A Distributed Numerical Approach to Interference Alignment and Applications to Wireless Interference Networks

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Abstract—Recent results establish the optimality of interference alignment to approach the Shannon capacity of interference networks at high SNR. However, the extent to which interference can be aligned over a finite number of signalling dimensions remains unknown. Another important concern for interference alignment schemes is the requirement of global channel knowledge. In this work, we provide examples of iterative algorithms that utilize the reciprocity of wireless networks to achieve interference alignment with only local channel knowledge at each node. These algorithms also provide numerical insights into the feasibility of interference alignment that are not yet available in theory.

Index Terms—Beamforming, capacity of wireless networks, degrees of freedom, interference alignment, interference channels, interference management, interference networks, iterative algorithms, nash equilibrium, relays.

I. INTRODUCTION

THE recent emergence of the idea of interference alignment for wireless networks has shown that the capacity of wireless networks can be much higher than previously believed [1]. The canonical example of interference alignment is a communication scenario where, regardless of the number of interferers, every user is able to access one half of the spectrum free from interference from other users [1]. For the Gaussian interference channel with K interfering transmitter-receiver pairs with each transmitting and receiving node having M antennas each, and with random, time-varying channel coefficients drawn from a continuous distribution, [1] characterizes the network sum capacity as

$$C_\Sigma(SNR) = \frac{KM}{2}\log(SNR) + o(\log(SNR)) \quad (1)$$

so that the capacity per user (i.e., transmitter-receiver pair) is $\frac{M}{2}\log(SNR)+o(\log(SNR))$. Here SNR is defined as the total transmit power of all the transmitters in the network when the local noise power at each receiving node is normalized to unity. The $o(\log(SNR))$ term, by definition, becomes negligible compared to $\log(SNR)$ at high SNR. Therefore, the accuracy of the capacity approximation in (1) approaches 100% at high SNR. Since the capacity of a single user with M antennas at both the transmitter and the receiver, in the absence of all interference is $M\log(SNR) + o(\log(SNR))$, the main result of [1] may be summarized as:

"At high SNR, every user in a wireless interference network is (simultaneously and almost surely) able to achieve approximately (within $o(\log(SNR))$) one half of the capacity that he could achieve in the absence of all interference."

The capacity-optimal achievable scheme within $o(\log(SNR))$ is shown to be the interference alignment scheme. Interference alignment on the K user interference channel refers to the idea of constructing signals in such a way that they cast overlapping shadows over one half of the signal space observed by each receiver where they constitute interference, leaving the other half of the signal space free of interference for the desired signal. This approach reveals that the cake-cutting view of spectrum allocation between co-existing wireless systems is fallacious. This is because the result implies that everyone gets "half the cake", hence defeating the purpose of the cake-cutting perspective.

Interference alignment schemes are presented in [1] in the form of closed form expressions for the transmit precoding matrices. However, these solutions suffer from three significant drawbacks. Firstly, these closed form expressions require global channel knowledge which can be an overwhelming overhead in practice. Secondly, closed form solutions have only been found in certain cases. In general, analytical solutions to interference alignment problem are difficult to obtain and even the feasibility of interference alignment over a limited number of signalling dimensions is an open problem. Thirdly, the results of [1] are meaningful only at high SNR, and to the best of our knowledge, their performance at low/moderate SNR is unknown. In this paper we explore distributed numerical interference alignment algorithms to (partially) address these drawbacks. In particular we design numerical iterative interference alignment algorithms which accomplish the objectives listed below for the K user Gaussian interference channel with arbitrary number of antennas at each transmitter and receiver.

- Require only local channel knowledge at each node. Specifically, each receiver is assumed to know only the channel to its desired transmitter and the covariance matrix of its effective noise (consisting of the AWGN and the interference from all other users).
- Provide numerical insights into the feasibility of alignment, in particular, with limited signalling dimensions.
- Improve the performance of interference alignment algorithms at low/moderate SNR.

Note that unlike [1], the algorithms designed here are presented in the context of the interference channel, where different users can have different number of antennas. Further, unlike in [1], the numerical algorithms do not necessarily require frequency-selectivity. In fact, the algorithms are designed to obtain insights into the limits of interference alignment in such cases which have not been studied in earlier works.

We propose iterative algorithms that take an altruistic approach to interference management and utilize only the local side information available naturally due to the reciprocity of wireless networks. The two key properties can be summarized as follows.

- *Altruistic Principle:* Unlike selfish approaches studied in prior work where each transmitter tries to maximize his own rate by transmitting along those signalling dimensions where his desired receiver sees the least interference, we follow an unselfish approach where each transmitter primarily tries to minimize the interference to unintended receivers. It must be noted that this principle is a cognitive approach, since, in cognitive radios, one of the fundamental goals of design is avoiding interference at the unintended receivers [2], [3]. The altruistic principle is found to lead to interference alignment, and is thus capable of approaching network capacity at high SNR.
- *Reciprocity:* For a given transmitter, learning how much interference is caused at unintended receivers can require too much side information, and is one of the key challenges for cognitive radio systems. However, this information is naturally available because of the reciprocity of the channel for networks where two-way communication is based on time-division duplex operation with synchronized time-slots. Due to reciprocity, the signalling dimensions along which a receiving node sees the least interference from other users are also the same signalling dimensions along which this node will cause the least interference to other nodes in the reciprocal network where all transmitters and receivers switch roles.

We next review some optimization approaches for interference networks in existing literature.

II. INTERFERENCE OPTIMIZATION APPROACHES

The optimality of interference alignment schemes at high SNR is interesting because these schemes treat all interference as noise and require no multiuser detection. Achievable schemes based on treating interference as noise have been explored extensively over the last decade. Prominent among these are the interference avoidance and iterative waterfilling
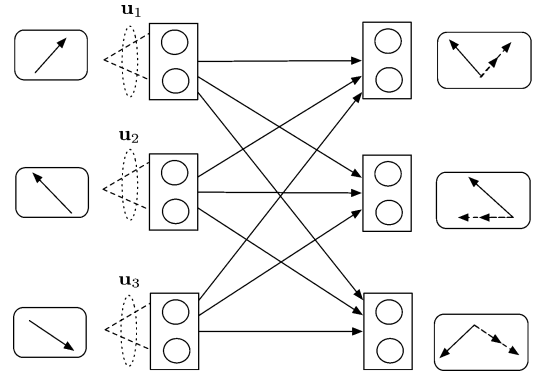

Fig. 1. Interference alignment solution for the three-user interference channel with two antennas at each node. The dashed lines at the receivers indicate the direction of the aligned interference.

algorithms where each transmitter acts selfishly to align its transmissions along those directions where its desired receiver sees the least interference [4]–[9], and network duality approaches [10]–[17] that are based on the reciprocity of the wireless propagation channel.

A. Interference Avoidance and Iterative Waterfilling

Iterative algorithms are commonly used for various resource allocation problems, such as interference avoidance and iterative waterfilling. However, the philosophy of interference alignment is quite distinct from both iterative waterfilling and interference avoidance. With iterative waterfilling/interference avoidance algorithms [4], [18]–[20], each transmitter tries to do what is best for his own receiver, i.e., each transmitter allocates its power in a manner best suited for his desired receiver. With interference alignment each transmitter tries to minimize the interference he causes to other receivers. The interference alignment schemes in [1], [21]–[23] show that for interference networks, the "do no harm" approach is much more powerful, and is in fact capacity-optimal within $o(\log(SNR))$, than the "help yourself" approach of interference avoidance and iterative waterfilling schemes.

From a game-theoretic perspective, interference avoidance and iterative waterfilling algorithms lead to a stable operating point commonly known as the Nash equilibrium. At Nash equilibrium, there is no incentive for any user to unilaterally change his transmit strategy. Interestingly, interference alignment is not a Nash equilibrium point if the goal of each user is to maximize his own rate. Fig. 1 shows the interference alignment solution for the 3 user interference channel, where each user has two antennas. Notice that interfering signals are co-linear at each receiver while the desired signal may not be exactly orthogonal to the interference—a price paid for interference alignment. It can be easily observed that the interference alignment solution is not a Nash equilibrium point. Fixing the transmit strategy for users 2 and 3, the best strategy for user 1 is to choose $\mathbf{u}_1$ such that his signal is orthogonal to the interference at receiver 1. Although this strategy is good for user 1, it will destroy interference alignment at receivers 2 and 3. Thus, Fig. 1 clearly highlights the difference between the optimal strategy of interference alignment and the selfish strategy of iterative waterfilling or interference avoidance. Thus, interestingly, in the case of the K-user interference network with M antennas at each node, the

optimal solution at high SNR is not a Nash equilibrium, indicating the latter's inefficiency in interference networks.

Iterative schemes have also been used to implicitly achieve interference alignment on the 2 user X channel in [24]–[26], [31]. However, for the 2 user X channel interference alignment can be explicitly achieved with roughly the same amount of channel knowledge as required by the iterative schemes, without the need for an iterative process [21]. The iterative schemes of [24] are specialized for the 2 user X channel and generalizations to X networks and interference networks with more than 2 users are not straightforward.

