DYNAMICAL SYSTEMS WITH POLYGONS

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1. Introduction

In this paper, we consider various dynamical systems on polygons. Specifically, we assign numerical values to the vertices of polygons, dictate an iteration rule (that specifies the next polygon in a sequence given the current polygon), and then study the behavior of these sequences of polygons. Specifically, we study three distinct dynamical systems: a rule of absolute difference of adjacent vertices on a square, a rule of absolute difference of adjacent vertices on a triangle, and a rule of addition in $\mathbb{Z}_n$ on a square.

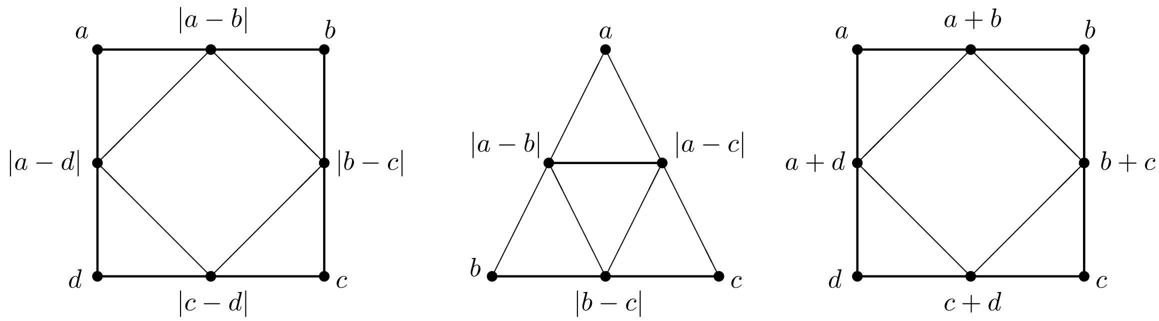

Figure 1. The larger polygon shows the values of the first polygon $x_0 = (a, b, c, d)$ or $x_0 = (a, b, c)$, and then the update rule is shown for the next (smaller) polygon x_1. This process continues recursively.

Consider a square marked with four numerical values (a_0, b_0, c_0, d_0) at each vertex. We consider the sequence of squares $x_k = (a_k, b_k, c_k, d_k)$ generated by the following game: in one step, write at the midpoint of each edge of (a_k, b_k, c_k, d_k) the absolute value of the difference of numbers at the endpoints of that edge, then draw a smaller square $x_{k+1} = (|a_k - b_k|, |b_k - c_k|, |c_k - d_k|, |a_k - d_k|)$ by connecting those midpoints. We define the length of a game $L(x_0) = n$ to be the smallest n such that $x_n = (a_n, b_n, c_n, d_n) = (0, 0, 0, 0)$. For example, a game that has the sequence of squares $(1, 3, 9, 7) \to (2, 6, 2, 6) \to (4, 4, 4, 4) \to (0, 0, 0, 0)$ is a finite game with length of 3. Throughout this paper, we study the 3 different games shown above and their lengths.

In Section 2, we study the game in the integer case where each vertex of the square has a positive integer or zero. We show the integer game is finite and more specifically find an explicit upper bound for the length of the game. Then we extend to the real case and show that all real games converge to $(0, 0, 0, 0)$. However, we show there exists infinite real game by showing a unique example.

In Section 3, we change the shape of the game to a triangle, while keeping the same iteration rule (absolute difference). In section 3.1, we prove any triangle $x_0 = (a_0, b_0, c_0) \in \mathbb{Z}^3$ never converges to $(0, 0, 0)$, but cycles on $(m, m, 0)$. We examine the length of integer games, where we define length $L(x_0)$ to be the number of iterations it takes for triangle x_0 to equal $(m, m, 0)$. Then, in section 3.2, we examine the real case to find games of infinite length that converge to $(0, 0, 0)$ or $(m, m, 0)$. Although a full characterization of games hasn't completed yet, we conjecture there are many games for which convergence to $(0, 0, 0)$ is guaranteed, and convergence to $(x, x, 0)$ is also guaranteed for other games.

In Section 4, we consider a square where the update rule is addition; specifically, it is addition in $\mathbb{Z}_n$, the integers modulo n. That is, $(a, b, c, d) \to (a + b, b + c, c + d, d + a) \pmod{n}$, where $\pmod{n}$ denotes the remainder when each element is divided by n (an element of $\{0, 1, \ldots, n - 1\}$). We specify all initial squares when working in $\mathbb{Z}_n$ which give sequences that eventually reach $(0, 0, 0, 0)$ (or "converge"), and we give a bound on the length of these games. Additionally, we study the periodic behavior of non-converging sequences, and specify all initial squares which eventually reappear in the sequence of squares in $\mathbb{Z}_n$ are determined for odd n.

Section 2.1 was written by Chae Won Lee. The first approach in Section 2.2 was written by Skyler Seto, and the second approach was written by Chae Won Lee. Section 3 was written by Skyler Seto, and Section 4 was written by Eli Ross.

2. Square Game

2.1 The Integer Case

Definition 2.1. (Square Game in the Integer Case) Given the game with initial square $x_0 = (a_0, b_0, c_0, d_0)$ where $a_0, b_0, c_0, d_0 \in \mathbb{N} \cup 0$, the kth square is given by $x_k = (|a_{k-1} - b_{k-1}|, |b_{k-1} - c_{k-1}|, |c_{k-1} - d_{k-1}|, |a_{k-1} - d_{k-1}|)$. The length of the game $L(x_0) = \min\{k | x_k = (0, 0, 0, 0)\}$.

In this section, we first observe the length of the game is preserved when the square is transformed by reflection, rotation, and multiplication by a constant. It is easy to also observe that the maximum number of the square decreases at every step since the absolute difference between two positive numbers cannot be greater than both positive numbers; that is, $max(a, b, c, d) \geq max(|a-b|, |b-c|, |c-d|, |a-d|)$. Given these facts, we find an upper bound of $L(a, b, c, d)$ using an algorithm that counts the steps based on each game having only even numbers within four steps.

Fact 2.2. *Given a constant $l > 0$ and $(a_0, b_0, c_0, d_0) \in \mathbb{R}^4$, we have the kth square in the sequence, generated by $(l \cdot a_0, l \cdot b_0, l \cdot c_0, l \cdot d_0)$, is given by $(l \cdot a_k, l \cdot b_k, l \cdot c_k, l \cdot d_k)$. In particular, $L(l \cdot a_0, l \cdot b_0, l \cdot c_0, l \cdot d_0) = L(a_0, b_0, c_0, d_0)$.*

Fact 2.3. *Rotation of the square (a, b, c, d) by 90 degrees in the counterclockwise direction to (b, c, d, a) preserves the length of the game; that is, $L(a, b, c, d) = L(b, c, d, a)$.*

Fact 2.4. *Reflection of the square (a, b, c, d) across diagonals preserves the length of the game; that is, $L(a, b, c, d) = L(a, d, c, b) = L(c, b, a, d)$.*

Given an integer square (a, b, c, d), let the *parity* p denote which numbers are even and which are odd. For example, the number square $x = (2, 1, 1, 2)$ corresponds to $p = (e, o, o, e)$.

Proposition 2.5. Within four steps, the *parity* of any integer game will be (e, e, e, e).

Proof. Because of of the rotation and reflection invariance of Fact 2.3 and 2.4, we only need to consider the cases shown in the diagram of Figure 2. For example, the number square $(2, 2, 1, 1)$ has the same length of game as $(1, 2, 2, 1)$ by Fact 2.3, and so corresponds to *parity* $p = (o, e, e, o)$ in the diagram. Then the *parity* of the game goes from (o, e, e, o) to (o, e, o, e) to (o, o, o, o) and finally to (e, e, e, e). Hence, any square goes to all evens within four steps as shown in Figure 2. □

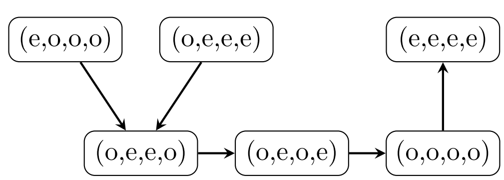

Figure 2. Parity of any integer game goes to all evens

Then given an arbitrary integer square (a, b, c, d), we can repeat the following two steps until all numbers of the square are less than 2.

1) Parity-reduction: Take less than four steps until the square is (e, e, e, e) by Proposition 2.5.

2) Divide-by-two: Divide the all even numbers by 2 without changing the length of the game by Fact 2.2.

Theorem 2.6. *Any integer square game finishes in a finite number of steps.*

Proof. Given finite initial numbers a, b, c, d, we repeat the parity-reduction and divide-by-two step of the algorithm a finite number of times until all numbers of the square are less than 2. Then by parity reduction, we find $(0, 0, 0, 0)$. $\square$

Corollary 2.7. Any rational square game finishes in a finite number of steps.

Proof. Given any rational square $(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ where $p_i \in \mathbb{N} \cup 0$ and $q_i \in \mathbb{N}$, multiply each number by $q_0 \cdot q_1 \cdot q_2 \cdot q_3$ without changing the length of the game by Fact 2.2. Because $L(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) = L(p_0 \cdot q_1 q_2 q_3, p_1 \cdot q_0 q_2 q_3, p_2 \cdot q_0 q_1 q_3, p_3 \cdot q_0 q_1 q_2)$, any rational square game is finite by Proposition 2.5. $\square$

Furthermore, computing the maximum number of times we repeat the parity-reduction and divide-by-two steps provides an upper bound on the length of the game.

Theorem 2.8. *Given $a, b, c, d \in \mathbb{N} \cup 0$, the upper bound for the $L(a, b, c, d)$ is $4 \cdot \lceil \log_2 max(a, b, c, d) \rceil$ where $\lceil \cdot \rceil$ is the smallest integer not less than $\cdot$.*

Proof. Given the finite initial numbers a, b, c, d, we only need to take a finite i steps until all numbers of the square are less than 2. This implies that i is the smallest integer such that

$$\frac{max(a, b, c, d)}{2^i} \leq 1$$

or $i \geq \log_2 \max(a, b, c, d)$. Therefore the number of times we take the divide-by-two step is bounded from above by $\lceil \log_2 max(a, b, c, d) \rceil$. And between the divide-by-two steps, there are at most four steps by Proposition 2.5. Hence, we take up to $4 \cdot \lceil \log_2 max(a, b, c, d) \rceil$ steps to end the game. $\square$

2.2. The Real Case

We proved in Section 2.1 that the game with integers converges to the same square, $(0, 0, 0, 0)$ after a finite number of steps. Naturally, we would like to see what happens when we remove the integer restriction and allow any real numbers.

