STABILIZATION OF MARKOWITZ PORTFOLIOS

Applying L1 and L2 regularization to the minimum variance and maximum Sharpe ratio portfolio

Abstract

This paper empirically assesses the stabilization of minimum variance and maximum Sharpe ratio Markowitz portfolios caused by regularization, given their widely noted vulnerability to input estimation errors. Using rolling-window backtests of daily industry portfolio returns (2000–2024), we demonstrate that both L1 and L2 regularization improve portfolio weight stability. For minimum variance portfolios, moderate penalization slightly improves risk measures, whereas overly aggressive shrinkage is counterproductive. On the other hand, max Sharpe ratio portfolios are helped significantly by added regularization, in particular L1, and are seen to enhance Sharpe ratios and reduce risk by inducing sparsity and mitigating overfitting intrinsic in unpenalized estimates. L2 effects are analogous but weaker. The results suggest that although robustness is enhanced through regularization, its most suitable application is strategy-dependent with type and strength requiring careful calibration to the portfolio goal.

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18th May 2025

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Abstract

This paper empirically studies the stabilizing effects of L1 and L2 regularization on classical Markowitz portfolio solutions, including the minimum variance and maximum Sharpe ratio portfolios. Although the mean-variance optimization theoretical framework provides an elegant solution to portfolio choice, it is plagued by severe sensitivity to estimation risk in expected returns and covariances that tends to produce extreme and unstable asset allocation. With a sample of daily returns (2000–2024) of ten industry portfolios from the Fama and French database, we assess the effect of different regularization strengths in a rolling-window backtest setting that simulates real investment conditions.

Both L1 and L2 regularization on the minimum variance portfolio decrease the standard deviation of portfolio weights and improve temporal stability. Weak L1 has a modest increase in out-of-sample Sharpe ratios and Value at Risk (VaR), with progressively larger penalties increasingly dampening diversification, resulting in worsening risk-adjusted performance. Likewise, L2 regularization realizes minimum volatility at moderate strengths, but over-shrinkage diminishes return prospects and inflates risk metrics, counteracting the portfolio's defensive nature.

On the optimal Sharpe ratio portfolio, L1 regularization imposes a stronger and more permanent positive impact. Greater sparsity not only optimizes the Sharpe ratio but also reduces volatility and downside risk appreciably without retrenching long-term returns. The max Sharpe ratio strategy is however by definition more susceptible to shifts in expected returns, which manifests as larger turnover even under heavy penalization. L2 regularization creates similar patterns but keeps all the assets, which results in smoother weight behaviours and comparatively less dramatic performance improvement.

In general, the findings show that both regularization methods can mitigate estimation risk and enhance empirical robustness, but the impact is strategy dependent. The minimum variance portfolio is optimally enhanced by mild penalization to suppress weight instability, but the maximum Sharpe ratio portfolio experiences continued performance enhancement under more severe regularization because of an overfitting when the weights are estimated without regularization. These results point to the need for tailoring regularization strength and type to the underlying portfolio goal when using penalized mean-variance optimization in practice.

JEL Classification: G11

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List of Abbreviations

CAGR	Compounded annual growth rate
CVaR	Conditional Value at Risk
Lasso	Least absolute shrinkage and selection operator
MVP	Minimum variance portfolio
MSRP	Maximum Sharpe ratio portfolio
OLS	Ordinary Least Squares
SLSQP	Sequential least squares programming
STD	Standard deviation
VaR	Value at Risk

1 Introduction

Markowitz's mean-variance portfolio theory has been the standard portfolio optimization model for decades. By casting investment decision as a trade-off between portfolio variance and expected return, the theory offers a mathematically tractable and theoretically appealing paradigm for building efficient portfolios. Yet, for all its theoretical elegance, the classical Markowitz model is badly flawed when implemented in practice, especially under estimation uncertainty and high-dimensional settings.

One of the most common criticisms of classical mean-variance optimization is its sensitivity to input parameters (Petukhina et al., 2023). The model demands accurate forecasts of expected returns, variances, and covariances, yet the estimates are notoriously noisy and error-prone in empirical practice. Even small changes to these inputs will cause the resulting portfolio weights to assume drastically different and extreme values, making the portfolios sensitive to the changes in data and impossible to implement. These instabilities are particularly trouble-some in high-dimensional environments, when the number of assets is close to or larger than the number of observations, so that the covariance matrices are ill-conditioned and overfitting occurs.

To overcome these limitations, the application of regularization methods, namely L1 (lasso) and L2 (ridge) penalties, has been shown to be an effective solution. Sparsity is induced by L1 regularization by shrinking some of the portfolio weights to exactly zero, thereby simplifying the allocation and possibly decreasing transaction costs. In contrast, L2 regularization shrinks all portfolio weights equally towards zero but in a smooth manner, enhancing weight stability while maintaining the presence of all the assets. Both approaches offer a means to stabilize the optimization procedure, diminish overfitting, and improve out-of-sample performance in the presence of noisy or finite data.

In this work, we analyse the impact of L1 and L2 regularization on the stability and performance of two popular portfolio strategies of the Markowitz model: the minimum variance portfolio and the maximum Sharpe ratio portfolio. Using a dataset containing daily industry returns in the period from 2000-2024 from the Fama and French database, this study examines the effect of various degrees of regularization strength on portfolio performance metrics including return, volatility, Sharpe ratio, Value at Risk (VaR), and asset weight stability. The study is conducted within the framework of a rolling-window backtesting setup that simulates real portfolio rebalancing and continuous market conditions.

1.1 Research Questions

The current research is directed by a series of primary research questions that try to clarify the empirical implications of regularization in typical Markowitz portfolio models. Firstly, it examines the impact of L1 regularization, which enforces sparsity through imposition of penalties on portfolio absolute weights, on stability and risk-adjusted performance of minimum variance and maximum Sharpe ratio portfolios. The goal is to determine whether the application of a sparsity-inducing penalty leads to more stable portfolios in terms of volatility control, turnover reduction, and Sharpe ratio maximization in the presence of dynamic market conditions.

In parallel, the study explores the performance of L2 regularization, which imposes a penalty on the squared magnitude of the weights and thereby enforces weight shrinkage without eliminating assets entirely. This form of regularization is evaluated with respect to its ability to enhance the robustness of portfolio allocations, measured by the volatility of the portfolio weights, especially in the presence of estimation uncertainty, while preserving a higher degree of diversification than L1-based approaches.

Another objective is to identify under which specific conditions L1 and L2 regularization provide considerable advantages over standard, unregularized mean-variance optimization. More precisely, the current study asks if the benefits of regularization, i.e., reduced sensitivity in estimation, better out-of-sample performance, and better protection against downside risk, are particularly worthwhile during periods of heightened market volatility or structural change.

This comprehensive study aims at improving the understanding of the effective application of L1 and L2 regularization techniques in a bid to improve the applied value and robustness of Markowitz-based portfolio strategies.

1.2 Scope

The research examines empirical comparison of L1 and L2 regularization techniques as a means to enhance the stability and empirical feasibility of Markowitz portfolio optimization. The key objective entails a systematic comparison of traditional mean-variance portfolios and their regularized versions in two common portfolio strategies: minimum variance portfolio and maximum Sharpe ratio portfolio. The test is performed on 10 industry portfolios from the Fama and French database with daily returns over 25 years (2000-2024), which provides a thorough assessment under various market conditions, including several financial crises and decades-long growth periods.

The paper experiments with an optimization backtesting scheme that simulates actual portfolio rebalancing through the optimization process, which is carried out every 126 trading days (approximately six months), according to new return data. Within this scheme, portfolios are built by the basic Markowitz model and then extended to add L1 or L2 regularizing terms to the objective function. The regularization intensity is systematically changed on a predefined grid of penalty parameters to examine the sensitivity of the key performance metrics, i.e., annualized return, volatility, Sharpe ratio, VaR, and asset weight stability, to the extent of regularization.

The study deliberately adopts a fixed-grid approach to regularization strength rather than optimizing hyperparameters. While hyperparameter tuning (e.g., via grid search) could identify the most effective penalty level for a given strategy and market environment, this task introduces considerable complexity and risk of overfitting to historical data. By avoiding this additional layer of model selection, the analysis isolates the structural impact of regularization strength on portfolio behaviour, allowing for a clearer interpretation of general patterns and trade-offs.

Furthermore, the paper focuses exclusively on linear regularization techniques applied within the traditional mean-variance framework. It does not extend to alternative risk measures such as Conditional Value at Risk (CVaR), downside deviation, or drawdown-based metrics. Nor does it incorporate advanced estimation techniques such as shrinkage estimators for the covariance matrix (e.g., Ledoit-Wolf), Bayesian frameworks, or robust optimization methods (see chapter 2.5 Related Work). These approaches are acknowledged as important complementary tools in the literature but fall outside the scope of the present study in order to maintain a clear and focused comparative analysis of L1 and L2 penalization. Additionally, the analysis of portfolios assumes portfolios with full capital allocation and does not examine the effects of leverage constraints as well as transaction costs.

The scope of the paper is intentionally narrow and focuses on the impact of L1 and L2 regularization on the stability, performance and composition of Markowitz portfolios under controlled assumptions. This targeted focus enables the identification of empirically robust patterns and offers practical insights.

2 Background

2.1 Markowitz Portfolio Theory

Harry Markowitz published his seminal work “Portfolio Selection” in the Journal of Finance in the year 1952 and laid the theoretical foundation for asset allocation in the modern era by quantifying the portfolio building process by implementing an optimal trade-off between expected return and risk. The theory's most significant contribution was its recognition that portfolio risk depends on the correlations between asset returns as well as the weighted average of the risks associated with individual assets. Markowitz essentially revolutionized investment management by demonstrating that diversity can reduce portfolio risk without necessarily sacrificing expected return. In recognition of his pioneering contributions to financial economics, Markowitz was awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1990 (The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1990 - NobelPrize.org, n.d.).

Markowitz Portfolio Theory is essentially a normative theory that prescribes the way in which rational risk-averse investors ought to allocate capital to a particular portfolio of risky assets in order to make the best trade-off between risk and expected return. The model assumes that investors base decisions on portfolios solely on the expected value and variance (or standard deviation) of returns, thereby inducing so-called mean-variance optimization problem. The single most significant concept of the theory is that, due to imperfect correlations among asset returns, it is possible for an investor to construct portfolios whose overall risk is lower than the weighted average risk of individual assets - a phenomenon known as the diversification effect.

The efficient frontier is the result of mean-variance optimization and comprises a convex set of portfolios that provide the best expected return for any particular level of risk. Portfolios below the efficient frontier are not optimal because they provide lower returns for the same amount of risk. Portfolios on the efficient frontier are the most desirable available investment opportunities within the model's constraints, for a given level of risk or return. A person's ultimate selection of portfolio will depend on the specific risk preference he or she has, which is conventionally stated by appeal to indifference curves or a utility function.

Optimization, in the original Markowitz model, is accomplished by portfolio variance minimization subject to a constraint on expected return (or, equivalently, expected return maximization subject to a risk constraint). The solution requires estimation of expected returns, variances, and covariances of all candidate assets. The reliance on estimation poses practical challenges, particularly in settings that exhibit high dimensionality, where estimation errors can significantly compromise the stability of the resulting portfolio weights (Petukhina et al., 2023). These limitations have motivated further research aimed at improving both the robustness and empirical performance of mean-variance portfolios, including the use of regularization techniques and other risk estimation measures (Markowitz, 1952).

2.1.1 Assumptions

Markowitz Portfolio Theory relies on several fundamental assumptions that form the analytical foundation for its framework of portfolio optimization. Central to the theory is the assumption that investors are rational and risk-averse, meaning they seek to maximize expected returns for a given level of risk or, equivalently, minimize risk for a given expected return. Risk, as is standard in classical financial theory, is quantified by the standard deviation (volatility) of portfolio returns. The theory further assumes that financial markets are frictionless and informationally efficient in the sense of the e

Efficient market hypothesis, which posits that all available and relevant information is instantaneously incorporated into asset prices. As a result, there are no arbitrage opportunities systematically, and expected returns depend only on systematic risk exposure (Markowitz, 1952).

Furthermore, the Markowitz model presumes assets' returns are normally distributed, a distributional assumption that makes mean-variance analysis possible as it implies the first two statistical moments (mean and variance) are all that are needed to characterize the return distribution. While this is an analytically convenient assumption, empirically it is a problem since financial return series are known to be fat-tailed, have excess kurtosis, and be skewed. The normality assumption, though, makes the optimization problem tractable and ensures that the efficient frontier can be obtained analytically (Karandikar & Sinha, 2012).

Another fundamental assumption is that investors make portfolio decisions exclusively on the basis of the relationship between expected returns and the volatility of the returns and disregard higher-order preferences concerning skewness (asymmetry) and kurtosis (tail risk). Another assumption is that the theory assumes that assets are infinitely divisible, meaning that investors can hold any proportion of a security, which permits continuous and exact portfolio rebalancing. In addition, it is presumed that there are no transaction costs, taxes, or any other market frictions, meaning trading occurs costless and instantaneously. It is also presumed that investors share homogeneous expectations of asset returns, volatilities, and correlations, thereby all investors will arrive at the same efficient frontier (Markowitz, 1952).

While these assumptions abstract significantly from the complexities of real-world financial markets, they provide a coherent and internally consistent foundation for the development of normative portfolio selection models. Therefore, extensions and modifications of the original Markowitz model have sought to relax some of these assumptions to enhance its empirical relevance and practical applicability, particularly in the context of portfolio stability and robustness under estimation uncertainty (see chapter 2.5 Related work).

2.1.2 Portfolio Selection Process

Firstly, an investor defines an investment universe containing all assets which could potentially be part of the final portfolio he would like to construct. A mixture between different asset classes such as bonds, stocks and commodities is possible as long as the returns of the different assets as well as the covariance between them can be calculated, which requires the historical time series of price movements or returns for each asset. Once the investment universe is defined, the investor formulates the portfolio optimization problem within the mean-variance framework. Let N denote the number of assets within the investors universe and let the vector of portfolio weights be represented by $w = (w_1, w_2, \dots, w_N)^T$, where each weight w_i corresponds to the proportion of the capital allocated to the asset i. The expected return of the portfolio is computed as the weighted sum of the individual asset returns, incorporating both capital gains and income components such as dividends or interest payments. Formally, the expected portfolio return, denoted as $E[r_p]$, is given by:

$$E[r_p] = w^T \mu$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)^T$ represents the vector of expected asset returns. The risk of the portfolio is quantified by the variance of returns, computed as

$$\sigma_p^2 = w^T \Sigma w$$

where Σ is the $N \, x \, N$ covariance matrix calculated from the historic asset returns. The investor's objective is to determine the optimal weights which either minimize the portfolio variance or maximize the portfolio's Sharpe ratio a measure of risk-adjusted return that evaluates the excess return of a portfolio per unit of risk (Fernando, 2024) and can be calculated as follows:

$$Sharpe\ Ratio = \frac{r_p - r_f}{\sigma_p}$$

where r_p is the portfolio return, r_f is the risk-free rate and σ_p is the volatility of the portfolio.

One approach to portfolio selection is constructing the minimum variance portfolio, which seeks to minimize total risk without directly considering the expected returns of the portfolio. The portfolio weights ŵ are determined by solving an optimization problem with minimizes the portfolio variance:

$$\hat{w} = \arg min_w \ w^T \Sigma w$$

If no short positions in the portfolio are allowed the optimization problem is extended by a constraint that all individuals weights w_i must be larger than 0. Additionally, another constraint is added, if no leverage is allowed, so that all weights w_i must sum up to 1.

The minimum variance portfolio achieves the lowest possible volatility given the investor's investment universe and lies on the leftmost boundary of the efficient frontier (see chapter 2.1.3 Efficient Frontier). While it is particularly suitable for highly risk-averse investors, it may lead to concentrated allocations in low-volatility assets, limiting diversification.

Alternatively, the investor may construct the optimal portfolio, also referred to as the tangency portfolio, which maximizes the Sharpe ratio and provides the highest risk-adjusted return. The weights of this portfolio are obtained by solving the following optimization problem:

$$\hat{w} = \arg max_w \ \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}}$$

The optimization problem may be extended by the same constraints as for the minimum variance portfolio, depending on if short selling and leveraging are allowed or not.

The constructed portfolio lies at the tangency point between the efficient frontier and the capital market line, making it an ideal choice for investors seeking an optimal balance between return and risk. By leveraging or deleveraging this portfolio, investors can adjust their exposure to match their individual risk tolerance.

2.1.3 Efficient Frontier

The efficient frontier represents the set of optimal portfolios that offer the highest expected return for a given level of risk or, equivalently, the lowest risk for given level of expected return. It is derived from the mean-variance optimization framework and is constructed by solving the constrained optimization problem of minimizing portfolio variance for different levels of expected return (first constraint). Formally, the efficient frontier consists of portfolios that satisfy the following optimization problem.

$$arg\ min_w\ w^T \Sigma w$$

$$subject\ to$$

$$w^T \mu = E[r_p]$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0\ \forall i$$

The second constraint ensures full capital allocation, while the third one (non-negativity) imposes a long-only portfolio restriction, though extensions allow for short selling.

The efficient frontier takes the form of an upward-sloping concave curve in risk-return space, with the x-axis representing portfolio volatility (standard deviation) and the y-axis representing expected return. Portfolios lying below the efficient frontier are suboptimal, as they provide lower expected returns given the same level of risk. Conversely, portfolios above the frontier are theoretically unattainable given the asset universe. The minimum variance portfolio marks the leftmost point on the efficient frontier, representing the portfolio with the lowest possible risk. As one moves along the frontier, portfolios exhibit increasing expected returns but also greater levels of risk. A key implication of the efficient frontier is that any rational investor should hold a portfolio located on this frontier, as any deviation below it would imply an inefficient allocation of capital. When a risk-free asset is introduced, the capital market line emerges, representing the combination of the risk-free asset and the tangency portfolio, which maximizes the Sharpe ratio. This leads to the conclusion that in equilibrium, all investors should hold some combination of the risk-free asset and the tangency portfolio, reinforcing the foundational principle of modern portfolio theory that diversification improves risk-adjusted returns (Gajendrakar, 2022).

2.1.4 Limitations

Although Markowitz Portfolio Theory gives the theoretical underpinning for the construction of optimal portfolios through mean-variance optimization, there are practical limitations to the theory. One problem in Markowitz Portfolio Theory is the generation of wildly oscillating and extreme weight portfolios, particularly in the presence of inaccurate or noisy return and covariance estimates. The instability that is observed arises from the reliance on past data for estimation of the covariance matrix and expected returns, which can be imprecise or unstable over time. Thus, small changes in the input parameters, such as expected returns or covariances, can lead to very large changes in the optimal portfolio weights. The problem is especially acute in portfolios that consist of a large number of assets. Where the covariance matrix is poorly estimated, particularly in situations of many assets compared to observations, Markowitz Portfolio Theory will give rise to portfolios with extremely concentrated positions in a small number of assets or excessively large or small positions for some assets. In some cases, the optimization algorithm can attribute weights that are near zero or very large to some assets, resulting in portfolios that are practically difficult to realize in real-world applications. For example, in a portfolio consisting of stocks, bonds, and commodities, the Markowitz Portfolio Theory can attribute an unrealistically high weight to a single asset or asset class, indicating an unbalanced and highly risk-prone portfolio (Petukhina et al., 2023).

Besides, it also takes asset covariances and returns to be constant through time in the sense that, once estimated, the model presumes their constancy going forward, a situation hardly found in reality (Gajendrakar, 2025). Financial markets are affected by phenomena such as volatility clustering, regime change, and other time-varying dynamics, thereby making the utilization of past information for forming future investment choices challenging. Volatility in the results rises in the event of severe disruptions to markets or times underpinned by increased uncertainty, for at such times quantified parameters are vulnerable to becoming outdated or skewed instantly and thus enhance the likelihood of over-allocation.

These limitations point to a fundamental flaw in Markowitz Portfolio Theory: the lack of a procedure for balancing stability and robustness in designing the optimal portfolios. The result is that the optimal portfolios will often exhibit significant sensitivity to input data, resulting in portfolios that are either infeasible or unsuitable for actual investment applications. To handle these problems, both researchers and practitioners have endeavoured to employ techniques that serve to regularize the optimization procedure, thereby lessening the sensitivity of the model to minimal variations in input data and alleviating the problem of disproportionate weight assignments.

In the following section, we will explore the concept of regularization, a technique that introduces penalties to the portfolio optimization problem in order to enhance stability, reduce overfitting, and provide more practical portfolio allocations. Regularization techniques such as L1 and L2 penalties have been used in portfolio optimization to address the instability and extreme weight allocations inherent in Markowitz Portfolio Theory, offering a more robust approach to constructing optimal portfolios.

2.2 Optimization Problem Solvers

In portfolio optimization, numerical methods are commonly employed to solve optimization problems such as minimizing portfolio variance or maximizing the Sharpe ratio, especially when closed-form solutions do not exist or are impractical. One of the most common used optimization algorithms for such tasks is Sequential Least Squares Programming (SLSQP), a constrained optimization method that iteratively adjusts portfolio weights to find a minimum or maximum of a given objective function. The SLSQP algorithm is particularly suitable for portfolio optimization due to its ability to handle both equality and inequality constraints, such as the requirement for fully invested portfolios ($\sum_{i=1}^{N} w_i = 1$) and no short selling constraints ($w_i \geq 0$) (Walavalkar, 2024).

The general process in the solution of the portfolio optimization problem for SLSQP is to define an objective function, i.e., portfolio variance for the minimum variance portfolio or negative Sharpe ratio for the optimal portfolio, then utilize the solver to minimize the function. The algorithm initializes with a starting estimate for the portfolio weights and then updates the portfolio weights iteratively based on the estimate of the Jacobian matrix and the solution of a least squares problem at each iteration. The process is repeated until convergence is reached, usually when the variations in the objective function or in the portfolio weights fall below a certain threshold.

But numerical optimization techniques such as SLSQP suffer from several limitations. One major problem is that these kinds of algorithms get stuck at local minima or maxima, especially with complicated, nonlinear objective functions with numerous local optima. This issue is happening due to the reason that the solver is taking the initial point of departure (portfolio weights) to initiate its search process; hence, if this initial point is in the vicinity of a local extremum, then the algorithm might end up discovering a suboptimal solution that doesn't correspond to the global optimum (Joshy & Hwang, 2024).To mitigate this issue, one can attempt to change the initial portfolio weights by selecting multiple random starting points and running the optimization from each of them. By comparing the results, the investor can identify whether the algorithm has found a global or local optimum. In some cases, running the optimization multiple times with different starting points can provide a better overall solution.

Besides, some of the optimization algorithms, including differential evolution, may be utilized to avoid SLSQP limitations. Differential evolution is a population-based optimization method that is less prone to initial conditions and local minima. The algorithm works by keeping a population of candidate solutions and recursively refining them in a way that mimics the natural selection laws. The algorithm is both exploiter and explorer since it applies difference vectors to adjust individuals within the population and hence is ideally suited for tackling complex multimodal optimization problems. In contrast to gradient-based solvers such as SLSQP, differential evolution is more stable with regards to local minima and can offer better global solutions, especially where there are many assets and sophisticated non-linearities (Shaji, 2022).

By utilizing such algorithms, portfolio optimization can be performed more effectively, ensuring that the resulting solution is not only feasible but also optimally aligned with the investor's objectives. While SLSQP remains a powerful tool for solving constrained portfolio optimization problems, exploring alternative algorithms like differential evolution can enhance the robustness and reliability of the optimization process, reducing the likelihood of suboptimal solutions.