B. Network Duality

Another approach taken in prior work is to exploit the duality relationships enabled by the reciprocity of the propagation channel. For example, network duality ensures that the same set of signal to interference and noise ratios (SINRs) can be achieved in the original and the reciprocal network with the same total transmit power [10], [11]. Network duality is used in [11], [12] to minimize the total transmit power required to support a feasible rate vector. Reciprocity of propagation channels is used in [13] for the optimal frequency allocation problem.

In this work we provide iterative algorithms to achieve interference alignment on wireless interference channels. These algorithms combine elements of all the above-mentioned approaches, especially [24] and [13].

III. SYSTEM MODEL

Consider the K-user MIMO interference channel where the kth transmitter and receiver are equipped with $M^{[k]}$ and $N^{[k]}$ antennas respectively. Note that the antennas could represent symbol extensions in time or frequency as well. However, if the antennas correspond to symbol extensions over orthogonal dimensions (time, frequency slots) then the channel matrices will have a diagonal structure. The channel is defined as:

$$Y^{[k]}(n) = \sum_{l=1}^{K} \mathbf{H}^{[kl]}(n)X^{[l]}(n) + Z^{[k]}(n), \quad \forall k \in \mathcal{K}$$

where, at the nth channel use, $Y^{[k]}(n), Z^{[k]}(n)$ are the $N^{[k]} \times 1$ received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at receiver k, $X^{[l]}(n)$ is the $M^{[l]} \times 1$ signal vector transmitted by transmitter l, and $\mathbf{H}^{[kl]}(n)$ is the $N^{[k]} \times M^{[l]}$ matrix of channel coefficients between transmitter l and receiver k. In the above equation, $\mathcal{K} = \{1, 2, \ldots, K\}$ represents the set of users. The transmit power at transmitter l is $\mathrm{E}[||X^{[l]}||^2] = P^{[l]}$.

For the K user interference channel defined above, we also define a reciprocal channel, where the role of transmitters and receivers are switched. For every variable on the original channel, the corresponding variable on the reciprocal channel is denoted with a left arrow on top. The reciprocal channel is defined as:

$$\overleftarrow{Y}^{[k]}(n) = \sum_{l=1}^{K} \overleftarrow{\mathbf{H}}^{[kl]}(n)\overleftarrow{X}^{[l]}(n) + \overleftarrow{Z}^{[k]}(n), \quad \forall k \in \mathcal{K}$$

where, at the nth channel use, $\overleftarrow{Y}^{[k]}(n), \overleftarrow{Z}^{[k]}(n)$ are the $M^{[k]} \times 1$ received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at receiver k which is equipped with $M^{[k]}$ antennas, $\overleftarrow{X}^{[l]}(n)$ is the $N^{[l]} \times 1$ signal vector transmitted by transmitter l, and $\overleftarrow{\mathbf{H}}^{[kl]}(n) = \mathbf{H}^{[lk]\dagger}(n)$ is the $M^{[k]} \times N^{[l]}$ matrix of channel coefficients between transmitter l and receiver k. The transmit powers at transmitter l on the reciprocal channel is $\mathrm{E}[||\overleftarrow{X}^{[l]}||^2] = \overleftarrow{P}^{[l]}$. The channel use index n is henceforth suppressed to avoid cumbersome notation.

Remark: It must be noted that the reciprocal network is a theoretical apparatus constructed, since it is useful for developing numerical interference alignment algorithms. The reciprocity property of the physical channel is useful from a practical perspective in distributed implementation the algorithms, as discussed later in Section V-B. However, the algorithms can be relevant, for instance in checking the feasibility of alignment (discussed in the next section), even when the physical channels do not satisfy reciprocity.

Notation: We use lower case for scalars, upper case for vectors and bold font to denote matrices. $\mathbf{A}_{\star d}$ represents the dth column of matrix $\mathbf{A}$. $\mathbf{I}_d$ represents the $d \times d$ identity matrix. Similarly, $\mathbf{0}_{M \times N}$ is used to indicate the $M \times N$ zero matrix. $\mathrm{Tr}[\mathbf{A}]$ denotes the trace of the matrix $\mathbf{A}$ and $\mathbf{A}^\dagger$ is the conjugate transpose of matrix $\mathbf{A}$. Finally, $\mathcal{K} \triangleq \{1, 2, \ldots, K\}$ is the index set of K users.

IV. INTERFERENCE ALIGNMENT OVER LIMITED DIMENSIONS—AN OPEN PROBLEM

References [1] and [22] present interference alignment schemes that are constructed over symbol extensions of time-varying channels. It is shown that by using *long* symbol extensions the degrees of freedom achieved per dimension approach arbitrarily close to the theoretical outerbound, thereby establishing the degrees of freedom of time-varying interference and X networks. However, the extent to which interference can be aligned over a *limited* number of dimensions remains an open problem. As a consequence, the maximum number of degrees of freedom that can be achieved through alignment of interference signal vectors is not known in general. Note that interference alignment can also be accomplished in terms of signal *levels* rather than signal vectors by using structured codes (multilevel and lattice codes) as shown in [27], [28]. However, in this work, our focus is on interference alignment through signal vectors. We first review the interference alignment problem and the reciprocity property of interference alignment.

Let $d^{[k]} \leq \min(M^{[k]}, N^{[k]}), k \in \mathcal{K}$ denote the degrees of freedom for user k's message.

Precoding at Transmitter: Let $\mathbf{V}^{[k]}$ be an $M^{[k]} \times d^{[k]}$ matrix whose columns are the orthonormal basis of the transmitted

signal space of user k. Mathematically, the transmitted signal vector of user k is given by

$$X^{[k]} = \sum_{d=1}^{d^{[k]}} \mathbf{V}_{[\star d]}^{[k]} \bar{X}_d^{[k]} = \mathbf{V}^{[k]} \bar{X}^{[k]}, \bar{X}^{[k]} \sim \mathcal{N}\left(0, \frac{P^{[k]}}{d^{[k]}} \mathbf{I}_{d^{[k]}}\right) \quad (2)$$

Each element of the $d^{[k]} \times 1$ vector $\bar{X}^{[k]}$ represents an independently encoded Gaussian codebook symbol with power $\frac{P^{[k]}}{d^{[k]}}$ that is beamformed with the corresponding vector of $\mathbf{V}^{[k]}$.

Interference Suppression at Receiver: Let $\mathbf{U}^{[k]}$ be an $N^{[k]} \times d^{[k]}$ matrix whose columns are the orthonormal basis of the interference-free desired signal subspace at receiver k. The kth receiver filters its received signal to obtain

$$\bar{Y}^{[k]} = \mathbf{U}^{[k]\dagger} Y^{[k]} \quad (3)$$

If interference is aligned into the null space of $\mathbf{U}^{[k]}$ then the following condition must be satisfied:

$$\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} = \mathbf{0}, \forall j \neq k \quad (4)$$
$$\mathrm{rank}\left(\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}\right) = d_k \quad (5)$$

In other words the desired signals are received through a $d^{[k]} \times d^{[k]}$ full rank channel matrix

$$\bar{\mathbf{H}}^{[kk]} \triangleq \mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}$$

while the interference is completely eliminated. The effective channel for user k is then expressed as

$$\bar{Y}^{[k]} = \bar{\mathbf{H}}^{[kk]} \bar{X}^{[k]} + \bar{Z}^{[k]} \quad (6)$$

where $\bar{Z}^{[k]} \sim \mathcal{N}(\mathbf{0}_{d^{[k]} \times 1}, \mathbf{I}_{d^{[k]}})$ is the effective $d^{[k]} \times 1$ AWGN vector at receiver k. The rate[1] achieved on this channel is

$$R^{[k]} = \log\left|\mathbf{I}_{d^{[k]}} + \frac{P^{[k]}}{d^{[k]}} \bar{\mathbf{H}}^{[kk]} \bar{\mathbf{H}}^{[kk]\dagger}\right| \quad (7)$$
$$= d^{[k]} \log\left(P^{[k]}\right) + o\left(\log\left(P^{[k]}\right)\right). \quad (8)$$

Thus, $d^{[k]}$ degrees of freedom are achieved by user k. The sum rate achieved over the interference channel is the sum of the rates achieved by all the users, i.e., $\sum_{k=1}^{K} R^{[k]}$. Similarly, $\sum_{k=1}^{K} d^{[k]}$ is defined to be the *total* number of degrees of freedom achieved over the channel.

Remark: The interference alignment conditions of (4), (5) are equivalent to the condition that the desired signals are linearly independent of the interference (as stated in [1], [21]). To see this, first consider the case where (4), (5) hold. Then, note that $\mathbf{U}^{[k]}$ is a matrix whose column-span contains the entire null-space of the interference at receiver k. Therefore, if, at

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[1]It must be noted that, the rate in (8) is normalized in time and frequency (i.e., number of symbols used), but *not* by the number of spatial dimensions (i.e., antennas).

receiver k, certain desired vectors are linearly dependent with the interference, then $\mathbf{U}^{[k]}$ would null those desired vectors and the matrix $\mathbf{U}^{[k]} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}$ would have less than a full rank of $d^{[k]}$ contradicting (5). Thus, (4) and (5) imply that the desired vectors are linearly independent of the interference. Conversely, if the desired vectors are known to be linearly independent of the interference, then $\mathbf{U}^{[k]}$ can be chosen to be precisely the null space of the interference at receiver k to ensure that (4), (5) are satisfied.

