Sample ACVFs / ACFs

Distribution & White Noise detection

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Discrete Time Series Analysis

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Sample ACVFs

Let $\{x_t\}$ be an **observed, N**-length, **stationary** time series.

- 🕑 The standard **ACVF estimator** is $\hat{\gamma}(h) = \frac{1}{N}\sum_{t=1}^{N-h}(x_t - \bar{x})(x_{t+h} - \bar{x})$
- 🕑 Can only compute for $|h| < N$
- 🕑 Don't trust this for lags **h** near **N** (or **-N**).

General Rule: Stick to lags less than **log(N)**, or less than **N/4**

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ACVFs and Seasonality

$$\hat{\gamma}(h) = \frac{1}{N}\sum_{t=1}^{N-h}(x_t - \bar{x})(x_{t+h} - \bar{x})$$

If {Xt} has a cyclic component, $\hat{\gamma}_X$ should show a similar cycle.

Extreme example: $x(t) = \cos(2\pi t/d)$. For $|h| \geq 0$,

$$\begin{aligned}\hat{\gamma}(h+kd) &= \frac{1}{N}\sum_{t=1}^{N-(h+kd)}(x_t - \bar{x})(x_{t+(h+kd)} - \bar{x}) \\ &= \frac{1}{N}\sum_{t=1}^{N-(h+kd)}(x_t - \bar{x})(x_{t+h} - \bar{x})\end{aligned}$$

$\hat{\gamma}(h+kd)$ will tend to be smaller than $\hat{\gamma}(h)$ as k increases

This is the original expression of $\boxed{\hat{\gamma}(h)}$, but with fewer terms

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ACVFs and Seasonality:

$$\hat{\gamma}(h) = \frac{1}{N}\sum_{t=1}^{N-h}(x_t - \bar{x})(x_{t+h} - \bar{x})$$

Random component → $\hat{\gamma}(h)$ is approximate.

Too much random → noise drowns out cyclic component

Suppose $\{X_t\}$ has the classical decomposition (⋆)

Its **theoretical** ACVF is

$$\begin{aligned}\gamma_X(h) &= \mathrm{Cov}(m_t + s_t + Y_t, m_{t+h} + s_{t+h} + Y_{t+h}) \\ &= \mathrm{Cov}(Y_t, Y_{t+h}) = \gamma_Y(h)\end{aligned}$$

X and Y have the same theoretical ACVF,

But when observed, the **sample** ACVFs can be very different!

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Properties of the Sample ACVF

Also, wait, why do we divide by N?

1) Symmetry: $\gamma(h) = \gamma(-h)$

2) Still positive semidefinite

Let $y_t = x_t - \bar{x}$

Then define **A** :=
$$\begin{bmatrix} 0 & \cdots & 0 & y_1 & y_2 & \cdots & y_n \\ \vdots & & y_1 & y_2 & y_3 & \cdots y_n & 0 \\ \vdots & & \iddots & & & \iddots & \vdots \\ 0 & \iddots & & & \iddots & & \vdots \\ y_1 & \cdots & & y_n & 0 & \cdots & 0 \end{bmatrix}$$

$(i,j)^{th}$ entry of AA^T is $\gamma_X(j-i)$

→ Divide by **N** $\frac{1}{N}AA^T = \hat{\Gamma}_N$

$$a^T\hat{\Gamma}_N a = a^T\left(\frac{1}{N}AA^T\right) = \frac{1}{N}(A^Ta)^T(a^Ta) = \frac{1}{N}||A^Ta||^2 \geq 0$$

So that's why we divide by **N** instead of **N-h**. Also these properties extend to $\hat{\rho}$.

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Distribution of $\hat{\rho}$

Let **m** be some maximum time-lag.

Let $\hat{\rho} = \big(\hat{\rho}(1), \ldots, \hat{\rho}(m)\big)$, where $\hat{\rho}(h)$ is the sample ACF at lag **h**

Distribution (large **N**): $\hat{\rho} \sim \mathcal{MVN}(\rho, \frac{1}{N} W)$

Covariance Matrix:

Mean vector: $\rho = \big(\rho(1), \ldots, \rho(m)\big)$

$$W[i,j] = \sum_{k=1}^{\infty} \big[\rho(k+i) + \rho(k-i) + \rho(i)\rho(k)\big] \times \big[\rho(k+j) + \rho(k-j) + \rho(j)\rho(k)\big]$$

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Example W | White Noise Detection

Consider $X_t \sim \text{wn}(0, \sigma^2)$.

