

# Time and Space Efficient Spectral Clustering via Column Sampling

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## Abstract

*Spectral clustering is an elegant and powerful approach for clustering. However, the underlying eigen-decomposition takes cubic time and quadratic space w.r.t. the data set size. These can be reduced by the Nyström method which samples only a subset of columns from the matrix. However, the manipulation and storage of these sampled columns can still be expensive when the data set is large. In this paper, we propose a time- and space-efficient spectral clustering algorithm which can scale to very large data sets. A general procedure to orthogonalize the approximated eigenvectors is also proposed. Extensive spectral clustering experiments on a number of data sets, ranging in size from a few thousands to several millions, demonstrate the accuracy and scalability of the proposed approach. We further apply it to the task of image segmentation. For images with more than 10 millions pixels, this algorithm can obtain the eigenvectors in 1 minute on a single machine.*

## 1. Introduction

The family of spectral clustering algorithms has been widely used in diverse areas such as data mining [10, 5], signal processing [1] and computer vision [11, 12]. Given  $n$  data points, clustering is based on the eigen-decomposition of an  $n \times n$  matrix that is derived from the pairwise similarities of these points. For example, the normalized cut algorithm [12], which is the focus of this paper, computes the eigenvectors corresponding to the  $k$  smallest eigenvalues of the normalized Laplacian matrix. Using the  $k$  respective components of the data points in these eigenvectors, another clustering algorithm (e.g., the  $k$ -means) is then invoked to form the final clusters. However, the requirement of computing and storing the affinity matrix, and the  $\mathcal{O}(n^3)$  complexity of decomposing the Laplacian matrix, severely restrict the scalability of spectral clustering to large data sets [12].

As only several eigenvectors are required in the procedure, a general approach to alleviate this problem is by using low-rank matrix approximations, among which the

Nyström method [4, 17] is the most popular. It samples  $m \ll n$  columns from the original  $n \times n$  matrix, and then forms a low-rank approximation of the full matrix by using the correlations between the sampled columns and the remaining  $n - m$  columns. As only a portion of the full matrix is computed and stored, the Nyström method can reduce the time and space complexities significantly. Fowlkes *et al.* successfully applied this to spectral clustering for image segmentation [6]. Besides this, the Nyström method has also been popularly used for tasks such as Gaussian processes [17] and manifold learning [15].

However, when the data set is huge, even the Nyström method can suffer from storage problems. Specifically, since the sampled columns have to be frequently used in the computational procedure, typically they have to be stored in the main memory. For data sets containing millions of samples, even storing only 1% of the columns (in double precision) takes about 80GB.

Another problem with the Nyström method is that the eigenvectors obtained are not orthogonal in general. As has been shown in [8], the lack of orthogonality can sometimes adversely affect the approximation quality. Thus, an additional orthogonalization step may have to be performed. However, in image segmentation, this is further complicated by that the affinity matrix used is typically sparsified and not positive semidefinite (psd). The corresponding orthogonalization procedure proposed in [6] then takes cubic time, making it impractical to implement in practice.

In this paper, we propose a time- and space-efficient spectral clustering algorithm based on the Nyström method. However, in contrast to the approach in [6], we directly compute a rank- $k$  approximation of  $M = I - L$  (where  $I$  is the identity matrix and  $L$  is the normalized Laplacian matrix) using the sampled columns of the affinity matrix. With just one single pass on the sampled columns, both the approximated eigenvectors of  $M$  and the affinity degrees can be computed simultaneously. Thus, these sampled columns no longer have to be stored, alleviating the storage problem when  $n$  is very large. We further propose a general algorithm to orthogonalize the approximated eigenvectors, irrespective of whether the underlying matrix is psd or not.

Our contributions are twofold. First, we reduce the time

and space complexities of the existing Nyström-based spectral clustering algorithm [6]. Theoretical analysis shows that the proposed algorithm can exactly recover the degrees of the sampled points. Experimentally, the algorithm can scale to much larger data sets (with millions of samples) than the existing algorithms. Second, we apply the proposed algorithm to the segmentation of images with very sparse affinity matrices. We demonstrate its scalability on images with more than 10 million pixels. To our best knowledge, this is the first attempt to scale spectral clustering to such a large image segmentation task.