2.3 Regularization Methods

2.3.1 L1 Regularization

L1 regularization, also known as lasso (least absolute shrinkage and selection operator), is a technique widely used in optimization and machine learning to improve model performance by preventing overfitting and ensuring sparsity in the solution. The core idea behind it is to add a penalty term to the objective function that constrains the magnitude of the model parameters, leading to solutions where many of the parameters are exactly zero. This characteristic makes L1 regularization particularly useful when seeking simpler, more interpretable models with fewer non-zero coefficients. The L1 penalty term is defined as the sum of the absolute values of the model parameters:

$$\lambda \sum_{i=1}^{N} |w_i|$$

where w_i represents the weight or parameter associated with asset i, and λ is a hyperparameter that controls the strength of the regularization. Larger values of λ enforce greater sparsity by driving more weights to zero, effectively eliminating certain assets from the portfolio, while smaller values allow for more flexibility in the weight allocation (Trotta, 2025).

In portfolio optimization, L1 regularization is applied to mitigate the problem of extreme weight allocations and instability that arises from traditional mean-variance optimization, such as the Markowitz framework. The addition of an L1 penalty term to the portfolio optimization problem helps to ensure that the resulting portfolio is both more stable and more diversified by forcing some of the portfolio weights to be exactly zero. This results in a sparser portfolio, where only the most important assets are retained, and less relevant or highly volatile assets are effectively excluded from the final allocation.

The objective function for the minimum variance portfolio is extended by the L1 penalty term can be written as:

$$arg\ min_w\, w^T \Sigma w + \frac{1}{N} \lambda \sum_{i=1}^{N} |w_i|$$

$$subject\ to$$

$$\sum_{i=1}^{N} w_i = 1$$

The added term penalizes large or non-zero weights, encouraging the solver to select fewer assets. To increase the comparability of the regularization strength when the number of assets which are present within the investment universe varies, the $\frac{1}{N}$ term where N denominates the number of assets, is added. To ensure that the solution contains not only weights which are close or equal to zero, the constraint is added.

The introduction of L1 regularization results in a portfolio that not only minimizes risk but also avoids over-concentration in any single asset, leading to a more diversified and robust allocation.

Similarly, L1 regularization can be applied to the optimization problem for determining the maximum Sharpe ratio portfolio, which seeks to maximize the risk-adjusted return. The objective function for this problem, with L1 regularization, is given by:

$$\arg max_w \frac{w^T\mu - r_f}{\sqrt{w^T\Sigma w}} - \frac{1}{N}\lambda\sum_{i=1}^{N}|w_i|$$

$$subject\ to$$

$$\sum_{i=1}^{N} w_i = 1$$

Note that in the case of the maximum Sharpe ratio portfolio the term is subtracted and not added as done for the minimum variance portfolio since this is a maximization problem meanwhile the optimization problem for the minimum variance portfolio is a minimization problem.

By incorporating the L1 regularization, the optimization process not only seeks to maximize the Sharpe ratio but also promotes portfolio sparsity, ensuring that the final allocation consists of a limited number of assets that contribute most to the risk-adjusted return.

2.3.2 L2 Regularization

L2 regularization, also known as ridge regression in the context of statistical modelling, is a widely used technique in optimization and machine learning that aims to improve model stability and prevent overfitting. Unlike L1 regularization, which encourages sparsity by setting some coefficients exactly to zero, L2 regularization distributes shrinkage across all parameters, reducing their magnitude while maintaining nonzero weights. This property makes L2 regularization particularly useful when the goal is to mitigate sensitivity to small changes in input data while preserving all variables in the model. The L2 penalty term is defined as the sum of the squared values of the model parameters:

$$\lambda\sum_{i=1}^{N} {w_i}^2$$

By introducing this penalty term, L2 regularization discourages extreme weight values, effectively stabilizing the optimization process and reducing the likelihood of over-reliance on a small number of assets (Trotta, 2025).

In the context of portfolio optimization, L2 regularization is introduced to address the instability and extreme weight allocations commonly observed in Markowitz optimization. By discouraging extreme portfolio weights and promoting well-distributed allocations, L2 regularization can improve robustness and ensures that the optimized portfolio is more resistant to estimation errors in expected returns and the covariance matrix.

Similar to the extension the objective functions using the L1 regularization term, the objective functions for the minimum variance and maximum Sharpe ratio portfolio can be extended by the L2 regularization term resulting in

$$arg\ min_w\ w^T \Sigma w + \frac{1}{N} \lambda \sum_{i=1}^{N} {w_i}^2$$

$$subject\ to$$

$$\sum_{i=1}^{N} w_i = 1$$

for the minimum variance portfolio and

$$\arg max_w\ \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}} - \frac{1}{N} \lambda \sum_{i=1}^{N} {w_i}^2$$

$$subject\ to$$

$$\sum_{i=1}^{N} w_i = 1$$

for the maximum Sharpe ratio portfolio. As done for L1 regularization, the $\frac{1}{N}$ term is added to increase the comparability of the regularization strength.

L2 regularization addresses several critical issues associated with traditional Markowitz optimization, particularly the problem of instability in portfolio weights. By penalizing extreme allocations, L2 regularization ensures that small estimation errors in asset returns and covariances do not lead to drastic changes in portfolio composition. This improves the robustness of the optimized portfolio, making it more stable over time and reducing turnover costs associated with frequent portfolio rebalancing.

Furthermore, compared to L1 regularization, which enforces sparsity by setting some weights to zero, L2 regularization retains all assets in the portfolio but shrinks their weights, leading to a smoother and more continuous allocation structure. This property is particularly advantageous when an investor seeks diversification while avoiding concentration risk in a small number of assets. Additionally, the well-conditioned nature of L2-regularized solutions helps mitigate numerical instability that may arise in cases where the covariance matrix is near-singular or poorly estimated.

2.3.3 Other Approaches

While L1 and L2 regularization represent powerful techniques to address instability, overfitting, and extreme weight allocations in classical mean-variance optimization, several alternative approaches have been proposed in the literature to further enhance portfolio stability and performance in the presence of estimation error and market uncertainty.

One such method is robust optimization, which explicitly accounts for uncertainty in the input parameters - namely expected returns and the covariance matrix - by incorporating worst-case scenarios within a predefined uncertainty set. Robust optimization techniques aim to construct portfolios that perform well not only under the estimated parameters but also under adverse deviations, thereby reducing sensitivity to estimation risk and improving out-of-sample performance (Palomar, 2025 p. 380-390).

Another promising class of techniques involves the use of shrinkage estimators, particularly for the covariance matrix. Shrinkage methods, such as the Ledoit-Wolf estimator, combine the sample covariance matrix with a structured target matrix (e.g., the identity matrix or a constant correlation matrix) to obtain more stable and well-conditioned estimates. These improved covariance estimates are less sensitive to noise in historical data, which in turn leads to more stable and reliable portfolio weights (Ledoit & Wolf, 2004).

Bayesian approaches offer another alternative by treating the input parameters of the optimization problem as random variables with prior distributions, rather than fixed estimates. This framework allows for the incorporation of investor beliefs and model uncertainty into the optimization process, producing posterior distributions for the optimal weights that naturally balance return, risk, and estimation uncertainty. Bayesian portfolio optimization methods often yield smoother and more robust allocations compared to traditional point-estimate-based approaches (Avramov & Zhou, 2010).

Resampling techniques, such as the resampled efficient frontier (REF), address the sensitivity of optimal portfolio weights to input estimates by generating multiple simulated versions of expected returns and covariances through Monte Carlo methods. The portfolio weights are then averaged across all simulations, resulting in an allocation that is less affected by estimation noise and more stable over time (Chen, 2013).

Lastly, constraints-based methods can be applied to directly control undesirable features in portfolio solutions. For example, imposing upper and lower bounds on individual asset weights, sectoral exposure limits, or turnover constraints can effectively curb concentration risk and reduce rebalancing costs. While these methods do not rely on penalization terms in the objective function, they offer intuitive and transparent mechanisms to enforce practical considerations and improve robustness (Jin et al., 2021).

Collectively, these alternative approaches can be used either independently or in combination with L1 and L2 regularization to construct portfolios that are not only theoretically optimal but also empirically stable, interpretable, and aligned with investor preferences and real-world constraints.

2.4 Related Work

Jagannathan and Ma (2003) demonstrate that imposing simple constraints, such as limiting short sales or bounding portfolio weights, can substantially reduce estimation error and lead to improved out-of-sample performance. Their work highlights the unintended regularizing effect of such constraints, which effectively shrink the solution space and mitigate the instability of the classic Markowitz framework. This observation is closely related to the present study, where explicit regularization terms are introduced to stabilize portfolio weights in the presence of noisy input parameters.

DeMiguel, Garlappi, and Uppal (2009) examine the performance of optimized portfolios relative to a naive allocation strategy. They find that portfolios derived from mean-variance optimization are often outperformed by simple equally weighted portfolios, primarily due to estimation errors in mean and covariances. Their findings underscore the sensitivity of the Markowitz solution to input uncertainty and motivate the development of regularized formulations that can produce more robust and reliable allocations, as proposed in this work.

Di Lorenzo et al. (2011) also address the L0-regularized portfolio selection problem but take a different route: instead of relying on convex relaxations, they propose a smooth concave reformulation of the L0 objective. They use a hybrid optimization strategy that combines Monotonic Basin Hopping with a reduced-dimension Frank–Wolfe method. Their results on large benchmark datasets (e.g., FTSE 100, S&P 500, NASDAQ) show that this direct approach to L0 regularization yields sparser portfolios with better out-of-sample Sharpe ratios than those based on L0-penalties or classical Markowitz weights.

A widely adopted alternative for sparse portfolios is L1 regularization, which encourages sparse solutions by shrinking small weights toward zero. In portfolio optimization, this not only reduces the number of active assets but also improves stability by dampening the influence of estimation error. Since the introduction of Lasso into the portfolio context by Brodie et al. (2009), L1 penalties have become a standard tool for regularization in large-scale allocation problems. Brodie et al. (2009) reformulate the Markowitz problem as a constrained least squares optimization with an L1 penalty on the weights. This modification promotes portfolio sparsity by shrinking small weights to zero, resulting in simpler and more interpretable asset allocations and offer a more robust alternative to the classical Markowitz solution, particularly when asset returns are noisy or highly correlated. Their work marks a turning point by linking the statistical literature on lasso to financial portfolio selection.

Building on this foundation, Fan et al. (2008) provide a theoretical foundation for applying an L1 norm constraint as a gross-exposure control in large portfolios. They show that this L1 constraint limits the accumulation of estimation errors in the covariance matrix and expected returns, which otherwise scales unfavourably with the number of assets. Using the LARS-Lasso algorithm, they compute the solution path efficiently and demonstrate that moderate exposure limits yield portfolios with more stable risk characteristics and improved out-of-sample performance compared to unconstrained Markowitz allocations. Their findings not only support the use of sparsity-inducing penalties in large-scale asset allocation but also clarify why constrained portfolios outperform unconstrained ones.

Corsaro and De Simone (2019) take a dynamic approach by proposing an adaptive Bregman iteration method for tuning the L1 penalty based on predefined financial targets such as maximum allowed short positions. Their approach is extended to a multi-period Markowitz framework by Corsaro et al. (2020), who combine Lasso regularization with a time-consistent risk measure to address long-term asset allocation problem with rebalancing over time. By applying an L1 penalty across multiple time periods they address the additional challenge of time-consistency in multi-period settings, which they solve using a separable dynamic risk measure. Their modified Bregman iteration method adaptively adjusts the L1 penalty to meet explicit financial targets, such as limiting short positions or reducing transaction costs. Compared to elastic net-based approaches like Ho et al. (2015) or Yen and Yen (2010), which balance L1 and L2 penalties, Corsaro et al. (2020) remain focused solely on L1, arguing that sparsity and interpretability are especially critical in dynamic portfolio settings. Their empirical results show that the method achieves substantial risk reductions and cost savings relative to naïve strategies, reinforcing the practical value of L1 regularization even in complex, long-horizon investment scenarios. In both cases, the adaptively tuned penalty allows practitioners to control sparsity and transaction costs more precisely than with fixed regularization schemes.

Other extensions focus on structured sparsity. Zhang et al. (2022) introduce a composite penalty combining L1 and L2,1 norm. The L2,1 penalty, also known as group lasso, leverages group structures among assets, enforcing sparsity across groups and diversification within selected groups. This group lasso enables group selection among clustered assets, such as sectors or regions, while maintaining diversification within selected groups. This allows the model to encode economic structure directly into the regularization framework and improve interpretability and performance, especially when group relationships among assets are economically meaningful. Similarly, Li (2015) reformulates the problem as a regression task and incorporates an L1 constraint directly on the portfolio weights, aligning the objective with shrinkage-based learning techniques while enabling direct control over sparsity levels. This approach enables more interpretable and structured portfolios, particularly in settings where assets naturally cluster (e.g., by sector or region).

While L1 regularization targets sparsity by shrinking some weights exactly to zero, L2 regularization penalizes the squared magnitudes of all portfolio weights. Rather than eliminating positions, it encourages smooth shrinkage across all assets, reducing the magnitude of extreme weights. Yen and Yen (2010) propose a direct application of L2 regularization to stabilize portfolio weights in a mean-variance framework. Their approach explicitly introduces additional L2 constraints to mitigate sensitivity to covariance estimation, particularly in markets characterized by volatility clustering or structural breaks. They demonstrate that adding an L2 penalty term smooths the weight profile and significantly reduces turnover and rebalancing costs, while maintaining comparable risk-adjusted performance. Their findings reinforce the practical value of L2 regularization not just as a theoretical fix but also as a means of improving portfolio implementability.

Carrasco and Noumon (2010) advance this perspective by framing the portfolio allocation problem as an ill-posed inverse problem. They evaluate four regularization techniques: Ridge, spectral cut-off, Landweber-Fridman, and lasso; and use generalized cross-validation to tune their parameters. Simulation and empirical analysis on Fama-French industry portfolios show that L2-based methods, particularly Ridge and Landweber-Fridman, significantly improve both in-sample and out-of-sample Sharpe ratios compared to the unregularized Markowitz and the 1/N benchmark. They conclude that ridge regularization offers more robust and reliable out-of-sample performance, as it better controls estimation-driven instability.

Yen and Yen (2014) investigate elastic net regularization for minimum variance portfolio optimization. By combining L1 and L2 penalties in a fixed-weight scheme, they stabilize and promote sparsity large portfolios. Their coordinate-wise descent algorithm efficiently handles high dimensional inputs and achieves improvements in turnover, short exposure, and risk-adjusted performance relative to unconstrained benchmarks. Their results underscore the benefits of jointly promoting sparsity and weight stability.

Shrinkage estimators aim to improve the quality of the input parameters, i.e. the expected return vector and the covariance matrix, before optimization. These techniques belong to the category of input regularization and are particularly effective when the sample size is small relative to the number of assets, which leads to noisy and unstable estimates. A key contribution in this field is Ledoit and Wolf (2004), who propose a shrinkage estimator for the covariance matrix. They blend the sample covariance matrix with a structured target, such as the identity matrix or a single-factor model. This convex combination stabilizes the eigen structure of the covariance matrix, and results in a better-conditioned input for portfolio optimization. Their estimator is computationally simple, asymptotically consistent, and reduces estimation risk.

Shrinkage estimators and weight regularization can be combined in a dual approach as they are complementary tools. While methods like Lasso and Ridge modify the portfolio weights to achieve sparsity or stability, shrinkage estimators enhance the input accuracy, reducing the likelihood that the optimizer will overreact to noise. This dual approach has become increasingly popular in portfolio optimization. For instance, Ho et al. (2015) combine shrinkage with elastic net penalties. While L1 regularization promotes sparsity and L2 regularization enhances stability, combining the two can often yield better practical performance. This is true particularly in high-dimensional settings where both estimation error and overfitting are concerns. The elastic net penalty, which adds both L1 and squared L2 terms to the objective function, seeks to strike this balance by producing portfolios that are sparse but not overly brittle to small changes in the data.

Wu et al. (2024) propose four strategies that impose L1 and L2 norm constraints on the weights to promote sparsity and stability. Their most general approach, combining both norms (Rule IV), is equivalent to elastic net regularization and aims to balance the benefits of Lasso (sparse portfolios) and Ridge (stable weights). Using penalized least squares and cross-validation, their empirical results show that the elastic net approach consistently outperforms pure L1 (Rule I) and pure L2 approaches across multiple datasets. The elastic net variant consistently achieves the best trade-off between return and risk. For instance, in one dataset with 100 stocks, Rule IV achieved 30–40% lower variance and half the turnover of the classical Markowitz portfolio, while maintaining a similar or higher Sharpe ratio. Rule I (L1 only) led to the sparsest portfolios but suffered from higher volatility in some cases. The paper demonstrates a practical advantage of combining L1 and L2 constraints, especially when transaction costs and portfolio interpretability are important.

Ho et al. (2015) introduce a weighted elastic net penalty, allowing each asset to receive a custom regularization strength in both the L1 and L2 components. Crucially, they calibrate these weights using a bootstrap-based estimate of parameter uncertainty (e.g. individual asset-level risk), effectively tailoring the penalty to the statistical reliability of each asset's input data. Compared to unweighted elastic net or weighted Lasso (as in Fastrich et al., 2015), their approach improves both Sharpe ratio and computational efficiency. The authors also contribute a fast adaptive-support split-Bregman algorithm, which exploits the sparsity induced by the L1 term for rapid convergence in large-scale problems.

Yen and Yen (2014) also focus on elastic net regularization, but in the context of minimum variance portfolio optimization. They apply a weighted combination of L1 and squared L2 penalties and develop a highly efficient coordinate-wise descent algorithm to solve the problem. Unlike Ho et al. (2015), who calibrate weights via bootstrap-based estimation risk, Yen & Yen (2014) focus purely on variance minimization and use fixed-weight regularization to stabilize and sparsify large portfolios. Their empirical tests on Fama–French industry and style portfolios shows that elastic net penalization reliably reduces variance, turnover, and short exposure, while maintaining or improving Sharpe ratios relative to the unconstrained minimum variance portfolio and $\frac{1}{N}$ benchmarks. Compared to Zhao et al. (2021), who argue that standard elastic net lacks formal control over short-selling, Yen and Yen's results demonstrate its practical effectiveness.

Zhao et al. (2021) build on the idea of combining sparsity and stability in portfolio optimization by introducing the L1,2 norm regularization. Unlike the L2,1 norm used by Zhang et al. (2022), which promotes group-level sparsity by applying L2 within groups and L1 across groups, the L1,2 norm targets unstructured portfolios and acts directly on individual weights. The key theoretical advantage over elastic net is that L1,2 retains a provable bound on short positions. Zhao et al. (2021) show that this formulation retains sparsity and improves numerical stability, but unlike standard elastic net, it also preserves theoretical bounds on short exposure. This is particularly relevant in constrained or regulatory settings where controlling short positions is critical.
Wu et al. (2024) analyse multiple regularization strategies, including separate and combined L1 and L2 constraints. Their most general approach, equivalent to the elastic net, balances sparsity and stability by penalizing both the absolute and squared magnitudes of asset weights. Empirical tests across several datasets show that this hybrid formulation consistently reduces variance and turnover without sacrificing Sharpe ratio performance.

In summary, the related literature reveals a strong and consistent emphasis on addressing the estimation-driven instability of the classical Markowitz framework through regularization techniques. Constraints on portfolio weights, shrinkage of input parameters, and penalization of the optimization objective all serve to improve robustness and out-of-sample performance. A wide range of methods, from implicit regularization via constraints (e.g., short-sale limits) to explicit L1, L2, and hybrid elastic net penalties, have been proposed to either sparsify or stabilize portfolio allocations. Notably, L1 penalties induce sparsity and improve interpretability, while L2 penalties promote smoother weight profiles and reduce turnover. More recent advances incorporate adaptive, asset-specific, or structurally aware penalties that align regularization with economic intuition or data characteristics. In parallel, shrinkage estimators enhance input stability, offering a complementary layer of regularization. The convergence of these strands in modern portfolio research underscores the necessity of incorporating regularization, whether on inputs or weights, as a central design element in reliable, high-dimensional asset allocation frameworks. This study builds upon these contributions by systematically evaluating the stabilizing effects of L1 and L2 penalties in the context of minimum variance and maximum Sharpe ratio portfolios, thereby contributing to the ongoing effort to reconcile theoretical optimality with practical robustness.

3 Methodology

In the following section the used methodology is described including a description of the dataset, the exact way how the portfolios were constructed as well as how the performance of the portfolios is evaluated. Furthermore, to fulfil the requirements set by the Zürcher Hochschule für Angewandte Wissenschaften (Zürcher Hochschule für Angewandte Wissenschaften, 2024), in the following we will critically reflect our use of generative AI tools. Generative AI tools, especially ChatGPT (a detailed list can be found in the AI Use and Disclaimer section at the end of this paper), have been used to enhance clarity, coherence as well as to improve grammar and provide a consistent writing style by providing text proposals, making it easier for the audience to understand the content of the paper. No personal data, business or trade secrets have been processed by any AI tool. The tools were not used to structure the paper, doing research, analysing data as well as the results and has not contributed to the intellectual output in another way than described above. The use of AI was allowed by the corresponding supervisor.

3.1 Dataset

The dataset used in this study is obtained from the Fama and French database and was downloaded using the pandas datareader (*Fama-French Data (Ken French's Data Library) - Pandas-datareader 0.10.0 Documentation*, n.d.). The Fama and French database is a widely recognized source for empirical asset pricing research. The utilized dataset consists of historical daily returns for 10 industry portfolios spanning the period from 1st January 2020 to 31st December 2024 covering a time period of 25 years and amounting to 6,289 observations. The dataset covers a broad range of economic conditions, including the dot-com bubble (2000), the global financial crisis (2008), and the COVID-19 pandemic (2020), ensuring that the analysis captures both periods of economic expansion and severe market downturns.

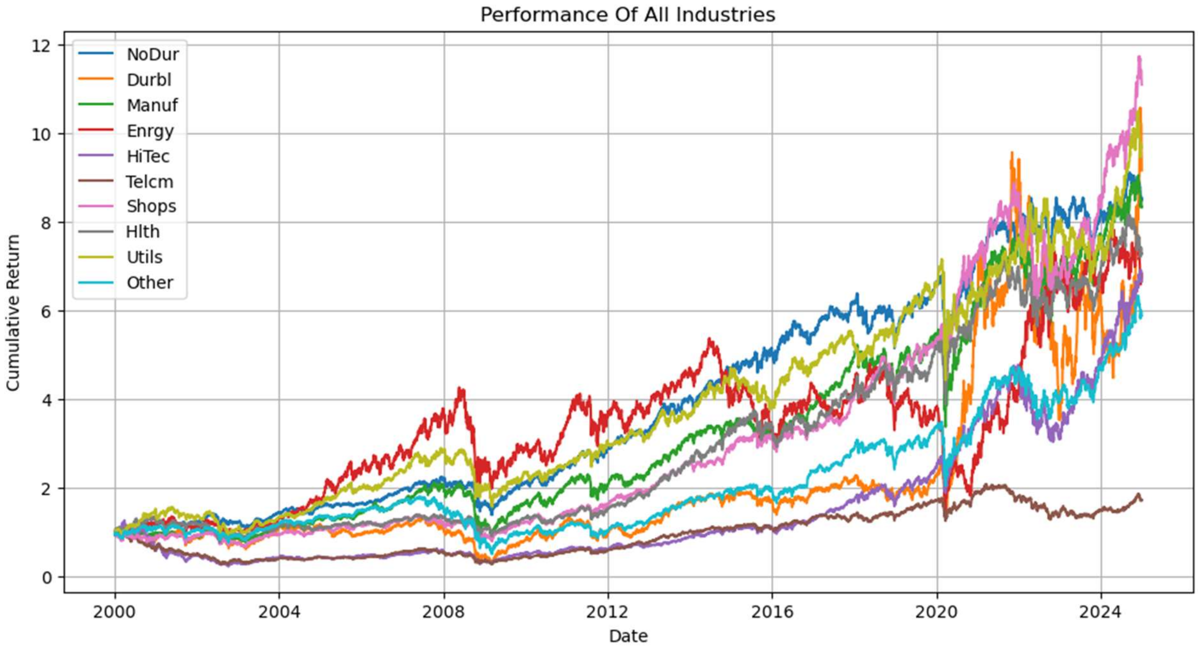

Figure 1: Performance of all industries

Over this period, the average compound annual growth rate (CAGR) across all industries was 8.12%, while the average annualized volatility stood at 22.31%. However, substantial variation exists across industries, both in terms of performance and risk. Certain sectors achieved particularly strong long-term performance, with cumulative returns exceeding 800% over the 25-year period. The highest-performing industries were Shops (+1010%), Utilities (+867%), and Durable Goods (+817%). These sectors benefited from consistent consumer demand, long-term capital investment cycles, and structural tailwinds, such as demographic shifts and technological innovation in retail and energy infrastructure.

