Brockwell & Davis | Section 1.3.3 | General approach to time series modelling

Modeling X_t is like replacing ink in a pen. $$X_t = m_t + s_t + Y_t \qquad (\star)$$

Remove the **body** which gives it its **shape** $$X_t - \hat{m}_t = s_t + Y_t$$

Polynomial regression
MA smoothing
Exponential smoothing

Take off the **spring** pushing it **back & forth** $$X_t - \hat{m}_t - \hat{s}_t = Y_t$$

Harmonic regression
Season (S1) method

Look at the **residual** ink.
Get ink tube that **fits our model of pen** $$X_t - \hat{m}_t - \hat{s}_t = \hat{Y}_t$$

Various methods
you'll soon learn ❤

Put the **spring** on the new tube $$X_t - \hat{m}_t = \hat{s}_t + \hat{Y}_t$$

Put the **body** of the pen back on $$X_t = \hat{m}_t + \hat{s}_t + \hat{Y}_t$$

= unknown

= estimated

Stationarity

Time Series in the style of M.C. Escher

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

Skyepaphora Griffith, Queen's University

We learned something just now, in Time Series 🕛

Correlation coefficient:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

Correlation = Covariance,
standardized to scale of [-1, 1]

What do we tell quin?

Covariance Matrix Γ_N

Random Vector: $\vec{X}_N = (X_1, X_2, \ldots, X_N)$ *such as the time series* ***X*t**

Covariance matrix describes **pairwise covariances:** $\Gamma_N[i,j] = \mathrm{Cov}(X_i, X_j)$

$$\begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$$

Correlation Matrix R_N

- 🕑 Also symmetric
- 🕑 All diagonal entries = 1

Same idea as Γ_N, except entries are

$$R_N[i,j] = \rho(X_i, X_j)$$

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Properties of Γ_N | Proof if time permits

1) Symmetric (Recall: covariance is symmetric)

2) Non-negative Definite: $a^T \Gamma_N a \geq 0 \qquad a \in \mathbb{R}^N$

Proof:

$$0 \leq \mathrm{Var}(a^T \vec{X}) = \mathrm{Cov}(a^T \vec{X}, a^T \vec{X})$$

$$= \mathrm{Cov}\left(\sum_{i=1}^{N} a_i \vec{X}_i, \sum_{i=1}^{N} a_i \vec{X}_i \right)$$

$$= \sum_{\substack{i=1 \\ j=1}}^{N} a_i a_j \mathrm{Cov}(X_i, X_j) = a^T \Gamma_N a$$

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Stationarity | Motivation

- Γ_N is a basic quantity we want to estimate from our **{X$_t$}**
- Virtually impossible to do this without further assumptions

Example: how do we estimate $\mathrm{Cov}(X_i, X_j)$?

We only have one observation (x_i, x_j)

from the joint distribution of (X_i, X_j)

Strong Stationarity

Definition: A time series $\{X_t\}$ is **strongly stationary** if

$\forall k \geq 1$ and $h \in \mathbb{Z}$, the joint distribution of $(X_{i_1}, \ldots, X_{i_k})$

is the same as the joint distribution of $(X_{i_{1+h}}, \ldots, X_{i_{k+h}})$

- 🕑 When $k = 1$: strongly stationary = identically distributed
- 🕑 Strong stationarity is rarely verifiable in practice
- 🕑 Not necessary for estimating second order stats (Var, Cov)

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Weak Stationarity

Definition: A time series **X** is **weakly stationary** if

1) $\mathrm{E}[X_t] = \mu \qquad \forall t \in \mathbb{Z}$

2) $\mathrm{Var}(X_i) = \sigma^2 < \infty$

3) $\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_{i+h}, X_{j+h}) \qquad \forall i, j, h \in \mathbb{Z}$

Weak Stationarity | Notes $X_t = \boxed{m_t} + \boxed{s_t} + \boxed{Y_t} \quad (\star)$

Assume all models for $\boxed{Y_t}$ are weakly stationary (unless I say otherwise)

"Stationary" will be shorthand for "weakly stationary"

Strong stationarity $\Longrightarrow$ / $\not\Longleftarrow$ **Weak stationarity**

For **stationary** time series, Γ_N can be expressed as function of 1 variable: **lag**

$$\gamma_X(h) = \mathrm{Cov}(X_t, X_{t+h}) \quad \forall\, t \in T_N$$

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Stationarity | A cute visualization

For your assignment 🕑

you will determine whether a given $\{X_t\}$ is stationary

Use **covariance formulas & properties** from lecture

to derive a closed form expression for **$Cov(X_t, X_{t+h})$**

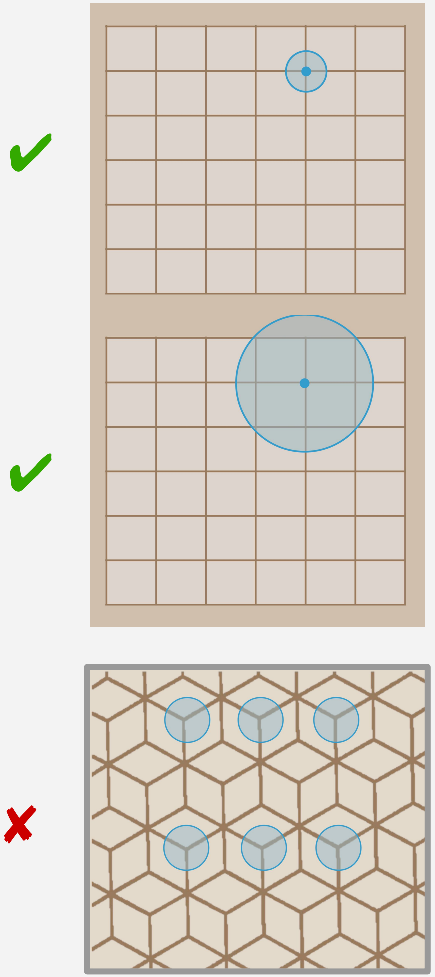
as a **function** of **t** and/or **h** (not X_t)

Can you write this function in terms of **h** only? (Not **t**!)

If so, you've shown $\{X_t\}$ is **stationary.**

If not, explain why, and conclude $\{X_t\}$ is **nonstationary**

~ Then you get a Star ★



We learned something today, in Time Series 🕛

Observations or RVs within the **same** time series can be **correlated with each other.**

We can describe the whole situation using a **matrix**

If the correlations depend on the **distance between** time points, but **not time itself,** then the series is called **stationary.**

Stationarity kinda feels like an M.C. Escher tesselation

What do we tell quin?