Remark: For interference alignment and the achievability of the degrees of freedom, it suffices if each of the beamforming matrices $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}$ has linearly independent column vectors. In the formulation above, we assume the additional property that these matrices have orthonormal column vectors. Note that this assumption of orthonormality does not affect the feasibility of interference alignment, and it naturally leads to the iterative algorithm to be presented in this paper.

A. Feasibility of Alignment

Given the channel matrices $\mathbf{H}^{[kj]}, k, j \in \mathcal{K}$, we say that the degrees of freedom allocation $(d^{[1]}, d^{[2]}, \ldots, d^{[K]})$ is feasible if there exist transmit precoding matrices $\mathbf{V}^{[k]}$ and receive interference suppression matrices $\mathbf{U}^{[k]}$

$$\mathbf{V}^{[k]} : M^{[k]} \times d^{[k]}, \quad \mathbf{V}^{[k]\dagger} \mathbf{V}^{[k]} = \mathbf{I}_{d^{[k]}} \quad (9)$$
$$\mathbf{U}^{[k]} : N^{[k]} \times d^{[k]}, \quad \mathbf{U}^{[k]\dagger} \mathbf{U}^{[k]} = \mathbf{I}_{d^{[k]}} \quad (10)$$

such that

$$\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} = \mathbf{0}_{d^{[k]} \times d^{[j]}}, \forall j \neq k \quad (11)$$
$$\mathrm{rank}\left(\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}\right) = d_k, \forall k \in \mathcal{K}. \quad (12)$$

The solution to the feasibility problem is not known in general. In other words, given a set of randomly generated channel matrices and a degree-of-freedom allocation $(d^{[1]}, d^{[2]}, \ldots, d^{[K]})$, it is not known if one can almost surely find transmit and receive filters that will satisfy the feasibility conditions. The distributed interference alignment algorithm developed in this paper will be useful in obtaining numerical insights into this open problem.

B. Reciprocity of Alignment

An interesting observation from the problem formulation above is the duality relationship between interference alignment on a given interference channel and its reciprocal channel obtained by switching the direction of communication. Specifically, let $\overleftarrow{\mathbf{V}}^{[k]}, \overleftarrow{\mathbf{U}}^{[k]}$ denote the transmit precoding filters and the receive interference suppression filters on the reciprocal channel. The feasibility conditions on the reciprocal channel are

$$\overleftarrow{\mathbf{V}}^{[k]} : N^{[k]} \times d^{[k]}, \quad \overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{I}_{d^{[k]}} \quad (13)$$
$$\overleftarrow{\mathbf{U}}^{[k]} : M^{[k]} \times d^{[k]}, \quad \overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{I}_{d^{[k]}} \quad (14)$$

such that

$$\overleftarrow{\mathbf{U}}^{[j]\dagger} \overleftarrow{\mathbf{H}}^{[jk]} \overleftarrow{\mathbf{V}}^{[k]} = \mathbf{0}_{d^{[k]} \times d^{[j]}}, \forall j \neq k \quad (15)$$

$$\operatorname{rank}\left(\overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{H}}^{[kk]} \overleftarrow{\mathbf{V}}^{[k]}\right) = d_k, \forall k \in \mathcal{K}. \quad (16)$$

Suppose we set $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}, \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{V}^{[k]}$. Then the feasibility conditions on the reciprocal channel become identical to the original feasibility conditions. Thus, the following observation can be made:

Reciprocity of Alignment: *Since the feasibility conditions are identical, if the degrees of freedom allocation $(d^{[1]}, d^{[2]}, \ldots, d^{[K]})$ is feasible on the original interference network then it is also feasible on the reciprocal network (and vice versa). Interference alignment on the reciprocal interference network is simply achieved by choosing the transmit filters and the receive filters on the reciprocal channel as the receive filters and the transmit filters (respectively) of the original channel.*

Reciprocity of alignment is a key property used for distributed interference alignment algorithms, described in the next section.

V. DISTRIBUTED ALGORITHM FOR INTERFERENCE ALIGNMENT

In this section we construct distributed interference alignment algorithms for the interference channel with multiple-antenna nodes and no symbol extensions. Continuing with the system model of Section IV, this implies that the relevant interference alignment feasibility condition is (11) while (12) is automatically satisfied. Basically, (11) requires that at each receiver, all interference is suppressed, leaving as many interference-free dimensions as the degrees of freedom allocated to that receiver.

Since we are interested in distributed algorithms, we start with arbitrary transmit and receive filters $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}$ and iteratively update these filters to approach interference alignment. The quality of alignment is measured by the power in the *leakage interference* at each receiver, i.e., the interference power remaining in the received signal after the receive interference suppression filter is applied. The goal is to achieve interference alignment by progressively reducing the leakage interference. If the leakage interference converges to zero, then interference alignment is feasible.

The total interference leakage at receiver k due to all undesired transmitters ($j \neq k$) is given by

$$I^{[k\star]} = \operatorname{Tr}\left[\mathbf{U}^{[k]\dagger} \mathbf{Q}^{[k]} \mathbf{U}^{[k]}\right] \quad (17)$$

where

$$\mathbf{Q}^{[k]} = \sum_{j=1, j \neq k}^{K} \frac{P^{[j]}}{d^{[j]}} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} \mathbf{V}^{[j]\dagger} \mathbf{H}^{[kj]\dagger} \quad (18)$$

is the interference covariance matrix at receiver k.

Similarly, in the reciprocal network, let $\overleftarrow{P}^{[k]} > 0$ be the power constraint at transmitter k. Then, the total interference leakage at receiver j due to all undesired transmitters ($k \neq j$) is given by

$$\overleftarrow{I}^{[j\star]} = \operatorname{Tr}\left[\overleftarrow{\mathbf{U}}^{[j]\dagger} \overleftarrow{\mathbf{Q}}^{[j]} \overleftarrow{\mathbf{U}}^{[j]}\right] \quad (19)$$

where

$$\overleftarrow{\mathbf{Q}}^{[j]} = \sum_{k=1, k \neq j}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \overleftarrow{\mathbf{H}}^{[jk]} \overleftarrow{\mathbf{V}}^{[k]} \overleftarrow{\mathbf{V}}^{[k]\dagger} \overleftarrow{\mathbf{H}}^{[jk]\dagger} \quad (20)$$

is the interference covariance matrix at receiver j.

The iterative algorithm alternates between the original and reciprocal networks. Within each network only the receivers update their interference suppression filters to minimize their total leakage interference.

Step I: In the original network, each receiver solves the following optimization problem.

$$\min_{\mathbf{U}^{[k]}: N^{[k]} \times d^{[k]}, \quad \mathbf{U}^{[k]} \mathbf{U}^{[k]\dagger} = \mathbf{I}_{d^{[k]}}} I^{[k\star]}. \quad (21)$$

In other words, receiver k chooses its interference suppression filter $\mathbf{U}^{[k]}$ to minimize the leakage interference due to all undesired transmitters. The $d^{[k]}$-dimensional received signal subspace that contains the least interference is the space spanned by the eigenvectors corresponding to the $d^{[k]}$ smallest eigenvalues of the interference covariance matrix $\mathbf{Q}^{[k]}$. Thus, the $d^{[k]}$ columns of $\mathbf{U}^{[k]}$ are given by:

$$\mathbf{U}^{[k]}_{\star d} = \boldsymbol{\nu}_d\left[\mathbf{Q}^{[k]}\right], \quad d = 1, \ldots, d^{[k]} \quad (22)$$

where $\boldsymbol{\nu}_d[\mathbf{A}]$ is the eigenvector corresponding to the dth smallest eigenvalue of $\mathbf{A}$.

Step II: The second step is identical to the first step, but performed in the reciprocal network. Consider the reciprocal network obtained by reversing the roles of the transmitters and the receivers. The transmit precoding matrices in the reciprocal network, $\overleftarrow{\mathbf{V}}^{[k]}$, are the receive interference suppression matrices $\mathbf{U}^{[k]}$ from the original network that were determined in Step I. Each receiver in the reciprocal network solves the following optimization problem:

$$\min_{\overleftarrow{\mathbf{U}}^{[j]}: M^{[j]} \times d^{[j]}, \quad \overleftarrow{\mathbf{U}}^{[j]} \overleftarrow{\mathbf{U}}^{[j]\dagger} = \mathbf{I}_{d^{[j]}}} \overleftarrow{I}^{[j\star]}. \quad (23)$$

Similar to Step I, the $d^{[j]}$ columns of $\overleftarrow{\mathbf{U}}^{[j]}$ are given by

$$\overleftarrow{\mathbf{U}}^{[j]}_{\star d} = \boldsymbol{\nu}_d\left[\overleftarrow{\mathbf{Q}}^{[j]}\right], \quad d = 1, \ldots, d^{[j]}. \quad (24)$$

The receive interference suppression filters in the reciprocal network are then used as the transmit precoding matrices in the original network, and the algorithm returns to Step I. The iterations continue in this manner until the algorithm converges.

