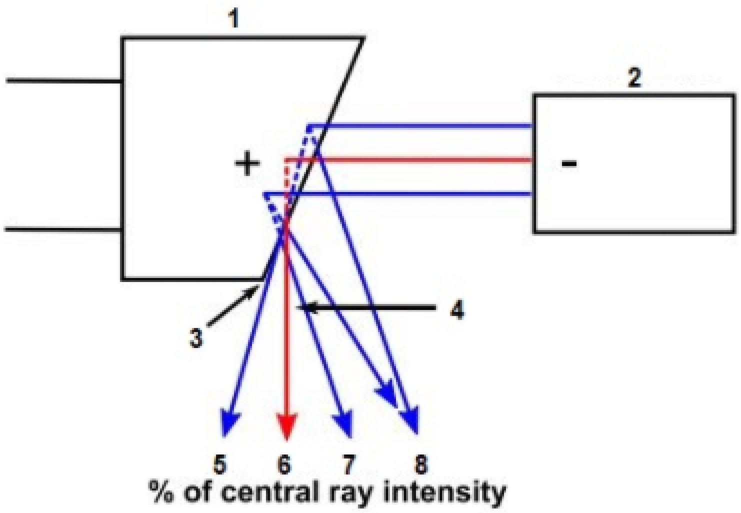
Match the following definitions to the corresponding distortion:

Intensity distortion passt zu **is a distortion which for instance is caused by defect pixels and illumination differences.**

Geometric distortion passt zu **is a distortion caused by the imaging device due to its inability to map correctly.**

Heel effect

The heel effect causes a gray level ramp in X-ray images, which is demonstrated in this image. Can you add the descriptions?



1: **Anode oder anode oder ANODE**

2: **Cathode oder cathode oder CATHODE**

3: **Anode Heel oder anode heel oder ANODE HEEL oder Anode heel oder anode Heel**

4: **Central Ray oder central ray oder CENTRAL RAY oder Central ray oder central Ray**

5: **less than 100 %**

6: **100 %**

7: **more than 100 %**

8: **highest percentage in this picture**

Examples for image distortion

Select the correct statements

Examples for image distortion are

- None of the options is right
- Noise
- Acquisition specific artifacts

Artifacts

Match the artifacts to the different imaging methods.

X-ray imaging	**passt zu**	**Image distortion**	
X-ray imaging	passt zu	Defect pixels	
X-ray imaging	passt zu	Heel effect	
Magnetic resonance imaging	passt zu	Intensity inhomogeneities	
Endoscopy	passt zu	Heterogeneous illumination	
Endoscopy	passt zu	Specular reflection	
Molecular imaging	passt zu	Noise	

Efficient evaluation of polynomials

We want to know the computational complexity of the evaluation of a certain polynomial.

Assume $d = 2$ and a mapping of points (x',y') from the undistorted image to the x-coordinate is given by $x = X(x', y') = \sum_{i=0}^{d} \sum_{j=0}^{d-i} u_{i,j} x'^{i} y'^{j}$

How many multiplications does a naive evaluation of this polynomial take?

d(d+1)(2d+1)/6+d(d+1)/2

How can the number of multiplications for the evaluation be efficiently reduced?

Horner scheme

How many multiplications are necessary to evaluate the polynomial with the latter method?

d(d+1)/2+d

Difference between interpolation and regression

Match the correct description to the given terms:

Interpolation	**passt zu**	**Unknown data between observed data is estimated.**	
Interpolation	passt zu	Each sampling point lies on the estimated function.	
Regression	passt zu	A mathematical relationship between multiple variables approximating the given data is computed.	
Regression	passt zu	Sampling points are not required to lie on the estimated curve.	

Rule of thumb

Image undistortion is not a straightforward task. If implementing, we need to consider at least one problem. Let us assume we want to generate an undistorted image**I1** from a distorted image**I0** First, we need to determine a **mapping oder Mapping oder MAPPING** between **I1**and **I0**. Next, we have to **interpolate oder Interpolate oder INTERPOLATE** between intensities of neighboring pixels, because lattice points of the undistorted image are **not necessarily mapped** to lattice points in the distorted image.

Always **sample oder SAMPLE oder Sample** in the space of your **output** !

Furthermore, we need to find a **robust and reliable** estimation of parameters and to develop **efficient and robust** algorithms for applicable image undistortion.

Defect Pixel

Defect pixel interpolation basic approach

Sort the following steps for a basic algorithm for defect pixel interpolation using bandlimitation.

1. Compute FT $G(\xi)$ of input signal $g(n)$
2. Set $G(\xi) = 0$ where ξ is beyond the bandlimits
3. Compute inverse FT of corrected $G(\xi)$
4. Replace defect samples in $g(n)$ with values of the bandlimited signal
5. Repeat all steps until changes are below a threshold

Disadvantages of Flat Panel Detectors

Choose the disadvantages of Flat Panel Detectors

- Slow readout
- Expensive technology
- High rejection rate in production
- Elimination of defects with digital image processing

Advantages of Flat Panel Detectors

Choose the advantages of Flat Panel Detectors

- Simple assembly and readout
- Higher contrast resolution
- Not sensitive to magnetic fields
- More robust with respect to under- and over-exposure
- Optimization of the clinical workflow

Defect pixel interpolation

Select the correct options and fill in missing words.
Defect pixels are caused by **Defect detector cells** . We use a **multiplicative** model where the **multiplication** of a **mask image** and the (unknown) ideal image results in the **observed image** .

Defect pixel interpolation, general methods

Which of the following is a true statement regarding defect pixel interpolation?

- For defect pixel interpolation we can use filtering in spatial domain.
- For defect pixel interpolation we can use filtering in frequency domain.
- Non-linear filtering can be used to fill in defect pixels
- Defect pixels often appear with strong edges in acquired images.