Conversely, other sectors lagged significantly behind in terms of total return. Telecommunications posted the weakest performance, with a cumulative return of just +72%, followed by the residual "Other" category (+489%) and High Technology (+566%). The underperformance of these sectors may be attributed to intensifying competition, technological obsolescence, and, in some cases, overvaluation during prior speculative phases, such as the dot-com era for the High Technology sector.

Differences in risk exposure were also pronounced across industries. The most volatile sectors included Durable Goods (31.27%), Energy (28.55%), and High Technology (26.47%), reflecting sensitivity to macroeconomic cycles, commodity price shocks, and innovation-driven valuation swings. In contrast, Nondurable Goods (15.19%), Health (18.05%), and Utilities (19.04%) ranked among the least volatile industries. These sectors typically involve essential consumer goods or regulated services, resulting in more stable cash flows and lower exposure to market fluctuations. Overall, the dataset provides a broad and heterogeneous cross-section of industry-level return dynamics, offering a robust empirical foundation for evaluating portfolio optimization techniques under varying market conditions.

To ensure the validity of the portfolio optimization framework, all time series of historical industry returns were tested for stationarity, a fundamental property in time series analysis that implies statistical characteristics such as mean, variance, and autocorrelation remain constant over time. Stationarity is a crucial assumption in financial modelling, as non-stationary time series may lead to unreliable estimations of risk and return. The Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test was employed to assess stationarity at a 95% confidence level. This test evaluates the null hypothesis that a time series is stationary against the alternative hypothesis that it contains a unit root, indicating non-stationarity (Krishnan, 2022). The results confirmed that all industry return time series are stationary, ensuring that the dataset is suitable for mean-variance optimization and further statistical analysis without the need for additional transformations such as differencing or detrending.

3.2 Portfolio Construction

In order to evaluate the effect of regularization on portfolio performance, we construct and compare a comprehensive set of portfolio types. These include: (i) the naively diversified portfolio ($\frac{1}{N}$), (ii) the classical minimum variance portfolio, (iii) minimum variance portfolios with L1 and L2 regularization, (iv) the classical maximum Sharpe ratio portfolio, and (v) maximum Sharpe ratio portfolios with L1 and L2 regularization. This framework allows for a systematic assessment of how the introduction of sparsity (via L1 penalty) and shrinkage (via L2 penalty) affects portfolio weights and overall, out-of-sample performance.

For both the minimum variance and the maximum Sharpe ratio portfolios, the optimal asset weights are derived by solving the respective constrained optimization problems as formally defined in chapter 2 allowing short positions but no leverage (sum of weights equals one). The optimization relies on the empirical covariance matrix of asset returns, which is estimated using a rolling window comprising the most recent 126 daily observations, equivalent to approximately six months of historical trading data. This rolling estimation framework is updated (rebalanced) at intervals of 126 trading days, thereby enabling a periodic re-optimization of portfolio weights. Such a procedure allows the portfolio allocation to adapt to evolving market conditions and reflect the most up-to-date information available at each rebalancing point.

Figure 2: CBOE 10-year interest rate

To account for the time-varying nature of the risk-free rate in Sharpe ratio calculations, the CBOE 10-Year Treasury Yield was employed as a proxy. The yield exhibits significant temporal variation over the sample period, reflecting shifts in macroeconomic conditions, monetary policy regimes, and investor sentiment. Notably, the rate reached its peak in the year 2000 at a level of 6.78%. In contrast, it fell to a historic low of 0.50% shortly before the onset of the COVID-19 pandemic in early 2020, driven by aggressive monetary easing and flight-to-safety dynamics before it rose again to fight inflation within the US (Ihrig & Waller, 2024). These fluctuations underscore the importance of incorporating a dynamic risk-free rate when evaluating risk-adjusted performance, as fixed-rate assumptions may distort the relative attractiveness of portfolio strategies across different market environments.

To examine the impact of regularization strength on portfolio outcomes, we systematically vary the penalty parameters for both L1 and L2 regularization. Specifically, we consider a grid of regularization coefficients ranging from 0 to 1 ($\lambda \epsilon$ [0, 0.001, 0.0025, 0.005, 0.01, 0.025, 0.05, 0.1, 0.25, 0.5, 1]), enabling a granular comparison of performance across different levels of penalization. This variation facilitates a detailed investigation into the trade-offs between risk-adjusted return, portfolio sparsity, and turnover introduced by regularization.

The out-of-sample performance of all portfolio strategies is evaluated over a rolling horizon, based on the rebalanced weights determined at each iteration of the 126-day update cycle. This approach closely mimics a realistic investment scenario, ensuring that the analysis captures the temporal dynamics of return distributions and the effect of regularization in a continuously evolving market environment.

3.3 Portfolio Performance Evaluation

To comprehensively assess and compare the effectiveness of the different portfolio strategies, we employ a multi-dimensional performance evaluation framework incorporating both return-based and risk-adjusted metrics. Specifically, we track the following indicators over the out-of-sample evaluation horizon: cumulative portfolio return, annualized return volatility, and realized Sharpe ratio and VaR. Each metric is computed at every rebalancing point, thereby generating time series for all performance indicators, which enables a detailed temporal analysis of each portfolio's behaviour under varying regularization regimes.

Cumulative portfolio return is calculated as the compounded product of one-period returns realized by each strategy over time. This metric provides a direct measure of total wealth accumulation, enabling an absolute comparison of portfolio profitability across different strategies and regularization strengths.

Annualized volatility is employed as a conventional risk metric, representing the standard deviation of daily portfolio returns scaled to an annual basis. It offers insights into the overall risk exposure of each strategy and is especially useful when interpreting the trade-off between return and variability introduced by regularization techniques.

Realized Sharpe ratio serves as the primary risk-adjusted performance measure. It is computed at each rebalancing interval as the ratio of the average excess daily return (over the risk-free rate) to the standard deviation of daily returns, scaled by the square root of 252 to reflect annualization. This metric enables the comparison of performance across portfolios on a risk-adjusted basis and allows for an assessment of how L1 and L2 penalties influence the efficiency of portfolio allocations in terms of return per unit of risk.

VaR is utilized as a forward-looking risk measure that quantifies the potential loss of a portfolio under normal market conditions over a specified time horizon and confidence level. In this study, we employ the one-day 95% VaR, computed empirically based on historical returns of the portfolio over the previous 252 trading days. Formally, the VaR at the 95%

confidence level corresponds to the fifth percentile of the portfolio return distribution and represents the maximum expected loss with 95% certainty. The inclusion of VaR is motivated by its wide adoption in both academia and industry as a coherent risk measure that captures downside risk, a dimension not fully addressed by volatility alone. It is particularly relevant when evaluating portfolios under regularization, as sparse or concentrated portfolios may exhibit asymmetric risk characteristics.

By calculating each of these metrics prior to every portfolio rebalancing, we construct consistent time series that capture the evolution of performance characteristics under dynamic market conditions. This approach allows for both pointwise and cumulative evaluations and provides a robust foundation for statistical comparisons between regularized and unregularized portfolio strategies.

4 Results

4.1 Performance Of Classical Markowitz Portfolios

This chapter provides a comprehensive empirical evaluation of classical Markowitz portfolio strategies, focusing on their performance, risk characteristics, and structural dynamics in the absence of regularization. By employing a rolling window out-of-sample backtesting approach from 2000 to 2024, the analysis contrasts three distinct portfolio constructions: the Maximum Sharpe Ratio portfolio, the Minimum Variance portfolio, and a benchmark Naively Diversified portfolio. The chapter examines cumulative returns, volatility profiles, VaR, Sharpe ratios, and portfolio weight dynamics to assess the trade-offs between theoretical optimality and practical implementability. Particular attention is paid to the behaviour of each strategy during periods of elevated market stress, such as the dot-com aftermath, the Global Financial Crisis, and the COVID-19 pandemic. The findings highlight the strengths and limitations of each approach and provide insight into their suitability under varying market conditions.

To assess the empirical effectiveness of classical Markowitz portfolio strategies, an out-of-sample backtest was conducted using a rolling window methodology as described in chapter 3 Methodology. Specifically, three portfolio strategies were evaluated: (i) the maximum Sharpe ratio portfolio, (ii) the minimum variance portfolio, and (iii) a benchmark naively diversified portfolio with equal weights. The optimization procedures were based on historical return data from the preceding 126 trading days (approximately six months) and rebalanced at the same frequency. Importantly, no form of regularization was applied, and short-selling was permitted in all strategies.

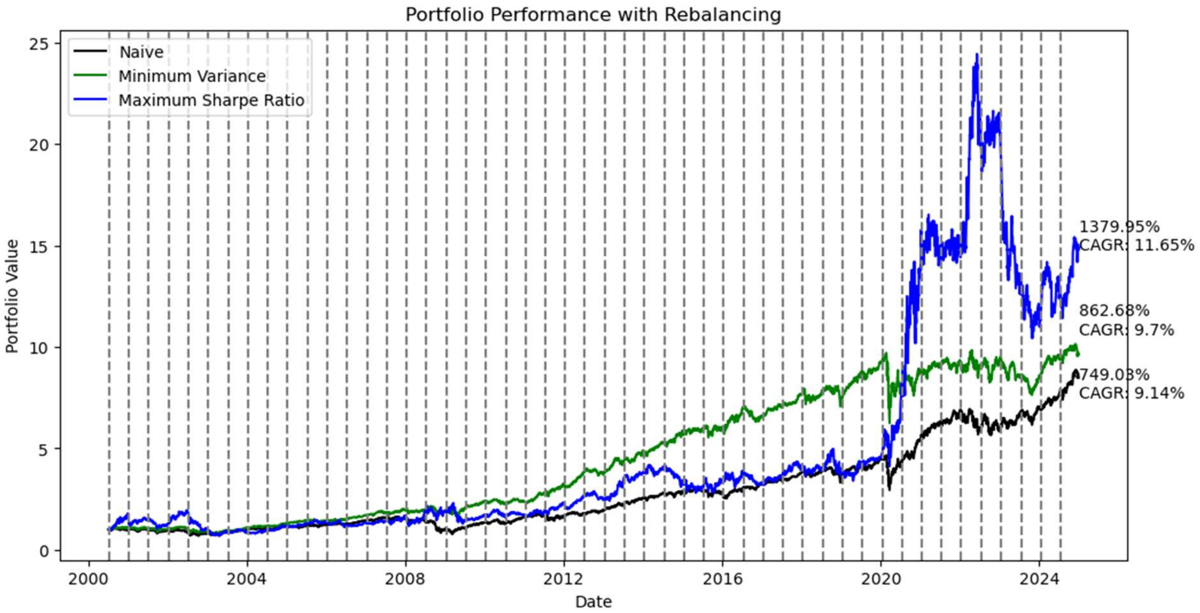

Figure 3: Performance of classical Markowitz portfolios.

The results, displayed in figure 3, clearly illustrate the differential performance characteristics of the three portfolio strategies over the period from 2000 to 2024 and the vertical dashed lines indicate the rebalancing dates. The maximum Sharpe ratio portfolio exhibited the highest cumulative return of 1379.95%, corresponding to a compound annual growth rate (CAGR) of 11.65%. However, this performance came at the cost of substantial volatility, as reflected by its pronounced fluctuations, particularly following the COVID-19 pandemic in early 2020. While it ultimately delivered superior long-term returns, this portfolio was consistently outperformed by the minimum variance portfolio over an extended period spanning from the aftermath of the Global Financial Crisis in 2009 until the onset of the pandemic in 2020.

The minimum variance portfolio achieved the second-best cumulative return of 862.68%, translating to a CAGR of 9.70%. Its performance was notably stable, especially during periods of heightened market uncertainty. The portfolio demonstrated a in comparison to the maximum Sharpe ratio portfolio smoother growth trajectory and experienced lower drawdowns, affirming its defensive characteristics and lower exposure to systematic risk. Remarkably, it maintained a leading position relative to the maximum Sharpe ratio portfolio for over a decade.

In contrast, the naively diversified portfolio, constructed by assigning equal weights to all assets without regard to risk or return forecasts, underperformed both optimized strategies. It yielded a cumulative return of 749.03% (CAGR: 9.14%), making it the least effective strategy in terms of absolute performance. Nonetheless, its relatively consistent path and moderate volatility suggest that naive diversification offers a simple yet resilient baseline against which more sophisticated approaches can be evaluated.

All three strategies experienced a marked decline during the onset of the COVID-19 crisis in early 2020. However, only the maximum Sharpe ratio portfolio managed to capitalize meaningfully on the subsequent market rebound, reflecting its greater exposure to high-beta assets. This behaviour further reinforces the trade-off between risk and return embedded within the Markowitz framework, wherein higher return potential is inherently accompanied by increased exposure to tail risk.

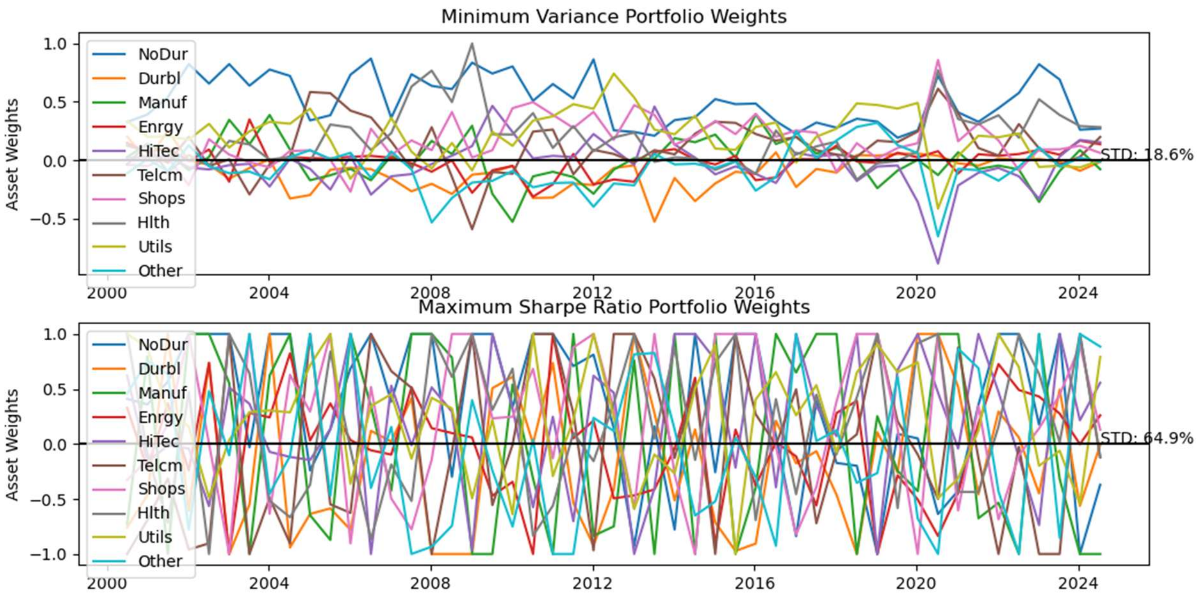

Figure 4: Classic Markowitz portfolio asset weight changes

Figure 4 illustrates the evolution of portfolio weights over time for the three strategies under investigation. A clear distinction emerges in the stability and variability of asset allocations across the different approaches. The minimum variance portfolio exhibits relatively stable weight dynamics, with gradual shifts in allocation across sectors and limited occurrence of extreme weight changes. This smooth evolution is indicative of a lower turnover strategy and is further supported by the standard deviation of portfolio weights over the sample period, which amounts to 18.6%. Such stability implies lower transaction costs in practical implementation, making this approach particularly attractive for investors with cost sensitivities or trading frictions.

In contrast, the maximum Sharpe ratio portfolio demonstrates significantly more volatile weight trajectories, with frequent and abrupt reallocations between assets at each rebalancing interval. The pronounced fluctuations suggest a highly reactive strategy that is more sensitive to short-term changes in expected returns and covariances. This behaviour is quantitatively reflected in the standard deviation of portfolio weights, which reaches 64.9% - a value more than three times that of the minimum variance counterpart. While this dynamic allocation may enhance theoretical return potential, it comes at the expense of substantially higher transaction costs and greater exposure to estimation error.

As expected, the naively diversified portfolio maintains constant weights across all assets throughout the entire observation period. Each asset is assigned an equal share, and no rebalancing occurs, resulting in a standard deviation of zero. This feature underscores the strategy's simplicity and transparency but also highlights its lack of responsiveness to changing market conditions.

Collectively, the comparison underscores the trade-off between optimality and implementability. While the maximum Sharpe ratio strategy may offer superior risk-adjusted returns in theory, the cost of maintaining such a portfolio, due to high turnover and unstable allocations, may significantly diminish its practical appeal. Conversely, the minimum variance portfolio strikes a balance between performance and operational feasibility, owing to its comparatively stable composition.

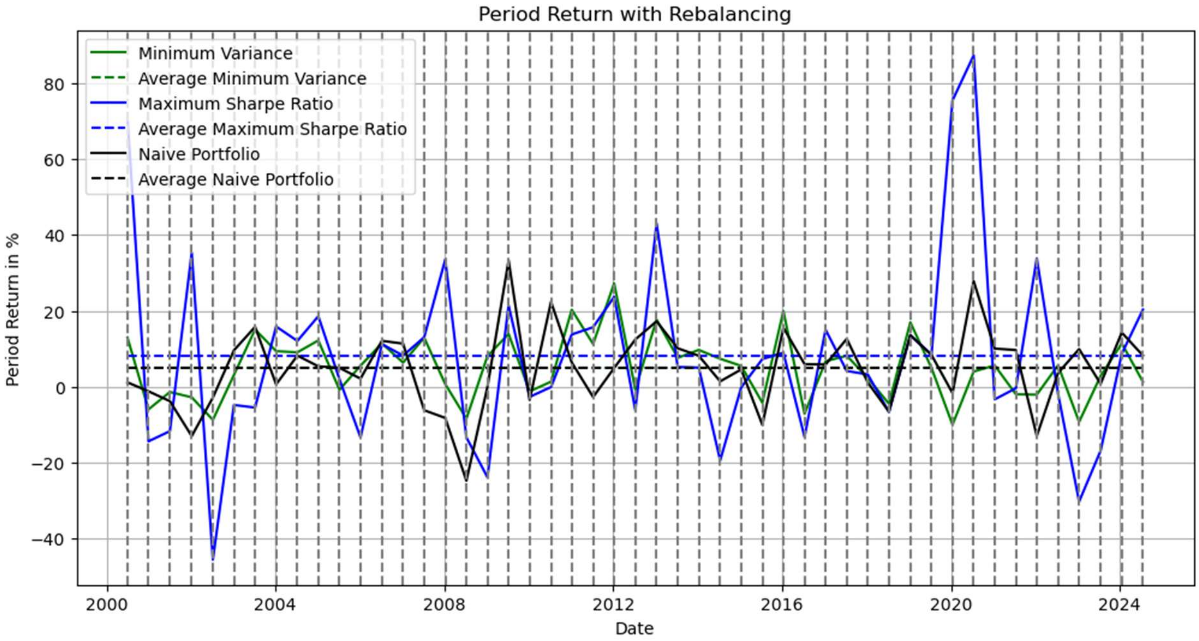

Figure 5: Classic Markowitz portfolios period returns

Figure 5 displays the realized returns for each 126-trading-day rebalancing period across the minimum variance, maximum Sharpe ratio, and naively diversified portfolios over the sample period from 2000 to 2024. On average, the maximum Sharpe ratio portfolio achieved the highest per-period returns at 8.22%. However, this performance came at the cost of significantly greater return dispersion. The return amplitude for this strategy ranged from a substantial post-COVID-19 rebound of +87.31% to a steep loss of -45.49% during the aftermath of the dot-com bubble and the September 11 terrorist attacks. This wide fluctuation underscores the strategy's pronounced exposure to market extremes.

In contrast, both the minimum variance and naively diversified portfolios exhibited markedly lower return volatility. The minimum variance portfolio achieved an average rebalancing-period return of 5.13%, while the naive portfolio returned 4.98% on average, both notably lower than the maximum Sharpe ratio portfolio, but with significantly more contained variability. These results reaffirm the trade-off between risk and return, with more conservative strategies offering improved stability at the expense of reduced upside potential.

The correlations between the realized returns of the portfolios were generally weak, indicating limited co-movement in short- to medium-term performance. The correlation between the minimum variance and maximum Sharpe ratio portfolios was 0.25, suggesting only a mild linear association. The correlation between the minimum variance and naive portfolios was somewhat stronger at 0.44, while the weakest relationship was found between the maximum Sharpe ratio and naive portfolios at just 0.16. These weak correlations imply that the portfolios respond differently to market conditions over the rebalancing horizon, likely reflecting the distinct nature of their construction methodologies and sensitivity to different sources of return.