The iterative procedure is summarized in Algorithm 1. A pictorial representation is shown in Fig. 2.

Algorithm 1 Iterative Interference Alignment

1: Start with arbitrary precoding matrices $\mathbf{V}^{[j]} : M^{[j]} \times d^{[j]}, \mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger} = \mathbf{I}_{d[j]}$.

2: Begin iteration

3: Compute interference covariance matrix at the receivers:

$$\mathbf{Q}^{[k]} = \sum_{j=1,j\neq k}^{K} \frac{P^{[j]}}{d^{[j]}} \mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}$$

4: Compute the interference suppression matrix at each receiver:

$$\mathbf{U}^{[k]}_{\star d} = \boldsymbol{\nu}_d[\mathbf{Q}^{[k]}], \quad d = 1,\ldots,d^{[k]}$$

5: Reverse the communication direction and set $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}$.

6: Compute interference covariance matrix at the new receivers:

$$\overleftarrow{\mathbf{Q}}^{[j]} = \sum_{k=1,k\neq j}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \overleftarrow{\mathbf{H}}^{[jk]}\overleftarrow{\mathbf{V}}^{[k]}\overleftarrow{\mathbf{V}}^{[k]\dagger}\overleftarrow{\mathbf{H}}^{[jk]\dagger}$$

7: Compute the interference suppression matrix at each receiver:

$$\overleftarrow{\mathbf{U}}^{[j]}_{\star d} = \boldsymbol{\nu}_d[\overleftarrow{\mathbf{Q}}^{[j]}], \quad d = 1,\ldots,d^{[k]}$$

8: Reverse the communication direction and set $\mathbf{V}^{[k]} = \overleftarrow{\mathbf{U}}^{[k]}$.

9: Continue till convergence.

A. Proof of Convergence

We now show that the algorithm must converge. The proof also highlights the intuition behind the algorithm.

We define a metric called the weighted leakage interference (WLI) as in (25) and (26) at the bottom of the page. We show

that each step in the algorithm reduces the value of WLI. Since WLI is bounded below by zero, this implies that the algorithm must converge. Note that an interference alignment solution corresponds to WLI= 0.

The WLI associated with receiver k can be written as (27) and (28), at the bottom of the page. Therefore, the value of $\mathbf{U}^{[k]}$ computed in Step 4 to minimize $I^{[k\star]}$ also minimizes $I_w^{[k\star]}$. Since $I_w = \sum_{k=1}^{K} I_w^{[k\star]}$, we have (29)

$$\begin{aligned}
\min_{\mathbf{U}^{[1]},\mathbf{U}^{[2]},\ldots,\mathbf{U}^{[K]}} I_w &= \min_{\mathbf{U}^{[1]},\mathbf{U}^{[2]},\ldots,\mathbf{U}^{[K]}} \sum_{k=1}^{K} I_w^{[k\star]} \\
&= \sum_{k=1}^{K} \left[\min_{\mathbf{U}^{[k]}} I_w^{[k\star]} \right] \\
&= \sum_{k=1}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \left[\min_{\mathbf{U}^{[k]}} I^{[k\star]} \right].
\end{aligned}$$

In other words, given the values of $\mathbf{V}^{[j]}, j \in \{1,2,\ldots,K\}$, Step 4 minimizes the value of I_w over all possible choices of $\mathbf{U}^{[k]}, k \in \{1,2,\ldots,K\}$. In particular, Step 4 can only reduce the value of I_w.

The weighted leakage interference associated with transmitter j is

$$\begin{aligned}
I_w^{[\star j]} &= \frac{P^{[j]}}{d^{[j]}} \sum_{k=1}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}\mathbf{U}^{[k]} \right] \\
&= \frac{P^{[j]}}{d^{[j]}} \sum_{k=1}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[\overleftarrow{\mathbf{V}}^{[k]\dagger}\overleftarrow{\mathbf{H}}^{[jk]\dagger}\overleftarrow{\mathbf{U}}^{[j]}\overleftarrow{\mathbf{U}}^{[j]\dagger}\overleftarrow{\mathbf{H}}^{[jk]}\overleftarrow{\mathbf{V}}^{[k]} \right] \\
&= \frac{P^{[j]}}{d^{[j]}} \sum_{k=1}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[\overleftarrow{\mathbf{U}}^{[j]\dagger}\overleftarrow{\mathbf{H}}^{[jk]}\overleftarrow{\mathbf{V}}^{[k]}\overleftarrow{\mathbf{V}}^{[k]\dagger}\overleftarrow{\mathbf{H}}^{[jk]\dagger}\overleftarrow{\mathbf{U}}^{[j]} \right] \\
&= \frac{P^{[j]}}{d^{[j]}} \mathrm{Tr}\left[\overleftarrow{\mathbf{U}}^{[j]\dagger}\overleftarrow{\mathbf{Q}}^{[j]}\overleftarrow{\mathbf{U}}^{[j]} \right].
\end{aligned}$$

Therefore, the value of $\overleftarrow{\mathbf{U}}^{[j]}$ computed in Step 7 to minimize $\overleftarrow{I}^{[j\star]}$ also minimizes $I_w^{[\star j]}$. Since $I_w = \sum_{k=1}^{K} I_w^{[\star j]}$, it is easily seen that Step 7 can also only reduce the value of I_w. Since

$$I_w = \sum_{k=1}^{K} \sum_{j=1,j\neq k}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} I^{[kj]} \tag{25}$$

$$= \sum_{k=1}^{K} \sum_{j=1,j\neq k}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \frac{P^{[j]}}{d^{[j]}} \mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}\mathbf{U}^{[k]} \right] \tag{26}$$

$$I_w^{[k\star]} = \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \sum_{j=1,j\neq k}^{K} \frac{P^{[j]}}{d^{[j]}} \mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}\mathbf{U}^{[k]} \right] \tag{27}$$

$$= \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{Q}^{[k]}\mathbf{U}^{[k]} \right] = \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} I^{[k\star]} \tag{28}$$
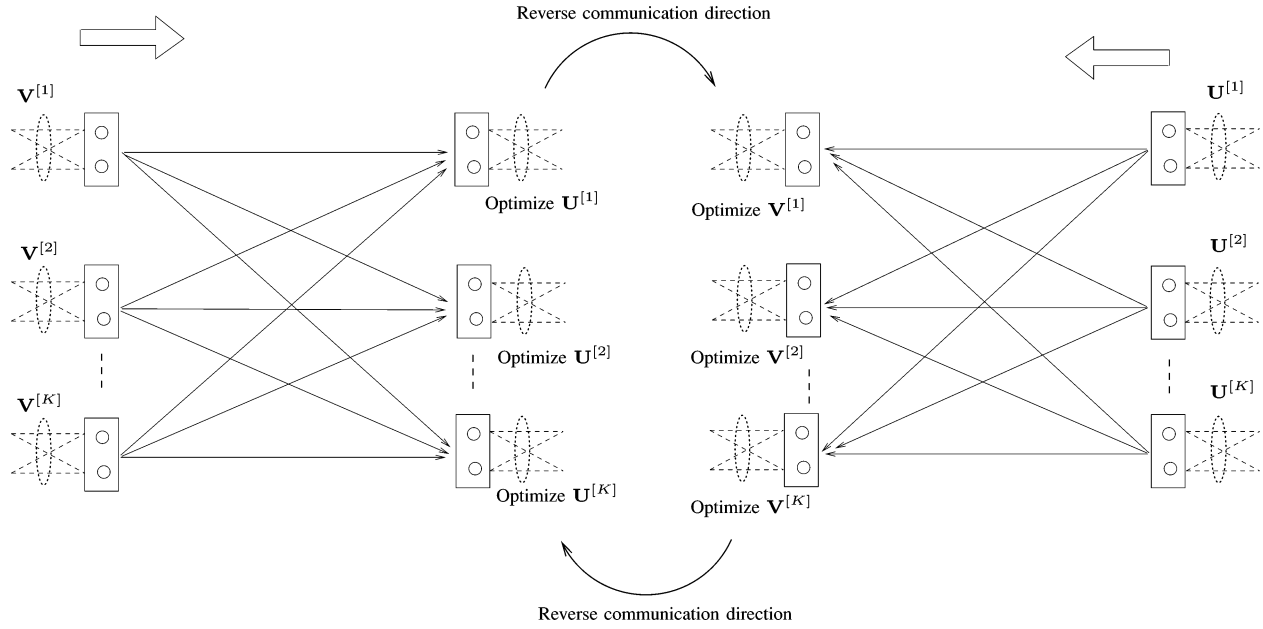
Fig. 2. Pictorial representation of the iterative interference alignment algorithm where the receive directions are optimized to minimize interference power at the receivers. Each link (arrow) represents a MIMO channel. The transmit power per node is P in both directions.

the value of I_w is monotonically reduced after every iteration, convergence of the algorithm is guaranteed.