Frequency Domain Defect Pixel Interpolation

Sort the Interpolation Algorithm steps

1. compute FT of input signal) $g(n)$)
2. initialize)$\hat{F}^{(0)}(k) = 0$),)$G^{(0)}(k) = G(k)$),)$i = 1$)
3. randomly select a line pair)$G^{(i-1)}(s)$),)$G^{(i-1)}(N-s)$)for)$s \neq 0$)
4. estimate)$\hat{F}^{(i)}(s), \hat{F}^{(i)}(N-s)$) using (FT-EST)
5. update spectrum)$\hat{F}^{(i)}(k)$), compute error)Δ_ε)
6. UNTIL error is below a threshold
7. Compute inverse FT of $\hat{F}^{(i)}(k)$

MR Inhomogeneities

Homomorphic unsharp masking (rand)

Homomorphic unsharp masking using the multiplicative model means applying the following relation to compute f:

$$f_{i,j} = \frac{\mu}{\mu_{i,j}} \cdot g_{i,j}$$

$$\text{Let } g = \begin{pmatrix} 7 & 5 & 4 & 1 \\ 6 & 3 & 10 & 6 \\ 10 & 6 & 4 & 10 \\ 5 & 10 & 6 & 1 \end{pmatrix}.$$

(Please give the values in fractions (e.g., 1/16) and not in floats.)

Compute the global mean: $\mu = \frac{94}{16}$
Now compute two local means with a neighbourhood containing 9 pixels (the indexing starts at 1):
$\mu 2, 2 = \frac{55}{9}$
$\mu 3, 3 = \frac{56}{9}$
Finally, compute the corresponding values of f:
$f2, 2 = \frac{2538}{880}$
$f3, 3 = \frac{3384}{896}$
Perfect! Using the formula for homomorphic unsharp masking, you removed the differences between the global and local means which are caused by the bias field.

Computation of entropy and KL divergence

Compute the entropies and both values of the KL divergence and put your result into the blanks.

We have two images and their corresponding intensity histograms are given by 3 bins:
p=(0.2,0.7,0.1)
q=(0.1,0.5,0.4)
Compare the entropy for both, and compute the KL divergence for both combinations. (Use the natural logarithm and a dot as decimal separator.)

Type your results here:
H(p)= 0.8018

Well done! The entropy measures the amount of disorder in the image.

H(q)=0.9433

Well done! The entropy measures the amount of disorder in the image.

L(p,q)=0.2355

Exactly! The KL divergence measures similarity.

KL(q,p)=0.3170

Exactly! The KL divergence measures similarity.

Causes for IIH

What are the three major causes for intensity inhomogeneities in MR imaging?

- Patient motion
- Inhomogeneity of the static main field
- Non-uniform radio-frequency

Pros and Cons of MRI

Which of the following statements are advantages and which are disadvantages of MRI? Match them accordingly.

Pro	passt zu	Patient care	
Pro	passt zu	Pre-/intraoperative guidance for intervention	
Pro	passt zu	Enables the usage of contrast agents	
Pro	passt zu	Discrimination of soft tissues	
Pro	passt zu	Functional imaging modality (diffusion, perfusion, flow imaging)	
Pro	passt zu	High spatial resolution	
Con	passt zu	Inhomogeneities caused by RF coil	
Con	passt zu	Intensity-based segmentation fails	
Con	passt zu	IIH produce spatial changes in tissue statistics	
Con	passt zu	Expensive devices (investment and maintenance)	
Con	passt zu	Unpredictable inhomogeneity variations	
Con	passt zu	Dangerous when approached with ferro magnetic objects	

Relation between low- and high-pass filters

Let $g(x)$ the input image, $h(x)$ a high-pass filter and $l(x)$ the low-pass filter for which holds: $H(k) = 1 - L(k)$ with $H(k) = FT(h(x))$ and $L(k) = FT(l(x))$. FT denotes the fourier transform, $FT^{-1} the inverse fourier transform, and$G(k)=FT(g(x))$.

Show that the relation $g(x) * h(x) = g(x) - g(x) * l(x)$ between low- and high-pass filtering holds!

Notation: $f(x) * g(x)$= integrate(f(y) * g(x - y), y, minf, inf).

$$
\begin{aligned}
&g(x) * h(x) = \\
&= FT - 1(FT(g(x) * h(x))) = \\
&= FT - 1(G(k) * H(k)) = \\
&= FT - 1(G(k) * (1 - L(k))) = \\
&= FT - 1(G(k) - G(k) \cdot L(k)) = \\
&= g(x) - int(l(x - y) * g(y), y, mint, inf)) = \\
&= g(x) - g(x) * l(x)
\end{aligned} \tag{1}
$$

Properties of KL divergence

Match the properties of KL divergence with the corresponding definition of the properties:

Continuity	**passst zu**	**KL(p,q) → 0 ⇒ p → q**	
Definiteness	passt zu	KL(p,q) = 0 ⇔ p = q	
Asymmetry	passt zu	KL(p,q) is not equal to KL(q,p)	
Non-negativity	passt zu	KL(p , q) ≥ 0	

Relation between KL divergence and entropy

Write down the continuous definition of the Kullback-Leibler (KL) divergence between two discrete probability density functions p and q. Show its relation to entropy.

$KL(p, q) = \int_{-\infty}^{\infty} p(x) * log(\frac{p(x)}{q(x)})d_x$

$= \int_{-\infty}^{\infty} p(x) * ln(q(x))d_x - (- \int_{-\infty}^{\infty} p(x) * ln(p(x)))d_x$

$= H(p, q) - H(p)$

Reason for image enhancement in MRI

Select the right option:

Image enhancement algorithms are applied to perform **bias field** correction.

Bias field models

MRI inhomogenities are often modeled by a pixelwise gain field $b_{i,j}$. For $b_{i,j}$ different mathematical models can be used, where $g_{i,j}$gi,j are the observed intensities and $n_{i,j}$ is additive Gaussian noise. Which of the following models are NOT commonly used to model MRI inhomogeneities?

