Autocovariance Functions (ACVFs)

& Autocorrelation Functions (ACFs)

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

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Covariance Matrices of Stationary Processes

The covariance matrix Γ_N
of a **stationary** time series is **Toeplitz**

Toeplitz: Identical entries along diagonals

There are only **N** distinct values in Γ_N

We can capture all info about Γ_N
in one **N**-length vector (or… function?)

$$\begin{matrix} a & b & c & d \\ b & a & b & c \\ c & b & a & b \\ d & c & b & a \end{matrix}$$

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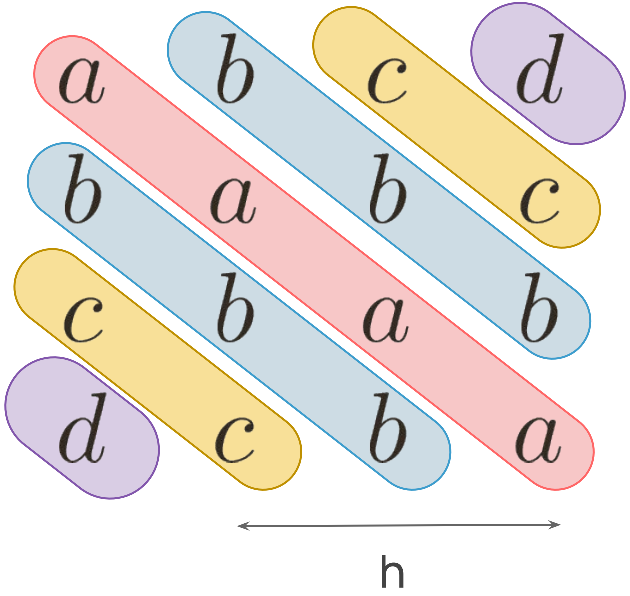
Autocovariance Functions | ACVFs

Covariance Kernel: $\gamma_X(t, t+h) \stackrel{\text{def}}{=} \mathrm{Cov}(X_t, X_{t+h}) \qquad t \in T, h \in \mathbb{Z}$

$$= \mathrm{Cov}(X_s, X_{s+h}) \qquad \forall s \in T$$



The time difference h is called “lag”

Autocovariance Function:

$$\gamma_X(h) \stackrel{\text{def}}{=} \mathrm{Cov}(X_t, X_{t+h}) \qquad t \in T, h \in \mathbb{Z}$$

$$\gamma_X(0, h)$$

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ACVFs and Stationarity

To show X_t is stationary:

1) Show that $\mathrm{E}[X_t]$ is time independent
2) Show that $\mathrm{Cov}(X_t, X_{t+h})$ is time independent

This will show up in your assignments and stuff. A lot.

If I ask you to find the ACVF of X_t, you can assume X_t is stationary

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Autocorrelation Functions (ACFs)

Denoted $\rho_X(h)$, or just $\rho(h)$

ACVFs are easy to standardize!

$$\rho(h) = \rho(X_t, X_{t+h}) = \frac{\mathrm{Cov}(X_t, X_{t+h})}{\sqrt{\mathrm{Var}(X_t)\mathrm{Var}(X_{t+h})}}$$

$$= \frac{\gamma(h)}{\sqrt{\gamma(0)\gamma(0)}} = \frac{\gamma(h)}{\gamma(0)}$$

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Properties of ACVFs and ACFs

1) $\gamma_X(0)$ is the variance $\mathrm{Var}(X_t) \quad \forall t \in T$ (independent of t)

2) γ_X is an even function. So is ρ_X, by extension.

$$\gamma_X(-h) = \mathrm{Cov}(X_t, X_{t-h})$$

$$= \mathrm{Cov}(X_{t-h}, X_t)$$

$$= \mathrm{Cov}(X_t, X_{t+h}) = \gamma_X(h)$$

3) $\rho_X(0) = 1$. Always.

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Example: White Noise

Definition:
A time series **{Xt}** is called **white noise** if

1) It's weakly stationary (assume mean = 0, WLOG)
2) All X_t are pairwise uncorrelated: $\rho_X(h) = 0 \quad \forall h \neq 0$

If the mean is \mu and the variance is \s^2, we write X_t ~ wn(,)

$$\textbf{i.i.d. Time Series} \;\; \underset{\not\Longleftarrow}{\Longrightarrow} \;\; \textbf{White noise}$$

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