2 Global Optimization

2.1 An abstract description of the global optimization problem

Let $\mathcal{F}$ be a finite family of segment sequences. The global optimization problem is to find

$$\max_{s \in \mathcal{F}} \max_{\boldsymbol{\Delta} \in \{-1,0,1\}^{|s|}} \left(\min_{\boldsymbol{\gamma} \in [-1,1]^{|s|}} \quad c_{s,\boldsymbol{\Delta}}^T \cdot \boldsymbol{\gamma} \quad \text{subject to} \quad A_{s,\boldsymbol{\Delta}} \cdot \boldsymbol{\gamma} \geq 0 \right)$$

Here, the inner minimization problem is a linear program, where $A_{s,\boldsymbol{\Delta}}$ is a matrix of constraints that depends on s and $\boldsymbol{\Delta}$. Similarly, $c_{s,\boldsymbol{\Delta}}$ is a vector of costs that depends on s and $\boldsymbol{\Delta}$.

2.2 Specifying the inner minimization problem

Here, we specify how $A_{s,\boldsymbol{\Delta}}$ and $c_{s,\boldsymbol{\Delta}}$ are defined. Let $s = s_1 \hookrightarrow \cdots \hookrightarrow s_n$ be a sequence of segments with total number of transitions m. Let $\boldsymbol{\Delta} \in \{-1,0,1\}^m$ be a vector of input perturbations.

We define some notation as follows:

- Let t_j^i denote the jth transition in segment i, and ε_j^i denote the noise added to the input before that transition.
- Let Δ_j^i denote the entry of $\boldsymbol{\Delta}$ that corresponds to the input perturbation for the jth transition in segment i.
- Let γ_j^i denote the entry of $\boldsymbol{\gamma} \in [-1,1]^m$ that corresponds to the coupling shift for the jth transition in segment i – this is to be determined by the inner minimization problem.

Then, the minimization problem over $\boldsymbol{\gamma}$ is as follows:

$$\min_{\boldsymbol{\gamma} \in [-1,1]^m} \quad \sum_{i=1}^{n} \sum_{j=1}^{|s_i|} |\gamma_j^i - \Delta_j^i| \varepsilon_i$$

$$\begin{array}{lll}
\text{subject to} & \gamma_k^i \leq \gamma_0^i & \text{if } t_k^i \text{ has guard } < \\
& \gamma_k^i \geq \gamma_0^i & \text{if } t_k^i \text{ has guard } \geq \\
& & \\
& \gamma_0^i \leq \gamma_0^k & \text{if } s_k \hookrightarrow s_i \text{ and guard}(s_i) \text{ is } < \\
& \gamma_0^i \geq \gamma_0^k & \text{if } s_k \hookrightarrow s_i \text{ and guard}(s_i) \text{ is } \geq \\
& & \\
& \gamma_k^i = 0 & \text{if } t_k^i \text{ outputs } \texttt{insample} \\
& \gamma_k^i = \Delta_k^i & \text{if } t_k^i \text{ belongs to a cycle}
\end{array}$$

This can be rewritten as a linear program using standard techniques, producing a constraint matrix $A_{s,\boldsymbol{\Delta}}$ and a cost vector $c_{s,\boldsymbol{\Delta}}$.

2.3 Suspected ways to simplify the problem

- We might be able to determine the maximizing $\boldsymbol{\Delta}$ in the second minimization problem in linear time given $s \in \mathcal{F}$.
  - I suspect this is true since I see that the maximizing $\boldsymbol{\Delta}$ always has $\Delta_j^i = -1$ if t_j^i has guard $\geq$, and $\Delta_j^i = 1$ if t_j^i has guard $<$. In the case that t_j^i has guard *true* and is an assignment transition, the value of Δ_j^i seems to depend on the costs ε_j^i in the segment s_i.
  - If this is true, we need not check exponentially many $\boldsymbol{\Delta}$ in the second maximization problem.
- It might be possible to solve a local minimization problem over segments instead of segment sequences, and then use the results to solve a global constraint system that is much smaller than the one described above.