- [ ] $\mathrm{M}_1\colon\ g_{i,j} = f_{i,j} \cdot b_{i,j} + n_{i,j}$
- [x] $\mathrm{M}_2\colon\ \log(g_{i,j}) = \log(f_{i,j}) \cdot \log(b_{i,j})$
- [x] $\mathrm{M}_3\colon\ g_{i,j} = f_{i,j} + b_{i,j}$
- [x] $\mathrm{M}_4\colon\ \log(g_{i,j}) = \log(f_{i,j} \cdot b_{i,j} + n_{i,j})$
- [x] $\mathrm{M}_5\colon\ g_{i,j} = f_{i,j} + n_{i,j}$
- [ ] $\mathrm{M}_6\colon\ \log(g_{i,j}) = \log(f_{i,j} \cdot b_{i,j})$

Homomorphic unsharp masking

Homomorphic unsharp masking requires the computation of the **global mean** value μ of the intensity distorted image and **local mean** values $\mu_{i,j}$ evaluated in a neighborhood of each pixel. If the multiplicative model is used, the estimated **intensity corrected** value $f_{i,j}$ is then computed pixelwise in the following manner: $f_{i,j} = \frac{\mu}{\mu_{i,j}} g_{i,j}$.

Compute bias field correction (rand)

Assume you are given the distorted image $g = \begin{pmatrix} 28 & 64 & 9 \\ 6 & 20 & 10 \\ 3 & 5 & 77 \end{pmatrix}$ and you somehow measured the gain field $b = \begin{pmatrix} 4 & 8 & 1 \\ 1 & 2 & 10 \\ 3 & 1 & 7 \end{pmatrix}$. Compute the corrected image.

$$\mathbf{f} = \begin{matrix} 7 & 8 & 9 \\ 6 & 10 & 1 \\ 1 & 5 & 11 \end{matrix}$$

Perfect!

$$f = \begin{pmatrix} 7 & 8 & 9 \\ 6 & 10 & 1 \\ 1 & 5 & 11 \end{pmatrix}$$

with $f_i = \frac{b_i}{g_i}$

Particle properties

Which of the following are fundamental properties of particles?

- mass
- charge
- spin

Polynomial surface fitting

Describe the basic idea of polynomial surface fitting by sorting the steps into a reasonable order.

1. The logarithmized image is considered as a 2-D function.
2. Fit a parametric, smooth surface to the logarithm of the intensity values.

3. Estimate the parameters by minimizing the distance of the surface points to the logarithmic image intensities.
4. The resulting surface is then subtracted from the logarithmic image.

Fuzzy partition matrix (rand)

Complete the given partition matrix for fuzzy C-means clustering.

0.2	1	0.1	0.0	0.3
0.6 wurde wie folgt interpretiert: 0.6	0 wurde wie folgt interpretiert: 0	0.0	0.1	0.2 wurde wie folgt interpretiert: 0.2
0.2	0 wurde wie folgt interpretiert: 0	0.3 wurde wie folgt interpretiert: 0.3	0.0	0.2
0.0	0 wurde wie folgt interpretiert: 0	0.6	0.9 wurde wie folgt interpretiert: 0.9	0.3

Well done! The matrix has to fulfill the probability constraint, which means that the sum over each row has to be 1.

How many clusters do we have here? 4 wurde wie folgt interpretiert: 4

And how many data points? 5 wurde wie folgt interpretiert: 5

Exactly! The number of rows is the number of clusters, the number of columns corresponds to the number of data points.

Components of an MR system

Write down which four components an MR system consists of.

Remark: Please just write the noun which consists of two or at most three single words (no articles / several variants are accepted).

1. main magnet
2. magnetic field gradient system
3. radio frequency system
4. imaging system

Motivation for regularization

Regularization is needed to incorporate **local** dependencies of the image data points.

Frequency filtering

When we design a high pass filter to eliminate IIH, the bias field is considered to be a **low frequency oder Low Frequency oder LOW FREQUENCY** component in the image.

Reconstruction Basics

Backprojection: Simple Example (rand)

Calculate the backprojected image B:

Consider the following 2×2projected image:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 6 & 9 \end{pmatrix}$$

Projection matrix **A** = $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

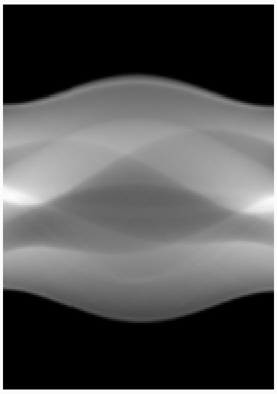
Build the projected image vector **P** = $\begin{pmatrix} 4 \\ 11 \\ 6 \\ 9 \end{pmatrix}$ and compute backprojected image vector **B**

B = $\begin{matrix} 10 \\ 17 \\ 13 \\ 20 \end{matrix}$

Good Job!

The backprojected image vector **B** = $\begin{pmatrix} 10 \\ 17 \\ 13 \\ 20 \end{pmatrix}$

CT Projection



Complete the sentence:

A set of many such projections under different angles organized in 2D is called **sinogram** .

In **X-ray CT** , the **line** integral represents the total attenuation of the beam of x-rays as it travels in a straight line through the object.

Backprojection: mathematical formulation

Which of the following are equivalent formulations of backprojection?

- [x] $b(x,y) = \int_0^{\pi} p(s,\theta) \mid_{s=x\cos(\theta)+y\sin(\theta)} d\theta$
- [x] $b(x,y) = \frac{1}{2}\int_0^{2\pi} p(x\cos(\theta)+y\sin(\theta),\theta) d\theta$

Understanding X-ray projections

Which of the following statements is true?

None of them are true.

- Tomography is a technique based on a single projection view.
- The Beer-Lambert law states that the ray energy is exponentially increasing when passing an object.
- The measured intensity at a detector pixel can be regarded as the integral of attenuation at a single point in the object over all projection views.
- 3-D projection data is usually stored in a star shaped pattern.

History of CT geometries

Sort the geometries in the order of historical appearance/usage.