Definition 2.9. (Square Game in the Real Case) Given the game with initial square $x_0 = (a_0, b_0, c_0, d_0) \in \mathbb{R}^4$, the kth square is given by $x_k = (|a_{k-1} - b_{k-1}|, |b_{k-1} - c_{k-1}|, |c_{k-1} - d_{k-1}|, |a_{k-1} - d_{k-1}|)$. The game is played bu the rules of the integer case.

In this section, we will prove that any square with real numbers still converges to $(0, 0, 0, 0)$, but not all games converge in a finite number of steps. We will also show one game that takes an infinite number of steps to reach $(0, 0, 0, 0)$.

By reflection and rotation invariance of Facts 2.2 and 2.3 which are still valid, we only need to consider the length of the game for three cases.

Lemma 2.10. *Given a real square (a, b, c, d), there exists a combination of rotation and reflection that gives the transformed square (a', b', c', d') which satisfies $L(a, b, c, d) = L(a', b', c', d')$ and one of the following conditions holds:*

$$a' \geq c' \geq b' \geq d'$$
$$a' \geq b' \geq d' \geq c'$$
$$a' \geq b' \geq c' \geq d'$$

Proof. By rotation invariance (Fact 2.3), we can let the maximum number of the square be at the upper left corner a. By reflection invariance (Fact 2.4), we can let $b \geq d$. With these two conditions fixed, there are only three cases listed above that need to be considered. $\square$

Lemma 2.11. *Given (a, b, c, d), the game starting at (a, b, c, d) is finite if one of the following conditions holds:*

$$a \geq c \geq b \geq d$$
$$a \geq b \geq d \geq c$$

Proof. We show the game that starts at (a, b, c, d) is finite by the sequence of squares of the game that converge to zero. If the condition $a \geq c \geq b \geq d$ holds, then the sequence of squares is:
$(a, b, c, d) \rightarrow$
$(a-b, c-b, c-d, a-d) \rightarrow$
$(a-c, b-d, a-c, b-d) \rightarrow$
$(|a-c-b+d|, |a-c-b+d|, |a-c-b+d|, |a-c-b+d|) \rightarrow$
$(0, 0, 0, 0)$
If the condition $a \geq b \geq d \geq c$ holds, then the sequence of squares is:
$(a, b, c, d) \rightarrow$
$(a-b, b-c, d-c, a-d) \rightarrow$
$(|a-2b+c|, b-d, |a-2b+c|, b-d) \rightarrow$
$(||a-2b+c|-(b-d)|, |b-d-|a-2b+c||, ||a-2b+c|-(b-d)|, |b-d-|a-2b+c||) \rightarrow$
$(0, 0, 0, 0)$

□

Theorem 2.12. *Let $x_0 = (a_0, b_0, c_0, d_0)$ be the initial square and x_i be the square after i iterations. Then we have*

$$\lim_{i\to\infty} x_i = (0, 0, 0, 0)$$

Proof. For a given square x_0, consider x_i. There are only 3 cases of the x_i that need to be considered by Lemma 2.10:

(1) $x_i = (a_i, b_i, c_i, d_i)$ where $a_i \geq c_i \geq b_i \geq d_i$
(2) $x_i = (a_i, b_i, c_i, d_i)$ where $a_i \geq b_i \geq d_i \geq c_i$
(3) $x_i = (a_i, b_i, c_i, d_i)$ where $a_i \geq b_i \geq c_i \geq d_i$

We showed in Lemma 2.11 that the first two cases converge quickly in a finite number of steps, so it suffices to prove the last case converges to $(0, 0, 0, 0)$. We assume we always have the configuration $a_i \geq b_i \geq c_i \geq d_i$, otherwise the game converges in a short number of steps.

We would like to show the two following properties now:

(1) The square sequence satisfies the recursion formula
$a_{i+1} = a_i - d_i$
$b_{i+1} = a_i - b_i$
$c_{i+1} = b_i - c_i$
$\mathrm{d}_{i+1} = c_i - d_i$
(2) $a_i, b_i, c_i, d_i \rightarrow 0$ as $i \rightarrow \infty$

To show the first property, consider $(a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1})$ that maintains the configuration. $a_i - d_i$ is the largest number of the next square, so clearly $a_{i+1} = a_i - d_i$. Similarly, we know $b_i - c_i$ is the 3rd largest number, since considering Figure 1, $b_i - c_i$ is the non adjacent vertex to a_{i+1}, and if we keep the same configuration, $c_{i+1} = b_i - c_i$.

Now we examine b_{i+1} and d_{i+1}. Since $a_i - b_i = (b_i + c_i + d_i) - b_i = c_i + d_i > c_i - d_i$, we have that $b_{i+1} = a_i - b_i$ and $d_{i+1} = c_i - d_i$ to maintain configuration $b_{i+1} \geq d_{i+1}$. This completes the proof of the first property.

It is derived that assuming $i \geq 1$, after one iteration we have $(a_i - d_i, a_i - b_i, b_i - c_i, c_i - d_i)$ and $a_i - d_i = (a_i - b_i) + (b_i - c_i) + (c_i - d_i)$.

To show the second property, from the result above, we have that $a_n = a_{n-1} - d_{n-1} = a_{n-2} - d_{n-1} - d_{n-2} = \cdots = a_0 - \sum_{i=0}^{n-1} d_i$. The sequence of a_n is decreasing and bounded below by 0. Because a_n is non-negative we have $\lim_{n\to\infty} \sum_{i=0}^{n-1} d_i$ exists and is bounded by a_0, so d_n must become arbitrarily small as $n \rightarrow \infty$.

So, we must have $\lim_{i\to\infty} d_i = 0$. We can use the fact that $d_{i+1} = c_i - d_i$ to conclude that $\lim_{i\to\infty} c_i = 0$ as well. By similar reasoning $a_i, b_i \rightarrow 0$. So $a_i, b_i, c_i, d_i \rightarrow 0$.

□

It's not hard to see from the proof of Theorem 2.12 that to keep the cyclic order at every step, the numbers of the square must satisfy $a_i = b_i + c_i + d_i$. Therefore if there is an infinite game, it must satisfy the conditions $a_i > b_i > c_i > d_i$ and $a_i = b_i + c_i + d_i$ at every step. For the remainder of the section, we will find such an example using the linear algebra approach and use a vector notation that defines a square (a, b, c, d) as a column vector $(a, b, c, d)^T$.

Given the 0th square $x = (a_0, b_0, c_0, d_0)^T \in \mathbb{R}^4$ where $a_0 = b_0 + c_0 + d_0$ and $a_0 \geq b_0 \geq c_0 \geq d_0$, the property (1) can be interpreted as $x_{i+1} = Mx_i$ where $M = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

Theorem 2.13. *Any constant multiple of* $(6.22226, 3.38298, 1.83929, 1.)^T$ *is an infinite game among real number square games.*

Proof. Given an initial real square $x = (a_0, b_0, c_0, d_0)^T$ that generates squares that satisfy conditions $a_i > b_i > c_i > d_i$ and $a_i = b_i + c_i + d_i$ at every step, it is natural to consider the eigenpair of M.

We will disregard the complex eigenvalues λ_1, λ_2 of M because the corresponding eigenvectors are also complex and so do not apply to the real square game.

We then consider eigenvalue $\lambda_3 = 0$ and its associated eigenvector $(1, 1, 1, 1)^T$. For $\lambda_3 = 0$, $M^n x = \lambda_3^n x$ is true for a square $x = (a, b, c, d)^T$ where $a = b = c = d$. So, the game trivially ends in the next step.

Finally, we consider $\lambda_4 \approx 0.839$ and its associated eigenvector $(6.22226, 3.38298, 1.83929, 1.)$. For $0 < \lambda_4 < 1$, $M^n x = \lambda_4^n x \neq 0$ is true for x that is the eigenvector of λ_4.

Hence, a constant of multiple of the eigenvector of λ_4 gives an infinite game. □

3. Behavior of Triangle Games with Subtraction

Naturally, we would like to vary the rules of the game and consider other possible games. One easy extension is to consider different polygons. In this section, the game uses triangles with the same subtraction rule as in the square case. Before we precisely define our game, we first explain our notatino for the section

Definition 3.1. The standard form for a triangle will be written $x = (a, b, c)$ where $a \geq b \geq c$ and $a, b, c \in \mathbb{R}$.

We are allowed to express the triangle in this way because all triangles with vertices a, b, c are just rotations of one another. Unless otherwise stated, when we refer to triangle x, we will assume it is in standard form.

We also denote the initial triangle as $x_0 = (a_0, b_0, c_0)$, and denote a triangle at the i^{th} iteration as $x_i = (a_i, b_i, c_i)$. We define our game formally below.

Definition 3.2. (Triangle Game with Subtraction in $\mathbb{R}$)
Given an initial triangle $x_0 = (a_0, b_0, c_0), a_0, b_0, c_0 \in \mathbb{R}$, the kth triangle in the game is given by $x_k = (a_{k-1} - c_{k-1}, \max\{a_{k-1} - b_{k-1}, b_{k-1} - c_{k-1}\}, \min\{a_{k-1} - b_{k-1}, b_{k-1} - c_{k-1}\})$.

Note: In this section, we first study the special case where $a_0, b_0, c_0 \in \mathbb{Z}$, *then extend the game to* $a_0, b_0, c_0 \in \mathbb{R}$.