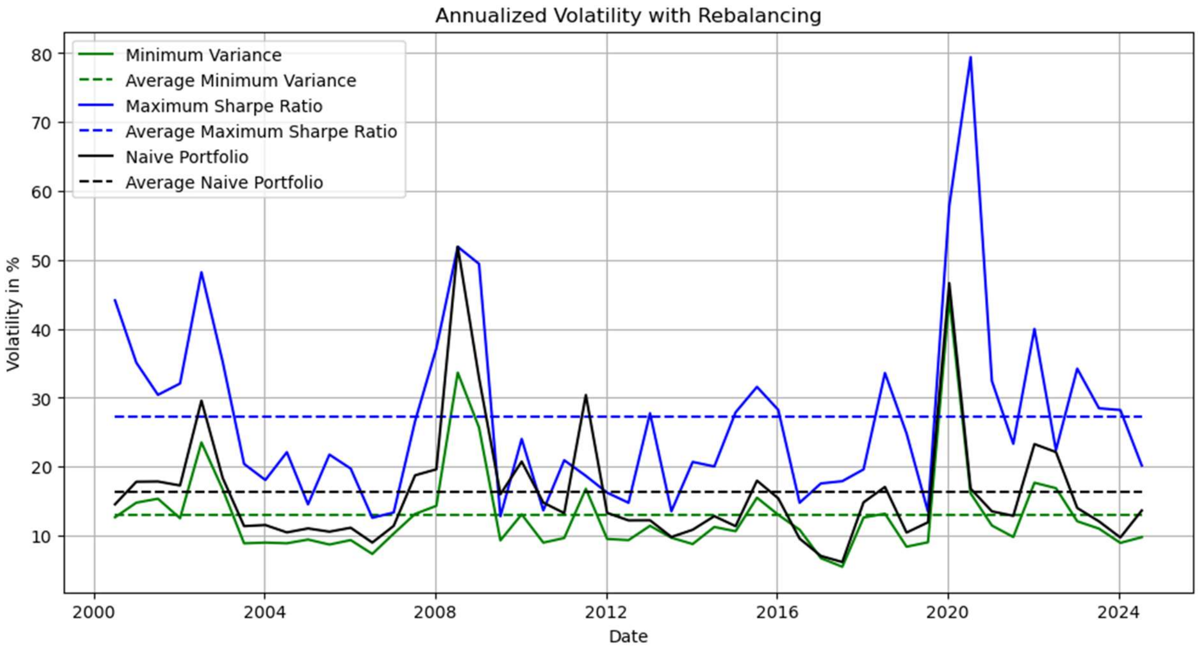

Figure 6: Classic Markowitz portfolios volatility

Figure 6 presents the annualized realized volatilities of the minimum variance, maximum Sharpe ratio, and naively diversified portfolios over the period 2000 to 2024. The maximum Sharpe ratio portfolio consistently exhibits the highest volatility throughout the examined period, with a single exception at the end of 2009, averaging 27.18%. This level of risk is more than twice that of the minimum variance portfolio, which maintained the lowest volatility on average at 12.97%, deviating from this trend only once in 2017. The naively diversified portfolio lies between the two in terms of risk, with an average volatility of 16.29%. These findings underscore the inherent risk trade-off embedded in return-optimizing strategies such as the maximum Sharpe portfolio, which sacrifices stability for potential performance.

The leftmost portion of the graph reflects elevated volatility during the early 2000s, consistent with the prolonged market turbulence following the dot-com crash. Notably, three distinct spikes in volatility are visible and align with major exogenous shocks: the 2003 Iraq War, the 2008–2009 Global Financial Crisis, and the COVID-19 pandemic in 2020. These events exerted systemic impacts across global markets, triggering substantial increases in portfolio volatility, with the maximum Sharpe and naive portfolios responding more acutely than the minimum variance portfolio. This highlights the superior downside protection of the volatility-minimizing strategy during times of market distress.

Furthermore, the volatilities of the three portfolios demonstrate strong co-movement. The correlation between the minimum variance and maximum Sharpe ratio portfolios is 0.70, indicating a substantial positive relationship. Even more pronounced is the correlation between the minimum variance and naively diversified portfolios, which stands at 0.94. This exceptionally high correlation suggests that the volatility profiles of these two strategies are not only synchronized over time but also likely influenced by similar market-wide risk factors. These results imply that while diversification and optimization approaches differ in methodology, their realized volatilities are often jointly shaped by prevailing macroeconomic and geopolitical conditions.

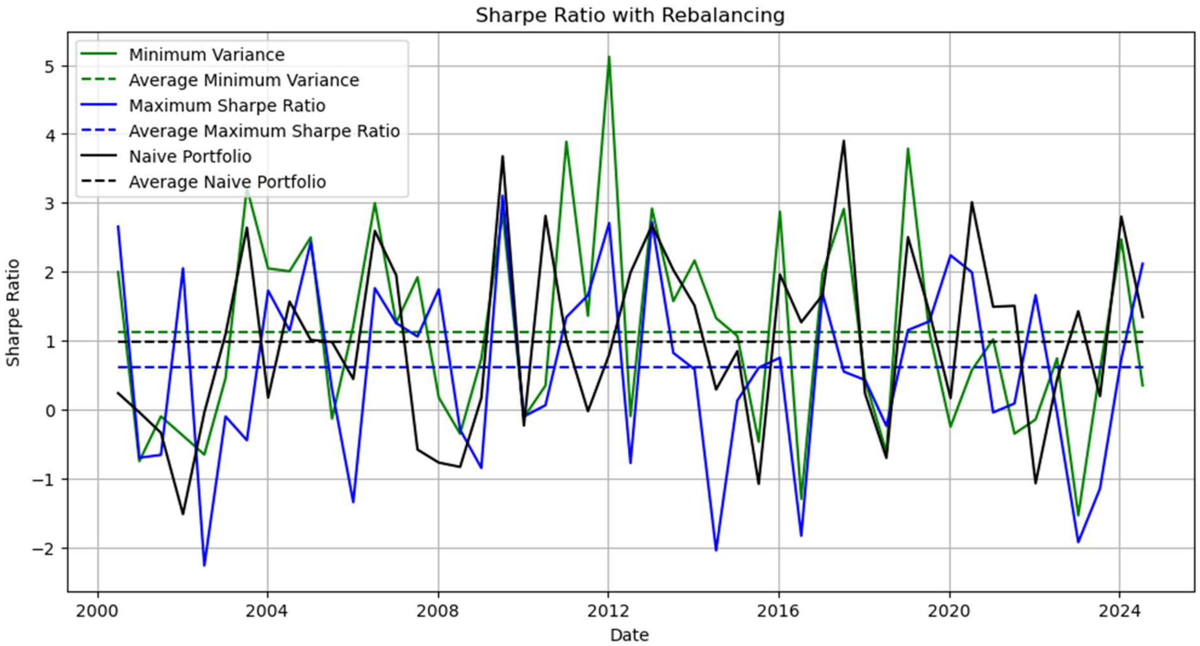

Figure 7: Classic Markowitz portfolios Sharpe ratios

Figure 7 illustrates the time series of Sharpe ratios for three portfolio strategies over the period from 2000 to 2024, with regular rebalancing. Contrary to theoretical expectations, the maximum Sharpe ratio portfolio, which is explicitly optimized for Sharpe ratio maximization, exhibits the lowest average Sharpe ratio of 0.60 across the observed period. This counterintuitive outcome can primarily be attributed to its pronounced volatility, as shown in figure 6. In contrast, the minimum variance portfolio achieves the highest average Sharpe ratio of 1.11, closely followed by the naive portfolio at 0.99. Both portfolios demonstrate more stable performance over time, suggesting that lower return volatility plays a more critical role in Sharpe maximization in practice than return magnitude alone. Furthermore, a moderate correlation of 0.51 between the minimum variance and maximum Sharpe portfolios indicates a partial co-movement in risk-adjusted returns. Similarly, the correlation between the minimum variance and naive portfolio stands at 0.50, reinforcing their similar volatility profiles. Meanwhile, the weak correlation of only 0.17 between the maximum Sharpe and Naive portfolios underscores the erratic behaviour of the Sharpe-optimal portfolio relative to more conservative strategies. These findings emphasize the trade-offs between theoretical optimization and empirical robustness, and they suggest that simpler or volatility-focused strategies may offer more consistent risk-adjusted performance in practice.

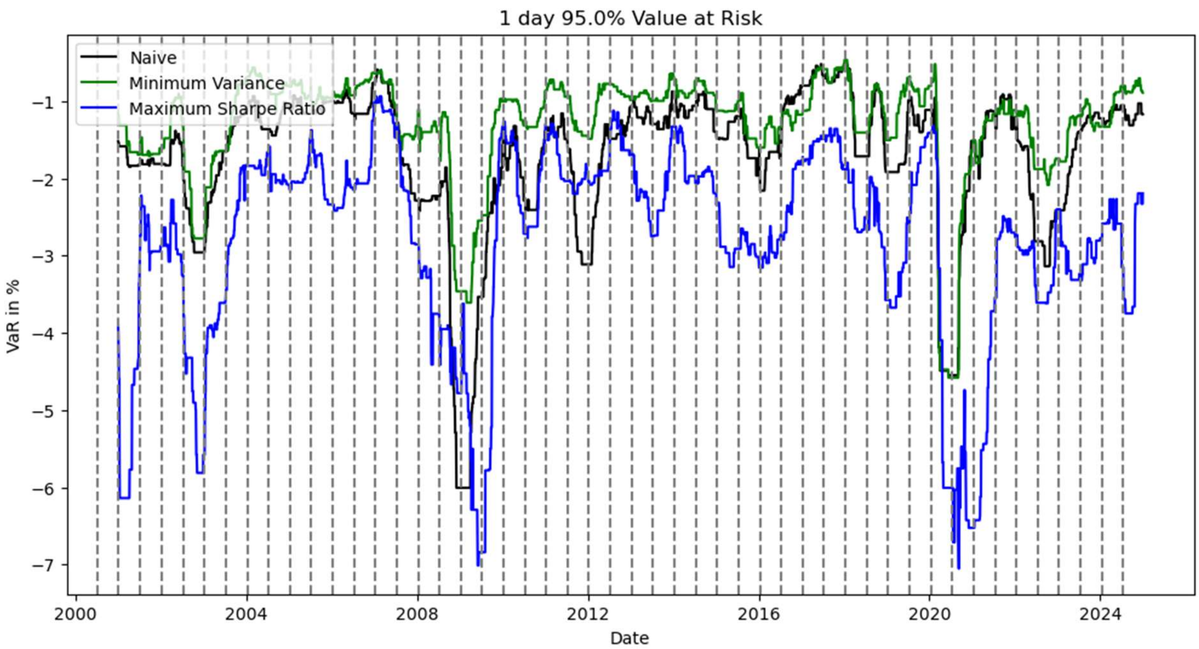

Figure 8: Classic Markowitz portfolios Value at Risk

Figure 8 illustrates the 1-day 95% VaR estimates for the three portfolio strategies across the full out-of-sample period. As expected, the minimum variance portfolio consistently exhibits the lowest downside risk, as reflected by its VaR estimates, which remain comparatively subdued throughout most of the sample period. With only a few exceptions, such as during certain low-volatility phases, the naively diversified portfolio occasionally achieves slightly lower VaR values. These rare instances underscore that naive diversification can, under specific market conditions, incidentally, generate portfolios with favourable risk characteristics, albeit not by design.

In contrast, the maximum Sharpe ratio portfolio is characterized by substantially higher VaR values across nearly all time periods, confirming its inherently higher exposure to volatility. This observation is consistent with earlier findings from the performance and volatility plots, which indicated that the portfolio with the highest Sharpe ratio also exhibited the greatest variability in returns. The elevated VaR is a direct consequence of the optimization procedure, which often results in highly concentrated and leveraged positions, thereby amplifying downside risk.

Consistent with the findings on portfolio volatility, the VaR estimates also exhibit strong correlations across the different portfolio strategies. Specifically, the correlation between the maximum Sharpe ratio portfolio and the naively diversified portfolio amounts to 0.62, indicating a moderate positive relationship. Notably, the correlation between the minimum variance portfolio and the naively diversified portfolio is even higher, reaching 0.89. This substantial correlation suggests that, despite their differing optimization objectives, these two portfolios are similarly exposed to downside risk, potentially reflecting shared sensitivities to broader market dynamics.

Across all three strategies, a common pattern emerges: VaR estimates increase markedly during systemic crisis periods, reflecting elevated market uncertainty and volatility. This pattern is particularly evident during the bursting of the dotcom bubble in the early 2000s, the global financial crisis of 2008–2009, and the COVID-19 pandemic shock around 2020–2021. These episodes demonstrate the procyclical nature of portfolio risk, where estimated losses rise substantially during periods of market stress, regardless of the underlying portfolio construction methodology. This emphasizes the importance of incorporating dynamic risk management frameworks when evaluating and implementing portfolio strategies in practice.

The maximum Sharpe ratio portfolio delivers the highest cumulative return (1379.95%; CAGR: 11.65%) and average rebalancing-period return (8.22%). However, its superior long-term performance is coupled with substantial risk, including the highest realized volatility (27.18%) and the widest range in period returns (+87.31% to –45.49%). Its unstable portfolio weights (standard deviation of 64.9%) further indicate high turnover and exposure to estimation error, diminishing its practical viability despite theoretical appeal. The minimum variance portfolio, while yielding lower cumulative returns (862.68%; CAGR: 9.70%), exhibits pronounced stability. It outperformed the Sharpe-optimal portfolio for over a decade (2009–2020), maintained the lowest volatility (12.97%), and achieved the highest average

Sharpe ratio (1.11). Its portfolio weights were notably stable (standard deviation: 18.6%), implying reduced transaction costs and robustness under turbulent market conditions. The naively diversified portfolio, although underperforming in absolute terms (749.03%; CAGR: 9.14%), demonstrated moderate volatility (16.29%) and a Sharpe ratio (0.99) comparable to the minimum variance strategy. Its simplicity and zero turnover make it an efficient and transparent benchmark, particularly in environments with high model risk or transaction frictions.

Importantly, all strategies exhibited procyclical risk behaviour, with elevated volatility and VaR estimates during crisis episodes such as the dot-com crash, the global financial crisis, and the COVID-19 pandemic. The minimum variance portfolio consistently provided the strongest downside protection, whereas the maximum Sharpe ratio portfolio was most vulnerable to tail events due to its high concentration and reactivity. In summary, while the maximum Sharpe ratio portfolio demonstrates superior theoretical performance, the minimum variance portfolio strikes a more effective balance between risk-adjusted return and practical implementability. The findings underscore the necessity of considering estimation risk, portfolio stability, and transaction costs when applying classical portfolio theory in real-world settings.

4.2 Impact of L1 Regularization on the Minimum Variance Portfolio

This section investigates the impact of L1 regularization strength ($\lambda \in [0, 1]$) on the empirical performance of the minimum variance portfolio across four key metrics: average annual return, annualized volatility, annualized Sharpe ratio, and average daily VaR. The results, visualized in Figure 9, reveal nuanced trade-offs between regularization strength and portfolio risk-return characteristics.

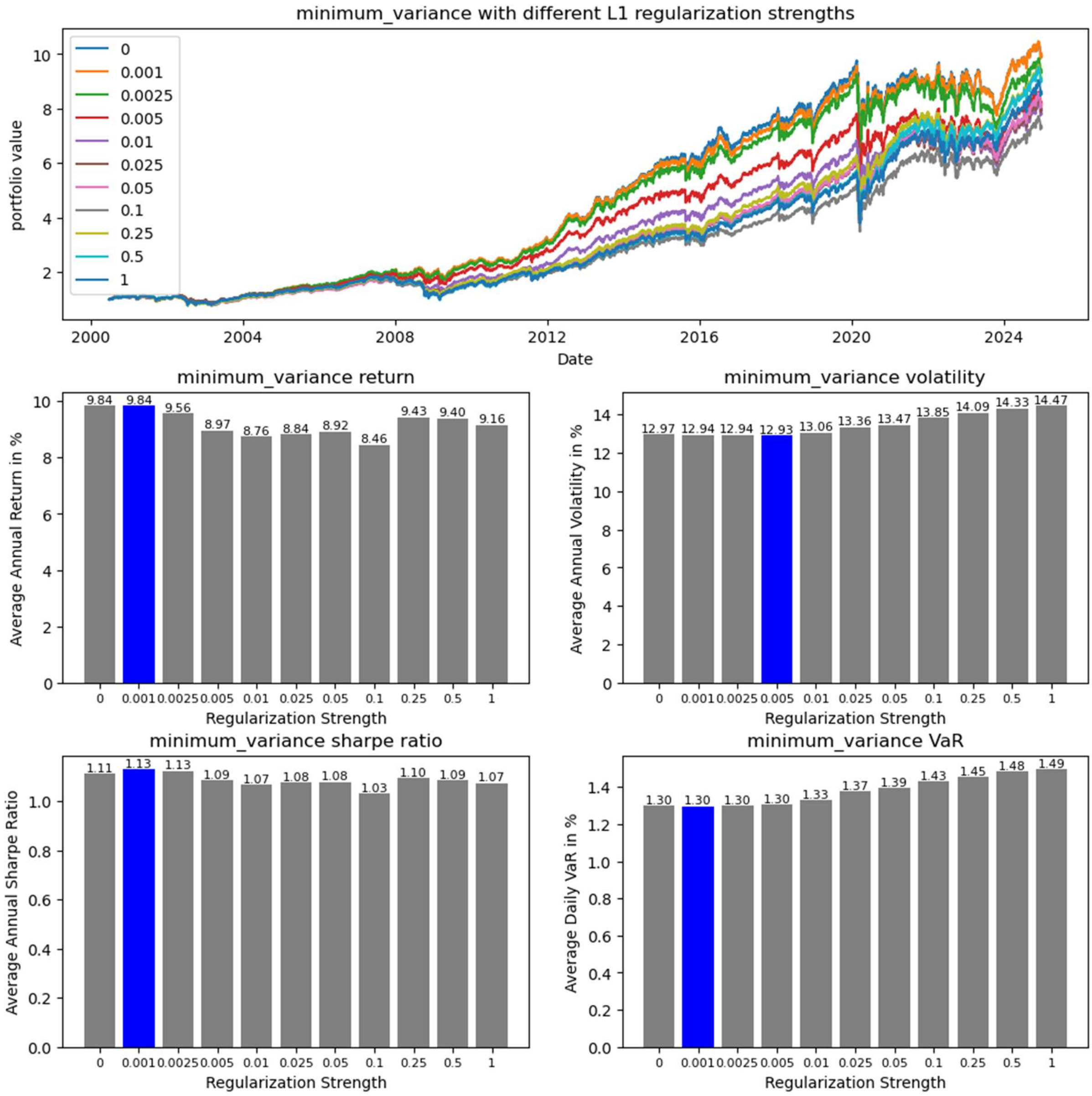

Figure 9: Influence of L1 regularization on the MVP

The analysis reveals that the average annual return exhibits a non-monotonic relationship with regularization strength. The highest return is achieved with a weak L1 penalty of λ = 0.001, yielding an average annual return of 9.84%. Notably, this performance is nearly identical to the unregularized portfolio (λ = 0), which trails by a mere 0.006 percentage points. As λ increases beyond 0.001, returns decrease monotonically, with a minimum value of 8.46% at λ = 0.1, representing a decline of 1.38 percentage points relative to the peak. This pattern suggests that minimal sparsity-inducing penalties may help mitigate noise in the covariance estimation process without substantially distorting the optimal allocation, while higher regularization introduces suboptimal constraints that deteriorate performance.

Furthermore, the proximity in performance between the portfolios at λ = 0 and λ = 0.001 highlights the sensitivity of return enhancement to model and market conditions.

In contrast to returns, the volatility of the minimum variance portfolio demonstrates a U-shaped response to increasing regularization. Introducing small L1 penalties in the range λ ∈ [0.001, 0.005] marginally reduces volatility, with the lowest level observed at λ = 0.005, where annualized volatility drops to 12.93% - 0.04 percentage points below the unregularized benchmark (12.97%). This suggests a stabilizing effect of mild regularization, potentially stemming from the exclusion of weakly contributing or noisy assets. However, as the regularization strength exceeds λ ≥ 0.01, volatility increases consistently. This positive trend is captured by a linear regression model yielding a R^2 value of 0.706 and can be expressed by:

$$\sigma(\lambda) = 13.2091 + 1.5955 * \lambda$$

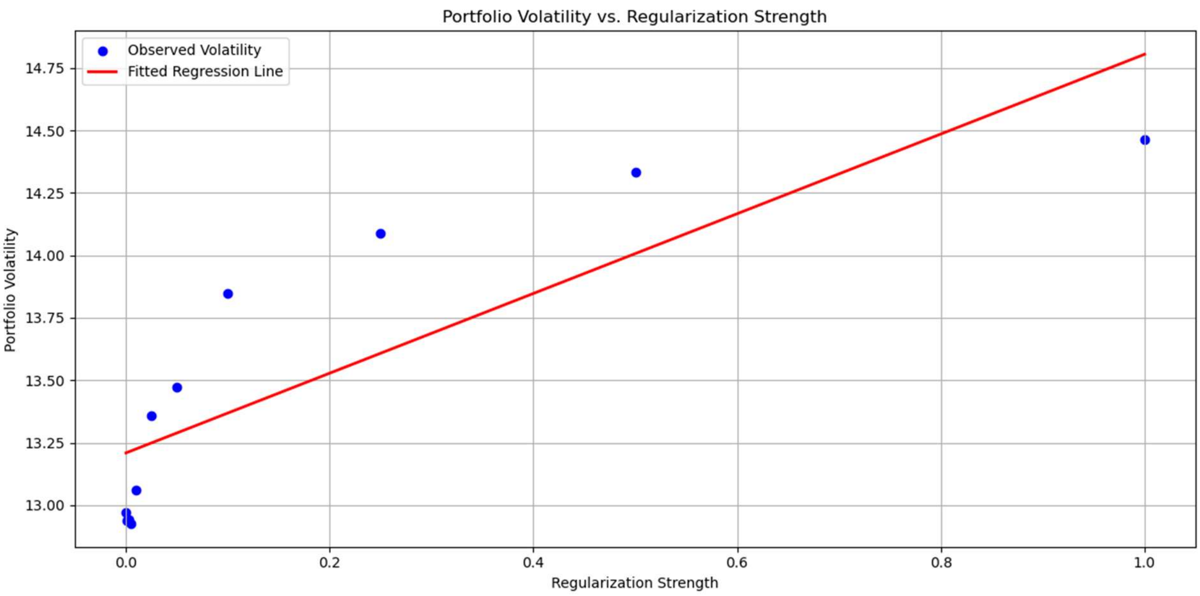

Figure 10: Influence of L1 regularization on the volatility of the MVP

indicating that each unit increase in λ is associated with a 1.60 percentage point increase in portfolio volatility. These findings highlight the delicate balance between beneficial sparsity and harmful over-penalization in risk management. To account for the quasi-logarithmic scaling of the regularization strengths, we attempted to apply a logarithmic transformation to the regularization strengths prior to fitting the linear regression model. However, this transformation did not lead to an improvement in the associated R^2 values. This is true for the upcoming linear regression models with the exception of the models describing the relation between regularization strength and the volatility of the portfolio weights.

The Sharpe ratio results mirror the trade-offs observed in the return and volatility metrics. Mild regularization appears beneficial, with λ = 0.001 and λ = 0.0025 both yielding the highest observed Sharpe ratio of 1.13, improving upon the unregularized benchmark of 1.11. This improvement reflects the modest reduction in volatility without significant sacrifice in return. However, for λ > 0.0025, Sharpe ratios decline, driven either by reduced returns or increased volatility. Unlike volatility, this relationship is less structured and more idiosyncratic. A linear regression model yields a low coefficient of determination ($R^2 \approx 0.047$), indicating a weak and noisy association between regularization strength and risk-adjusted performance. This irregularity may reflect the complex interaction between asset inclusion/exclusion and regime-dependent return dynamics.

Lastly, the VaR metric further reinforces the findings on volatility. For λ ∈ [0.001, 0.01], the portfolio experiences a marginal decline in average daily VaR, reaching the lowest value of 1.30% at both λ = 0.001 and λ = 0.0025. These differences, although minor and only present at the third decimal place, suggest a slight improvement in downside risk control. Beyond λ = 0.01, VaR increases consistently, peaking at 1.49% for λ = 1. This monotonic rise again reflects the trade-off between sparsity and diversification: excessive regularization limits the portfolio's ability to spread risk, leading to greater exposure to tail events.