1. parallel beam geometry
2. fan beam geometry
3. cone beam geometry

Analytic reconstruction (rand)

Build the system matrix and compute the original image:

Consider the following 2×2image:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

Given projection values from four angles:

$p_0 : x + z = 7, w + y = 3$

$p_{\frac{\pi}{4}} : w + z = 4$

$p_{\frac{\pi}{2}} : w + x = 9, y + z = 1$

$p_{\frac{3\pi}{4}} : y + x = 6$

build the system matrix and compute the 2×2 image values. Check your result.

The original image is:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix}$$

Good Job!

The Original Image is $\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix}$

Reconstruction steps

Write down which three steps have to be part of a reconstruction algorithm.

1. Projection
2. Backprojection
3. Filtering

Parallel Beam Reconstruction

Complexity of filtering

Given the number of projections P and N image points to reconstruct, what is the total complexity for filtering in parallel beam reconstruction?

◉ O(log(N) x P x N)

Name the window functions

Write down the three window functions that you have studied in this unit.
(Use the form "[name] window".)

1. cosine window
2. shepp-logan window
3. rectangular window

Central slice theorem for convoluted functions (opt)

Let $g(x, y)$ and $h(x, y)$ be two 2-D functions, and their Radon transforms are $q(s, t)$ and $r(s, t)$, respectively. If $f(x, y)$ is the 2-D convolution of $g(x, y)$ and $h(x, y)$, use the central slice theorem to prove that the Radon transform $p(s, t)$ of $f(x, y)$ is the convolution of $q(s, t)$ and $r(s, t)$ with respect to variable s. FT denotes the Fourier transform, FT^{-1} the inverse Fourier transform, $P(w, t) = FT(p(s, t))$ and $F(w \cdot cos(t), w \cdot sin(t)) = FT(f(x, y))$, correspondingly for the remaining functions.

Remark: If you compare the notation to the lecture, t corresponds to θ and w corresponds to ω.

$$\begin{gathered}
p(s,t) = FT-1(P(w,t)) = \\
= FT-1(F(w \cdot cos(t), w \cdot sin(t)) = \\
= FT-1(FT(f(x,y))) = \\
= FT-1(FT(g(x,y) * h(x,y))) = \\
= FT-1(FT(g(x,y)) \cdot FT(h(x,y))) = \\
= FT-1(G(w \cdot cos(t), w \cdot sin(t)) \cdot H(w \cdot cos(t), w \cdot sin(t))) = \\
= FT-1(G(cos(t) * w, sin(t) * w)) * FT-1(H(cos(t) * w, sin(t) * w)) = \\
= q(s,t) * r(s,t)
\end{gathered} \tag{2}$$

Fourier Transform

The concept of the Fourier transform is based on the fact that it is possible to form a function p(s) as a **weighted summation** of a series of **sin and cos** terms of various **frequencies** , ω_k, with a weighting function $P(\omega)$.

Filtered backprojection

To which part of the algorithm do the different parts of this term correspond?

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} p(s,\theta) e^{-2\pi i \omega s} \, \mathrm{d}s \right) |\omega| \, e^{2\pi i \omega (x \cos\theta + y \sin\theta)} \, \mathrm{d}\omega \, \mathrm{d}\theta$$

Red: 1-D FT of rows/columns in sinogram,

Green: filtering (multiplication with ramp filter)

Blue: backprojection

Variety of reconstruction algorithms

Which of the following model valid reconstruction methods?

- Derivative -> Backprojection -> Hilbert Transform
- 1-D Ramp Filter with Fourier Transform -> Backprojection
- Hilbert Transform -> Backprojection -> Derivative

- Backprojection -> 2-D Ramp Filter with Fourier Transform
- Hilbert Transform -> Derivative -> Backprojection
- Backprojection -> Hilbert Transform -> Derivative

Hilbert transform

Multiplication in frequency space with $-i\ sgn(\omega)$ is a **Hilbert transform** , i.e., equivalent to a **convolution** with $h(s) = \frac{1}{\pi s}$.

Derivation of filtered backprojection

Assume you want to show how the classic filtered backprojection formula is derived. For that purpose, sort the different formulas such that there is a logical transition from the Fourier transformation to the final algorithm.

1. $f(x, y) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} F(u, v) e^{2\pi i(xu+yv)}\, \mathrm{d}u\, \mathrm{d}v$
2. $u = \omega \cos\theta, \quad v = \omega \sin\theta, \quad \omega \geq 0,$
3. $J_{(u,v)}(\omega, \theta) = \begin{vmatrix} \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial \omega} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\omega \sin\theta \\ \sin\theta & \omega \cos\theta \end{vmatrix} = |\omega|\,.$
4. $f(x, y) = \int\limits_{0}^{2\pi} \int\limits_{0}^{\infty} F_{\text{polar}}(\omega, \theta)\, |\omega|\, e^{2\pi i \omega(x\cos\theta + y\sin\theta)}\, \mathrm{d}\omega\, \mathrm{d}\theta$
5. $F_{\text{polar}}(\omega, \theta) = P(\omega, \theta)$ for all $\omega \geq 0,\ \theta \in [0, 2\pi).$
6. $P(\omega, \theta) = \int\limits_{-\infty}^{\infty} p(s, \theta) e^{-2\pi i \omega s}\, \mathrm{d}s.$
7. $f(x, y) = \int\limits_{0}^{2\pi} \int\limits_{0}^{\infty} \left(\int\limits_{-\infty}^{\infty} p(s, \theta) e^{-2\pi i \omega s}\, \mathrm{d}s \right) |\omega|\, e^{2\pi i \omega(x\cos\theta + y\sin\theta)}\, \mathrm{d}\omega\, \mathrm{d}\theta$

Ram-Lak filter

Which of the following is an advantage of the Ram-Lak filter?

- Its spatial sampling conforms with the detector grid points.