For the convergence of the algorithm, the value of $\overleftarrow{P}^{[k]}$ can be any arbitrary positive value. In general, $\overleftarrow{P}^{[k]}$ is a design parameter, and its choice so that the sum-rate (or any other metric) is optimized is an open problem.

The following observations summarize the intuition behind the iterative algorithm.

1) Dimensions along which a receiver sees the least interference from other nodes are also the dimensions along which it causes the least interference to other nodes in the reciprocal network where it functions as a transmitter.

2) The weighted leakage interference is unchanged in the original and reciprocal networks if the transmit and receive filters are switched.

Remark: While the algorithm minimizes leakage interference at every iteration and is guaranteed to converge, convergence to global minimum is not guaranteed due to the nonconvex nature of the interference optimization problem. Numerical results for the performance of the algorithm are presented in the next section.

B. Distributed Implementation of the Algorithm

The reciprocal property of the wireless channels, combined with the fact that the interference covariance matrices can be naturally learnt at the receivers, enables a distributed implementation of the above algorithm. For example, consider step 3 of the algorithm. While the covariance matrix $\mathbf{Q}^{[k]}$ is a function of all the channels and transmit precoding vectors, this matrix can,

in fact, be estimated by receiver k in a distributed manner. In particular, when the signal transmitted by the transmitter is as given in (2), the signal received at receiver k can be expressed as

$$\mathbf{Y}^{[k]} = \sum_{j=1}^{K} \frac{P^{[j]}}{d^{[j]}} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} \bar{X}^{[k]} + \mathbf{Z}^{[k]}.$$

If receiver k is aware of the effective channel from transmitter k, $\mathbf{H}^{[kk]}\mathbf{V}^{[k]}$, then the receiver can learn the effective covariance matrix $\mathbf{Q}^{[k]}$ from $\mathbf{Y}^{[k]}$ using a channel training phase. For example, if during a sufficiently long training phase, the training sequences $\bar{X}^{[j]}, j = 1, 2, 3$, are chosen to be independent zero-mean unit-variance i.i.d Gaussian sequences, then $\mathbf{Q}^{[k]}$ can be estimated at receiver k by evaluating $\left(\mathbf{Y}^{[k]} - \frac{P^{[j]}}{d^{[j]}}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}\bar{X}^{[k]} - \mathbf{I}_{N^{[k]}}\right)\left(\mathbf{Y}^{[k]} - \frac{P^{[j]}}{d^{[j]}}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}\bar{X}^{[k]} - \mathbf{I}_{N^{[k]}}\right)^{\dagger}$ and averaging over the training phase. Further, if the physical channels gains satisfy reciprocity, then the transmitters can learn $\overleftarrow{\mathbf{Q}}^{[k]}$ in a similar manner over the reciprocal channel in step 6. Therefore, for implementation of the algorithm, the receivers are required to learn the effective channel of the desired transmitter, and the covariance matrix of the interference. This reduces the overhead as compared to the analytical solution of [1], where all the receeivers require knowledge of the $K(K-1)$ channel matrices representing the interfering links in the K user interference channel. It must be noted that while the reciprocity of the wireless channel enables a distributed implementation, the algorithm is relevant, from a theoretical perspective, even in cases where the wireless channel is not reciprocal. In such scenarios, the reciprocal channel maybe used as a theoretical tool for examining the feasibility of interference alignment.

C. Max-SINR Algorithm

The algorithm presented above seeks perfect interference alignment. In particular it seeks to create an interference-free subspace of the required number of dimensions, that is designated as the *desired* signal subspace. However, note that interference alignment makes no attempt to maximize the desired signal power within the desired signal subspace. In fact the algorithm described above does not depend at all on the direct channels $\mathbf{H}^{[kk]}$ through which the desired signal arrives at the intended receiver. Therefore, while the interference is eliminated within the desired space, no coherent combining gain (array gain) for the desired signal is obtained with interference alignment. While this is optimal as all signal powers approach infinity, it is not optimal in general at intermediate SNR values. Therefore, other algorithms may be designed which will perform better than the interference alignment algorithm at intermediate SNR values.

In this section we consider one such natural extension of the interference alignment algorithm where the receive filters $\mathbf{U}^{[k]}$ and $\overleftarrow{\mathbf{U}}^{[k]}$ are chosen to maximize SINR at the receivers instead of only minimizing the leakage interference. While there is no loss of generality in assuming orthogonal precoding vectors for the streams sent from the same transmitter as far as interference alignment is concerned, orthogonal precoding vectors are in general suboptimal for SINR optimization. We, therefore, no longer assume that the columns of $\mathbf{V}^{[k]}$ (the transmit precoding vectors) are mutually orthogonal. We also identify the columns of $\mathbf{U}^{[k]}$ to be the specific combining vectors for the corresponding desired data stream, so that they are not necessarily orthogonal either. With these modified definitions, the SINR of the lth stream of the kth receiver is

$$\text{SINR}_{kl} = \frac{\mathbf{U}_{\star l}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}_{\star l}^{[k]} \mathbf{V}_{\star l}^{[k]\dagger} \mathbf{H}^{[kk]\dagger} \mathbf{U}_{\star l}^{[k]}}{\mathbf{U}_{\star l}^{[k]\dagger} \mathbf{B}^{[kl]} \mathbf{U}_{\star l}^{[k]}} \frac{P^{[k]}}{d^{[k]}} \tag{29}$$

where

$$\mathbf{B}^{[kl]} = \sum_{j=1}^{K} \frac{P^{[j]}}{d^{[j]}} \sum_{d=1}^{d^{[j]}} \mathbf{H}^{[kj]} \mathbf{V}_{\star d}^{[j]} \mathbf{V}_{\star d}^{[j]\dagger} \mathbf{H}^{[kj]\dagger}$$
$$- \frac{P^{[k]}}{d^{[k]}} \mathbf{H}^{[kk]} \mathbf{V}_{\star l}^{[k]} \mathbf{V}_{\star l}^{[k]\dagger} \mathbf{H}^{[kk]\dagger} + \mathbf{I}_{N^{[k]}}. \tag{30}$$

The unit vector $\mathbf{U}_{\star l}^{[k]}$ that maximizes SINR_{kl} is given by

$$\mathbf{U}_{\star l}^{[k]} = \frac{\left(\mathbf{B}^{[kl]}\right)^{-1} \mathbf{H}^{[kk]} \mathbf{V}_{\star l}^{[k]}}{\left\|\left(\mathbf{B}^{[kl]}\right)^{-1} \mathbf{H}^{[kk]} \mathbf{V}_{\star l}^{[k]}\right\|}. \tag{31}$$

The steps of the iteration are given in Algorithm 2.

Max-SINR algorithm

1: Start with any $\mathbf{V}^{[k]} : M^{[k]} \times d^{[k]}$, columns of $\mathbf{V}^{[k]}$ are linearly independent unit vectors.
2: Begin iteration
3: Compute interference plus noise covariance matrix for $\mathbf{B}^{[kl]}$ for stream l at receiver k according to (30), $\forall k \in \{1, 2, \ldots, K\}, l \in \{1, 2, \ldots, d^{[k]}\}$.
4: Calculate receive combining vectors $\mathbf{U}_{\star l}^{[k]}$ at receiver k according to (31), $\forall k \in \{1, 2, \ldots, K\}, l \in \{1, 2, \ldots, d^{[k]}\}$.
5: Reverse the communication direction and use the receive combining vectors as precoding vectors: $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}, \forall k \in \{1, 2, \ldots, K\}$.
6: In the reciprocal network, compute interference plus-noise covariance matrix $\overleftarrow{\mathbf{B}}^{[kl]}$ for stream l at receiver k, $\forall k \in \{1, 2, \ldots, K\}, l \in \{1, 2, \ldots, d^{[k]}\}$.
7: Calculate receive combining vectors $\overleftarrow{\mathbf{U}}_{\star l}^{[k]}$, $\forall k \in \{1, 2, \ldots, K\}, l \in \{1, 2, \ldots, d^{[k]}\}$.
8: Reverse the communication direction and use the receive combining vectors as precoding vectors: $\mathbf{V}^{[k]} = \overleftarrow{\mathbf{U}}^{[k]}, \forall k \in \{1, 2, \ldots, K\}$.
9: Repeat until convergence, or the number of iterations reaches a limit defined earlier.

Remark: It must be noted that the distributed implementation explained previously is the context of Algorithm 1 is also applicable for the Max SINR algorithm.

Remark: Intuitively, for the Max-SINR algorithm, the power chosen in the reciprocal network must be comparable to the power in the original network. This is because, if the power in the original network is high (as compared to the noise power), the performance of the system is limited by interference and optimal solution should align the interference. However, if the power in the reciprocal network $\overleftarrow{P}$ is chosen to be small as compared to the noise power, then the dual network optimizes beamforming directions at its receivers virtually ignoring interference since it operates in a noise limited regime. Since the receive beamforming directions in the reciprocal network are the transmit beamforming directions in the original network, this would lead to a suboptimal solution overall. Note that this intuition does not necessarily apply to Algorithm 1, since noise is ignored in the algorithm. An analytical treatment of the choice of $\overleftarrow{P}$, as mentioned before, is an open problem.