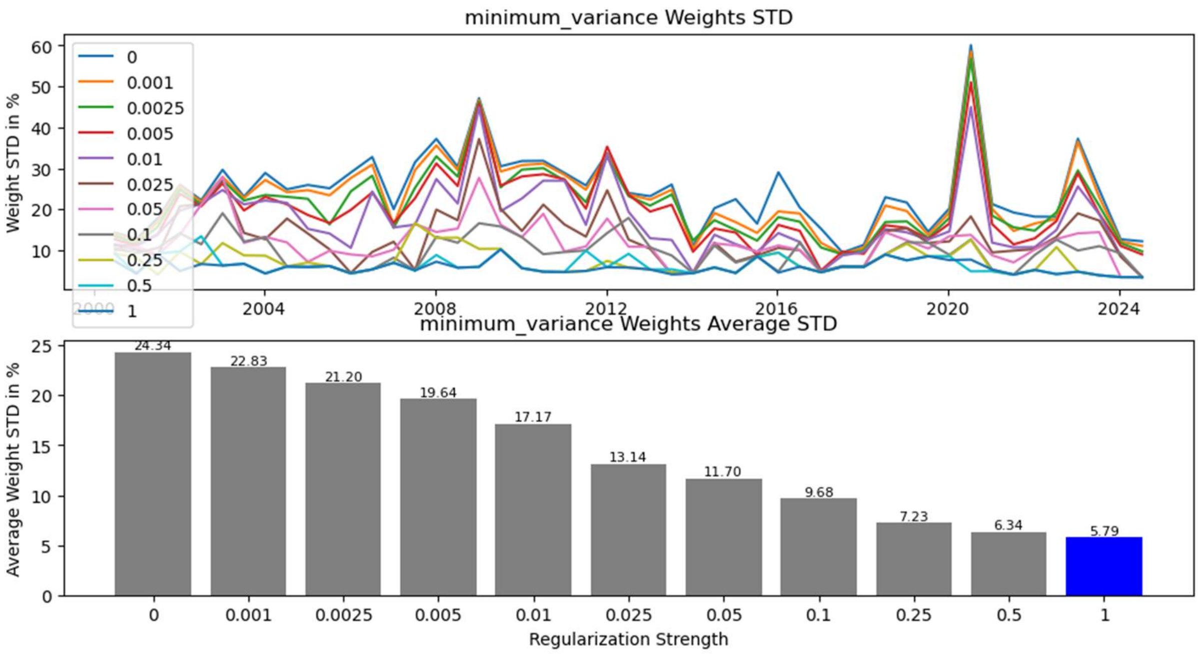

Figure 11: Influence of L1 regularization on the asset weights STD of the MVP

The analysis presented in figure 11 illustrates the evolution and variability of asset weights in the minimum variance portfolio under varying levels of L1 regularization. The upper panel clearly demonstrates that portfolio weights are subject to substantial fluctuations during periods of financial turmoil, notably at the end of the global financial crisis in 2009 and during the COVID-19 pandemic in 2020, as evidenced by pronounced spikes in the standard deviation of portfolio weights. These periods of heightened market stress induce significant rebalancing activity in the absence of regularization. However, as L1 regularization strength increases, the temporal volatility of asset weights visibly declines. This pattern indicates that stronger regularization imposes greater sparsity and stability in portfolio composition, thereby reducing turnover and potential transaction costs associated with frequent rebalancing.

The lower panel reinforces this observation by quantifying the average standard deviation of asset weights across the full sample period. A strong inverse relationship emerges between the regularization strength and the average weight standard deviation, with the unregularized portfolio exhibiting the highest average standard deviation (24.34%), and the most heavily regularized portfolio (strength = 1) displaying the lowest (5.79%). This relationship can be expressed as a logarithmic transformed regression of form:

$$\sigma_{weigh}\ (\lambda) = 7.5512 - 1.5936 * log(\lambda)$$

where the linear regression model produces a R^2 value of 0.791 indicating strong explanatory power.

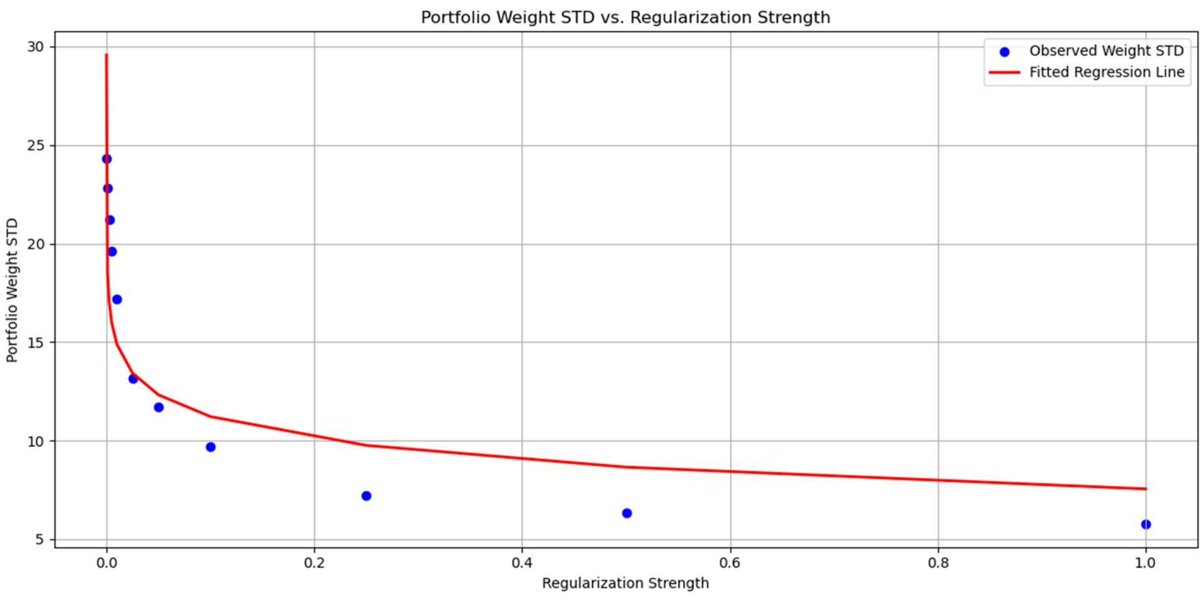

Figure 12: Influence of L1 regularization on the asset weights STD of the MVP OLS model

This systematic decline in weight variability highlights the effectiveness of L1 penalization in promoting temporal stability in portfolio allocation. The consistent and monotonic nature of this relationship further supports the conclusion that L1 regularization serves as a robust mechanism for controlling the sensitivity of portfolio weights to market dynamics, ultimately contributing to more stable and potentially more cost-efficient investment strategies.

The analysis of the minimum variance portfolio under varying L1 regularization strengths reveals a nuanced trade-off between performance and stability. Weak regularization (λ ≈ 0.001–0.005) provides marginal improvements in return, volatility, Sharpe ratio, and VaR, suggesting minor efficiency gains without significantly altering the risk profile. However, higher levels of regularization consistently degrade performance across all metrics. Complementary analysis of the portfolio's weight dynamics demonstrates that L1 regularization effectively enhances temporal stability by reducing fluctuations in asset allocations, particularly during periods of market stress. This stabilization effect is approximately linear and becomes more pronounced as λ increases, indicating that stronger regularization reduces rebalancing frequency and may lower transaction costs. Overall, L1 regularization offers modest benefits in terms of return-risk trade-offs while substantially improving portfolio stability and manageability.

4.3 Impact of L1 Regularization on the Maximum Sharpe Ratio Portfolio

To assess the effect of L1 regularization on the maximum Sharpe ratio portfolio, we investigated a range of regularization strengths $\lambda \epsilon [0, 200]$ and evaluated their impact on four core performance metrics: average annual return, average annual volatility, average annual Sharpe ratio, and average daily VaR. The initial investigation of regularization strengths within the narrower range $\lambda \epsilon [0, 1]$ revealed that all evaluated metrics reached their local optimum at the upper boundary $\lambda = 1$. Motivated by this finding, we extended our analysis to higher regularization strengths, up to $\lambda = 200$, to explore whether further performance improvements could be achieved.

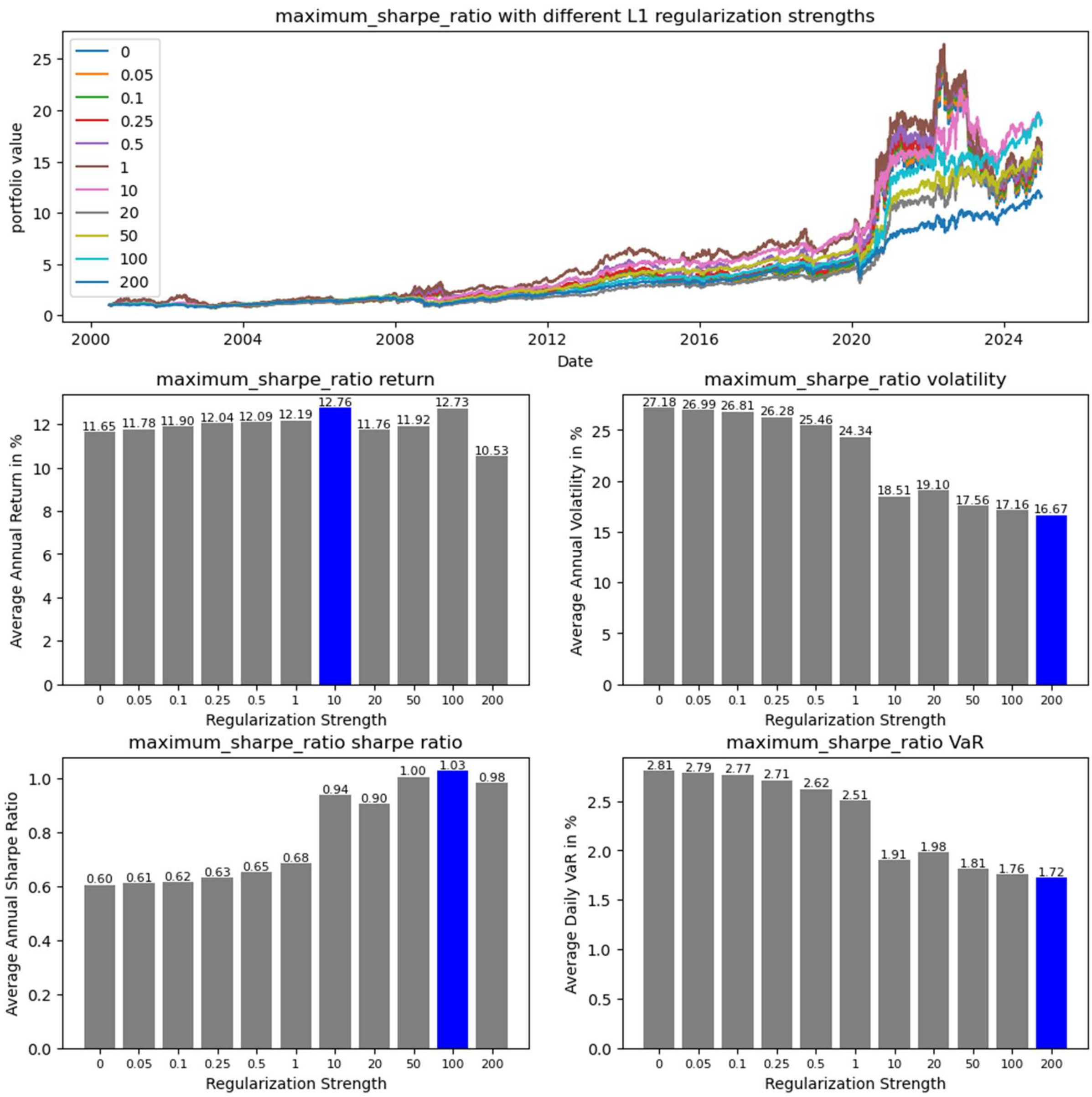

Figure 13: Influence of L1 regularization on the MSRP

The average annual return of the portfolio exhibited a non-linear relationship with the regularization strength. The baseline portfolio without regularization ($\lambda = 0$) yielded an annual return of 11.65%. As λ increased, returns also rose, peaking at $\lambda = 10$ with an annual return of 12.76%. This represents a relative improvement of approximately 9.3% compared to the unregularized portfolio. Notably, further increases in λ beyond 100 led to a decline in returns, suggesting diminishing benefits of excessive regularization. While $\lambda = 10$ delivered the highest overall return, it is important to note that this portfolio was outperformed by others, such as $\lambda = 1$, for most of the sample period. However, beginning in 2023, portfolios with low or no regularization experienced substantial drawdowns, whereas portfolios with stronger regularization demonstrated improved resilience, which allowed the $\lambda = 10$ portfolio to outperform by the end of the sample.

L1 regularization had a pronounced dampening effect on portfolio volatility. The unregularized portfolio had an average annual volatility of 27.18%. As regularization strength increased, volatility decreased consistently, reaching a low of 16.67% for $\lambda = 200$. This reduction of approximately 10.51 percentage points highlights the stabilizing influence of L1 regularization on portfolio risk. Again, a regression model was fitted to determine the effect, yielded a comparable low R^2 value of 0.500 and can be expressed as:

$$\sigma(\lambda) = 24.1158 - 0.0503 * \lambda$$

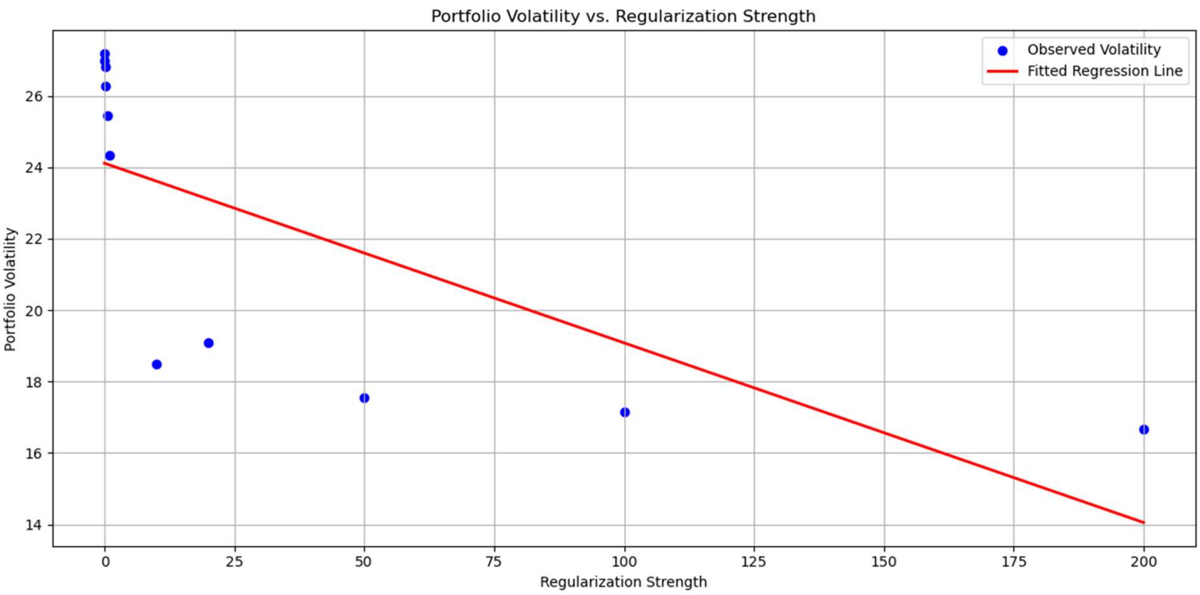

Figure 14: Influence of L1 regularization on the volatility of the MSRP portfolio

The volatility appears to converge around 16.1% for high values of λ, indicating a saturation effect where further increases in sparsity do not significantly improve volatility.

The average annual Sharpe ratio also benefitted from increasing L1 regularization. The baseline portfolio achieved a Sharpe ratio of 0.60, which almost increased monotonically up to a peak of 1.03 at $\lambda = 100$. This gain of 0.43 reflects the joint impact of higher returns and reduced volatility. For $\lambda > 100$, the Sharpe ratio declined slightly and fluctuated around a value of 1.0, suggesting a stabilization effect similar to that observed in the volatility metric. The results imply that moderate-to-strong L1 regularization can enhance risk-adjusted performance, but that excessive sparsity may eventually constrain potential returns.

VaR followed a similar pattern as volatility. The unregularized portfolio had an average daily VaR of 2.81%. This figure declined significantly with increasing λ, reaching a minimum of 1.72% at $\lambda = 200$, representing an improvement of 1.09 percentage points. For $\lambda \geq 200$, VaR values converged near 1.7%, mirroring the plateau effect seen in volatility. This decline in downside risk is partially attributable to the fact that highly regularized portfolios did not participate in the pronounced drawdown phase observed in 2023, which disproportionately affected portfolios with lower regularization.

Across all four metrics, L1 regularization exhibits a strong non-linear effect, improving performance up to a certain threshold and then plateauing or mildly deteriorating beyond it. This behaviour undermines the suitability of simple linear regression models for capturing the relationship between λ and performance. Indeed, regressions of each metric on λ produced near-zero R^2 values, indicating that linear models are inadequate for describing the complex dynamics introduced by L1 regularization.

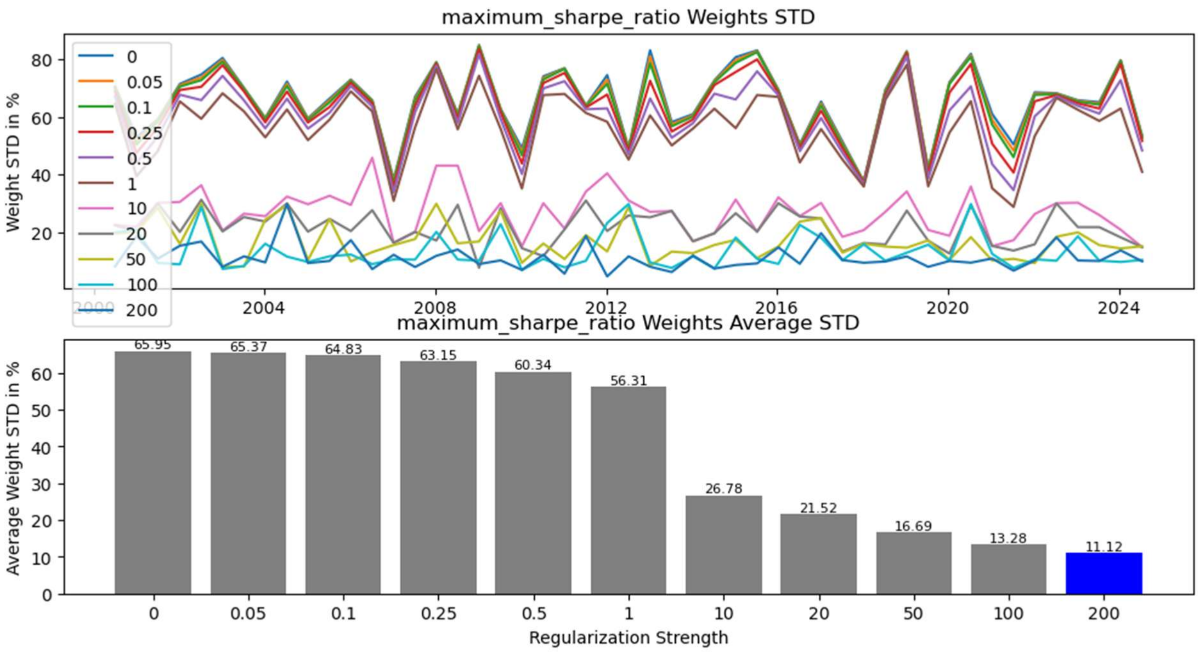

Figure 15: Influence of L1 regularization on the asset weights STD of the MSRP

An additional perspective on the effects of L1 regularization is provided by examining the standard deviation of portfolio weights over time, which serves as a proxy for the stability and turnover of the portfolio composition. As theoretically expected, increasing the regularization strength λ leads to a systematic reduction in weight volatility. Portfolios with comparable low or no regularization (e.g., $\lambda \leq 1$) exhibit average weight standard deviations of approximately 60%, indicating frequent and substantial shifts in asset allocation. In contrast, portfolios subject to stronger regularization $\lambda \geq 100$ display markedly lower weight volatility, thereby contributing to the observed reduction in overall portfolio volatility. This pattern underscores the sparsity-inducing nature of L1 regularization, which promotes more stable and parsimonious portfolio structures. The effect can be captured by a logarithmic transformed linear regression model of form:

$$\sigma_{weight}(\lambda) = 41.6518 - 3.4479 * log(\lambda)$$

yielding an R^2 value of 0.597 indicating moderate explanatory power.

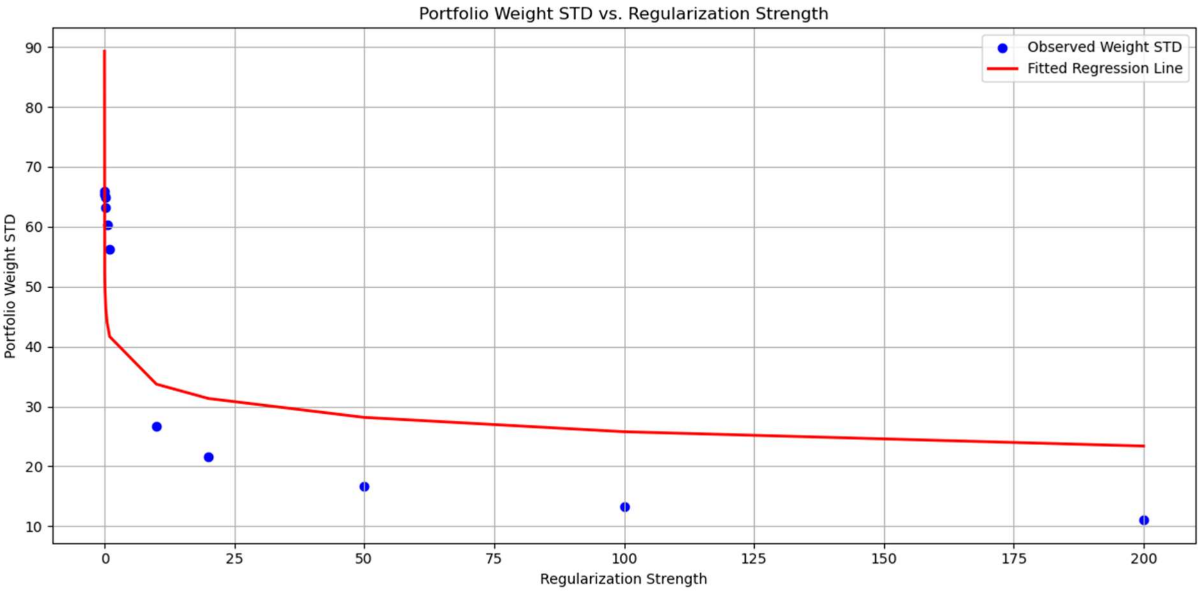

Figure 16: Influence of L1 regularization on the asset weights STD of the MSRP OLS model

Despite this general decline in weight volatility, pronounced spikes are still observable during periods of financial stress across all levels of regularization. Interestingly, these fluctuations are not limited to low λ portfolios, additional volatility spikes appear even at high regularization strengths. This phenomenon reveals that the maximum Sharpe ratio portfolio, unlike its minimum variance counterpart, is more sensitive to changes in expected returns and thus experiences higher turnover at each rebalancing point. As a result, the maximum Sharpe ratio strategy exhibits inherently higher instability in weight allocations, even under strong regularization, suggesting that the drive for return maximization induces more aggressive shifts in portfolio composition compared to volatility-minimizing strategies.

