Forecasting - Part II

h-Step ahead prediction for **MA(q)** processes

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

Skyepaphora Griffith, Queen's University

We learned something last time, in Time Series 🕛

Linear Predictors: we want to predict **h** steps into the future

Autoregressive processes have special, recursive forecasting properties

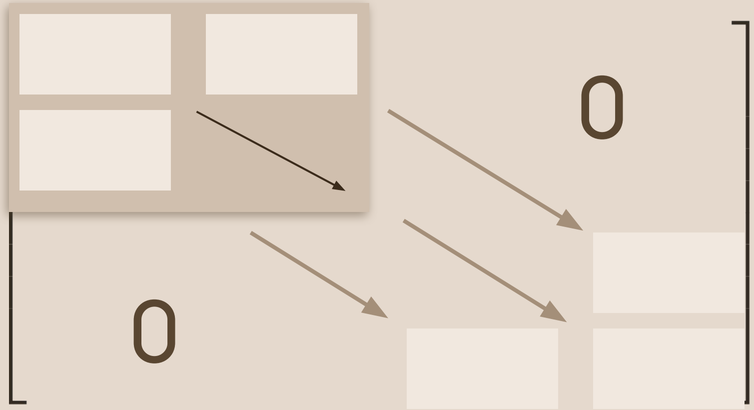
What did we tell quin?

MA(1) | 1-step ahead

$$X_t = Z_t + \theta Z_{t-1}$$

$$Z_t \sim wn(\sigma^2)$$

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2\theta & |h| = 1 \\ 0 & |h| > 1 \end{cases}$$



$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \gamma_X(1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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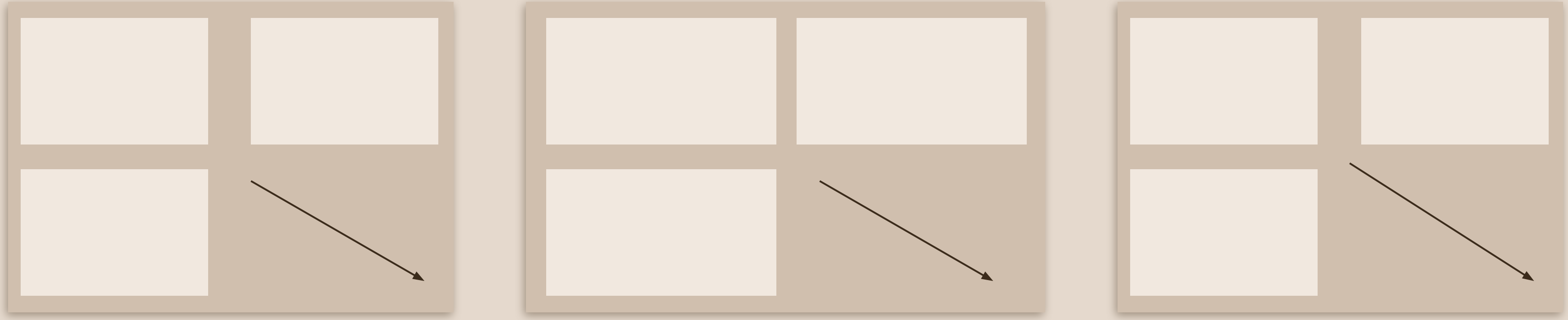

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Solving these equations exactly for $a_1, \ldots, a_N$ is curiously difficult

1) Solve numerically
2) Approximation

$$\begin{bmatrix} 1 & \rho(1) & & 0 \\ \rho(1) & \ddots & \ddots & \\ & \ddots & \ddots & \rho(1) \\ 0 & & \rho(1) & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \rho(1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

MA(1)

1-step ahead | Solve for N=2

$$a_1 + \rho_1 a_2 = \rho_1$$

$$\rho_1 a_{i-1} + a_i + \rho_1 a_{i+1} = 0 \qquad 1 < i < N$$

$$\rho_1 a_{N-1} + a_N = 0$$

MA(1) 1-step ahead | Approximation

Wild Idea: compute best LP of $X_{N=1}$ from infinitely many past values

Our hope: When we truncate the solution to **available** past values, the difference won't be too large.

To justify: Values of X far into the past are uncorrelated with $X_{N=1}$
So for large N, we may consider this

MA(1) | forecasting from the infinite past

Operator $\tilde{P}_N$ has properties analogous to P_N
(Properties 1-4 from last week)

MSE :

$$\mathrm{E}\left[\left(Y - (a_0 + \sum_{i=1}^{\infty} a_i X_{N+1-i})\right)^2\right]$$

① Linearity: If U and V are random variables and α_1, α_2, and β are constants, then

$$\hat{P}_n(\alpha_1 U + \alpha_2 V + \beta) = \alpha_1 \hat{P}_n U + \alpha_2 \hat{P}_n V + \beta$$

This extends recursively to

$$\hat{P}_n\left(\sum_{i=1}^{\infty} \alpha_i U_i + \beta\right) = \sum_{i=1}^{\infty} \alpha_i \hat{P}_n U_i + \beta \,.$$

② The residual from the prediction is uncorrelated with all the predictors, i.e., $E[(Y - \hat{P}_n Y) X_j] = 0$ for all $j = n, n-1, \ldots$

③ If Y is uncorrelated with all the predictors then $\hat{P}_n Y = E[Y]$.

④ If Y is a linear combination of the predictors, i.e.,

$$Y = \sum_{i=1}^{\infty} \alpha_i X_{n+1-i} + \beta \,, \text{ then}$$

$$\hat{P}_n Y = \sum_{i=1}^{\infty} \alpha_i X_{n+1-i} + \beta \,.$$

Next time: invertibility

$$X_{N+1} = Z_{N+1} + \theta Z_N$$

$$\tilde{P}_N X_{N+1} = \tilde{P}_N Z_{N+1} + \tilde{P}_N \theta Z_N$$

$\mathbf{Z_N}$ is uncorrelated with all predictors $\mathbf{X_N}, \ldots, \mathbf{X_1}$ → $\tilde{P}_N Z_{N+1} = 0$

$\mathbf{Z_N}$ is not uncorrelated with $\mathbf{X_N}$ → we can't say $\tilde{P}_N \theta Z_N = 0$

Approach: treat **X** as an ARMA(1,1) process, solve for **Z** in terms of **X**

To do this, we must invert the operator $\boldsymbol{\theta}$**(B)**

Definition: An ARMA(1,1,) process with MA coefficient $\boldsymbol{\theta}$ is <u>invertible</u> if $|\theta| < 1$

We learned something today, in Time Series 🕛

What do we tell quin?