Then $\hat{\rho}(1), \ldots, \hat{\rho}(m)$ are approximately independent $\mathcal{N}(0, 1/N)$ RVs

To test if $\{x_t\}$ is white noise, we can test $\hat{\rho}$

For a given (nonzero) lag h,

Null Hypothesis: $H_0 : \rho(h) = 0$

Alternative Hypothesis: $H_A : \rho(h) \neq 0$

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Example W | White Noise Detection

- Under the **wn(0, σ^2)** hypothesis, $\hat{\rho}(i) \dot{\sim} N(0, 1/N)$
- Reject **H$_A$** if $|\hat{\rho}(h)|$ is too large, say, **> c**
- Probability of type 1 error (under the null):

$$\begin{aligned} H_0 &: \rho(h) = 0 \\ H_A &: \rho(h) \neq 0 \end{aligned}$$

$$P(|\hat{\rho}(h)| > c) = P\left(\frac{|\hat{\rho}(h)|}{1/\sqrt{N}} > \frac{c}{1/\sqrt{N}}\right) \approx P\left(|Z| > \frac{c}{1/\sqrt{N}}\right)$$

where $Z \sim \mathcal{N}(0, 1)$

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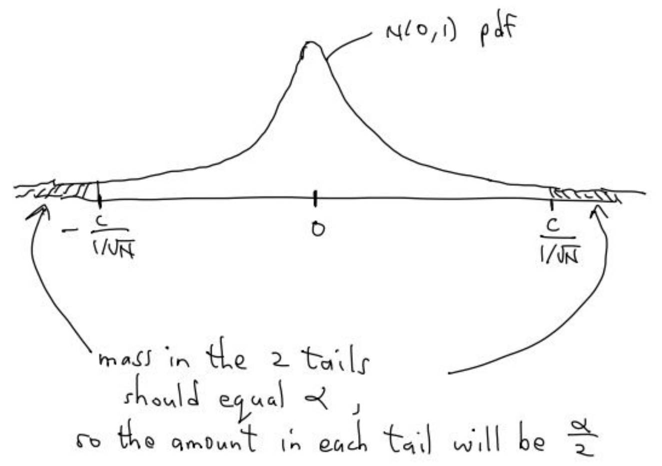
Example W | White Noise Detection

- Now we want a **threshold** for P(type I error): some $\alpha \in (0,1)$

  Typical choices: [α = 0.05] or [α = 0.01]

- Set $\frac{c}{1/\sqrt{N}}$ equal to $\left(1-\frac{\alpha}{2}\right)$ quantile of the **N(0,1)** distribution



For $\alpha = 0.05$, we want $\frac{c}{1/\sqrt{N}}$

to = the 0.975-quantile (**≈1.96**)

$$\rightarrow \quad c = \frac{1.96}{\sqrt{N}}$$

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Example W | White Noise Detection

Conclusion:

🕑 To test if {X_t} is **wn(0, σ^2)**, at the $\alpha = 0.05$ level, compute $\hat{\rho}(h)$ for some set of lags {**h = 0, 1, …, m**}

🕑 $|\hat{\rho}(h)| > 1.96/\sqrt{N}$ → Reject the null hypothesis {X_t} ~ **wn(0, σ^2)**

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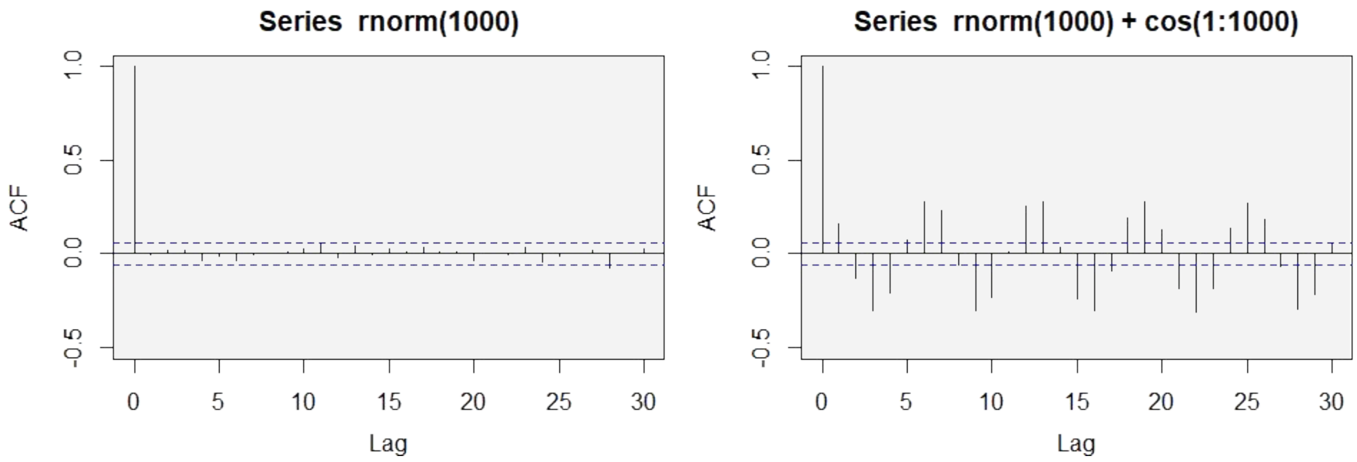
ACFs in R



- Function `acf()` plots 2 horizontal dashed blue lines at $\pm 1.96/\sqrt{N}$
- $\hat{\rho}(h)$ that fall outside blue lines = evidence $\{X_t\}$ is not **wn(0, σ^2)**

Interpretation: If we compute $\hat{\rho}$ for white noise at **m** different lags,

We expect $\hat{\rho}$ to fall outside the blue lines M x (alpha) times

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We learned something today, in Time Series 🕛

What do we tell quin??