The analysis of L1 regularization on the maximum Sharpe ratio portfolio reveals a complex but consistent influence on key portfolio performance metrics. Moderate to strong regularization strengths $(\lambda \epsilon [0, 100])$ enhance portfolio characteristics by simultaneously increasing average annual returns and reducing both volatility and downside risk, culminating in a peak Sharpe ratio at $\lambda = 100$. Beyond this point, returns begin to decline slightly, and risk metrics converge toward stable levels, indicating diminishing marginal benefits of further sparsity. Importantly, the superior performance of highly regularized portfolios during recent market downturns, particularly in 2023, highlights their robustness under stress. In parallel, the standard deviation of portfolio weights decreases significantly with increasing regularization, contributing to lower volatility; however, even with high λ, weight instability remains elevated compared to the minimum variance portfolio, reflecting the inherently dynamic nature of return-maximizing strategies. Overall, L1 regularization serves as an effective mechanism for improving the stability and risk-adjusted performance of maximum Sharpe ratio portfolios, though the benefits plateau beyond moderate regularization strengths.

4.4 Impact of L2 Regularization on the Minimum Variance Portfolio

This section analyses the impact of L2 regularization strength (λ) on the performance of the minimum variance portfolio, again assessed through the four key performance metrics: annualized return, annualized volatility, Sharpe ratio, and average daily VaR.

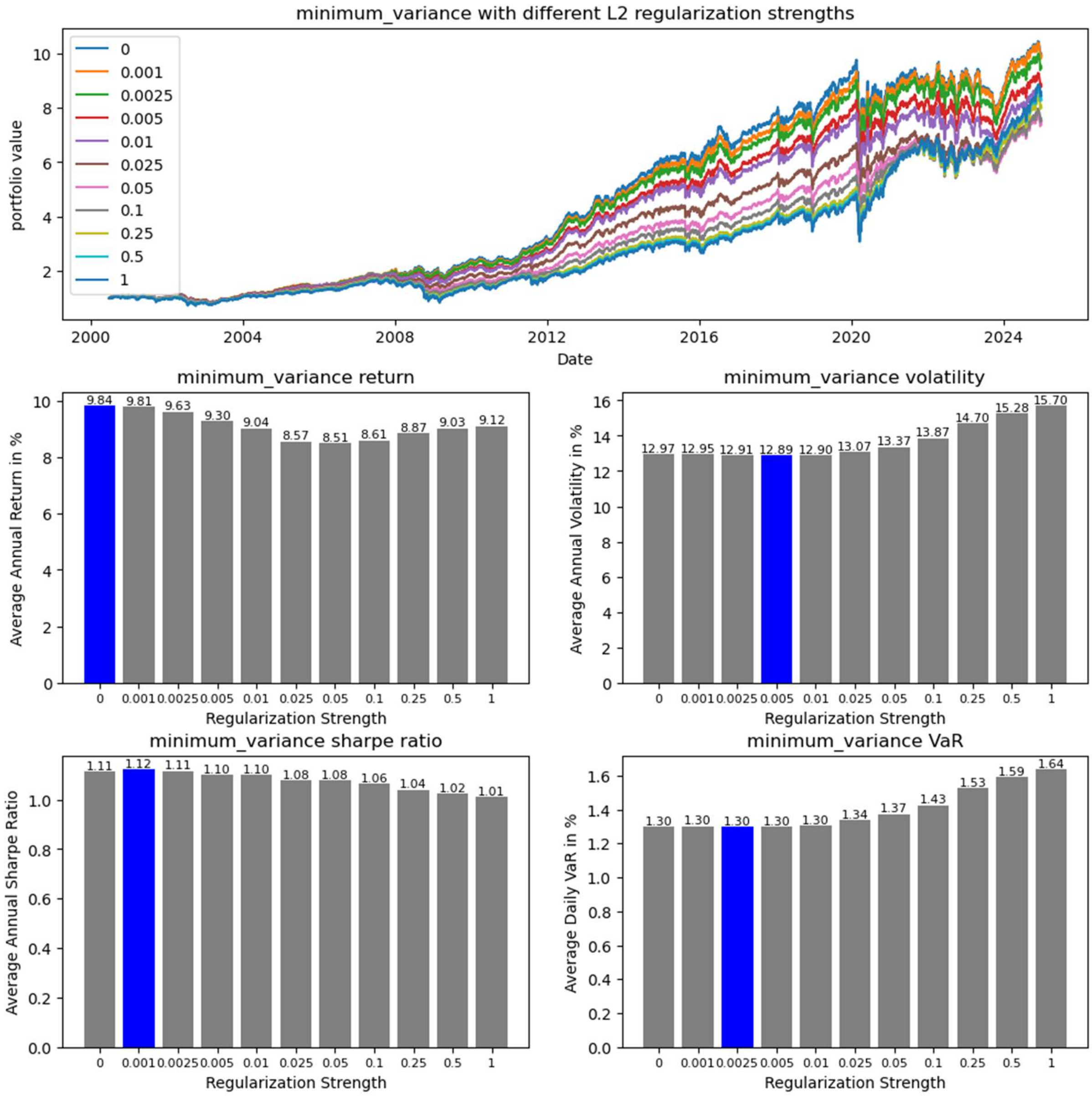

Figure 17: Influence of L2 regularization on the MVP

The results shown in figure 17 indicate a non-linear relationship between L2 regularization strength and portfolio returns. The benchmark portfolio, constructed without any L2 regularization (λ = 0), achieves the highest average annual return of 9.84%, thereby outperforming all regularized portfolios. Introducing weak regularization (λ = 0.001) leads to a marginal decline in performance, with the average annual return decreasing to 9.81%. This downward trend continues, and the return reaches its lowest point of 8.57% at λ = 0.05. Interestingly, further increasing the regularization strength beyond this threshold results in a recovery of portfolio returns, though none of the portfolios with λ > 0 surpass the performance of the unregularized benchmark. These findings suggest that while moderate levels of regularization may reduce overfitting, they can also impair the return-generating potential of minimum variance portfolios by excessively penalizing portfolio weights.

With respect to risk, as measured by annualized volatility, the addition of weak L2 regularization initially proves beneficial. The minimum variance portfolio with λ = 0.005 achieves the lowest volatility of 12.89%, representing a reduction of 0.08 percentage points relative to the benchmark portfolio. Similar but smaller volatility reductions are observed for λ = 0.001 and λ = 0.0025. However, for regularization strengths λ ≥ 0.01, volatility begins to increase monotonically. This trend is well-described by a linear regression model with a coefficient of determination $R^2 = 0.844$ and of form:

$$\sigma(\lambda) = 13.1457 + 3.0768 * \lambda$$

suggesting a strong linear relationship between L2 penalty strength and portfolio volatility. The model estimates that each unit increase in λ results in a 3.0768 percentage point increase in volatility, indicating that higher regularization strengths diminish the portfolio's risk-reduction effectiveness.

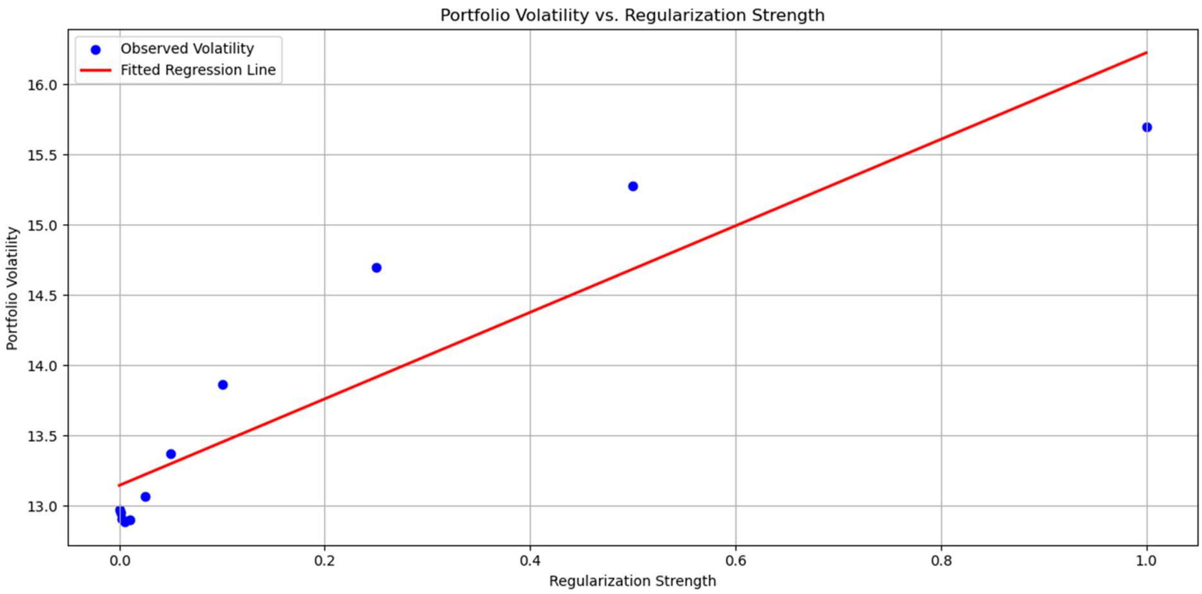

Figure 18: Influence of L2 regularization on the volatility of the MVP

The Sharpe ratio, which captures risk-adjusted returns, initially benefits slightly from weak regularization. Specifically, the portfolio with λ = 0.001 exhibits a marginal improvement in Sharpe ratio relative to the unregularized benchmark. However, stronger regularization (λ ≥ 0.005) leads to a consistent decline in Sharpe ratio. This deterioration is attributed to the combined effect of decreasing returns and increasing volatility, both of which are unfavourably impacted by higher regularization strengths. The inverse relationship between λ and the Sharpe ratio is captured by a linear regression model with $R^2 = 0.731$, which estimates a decline in Sharpe ratio of −0.1049 per unit increase in λ and can be expressed as:

$$Sharpe\ Ratio(\lambda) = 1.0954 - 0.1049 * \lambda$$

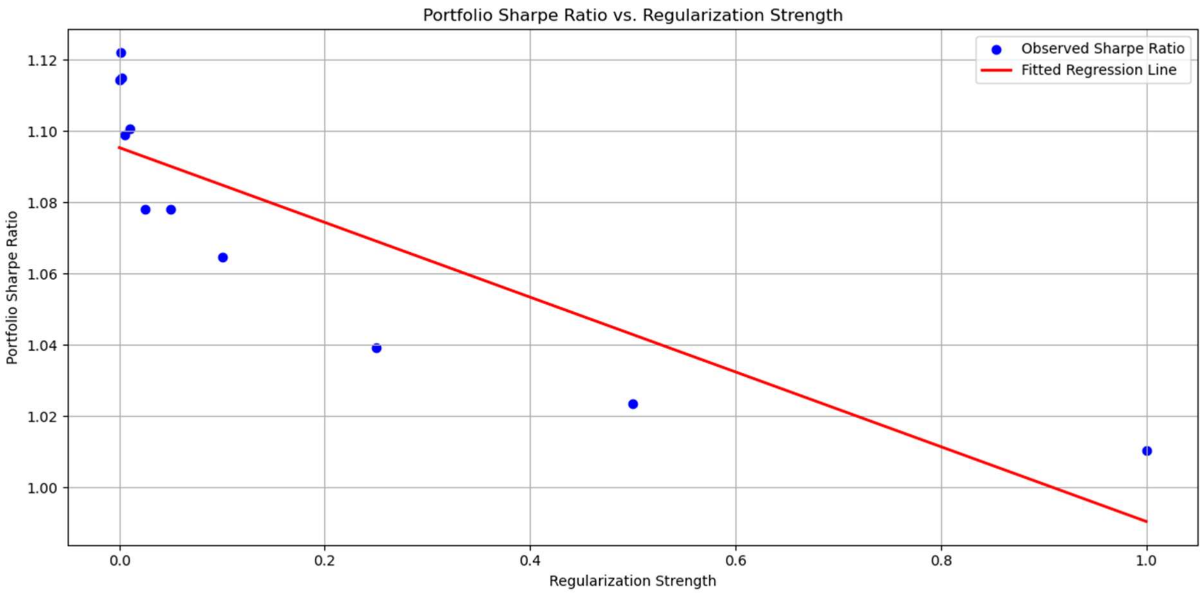

Figure 19: Influence of L2 regularization on the Sharpe ratio of the MVP

Lastly, the analysis of average daily VaR reveals that L2 regularization has a negligible effect on downside risk for small regularization strengths. Portfolios with $\lambda \leq 0.01$ exhibit an average daily VaR of approximately 1.30%, closely matching that of the benchmark portfolio. However, for $\lambda > 0.01$, VaR gradually increases, reflecting a deterioration in tail risk protection. This suggests that higher L2 penalties may inadvertently allocate weight to riskier assets or diminish diversification, thereby increasing exposure to extreme losses.

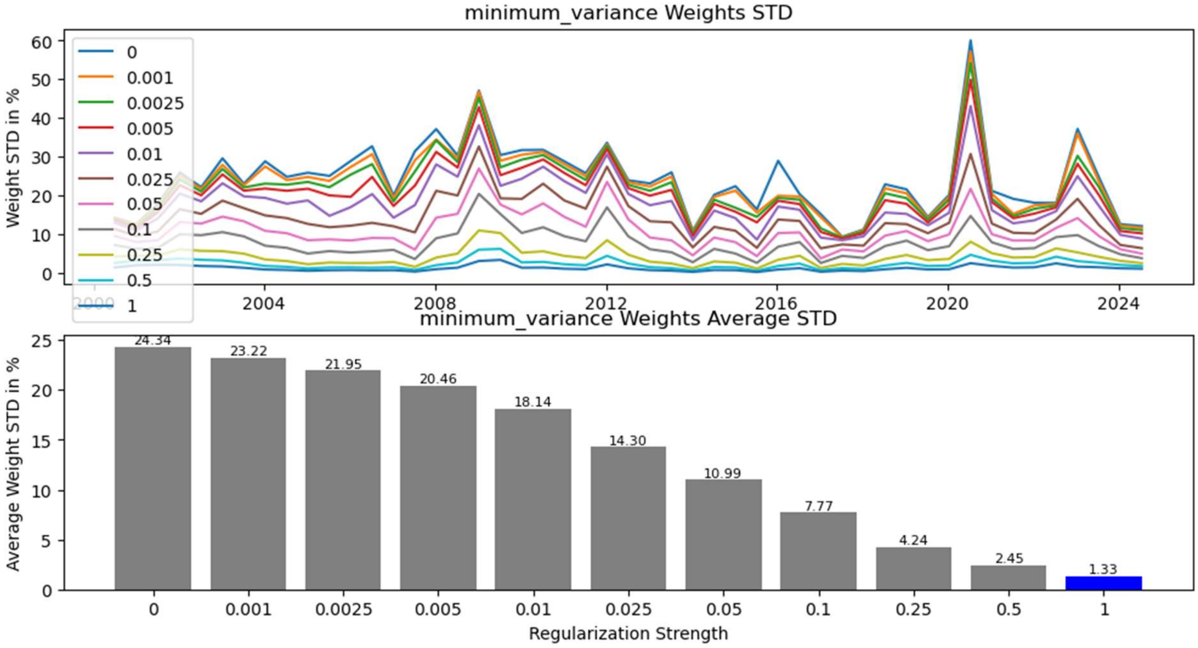

Figure 20: Influence of L2 regularization on the asset weights STD of the MVP

An additional perspective on the impact of L2 regularization emerges when examining the evolution and variability of portfolio weights over time. The behaviour of asset weights in the minimum variance portfolio under varying levels of L2 regularization closely mirrors patterns observed under L1 regularization. In both cases, pronounced spikes in portfolio weights appear during periods of heightened market stress, notably during the COVID-19 pandemic, suggesting a common sensitivity of optimization-based portfolios to financial turmoil. However, the stabilizing effect of regularization becomes more pronounced as the strength increases. In particular, L2 regularization appears to exert a slightly stronger dampening effect on weight volatility compared to L1 regularization when the same regularization strength is applied. This finding indicates that L2 regularization more effectively penalizes extreme weight shifts, promoting smoother portfolio dynamics.

This interpretation is substantiated by the lower panel of the analysis, which presents the average standard deviation of asset weights across the full sample period. The results reveal a strong inverse relationship between regularization strength and weight variability, with the unregularized portfolio exhibiting the highest average standard deviation of 24.34%, while the most heavily regularized portfolio ($\lambda = 1$) shows a markedly lower value of 1.33%. This decline follows an approximately logarithmic transformed linear trend, which can be described by the equation:

$$\sigma_{weight}(\lambda) = 5.1915 - 1.9309 * log(\lambda)$$

with an associated coefficient of determination $R^2 = 0.733$, indicating moderate to strong explanatory power.

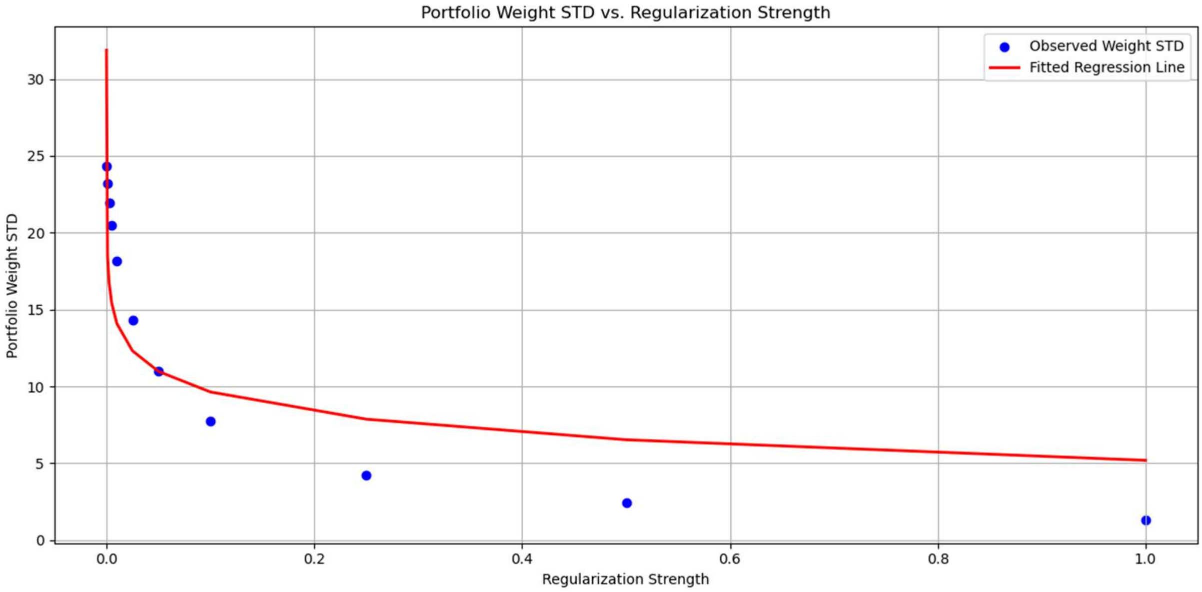

Figure 21: Influence of L2 regularization on the asset weights STD of the MVP OLS model

This systematic reduction in weight volatility underscores the effectiveness of L2 regularization in enhancing temporal stability in portfolio allocations, thereby mitigating the sensitivity of optimized portfolios to estimation error and market fluctuations.

The application of L2 regularization to the minimum variance portfolio reveals a clear trade-off between performance and stability. While the unregularized portfolio achieves the highest average annual return of 9.84%, the introduction of weak regularization ($\lambda = 0.001$) results in a slight reduction in returns, with further declines observed up to $\lambda = 0.05$. Although returns begin to recover beyond this point, they never surpass those of the benchmark portfolio. In terms of risk, weak regularization initially reduces portfolio volatility, with the lowest annualized volatility of 12.89% occurring at $\lambda = 0.005$. However, stronger regularization leads to a steady increase in volatility, a relationship well captured by a linear model with strong explanatory power ($R^2 = 0.844$). The Sharpe ratio follows a similar pattern, showing marginal improvement under weak regularization but declining with higher values of λ due to the combined effect of reduced returns and elevated risk. Furthermore, VaR remains stable for small regularization strengths but begins to rise once λ exceeds 0.01, indicating increased downside risk at higher penalty levels.

In addition to performance metrics, the analysis of asset weight dynamics highlights the stabilizing effect of L2 regularization. Similar to L1 regularization, weight volatility exhibits noticeable spikes during periods of financial stress, such as the COVID-19 pandemic. However, L2 regularization exerts a slightly stronger dampening effect on weight variability for the same level of regularization strength. This is quantitatively supported by a linear relationship between regularization strength and the average standard deviation of asset weights, with values declining from 24.34% ($\lambda = 0$) to 1.33% ($\lambda = 1$). The regression model explaining this trend ($R^2 = 0.582$) underscores the capacity of L2 penalization to enhance the temporal stability of portfolio allocations. Overall, these findings demonstrate that while L2 regularization may modestly reduce return performance, it provides significant benefits in terms of risk control and weight stability, particularly under volatile market conditions.

4.5 Impact of L2 Regularization on the Maximum Sharpe Ratio Portfolio

This section analyses the effect of varying L2 regularization strengths on the performance of the maximum Sharpe ratio portfolio. The analysis is based on four key performance metrics: annualized return, annualized volatility, annualized Sharpe ratio, and average daily VaR. Regularization strengths λ range from 0 (no regularization) to 200.