VI. PERFORMANCE RESULTS AND APPLICATIONS

Consider the 3 user interference channel where each node is equipped with 2 antennas and all channel coefficients are i.i.d. zero mean unit variance circularly symmetric complex Gaussian. Assume that each user has identical power with $P^{[j]} = P, j = 1, 2, 3$—note that with these power constraints and the noise power at each receiver normalized to unity, P
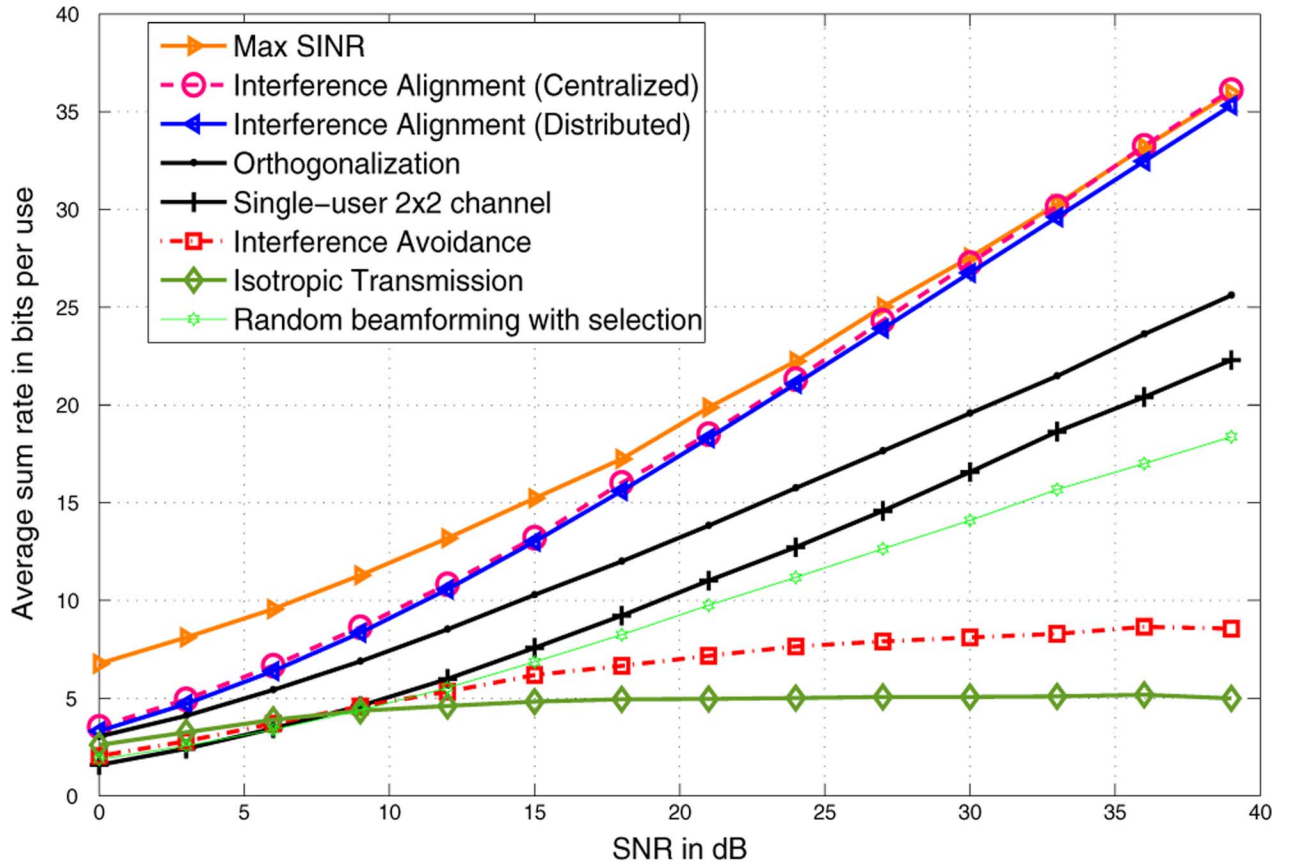
Fig. 3. Performance of the distributed interference alignment algorithm for the three-user interference channel with two antennas at each node.

indicates the signal-to-noise ratio in the network. As shown in Fig. 1 with interference alignment each user achieves 1 degree of freedom. In Fig. 3 we compare the performance of the following achievable schemes versus P (i.e., SNR).

1) The distributed iterative interference alignment algorithm (Algorithm 1 of the previous section).
2) The distributed Max-SINR algorithm (Algorithm 2 of the previous section).
3) The centralized theoretical closed form solution of [1].
4) An achievable scheme with users orthogonalized, with equal time sharing for the users, and with power $3P$ per node.
5) The rate of the single-user channel, i.e., the rate of the 2×2 MIMO channels.
6) The greedy interference avoidance algorithm of [29].
7) Isotropic transmission, where each transmitter sends 2 streams of equal power without regard to the channel information.
8) Random beamforming vectors with selection—Each user generates a beam randomly, and transmits along the generated beam. A number of such solutions are generated and the maximum rate achieved among these random solutions is plotted. The number of solutions is equal to the number of iterations used by the distributed algorithms, so that algorithms of (roughly) equal complexity are compared. Note that as the number of solutions used increases, this strategy would approach a brute-force search over the entire space based on the Monte-Carlo philosophy.

The performance is measured by the sum rate achieved over the interference channel, i.e., the sum of the rates achieved by the 3 users, measured in bits per channel use. Further, it must be noted that the number of spatial dimensions (i.e., antennas) in all the nodes is equal to two, in all the systems listed above. The degrees of freedom advantage of certain achievable schemes such as the theoretical solution of [1] over others such as or-thogonalization is due to interference alignment. From the plot of Fig. 3, it can be seen that the distributed iterative interfer-ence alignment and Max-SINR algorithms perform very close to the theoretical solution at high SNR. In particular, both algo-rithms achieve an almost identical capacity scaling to the the-oretical case, indicating that they achieve an optimal 1 degree for each user, or equivalently a *total* of 3 degrees of freedom over the channel. More importantly, the distributed algorithms provides significant benefits over the orthogonalization over the entire SNR range considered. It is expected that the interference alignment algorithms outperform orthognalization at high SNR because the latter only achieves $2/3$ degrees of freedom per user, or equivalently a total of 2 degrees of freedom over the channel. However, the plot of Fig. 3 is particularly interesing becuase the iterative Max-SINR algorithm significantly outperform orthog-onalization and at interference avoidance even at moderate and low SNR. We next explore the application of the algorithm for verification of feasibility of interference alignment.

A. Feasibility of Interference Alignment

While the iterative algorithm is useful for circumventing the need for global channel knowledge, it can also be used
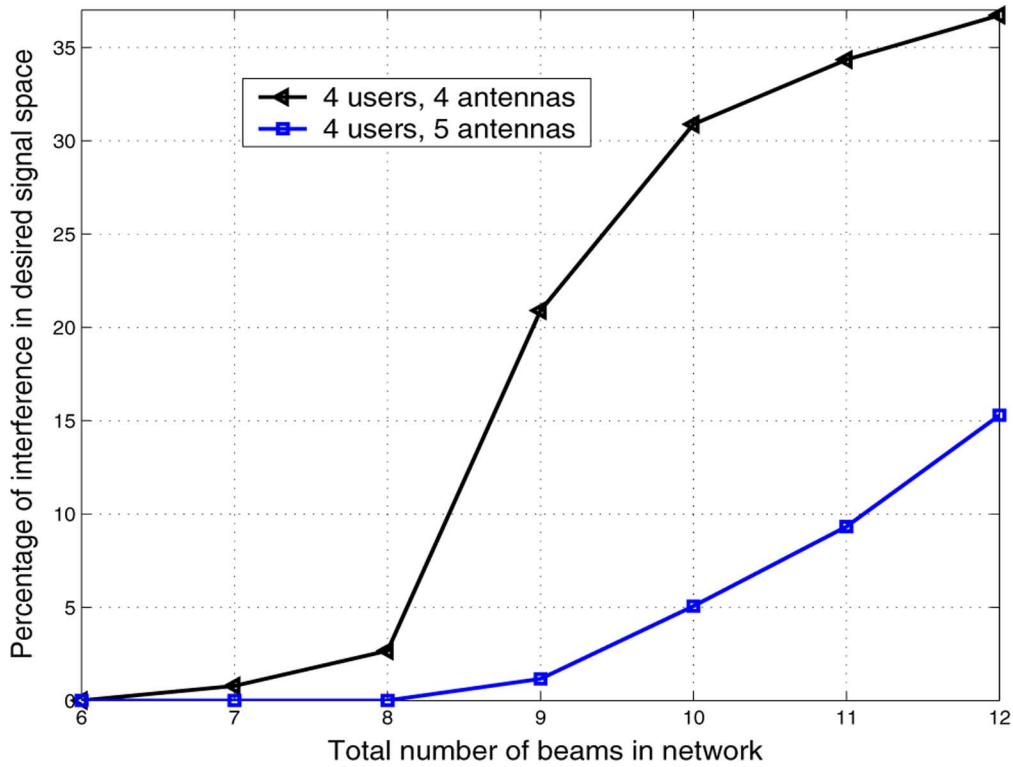
Fig. 4. Percentage of interference power in desired signal space as a function of the total number of data streams in the network.

to check theoretical feasibility of interference alignment for a given number of streams per user. Let $\left(d^{[1]}, d^{[2]}, \ldots d^{[k]}\right)$ denote the number of transmit streams of the users. For perfect interference alignment $\sum_{j=1}^{d^{[k]}} \lambda_j[\mathbf{Q}^{[k]}] = 0$ at receiver k where $\lambda_j[\mathbf{A}]$ denotes the jth smallest eigenvalue of $\mathbf{A}$. Note that $\sum_{j=1}^{d^{[k]}} \lambda_j[\mathbf{Q}^{[k]}]$ indicates the interference power in the desired signal space.

