The Backshift Operator

[13_x, 13c - mostly 13c]

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

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Proposition 2.2.1

Proposition 2.2.1 | Statement

$\{\psi_j\}$: Linear filter

$\{Y_t\}$: Input

- 🕑 Stationary
- 🕑 Zero-mean
- 🕑 ACVF: $\gamma_Y(h)$

~ ConVoLuTioN ~ →

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$$

$\{X_t\}$: Output

- 🕑 Stationary 1 2 3
- 🕑 Zero-mean
- 🕑 ACVF $\gamma_X(h)$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h-k+j)$$

If input $\{Y_t\}$ is wn(σ^2) ,

i.e.) $\{X_t\}$ is a linear process

then $\gamma_X(h)$ reduces to

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$$

Proposition 2.2.1

Proving $E[X_t] = 0$

☐ Stationarity 1) time-invariant mean

$$E[X_t] = E\left[\boxed{\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}}\right]$$

$$= \sum_{j=-\infty}^{\infty} \psi_j E[Y_{t-j}]$$

$$= \sum_{j=-\infty}^{\infty} \psi_j \times 0$$

$$= 0$$

We can bring the **E** inside the sum because the sum (highlighted) is finite with probability 1

TLDR: Linearity of **E**

Proposition 2.2.1

Deriving $\gamma_X(h)$

- ☐ **Stationarity 2)** finite variance
- ☐ **Stationarity 3)** time-invariant ACVF

$$\gamma_X(h) = E[\qquad\qquad]$$

$E[X_t] = E[X_{t+h}] = 0$

$$= E\left[\left(\sum \qquad\right)\left(\sum \qquad\right)\right]$$

$$= \sum\sum \qquad\qquad = \sum$$

🕑 Covariance independent of t

🕑 Yt is stationary → $\gamma_Y(h)$ finite for all **h** → Variance $\gamma_X(0)$ is finite

Absolute summability

Proposition 2.2.1 | Linear Processes

If the input $\{Y_t\}$ is a $\text{wn}(\sigma^2)$ process,

$$\gamma_Y(h-k+j) = \begin{cases} \\ \end{cases}$$

For each j:

Therefore:

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty}$$

Why do we care? | Attacking AR(1) with Prop. 2.2.1

{Xt} is a linear filter output

Nay, a linear process!

Is it causal?

Yes.

By Proposition 2.2.1, Xt is:

- 🕑 Causal
- 🕑 Stationary
- 🕑 Linear Process
- 🕑 With known ACVF

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$$

Why do we care? | Getting the ACVF with Prop. 2.2.1

By Proposition 2.2.1, Xt is:

- Causal
- Stationary
- Linear Process
- With known ACVF

$$\boxed{\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2}$$

Why do we care? | Another reason we need |phi| < 1

The ACF is then $\rho_X(h) = \phi^{|h|}$

What about the case when $|\phi| = 1$?

Then $X_t = \phi X_{t-1} + Z_t$ gives

$$Var(X_t) = Var(X_{t-1}) + \sigma^2$$

If $\{X_t\}$ was stationary then $Var(X_t) = Var(X_{t-1})$ we would get $\sigma^2 = 0$. So, other than this trivial case, this is a contradiction to $\sigma^2 > 0$. So $\{X_t\}$ is not stationary.

By Proposition 2.2.1, Xt is:

- Causal
- Stationary
- Linear Process
- With known ACVF

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$$

The Backshift Operator

The Backshift Operator

- Basic operation:
- Powers
  extend/contract timeshift

Why do we care?

- We can **express a linear filter** as a linear combo of powers of B
- Expressions involving B can be **treated as regular expressions** and used strategically - let's revisit our AR(1) example.

Linear Filters in terms of B

A linear filter $\{a_j\}$ can be written as an operator in terms of B:

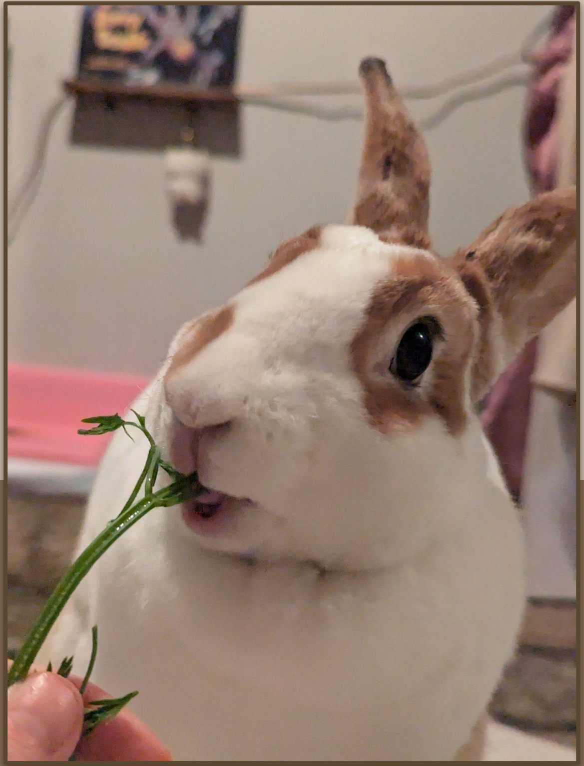
This is exactly the filter's output process, $\{X_t\}$

Example: AR(1)

By Proposition 2.2.1, X_t is:

- 🕑 Causal
- 🕑 Stationary
- 🕑 Linear Process
- 🕑 With known ACVF

We learned something Today, in Time Series 🕛



What do we tell quin??