Workshop 3: MA and AR Simulations

STAT 464/864 - Fall 2024
Discrete Time Series Analysis
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Setup

As usual, we need to load the `itsmr` package from the textbook. We're also going to load `knitr`, which should be installed by default. (Else, you can install it by running the code `install.packages("knitr")` in the CONSOLE.)

```
library(itsmr)
library(knitr)
set.seed(420)  # we're also setting a *randomness* seed.
```

Data

We'll be making our own data, today, according to the following models:

$$\begin{aligned} \text{MA}(2): \quad X_t &= Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} \\ Z_t &\sim wn\left(\sigma_Z^2\right) \end{aligned}$$

$$\{\theta_1,\, \theta_2,\, \sigma_Z^2\} := \{0.25,\, 0.75,\, 4\}$$

$$\begin{aligned} \text{AR}(2): \quad Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + W_t \\ W_t &\sim wn\left(\sigma_W^2\right) \end{aligned}$$

$$\{\phi_1,\, \phi_2,\, \sigma_W^2\} := \{0.25,\, 0.75,\, 4\}$$

For integer $t \in \{1, \ldots, N = 500\}$

Creating an MA(2) using ITSMR

We want to simulate the series

$$X_t = Z_t + (0.25)Z_{t-1} + (0.75)Z_{t-2}$$

Simulation

The easy way to do this is by using the functions `sim` and `specify`, from ITSMR.

```
# --- Specify Parameters
theta <- c(0.25, 0.75)
s     <- 2
N     <- 500

# --- Simulate!
model <- specify(ma = theta,  # give it your MA parameters
                 sigma2 = 4)  # noise variance

x     <- sim(n = N,      # number of simulations
             a = model)  # specify ARMA model
```

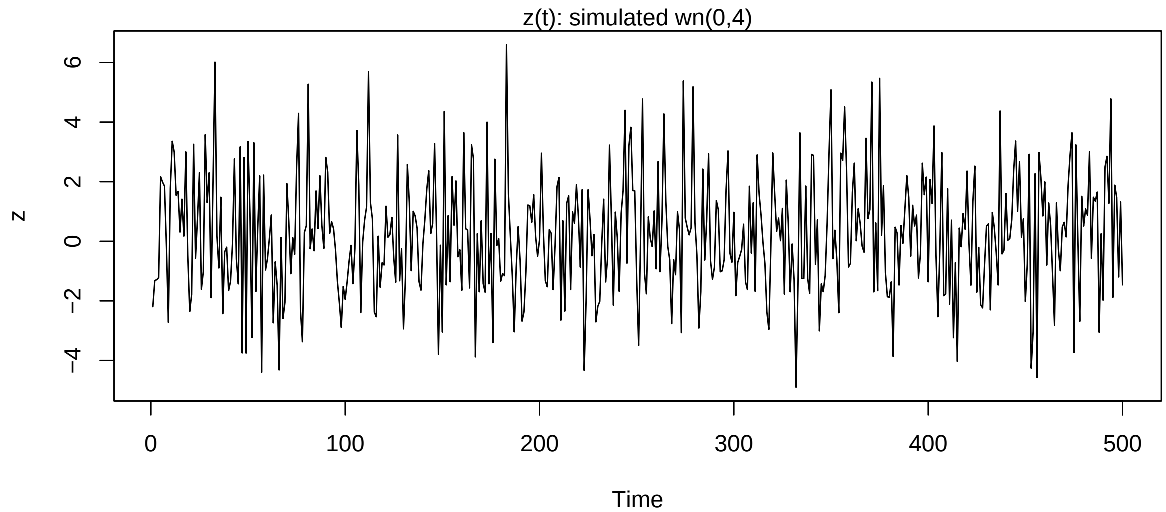
Table 1: This table was created using the `kable` function.