Using the algorithm, we plot in Fig. 4, the percentage of interference in the desired signal space versus the total number of transmit streams in the network. The fraction of interference in the desired signal space of receiver k is defined as

$$p_k = \frac{\sum_{j=1}^{d^{[k]}} \lambda_j\left[\mathbf{Q}^{[k]}\right]}{\mathrm{Tr}\left[\mathbf{Q}^{[k]}\right]}. \qquad (32)$$

When the fraction of interference in desired signal space is zero (within numerical errors), then interference alignment is feasible, but not *vice-versa*. Fig. 4 suggests that interference alignment is feasible on the four user interference channel with 5 antennas at each node when each transmitter sends two streams. Note that a nonzero interference leakage when the number of beams is increased does not necessarily imply the infeasibility of interference alignment. This is because, our numerical algorithm does not necessarily converge to a global optimum. Therefore, while the upperbound on the degrees of freedom for this network is 10, the plot suggests that atleast 8 degrees of freedom $(d^{[1]} = d^{[2]} = d^{[3]} = d^{[4]} = 2)$ can be achieved without channel extension. Similarly for the 4 antenna case, the plot indicates that interference alignment is possible for atleast a total of 6 streams in the 4 user interference network with only 4 antennas at each node, while the degrees of freedom outerbound is

8. While the results of this section clearly illustrate the benefits of iterative algorithms, there are cases where these iterative algorithms are *not* applicable. We describe such cases in the next section.

VII. LIMITATIONS OF THE ITERATIVE ALGORITHMS

The iterative algorithms are not effective in finding interference alignment solutions for channels which require symbol extensions. It must be noted that long symbol extensions are required in general to achieve the optimal degrees of freedom for time-varying channels with single-antenna nodes. Therefore, the algorithms are specifically not effective for SISO interference channels if they are time-varying, even though interference alignment solutions indeed exist. The reason behind the inapplicability of the iterative algorithms to the SISO case may be understood as follows.

Symbol extensions over orthogonal dimensions produce structured (diagonal or block diagonal) matrices for which both conditions (12) and (11) are nontrivial. To see this, we first note that the iterative algorithms aim to align interference since the focus on designing precoding and nulling vectors to minimize the total interference leakage. In other words, they focus on finding $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}$ satisfying (11). When the MIMO channel is considered without channel extensions, then all the elements of the channel matrices are randomly and independently generated from continuous distributions and condition (12) will also be satisfied with probability 1. This is because the direct channel matrices $\mathbf{H}^{[kk]}$ do not appear in condition (11). So the choice of transmit and receive filters $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}, k \in \mathcal{K}$ to satisfy (11) does not depend on the direct channel matrices $\mathbf{H}^{[kk]}$. Since $\mathbf{H}^{[kk]}$ is independent of $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}$ and all its elements are

randomly generated from a continuous distribution (i.e., it lacks any special structure), the product matrix $\mathbf{U}^{[k]T}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}$ has full rank with probability 1. Thus, for random MIMO channels without time-extensions, if (11) can be satisfied then (12) is automatically satisfied almost surely as well.

However, if time-extensions are considered then the channel matrices may have a diagonal or block diagonal structure and (12) cannot be taken for granted. For instance, consider a scenario where all nodes have 1 antenna and a two symbol extension is chosen. Over a channel with two-symbol extensions, all inputs and outputs are 2-D vectors, and the channel is a 2×2 diagonal matrix—the diagonal property of the matrix arising from the fact that we do not model intersymbol interference. Thus, each matrix $\mathbf{H}^{[jk]}$ is of the form

$$\mathbf{H}^{[jk]} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}.$$

Also, let $d^{[k]} = 1$ so that $\mathbf{U}^{[k]}$ and $\mathbf{V}^{[k]}$ are 4×1 column vectors. Now if

$$\mathbf{U}^{[k]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{V}^{[k]} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that (11) is satisfied, then, even though the nonzero entries of $\mathbf{H}^{[kk]}$ are independent of $\mathbf{U}^{[k]}$ and $\mathbf{V}^{[k]}$, the value of $\mathbf{U}^{[k]T}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}$ is equal to 0 and, therefore, does not have the desired rank of 1. This issue is also discussed in Section IV-D of [1] to show the necessity of arbitrarily long symbol extensions for interference alignment.

A second case where a similar problem arises is in MIMO channels with symbol extensions and constant (i.e., not time-varying/frequency-selective) channel co-efficients. To see this consider a K user interference channel with 2 antennas at each transmitter and receiver, where a two symbol extension is chosen. Since a two-symbol extension is chosen, the inputs and outputs are 4-D vectors. Further, without intersymbol interference, the channel matrices $\mathbf{H}^{[jk]}$ are 4×4 block diagonal matrices of the form

$$\mathbf{H}^{[jk]} = \begin{bmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ 0 & 0 & h_{21} & h_{22} \end{bmatrix}.$$

Also, let $d^{[k]} = 1$ so that $\mathbf{U}^{[k]}$ and $\mathbf{V}^{[k]}$ are 2×1 column vectors. Now if

$$\mathbf{U}^{[k]} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{V}^{[k]} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

so that (11) is satisfied, then, even though the nonzero entries of $\mathbf{H}^{[kk]}$ are independent of $\mathbf{U}^{[k]}$ and $\mathbf{V}^{[k]}$, the value of
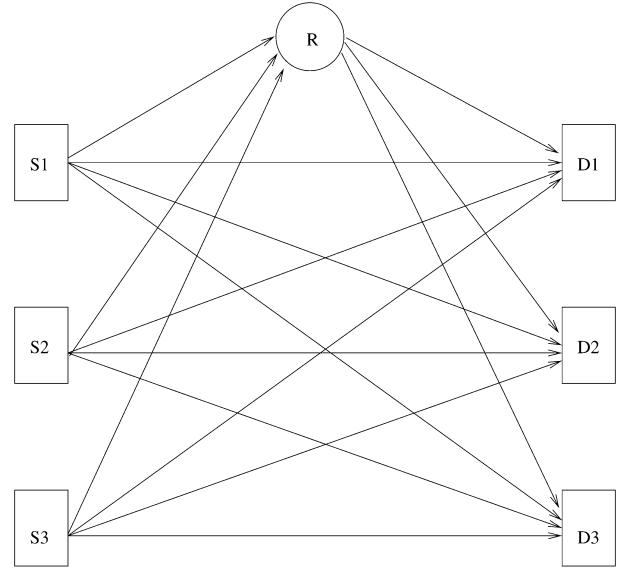

Fig. 5. Interference relay channel.

$\mathbf{U}^{[k]T}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}$ is equal to 0 and, therefore, does not have the desired rank of 1. Thus, in general, the applicability of the iterative algorithms is limited in cases where the channels may have some special structure. Due to these difficulties, it is not trivial to apply the numerical interference alignment algorithms to cases with symbol extensions. Since the interference alignment solutions for SISO channels require symbol extensions, the design and application of numerical interference alignment algorithms for the SISO case is an important open problem. In the next section, we describe an alternate (analytical) technique to overcome the interference alignment feasibility issue, using relays.

VIII. INTERFERENCE ALIGNMENT WITH LIMITED DIMENSIONS USING A RELAY

In this section, we give an example of how long symbol extensions may be avoided by the use of relays. Note that [23] has shown that relays cannot increase the degrees of freedom for time-varying wireless networks. However, as we show in this section, relays can be very useful by reducing the size of the signalling space over which interference alignment can be accomplished. The key idea is to employ relays to create a *virtual* MIMO system. Consider an interference relay channel with three sources, three destinations and a half-duplex relay (node 0) as shown in Fig. 5. Recall that in the absence of relays this network is shown to approach the upperbound of $3/2$ degrees of freedom per orthogonal dimension in the asymptotic limit of infinitely long symbol extensions [1]. However, we show that with relays only two time slots are required to achieve the outerbound, i.e., $3/2$ degrees of freedom.

