The Terrace | Study Guide ⋆

Trends and Seasonality: Matching

Draw a line from each definition to the corresponding trend estimate.

Moving Average Smoother	$\hat{m}_t = \sum_{j=0}^{t-2} \alpha(1-\alpha)^j x_{t-j} + (1-\alpha)^{t-1} x_1$
Polynomial Regression	$\hat{m}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$
Exponential Smoother	$\hat{m}_t = a_0 + a_1 t + a_2 t^2 + \ldots + a_p t^p$ Where $\{a_0, a_1, \ldots, a_p\}$ minimize: $\sum_{t=1}^{N} \left(x_t - (a_0 + \ldots + a_p t^p)\right)^2$

Trends and Seasonality: Questions

1. What does polynomial regression do?

2. What does MA smoothing do? What's a "low-pass filter?"

3. What does exponential smoothing do?

4. What does harmonic regression do?

5. What is the S1 method? How does it differ from harmonic regression? (Consider how these techniques make different assumptions about the data.)

White Noise Hypothesis Testing

Suppose a sample ACF plot shows $\hat{\rho}_x(h)$ for $h = 0, \dots, 30$.
You wish to test if the times series $\{x_t\}$ is white noise, at the $\alpha = 0.05$ level.

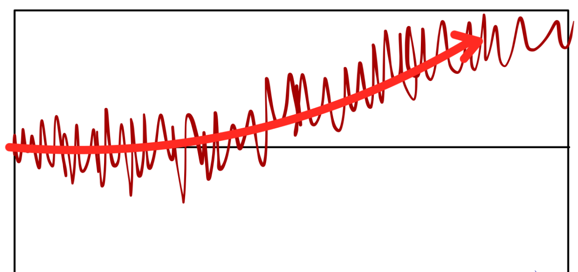
1. Describe the null hypothesis in words. Then write H_0 explicitly, in terms of $\rho(h)$.
2. Describe the alternative hypothesis in words. Then write H_A explicitly, in terms of $\rho(h)$.
3. How many times is $\hat{\rho}_x$ allowed to enter the plot's *rejection region* before we reject H_0?

Stationarity (Theory)

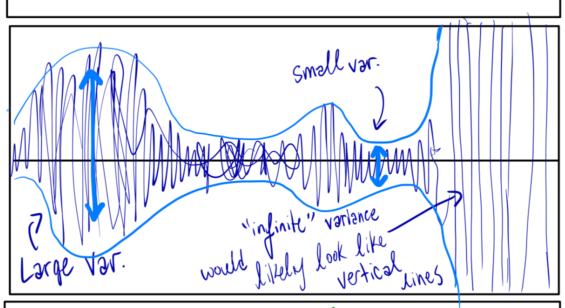
1. Name the three requirements for (weak) stationarity; call them C1,C2, and C3.
2. Sketch examples of the following time series, by hand. I've provided some hints.

   a. A stationary time series
   b. A time series breaking stationarity condition C1
   c. A time series breaking C2
   d. A time series breaking C3

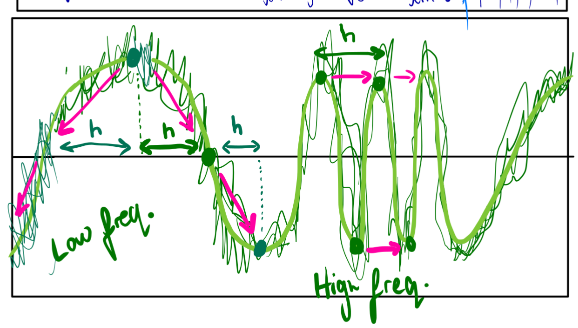
Over time, A changing mean looks like this



Over time, changing variance looks like this



Over time, changing covariance $Cov[X_t, X_{t+h}]$ (fixed h) looks like this



The Fountain | Study Guide $\star$

Linear Filters: Questions

1. What is a causal filter? What is a causal process?
2. What is a linear process?
3. Express an output process X_t in terms of a linear filter $\{a_j\}_{j\in\mathbb{Z}}$ and input process Y_t.
4. Design a causal filter $\{a_j\}_{j\in\mathbb{Z}}$, and express it as a piecewise function of j.

Proposition 2.2.1 (Covered in the "Backshift Operator" lecture)

Let Y_t be a zero-mean, stationary series, with ACVF $\gamma_Y(h)$ and variance σ_Y^2 Let X_t be the output of a linear filter with input series Y_t and coefficients $\{a_j\}_{j\in\mathbb{Z}}$

1. What is $E[X_t]$?
2. Is X_t stationary?
3. What is γ_X, the ACVF of X_t, if it exists?
4. What is σ_X^2, the variance of X?
5. Assume the input Y_t is white noise. Use proposition 2.2.1 to express σ_X^2 in terms of σ_Y^2.