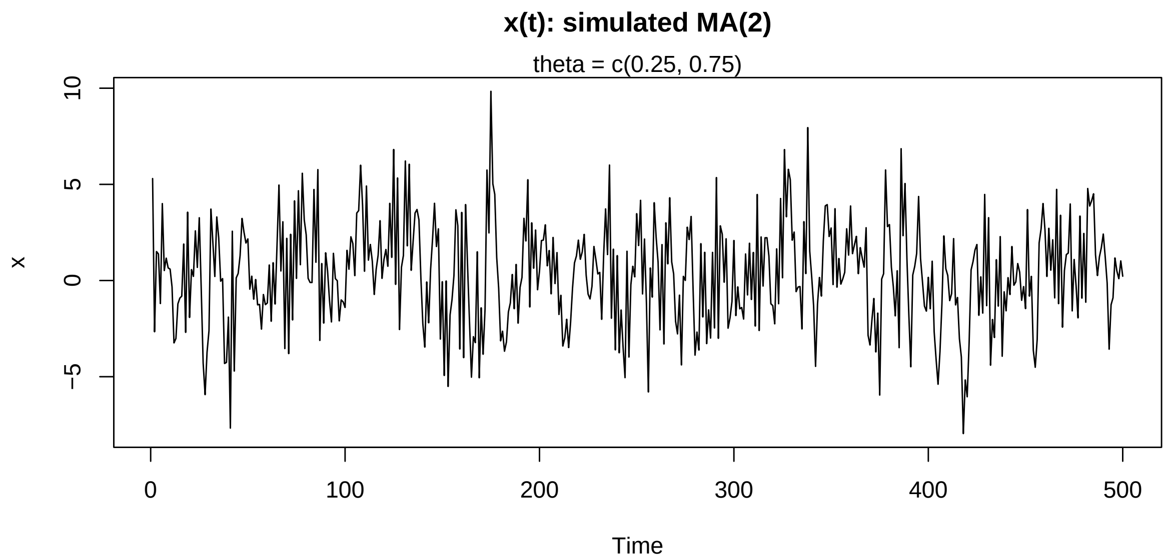
phi	theta	sigma2
0	c(0.25, 0.75)	4

Plotting

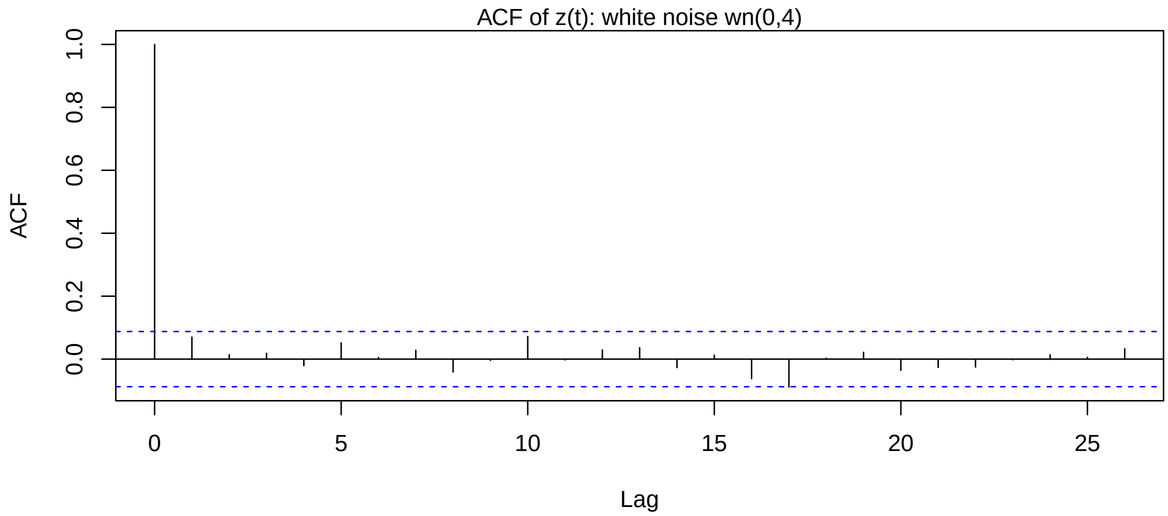
Let's plot x_t and compare it to plain white noise. We'll also plot the corresponding ACFs. Since this is simulated data, there are no units or scientific details to specify.

```
par(mfrow = c(2,1), mar = c(4,4,4,1))
z <- rnorm(N, sd=2)

# --- Time plots
plot.ts(z, main = "")
mtext("z(t): simulated wn(0,4)") # subtitle

plot.ts(x, main = "x(t): simulated MA(2)")
mtext(  paste("theta =", list(theta)))  # puts theta vals in the subtitle
```