Consider the following two slot protocol. In the first slot, the relay is silent and the received signal at destination j is given by

$$y^{[j]}(1) = \sum_{i=1}^{3} h^{[ji]}(1)x^{[i]}(1) + z^{[j]}(1), \quad j = 1, 2, 3. \quad (33)$$
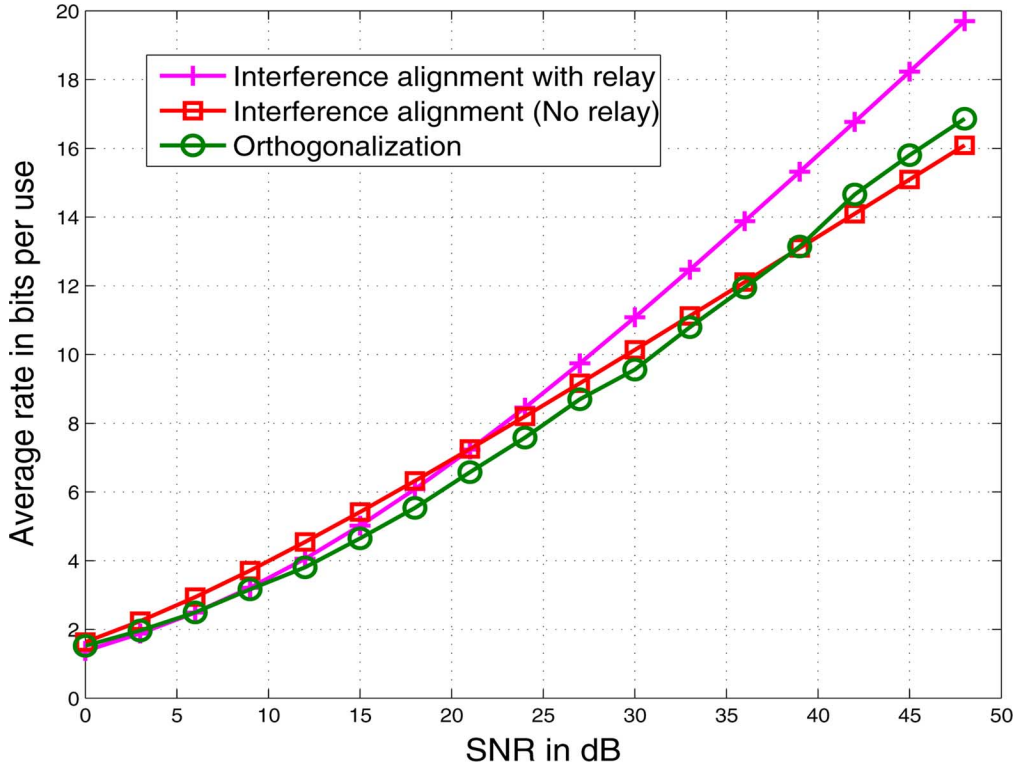
Fig. 6. Performance of the interference alignment schemes with time slot extension for the three user interference channel.

The received signal at the relay can be expressed as

$$y^{[0]}(1) = \sum_{i=1}^{3} h^{[0i]}(1)x^{[i]}(1) + z^{[0]}(1).$$

In the second slot, the relay transmits a scaled version of its received symbol while source i transmits $x^{[i]}(2)$. The received signal at the jth destination node in the second slot ($j = 1, 2, 3$)

$$y^{[j]}(2) = \sum_{i=1}^{3} h^{[ji]}(2)X^{[i]}(2) + h^{[j0]}(2)\beta y^{[0]}(1) + z^{[j]}(2) \quad (34)$$

Let $Y^{[j]} = [y^{[j]}(1)y^{[j]}(2)]^T$ and $X^{[i]} = [x^{[i]}(1)x^{[i]}(2)]^T$. In other words, the two inputs and the two outputs corresponding to the two slots are combined and expressed as 2-D vectors. On observing (33), (34), it can be noted that the dependence of the outputs $Y^{[j]}, j = 1, 2$, on the inputs $X^{[i]}, i = 1, 2$ is linear so that, we can combine them to obtain an equivalent vector representation as

$$Y^{[j]} = \sum_{i=1}^{3} \mathbf{H}^{[ji]}X^{[i]} + Z^{[j]} \quad (35)$$

where

$$\mathbf{H}^{[ji]} = \begin{bmatrix} h^{[ji]}(1) & 0 \\ \beta h^{[j0]}(2)h^{[0i]}(1) & h^{[ji]}(2) \end{bmatrix}$$

and

$$Z^{[j]} = \begin{bmatrix} z^{[j]}(1) \\ \beta h^{j0}(2)z^{[0]}(1) + z^{[j]}(2) \end{bmatrix}.$$

It must be noted that, while arbitrarily long symbol extensions are used in [1] in most cases, for the MIMO three user interference channel, finite symbol extensions are sufficient. In particular, if the number of antennas at each node is equal to 2 (or any even number), then symbol extensions are not required. Now, the above equations imply that, over two time slots, the relay network reduces to a three user MIMO interference channel with (nondiagonal) structure on the channel matrix. Since the channel matrix is nondiagonal (unlike symbol extensions in the absence of a relay) it is easy to verify that when the channels are random and independent of each other, a multiplexing gain of $\frac{3}{2}$ is achieved with probability 1. The advantage of this scheme is that it requires only two time-slots to achieve $3/2$ degrees of freedom per time-slot, whereas in the absence of the relay, we simply have the 3-user SISO interference channel which requires arbitrarily large number of time-slots [1].

In Fig. 6 we plot the performance of the interference alignment schemes for the case of time extensions. For the interference alignment scheme with relay, the transmit power per node is P. That is, the three transmitters and the relay have a total power of $4P$. The transmit power for the orthogonal scheme where only transmitter is active at a time is $4P$. The interference alignment scheme without relay achieves a multiplexing gain of 4 in 3 time slots [1]. Here transmitter 1 sends two streams while transmitters 2 and 3 send one stream each. The transmit power per stream is P adding up to $4P$ for the scheme. We can also increase the number of time slots for channel extension to improve the multiplexing gain. However, it requires very high P to outperform the orthogonal scheme. It can be seen that adding an extra relay helps in achieving the multiplexing gain of $\frac{3}{2}$ in two slots. Further the performance improves at low P

as well. The performance of the algorithm can be improved by optimizing the power allocation at the relay. Simulations reveal that the application of iterative algorithms for the interference channel with relays lead to poor performance, even as comparable to orthogonalization. While the fundamental reason for this poor performance is not clear, the distributed nature of the algorithm in combination with the lower-triangular nature of the matrices seem to lead to infeasible interference alignment solutions which satisfy (11), but not (12).

It must be stressed that the main idea behind the interference alignment scheme with the relay is to show that there are benefits in employing relays in making alignment feasible with limited signalling dimensions. The idea of using relays to enhance the feasibility alignment schemes has also been extended to X channels in [30]. In particular, the reference shows that the relay obviates the need of frequency selectivity and time-varying nature of the channel interference alignment over certain X channels. It must be noted that we need the channel to be time-varying (or frequency-selective) in our case. This is because, if the channel is not time-varying, then each channel matrix is a 2×2 lower triangular matrix with identical diagonal elements. That means that the vector $[0 \quad 1]^T$ is the only eigen vector for all the channel matrices, and then the solution of [1] would not be applicable (See discussion in Section IV-D in [1]). However, with the time-varying channels, different channels have different (non-collinear) sets of eigen vectors with probability 1 and the solution of the MIMO 3-user interference channel [1] is applicable here.

IX. CONCLUSION

We developed a distributed numerical approach to interference alignment in interference channels to complement the recently developed analytical approaches [1]. Interference alignment is found to be achievable through iterative algorithms based on network reciprocity and the "minimize interference to others" approach. Numerical comparisons to orthogonal schemes, simultaneous transmission schemes and selfish interference avoidance schemes show that the benefits of distributed interference alignment algorithm are significant and close to the theoretical predictions. Further, the algorithms provide insights into the feasibility of alignment, where theoretical results are not available, particularly in cases where the number of signalling dimensions are limited. In general, as mitigating interference is the fundamental problem of wireless networks, the 'do no harm' approach based algorithms have enormous applications in wireless networks.

The paper leads to several interesting open problems. Firstly, there are several algorithmic aspects of the problem that remain to be investigated, such as the rate of convergence as the network size increases. The convergence of the Max-SINR algorithm is also an open question. Secondly, while the algorithms are applicable in certain cases, they are not applicable when the channels have a diagonal/block-diagonal/lower-triangular structure. This means that the numerical insights are not applicable in channels where time-extensions are used. The development of numerical algorithms for these cases remains an open problem. Finally, the paper motivates a more thorough investigation of the idea of using relays to simplify alignment

schemes. In particular, the example described in this paper restricts itself to the 3-user time-varying/frequency-selective interference channel. The paper opens the question of whether, and to what extent, relays can simplify alignment schemes in larger interference channels, or in channels which are not time-varying/frequency-selective. Partial results in this direction [30] for X channels suggest that this is a promising direction of research.

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