ARMA(1,1)

Intro to Autoregressive Moving Average Processes

STAT 464 / 864 | Fall 2024

Discrete Time Series Analysis

Skyepaphora Griffith, Queen’s University

Autoregressive Moving Average Process | Order (1,1)

ARMA (1,1)

(🏋🏽)

For some parameters

ARMA(1,1) and the Backshift Operator

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$\boxed{\phi(B)X_t = \theta(B)Z_t}$$

$$\rightarrow X_t =$$

$$=$$

$$= (1 + \theta B) \qquad = \left(\qquad \right) Z_t$$

ARMA(1,1) and the Backshift Operator

$$(🏋) \quad X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$$

$$\phi(B) \stackrel{\text{def}}{=} 1 - \phi B$$

$$\theta(B) \stackrel{\text{def}}{=} 1 + \theta B$$

$$\phi(B) X_t = \theta(B) Z_t$$

$$X_t = \left(\sum_{j=0}^{\infty} (\phi B)^j + \theta \sum_{j=0}^{\infty} \phi^j B^{j+1} \right) Z_t$$

$$= \left(\qquad\qquad\qquad\qquad \right) Z_t$$

$$= \left(\qquad\qquad\qquad \right) Z_t =$$

ARMA(1,1) | Filter

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

$$\{a_j\} = \begin{cases} \\ \end{cases}$$

If then $\{X_t\}$ is just $\{Z_t\}$

ARMA(1,1) | ACVF

(🏋️) $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

Filter

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$

since **{Y_t}** is a zero-mean **wn(σ^2)** process,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sigma^2 a_j a_{j+h} =$$

Deriving the ACVF of an ARMA(1,1) | h = 0

$$\gamma_X(0) = \sum_{j=0}^{\infty}$$

$$=$$

$$=$$

Filter

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$

General ACVF, by Prop. 2.2.1

$$\gamma_X(h) = \sum_{j=0}^{\infty} \sigma^2 a_j a_{j+h}$$

Deriving the ACVF of an ARMA(1,1) | h ≥ 0

$$\frac{\gamma_X(h)}{\sigma^2} = \sum_{j=0}^{\infty} a_j a_{j+h} = \qquad \sum_{j=1}^{\infty} a_j a_{j+h}$$

$$=$$

$$= (\phi+\theta)\phi^{h-1} + (\phi+\theta)^2$$

$$= (\phi+\theta)\phi^{h-1} + \qquad =$$

Filter

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi+\theta)\phi^{j-1} & j \geq 1 \end{cases}$$

Problem Setup

Let **Y** be an RV with finite variance

Consider some sequence of RVs $\mathbf{W} = \{\mathbf{W}_N, \ldots, \mathbf{W}_1\}$

We want **W** to give a good prediction of **Y** (function of **W** that is “close” to **Y**)

We’ll measure that “closeness” using **Mean Squared Error (MSE)**

Linearity of our Prediction

$$MSE \stackrel{\text{def}}{=} \mathrm{E}\Big[\big(Y - \underbrace{g(W_1, \ldots, W_N)}_{\text{prediction}}\big)^2\Big]$$

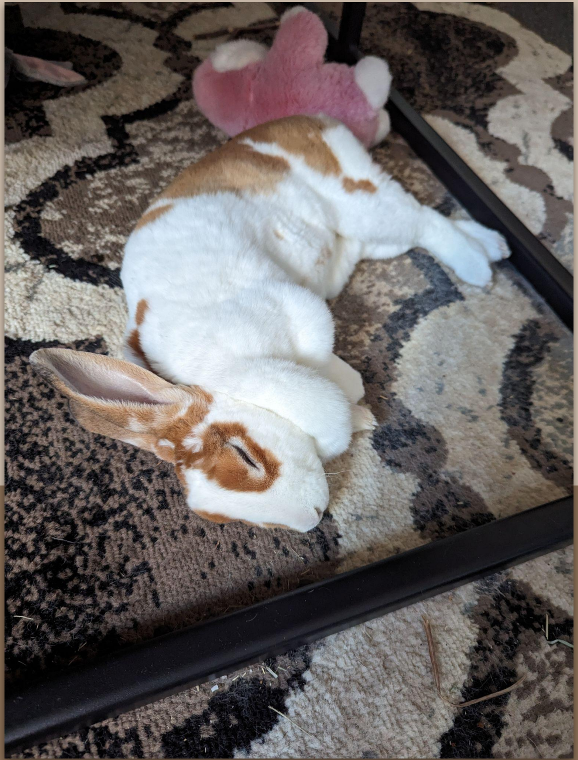
In time series, we typically don't estimate beyond 2nd order properties (Example: variance / covariance)

Optimal Prediction: $\mathbf{E[Y \mid W_1, \ldots, W_N]}$

Only have 2nd order properties?
→ optimal prediction not computable without joint distribution of Y and $\{W_t\}$

If we specify the prediction to be a linear function **g**, of **W**,
We can compute optimal prediction using only 2nd order properties

We learned something Today, in Time Series 🕛



What do we tell quin?