Fourier Slice Theorem Proof

The 1-D Fourier transform	**passt zu**	$P(\omega,\theta) = \int_{-\infty}^{\infty} p(s,\theta) e^{-2\pi i \omega s}\,\mathrm{d}s$	
Using the definition of the projection	passt zu	$P(\omega,\theta) = \int_{-\infty}^{\infty} \iint_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - s)\mathrm{d}x\mathrm{d}y\; e^{-2\pi i\omega s}\mathrm{d}s$	
Rearranging the order of the integrals	passt zu	$P(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y) \int_{-\infty}^{\infty} \delta(x\cos\theta + y\sin\theta - s)\; e^{-2\pi i\omega s}\mathrm{d}s\mathrm{d}x\mathrm{d}y$	
After elimination of the delta function	passt zu	$P(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y) e^{-2\pi i(x\cos\theta + y\sin\theta)\omega}\mathrm{d}x\mathrm{d}y$	
Using the definition from the slides we rewrite the 2-D Fourier transform, then we obtain	passt zu	$P(\omega,\theta) = \iint_{-\infty}^{\infty} f(x,y) e^{-2\pi i(xu+yv)}\vert_{u=\omega\cos\theta,\; v=\omega\sin\theta}\mathrm{d}x\mathrm{d}y$	
Results in the proposed statement	passt zu	$P(\omega,\theta) = F(\omega\cos\theta, \omega\sin\theta) = F_{polar}(\omega,\theta)$	

Ramp vs Ram-Lak

The DC shift of the ramp filter is	**passt zu**	**zero.**	
The DC shift of the Ram-Lak filter is	passt zu	nonzero.	

Fourier slice theorem

It states that the **1-D Fourier transform** $P(\omega,\theta)$ of a **projection** $p(s,\theta)$ in parallel beam geometry for a fixed **rotation angle** θ is identical to the **1-D profile** through the origin of the **2-D Fourier transform** $F(\omega cos\theta, \omega sin\theta)$ of the irradiated object .

Lines in central slice theorem

Let us project a 2-D function $f(r)$ onto a 1-D linepunder the angle θ and do a 1-D Fourier transform of this projection. Then let us take the same function and do a 2-D Fourier transform first. If we select a slices through the **origin oder Origin oder ORIGIN** of the Fourier domain under the same angle θ (w.r.t. both real axes, so to speak **parallel** to the projection linep) we get **the same** result as in the first computation.

Fan Beam Reconstruction

Water cylinder assumption

Match the corresponding facts:

The correction method using the water cylinder assumption works well	**passt zu**	**for many clinically interesting objects (head, abdomen, etc...).**	
The correction method using the water cylinder assumption works well	passt zu	if a water cylinder is imaged.	
The correction method using the water cylinder assumption works not so well	passt zu	for scans of both knees simultaneously.	
The correction method using the water cylinder assumption works not so well	passt zu	if the water cylinder assumption is violated.	

Short scan properties

Which of the following statements holds true for a short scan?

- PSF is not uniform.
- Reconstruction resolution changes over the image.
- The Parker weights allow a smooth weighting transition between redundant and singular data.

Cosine filtering

Why do we need cosine filtering? Give the most specific answer possible.

- The Jacobian of the parameter transformation is D cos γ.

Truncation correction

Which of the given options is NOT a truncation correction method?

- [ ] Defect pixel extrapolation
- [ ] Heuristic extrapolation
- [ ] Water cylinder assumption
- [ ] Use of prior knowledge
- [ ] Use of a semi-transparent filter
- [x] Water cube assumption
- [x] Deterministic extrapolation
- [x] Use of a semi-opaque filter
- [x] ATRACTIVE filtering

FBP

Arrange the following steps based on the FBP for the equiangular case:

Perform cosine weighting: $g_1(\gamma,\beta) = g(\gamma,\beta) cos\gamma$.

Apply fan beam filter: $g_2(\gamma',\beta) = (g_1 * h_{fan})(\gamma',\beta)$.

Backproject with distance weight: $f(r,\varphi) = \int_0^{2\pi} \frac{1}{D^2} g_2(\gamma',\beta) d\beta$.

Number of short scan projections

Compute the correct number of projections:

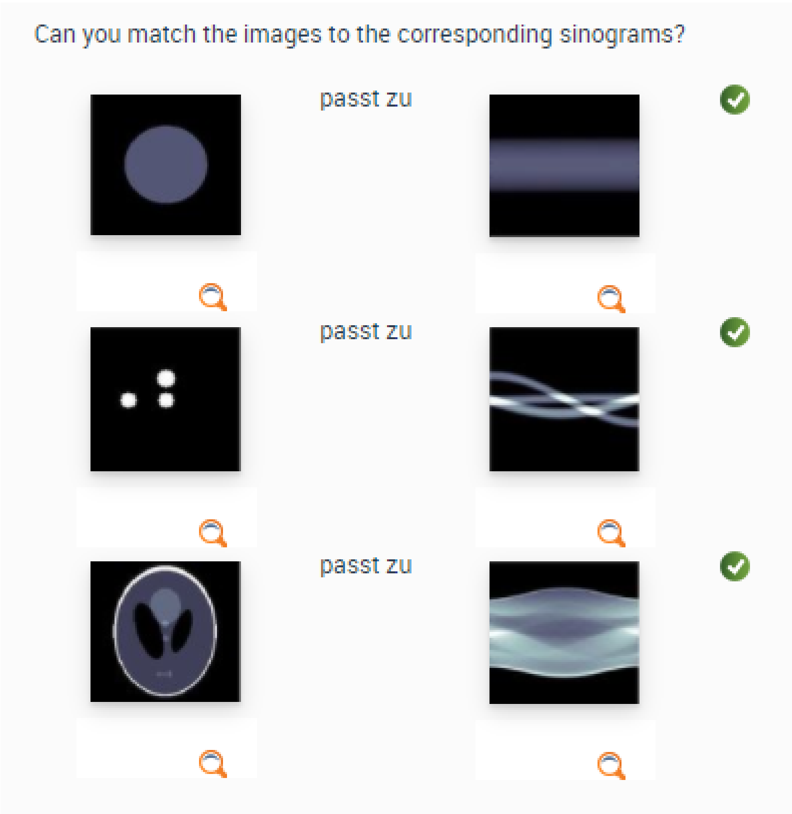
A CT acquisition of a heart is reconstructed by fan beam reconstruction with a fan angle $\delta = 11°$. It is performed by a short-scan. Projections are taken with $\triangle\gamma = 2°$. At least how many projections have to be acquired to reconstruct the image?

