Lecture Pattern Analysis

Part 24: MRF Inference via Min Cuts

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Overview

- The MRF inference task is to find optimal label assignments

$$x_1^*, \ldots, x_N^* = \underset{x_1,\ldots,x_N}{\operatorname{argmax}} \frac{1}{Z} \exp\left(-\sum_i E(x_i, y_i) - \sum_{i,j} E(x_i, x_j) \right) \quad (1)$$

  (where we limited the maximum clique size to 2, i.e., each term includes at most two hidden variables)

- This is equivalent to the minimization of the sum of energy terms

$$x_1^*, \ldots, x_N^* = \underset{x_1,\ldots,x_N}{\operatorname{argmin}} \sum_i E(x_i, y_i) + \sum_{i,j} E(x_i, x_j) \quad (2)$$

- The idea of MRF inference via graph cuts[1] is to
  - encode these energy terms in a specialized graph,
  - such that that graph's minimum cut also minimizes the sum of energy terms

[1] The literature reference for this lecture is the paper by Kolmogorov and Zabih, which is uploaded to studOn

Constraints and Benefits

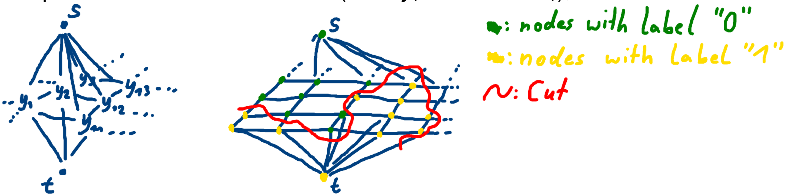
- The construction requires
  - binary labels,
  - maximum clique size of 2, and
  - pairwise energy terms must satisfy the **submodularity condition**

$$E(0,0) + E(1,1) \leq E(0,1) + E(1,0) \tag{3}$$

- Under these constraints, the algorithm finds
  - a globally optimal labelling
  - in polynomial time
- The **α-expansion algorithm** extends the method to non-binary labelings
- α-expansion is only locally optimal, but within a guaranteed margin around the global optimum

Construction Idea

- Graph with all hidden variables (here y_i instead of x_i), source s and sink t:



- A cut separates each node from either s or t (but never from both)
- For example in a 0-1 labelling task, identify s with label "0", and t with label "1"
- Each node (=hidden variable) receives the label that it is connected to
- Smart construction of the edge weights sets the minimal s-t cut identical with the minimal-energy binary labeling in the MRF

Additivity of Graphs and Encoding of Unary Energy Terms

- We can encode each energy term independently, because the min cut construction is homomorphic under graph composition:
  - if G_1 encodes $\min(E_1)$ and G_2 encodes $\min(E_2)$,
  - then $(G_1 \cup G_2)$ encodes $\min(E_1 + E_2)$
- Graph construction for a single unary term $E(x_i, y_i)$, with $s = 0$, $t = 1$:

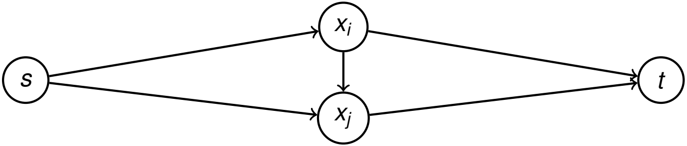
$$\begin{tikzcd} s \arrow[r, "E(1, y_i)"] & x_i \arrow[r, "E(0, y_i)"] & t \end{tikzcd}$$

- For example, a cut between s and x_i assigns the label $x_i = t = 1$. Hence, the cost is $E(1, y_i)$, which relates 1 to observation y_i
- With the goal to search for the **minimum** cut, we can construct an equivalent smaller graph by relating $E(1, y_i)$ and $E(0, y_i)$
- If, for example, $E(1, y_i) > E(0, y_i)$, then the equivalent graph is

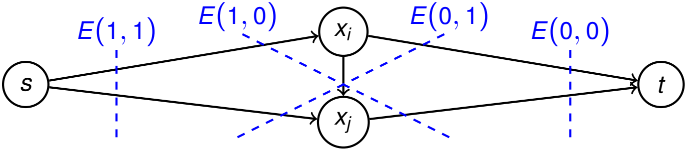
$$\begin{tikzcd} s \arrow[r, "E(1, y_i) - E(0, y_i)"] & x_i & t \end{tikzcd}$$

Encoding of Pairwise Energy Terms (1/2)

- Graph construction for a single pairwise term $E(x_i, x_j)$, with $s = 0$, $t = 1$:



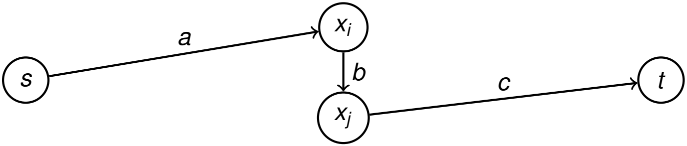
- Possible cuts and associated costs:



- This graph has 5 edges, but only 4 terms $E(0,0)$, $E(0,1)$, $E(1,0)$, $E(1,1)$
- Hence, let us use immediately write the reduced graph

Encoding of Pairwise Energy Terms (2/2)

- For example, the likely case that $E(1,0) > E(0,0)$ and $E(1,0) > E(1,1)$ gives



with edge weights a, b, c obtained from the linear system of equations

$$a + k = E(1,1) \tag{4}$$

$$c + k = E(0,0) \tag{5}$$

$$b + k = E(0,1) \tag{6}$$

$$a + c + k = E(1,0) \tag{7}$$

with constant offset k, and without b in Eqn. 7 since this is a backward edge

Graph Cut Inference for Binary and Non-Binary Labels

- The full graph is constructed by summing the subgraphs of all energy terms
- Min cut on that full graph finds in **polynomial time** a solution that is **globally optimal** for binary labels
- Non-binary labellings can be found via α-expansion:
    1. From the set of labels, select one specific label α
    2. Fix hidden variables that already have label α
    3. Seek the lowest energy labelling that switches at least one other hidden variable to α
    4. Keep this "expanded" α labelling if its energy is lower than the current energy

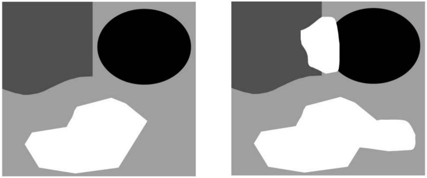


Fig. 1. An example of an expansion move. The labeling on the right is a white-expansion move from the labeling on the left.