Now, to motivate our discussion of the triangle, we have the following theorem:

Theorem 3.3. *For any triangle* $x_0 = (a_0, b_0, c_0)$ *if* $a_0, b_0, c_0 \in \mathbb{R}$, *the game does not converge to* $(0, 0, 0)$ *in a finite number of iterations.*

Proof. If the triangle becomes $(0, 0, 0)$, then at the previous iteration, it must have been (m, m, m). For some triangle (a_0, b_0, c_0) to reach (m, m, m), $a_n - c_n = a_n - b_n = b_n - c_n$ must be true, but this only happens when $a_n = b_n = c_n$. Now we show $a_n = b_n = c_n$ can never be true. We show this trivially by induction.

Initially, we have $a_n = b_n = c_n$. Suppose $a_i = b_i = c_i$, where $i < n$ for the triangle $x_{i-1} = (a_{i-1}, b_{i-1}, c_{i-1})$. We need $a_{i-1} - b_{i-1} = a_{i-1} - c_{i-1} = b_{i-1} - c_{i-1}$, which only happens if $a_{i-1} = b_{i-1} = c_{i-1}$. Since this would imply $a_0 = b_0 = c_0$, we must have that $a_n \neq b_n \neq c_n$, which completes the proof. □

Unlike the square, we see that for no distinct values of a, b, c, does the triangle converge to $(0,0,0)$ in a finite number of steps. In the rest of the section, we will discuss the long term behavior of the triangle game. Section 3.1 shows all triangles with integer vertices cycle on $(m, m, 0)$, where $m \neq 0$. Section 3.2 shows all triangles with real vertices converge to either $(0,0,0)$ or $(m, m, 0)$.

3.1. Cyclic Behavior of Triangles in $\mathbb{Z}$

Next, for any triangle $x_i = (a_i, b_i, c_i)$, let $D_i = a_i - c_i$. $D_i \in \mathbb{Z}^*$ is the maximum difference between any two vertices on the triangle at the i^{th} iteration.

In this section, we will use the maximum difference at each iteration, D_i, to show that $\forall a_0, b_0, c_0 \in \mathbb{Z}$ and $a_0 > b_0 > c_0$, $x_0 = (a_0, b_0, c_0) \to (m, m, 0)$, where $m \in \mathbb{Z}^+$ after a finite number of iterations. $(m, m, 0)$ then cycles. The triangle (a_0, b_0, c_0) trivially converges to $(0,0,0)$ after one step if $a_0 = b_0 = c_0$.

Remark: The reason we allow any negative integers for a_0, b_0, c_0 and later negative real numbers is because after one iteration, since we are taking the absolute value of the difference, they become positive integers or positive reals. We handle the initial triangle differently, but considering the triangle x_1 as the initial triangle won't have any change on the long term behavior of the game.

Theorem 3.4. *For any triangle $x_0 = (a_0, b_0, c_0)$ if $a_0 > b_0 > c_0$ and $a_0, b_0, c_0 \in \mathbb{Z}$, the game starting at x_0 reaches $(m, m, 0)$ after a finite number of iterations, then cycles on $(m, m, 0)$ where $m \in \mathbb{Z}^+$.*

Proof. To show this, it suffices to show that $\forall i \in \mathbb{Z}^+, D_{i+1} \leq D_i$. The reason for this is D_i cannot decrease at every iteration, since if it does, then for some finite N, $D_N = 0$, which cannot happen because $a_i \neq b_i \neq c_i$ by our initial assumption and by Theorem 3.1. So, at some iteration it must stop decreasing, and $D_{k+1} = D_k$, which implies $a_k - c_k = a_k - b_k$ or $a_k - c_k = b_k - c_k$, and $c_k = b_k$ or $a_k = b_k$. After one iteration, $x_{k+1} = (a_k, a_k, b_k)$ or $x_{k+1} = (a_k, b_k, b_k)$. After another iteration, we have $x_{k+2} = (a_k - b_k, a_k - b_k, 0)$.

Now we consider the general triangle (a_i, b_i, c_i). For this triangle we have $D_i = a_i - c_i$. After one iteration,

$$x_i \to x_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = \begin{cases} (a_i - c_i, a_i - b_i, b_i - c_i) & : a_i - b_i > b_i - c_i \\ (a_i - c_i, b_i - c_i, a_i - b_i) & : \text{ otherwise} \end{cases}$$

The maximum difference $D_{i+1} = a_{i+1} - c_{i+1}$ becomes

$$D_{i+1} = \begin{cases} a_i - c_i - (b_i - c_i) = a_i - b_i & : a_0 - b_0 > b_0 - c_0 \\ a_i - c_i - (a_i - b_i) = b_i - c_i & : \text{ otherwise} \end{cases}$$

Previously we had $D_i = a_i - c_i$, and $D_{i+1} = a_i - b_i$ or $b_i - c_i$.

$$D_i - D_{i+1} = \begin{cases} a_i - c_i - (a_i - b_i) & : a_0 - b_0 < b_0 - c_0 \\ a_i - c_i - (b_i - c_i) & : \text{ otherwise} \end{cases}$$

The difference is either $b_i - c_i$, which is greater than 0, or $a_i - b_i$ which is also greater than 0. So, $D_{i+1} < D_i$. Finally, we note that $(m, m, 0)$ cycles. $(m, m, 0) \to (m - 0, m - 0, m - m) = (m, m, 0)$.

□

For an upper bound of the length of the game, we consider the worst case triangle. The worst case is the triangle $(a, 1, 0)$. $(a, 1, 0) \to (a, a - 1, 1) \to (a - 1, a - 2, 1) \ldots$. In the last step, to preserve the ordering $a \geq b \geq c$, we took $b - c$ then $a - b$, and we would continue this process $a + 1$ times to get $(1, 1, 0)$. For the worst case game, our goal was to avoid (a, b, b) the terminating case as long as possible. It's obvious we accomplish this with the above example, since none of them are equal until $a - 1$ becomes 1, after a iterations, and $a - 1$ decreases by 1 each time, the smalles it can decrease for integer values.

We conclude (without proof) an upper bound on the length $L(x_0)$ of a game with initial triangle $x_0 = (a_0, b_0, c_0)$ would be $L(x_0) \leq a_0 + 1$.

3.2. Behavior of Triangles in $\mathbb{R}$

In this section, we extend our discussion of triangle games to real numbers. We will prove there are some games that converge to $(0, 0, 0)$, and other infinite games that converge to $(m, m, 0)$ and cycle. Here by converge to (w, y, z), we mean the

$$\lim_{n\to\infty}(a_n, b_n, c_n) = (w, y, z).$$

This can be slightly confusing because one might think we can't have convergence if we have periodic behavior. We will also provide examples of triangles with these behaviors. Again, as a reminder, our standard form for a triangle (a, b, c) will be where $a \geq b \geq c$ and $a, b, c \in \mathbb{R}$, which we are allowed to write because all triangles with the same vertices are just rotations of one another.

Interestingly, we note that for triangle (a_n, b_n, c_n), we don't know if $a_n - b_n > b_n - c_n$ after one iteration unlike in the infinite length square game where we always had the same expression for $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}$. This means we can't define a single linear operator, as was used to find the infinite game for the square, so we must use casework to discuss the different triangles that can arise after one iteration. To do this, we define temporary variables $a'_{n+1}, b'_{n+1}, c'_{n+1}$.

$$a'_{n+1} = a_n - c_n, b'_{n+1} = a_n - b_n, \text{ and } c'_{n+1} = b_n - c_n.$$

We have two cases for the triangle in standard form:

(1) $(a'_{n+1}, b'_{n+1}, c'_{n+1})$
(2) $(a'_{n+1}, c'_{n+1}, b'_{n+1})$

Our goal will be to classify the behavior of all triangles. The following four lemmas discuss the conditions necessary to stay in case 1 or case 2, the triangle after each iteration, and the long term behavior of triangles always in case 1 or case 2. Afterwards, Theorem 3.9 will show a triangle that switches between the two cases converges to either $(0, 0, 0)$ or $(m, m, 0)$. To show the properties of case 1 and case 2, we use induction. For case 3 , show that the values of the vertices decrease to a steady state solution and this solution is either $(0, 0, 0)$ or $(m, m, 0)$.

First, for simplicity, we consider the game after one iteration to be the starting game. The reason we do this is after one iteration, $(a_0, b_0, c_0) \to (a_0 - c_0, a_0 - b_0, b_0 - c_0)$ or $(a_0 - c_0, b_0 - c_0, a_0 - b_0)$, $a_1 = b_1 + c_1$, so we can just consider the triangle $(b_0 + c_0, b_0, c_0)$, and considering the triangle after one iteration doesn't really change the game's behavior in any way.

Case 1 $(a'_{n+1}, b'_{n+1}, c'_{n+1})$

To start our exploration of triangles in $\mathbb{R}$, we assume if we are in case 1 at the n^{th} iteration, then we have been in the configuration (a'_n, b'_n, c'_n),that is we always have $b'_n > c'_n$, up through time n.

In our short exploration, we begin by explicitly finding the first few iterations and examine the conditions necessary for $b'_n > c'_n$ to be true. The conditions and triangles from the first few iterations are then summarized and discussed in the table right before the lemma. Then, in the lemma below, we provide an explicit form for the triangle after n iterations, and the conditions necessary for the triangle to still be in case 1.

To begin, we start with the configuration $(b_0 + c_0, b_0, c_0)$. After two iterations,

$$x_0 = (b_0 + c_0, b_0, c_0) \to x_1 = (a_1, b_1, c_1) = (b_0, c_0, b_0 - c_0) \to x_2 = (a_2, b_2, c_2) = (c_0, b_0 - c_0, 2c_0 - b_0).$$

Since $b_1 > c_1$ must be true, we must have

$$c_0 > b_0 - c_0 \text{ or } b_0 < 2c_0.$$

Similarly, after another iteration, $b_2 > c_2$ must be true, and we have

$$b_0 - c_0 > 2c_0 - b_0 \text{ or } b_0 > \frac{3}{2}c_0.$$

Since both conditions need to be satisfied:

$$\frac{3}{2}c_0 < b_0 < 2c_0.$$

After two more iterations,

$$(c_0, b_0 - c_0, 2c_0 - b_0) \to x_3 = (a_3, b_3, c_3) = (b_0 - c_0, 2c_0 - b_0, 2b_0 - 3c_0)$$
$$(b_0 - c_0, 2c_0 - b_0, 2b_0 - 3c_0) \to x_4 = (a_4, b_4, c_4) = (2c_0 - b_0, 2b_0 - 3c_0, 5c_0 - 3b_0)$$

Again, since $b_3 > c_3$ and $b_4 > c_4$ must both be true, we must have

$$\frac{8}{5}c_0 < b_0 < \frac{5}{3}c_0.$$

The following table summarizes the derivation above.