The rest of this paper is organized as follows. Section 2 briefly reviews spectral clustering and the Nyström based approach. Section 3 then describes the proposed algorithm. Experimental results are presented in Section 4, and Section 5 gives some concluding remarks.

**Notations.** For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , we denote by  $A_k = U\Lambda U$  its rank- $k$  approximation, where  $\Lambda$  contains its  $k$  largest eigenvalues and  $U$  contains its  $k$  leading eigenvectors. Moreover,  $\text{diag}(v)$  is the matrix with the elements of  $v$  on the main diagonal.

## 2. Related Works

### 2.1. Spectral Clustering

Spectral clustering is based on analyzing the affinity matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  defined on a set of  $n$  data points  $X = \{x_1, \dots, x_n\}$ , where  $a_{ij} \geq 0$  is the similarity / affinity between  $x_i$  and  $x_j$ . In general, there are several ways to group  $X$  into clusters [16]. In this paper, we focus on the popular method of normalized cut [12]. It first computes the normalized Laplacian  $L = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$  of  $A$ , where  $D = \text{diag}(A1)$  is the degree matrix and  $1$  is the vector of all ones. Next, the  $k$  trailing eigenvectors of  $L$  are obtained, where  $k$  is usually not less than the number of clusters. Finally,  $k$ -means clustering is performed on these eigenvectors to obtain the clusters.

### 2.2. Nyström Method

The Nyström method [4, 17] has been commonly used to construct low-rank matrix approximations. Given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , it first samples  $m \ll n$  columns from  $A$ . Without loss of generality, we can assume these to be the first  $m$  columns of  $A$ , and thus  $A$  can be rewritten as

$$A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} \text{ and } A_{:1} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad (1)$$

where  $A_{:1} \in \mathbb{R}^{n \times m}$  is the submatrix containing the selected columns, and  $A_{11} \in \mathbb{R}^{m \times m}$  is the submatrix containing the intersection of the selected rows and columns. The Nyström approximation of  $A$  is then given by

$$\hat{A}_{\text{nys}} = A_{:1}A_{11}^{-1}A_{:1}^T. \quad (2)$$

### 2.3. Using Nyström in Spectral Clustering

The normalized cut requires the computation of the  $k$  trailing eigenvectors of the normalized Laplacian  $L = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ , which are the same as the  $k$  leading eigenvectors of the  $n \times n$  matrix

$$M = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}. \quad (3)$$

The direct eigen-decomposition of  $M$  takes  $\mathcal{O}(n^3)$  time, and can be expensive when  $n$  is large. In [6], this problem is alleviated by using the Nyström method to approximate the eigenvectors as follows. First, to avoid using the whole  $A$ , the degree matrix  $D$  is approximated as

$$\hat{D} = \text{diag}(\hat{A}_{\text{nys}}1) = \text{diag}(A_{:1}A_{11}^{-1}A_{:1}^T1), \quad (4)$$

by using the Nyström approximation of  $A$  in (2). Let  $D_{:1} = \text{diag}(A_{:1}^T1)$ . The submatrices in  $M$  corresponding to  $A_{11}$  and  $A_{:1}$  are

$$M_{11} = D_{:1}^{-\frac{1}{2}}A_{11}D_{:1}^{-\frac{1}{2}}, \quad M_{:1} = D^{-\frac{1}{2}}A_{:1}D_{:1}^{-\frac{1}{2}}. \quad (5)$$

Using  $\hat{D}$ ,  $M_{:1}$  can then be approximated as

$$\hat{M}_{:1} \simeq \hat{D}^{-\frac{1}{2}}A_{:1}\hat{D}_{:1}^{-\frac{1}{2}} = [M_{11}\hat{M}_{21}^T]^T,$$

and  $M$  as

$$\hat{M} = \hat{M}_{:1}M_{11}^{-1}\hat{M}_{:1}^T, \quad (6)$$

where  $\hat{M}_{:1}$  now contains the approximate eigenvectors of  $M$ .