Autocorrelation and Stationarity: calculations

1. One Star on the midterm will ask you to determine whether a series is stationary, and so forth. Review this kind of problem from Rooms 1 and 2, and review the Covariance slides from class. **I've decided to publish select solutions from a previous year's midterm,** so that you have a more concrete example to study from. (These are more difficult than what you will encounter on our midterm, as they came from a test with fewer problems.)
2. Know what the ACF of an MA process looks like on a plot (see workshop 3)
3. Know what the ACF of an AR process looks like on a plot
(return to workshop 3, and put your hand-crafted AR(1) into the `acf` function)
4. Consider what ACFs might look like for other kinds of series you've encountered in this class. Try to sketch them by hand on a plot.

Example Problems (with solutions)

1. Let $\{s_t\}$ be a seasonal component with period 10 and $\{Y_t\}$ be a weakly stationary process with mean 0 and ACVF $\gamma_Y(h)$. Let $X_t = s_t + Y_t$, $t \in \mathbb{Z}$.

(a) Is $\{X_t\}$ weakly stationary? If so, compute its ACVF in terms of $\gamma_Y(\cdot)$. [3]

(b) Let $U_t = X_t - X_{t-1}$. Is $\{U_t\}$ weakly stationary? If so, compute its ACVF in terms of $\gamma_Y(\cdot)$. If not, does $\{U_t\}$ still have a seasonal component? [3]

(a) $\{X_t\}$ is not weakly stationary since $E[X_t] = s_t$, which depends on t.

(b) $\{U_t\}$ is not weakly stationary. We have $E[U_t] = s_t - s_{t-1}$. If this was equal to some constant c for all t, where $c \neq 0$, then we would have $s_t - s_{t-10} = \sum_{i=0}^{9}(s_{t-i} - s_{t-i-1}) = 10c$, which contradicts $s_t - s_{t-10} = 0$, since $\{s_t\}$ has period 10. On the other hand, if $c = 0$ then we would have s_t equal to some constant for all t, which again contradicts that the period of $\{s_t\}$ is 10.

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3. Let $\{Z_t\}$ be a sequence of independent and identically distributed normal random variables with mean 0 and variance 1. Let $Y_t = Z_t + Z_{t-1}$ and $X_t = Y_t^2$. Find the variance of X_t and $\mathrm{Cov}(X_t, X_{t+h})$ for $h \geq 1$. Is the process $\{X_t\}$ weakly stationary? You may use the fact that $E[Z_t^4] = 3$. *Hint:* The covariances are all 0 for $h \geq 2$, but you should explain why. [10]

Solution: We have $X_t = Y_t^2 = Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}$. Since Z_t and Z_{t-1} are independent $E[Z_tZ_{t-1}] = E[Z_t]E[Z_{t-1}] = 0$, and so $E[X_t] = 2$ which does not depend on t. For $h \geq 1$,

$$\mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}, Z_{t+h}^2 + Z_{t+h-1}^2 + 2Z_{t+h}Z_{t+h-1}).$$

For $h \geq 2$ the covariance above is 0 for all t since the indices in $Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}$ are all different than the indices in $Z_{t+h}^2 + Z_{t+h-1}^2 + 2Z_{t+h}Z_{t+h-1}$, and the Z_t are i.i.d. For $h = 1$, we have

$$\mathrm{Cov}(X_t, X_{t+1}) = \mathrm{Cov}(Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}, Z_{t+1}^2 + Z_t^2 + 2Z_{t+1}Z_t).$$

Upon expanding the above covariance into the sum of 9 covariances using the linearity of the covariance operator, we get that the only nonzero covariance is $\mathrm{Cov}(Z_t^2, Z_t^2)$, again since the Z_t are i.i.d. That is, $\mathrm{Cov}(X_t, X_{t+1}) = \mathrm{Cov}(Z_t^2, Z_t^2) = E[Z_t^4] - \sigma^4 = 3 - 1 = 2$. This again does not depend on t. Finally, for $h = 0$,

$$\begin{aligned}
\mathrm{Cov}(X_t, X_t) &= \mathrm{Cov}(Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}, Z_t^2 + Z_{t-1}^2 + 2Z_tZ_{t-1}) \\
&= \mathrm{Cov}(Z_t^2, Z_t^2) + \mathrm{Cov}(Z_{t-1}^2, Z_{t-1}^2) + 4\mathrm{Cov}(Z_tZ_{t-1}, Z_t, Z_{t-1}) \\
&= 2(E[Z_t^4] - 1) + 4E[Z_t^2Z_{t-1}^2] \\
&= 6 + 2 = 8.
\end{aligned}$$

This does not depend on t. We have shown that $\{X_t\}$ is stationary with ACVF

$$\gamma_X(h) = \begin{cases} 8 & \text{if } h = 0 \\ 2 & \text{if } |h| = 1 \\ 0 & \text{if } |h| \geq 2 \end{cases}.$$