A total of **102** wurde wie folgt interpretiert: 102 projections is needed.

Truncation

Truncation introduces **artificial frequencies oder Artificial Frequencies oder ARTIFICIAL FREQUENCIES** into the reconstruction and causes a typical **artifact oder Artifact oder ARTIFACT** at the end of the field of view .

Sinogram matching



Short Scan Angle

Fan-beam minimum scan angle for a complete short scan depends on:

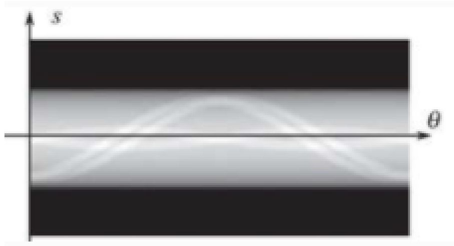
- the location of the object
- the size of the object

Phantom types

Numeric/simulated phantoms...	**passt zu**	**...originate from computer simulations.**	
Numeric/simulated phantoms...	passt zu	...are known exactly.	
Numeric/simulated phantoms...	passt zu	...have only limited realism.	
Real phantoms...	passt zu	...are designed with desired properties.	
Real phantoms...	passt zu	...are manufactured at a high accuracy.	
Real phantoms...	passt zu	...may be difficult to use.	
Real phantoms...	passt zu	...may still have a limited manufacturing accuracy.	

Sinogram

Look at the following figure:



What is this type of image called? Explain what information the image contains and how it can be obtained.

This image is called **sinogram oder Sinogram oder SINOGRAM** . Each line of the image contains the **projection oder Projection oder PROJECTION oder projection values oder Projection Values oder PROJECTION VALUES** of the object from an angle on a trajectory around the object. The image can be obtained by **Radon transform** .

Fan Beam Reconstruction Sampling

Researchers treat the flat detector fan-beam and curved detector fan-beam differently in reconstruction algorithm development.

Sampling:

Flat detector	**passt zu**	**ΔsΔs**	
Curved detector	passt zu	ΔγΔγ	

Phantoms

There are various phantoms available. Some of the scientific decision factors how to choose them are:

- Modality
- Application area
- Simulation
- Manufacturing precision
- Absorption properties
- Anatomical resemblance

Fan beam knowledge

Which of the following statements are true?

- The fan beam PSF is equivalent to the parallel beam PSF.
- A sinogram contains all information to reconstruct one slice.
- The PSF in fan beam geometry is shift invariant.

Rebinning

Match the LHS and RHS of the given equations on the basis of parallel beam to fan beam conversion.

θ	**passt zu**	**$\gamma + \beta$**	
s	passt zu	$D \sin\gamma$	
$p(s,\theta)$	passt zu	$g(\gamma, \beta)$	

3-D Reconstruction

Grangeat’s algorithm

Please arrange the following steps of Grangeat’s algorithm:

1. Form all possible line-integrals on each detector plane (all locations and orientations).
2. Compute the angular derivative.
3. Rebin the data to Radon space.
4. Take the derivative with respect to t.
5. Perform the 3-D Radon backprojection.

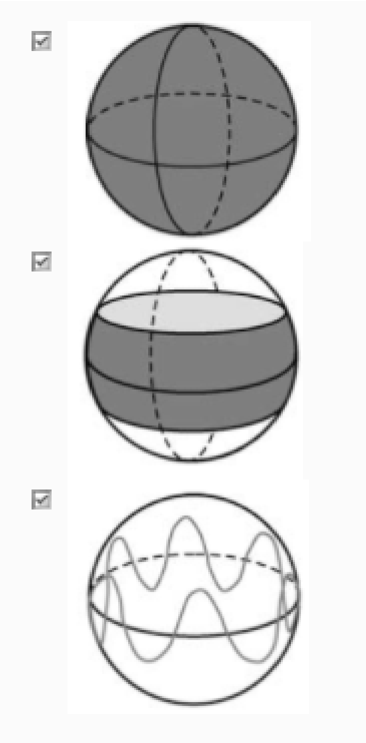
Cone beam final statements

Which of the following statements is applicable to cone beam geometry?

- The cone angle influences the strength of the cone beam artifact.
- Cone beam reconstruction suffers from scatter much more than fan beam reconstruction.
- With complete data, exact algorithms produce results without artifacts.

Orlov's condition

Which trajectory is fulfilling Orlov's condition:



Data completeness

A complete data set can be obtained if every great circle intersects the trajectory. This condition is known as:

- Orlov’s condition

Line- and plane-integrals

Name the transformations and specify the type of integral given by the formulas.

$$\boxed{\text{Radon transform}} = \int_{\boldsymbol{x}\cdot\boldsymbol{\theta}=s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \rightarrow \boxed{\text{plane-integral}}$$

$$\boxed{\text{X-ray transform}} = \int_{\mathbb{R}} f(\boldsymbol{x_0} + t\boldsymbol{\theta})\,\mathrm{d}t \rightarrow \boxed{\text{line-integral}}$$

Conditions for exact Feldkamp’s algorithm reconstruction

Conditions of Feldkamp’s algorithm for exact reconstruction:

- object is constant in the axial direction
- small cone angle
- cone-beam projection

Feldkamp’s algorithm

Which of the following statements are true?

- FDK is a commonly used cone beam reconstruction algorithm because it is fast and robust.
- FDK is based on a fan beam reconstruction algorithm with appropriate cosine weights.

FDK

Arrange the following steps based on the FDK algorithm:

1. Perform adjusted cosine weighting
2. Apply ramp filter for each detector row
3. Backproject with distance weight

Tuy's condition

Every **plane oder Plane oder PLANE** that **intersects oder Intersects oder INTERSECTS** the object of interest must contain a **cone beam focal point** . This condition is used to check **data completeness**

Katsevich

Which of the following statements on Katsevich's method are true?

