Concatenated Filters

And intro to ARMA(1,1) processes

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

Skyepaphora Griffith, Queen's University

Concatenation of 2 Filters | Idea

Consider two linear filters, $\{a_j\}$ and $\{b_j\}$.

Stationary input process for filter

Output of filter

Input process for filter

output of the filter

Concatenation of 2 Filters
Order doesn't matter

$$\{Y_t\} \longrightarrow \boxed{a_j} \longrightarrow \{X_t\} \longrightarrow \boxed{b_j} \longrightarrow \{W_t\}$$

$$X_t = \sum_{j\in\mathbb{Z}} a_j Y_{t-j}$$

$$W_t = \sum_{k\in\mathbb{Z}} b_k X_{t-k} = \sum_{k\in\mathbb{Z}} b_k \quad = \quad (\bowtie)$$

$$X_t' = \sum_{j\in\mathbb{Z}} b_j Y_{t-j}$$

Takeaway: Same final output regardless of order the filters are applied

$$W_t' = \sum_{k\in\mathbb{Z}} a_k X_{t-k}' = \sum_{k\in\mathbb{Z}} a_k \sum_{j\in\mathbb{Z}} b_j Y_{t-(k+j)} = \sum_{j,k\in\mathbb{Z}} a_k b_j Y_{t-(k+j)} = (\bowtie)$$

Filtered Filter Coefficients

$$(\bowtie) \qquad W_t = \sum_{j,k\in\mathbb{Z}} a_j b_k Y_{t-(k+j)}$$

$$W_t = \sum a_j b_k Y_{t-i} = \sum_{i\in\mathbb{Z}} \overbrace{\qquad\qquad}^{c_i} Y_{t-i} = \sum_{i\in\mathbb{Z}} c_i Y_{t-i}$$

Output $\{W_t\}$ is equivalent to applying filter $\{c_i\}$ to initial input $\{Y_t\}$

Concatenated filters and the Backshift Operator

$$\alpha(B)\beta(B)$$

$$= \qquad = \sum \sum a_j b_k B^i$$

$$= \sum_{i\in\mathbb{Z}} \left(\qquad \right) B^i = \sum_{i\in\mathbb{Z}} \quad B^i$$

$$\alpha(B) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} a_j B^j$$

$$\beta(B) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} b_j B^j$$

Autoregressive Moving Average Process | Order (1,1)

ARMA(1,1)

(🏋🏽)

For some parameters

ARMA(1,1) and the Backshift Operator

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$\boxed{\phi(B)X_t = \theta(B)Z_t}$$

$$X_t =$$

$$= \qquad Z_t$$

$$= (1 + \theta B) \qquad Z_t = \left(\qquad \right) Z_t$$

ARMA(1,1) and the Backshift Operator

$$(🏋️)\quad X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$$

$$X_t = \left(\sum_{j=0}^{\infty} (\phi B)^j + \theta \sum_{j=0}^{\infty} \phi^j B^{j+1} \right) Z_t$$

$$= \left(1 + \sum_{j=1}^{\infty} \qquad + \theta \sum_{j=1}^{\infty} \qquad \right) Z_t$$

$$= \left(1 + \qquad \sum_{j=1}^{\infty} \phi^{j-1} B^j \right) Z_t =$$

$$\phi(B) \stackrel{\text{def}}{=} 1 - \phi B$$

$$\theta(B) \stackrel{\text{def}}{=} 1 + \theta B$$

$$\phi(B) X_t = \theta(B) Z_t$$

$$\sum_{j=1}^{\infty} \phi^{j-1} B^j$$

ARMA(1,1) and the Backshift Operator

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

If $\phi = -\theta$, then $\{X_t\}$ is just $\{Z_t\}$

ACVF

Of an ARMA(1,1) Process

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

By proposition 2.2.1 : since **{Y$_t$}** is a zero-mean **wn(σ^2)** process,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sigma^2 a_j a_{j+h} =$$

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$

ACVF of ARMA(1,1) | h = 0

$$\gamma_X(h) = \sum_{j=0}^{\infty} \sigma^2 a_j a_{j+h}$$

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$