

# Concatenated Filters

And intro to ARMA(1,1) processes

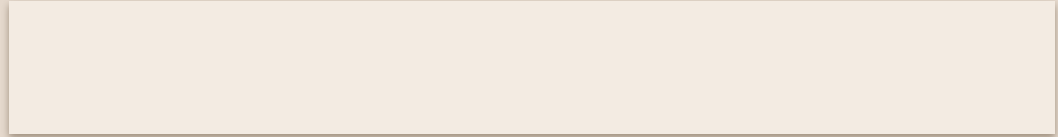
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Discrete Time Series Analysis

Skyepaphora Griffith, Queen's University

# Concatenation of 2 Filters | Idea

Consider two linear filters,  
 $\{a_j\}$  and  $\{b_j\}$ .



Stationary input process for filter

Output of filter

Input process for filter

output of the filter



## Concatenation of 2 Filters

### Order doesn't matter

$$\{Y_t\} \longrightarrow \boxed{a_j} \longrightarrow \{X_t\} \longrightarrow \boxed{b_j} \longrightarrow \{W_t\}$$

$$X_t = \sum_{j \in \mathbb{Z}} a_j Y_{t-j}$$

$$W_t = \sum_{k \in \mathbb{Z}} b_k X_{t-k} = \sum_{k \in \mathbb{Z}} b_k = \boxed{\phantom{\sum_{k \in \mathbb{Z}} b_k}} \quad (\boxtimes)$$

$$X'_t = \sum_{j \in \mathbb{Z}} b_j Y_{t-j} \quad \textbf{Takeaway:} \text{ Same final output regardless of order the filters are applied}$$

$$W'_t = \sum_{k \in \mathbb{Z}} a_k X'_{t-k} = \sum_{k \in \mathbb{Z}} a_k \sum_{j \in \mathbb{Z}} b_j Y_{t-(k+j)} = \sum_{j,k \in \mathbb{Z}} a_k b_j Y_{t-(k+j)} = (\boxtimes)$$

# Filtered Filter Coefficients

$$(\boxtimes) \quad W_t = \sum_{j,k \in \mathbb{Z}} a_j b_k Y_{t-(k+j)}$$

The diagram illustrates the simplification of the double sum equation. A box containing  $c_i$  is connected by a line to the double sum  $\sum_{j,k \in \mathbb{Z}} a_j b_k Y_{t-i}$ . This line then branches into two separate lines, each connecting to one of the terms in the double sum,  $a_j$  and  $b_k$ , effectively showing that the double sum can be reduced to a single sum over  $i$  with coefficients  $c_i$ .

$$W_t = \sum_{j,k \in \mathbb{Z}} a_j b_k Y_{t-i} = \sum_{i \in \mathbb{Z}} Y_{t-i} = \sum_{i \in \mathbb{Z}} c_i Y_{t-i}$$

Output  $\{W_t\}$  is equivalent to applying filter  $\{c_i\}$  to initial input  $\{Y_t\}$

# Concatenated filters and the Backshift Operator

$$\alpha(B)\beta(B)$$

$$= \sum \sum a_j b_k B^i$$

$$= \sum_{i \in \mathbb{Z}} \left( \quad \right) B^i = \sum_{i \in \mathbb{Z}} B^i$$

$$\alpha(B) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} a_j B^j$$

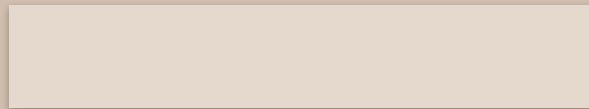
$$\beta(B) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} b_j B^j$$

# Autoregressive Moving Average Process | Order (1,1)

## ARMA(1,1)



For some parameters



# ARMA(1,1) and the Backshift Operator

(🏋️)  $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$\phi(B)X_t = \theta(B)Z_t$$

→  $X_t =$

$=$

$Z_t$

$= (1 + \theta B)$

$$Z_t = \left( \begin{array}{c} \\ \\ \end{array} \right) Z_t$$

# ARMA(1,1) and the Backshift Operator

(🏋️)  $X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$

$$X_t = \left( \sum_{j=0}^{\infty} (\phi B)^j + \theta \sum_{j=0}^{\infty} \phi^j B^{j+1} \right) Z_t$$

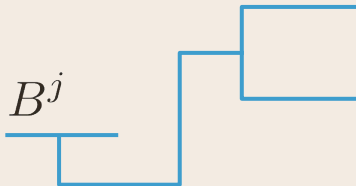
$$= \left( 1 + \sum_{j=1}^{\infty} \phi^j B^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} B^j \right) Z_t$$

$$= \left( 1 + \sum_{j=1}^{\infty} \phi^{j-1} B^j \right) Z_t =$$

$$\phi(B) \stackrel{\text{def}}{=} 1 - \phi B$$

$$\theta(B) \stackrel{\text{def}}{=} 1 + \theta B$$

$$\phi(B)X_t = \theta(B)Z_t$$

$$\sum_{j=1}^{\infty} \phi^{j-1} B^j$$




# ARMA(1,1) and the Backshift Operator

$$(\text{🏋️}) \quad X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

If  $\phi = -\theta$  , then  $\{X_t\}$  is just  $\{Z_t\}$

# ACVF

## Of an ARMA(1,1) Process

$$(\text{🏋️}) \quad X_t - \phi X_{t-1} = +Z_t + \theta Z_{t-1}$$

By proposition 2.2.1 : since  $\{Y_t\}$  is a zero-mean  $\text{wn}(\sigma^2)$  process,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sigma^2 a_j a_{j+h} =$$

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$

## ACVF of ARMA(1,1) | $\mathbf{h} = \mathbf{0}$

$$\gamma_X(h) = \sum_{j=0}^{\infty} \sigma^2 a_j a_{j+h}$$

$$\{a_j\} = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$