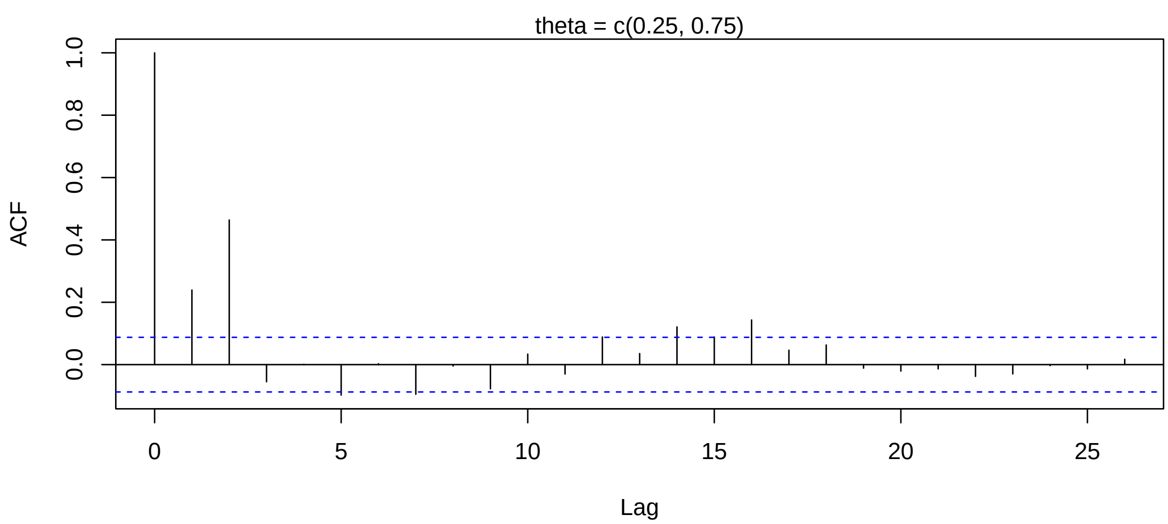





```
par(mfrow = c(2,1), mar = c(4,4,4,1))
z <- rnorm(N, sd=2)
# --- ACF plots
acf(z, main = "")
mtext("ACF of z(t): white noise wn(0,4)")

acf(x, main = "ACF of x(t): simulated MA(2)")
mtext(paste("theta =", list(theta) ))
```



ACF of x(t): simulated MA(2)



Time plots: Notice the MA(2) series looks *smoother* than the white noise series. This makes sense, as each observation is a linear combination of previous values.

ACF plots According to the the white noise hypothesis test, from class, we are allowed to enter the rejection region $\alpha \times \max\{h\} = 0.05 \times 26 = 1.3 \rightarrow 1$ time before we have to reject the null.

So, z_t passes the white noise test, but x_t does not. This makes sense: x_t is highly correlated with the previous two timepoints, and that correlation should be proportional to the MA(2) coefficients.

Creating an AR(2) from scratch

1. Create the white noise series $\{w_t\}_{t=1}^{500}$. This series is obviously *random*, but at each time (t) where we create a new observation $y_t = \text{blablabla } + w_t$, we need to be drawing from the *same* observed series.{w_t}.

   We'll create way more than N observations, because we want to iteratively establish the shape of our model before formally storing the data. Let's choose $M = 2500$.

2. We have to get our AR(2) started, somehow. The first observation y_1 is supposed to be made of past y_t values, but there is no y_0! So let's start with white noise – not our w_t series, but some other white noise v_t with the same variance σ_W^2. We call this process "seeding" the series.

   Again, this vector will be length M, not length N. We'll seed the first N entries of y_t with some noise unrelated to w_t.

