Brockwell & Davis | Section 1.3.3 | General approach to time series modelling

Modeling X_t is like replacing ink in a pen.

$$X_t = \boxed{m_t} + \boxed{s_t} + \boxed{Y_t} \qquad (\star)$$

Remove the **body** which gives it its **shape**

$$X_t - \boxed{\hat{m}_t} = \boxed{s_t} + \boxed{Y_t}$$

Polynomial regression
MA smoothing
Exponential smoothing

Take off the **spring** pushing it **back & forth**

$$X_t - \boxed{\hat{m}_t} - \boxed{\hat{s}_t} = \boxed{Y_t}$$

Harmonic regression
Season (S1) method

Look at the **residual** ink.
Get ink tube that **fits our model of pen**

$$X_t - \boxed{\hat{m}_t} - \boxed{\hat{s}_t} = \boxed{\hat{Y}_t}$$

Various methods
you'll soon learn ❤

Put the **spring** on the new tube

$$X_t - \boxed{\hat{m}_t} = \boxed{\hat{s}_t} + \boxed{\hat{Y}_t}$$

Put the **body** of the pen back on

$$X_t = \boxed{\hat{m}_t} + \boxed{\hat{s}_t} + \boxed{\hat{Y}_t}$$

= unknown

= estimated

Stationarity

Time Series in the style of M.C. Escher

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

Skyepaphora Griffith, Queen's University

We learned something Fri/Mon, in Time Series 🕛

Cauchy Schwarz Inequality

$$\mathrm{Cov}(X,Y)^2 \leq \mathrm{Var}(X)\mathrm{Var}(Y)$$

Inner product space?

Functions **E / Cov / Var** of linear combos

Modeling time series is like taking apart a pen

What do we tell quin?

Covariance Matrix Γ_N

Random Vector: $\vec{X}_N = (X_1, X_2, \ldots, X_N)$ *such as the time series* ***X**$_t$*

Covariance matrix describes **pairwise covariances:** $\Gamma_N[i,j] = \mathrm{Cov}(X_i, X_j)$

$$\begin{aligned}\Gamma_N[1,1] &= \mathrm{Cov}(X_1, X_1)\\ \Gamma_N[1,2] &= \mathrm{Cov}(X_1, X_2)\\ \Gamma_N[2,1] &= \mathrm{Cov}(X_2, X_1)\\ \Gamma_N[2,2] &= \mathrm{Cov}(X_2, X_2)\end{aligned}$$

$$\begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$$

Correlation Matrix R_N

- 🕑 Also symmetric
- 🕑 All diagonal entries = 1

Same idea as Γ_N, except entries are

$$R_N[i,j] = \rho(X_i, X_j)$$

Properties of Γ_N | Proof if time permits

1) Symmetric (Recall: covariance is symmetric)

2) Non-negative Definite: $a^T \Gamma_N a \geq 0 \qquad a \in \mathbb{R}^N$

Proof:

$$0 \leq \mathrm{Var}(a^T \vec{X}) = \mathrm{Cov}(a^T \vec{X}, a^T \vec{X})$$

$$= \mathrm{Cov}\left(\sum_{i=1}^{N} a_i \vec{X}_i, \sum_{i=1}^{N} a_i \vec{X}_i\right)$$

$$= \sum_{\substack{i=1 \\ j=1}}^{N} a_i a_j \mathrm{Cov}(X_i, X_j) = a^T \Gamma_N a$$

Stationarity | Motivation

- Γ_N is a basic quantity we want to estimate from our $\{X_t\}$
- Virtually impossible to do this without further assumptions

Example: how do we estimate $\mathrm{Cov}(X_i, X_j)$?

We only have one observation (x_i, x_j)

from the joint distribution of (X_i, X_j)

Strong Stationarity

Definition: A time series $\{X_t\}$ is **strongly stationary** if

$\forall k \geq 1$ and $h \in \mathbb{Z}$, the joint distribution of $(X_{i_1}, \ldots, X_{i_k})$

is the same as the joint distribution of $(X_{i_{1+h}}, \ldots, X_{i_{k+h}})$

- 🕑 When $k = 1$: strongly stationary = identically distributed
- 🕑 Strong stationarity is rarely verifiable in practice
- 🕑 Not necessary for estimating second order stats (Var, Cov)

Weak Stationarity

Definition: A time series **X** is **weakly stationary** if

1) $\mathrm{E}[X_t] = \mu \qquad \forall t \in \mathbb{Z}$

2) $\mathrm{Var}(X_i) = \sigma^2 < \infty$

3) $\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_{i+h}, X_{j+h}) \qquad \forall i, j, h \in \mathbb{Z}$

Weak Stationarity | Notes $X_t = \boxed{m_t} + \boxed{s_t} + \boxed{Y_t}$ $(\star)$

Assume all models for $\boxed{Y_t}$ are weakly stationary (unless I say otherwise)

"Stationary" will be shorthand for "weakly stationary"

$$\textbf{Strong stationarity} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \textbf{Weak stationarity}$$

For **stationary** time series, Γ_N can be expressed as function of 1 variable: **lag**

$$\gamma_X(h) = \text{Cov}(X_t, X_{t+h}) \quad \forall\, t \in T_N$$

Stationarity | A cute visualization

For your assignment 🕑

you will determine whether a given $\{X_t\}$ is stationary

Use **covariance formulas & properties** from lecture

to derive a closed form expression for **$Cov(X_t, X_{t+h})$**

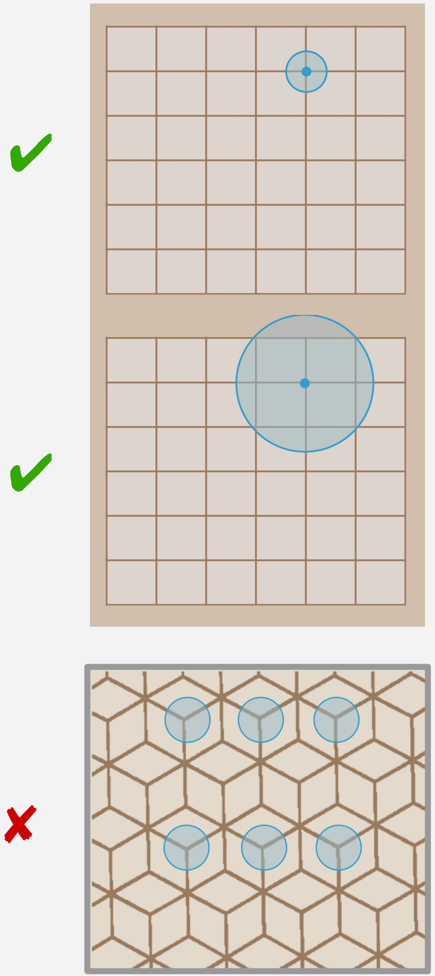
as a **function** of **t** and/or **h** (not X_t)

Can you write this function in terms of **h** only? (Not **t**!)

If so, you've shown $\{X_t\}$ is **stationary.**

If not, explain why, and conclude $\{X_t\}$ is **nonstationary**

~ Then you get a Star ❤



We learned something today, in Time Series 🕛

What do we tell quin?