Linear Filters

Introduction

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

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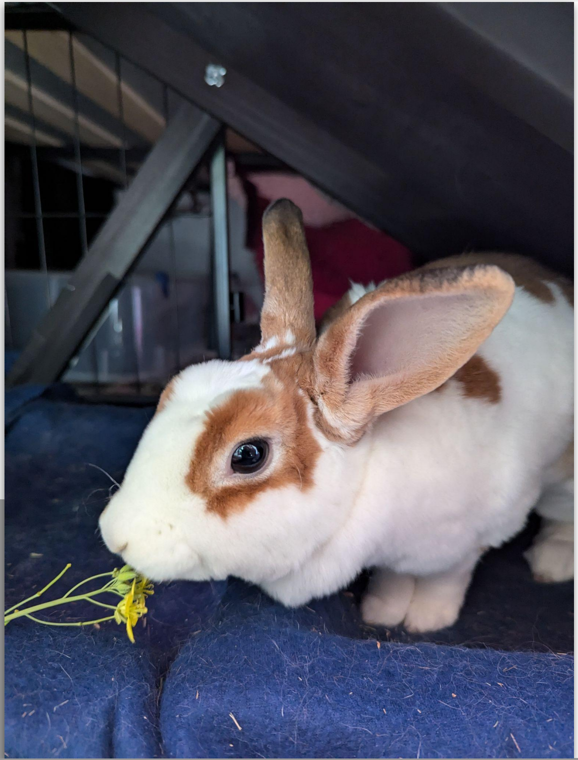
We learned something in Unit 1!



When you have a sequence of data taken over time,
that's called a **Time Series.**

Over time, this kind of data might show a **trend**.
We can estimate that trend by **fitting a polynomial,**
or by **smoothing** out the noise to reveal a clearer shape.

The other thing we're interested in is **repetition.**
We can model seasonal patterns by **fitting harmonic sinusoids,**
or by examining the **average behaviour across cycles.**

What's left over should just be **residual noise.**
We can **test** if the residual data is noisy enough by looking at its **self-similarities**, at different time-scales.

What do we tell quin?

Linear Filters

Definition: A linear filter is a sequence of coefficients $\{a_j\}_{j\in\mathbb{Z}}$

Satisfying $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ (absolute summability)

Finite Filter: Finitely many nonzero filter coefficients (in practice, all filters are finite)

Input process Y_t

Output process

$$X_t = \underbrace{\sum_{j=-\infty}^{\infty} a_j Y_{t-j}}_{\textit{Convolution}}$$

Linear Processes

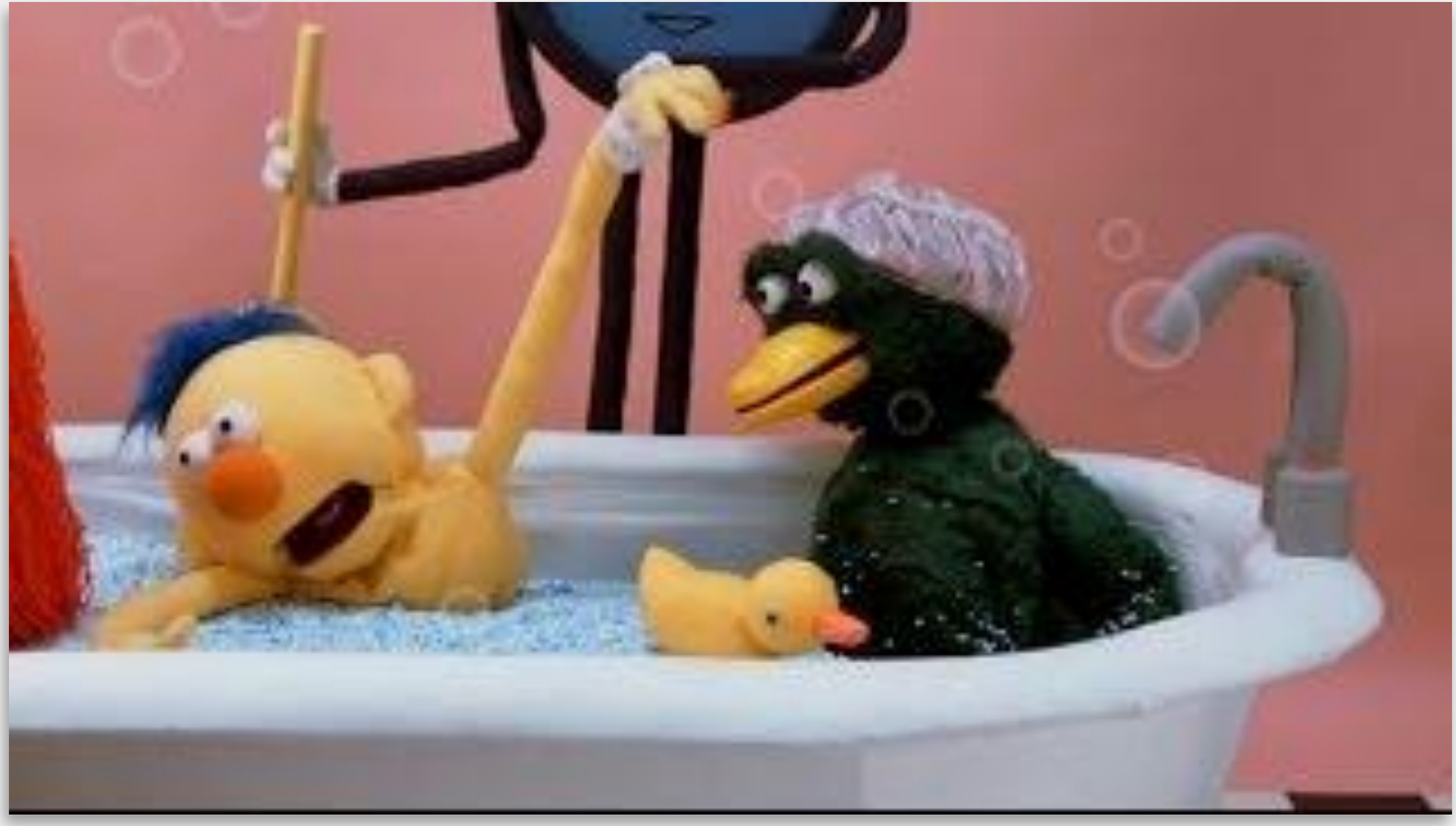
Definition: If input $\{Y_t\}$ is **wn**(σ^2) (Recall μ is zero, WLOG)