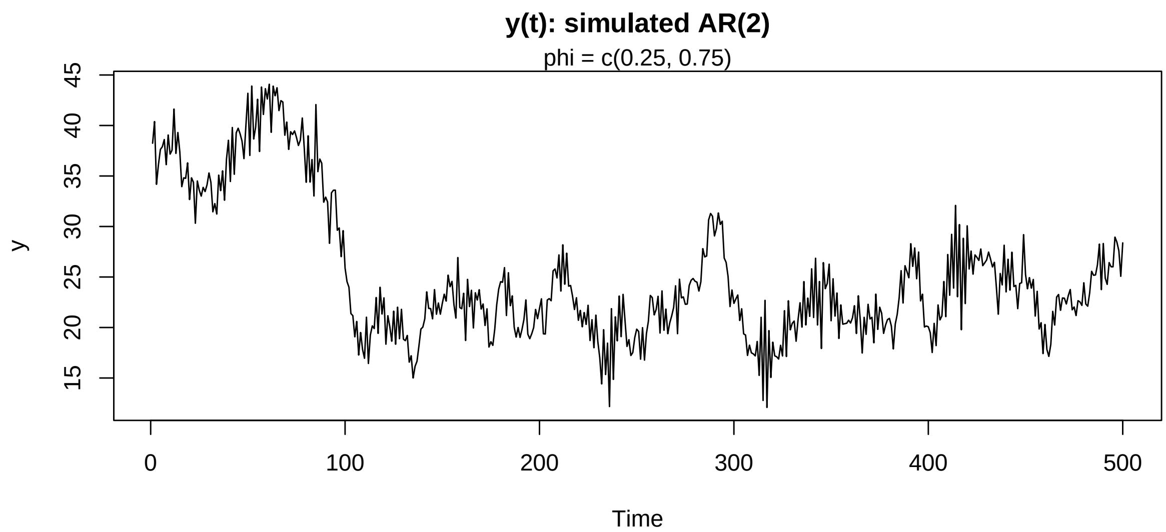
3. Use a `for` loop to fill the remaining "un-seeded" y_t vector according to the desired model.

4. Throw out the first 2000 points as "burn-in" values. The remaining 500 observations form our final simulation y_t.

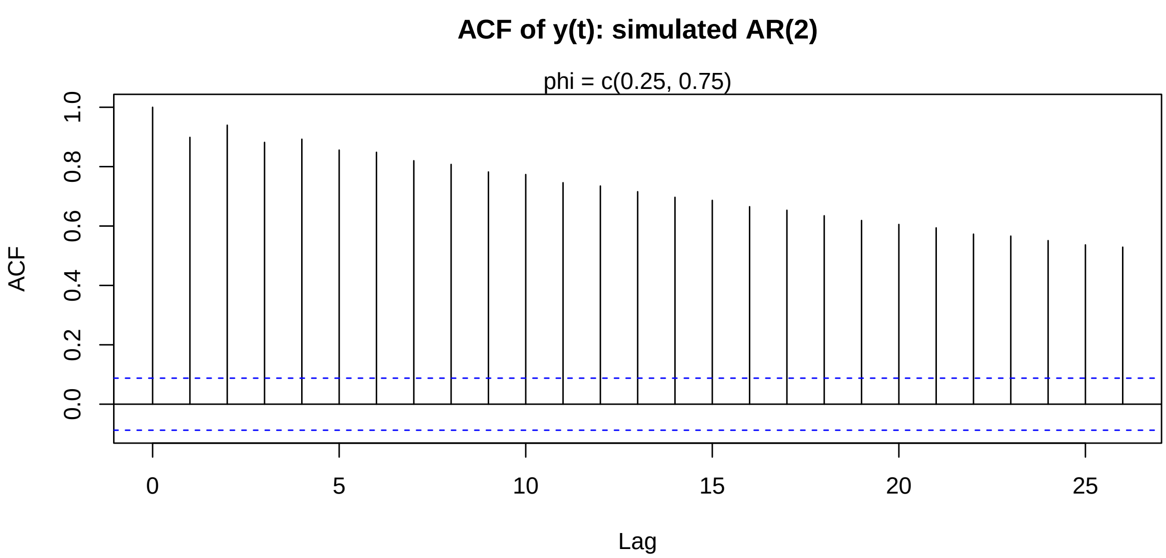
```
# --- Specify Parameters
phi <- c(0.25, 0.75)
s   <- 2
N   <- 500
M   <- 5*N

# --- 1. White noise w_t
w <- rnorm(M, sd = s)

# --- 2. Seed the series
v <- rnorm(N, sd = s)  # some other white noise
y <- c(v, rep(0,M-N))  # seed 1 to N, fill rest with 0's, total = M points

# --- 3. Simulate M-length AR(2)
for(t in (N+1):M) {
  y[t] <- phi[1]*y[t-1] + phi[2]*y[t-2] + w[t]   # our model!
}

# --- 4. Throw away first (M-N) points as 'burn-in'
y.full <- y  # store the full series for later in the workshop :)
w <- w[(M-N+1):M]
y <- y[(M-N+1):M]
```