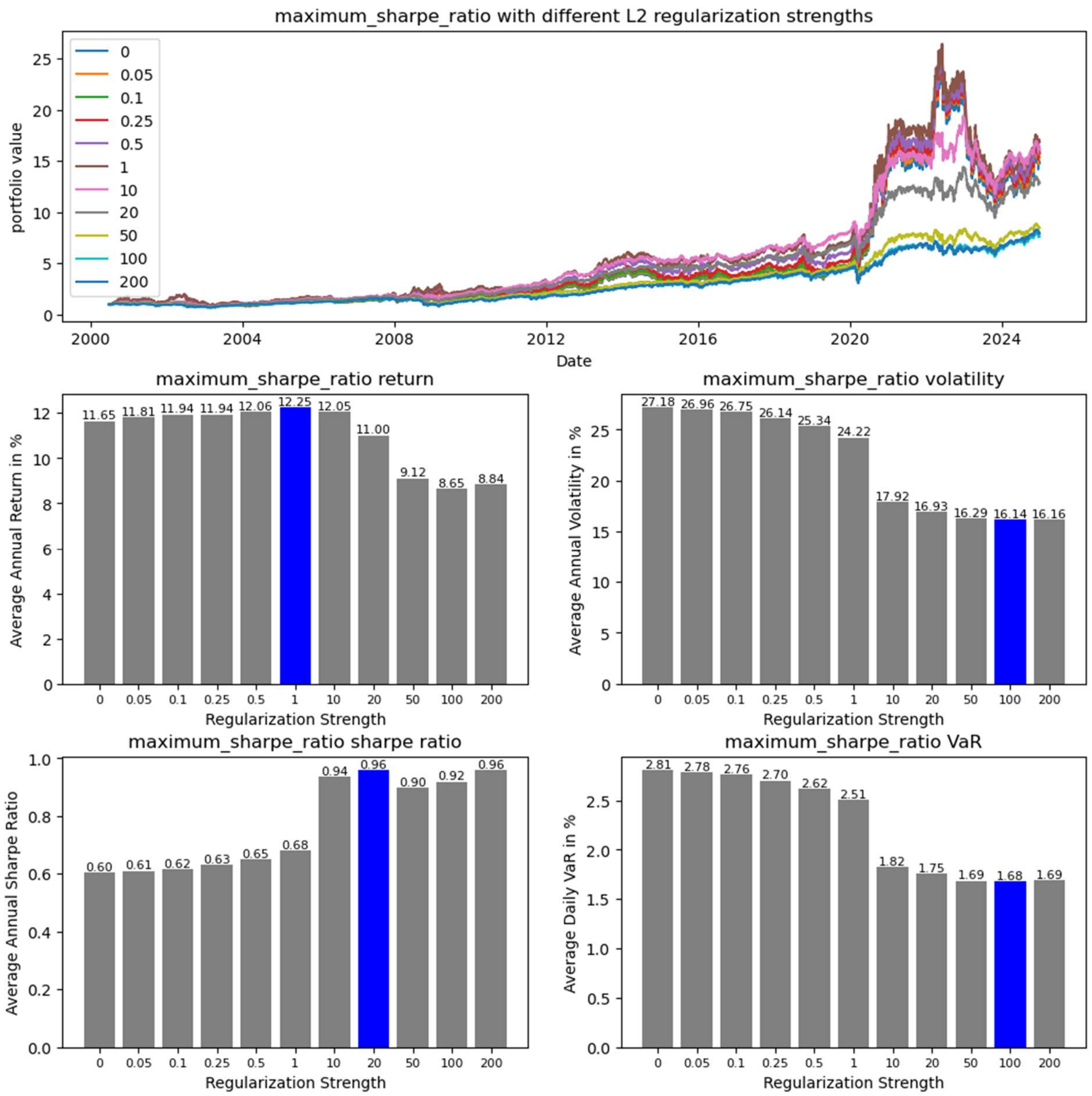

Figure 22: Influence of L2 regularization on the MSRP

The average annual return exhibits a clear non-linear relationship with increasing regularization strength. The highest average annual return of 12.25% is observed at $\lambda = 1$, outperforming the unregularized benchmark portfolio (11.65%) by 0.6 percentage points. Across the lower range of regularization strengths $\lambda \leq 1$, there is a consistent upward trend in performance, indicating that the inclusion of moderate L2 penalties encourages more robust portfolio estimates by mitigating the effects of estimation error in the covariance and expected returns. However, for $\lambda \geq 10$, a reversal in trend emerges. Returns gradually decline as regularization strength increases, forming a U-shaped pattern. This suggests that while mild L2 regularization can enhance portfolio performance through better generalization, excessive penalization dampens the optimization of the risk-return trade-off, resulting in suboptimal portfolios.

The effect of L2 regularization on portfolio volatility is more systematic. Volatility monotonically decreases as λ increases, reaching its minimum at $\lambda = 100$ with a value of 16.14%, which is 11.04 percentage points lower than the volatility of the unregularized benchmark. This reduction reflects the stabilizing influence of ridge regularization, which suppresses extreme weight allocations that can introduce excessive risk.

The relationship between volatility and regularization strength can be approximated by a linear regression model:

$$\sigma(\lambda) = 23.6618 - 0.0530 * \lambda$$

with $R^2 = 0.447$, indicating a moderate explanatory power. The estimated slope implies that each unit increase in λ reduces volatility by approximately 0.0530 percentage points.

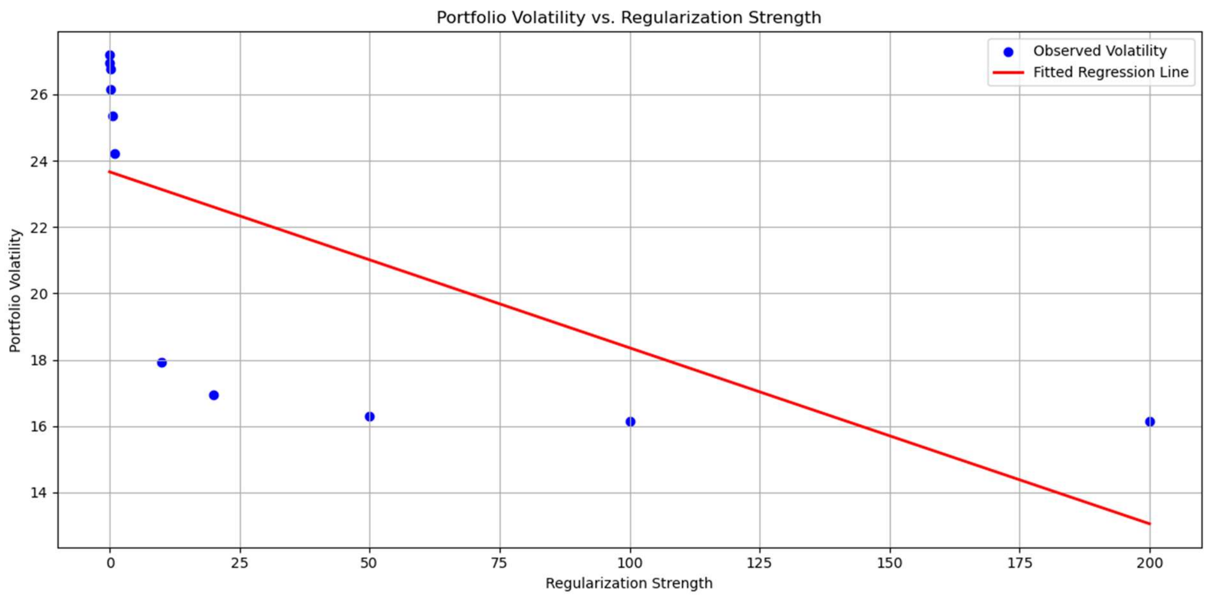

Figure 23: Influence of L2 regularization on the volatility of the MSRP

Given the concurrent rise in returns up to $\lambda = 1$ and decrease in volatility, the Sharpe ratio improves with increasing regularization strength, reaching a maximum of 0.96 at $\lambda = 20$. This constitutes an increase of 0.36 points over the benchmark Sharpe ratio of 0.60. Beyond $\lambda = 20$, the Sharpe ratio exhibits non-monotonic behaviour, with minor fluctuations indicative of irregularities likely driven by interaction effects between declining returns and volatility stabilization. A linear regression model of the Sharpe ratio on regularization strength captures the trend:

$$Sharpe\ Ratio(\lambda) = 0.7132 + 0.0016 * \lambda$$

with an $R^2 = 0.408$, again suggesting moderate explanatory power. The estimated coefficient indicates a Sharpe ratio increase of 0.0016 per unit increase in λ.

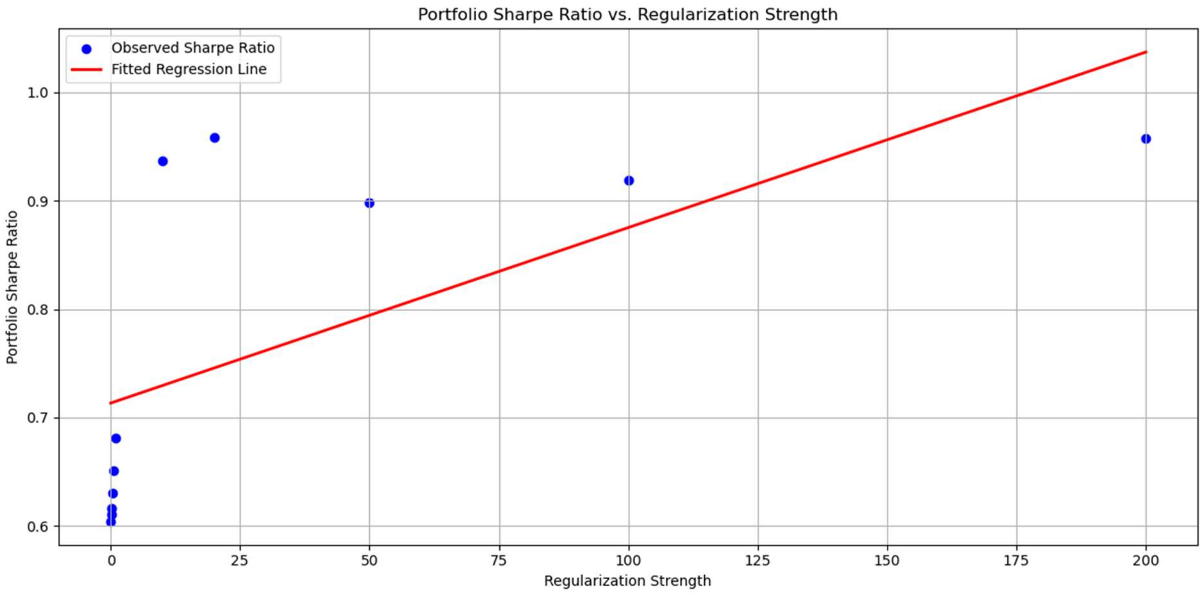

Figure 24: Influence of L2 regularization on the volatility of the MSRP

The daily VaR metric reflects similar behaviour to that observed under L1 regularization. As regularization strength increases, the average daily VaR decreases, reaching a minimum of 1.68 at $\lambda = 100$, which is 1.13 percentage points lower than the benchmark VaR. This supports the conclusion that regularization not only stabilizes portfolios but also reduces downside risk. The lowest VaR is observed in conjunction with the lowest volatility, reinforcing the interpretation that L2 regularization promotes more risk-averse portfolios by implicitly shrinking portfolio weights and suppressing leverage. At $\lambda = 100$, VaR increases slightly, indicating that overly restrictive regularization may undermine risk reduction benefits.

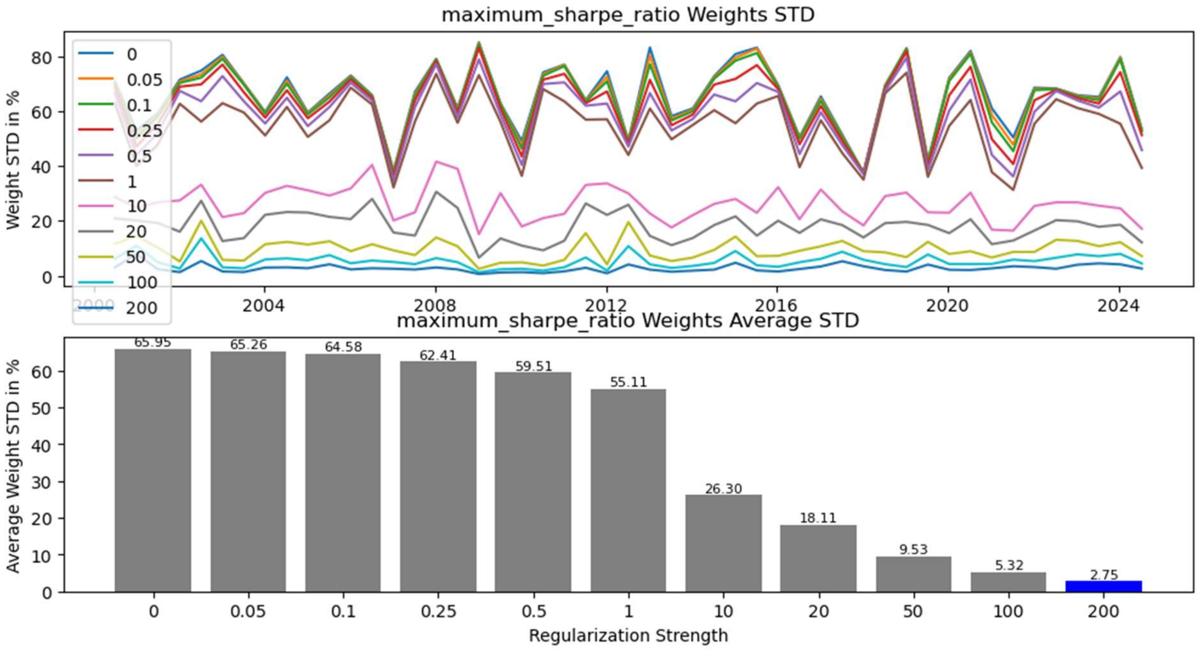

Figure 25: Influence of L2 regularization on the asset weights STD of the MSRP

The analysis of the standard deviation of the portfolio weights for the maximum Sharpe ratio portfolio with varying L2 regularization strengths reveals patterns consistent with those previously observed under L1 regularization. As theoretically expected, increasing the regularization strength leads to a marked reduction in the volatility of the asset weights. For low levels of regularization, weight standard deviations hover around 60%, indicating substantial reallocation between rebalancing periods. In contrast, portfolios with strong L2 penalties (λ ≥ 10) exhibit significantly lower weight volatility, underscoring the stabilizing effect of regularization. This reduction in allocation variability is a key contributor to the observed decline in overall portfolio volatility under stronger penalization.

A logarithmic transformed linear regression model of form can again describe this effect:

$$\sigma_{weight}(\lambda) = 38.7961 - 3.8800 * log(\lambda)$$

having moderate explanatory power with an associated R^2 value of 0.597.

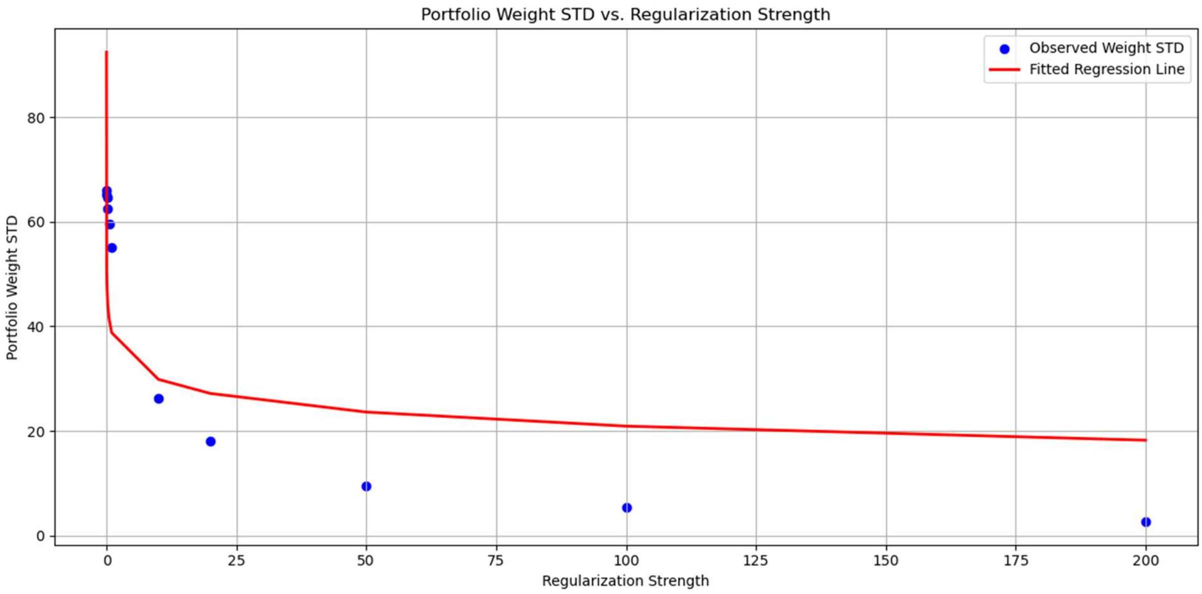

Figure 26: Influence of L2 regularization on the asset weights STD of the MSRP OLS model

Notably, while the characteristic spikes in weight volatility of the minimum variance portfolio during episodes of financial stress persist, additional spikes emerge across all levels of regularization in the maximum Sharpe ratio portfolio. This suggests that, despite the dampening effect of regularization, the maximum Sharpe ratio portfolio inherently exhibits greater sensitivity to market conditions and a higher tendency to reallocate aggressively at each rebalancing point, thereby displaying more structural instability compared to its minimum variance counterpart.

The analysis shows that strong L2 regularization ($\lambda \geq 0.1$) enhances the stability and risk profile of the maximum Sharpe ratio portfolio. Moderate strengths ($\lambda \approx 0.1$–1) improve performance by mitigating estimation error, yielding higher returns and Sharpe ratios. However, beyond $\lambda \geq 10$, further penalization leads to declining returns and reduced efficiency in the risk-return trade-off. Volatility and VaR consistently decrease with increasing λ, reflecting the dampening effect of regularization on risk and extreme allocations. As λ increases, the portfolio increasingly resembles the naively diversified portfolio due to the shrinkage of weights toward uniformity, a characteristic feature of L2 regularization. While strong regularization reduces weight volatility and enhances robustness, it must be calibrated carefully to avoid overly conservative allocations and loss of performance.

4.6 Comparative Analysis of L1 and L2 Regularization Effects on the Minimum Variance Portfolio

This section presents a comparative analysis of the empirical effects of L1 and L2 regularization on the minimum variance portfolio strategy. The analysis is based on key performance metrics, including average annual return, annualized volatility, annualized Sharpe ratio, average daily VaR, and portfolio weight stability, which is measured by the standard deviation of asset weights. Both L1 and L2 regularization techniques exhibit several shared effects when applied to the minimum variance portfolio:

1. Enhancement of portfolio weight stability: One of the primary benefits of both regularization methods is the enhancement of weight stability, leading to lower portfolio turnover and reduced rebalancing frequency. As seen in figures 11 and 20, there is a strong inverse relationship between the regularization strength (λ) and the standard deviation of portfolio weights. Increasing λ results in a consistent decrease in weight volatility for both L1 and L2, indicating more stable portfolio compositions over time.
2. U-shaped volatility response: The impact on portfolio volatility follows a similar U-shaped pattern for both L1 and L2 regularization (figures 9 and 117. For small values of λ (approximately 0.001 to 0.005), there is a slight decrease in volatility relative to the unregularized portfolio ($\lambda = 0$), suggesting some risk stabilization. However, as λ increases beyond a certain threshold ($\lambda \approx 0.01$), both methods result in higher portfolio volatility, indicating that excessive regularization impairs the portfolio's diversification benefits.
3. VaR pattern: The average daily 95% VaR responds similarly to volatility for both L1 and L2 regularization (figures 9 and 17). For $\lambda \leq 0.01$, there is minimal impact on downside risk. However, for $\lambda > 0.01$, VaR increases, suggesting that stronger regularization deteriorates tail risk protection, a pattern that aligns with the observed rise in volatility.
4. Limited performance enhancement: Both L1 and L2 regularization provide only marginal improvements in the core risk-return profile of the minimum variance portfolio. Any improvements in terms of volatility reduction or Sharpe ratio enhancement are observed only at very small values of λ ($\lambda \leq 0.005$).

Despite these similarities, there are several important differences between the effects of L1 and L2 regularization:

1. Impact on average annual return: A notable distinction is observed in the effect on average annual return. L1 regularization demonstrates a non-monotonic effect, with a slight peak in return at λ = 0.001, similar to the unregularized portfolio, followed by a gradual decline as λ increases (figure 9). In contrast, L2 regularization results in an immediate, monotonic decline in returns as λ increases from zero, reaching a minimum at λ = 0.05 before showing a slight recovery, although it never surpasses the unregularized portfolio's return (figure 17). This suggests that L2 penalization harms return-generating potential more significantly than L1 from the outset.
2. Impact on Sharpe ratio: Reflecting the differential effects on returns, L1 regularization shows a slightly greater potential for Sharpe ratio improvement (reaching a peak of 1.13 at λ = 0.001/0.0025) compared to the unregularized portfolio's Sharpe ratio of 1.11. On the other hand, L2 regularization provides only a small improvement at λ = 0.001, followed by a more consistent decline in Sharpe ratio for λ ≥ 0.005 (Figure 17). L2, therefore, seems to degrade risk-adjusted performance more systematically than L1 for the minimum variance portfolio.
3. Magnitude of volatility increase: While both regularization methods lead to an increase in volatility beyond a certain λ threshold, the rate of increase appears steeper under L2 regularization (regression slope ≈ 3.08) compared to L1 (slope ≈ 1.60). This suggests that L2's shrinkage mechanism has a more pronounced negative impact on diversification as regularization strength increases, leading to faster risk escalation.
4. Magnitude of weight stabilization: L2 regularization generally achieves more substantial weight stabilization at higher values of λ. For example, at λ = 1, the average standard deviation of asset weights is lower under L2 (1.33%) compared to L1 (5.79%). The linear regression model also reveals that the rate of weight stabilization is steeper for L2 (-21.74%) than for L1 (-18.55%), indicating that L2 tends to shrink all asset weights towards zero, while L1 results in some weights being exactly zero.
5. Underlying mechanism: The key difference between L1 and L2 regularization lies in the structure of their penalties. L1 regularization uses an absolute value penalty, inducing sparsity by setting some asset weights to zero. In contrast, L2 regularization applies a squared penalty, leading to weight shrinkage without eliminating assets entirely. This difference in penalty structure is likely the cause of the observed disparities in return profiles, Sharpe ratios, volatility, and weight stability.

In summary, both L1 and L2 regularization techniques contribute to the stability of minimum variance portfolio weights, effectively reducing portfolio turnover. However, they offer limited improvements in terms of risk-adjusted performance or volatility, with benefits being most apparent only at very small regularization strengths. Beyond a certain threshold, both methods lead to increased volatility and VaR, alongside a decrease in Sharpe ratio. The primary difference between L1 and L2 lies in their impact on returns, with L2 consistently underperforming the unregularized portfolio, while L1 shows marginal potential improvement at low penalization levels. Additionally, L2 induces greater weight stability at the cost of a steeper rise in volatility at higher regularization strengths. Ultimately, the choice between L1 and L2 for stabilizing minimum variance portfolios involves a trade-off between the desired level of performance degradation and the degree of weight stability, influenced by the distinct mechanisms of sparsity (L1) and shrinkage (L2).

4.7 Comparative Analysis of L1 and L2 Regularization Effects on the Maximum Sharpe Ratio Portfolio

This section provides a comparative analysis of the empirical effects of L1 and L2 regularization on the maximum Sharpe ratio portfolio. The evaluation is again based on the key portfolio performance metrics, including average annual return, annualized volatility, annualized Sharpe ratio, average daily VaR, and portfolio weight stability. Regularization strengths (λ) up to 200 are considered to fully capture the non-linear effects of penalization.

Both L1 and L2 regularization yield substantial and qualitatively similar benefits when applied to the maximum Sharpe ratio portfolio optimization problem:

1. Significant risk reduction: A pronounced and largely monotonic decline in annualized volatility and average daily VaR is observed as λ increases for both regularization methods (figures 13 and 22). Starting from a highly volatile and risk-prone baseline (Volatility ≈ 27.18%, VaR ≈ 2.81%), both L1 and L2 regularization achieve meaningful reductions in risk, converging toward similar minimum levels (volatility ≈ 16.1–16.7%, VaR ≈ 1.7%). This demonstrates their effectiveness in mitigating estimation error-induced instability commonly associated with the classical maximum Sharpe ratio portfolio.
2. Improvement in Sharpe ratio: Both regularization techniques lead to notable improvements in the annualized Sharpe ratio compared to the unregularized baseline (0.60). This improvement follows a non-linear pattern, initial increases in λ enhance

risk-adjusted returns, with a subsequent plateau or decline beyond optimal regularization levels (figures 13 and 22). This reinforces the utility of regularization in enhancing out-of-sample performance for maximum Sharpe ratio portfolios.

3. Enhanced portfolio weight stability: As regularization strength increases, both L1 and L2 methods systematically reduce the standard deviation of asset weights (Figures 12 and 16), indicating improved temporal consistency and lower turnover. Despite this stabilizing effect, the maximum Sharpe ratio portfolios, whether regularized or not, continue to exhibit greater weight fluctuations during periods of market stress compared to minimum variance portfolios, reflecting their inherently more return-oriented objective.
4. Non-linear relationships with λ: The response of performance metrics such as return and Sharpe ratio to increasing λ is distinctly non-linear for both methods. This non-linearity implies the existence of optimal penalization levels and invalidates simplistic interpretations of regularization as uniformly beneficial or harmful.