the output $\{X_t\}$ is called a **linear process**

Notes: We will be considering **time-invariant** filters

Assume input $\{Y_t\}$ is **stationary** and **zero-mean**

Causality



Causality

The past is far behind us, the future doesn't exist.

Definition: A filter is called **causal** if $a_j = 0$ for $j < 0$

Its output $\{X_t\}$ is called a **causal process**

Depends only on past/present values of input $\{Y_t\}$

Linear Filter $$X_t = \sum_{j=-\infty}^{\infty} a_j Y_{t-j}$$

Proposition 2.2.1

Proposition 2.2.1 | Statement

$\{\psi_j\}$: Linear filter

$\{Y_t\}$: Input

- Stationary
- Zero-mean
- ACVF: $\gamma_Y(h)$

~ ConVoLuTioN ~ $\longrightarrow$

$\{X_t\}$: Output

- Stationary
- Zero-mean
- ACVF $\gamma_X(h)$

1
2
3

If input $\{Y_t\}$ is wn(σ^2) ,

then $\gamma_X(h)$ reduces to

Proposition 2.2.1

✓ Proof $E[X_t] = 0$

✓ Stationarity 1) time-invariant mean

We can bring the **E** inside the sum because the sum (highlighted) is finite with probability 1

Proposition 2.2.1

✔ Deriving $\gamma_X(h)$
✔ Stationarity 2) finite variance
✔ Stationarity 3) time-invariant ACVF

🕑 Covariance independent of t

🕑 Yt is stationary → $\gamma_Y(h)$ finite for all **h** → Variance $\gamma_X(0)$ is finite

Absolute summability ↗

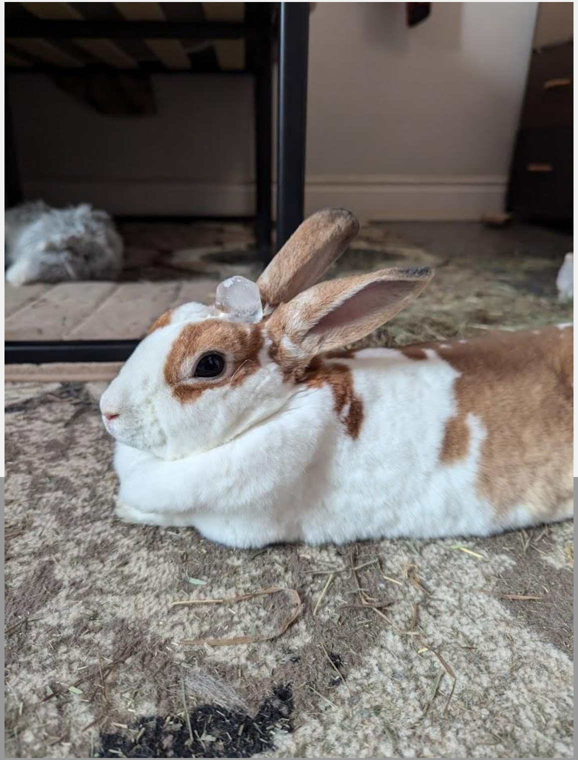
Proposition 2.2.1 | Linear Processes

If the input $\{Y_t\}$ is a $\mathbf{wn}(\sigma^2)$ process,

For each j:

Therefore:

We learned something Today, in Time Series 🕛



What do we tell quin?