Plotting

```
par(mar = c(4,4,3.5,1))

plot.ts(y, main = "y(t): simulated AR(2)")
mtext(paste("phi =", list(phi) ))
```




```
acf(y, main = "ACF of y(t): simulated AR(2)")
mtext(paste("phi =", list(phi) ))
```



Comparing the sample ACFs to the theoretical ACF

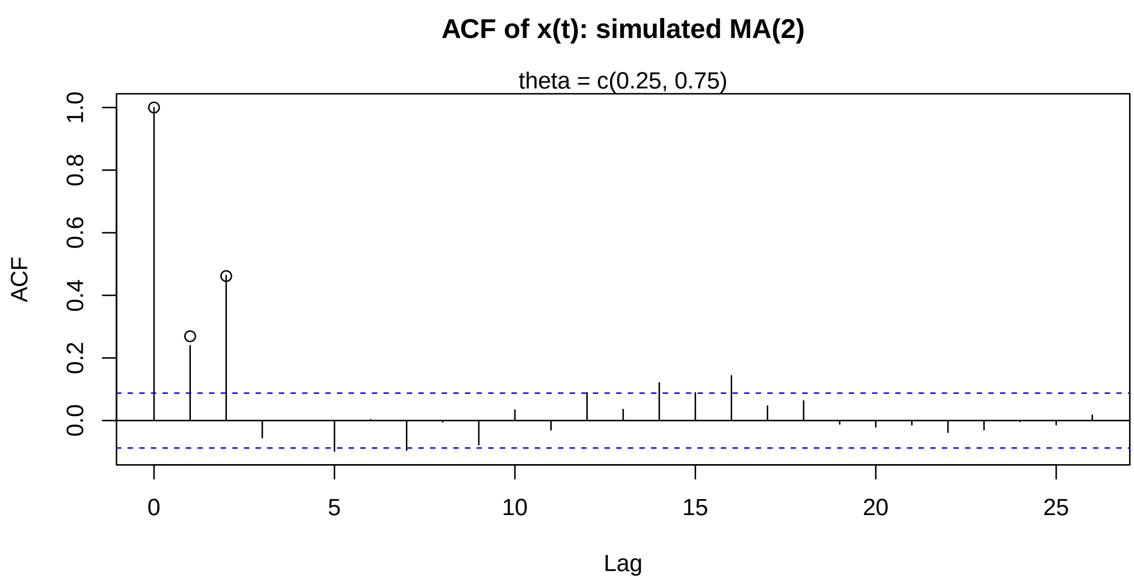
Recall from class, for an MA(2) process X_t,

$$\gamma_X(0) = \sigma_Z^2(1 + \theta_1^2 + \theta_2^2)$$
$$\gamma_X(1) = \sigma_Z^2(\theta_1 + \theta_1\theta_2)$$
$$\gamma_X(2) = \sigma_Z^2\,\theta_2$$

Note γ_X is symmetric, and it's zero for $|h| \geq 3$.
Thus, we can capture the entire behaviour of γ_X (and ρ_X) using only $h = 0, 1, 2$.

To get $\rho_X(h)$, we just standardize by dividing $\gamma_X(h)/\gamma_X(0)$.

```
rho.y <- c(s^2*(1 + theta[1]^2 + theta[2]^2),  # h=0
           s^2*(theta[1] + theta[1]*theta[2]), # h=1
           s^2*(theta[2]))                     # h=2

rho.y <- rho.y / rho.y[1]    # standardize by rho(0)

acf(x, main = "ACF of x(t): simulated MA(2)")
mtext(paste("theta =", list(theta) ))

points(0:2, rho.y)
```