$x_i = (a_i, b_i, c_i)$	$b'_i > c'_i$	upper or lower bound
$x_1 = (b_0, c_0, b_0 - c_0)$	$c_0 > b_0 - c_0$	$b_0 < 2c_0$
$x_2 = (c_0, b_0 - c_0, 2c_0 - b_0)$	$b_0 - c_0 > 2c_0 - b_0$	$b_0 > \frac{3}{2}c_0$
$x_3 = (b_0 - c_0, 2c_0 - b_0, 2b_0 - 3c_0)$	$2c_0 - b_0 > 2b_0 - 3c_0$	$b_0 < \frac{5}{3}c_0$
$x_4 = (2c_0 - b_0, 2b_0 - 3c_0, 5c_0 - 3b_0)$	$2b_0 - 3c_0 > c_0 - 3b_0$	$b_0 > \frac{8}{5}c_0$

We notice two interesting properties from the table. First, we see that the odd iterations give an upper bound on b_0, and the even iterations give a lower bound. Furthermore, the odd bounds are decreasing, and the even bounds are increasing, making the constraint on b_0 at each iteration stronger. Second, we notice an interesting pattern for the bounds on b_0 in terms of c_0. This pattern uses the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$. We would guess that the general constraint on b_0 after an even n iterations is $\frac{F_{i+2}}{F_{i+1}}c_0 < b_0 < \frac{F_{i+1}}{F_i}c_0$, where F_i is the i^{th} Fibonacci number indexed from 0 and i is even. The following lemma proves this fact, and also gives an expression for the triangle at the ith iteration.

Lemma 3.5. *Given triangle $x_0 = (b_0 + c_0, b_0, c_0)$, if $\forall k \in \mathbb{Z}^+$, $b'_k > c'_k$, then for $k \geq 2$,*

$$x_k = \big((-1)^k(F_{k-1}c_0 - F_{k-2}b_0), (-1)^k(F_{k-1}b_0 - F_k c_0), (-1)^k(F_{k+1}c_0 - F_k b_0)\big)$$

$$\textit{and for even } k \geq 2, \frac{F_{k+2}}{F_{k+1}}c_0 < b_0 < \frac{F_{k+1}}{F_k}c_0$$

Proof. We prove the lemma using induction. Since $\forall k$, $b'_k > c'_k$ we let $b_k = b'_k, c_k = c'_k$ always, and $a_k > b_k > c_k$. The reason why we require even k is that we get an upper bound on b_0 from the odd iterations, and a lower bound on b_0 from the even iterations. This is seen above in the 3rd column of the table. Neither of these cases is extremely helpful by themselves, so we need to consider them together for a closed interval bound on b_0.

After three iterations the triangle is $(F_1 b_0 - F_2 c_0, F_3 c_0 - F_2 b_0, F_3 b_0 - F_4 c_0)$, which constrains $b_0 < \frac{F_5}{F_4}c_0$ and the triangle after four iterations $(F_3 c_0 - F_2 b_0, F_3 b_0 - F_4 c_0, F_5 c_0 - F_4 b_0)$ constrains $\frac{F_6}{F_5}c_0 < b_0$. Then, after four iterations, we have $\frac{F_6}{F_5}c_0 < b_0 < \frac{F_5}{F_4}c_0$ must be true. We have already dealt with this case above (ro2s 4 and 5 of the table, but for simplicity we are using the F_n notation.

Assume at k (even) iterations the triangle is

$$\big((-1)^k(F_{k-1}c_0 - F_{k-2}b_0), (-1)^k(F_{k-1}b_0 - F_k c_0), (-1)^k(F_{k+1}c_0 - F_k b_0)\big)$$

and, $\dfrac{F_{k+2}}{F_{k+1}}c_0 < b_0 < \dfrac{F_{k+1}}{F_k}c_0$.

We will compute the $k+1$ and $k+2$ step to show the statement is true. Since $\forall k$, $a_k > b_k > c_k$ and $a_k - b_k > b_k - c_k$ under our assumption, then we have $a_{k+1} = a_k - c_k$, $b_{k+1} = a_k - b_k$, $c_{k+1} = b_k - c_k$, and

$$\begin{aligned}
a_{k+1} &= (-1)^k(F_{k-1}c_0 - F_{k-2}b_0) - (-1)^k(F_{k+1}c_0 - F_kb_0)\\
&= (-1)^k((F_{k-1} - F_{k+1})c_0 - (F_{k-2} - F_k)b_0)\\
&= (-1)^{k+1}(F_kc_0 - F_{k-1}b_0)\\
b_{k+1} &= (-1)^k(F_{k-1}c_0 - F_{k-2}b_0) - (-1)^k(F_{k-1}b_0 - F_kc_0)\\
&= (-1)^{k+1}(F_kb_0 - F_{k+1}c_0)\\
c_{k+1} &= (-1)^k(F_{k-1}b_0 - F_kc_0) - (-1)^k(F_{k+1}c_0 - F_kb_0)\\
&= (-1)^{k+1}(F_{k+2}c_0 - F_{k+1}b_0)
\end{aligned}$$

Since $b_{k+1} > c_{k+1}$, $(-1)(F_kb_0 - F_{k+1}c_0) > (-1)(F_{k+2}c_0 - F_{k+1}b_0)$ which implies $F_{k+2}b_0 < F_{k+3}c_0$ which gives $b_0 < \frac{F_{k+3}}{F_{k+2}}c_0$.

Computing the $k+2$ iteration gives us the other side of the bound.

$$\begin{aligned}
a_{k+2} &= (-1)^{k+1}(F_kc_0 - F_{k-1}b_0) - (-1)^{k+1}(F_{k+2}c_0 - F_{k+1}b_0) = (-1)^{k+2}(F_{k+1}c_0 - F_kb_0)\\
b_{k+2} &= (-1)^{k+1}(F_kc_0 - F_{k-1}b_0) - (-1)^{k+1}(F_kb_0 - F_{k+1}c_0) = (-1)^{k+2}(F_{k+1}b_0 - F_{k+2}c_0)\\
c_{k+2} &= (-1)^{k+1}(F_kb_0 - F_{k+1}c_0) - (-1)^{k+1}(F_kb_0 - F_{k+1}c_0) = (-1)^{k+2}(F_{k+3}c - F_{k+2}b)
\end{aligned}$$

$(F_{k+1}b - F_{k+2}c) > (F_{k+3}c - F_{k+2}b)$ which implies $F_{k+3}b > F_{k+4}c$ which confirms the induction. □

Lemma 3.6. *$\forall n$, if $b'_n > c'_n$, then*

$$\lim_{n\to\infty}(a_n, b_n, c_n) = (0, 0, 0)$$

Proof. If $\frac{F_{n+2}}{F_{n+1}}c_0 < b_0 < \frac{F_{n+1}}{F_n}c_0$ is always within the interval, $b_0 = \phi c_0$, where $\phi = \frac{1+\sqrt{5}}{2}$, and $\phi = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}$. If $b_0 = \phi c_0$, the triangle after k iterations becomes,

$$\begin{aligned}
&\big((-1)^k(F_{k-1}c_0 - F_{k-2}b_0), (-1)^k(F_{k-1}b_0 - F_kc_0), (-1)^k(F_{k+1}c_0 - F_kb_0)\big)\\
&= \big((-1)^k(F_{k-1}c_0 - F_{k-2}\phi c_0), (-1)^k(F_{k-1}\phi c_0 - F_kc_0), (-1)^k(F_{k+1}c_0 - F_k\phi c_0)\big),
\end{aligned}$$

which taking $k \to \infty$ by the definition of ϕ gives us $(0, 0, 0)$. □

Next we prove the behavior of case 2, and the conditions on b_0 necessary to stay in case 2.

Case 2 $(a'_{n+1}, c'_{n+1}, b'_{n+1})$

Lemma 3.7. *Given triangle $x_0 = (b_0 + c_0, b_0, c_0)$, if $\forall i \in \mathbb{Z}^+$, and $c'_i > b'_i$, then $x_i = (b_0 - (i-1)c_0, b_0 - ic_0, c_0)$, and $b_0 > (i+1)c_0$.*

Proof. Once again, we prove this inductively.

For $k = 1$,

$$x_0 = (b_0 + c_0, b_0, c_0) \to x_1 = (b_0 - 0 \cdot c_0, b_0 - c_0, c_0),\ \text{since } c'_k > b'_k$$

Since the triangle is in standard form, $b_0 - c_0 > c_0$ must be true, so $b_0 > 2c_0$.
For some k and the triangle after k iterations be $(b_0 - (k-1)c_0, b_0 - kc_0, c_0)$ and $b_0 > (k+1)c_0$. If we iterate once more, we will find the $k+1$ triangle and constraint for b_0. By assumption, we have $a_{k+1} = a_k - c_k$, $b_{k+1} = b_k - c_k$, and $c_{k+1} = a_k - b_k$.

$$\begin{aligned}
a_{k+1} &= a_k - c_k = b_0 - (k-1)c_0 - c_0\\
b_{k+1} &= b_k - c_k = b_0 - kc_0 - c_0\\
c_{k+1} &= a_k - b_k = b_0 - (k-1)c_0 - (b_0 - kc_0)
\end{aligned}$$

The triangle after $k+1$ iterations is $(b_0 - kc_0, b_0 - (k+1)c_0, c_0)$.

Because the triangle is in standard form, $b_{k+1} > c_{k+1}$, we need $b_0 - (k+1)c_0 > c_0$ or $b_0 > (k+2)c_0$, which completes the proof.

