Estimating Seasonal Components

Harmonic Regression & the S1 Method

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

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We learned something Fri/Mon, in Time Series 🕛

Moving average (MA) smoothing,
average of points around time of interest
gets rid of high frequencies (low pass filter)

Some basic intro quarto and r stuff
Made our first document
Quarto interprets *code as code* and *text as text!*

Exponential Smoothing
Only uses past values (can forecast)

What did we tell quin?

Harmonic Regression

$$X_t = \cancel{\boxed{m_t}} + \boxed{s_t} + \boxed{Y_t} \quad (\star)$$

Assume s_t has the form $$s_t = \sum_{k=1}^{M} a_k \cos\left(2\pi \frac{1}{d_k} t + \phi_k\right)$$

For the k[th] seasonal component,

a_k **Amplitude**

d_k **Period** $\frac{1}{d_k}$ = **Frequency**

ϕ_k **Phase**

(from textbook)

$$s_t = a_0 + \sum_{j=1}^{k} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)),$$

aj and bj are unknown parameters

λ_j = fixed frequencies of the form **2kπ/d** for integer **k**

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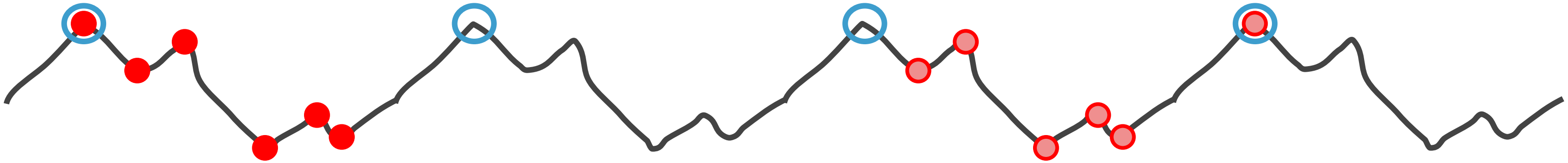
The S1 Method (See textbook: Brockwell & Davis)

- 🕑 This method doesn't assume any particular behaviour within a cycle (unlike **HR**)
- 🕑 Step 1 is based on the fact:

(True for average) $\frac{1}{d}\left(s_t + s_{t+1} + \cdots + s_{t+d-1}\right)$ is the same for every **t**



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The S1 Method: Step 1

Apply MA smoother with window length = d

d odd → set order **q** = **(d-1)/2**

d even

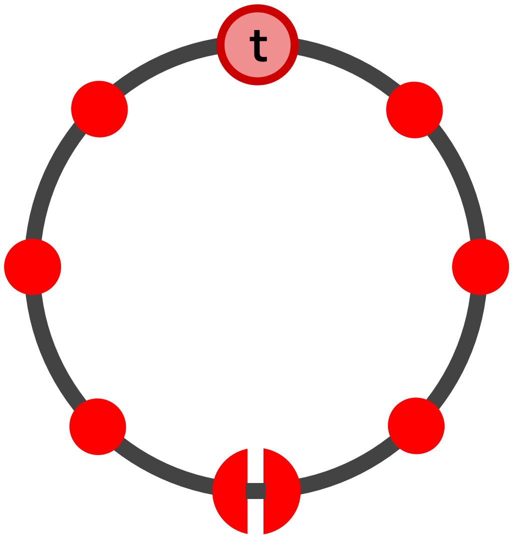
→ set **q** = **d/2**

→ set window length = **d+1**

→ modify MA smoother:

first/last value get weighted by **1/2**

$$s_{t-q} = s_{t+q}$$



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The S1 Method: Step 1

- The smooth at time **t** is:

$$\hat{m}_t = \begin{cases} \frac{1}{d}(x_{t-q} + \cdots + x_t + \cdots + x_{t+q}) & d \text{ odd} \\ \frac{1}{d}\Big(\frac{1}{2}x_{t-q} + x_{t-(q-1)} + \cdots + x_t \\ \quad + \cdots + x_{t+(q-1)} + \frac{1}{2}x_{t+q}\Big) & d \text{ even} \end{cases}$$