Despite these shared effects, several important differences emerge regarding the magnitude, nature, and optimal penalization strength of L1 versus L2 regularization:

1. Impact on average annual return: L1 regularization provides a more robust and persistent boost to portfolio returns. The maximum return of 12.76% occurs at λ = 10, compared to the unregularized benchmark of 11.65%. In contrast, L2 regularization achieves a lower peak return of 12.25% at λ = 1, followed by a steady decline below the baseline for λ ≥ 10 (Figures 13 and 22). This suggests that the sparsity introduced by L1 enables more effective identification of return-contributing assets at moderate levels of penalization, whereas L2's uniform shrinkage diminishes return potential more rapidly.
2. Sharpe ratio peaks and optimal λ: The maximum Sharpe ratio achieved under L1 (1.03 at λ = 100) exceeds that of L2 (0.96 at λ = 20), indicating L1's superior ability to enhance risk-adjusted performance. Importantly, the location of these peaks differs substantially, with L1 requiring a much higher λ to attain optimal performance. This suggests that L1's impact on return and volatility dynamics unfolds more gradually but yields superior overall outcomes when stronger penalization is permitted.
3. Risk minimization thresholds: While both regularization methods eventually achieve similar minimum volatility and VaR levels, the λ values required to reach these points differ. L1 achieves minimum volatility around λ = 10 and minimum VaR only at λ =

200, whereas L2 reaches both minimum risk levels at λ ≈ 100. This indicates that L2 delivers its full risk-reduction benefits at lower regularization strengths, while L1 continues to offer marginal tail-risk improvements at higher λ levels.

4. Mechanistic trade-offs and performance limits: The fundamental difference in penalty structure accounts for these divergent effects. L1 regularization, through its sparsity-inducing L1-norm, tends to eliminate irrelevant or unstable assets entirely, thereby concentrating weight on more robust return sources. In contrast, L2 regularization applies a uniform shrinkage to all weights, reducing estimation noise but at the expense of potential return suppression. Consequently, L1 is better suited for extracting signal under conditions of high estimation error, whereas L2 favours a more conservative shrinkage that stabilizes risk at the cost of reduced upside.

In conclusion, both L1 and L2 regularization techniques significantly mitigate the core weaknesses of the traditional maximum Sharpe ratio portfolio, namely its high volatility, poor out-of-sample Sharpe ratio, and unstable asset allocations. They both reduce risk and enhance stability but differ markedly in their impact on returns and the location of optimal λ values. L1 regularization offers superior peak returns and Sharpe ratios, though it requires higher penalization strengths (λ = 10 for peak return; λ = 100 for peak Sharpe ratio). L2 regularization achieves more rapid gains in risk reduction and modest Sharpe ratio improvements at lower λ (λ = 1–20), albeit with a lower ceiling on performance. These results suggest that L1 regularization may be preferable when maximizing risk-adjusted returns is the primary objective, particularly under strong regularization. In contrast, L2 regularization may be more suitable when moderate penalization is sufficient and a slightly lower, but more stable, performance is acceptable.

In general, the traditional maximum Sharpe ratio portfolio exhibits poor out-of-sample performance when evaluated using the Sharpe ratio, particularly in comparison to the minimum variance and naively diversified portfolios, both of which achieve significantly higher Sharpe ratios ex post. This discrepancy becomes especially apparent when contrasted with the in-sample expectation: after calibration, the maximum Sharpe ratio portfolio frequently attains Sharpe ratios exceeding 4, indicating seemingly superior performance. However, the stark contrast between in-sample and out-of-sample results strongly suggests overfitting, a phenomenon commonly encountered in machine learning contexts, where models capture noise rather than signal due to excessive flexibility during calibration. This inference is further substantiated by the high standard deviation of the portfolio weights, indicating unstable and erratic asset allocations that are particularly pronounced compared to the minimum variance portfolio. The introduction of regularization effectively counters this instability. In particular, L2 regularization imposes smoothness on the weight structure, gradually steering the solution towards that of the naively diversified portfolio as the penalization strength increases. This regularized behaviour enhances robustness and reduces sensitivity to input estimation errors, thereby improving out-of-sample reliability without sacrificing interpretability.

4.8 Recommendations for the Use of L1 and L2 Regularization

Based on the empirical findings from the comparative analysis of L1 and L2 regularization applied to both minimum variance and maximum Sharpe Ratio portfolios, this section provides practical guidance on when and how to employ these techniques. The decision to use regularization, as well as the choice between L1 and L2 and the appropriate regularization strength (λ), should be made considering the investor's objectives regarding return, risk, weight stability, transaction costs, and portfolio complexity.

For minimum variance portfolios, regularization is not necessary if the primary goal is to achieve a strong risk-return profile. The unregularized minimum variance portfolio typically already exhibits low volatility and relatively high Sharpe ratios. However, when stability of portfolio weights and minimization of turnover are important considerations, applying weak regularization can be beneficial. In this context, L1 regularization at low levels of λ (approximately between 0.001 and 0.005) appears slightly preferable. It offers some marginal improvements in Sharpe ratio while significantly enhancing the stability of portfolio weights, with only minimal adverse effects on returns or volatility. The sparsity induced by L1 may also help suppress estimation noise without sacrificing overall performance. L2 regularization, applied at similarly weak levels, is also a viable option. It tends to deliver comparable or even slightly greater weight stabilization, but it may result in a quicker onset of return degradation as λ increases. Stronger regularization (with λ greater than 0.01) should generally be avoided in the minimum variance context, as it leads to a consistent deterioration in portfolio performance, increasing volatility and VaR while reducing returns and Sharpe ratios. If transaction costs and turnover are not major concerns, then the classical unregularized minimum variance portfolio remains a robust and effective solution.

In contrast, for maximum Sharpe ratio portfolios, the application of regularization is highly recommended. Empirical evidence clearly shows that the unregularized maximum Sharpe ratio portfolio performs poorly out-of-sample, exhibiting excessive volatility, high downside risk, low realized Sharpe ratios, and unstable portfolio weights. Regularization, whether in the form of L1 or L2, substantially improves upon this baseline across all relevant dimensions. If the primary objective is to maximize risk-adjusted return, L1 regularization is the preferred approach. When λ is set around 100, L1 achieves the highest Sharpe ratio observed in the study (approximately 1.03), while also significantly reducing volatility and VaR. This comes at the cost of a relatively high regularization strength, which induces a sparser portfolio with fewer active positions. Such sparsity may be desirable for investors who value interpretability or seek to avoid over-diversification. L2 regularization, with λ set around 20, offers an effective alternative for investors who prefer to achieve meaningful performance improvements with a more moderate degree of penalization. Although the peak Sharpe ratio achieved by L2 is somewhat lower (approximately 0.96), it is still a substantial improvement over the unregularized baseline. Moreover, L2 maintains exposure to all assets and avoids the sparsity associated with L1, which may be advantageous in contexts where diversification is prioritized.

It is important to note that the optimal regularization strength is data-dependent and sensitive to the specific characteristics of the return and covariance structure. While this study identified $\lambda \approx 100$ for L1 and $\lambda \approx 20$ for L2 as high-performing values, these should be treated as empirical benchmarks rather than universally optimal choices. In practical applications, investors should calibrate λ using techniques such as cross-validation on historical data to align regularization strength with their particular performance objectives and risk preferences.

Beyond the specifics of each portfolio type, a few overarching considerations apply. In markets with high transaction costs, the use of regularization, especially for maximum Sharpe ratio portfolios, can help control turnover and thus reduce implementation costs. From a portfolio management perspective, L1 regularization is particularly well-suited for investors who value sparse, interpretable allocations, while L2 may appeal to those who favour smoother, more broadly diversified portfolios. Ultimately, the choice between L1 and L2 should be informed by the investor’s tolerance for estimation risk, preference for diversification, and performance targets. For aggressive, performance-driven strategies where maximizing the Sharpe ratio is paramount, L1 regularization offers the greatest potential, provided that strong penalization is acceptable. For more conservative strategies where moderate risk reduction and stable performance are prioritized, L2 regularization offers a compelling balance with lower complexity.

5 Limitations

The findings presented in this study are subject to several limitations that constrain the generalizability and robustness of the results. First and foremost, the validity of the results is inherently tied to the specific dataset and the time period under consideration. Both the historical window used to estimate the covariance matrix and the evaluation period for portfolio performance critically influence the outcome of the optimization. Altering these temporal parameters, whether in the context of recalculating portfolio weights or in determining the underlying covariance structure, could yield substantially different results, thereby limiting the extrapolation of the conclusions to other market environments or data regimes.

Furthermore, the optimization process relies on the SLSQP algorithm, which, while widely used, is an iterative optimizer and therefore susceptible to convergence at local minima. This characteristic implies that the solutions obtained for both the minimum variance and the maximum Sharpe ratio portfolios may not represent the true global optima. Consequently, the resulting portfolio allocations might not reflect the most efficient or stable configurations attainable under the specified regularization schemes.

An additional limitation arises from the methodological choice to employ a fixed grid of regularization strengths without performing a systematic hyperparameter tuning procedure. The absence of such tuning introduces the possibility that more optimal regularization parameters, those lying outside or between the points of the predefined grid, may exist and could potentially lead to improved portfolio stability or performance. This constraint reduces the flexibility of the regularization approach and may have impeded the discovery of more effective penalization levels, particularly in cases where the sensitivity of the optimization problem to the regularization strength is pronounced.

Collectively, these limitations underscore the importance of cautious interpretation of the results and suggest that further research is warranted to explore the stability and performance of regularized Markowitz portfolios under alternative data samples, optimization techniques, and tuning methodologies.

6 Conclusion

The empirical findings of this study show that introducing L1 and L2 regularization into the Markowitz optimization framework has a substantial impact on both the minimum variance and maximum Sharpe ratio portfolios. The effects, however, differ in strength and character depending on the portfolio strategy and the type and level of regularization applied.

For the minimum variance portfolio, regularization was not essential to achieve strong risk-adjusted performance. However, both L1 and L2 penalties improved the temporal stability of portfolio weights, with L2 having a slightly stronger smoothing effect. This stability benefit is particularly relevant for reducing turnover and implementation costs. Weak regularization proved most effective yielding improved weight stability without degrading returns or risk-adjusted performance. In contrast, strong regularization led to a decline in portfolio efficiency, with reduced returns and increased volatility.

For the maximum Sharpe ratio portfolio, regularization was found to be crucial. The unregularized version suffered from extreme weight sensitivity and poor out-of-sample Sharpe ratios. L1 regularization, particularly at moderate to strong levels, significantly improved the portfolio’s return, volatility profile, and downside risk. It also enhanced weight stability and sparsity, making the strategy more robust and interpretable. L2 regularization offered similar benefits in terms of reduced volatility and smoother allocations, but its impact on returns and Sharpe ratios was generally weaker compared to L1.

Overall, regularization is highly beneficial, especially for the maximum Sharpe ratio portfolio. For this strategy, L1 regularization is preferred when return optimization and sparsity are the main goals, while L2 may be suitable when smoother diversification is desired. For the minimum variance portfolio, weak regularization, either L1 or L2, can be helpful to improve weight stability without compromising performance, but strong penalization should be avoided.

These results highlight the practical value of regularization in portfolio construction and support its use as a tool to mitigate the well-known instabilities of classical Markowitz optimization. The choice between L1 and L2, and the level of regularization, should ultimately be guided by the investor’s objectives and risk tolerance.

7 Further Research Potential

Building on the limitations defined by the scope of this study and the empirical results obtained, several avenues for future research emerge that could significantly extend and deepen the understanding of regularization in portfolio optimization.

One of the most immediate directions for further investigation lies in the calibration of regularization parameters. This study employed a fixed grid of penalty strengths for both L1 and L2 regularization, allowing for the identification of general performance patterns. However, no attempt was made to determine the optimal value of the regularization parameter λ for a given strategy or market condition. Future research could focus on data-driven optimization of λ using techniques such as cross-validation, Bayesian optimization, or other model selection frameworks. These approaches would allow for a dynamic adjustment of penalization strength based on predictive accuracy or out-of-sample robustness and could improve both risk-adjusted returns and implementation efficiency.

Moreover, while this study evaluated portfolio risk using volatility and VaR, these measures do not fully capture the tail behaviour or asymmetry of return distributions. A promising extension would involve analysing the impact of regularization under alternative risk measures, such as CVaR, downside deviation, or drawdown-based metrics. Such risk metrics are increasingly relevant in regulatory and institutional investment settings and may reveal additional dimensions of how L1 and L2 penalties influence portfolio construction under more refined risk objectives.

In addition, this study relied on the sample covariance matrix for portfolio optimization, which is known to be highly sensitive to estimation error in high-dimensional settings. Future research could explore the interaction between regularization and advanced estimation techniques, such as shrinkage estimators (e.g., Ledoit-Wolf), Bayesian frameworks, or robust optimization methods. These approaches offer complementary mechanisms to improve estimation stability and may further amplify the benefits of regularization when used in tandem.

Another important extension involves the explicit modelling of transaction costs. While this study indirectly accounted for implementability through the volatility of portfolio weights, actual transaction costs were not incorporated into the optimization process. Future research could incorporate cost-aware objective functions or turnover constraints to evaluate how regularization affects net returns under realistic market frictions. This is particularly relevant for institutional investors and high-frequency strategies where trading costs materially affect portfolio outcomes.

A more nuanced examination of the effectiveness of regularization under different market regimes also represents a valuable area for exploration. Although this study spans a broad historical period encompassing multiple cycles of boom and bust, it does not distinguish between phases such as bull markets, bear markets, or high-volatility environments. Future work could investigate how the optimal regularization strength and the relative efficacy of L1 versus L2 vary across such regimes, potentially leading to regime-adaptive regularization frameworks.

Lastly, the analysis revealed several non-linear relationships between the strength of regularization and performance metrics, particularly for the maximum Sharpe ratio portfolio. These findings highlight the need for deeper theoretical modelling and empirical testing of non-linear effects, including the potential role of interaction terms, saturation thresholds, and regime-dependent sensitivities. A more granular understanding of these dynamics could inform the development of regularization schemes that adapt to structural breaks or changes in market complexity.

In sum, while this study confirms the stabilizing and performance-enhancing potential of L1 and L2 regularization, numerous extensions remain to be explored. These future research directions promise to refine the theoretical foundations, broaden the practical relevance, and enhance the robustness of regularized portfolio optimization in increasingly complex and uncertain financial environments.

References

AVRAMOV, D., & ZHOU, G. (2010). Bayesian Portfolio analysis. *Annual Review of Financial Economics*, *2*(1), 25–47. https://doi.org/10.1146/annurev-financial-120209-133947

BRODIE, J., DAUBECHIES, I., DE MOL, C., GIANNONE, D., & LORIS, I. (2009). Sparse and stable Markowitz portfolios. *Proceedings of the National Academy of Sciences*, *106*(30), 12267–12272. https://doi.org/10.1073/pnas.0904287106

CHEN, Y.-C. (2013). *Portfolio resampling on various financial models* (By University of California). https://escholarship.org/content/qt9391c2vj/qt9391c2vj_noSplash_aa406d0489a8e5982bd039bcfc06c2f5.pdf

CORPORATE FINANCE INSTITUTE (2024, May 22). *Efficient Frontier*. Corporate Finance Institute. https://corporatefinanceinstitute.com/resources/career-map/sell-side/capital-markets/efficient-frontier/

CORSARO, S., & DE SIMONE, V. (2019). Adaptive l1-regularization for short-selling control in portfolio selection. *Computational Optimization and Applications*, *72*(2), 457–478. https://doi.org/10.1007/s10589-018-0049-4

CORSARO, S., DE SIMONE, V., MARINO, Z., & PERLA, F. (2020). $$l_1$$-Regularization for multi-period portfolio selection. *Annals of Operations Research*, *294*(1–2), 75–86. https://doi.org/10.1007/s10479-019-03308-w

CARRASCO, M., NOUMON, NÈ., & UNIVERSITÈ DE MONTRÈAL. (2010). *Optimal Portfolio Selection using Regularization*. https://www.eco.uc3m.es/temp/port8.pdf

DEMIGUEL, V., GARLAPPI, L., & UPPAL, R. (2007). Optimal Versus Naive Diversification: How Inefficient is the 1/NPortfolio Strategy? *Review of Financial Studies*, *22*(5), 1915–1953. https://doi.org/10.1093/rfs/hhm075

FAMA-FRENCH DATA (KEN FRENCH'S DATA LIBRARY) - PANDAS-DATAREADER 0.10.0 DOCUMENTATION. (n.d.). https://pandas-datareader.readthedocs.io/en/latest/readers/famafrench.html

FAN, J., ZHANG, J., & YU, K. (2012). Vast portfolio selection with Gross-Exposure constraints. *Journal of the American Statistical Association*, *107*(498), 592–606. https://doi.org/10.1080/01621459.2012.682825

FASTRICH, B., PATERLINI, S., & WINKER, P. (2014). Constructing optimal sparse portfolios using regularization methods. *Computational Management Science*, *12*(3), 417–434. https://doi.org/10.1007/s10287-014-0227-5

FERNANDO, J. (2024, August 4). *Sharpe Ratio: Definition, Formula, and examples*. Investopedia. https://www.investopedia.com/terms/s/sharperatio.asp

GAJENDRAKAR, P. (2022, October 21). *Markowitz Model - What is It, Assumptions, Diagram, Formula*. Retrieved May 2, 2025, from https://www.wallstreetmojo.com/markowitz-model/

HO, M., SUN, Z., & XIN, J. (2015). Weighted Elastic net Penalized Mean-Variance Portfolio design and computation. *arXiv (Cornell University)*. https://doi.org/10.48550/arxiv.1502.01658

IHRIG, J., & WALLER, C. (2024, February 14). *The Federal Reserve's responses to the post-Covid period of high inflation*. https://www.federalreserve.gov/econres/notes/feds-notes/the-federal-reserves-responses-to-the-post-covid-period-of-high-inflation-20240214.html

JAGANNATHAN, R., & MA, T. (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance*, *58*(4), 1651–1683. https://doi.org/10.1111/1540-6261.00580

JIN, Y., QU, R., ATKIN, J., & ASAP GROUP. (2021). Constrained Portfolio Optimisation: the state-of-the-art Markowitz Models. In *ASAP Group, School of Computer Science, the University of Nottingham, Nottingham, UK* [Journal-article]. https://people.cs.nott.ac.uk/pszrq/files/ICORES16mv.pdf

JOSHY, A. J., & HWANG, J. T. (2024). PySLSQP: A transparent Python package for the SLSQP optimization algorithm modernized with utilities for visualization and post-processing. *The Journal of Open Source Software*, *9*(103), 7246. https://doi.org/10.21105/joss.07246

KARANDIKAR, R., & SINHA, T. (2012). Modelling in the spirit of Markowitz portfolio theory in a non-Gaussian world. *Current Science*, *103(6)*, 666–672. https://www.researchgate.net/publication/292563061_Modelling_in_the_spirit_of_arkowitz_portfolio_theory_in_a_non-Gaussian_world

KRISHNAN, S. (2022, April 11). What is Stationarity in Time Series? How it can be detected? | by Sandhya Krishnan | CodeX | Medium | CodeX. *Medium*. https://medium.com/codex/what-is-stationarity-in-time-series-how-it-can-be-detected-7e5dfa7b5f6b

LEDOIT, O., & WOLF, M. (2003). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, *88*(2), 365–411. https://doi.org/10.1016/s0047-259x(03)00096-4

LEDOIT, O., & WOLF, M. (2004). Honey, I shrunk the sample covariance matrix. *The Journal of Portfolio Management*, *30*(4), 110–119. https://doi.org/10.3905/jpm.2004.110

MARKOWITZ, H. (1952). Portfolio selection. *The Journal of Finance*, *7*(1), 77. https://doi.org/10.2307/2975974

PALOMAR, D. P. (2025). *Portfolio Optimization: Theory and Application*. Cambridge University Press. https://portfoliooptimizationbook.com/portfolio-optimization-book.pdf

PETUKHINA, A., KLOCHKOV, Y., HÄRDLE, W. K., & ZHIVOTOVSKIY, N. (2023). Robustifying Markowitz. *Journal of Econometrics*, *239*(2), 105387. https://doi.org/10.1016/j.jeconom.2022.12.006

SHAJI, R. (2022, January 5). Differential evolution | What it is | How it works | Medium. *Medium*. https://medium.com/@reshma_shaji/differential-evolution-what-it-is-and-how-does-it-work-81d3415c2367

THE SVERIGES RIKSBANK PRIZE IN ECONOMIC SCIENCES IN MEMORY OF ALFRED NOBEL 1990 - NOBELPRIZE.ORG. (n.d.). NobelPrize.org. https://www.nobelprize.org/prizes/economic-sciences/1990/summary/

TROTTA, F. (2025, January 16). *Understanding l1 and l2 Regularization*. Towards Data Science. https://towardsdatascience.com/understanding-l1-and-l2-regularization-93918a5ac8d0/

TU, J., & ZHOU, G. (2010). Markowitz meets Talmud: A combination of sophisticated and naive diversification strategies. *Journal of Financial Economics*, *99*(1), 204–215. https://doi.org/10.1016/j.jfineco.2010.08.013

WALAVALKAR, A. (2024, March 12). Mastering Portfolio Optimization with Sharpe Ratio and SLSQP. *Triumph.ai*. https://www.triumphai.in/post/portfolio-optimization-using-sharpe-ratio-and-slsqp

WU, Z., SUN, K., GE, Z., ALLEN-ZHAO, Z., & ZENG, T. (2024). Sparse portfolio optimization via ℓ1 over ℓ2 regularization. *European Journal of Operational Research*, *319*(3), 820–833. https://doi.org/10.1016/j.ejor.2024.07.017

YEN, Y., & YEN, T. (2013). Solving norm constrained portfolio optimization via coordinate-wise descent algorithms. *Computational Statistics & Data Analysis*, *76*, 737–759. https://doi.org/10.1016/j.csda.2013.07.010

ZHANG, N., CHEN, J., & DAI, G. (2021). Portfolio Selection with Regularization. *Asia Pacific Journal of Operational Research*, *39*(02). https://doi.org/10.1142/s0217595921500160

ZHAO, H., KONG, L., & QI, H. (2021). Optimal portfolio selections via l1,2-norm regularization. *Computational Optimization and Applications*, *80*(3), 853–881. https://doi.org/10.1007/s10589-021-00312-4

ZÜRCHER HOCHSCHULE FÜR ANGEWANDTE WISSENSCHAFTEN (2024, April). *Use of generative AI for graded assignments. https://www.zhaw.ch/storage/lsfm/studium/_formulare-merkblaetter/MB-Use-generative-AI-for-graded-assignments.pdf*

8 AI Use and Disclaimer

For transparency, we acknowledge that while the research, analysis as well as the structure of the paper are entirely our own work, we used the below listed AI tools throughout the process to enhance clarity, coherence and flow. These tools were employed to consistency in writing style, improve grammar and organize ideas effectively without altering the substance or original concepts presented in the text. Furthermore, we declare that this paper is our own unaided work. All direct or indirect sources used are acknowledged as references. This paper was not previously presented to another examination board and has not been published.

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Winterthur, 17.05.2025	Zürich, 17.05.2025	Zürich, 17.05.2025

List of AI tools used

OpenAI (2025). ChatGPT (version GPT-4-turbo & GPT-3.5). https://chatgpt.com/, used for: Generation of text proposals and grammar improvement