□

Lemma 3.8. *$\forall n$, if $b'_n < c'_n$, then*

$$\lim_{n\to\infty}(a_n, b_n, c_n) = (m, m, 0)$$

Proof. We can't have $b_0 > kc_0$ as $k \to \infty$ unless $c_0 = 0$. If $c_0 = 0$, we have the triangle $(b_0 - (k-1)c_0, b_0 - kc_0, c_0) = (b_0, b_0, 0)$.

□

Finally, using lemma 3.5 and 3.7 we can characterize the behavior of any triangle with values in $\mathbb{R}$.

Theorem 3.9. *If $a_0, b_0, c_0 \in \mathbb{R}$, $a_0 > b_0 > c_0$, then*

$$\lim_{n\to\infty}(a_n, b_n, c_n) = (0,0,0) \text{ or } (m, m, 0)$$

Proof. First, we will examine the long term behavior of (a_0, b_0, c_0) in 3 cases:

(1) $\forall k, b'_k > c'_k$
(2) $\forall k, c'_k > b'_k$
(3) The triangle switches between $b'_k > c'_k$ and $c'_k > b'_k$

Lemmas 3.6 and 3.8 show that cases 1 and 2 converge to $(0,0,0)$ and $(m,m,0)$. For the rest of this proof, we will assume we are in case 3.

So, we must have that the triangle oscillates between cases 1 and 2. In this case, we examine the behavior of the triangles in cases 1 and 2.

Since case 3 is complex, I will give a general overview, then prove convergence. Since there are only two possible triangles that are in either case 1 or case 2 always, most will fall into case 3. This means, they will switch from case 1 to case 2 and back from case 2 to case 1 satisfying $b'_k > c'_k$ and $c'_k > b'_k$ but not for all k. In particular, when we examine a boundary condition (some k where a switch between cases occurs), we will show that the a_i, b_i, c_i all decrease. Using this fact, we will then show it decreases to either $(0,0,0)$, or $(m,m,0)$.

While temporarily in case 1 ($k < \infty$), we have by inspection that $a_{k+2} = c_k$ and $b_{k+1} = c_k$. To see this, recall

$$\begin{aligned} a_{k+2} &= a_{k+1} - c_{k+1} = (b_k + c_k - c_k) - (b_k - c_k) = c_k \\ b_{k+1} &= b_k + c_k - b_k = c_k \end{aligned}$$

Because $c_k < b_k < a_k$, either a_k, b_k, c_k decrease, or they are equal. If two are equal, we have (a_0, b_0, b_0), which converges in 1 step, so we assume they decrease at each step.

While temporarily in case 2 ($k < \infty$), we have that on the $k+1$ iteration where it breaks the condition, $kc_0 < b_0 < (k+1)c_0$. Let $b_0 = kc_0 + r$ where $0 < r < c_0$. We rewrite the triangle

$$(b_0 + (k-1)c_0, b_0 - kc_0, c_0) \text{ as } (kc_0 + r - (k-1)c_0, kc_0 + r - kc_0, c_0) = (c_0 + r, c_0, r).$$

In this case, a_0, b_0, c_0 decrease as well, since $a_0 = kc_0 r + c_0 > kc_0 + r$, $b_0 = kc_0 > c_0$ and by construction $c_0 > r$.

Because $x_0 = (a_0, b_0, c_0)$ decreases after each oscillation, and is bounded below, we have that it must converge to some steady state $\chi = (a^*_\infty, b^*_\infty, c^*_\infty)$.

To show what this steady state is, we fix some $\epsilon > 0$ and choose some N_ϵ such that $x_{N_\epsilon} = (\alpha_1, \beta_1, \gamma_1)$, and $\alpha_1 \in [a^*_\infty, a^*_\infty + \epsilon]$, $\beta_1 \in [b^*_\infty, b^*_\infty + \epsilon]$, $\gamma_1 \in [c^*_\infty, c^*_\infty + \epsilon]$.

We examine what happens if we iterate once more on x_{N_ϵ}.

$\alpha_2 = \alpha_1 - \gamma_1$, taking the maximum and minimum differences over the intervals,

$$a^*_\infty - c^*_\infty - \epsilon \leq \alpha_2 \leq a^*_\infty + \epsilon - c^*_\infty, \text{ and } a^*_\infty \leq \alpha_2 \leq a^*_\infty + \epsilon$$

So, $a^*_\infty \leq \alpha_2 \leq a^*_\infty + \epsilon - c^*_\infty$, which implies $a^*_\infty + \epsilon - c^*_\infty \geq a^*_\infty$, which implies $c^*_\infty = 0$, since ϵ can be arbitrarily small.

We then have two options, either $\beta_2 = \alpha_1 - \beta_1$ or $\gamma_2 = \alpha_1 - \beta_1$. For formality, we show their derivatin below, however the method is the same as the previous calculation.

Examining $\beta_2 = \alpha_1 - \beta_1$, we find

$$a^*_\infty - b^*_\infty - \epsilon \leq \beta_2 \leq a^*_\infty + \epsilon - b^*_\infty \text{ and } b^*_\infty \leq \beta_2 \leq b^*_\infty + \epsilon,$$

which implies $\beta_2 = b^*_\infty$.

$$b^*_\infty \leq a^*_\infty + \epsilon - b^*_\infty \text{ and } b^*_\infty + \epsilon \geq a^*_\infty - b^*_\infty - \epsilon$$

$$2b^*_\infty - \epsilon \leq a^*_\infty \leq 2b^*_\infty + 2\epsilon \text{ and } a^*_\infty = 2b^*_\infty$$

Examining $\gamma_2 = \beta_1 - \gamma_1$, we find

$$b^*_\infty - c^*_\infty - \epsilon \leq \gamma_2 \leq b^*_\infty - c^*_\infty + \epsilon \text{ and } c^*_\infty \leq \gamma_2 \leq c^*_\infty + \epsilon.$$

$$c^*_\infty + \epsilon \geq b^*_\infty - c^*_\infty - \epsilon$$

Since $c^*_\infty = 0$, we have $b^*_\infty < 2\epsilon$, so $b^*_\infty = 0$ giving $a^*_\infty = b^*_\infty = c^*_\infty = 0$

Now we examine $\gamma_2 = \alpha_1 - \beta_1$.

$$a^*_\infty - b^*_\infty - \epsilon \leq \gamma_2 \leq a^*_\infty + \epsilon - b^*_\infty \text{ and } c^*_\infty \leq \gamma_2 \leq c^*_\infty + \epsilon$$

$$a^*_\infty - b^*_\infty - \epsilon \leq c^*_\infty + \epsilon \text{ and } c^*_\infty \leq a^*_\infty + \epsilon - b^*_\infty$$

$$c^*_\infty - \epsilon \leq a^*_\infty - b^*_\infty < c^*_\infty + \epsilon$$

Since $c^*_\infty = 0$, we have $-\epsilon \leq a^*_\infty - b^*_\infty \leq \epsilon$, which implies $a^*_\infty = b^*_\infty$ giving the steady state $(a^*_\infty, a^*_\infty, 0)$. □

Remark: Unfortunately, we have not been able to characterize which games converge to $(0, 0, 0)$ *and which games do not converge. The proof does not inform us of what games other than the special cases converge, however we suspect there are many for each, since it seems just as likely we have either case above as the steady state.*

4. Square Game with Addition in $\mathbb{Z}/n\mathbb{Z}$

In Sections 2 and 3, we considered games with a subtraction rule; in this section, we instead consider an addition rule. Specifically, we consider the game on a square with addition; that is, with an iterative step $(a, b, c, d) \to (a+b, b+c, c+d, d+a)$. In our analysis, we consider the position of the vertices to be fixed according to the above iterative step, rather than actively considering the rotations and reflections discussed in Fact 2.3 and Fact 2.4. It is easy to see that the only square which would remain constant under these rules is $(0, 0, 0, 0)$, so it is interesting to see which sequences of squares eventually reach $(0, 0, 0, 0)$.

However, naively using traditional addition (with positive values) would result in sequences of squares with growing, unbounded values. For example, consider the game beginning with the square $x_0 = (1, 1, 1, 1)$. The resulting sequence of squares would be

$$(1, 1, 1, 1) \to (2, 2, 2, 2) \to (4, 4, 4, 4) \to \cdots \to \left(2^k, 2^k, 2^k, 2^k\right) \to \cdots$$

whose elements clearly grow without bound; thus, any notion of convergence would be meaningless.

To remedy this issue, we instead consider addition in $\mathbb{Z}/n\mathbb{Z}$, which is the additive group of integers modulo n. In simple terms, each element is an integer in $\{0, 1, \ldots, n-1\}$, and the sum of multiple elements is taken to be the remainder when the sum is divided by n. For example, in $\mathbb{Z}/10\mathbb{Z}$, $9 + 7 = 6$ because 16 divided by 10 gives a remainder of 6.

To provide additional clarity, we consider the sequence of squares in $\mathbb{Z}/10\mathbb{Z}$ with $x_0 = (1, 9, 1, 4)$:

$$(1, 9, 1, 4) \to (0, 0, 5, 5) \to (0, 5, 0, 5) \to (5, 5, 5, 5) \to (0, 0, 0, 0) \to (0, 0, 0, 0) \to \cdots .$$

From this example, it is clear that we have the possibility of "convergence" to $(0, 0, 0, 0)$, similar to the games considered in Sections 2 and 3.

Throughout the remainder of this section, we write $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}_n$ for convenience. When n is an odd prime p, we will write $\mathbb{Z}_p$, but we warn advanced readers not to confuse this notation with the p-adic numbers.

We tackle two main questions about this game. In Subsection 4.1, we determine which initial squares generate a sequence of squares in $\mathbb{Z}_n$ that converges to $(0, 0, 0, 0)$, and we provide an upper bound on the length of such games. In Subsection 4.2, we study the long-term behavior of sequences of squares which do not converge, determining which initial squares eventually reappear in the sequence they generate. To begin our exploration, we formally define this new game as follows.

