The Fourier Transform

Familiar Identities:

$\mathcal{F}\{1\} = \delta(\omega)$

$\mathcal{F}\{e^{j\omega_0 t}\} = \delta(\omega - \omega_0)$

Problems in Lean:

$$\int_{\mathbb{R}} 1 e^{-j\omega t}\, dt \stackrel{?}{=} \delta(\omega)$$

This integral is undefined.

dirac delta not a function

Fourier to Fourier-Stieljes

Measure theory defines "measures" which make "weighted" integrals.
Measures define a "weight"

Ordinary measure:

Let $f: X \to Y$
$\mu: \mathcal{B}(X) \to \mathbb{R}$.

$\mu = 1$ is the "ordinary" measure.

Dirac Measure:

$$\delta_a(x) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise.} \end{cases}$$

$$\int f(t)\, d(\delta_a) = f(a) \; !$$

$$\mathcal{F}\{\mu\} = \int e^{-j\omega t}\, d\mu.$$

Tempered Distributions:

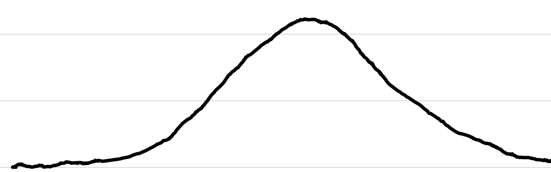
Let $S(\mathbb{R}^n)$ be the space of "Schwarz functions".

relatively "nice"

Bounded Support.

C^∞

Decays to 0 (and all derivatives).



guassians are nice examples.

Tempered Distributions:

Cont. Linea functionals on $S(\mathbb{R}^n)$.

$$T : S(\mathbb{R}^n) \to \mathbb{R}$$

Example: $\int f(t)\, d(\delta_a)$

$$= \langle \delta_a, f(t) \rangle$$

In general it is <u>sufficient</u> for a functional to be bounded by a polynomial.

The General Fourier Transform

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle \quad \forall \varphi \in S(\mathbb{R}^1)$$

Theorem: $\mathcal{F}\delta_0 = 1$.

Proof:

$$\langle \mathcal{F}\delta_0, \varphi \rangle = \langle \delta_0, \underbrace{\mathcal{F}\varphi}_{\hat{\varphi}(\omega)} \rangle$$

$$= \langle \delta, \hat{\varphi}(\omega) \rangle$$

$$= \int \hat{\varphi}(\omega)\, d\delta_0$$

$$= \hat{\varphi}(0)$$

$$= \int_{\mathbb{R}} \varphi(t) e^{-j\omega t}\, dt \Big|_{\omega=0}$$

$$\langle \mathcal{F}\delta_0, \varphi \rangle = \int_{\mathbb{R}} \varphi(t)\, dt$$

$$\int \varphi(t)\, d(\underbrace{\mathcal{F}\delta_0}_{\mu}) = \int \varphi(t)\, dt$$

$$\Rightarrow \mu = \mathcal{F}\delta_0 = 1.$$

Theorem: $\mathcal{F}\{e^{j\omega_0 t}\} = \delta_{\omega_0}$ "$= \delta(\omega-\omega_0)$"

$$\langle \mathcal{F}\{e^{j\omega_0 t}\}, \varphi \rangle = \langle e^{j\omega_0 t}, \mathcal{F}\{\varphi\} \rangle$$

$$= \langle e^{j\omega_0 t}, \hat{\varphi}(t) \rangle$$

$$= \int_{\mathbb{R}} e^{j\omega_0 t} \hat{\varphi}(t)\, dt$$

$$= 2\pi \cdot \frac{1}{2\pi} \int_{\mathbb{R}} e^{j\omega_0 t} \hat{\varphi}(t)\, dt$$

$$\int \varphi(t)\, d(\mathcal{F}\{e^{j\omega_0 t}\}) = 2\pi\, \varphi(\omega_0)$$

$$\int \varphi(t)\, d(\mathcal{F}\{e^{-j\omega_0 t}\}) = \int \varphi(t)\, d(2\pi\delta_{\omega_0})$$

$$\Rightarrow \mathcal{F}\{e^{-j\omega_0 t}\} = 2\pi\,\delta_{\omega_0}$$

Corollary $\mathcal{F}\{1\} = \mathcal{F}\{e^{-j\omega_0 t}\}\big|_{\omega_0 = 0}$

$$= 2\pi\,\delta_0$$