- **Effect of MA smoother:**

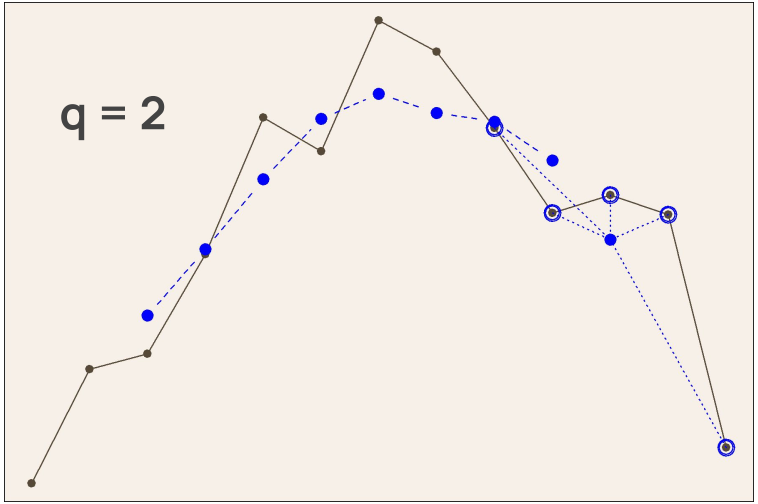
  eliminates s_t, dampens noise

- **Residuals:** $r_t = x_t - \hat{m}_t$

  Should be dominated by s_t

Compute r_t only for $q < t \leq N - q$

(**t** for which we have a full window of data)



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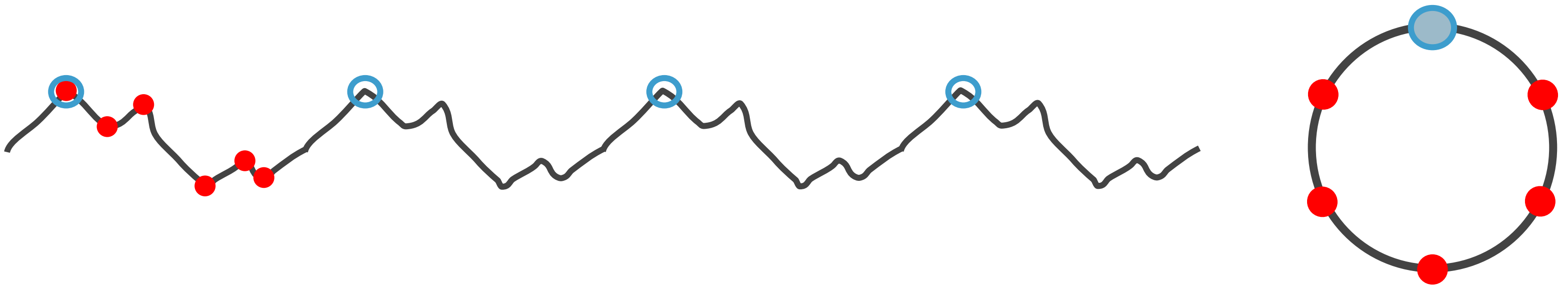
The S1 Method: Step 2

Use residuals to estimate seasonal component within 1 cycle

For **k** in **{1, . . . , d}** , compute w_k: the average of all residuals in the set

$$\left\{ \left(x_{k+jd} - \hat{m}_{k+jd}\right) \; ; \; q < k + jd \leq N - q \right\}$$



→ w_k estimates **k**$^{\text{th}}$ value of s_t in 1 cycle

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The S1 Method: Step 2

Use residuals to estimate seasonal component within 1 cycle

We're assuming

$\sum_{t=1}^{d} s_t = 0$

Final estimate **demeans** the w_k :

$$\hat{s}_k = w_k - \frac{1}{d}\sum_{j=1}^{d} w_j$$

- Now the $\{\hat{s}_1, \ldots, \hat{s}_d\}$ estimate one cycle.
- Clone it across the time span of the data, and you've estimated all of s_t

$$\hat{s}_k = \hat{s}_{k-d}, k > d.$$

Notes

1) again, you have to know **d** beforehand
2) Only works for 1 seasonal component

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Skye talk about this pls ❤

The *deseasonalized* data is then defined to be the original series with the estimated seasonal component removed, i.e.,

$$d_t = x_t - \hat{s}_t, \quad t = 1, \ldots, n. \tag{1.5.14}$$

Finally, we reestimate the trend from the deseasonalized data $\{d_t\}$ using one of the methods already described. The program ITSM allows you to fit a least squares polynomial trend $\hat{m}$ to the deseasonalized series. In terms of this reestimated trend and the estimated seasonal component, the estimated noise series is then given by

$$\hat{Y}_t = x_t - \hat{m}_t - \hat{s}_t, \quad t = 1, \ldots, n.$$

The reestimation of the trend is done in order to have a parametric form for the trend that can be extrapolated for the purposes of prediction and simulation.

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We learned something Today, in Time Series 🕛

What do we tell quin?