Definition 4.1. (Square Game with Addition in $\mathbb{Z}_n$) Given the game with a positive integer $n > 1$ and the initial square $x_0 = (a, b, c, d) \in \mathbb{Z}_n^4$, the kth square is given by $x_k^T = M^k(x_0^T) \pmod n$ where $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $\pmod n$ denotes that each element of the column vector $M^k(x_0^T)$ is replaced by its remainder when it is divided by n (an element of $\mathbb{Z}_n$), and x_i^T denotes the transpose of x_i.

We remark that $(a, b, c, d) \in \mathbb{Z}_n^4$ simply means that the four values in the initial square are elements of $\{0, 1, \ldots, n-1\}$. The linear operator M satisfies $M\big((a, b, c, d)^T\big) = (a + b, b + c, c + d, d + a)^T$, so left multiplying by M^k gives the kth square with traditional addition by the same logic considered with the linear operator in Section 2. Finally, taking the result modulo n gives the square when addition is considered in $\mathbb{Z}_n$. It is not hard to see that doing the modulo operation after applying M^k is equivalent to doing the modulo operation after each left-multiplication by M.

To show an application of Definition 4.1, we refer back to our previous example where $n = 10$ and $x_0 = (1, 9, 1, 4)$. We quickly verify our result for the second square in this sequence, $x_2 = (0, 5, 0, 5)$, via Definition 4.1:

$$\begin{aligned} x_2 &= \left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 9 \\ 1 \\ 4 \end{pmatrix} \right) \pmod{10} \\ &= \left(\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 9 \\ 1 \\ 4 \end{pmatrix} \right) \pmod{10} \\ &= \begin{pmatrix} 20 \\ 15 \\ 10 \\ 15 \end{pmatrix} \pmod{10} \\ &= \begin{pmatrix} 0 \\ 5 \\ 0 \\ 5 \end{pmatrix}, \end{aligned}$$

which corresponds to what we found in our example.

In order to help determine the behavior of sequences of squares in this game, we first compute the elements of the kth square given an initial square and value of n.

Lemma 4.2. *Consider the sequence of squares $x_0, x_1, \dots$ in $\mathbb{Z}_n$ with $x_0 = (a, b, c, d)$. For integers $k \geq 2$ that are multiples of 8, the kth square is given by*

$$x_k^T = \begin{pmatrix} 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(a-c) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(b-d) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(c-a) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(d-b) \end{pmatrix} \pmod{n}.$$

A proof of Lemma 4.2 is provided in the Appendix, and the value of x_k is determined for all k, not only multiples of 8. However, for most of our analysis, knowing the value of every 8th square will be sufficient, as seen in the remainder of this section. Now, we are prepared to determine which values of x_0 and n will give sequences of squares that eventually reach $(0, 0, 0, 0)$.

4.1. Convergence

Definition 4.3. A sequence of squares $x_0, x_1, \dots$, as given by Definition 4.1, converges if there exists a positive integer K such that $x_K = (0, 0, 0, 0)$. Equivalently, the sequence converges if there exists a positive integer K such that $x_k = (0, 0, 0, 0)$ for all $k \geq K$.

To understand why the two conditions given in Definition 4.3 are equivalent, note that $(0, 0, 0, 0) \to (0, 0, 0, 0) \to \cdots$, so if $x_K = (0, 0, 0, 0)$, all subsequent squares are also $(0, 0, 0, 0)$. We note that the length of a game is defined the same as in the previous sections, though we will use $L_n(x_0)$ to denote the length of a game in $\mathbb{Z}_n$ with initial square x_0. If a sequence of squares does not converge, the game has length $L_n(\cdot) = \infty$.

To provide an introduction to the problem of determining which sequences of squares converge, we consider games in $\mathbb{Z}_p$, where p is an odd prime, since we suspect that it will be simplest to consider remainders with respect to a prime.

Lemma 4.4. *For any odd prime number p, the sequence of squares $x_0, x_1, \dots$ in $\mathbb{Z}_p$ converges if and only if $x_0 = (a, p-a, a, p-a)$ for some $a \in \mathbb{Z}_n$. (Note that $p - 0 = 0$ since we are working in $\mathbb{Z}_p$.)*

Proof. Via Definition 4.3, if a sequence of squares converges, then $x_k = (0, 0, 0, 0)$ for all sufficiently large k. Hence, using Lemma 4.2, the sequence of squares converges if and only if there exists a k which is a multiple of 8 such that

$$x_k^T = \begin{pmatrix} 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(a-c) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(b-d) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(c-a) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(d-b) \end{pmatrix} \pmod{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

Adding the first and third elements of both sides of (4.1) tells us that if the sequence of squares converges, then

$$2^{k-1}(a+b+c+d) \equiv 0 \pmod{p}. \tag{4.2}$$

However, since $\gcd(2, p) = 1$, $2^{k-1}(a+b+c+d) \equiv 0 \pmod{p}$ if and only if

$$a+b+c+d \equiv 0 \pmod{p}. \tag{4.3}$$

Hence, if the sequence of squares converges, (4.3) must be satisfied. Substituting (4.3) into (4.1) gives the additional necessary conditions

$$2^{\frac{k}{2}-1}(a-c) \equiv 2^{\frac{k}{2}-1}(b-d) \equiv 0 \pmod{p}. \tag{4.4}$$

However, since $\gcd(2, p) = 1$, $2^{\frac{k}{2}-1}(a-c) \equiv 2^{\frac{k}{2}-1}(b-d) \equiv 0 \pmod{p}$ if and only if

$$a-c \equiv b-d \equiv 0 \pmod{p}. \tag{4.5}$$

Hence, if the sequence of squares converges, $x_0 = (a, b, c, d)$ must satisfy (4.3) and (4.5). Substituting (4.5) into (4.3) gives $a + b + a + b \equiv 2(a + b) \equiv 0 \pmod p$, so $a + b \equiv 0 \pmod p$. Since $a, b \in \mathbb{Z}_p$, this is equivalent to $b = p-a$. Thus, if a sequence of squares converges, $x_0 = (a, p-a, a, p-a)$ for some $a \in \mathbb{Z}_n$.

For the other direction of the proof, we note that if $x_0 = (a, p - a, a, p - a)$, then the sequence of squares converges. To see this, we note that the sequence in $\mathbb{Z}_p$ is

$$(a, p - a, a, p - a) \to (0, 0, 0, 0) \to \cdots .$$

This completes the proof. □

Remark 4.5. It is easily noted that the result of Lemma 4.4 also holds in $\mathbb{Z}_n$ for any odd value of n, since the proof of Lemma 4.4 only used the condition $\gcd(2, p) = 1$.

To tackle the general case, we cannot rely on using $\gcd(2, n) = 1$. Instead, we will use the following well-known result to solve more complicated modular equivalences.

Fact 4.6. *If $\alpha \cdot \beta \equiv \alpha \cdot \gamma \pmod n$ for integers α, β, γ with $\alpha \neq 0$, then $\beta \equiv \gamma \pmod{\frac{n}{\gcd(\alpha,n)}}$.*

Furthermore, it is useful to define n in terms of its even part and its odd part.

Definition 4.7. Suppose $n = 2^j \cdot \Omega(n)$ where j is a non-negative integer and $\Omega(n)$ is odd. Then, the odd part of n is $\Omega(n)$ and the even part of n is $2^j = \frac{n}{\Omega(n)}$.

For example, $60 = 2^2 \cdot 15$, so $\Omega(60) = 15$. Armed with Fact 4.6 and Definition 4.7, we determine, for arbitrary n, which x_0 give converging sequences of squares.

Theorem 4.8. *For any integer $n > 1$, the sequence of squares $x_0, x_1, \ldots$ in $\mathbb{Z}_n$ converges if and only if $x_0 = (a, b, c, d) \in \mathbb{Z}_n^4$ with $a + b \equiv c + d \equiv 0 \pmod{\Omega(n)}$, $a \equiv c \pmod{\Omega(n)}$, and $b \equiv d \pmod{\Omega(n)}$.*

Proof. Via Definition 4.3, if a sequence of squares converges, then $x_k = (0, 0, 0, 0)$ for all sufficiently large k. Hence, using Lemma 4.2, the sequence of squares converges if and only if there exists a k which is a multiple of 8 such that

$$x_k^T = \begin{pmatrix} 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(a-c) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(b-d) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(c-a) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(d-b) \end{pmatrix} \pmod n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.6}$$

Adding the first and third elements of both sides of (4.6) tells us that if the sequence of squares converges, then

$$2^{k-1}(a + b + c + d) \equiv 0 \pmod n \tag{4.7}$$

for all sufficiently large k. Dividing (4.7) by 2^{k-1} and applying Fact 4.6 gives

$$a + b + c + d \equiv 0 \pmod{\frac{n}{\gcd(2^{k-1},n)}} \tag{4.8}$$

However, if the sequence of squares converges, (4.7) must hold for arbitrarily large k, where

$$\gcd(2^{k-1}, n) = \gcd(2^{k-1}, \tfrac{n}{\Omega(n)} \cdot \Omega(n)) = \tfrac{n}{\Omega(n)} \tag{4.9}$$

since $2^{k-1} > \frac{n}{\Omega(n)}$, the even part of n, for very large k. Substituting (4.9) into (4.8), if the sequence of squares converges, we must have

$$a + b + c + d \equiv 0 \pmod{\Omega(n)}. \tag{4.10}$$

Substituting (4.10) into (4.6) gives the additional necessary conditions

$$2^{\frac{k}{2}-1}(a - c) \equiv 2^{\frac{k}{2}-1}(b - d) \equiv 0 \pmod n. \tag{4.11}$$

By a very similar argument, if the sequence of squares converges, we must have

$$a - c \equiv b - d \equiv 0 \pmod{\Omega(n)}. \tag{4.12}$$

Hence, if the sequence of squares converges, $x_0 = (a, b, c, d)$ must satisfy (4.10) and (4.12).