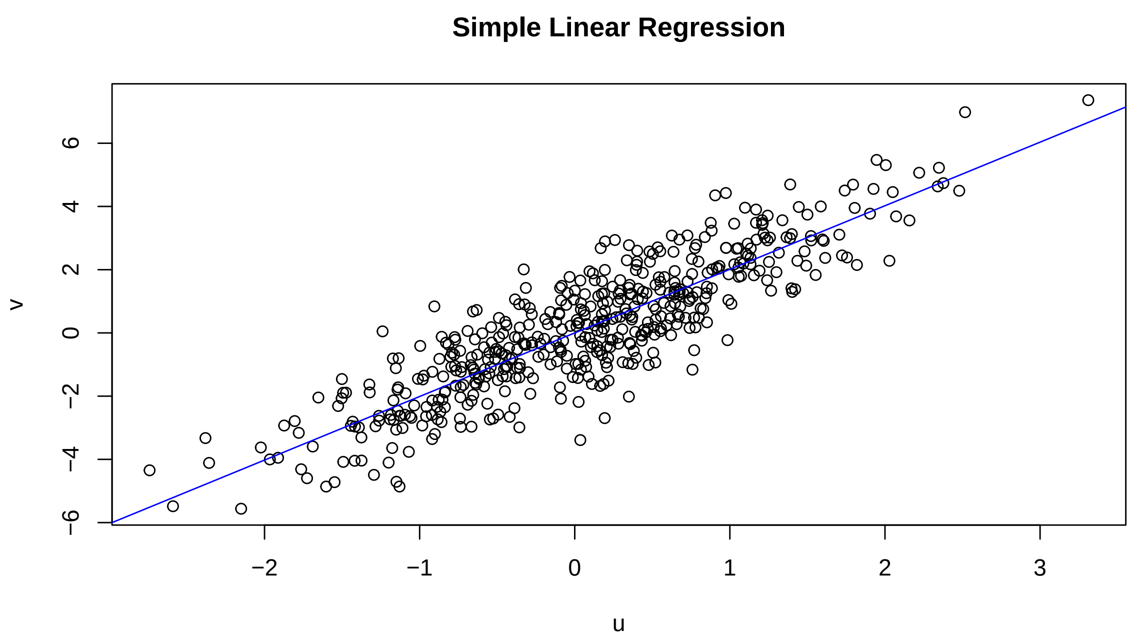
Linear Regression

Let's produce and plot the simple linear regression model:

$$y_i = \beta x_i + \varepsilon_i.$$

To avoid overwriting our time series, we'll call the predictor and response u, v instead of x, y.

```
u <- rnorm(N) # predictor (just using noise, here)
e <- rnorm(N) # noise vector
b <- 2        # true regression coefficient

v <- u*b + e  # response

# --- Create model
model1 <- lm(v ~ u -1)  # -1 removes intercept

# --- Plot line of best fit
plot(u,v, main = "Simple Linear Regression")
abline(model1, col = "blue")
```



Estimating the AR(2) coefficients ϕ_1 and ϕ_2

NOTE: on your assignment, you do NOT need to do this burn-in thing, with the huge `y.full` vector. Just use the indices suggested by the problem statements. I'm being thorough here for demonstration purposes.

```
model2 <-
  lm(y.full[(M-N+1):M] ~ y.full[(M-N):(M-1)] + y.full[(M-N-1):(M-2)] -1)

phi.hat <- summary(model2)$coef

rownames(phi.hat) <- c("phi 1", "phi 2")

kable(phi.hat)
```

	Estimate	Std. Error	t value	Pr(>|t|)
phi 1	0.2618219	0.0304571	8.596419	0
phi 2	0.7342062	0.0304123	24.141728	0

The estimates $\hat{\phi}_1$ and $\hat{\phi}_2$ are quite good. We can see that by comparing the absolute difference $|\hat{\phi} - \phi|$ to the standard error column, above. Or, we can look at the CIs directly:

```
# --- within 1.96 times the standard error?
result <- data.frame(
  estimate = phi.hat[,1],
  confint(model2),
  "within CI?" = ((phi.hat[,1] > confint(model2)[,1])
                &(phi.hat[,1] < confint(model2)[,2]))
)
colnames(result)[2:3] <- colnames(confint(model2))

kable(result)
```

	estimate	2.5 %	97.5 %	within.CI.
phi 1	0.2618219	0.2019817	0.3216621	TRUE
phi 2	0.7342062	0.6744539	0.7939585	TRUE

The estimates fall within the 95% confidence intervals! nice :)