- It is a filtered backprojection algorithm.
- A line that passes a point inside the helix and hits it two times within one pitch is called π-line.
- The Hilbert transform is part of this algorithm.

Modalities

Larmor frequency

The measured resonance pulse is at the **larmor frequency**

PET and SPECT summary

PET is based on the insertion of a **radioactive oder Radioactive oder RADIOACTIVE** substance into the patient's body. The radioactive decay causes the creation of **positrons** that emit two **photons** in opposite direction when hitting an **electron** . These events have to be detected simultaneously at **opposite oder Opposite oder OPPOSITE** directions which induces a **parallel** imaging geometry. Ultimately, PET imaging has lower **resolution oder Resolution oder RESOLUTION** than X-ray imaging. In comparison, in SPECT the detected ray energy consists of **collimated oder Collimated oder COLLIMATED** gamma rays and SPECT imaging has a **lower** resolution than PET imaging.

MR directional encoding

Match the method of encoding different directions:

Encoding in x-direction	**passt zu**	**Variation of the magnetic field during the read-out of the RF pulse**	
Encoding in y-direction	passt zu	Variation of the phase of the spins	
Encoding in z-direction	passt zu	Two coils with currents running in opposite direction	

Magnetic resonance imaging: read-out

This read-out space is called **k-space oder k space oder kspace oder K-Space oder K Space oder Kspace oder K-SPACE oder K SPACE oder KSPACE** . To reconstruct the image, the **inverse Fourier transform oder inverse fourier transform oder Inverse Fourier Transform oder INVERSE FOURIER TRANSFORM oder ift oder iFT oder IFT** is used to transform the data from the frequency domain to the spatial domain.

PET/SPECT emission

PET and SPECT use **gamma rays** emitted from radioactive substances inside the patient's body.

Ionizing radiation

Which of the following modalities does not use a form of ionizing radiation?

- Magnetic resonance imaging

Scanning geometries

List the different scanning geometries from Unit 51 for X-ray computed tomography!

- Parallel Beam
- Fan Beam
- Cone Beam
- Helical Scanning

Iterative Reconstruction

ART extensions

Complete the statements given here by filling in the gaps and selecting correct options.

Slow convergence is the main drawback of ART. Improvements can be achieved for example with one of the following methods:

- **Ordered subsets** describes a method to intelligently select the order of the update equations.
- **SART** : Compute multiple updates at the same time and combine the result.
- **SIRT** : Compute update once per iteration (use a single projection for all updates).

Emission model

What is a common distribution to describe photon emissions?

- Poisson distribution

Good hyperplanes

Complete the following statement:

In ART the **angle oder Angle oder ANGLE** between hyperplanes affects the **rate of convergence to** the solution. If they are nearly **orthogonal oder Orthogonal oder ORTHOGONAL** to each other, the method converges rapidly. In order to improve the convergence we can apply **orthogonalization methods** in advance to iterations.

TV-Norm

Which of the following statements is true?

- The TV-norm effects small areas showing high frequency variations to become smoother.

Projection onto a hyperplane (rand)

Let $x = \begin{pmatrix} 0 \\ 5 \\ 9 \end{pmatrix}$, $d = 3$ and $n = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$. Consider the hyperplane $\{c \in \mathbb{R}^3 | \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}^T c = 3\}$ and calculate the projection y of $\begin{pmatrix} 0 \\ 5 \\ 9 \end{pmatrix}$ onto the hyperplane.

$y =$

0

-55/29

181/29

ART example

The algebraic reconstruction technique (ART) is a practical application of the Kaczmarc method. To get an idea how it works, assume that the following two equations describe two lines in the solution space of a synthetic CT projection:

$$3y - x = 5,$$
$$11x + 4y = 19.$$

Compute two iteration steps using ART to find an approximate solution $\mathbf{X}^{(2)} \in \mathbb{R}^2$. Initialize your algorithm with $\mathbf{X}^{(0)} = (0, 0)^\top$ and use the exact order of equations as above for the system matrix (first iteration $\rightarrow$ use first equation).

Result:

System matrix $\mathbf{A} =$ (

-1

-1 wurde wie folgt interpretiert: -1

3

11

4

4 wurde wie folgt interpretiert: 4

)

Projection vector $\mathbf{P} =$ (

5

5 wurde wie folgt interpretiert: 5

19

)

First iteration $\mathbf{X}^{(1)} =$ (

-0.5

-0.5 wurde wie folgt interpretiert: -0.5

1.5

1.5 wurde wie folgt interpretiert: 1.5

)

Second iteration $\mathbf{X}^{(2)} =$ (

0.985

0.985 wurde wie folgt interpretiert: 0.985

2.04

2.04 wurde wie folgt interpretiert: 2.04

)

Gradient descent

Gradient descent algorithms are **iterative** methods to find a **minimum oder Minimum oder MINIMUM** of a given objective function. The idea behind it is that following the **negative** gradient direction leads to a **local** minimum.

Iterative Reconstruction Algorithms

Many algorithms can be used as iterative reconstruction, try to match the algorithm with corrosponding objective function:

Algebraic Reconstruction Technique	**passt zu**	$X^{k+1} = X^k + \frac{p_i - A_i X^k}{A_i A_i^\mathsf{T}} A_i^\mathsf{T}$	
Gradient Descent	passt zu	$X^{k+1} = X^k + \lambda \Delta$	
Linear Equations	passt zu	$X = A^{-1} P,$ $X = (A^\mathsf{T} A)^{-1} A^\mathsf{T} P,$ $X = A^\mathsf{T} (A A^\mathsf{T})^{-1} P,$	
Maximum-Likelihood Expectation-Maximization	passt zu	$x_j^{k+1} = \frac{x_j^k}{\text{backproject}(1)} \text{backproject}\left(\frac{p_i}{\text{project}(x_i^k)}\right)$	

Bilateral filtering

Bilateral filtering is a method for **edge preserving** **noise reduction** . Instead of simply averaging image values in dependence on their **geometric closeness** , also the **photometric similarity** of nearby pixels is considered.