In fact, it turns out that these conditions are sufficient to guarantee convergence. We begin by considering the value of $2^{k-2}(a+b+c+d)$ in (4.6). From (4.10), there exists an integer m such that $m \cdot \Omega(n) = a+b+c+d$. Consider the even part of n, $2^j = \frac{n}{\Omega(n)}$, and note that

$$2^{k-2}(a+b+c+d) = \underbrace{2^{k-2-j} \cdot 2^j}_{2^{k-2}} \cdot \underbrace{m \cdot \Omega(n)}_{a+b+c+d} = 2^{k-2-j} \cdot m \cdot n \tag{4.13}$$

where we have used the fact that $2^j \cdot \Omega(n) = n$. For convergence, we simply need a sufficiently large k that satisfies (4.6), so taking $k \geq j+2$, we have that $2^{k-2-j} \cdot m$ is an integer, so

$$2^{k-2}(a+b+c+d) \equiv 2^{k-2-j} \cdot m \cdot n \equiv 0 \pmod{n}. \tag{4.14}$$

Using essentially the same argument, we find that (4.12) implies

$$2^{\frac{k}{2}-1}(a-c) \equiv 2^{\frac{k}{2}-1}(b-d) \equiv 0 \pmod{n}. \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.6) shows that (4.6) is satisfied. Hence, we have shown that any $x_0 = (a, b, c, d)$ satisfying (4.10) and (4.12) will satisfy (4.14) and (4.15) for some (sufficiently large) k, and thus will give a sequence of squares that converges.

Finally, substituting (4.12) into (4.10) gives $a+b+a+b \equiv 2(a+b) \equiv 0 \pmod{\Omega(n)}$. Hence, $a+b \equiv 0 \pmod{\Omega(n)}$ and equivalently $c+d \equiv 0 \pmod{\Omega(n)}$. Hence, satisfying (4.10) and (4.12) is equivalent to satisfying the above conditions and (4.12). We remark that any one of the four conditions in this theorem is redundant given the other three. This completes the proof. □

It is easy to see that Lemma 4.4 follows directly from Theorem 4.8. Now, we have determined exactly which sequences of squares converge in $\mathbb{Z}_n$. We now analyze how many initial squares give converging sequences.

Corollary 4.9. The number of 4-tuples $x_0 \in \mathbb{Z}_n^4$ for which the sequence in $\mathbb{Z}_n$ with initial square x_0 converges is $\frac{n^4}{\Omega(n)^3}$. The probability that a random 4-tuple x_0 (chosen uniformly from $\mathbb{Z}_n^4$) gives a converging sequence is $\frac{1}{\Omega(n)^3}$.

Proof. Let $x_0 = (a, b, c, d)$. First, we choose a value of $a \in \mathbb{Z}_n$, for which there are n choices. Then, via Theorem 4.8, $c = a + m_1 \cdot \Omega(n)$ for some integer m_1. It is not hard to see that exactly $\frac{n}{\Omega(n)}$ of these values are unique elements in $\mathbb{Z}_n$. To help show this informally, take $a = 2$ and $n = 12$. Then, c can be any of $\ldots, -1, 2, 5, 8, 11, 14, \ldots$, but only $2, 5, 8, 11$ are in $\mathbb{Z}_n$, which is $\frac{12}{3} = 4$ choices. Similarly, $b = -a + m_2 \cdot \Omega(n)$ and $d = -c + m_3 \cdot \Omega(n)$ for some integers m_2 and m_3, and these relationships give exactly $\frac{n}{\Omega(n)}$ unique values in $\mathbb{Z}_n$ for each of b and d. Hence, the number of initial squares which give converging sequences in $\mathbb{Z}_n$ is $n \cdot \left(\frac{n}{\Omega(n)}\right)^3 = \frac{n^4}{\Omega(n)^3}$. Since there are n^4 possible values of x_0, it follows immediately that the probability of such an initial square is $\frac{1}{\Omega(n)^3}$. Interestingly, this is only dependent on the odd part of n. This completes the proof. □

Finally, we discuss the length of games in $\mathbb{Z}_n$.

Corollary 4.10.

(a) For an odd integer $n > 1$, a sequence of squares $x_0, x_1, \ldots$ in $\mathbb{Z}_n$ satisfies $L_n(x_0) = 0$ if $x_0 = (0, 0, 0, 0)$, $L_n(x_0) = 1$ if $x_0 = (a, n-a, a, n-a)$ for $a \neq 1$, and $L_n(x_0) = \infty$ otherwise.

(b) For an even integer $n > 1$, the game with initial square x_0 in $\mathbb{Z}_n$ satisfies

$$L_n(x_0) \leq 2 \cdot \log_2\left(\frac{n}{\Omega(n)}\right) + 2$$

if $x_0 = (a, b, c, d) \in \mathbb{Z}_n^4$ satisfies $a + b \equiv c + d \equiv 0 \pmod{\Omega(n)}$, $a \equiv c \pmod{\Omega(n)}$, and $b \equiv d \pmod{\Omega(n)}$. Otherwise, $L_n(x_0) = \infty$.

Proof. For (a), from the proof of Lemma 4.4 and Remark 4.5, we immediately find the length of all games in $\mathbb{Z}_n$ where n is odd. For (b), take $2^j = \frac{n}{\Omega(n)}$, so $j = \log_2\left(\frac{n}{\Omega(n)}\right)$. In the proof of Theorem 4.8, we showed that if $k \geq j+2$ (just before (4.14)), $\frac{k}{2} \geq j+1$ (in the summarized portion of the proof just before (4.15)), and x_0 satisfies the conditions for a converging sequence of squares, then $x_k = (0,0,0,0)$. The dominant constraint is $k \geq 2j+2$, so the length cannot exceed $2j+2$. To be completely rigorous, since we are taking $2j+2$ even and not necessarily a multiple of 8, we must invoke the extended form of Lemma 4.2 (see Appendix A.1) which shows that any even square can be expressed in the form of Lemma 4.2, up to rotation (which is good enough since we are considering convergence to $(0,0,0,0)$). □

The upper bound in (b) can be obtained. For example, consider $x_0 = (3,0,0,0)$ in $\mathbb{Z}_6$, which gives the sequence

$$(3,0,0,0) \to (3,0,0,3) \to (3,0,3,0) \to (3,3,3,3) \to (0,0,0,0) \to \cdots,$$

so $L_6((3,0,0,0)) = 4 = 2 \cdot \log_2\left(\frac{6}{3}\right) + 2$. In fact, for any given n, the upper bound is attained, as it is easy to show that $L_n((\Omega(n),0,0,0)) = 2 \cdot \log_2\left(\frac{n}{\Omega(n)}\right) + 2$. It would be interesting to try to determine an explicit form for $L_n(x_0)$ for even n in terms of x_0, and to analyze how often the upper bound is attained (perhaps using a computational tool). However, for the sake of brevity, this is left as a possible further exploration for future research or an especially interested reader.

We have now determined all (and how many) games that give converging sequences in $\mathbb{Z}_n$, as well as some properties of the length of these games. However, as seen in Corollary 4.9, these might not make up very many games overall. Therefore, we study what happens in games which do not converge.

4.2. Periodicity

To begin to study the behavior of games which do not converge, we provide two motivating examples in $\mathbb{Z}_5$:

$$\boxed{(1,3,4,2)} \to (4,2,1,3) \to \boxed{(1,3,4,2)} \to (4,2,1,3) \to \boxed{(1,3,4,2)} \to \cdots$$

$$(1,1,0,3) \to \boxed{(2,1,3,4)} \to (3,4,2,1) \to \boxed{(2,1,3,4)} \to (3,4,2,1) \to \boxed{(2,1,3,4)} \to \cdots$$

Both of these examples exhibit periodic behavior: after some point the sequence of squares began to repeat. This is not unique to these two examples.

Proposition 4.11. Every sequence of squares eventually becomes periodic.

Proof. This result follows from the Pigeonhole Principle. Note that there are only n^4 possible squares, so in the infinite sequence, some square must appear twice. Since the sequence starting at a given square is determined precisely by that square (e.g., the iteration process is memory-less), the sequence must be periodic starting at the first square that appears twice. □

We can formally define this concept of periodicity as follows.

Definition 4.12. A period, P, of a sequence of squares $x_0, x_1, \ldots$ in $\mathbb{Z}_n$ is a non-negative integer satisfying

$$M^Q x \pmod{n} = M^{Q+P} x \pmod{n}$$

for some non-negative integer Q. We call the smallest such P the *minimal period*.

Colloquially, this says that the sequence becomes periodic starting at the Qth square in cycles of length P. For example, we can apply Definition 4.8 to the examples in $\mathbb{Z}_5$ to see that the first example has $(P,Q) = (2,0)$ and the second example has $(P,Q) = (2,1)$. We also remind the reader that we take the vertex positions to be fixed, ignoring the symmetries discussed in Section 2.

In general, it is hard to explicitly determine the period of a sequence of this game. To illustrate this, we consider the following "simple" case.

Lemma 4.13. *Consider the sequence in $\mathbb{Z}_p$ (where p is an odd prime) with initial square $x_0 = (1,1,1,1)$. The minimal period of this sequence is $\text{ord}_p(2)$, the multiplicative order of 2 modulo p; that is, the least $k > 0$ such that $2^k \equiv 1 \pmod{p}$.*

Proof. It is easy to see that each element of x_k is given by $2^k \pmod{p}$. Fermat's Little Theorem states that $a^p \equiv a \pmod{p}$ for prime p, so $a^{p-1} \equiv 1 \pmod{p}$ if $\gcd(a,p) = 1$ via Fact 4.6. Hence, there exists a k such that $2^k \equiv 1 \pmod{p}$; specifically, this holds for $k = p-1$. Hence, the square $(1,1,1,1)$ will be the first to appear twice, and a period of this sequence will be $p-1$. However, this is not necessarily the minimal period. For example, note that for $p = 7$, we have the sequence

$$\boxed{(1,1,1,1)} \to (2,2,2,2) \to (4,4,4,4) \to \boxed{(1,1,1,1)} \to (2,2,2,2) \to (4,4,4,4) \to \boxed{(1,1,1,1)} \to \cdots$$

which has minimal period 3 (though $p-1 = 6$ is also a period). Indeed, $k = 3$ is the minimal solution for $k > 0$ of $2^k \equiv 1 \pmod{p}$, and in general, the minimal solution to $2^k \equiv 1 \pmod{p}$ is clearly the minimal period. However, the arithmetic function $(\text{ord}_p(2))$ cannot be described in any simple form. $\square$

Even this "simple" case for x_0 gave a rather tricky result! To create a more tractable problem that is within the scope of this paper, we seek to find which x_0 and n give *purely periodic* sequences; that is, sequences where Definition 4.12 has $Q = 0$ (e.g., the first square seen for a second time is the original square). For example, $x_0 = (1,1,1,1)$ for $n = 5$ is purely periodic with $P = 4$:

$$\boxed{(1,1,1,1)} \to (2,2,2,2) \to (4,4,4,4) \to (3,3,3,3) \to \boxed{(1,1,1,1)} \to \cdots$$

In a sense, these purely periodic sequences describe the long-term behavior of all sequences because all sequences of squares become periodic at some point, as given by Proposition 4.11.

