FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG
SCHOOL OF ENGINEERING

Lecture Pattern Analysis

Part 17: Introduction to Probabilistic Graphical Models

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Introduction

- We will now look deeper into relationships between random variables
- Probabilistic graphical models constitute the third part of PA[1]
- We are already familiar with probabilistic models:
  - Probabilistic models represent data properties in a probabilistic formulation
  - We usually start with a joint PDF of all random variables $p(\mathbf{x}_1, \dots, \mathbf{x}_N)$
  - Factorization via the product rule creates some smaller terms:

$$p(\mathbf{x}_1, \dots, \mathbf{x}_N) = p(\mathbf{x}_N | \mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \cdot \dots \cdot p(\mathbf{x}_2 | \mathbf{x}_1) \cdot p(\mathbf{x}_1) \tag{1}$$

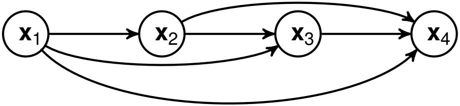
  - The order of factorization is arbitrary. For example, this is also correct:

$$p(\mathbf{x}_1, \dots, \mathbf{x}_N) = p(\mathbf{x}_3 | \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \dots, \mathbf{x}_N) \cdot \dots \cdot p(\mathbf{x}_2 | \mathbf{x}_1) \cdot p(\mathbf{x}_1) \tag{2}$$

[1] Literature reference for this video is Bishop Sec. 8 and Sec. 8.1 (just the first 4 pages, i.e., without Sec. 8.1.1 until 8.1.4)

Graphical Models

- A graph representation aims to improve the understanding of these relationships, and the reasoning (inference) on the variables
  - Directed edges represent **conditional probabilities** (the general class of such graphs is called Bayesian networks).
    We will study Hidden Markov Models (HMMs) as an important special case
  - Undirected edges represent joint probabilities. We will study Markov Random Fields (MRFs) as an important special case.
- Example graph for $p(\mathbf{x}_4 | \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \cdot p(\mathbf{x}_3 | \mathbf{x}_1, \mathbf{x}_2) \cdot p(\mathbf{x}_2 | \mathbf{x}_1) \cdot p(\mathbf{x}_1)$:

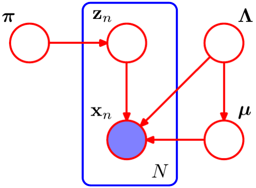


  where an edge expresses a conditional probability. In particular,
  - $\mathbf{x}_1$ conditions all variables $\rightarrow$ 3 outgoing edges
  - $\mathbf{x}_4$ depends on all other variables $\rightarrow$ 3 incoming edges
  - $\mathbf{x}_2$ depends on $\mathbf{x}_1$ and conditions $\mathbf{x}_3$, $\mathbf{x}_4$ $\rightarrow$ 1 incoming edge, 2 outgoing edges

Graphical Models and Probabilistic Inference

- As mentioned for the model selection on GMMs, inference on a full probabilistic model is oftentimes infeasible
- Assumptions on variable independence can solve this issue
- The associated graphical model has no edge between independent variables
- For example, the graphical model for GMM fitting via Bishops variational approximation is

Figure 10.5 Directed acyclic graph representing the Bayesian mixture of Gaussians model, in which the box (plate) denotes a set of N i.i.d. observations. Here $\boldsymbol{\mu}$ denotes $\{\boldsymbol{\mu}_k\}$ and $\boldsymbol{\Lambda}$ denotes $\{\boldsymbol{\Lambda}_k\}$.



- In this Chapter, we will investigate independence assumptions to obtain tractable models for sequences (HMMs) and label assignment tasks (MRFs)