Reason for non-uniqueness

Which of the following can be stated as reason for not finding a unique solution by inverting the projection system?

- Inconsistent data

Gradient descent step (rand)

Let $p = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $x^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = 1$. To solve the minimization problem with the objective function $||Ax - p|| = (Ax - p)^T (Ax - p)$, you can use the gradient descent method - do one step of this method with the given parameters.

$x^1 =$

7

20

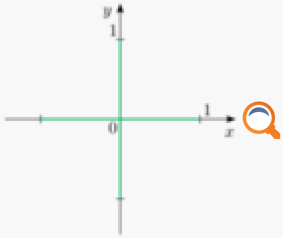
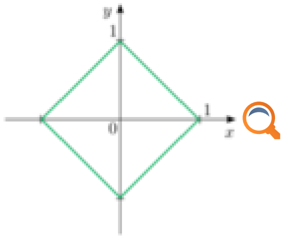
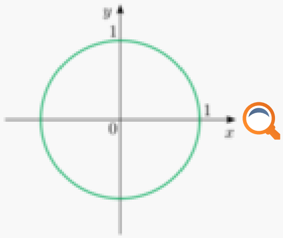
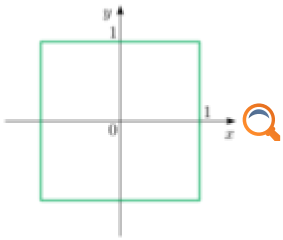
ML-EM meaning

Write out what the abbreviation ML-EM means.

ML-EM is the abbreviation for **maximum** **- likelihood - expectation - maximization**.

Lp-Norms

Match the schemes of the Lp-norms with the corresponding norms.

	**passt zu**	**L0**	
	passt zu	L1	
	passt zu	L2	
	passt zu	L∞	

Rigid Registration

Quaternion multiplication (rand)

Recall the multiplication rule for quaternions.

Given $r_1 = 2 + 9i + 0j + 3k$ and $r_2 = 7 + 4i + 4j + 1k$, calculate the product of r_1 and r_2.

$r_1 * r_2 =$ -25 wurde wie folgt interpretiert: $-25 +$ 59 wurde wie folgt interpretiert: $59\ i+$ 11 wurde wie folgt interpretiert: $11\ j+$ 59 wurde wie folgt interpretiert: $59\ k$

Perfect! The exact solution is $-25 + 59i + 11j + 59k$.

3-D Rotations

Match the different types of representations for 3-D rotations to the corresponding mathematical concept.

Rotation quaternion	passt zu	Quaternions	
Rotation matrices	passt zu	Euler angles	
Rodriguez formula	passt zu	Axis-angle representation	

Euler angles

What is a problem with the Euler angle representation of 3-D rotations?

- Matrix multiplication is not commutative
- This representation of rotations is not unique.
- The angles can only range from [-2π,2π].

Rigid registration

Complete the following statements correctly:

The computation of a transformation that maps corresponding **image points** of two images is known as point based image registration. In rigid registration the transformation consists of **rotation** and **translation** .

Complex numbers

Answer the following three questions successively. First solve them exact on paper and then put in rounded values (-> one decimal).

1. What is the complex number for a rotation by 45 degrees?

0.71 + 0.71 i

2. What complex number do you get if you rotate 2+2i by 45 degrees?

0 + 2.83 i

3. Finally, consider rotating 3+3i around the point 2+2i by 45 degrees, what do you get?

2 + 3.41 i

Formula of Rodrigues (rand, opt)

Use the formula of Rodrigues to compute the rotation matrix R with the following parameters:

Rotation axis $\boldsymbol{u} = \begin{pmatrix} 0.52 \\ 0.52 \\ 0.68 \end{pmatrix}$

Angle $\theta = 5.7$

$$R = \begin{matrix} 0.88 & 0.42 & -0.23 \\ -0.33 & 0.88 & 0.34 \\ 0.34 & -0.23 & 0.91 \end{matrix}$$

(Please use numbers with two decimal digits in your answer.)

Exactly! The correct solution to rotate with the angle 5.7 around the axis $\begin{pmatrix} 0.52 \\ 0.52 \\ 0.68 \end{pmatrix}$ is $\begin{pmatrix} 0.88 & 0.42 & -0.23 \\ -0.33 & 0.88 & 0.34 \\ 0.34 & -0.23 & 0.91 \end{pmatrix}$.

ICP Algorithm

Optimal translation

Given two 2-D point sets, compute the translation vector that minimizes the average square distance between one of the sets and the the other after translation.

Let the first curve be sampled by $\mathbf{p}_0 = (0,1)^\top$, $\mathbf{p}_1 = (2,2)^\top$, $\mathbf{p}_2 = (4,1)^\top$, $\mathbf{p}_3 = (6,2)^\top$, and the second curve by $\mathbf{q}_0 = (0.5, 0.5)^\top$, $\mathbf{q}_1 = (2,1)^\top$, $\mathbf{q}_2 = (4,0)^\top$, $\mathbf{q}_3 = (6,1)^\top$.

Minimize the distance $\sum_{i=0}^{3} \|\mathbf{p}_i - \mathbf{q}'_i\|^2$ where $\mathbf{q}'_i$ denote translated points $\mathbf{q}'_i = \mathbf{q}_i + \mathbf{t}$, $\mathbf{t} = (t_x, t_y)^\top$ (no rotation here!).The simplified goal function yields:

$\underset{\mathbf{t}}{\operatorname{argmin}}$ 4 t_x^2 + 4 t_y^2 + 1 t_x + -7 t_y + 3.5

Now compute the optimal translation vector (enter exact values):

$\mathbf{t} = ($ -0.125 , 0.875 $)^\top$

Geometric Data for ICP

Note three different types of geometric data that ICP can be used with:

1. point sets
2. faceted surfaces
3. implicit curves