Now, we determine which initial squares x_0 give purely periodic sequences in $\mathbb{Z}_n$ where n is odd.

Theorem 4.14. *The sequence $x_0, x_1, \ldots$ in $\mathbb{Z}_n$ where $n > 1$ is an odd integer generated by the initial square x_0 is purely periodic if and only if $x = (a,b,c,d)$ with $a + c \equiv b + d \pmod{n}$.*

Proof. The sequence is purely periodic if and only if $x_k = x_0$, or

$$x_k^T = \begin{pmatrix} 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(a-c) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(b-d) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(c-a) \\ 2^{k-2}(a+b+c+d) + 2^{\frac{k}{2}-1}(d-b) \end{pmatrix} \pmod{n} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{4.16}$$

for some k which is a multiple of 8 where this equation (derived from Lemma 4.2) represents a (possibly non-minimal) period of k. This is because if the minimal period is P, then $8P$ is also a period.

First, note that adding the first and third elements, as well as the second and fourth elements, in (4.16) tells us that if the sequence converges, then for all sufficiently large k,

$$2^{k-1}(a+b+c+d) \equiv a + c \equiv b + d \pmod{n}. \tag{4.17}$$

We then claim that any $x_0 = (a,b,c,d)$ that satisfies

$$a + c \equiv b + d \pmod{n} \tag{4.18}$$

from (4.18) will give a sequence of squares that is purely periodic, with (possibly non-minimal) period $k = 8 \cdot \phi(n)$, where $\phi(n)$ is Euler's totient function (the number of non-negative integers less than or equal to n that are relatively prime to n). For odd prime $n = p$, this is simply $k = 8(p-1)$. To prove this claim, we will rewrite the condition in (4.18) to match the form of (4.16). First, we apply Euler's theorem, which states that $t^{\phi(n)} \equiv 1 \pmod{n}$ where $\gcd(t,n) = 1$. This gives $2^{\phi(n)} \equiv 1 \pmod{n}$ for odd n. Hence, for $k = 8 \cdot \phi(n)$, we have

$$2^k \equiv 2^{8\cdot\phi(n)} \equiv (2^{\phi(n)})^8 \equiv 1 \pmod{n} \tag{4.19}$$

and

$$2^{\frac{k}{2}} = 2^{4\cdot\phi(n)} \equiv (2^{\phi(n)})^4 \equiv 1 \pmod{n}. \tag{4.20}$$

We can then rewrite (4.18) as

$$4a \equiv 3a + b - c + d \equiv (a + b + c + d) + 2(a - c) \pmod{n}, \tag{4.21}$$

and apply (4.20) and (4.21) to get

$$4a \equiv 2^k(a + b + c + d) + 2 \cdot 2^{\frac{k}{2}}(a - c) \pmod{n}. \tag{4.22}$$

Dividing (4.23) by 4 using Fact 4.6 gives

$$a \equiv 2^{k-2}(a + b + c + d) + 2^{\frac{k}{2}-1}(a - c) \pmod{n},$$

which is precisely the equation for the first element in (4.16). Repeating this process for the other three elements gives analogous results.

Hence, we have shown that a purely periodic sequence in $\mathbb{Z}_n$ must have $x_0 = (a, b, c, d)$ with $a+c \equiv b+d \pmod{n}$, and we have shown that there is a value of k such that the sequence starting with x_0 has period k. This completes the proof. □

Corollary 4.15. Let $n > 1$ be an odd integer. The number of 4-tuples $x_0 \in \mathbb{Z}_n^4$ for which the sequence in $\mathbb{Z}_n$ with initial square x_0 is purely periodic is n^3. The probability that a random 4-tuple x_0 (chosen uniformly from $\mathbb{Z}_n^4$) gives a purely periodic sequence is $\frac{1}{n}$.

Proof. Let $x_0 = (a, b, c, d)$. First, we choose a value of $a \in \mathbb{Z}_n$, for which there are n choices. Then, via Theorem 4.14, there are n choices for each of b and c, but then d is completely determined. Hence, the number of initial squares which give converging sequences in $\mathbb{Z}_n$ is n^3. Since there are n^4 possible values of x_0, it follows immediately that the probability of such an initial square is $\frac{1}{n}$. □

It seems reasonable to assume that Theorem 4.14 could be extended to $\mathbb{Z}_n$ for general n similarly to how Lemma 4.4 was extended to Lemma 4.8. However, this is left for further exploration in future research or as an exercise for an especially interested reader.

We have not directly addressed the behavior of sequences of squares which do not converge and are not purely periodic. Of course, they become periodic, but further research should be done to specify the precise periodic behavior (e.g., what square does it cycle on? what is its minimal period and corresponding value of Q as given in Definition 4.12?) of such sequences. Furthermore, we have not discussed the minimal periods for purely periodic sequences of squares. This is another area that merits further research.

References

No references were used for this paper.

Appendix

A.1. **Proof of Lemma 4.2**

Proof. We can find the eigenvalues of M and diagonalize M to find that

$$M^k = \begin{pmatrix} -1 & 1 & -i & i \\ 1 & 1 & -1 & -1 \\ -1 & 1 & i & -i \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^k & 0 & 0 \\ 0 & 0 & (1-i)^k & 0 \\ 0 & 0 & 0 & (1+i)^k \end{pmatrix} \begin{pmatrix} -1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ i/4 & -1/4 & -i/4 & 1/4 \\ -i/4 & -1/4 & i/4 & 1/4 \end{pmatrix}.$$

We then break this into cases based on the remainder when k is divided by 8 to get the following results:

$$k \equiv 0 \pmod{8}: \quad M^k = \begin{pmatrix} 2^{k-2} + 2^{\frac{k}{2}-1} & 2^{k-2} & 2^{k-2} - 2^{\frac{k}{2}-1} & 2^{k-2} \\ 2^{k-2} & 2^{k-2} + 2^{\frac{k}{2}-1} & 2^{k-2} & 2^{k-2} - 2^{\frac{k}{2}-1} \\ 2^{k-2} - 2^{\frac{k}{2}-1} & 2^{k-2} & 2^{k-2} + 2^{\frac{k}{2}-1} & 2^{k-2} \\ 2^{k-2} & 2^{k-2} - 2^{\frac{k}{2}-1} & 2^{k-2} & 2^{k-2} + 2^{\frac{k}{2}-1} \end{pmatrix}$$

$k \equiv 2 \pmod{8}$: shift rows down by 1 from above
$k \equiv 4 \pmod{8}$: shift rows down by 1 from above
$k \equiv 6 \pmod{8}$: shift rows down by 1 from above

So, for even k, they are of the same form, up to rotation. However, we find it easier to deal with a fixed vertex form, especially for Subsection 4.2 (Periodicity).

$$k \equiv 1 \pmod 8:\ M^k = \begin{pmatrix} 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} \\ 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} \\ 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} \\ 2^{k-2}+2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}-2^{\frac{k-1}{2}} & 2^{k-2}+2^{\frac{k-1}{2}} \end{pmatrix}$$

$k \equiv 3 \pmod 8$: shift rows up by 1 from above
$k \equiv 5 \pmod 8$: shift rows up by 1 from above
$k \equiv 7 \pmod 8$: shift rows up by 1 from above

So, similarly to even k, the odd k squares are of the same form, up to rotation. This completes the proof. □

A.2. **Probability Distribution for Length of Games.** Naturally, we suspect this upper bound is larger than the length of worst case games, because we might expect we don't actually have (e, e, e, o). We tested this empirically by finding the probability distribution for the length of games, and compared it to the theoretical bound found in Section 2.1.

Definition 4.16. The probability mass function (PMF) $f_X(u) = Pr\{X = u\}$.

Definition 4.17. The cumulative distribution function (CDF) $F_X(u) = Pr\{X \leq u\}$.

For us, $X = L(x)$, and $f_X(u)$ is the probability the length of a particular game x is u. $F_X(u)$ is the probability the length is at most u.

The following figures graph the PMF and CDF for the length of a game using all games (w, x, y, z) where $w, x, y, z \in Z$ and $w, x, y, z \in [0, 100]$.

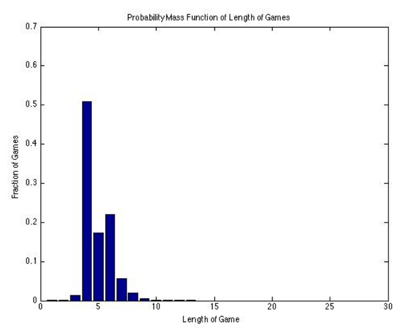

Figure 3. PMF for the Length of a Game

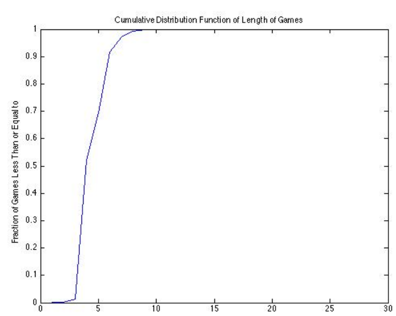

Figure 4. CDF for the Length of a Game

From the PMF and CDF, we have two interesting facts:

- The upper bound never occurs for any game.
- There is a large spike at 4, since case 1 of Lemma 2.10 converges in 4 steps. Furthermore, many games converge in a small number of steps making it unlikely that a long game occurs.