In general,  $\hat{M}_{:1}$  is not orthogonal. To obtain better clustering results, Fowlkes *et al.* [6] proposed the following orthogonalization procedure when  $A$  is psd. First, construct

$$S = M_{11} + M_{11}^{-\frac{1}{2}}\hat{M}_{21}^T\hat{M}_{21}M_{11}^{-\frac{1}{2}}. \quad (7)$$

Then, perform singular value decomposition (SVD) on  $S$  to obtain  $U\Lambda T^T$ . The orthogonal decomposition of  $\hat{M}$  can be obtained as

$$\hat{M} = V\Lambda V^T \quad \text{and} \quad V^TV = I,$$

where

$$V = \hat{M}_{:1}M_{11}^{-\frac{1}{2}}U\Lambda^{-\frac{1}{2}}. \quad (8)$$

The whole algorithm is summarized in Algorithm 1. When  $A$  is not psd, a more complicated two-step orthogonalization step has to be used [6].

The time complexity of Algorithm 1 is  $\mathcal{O}(nm^2 + m^3)$ . As for space, since  $A_{:1}$  and  $\hat{M}_{:1}$  need to be accessed more than once (e.g., in (4), (5), (6), (7), and (8)), typically they have to be stored in main memory so as to avoid duplicate computations. However, this leads to a space complexity of  $\mathcal{O}(nm)$ . In large-scale applications, even the sampling of a small subset of columns can result in the matrices  $A_{:1}$  and  $\hat{M}_{:1}$  being too large to fit into main memory. Moreover, when  $A$  is not psd, the two-step orthogonalization procedure, which has a time complexity  $\mathcal{O}(n^3)$ , is often impractical [6].

**Algorithm 1** Orthogonal Nyström approximation for a psd  $A$  [6].

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- 1:  $\hat{D} \leftarrow \text{diag}(A_{:1} A_{11}^{-1} A_{:1}^T \mathbf{1})$  and  $D_{:1} \leftarrow \text{diag}(A_{:1} \mathbf{1})$ .
  - 2:  $\hat{M}_{:1} = [M_{11} \hat{M}_{21}^T]^T \leftarrow \hat{D}^{-\frac{1}{2}} A_{:1} D_{11}^{-\frac{1}{2}}$ .
  - 3:  $S \leftarrow M_{11} + M_{11}^{-\frac{1}{2}} \hat{M}_{21}^T \hat{M}_{21} M_{11}^{-\frac{1}{2}}$ .
  - 4: SVD:  $S = U \Lambda V^T$ .
  - 5:  $V \leftarrow \hat{M}_{:1} M_{11}^{-\frac{1}{2}} U \Lambda^{-\frac{1}{2}}$ .
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### 3. Proposed Approach

Recall that spectral clustering only needs the  $k$  trailing eigenvectors of the normalized Laplacian  $L$ , or, equivalently, the  $k$  leading eigenvectors of  $M = I - L$ . Therefore, our goal is to find a rank- $k$  approximation  $M_k$  of  $M$ . Let the eigen-decomposition of  $M_k$  be  $U \Lambda U^T$ , where  $\Lambda$  contains the  $k$  largest eigenvalues of  $M$ , and  $U$  contains the corresponding eigenvectors. As  $M_k = MM_k^{-1}M$ , so on using (3), we have

$$\begin{aligned} M_k &= D^{-1/2} A D^{-\frac{1}{2}} M_k^{-1} D^{-\frac{1}{2}} A^T D^{-1/2} \\ &= D^{-1/2} A S_A A^T D^{-1/2}, \end{aligned} \quad (9)$$

where

$$S_A = D^{-\frac{1}{2}} M_k^{-1} D^{-\frac{1}{2}} \quad (10)$$

is a rank- $k$  matrix. Note that although this  $M_k$  is the best rank- $k$  approximation of  $M$ , (9) requires the use of the whole affinity matrix  $A$ . This can be too expensive for large data sets.

#### 3.1. A Good Approximate Degree Matrix

Note that  $AS_A A^T$  in (9) plays the role of the affinity matrix  $A$  in (3). Since the handling of  $A$  is expensive for large data sets, we consider in the following an approximation which only involves the  $m$  sampled columns  $A_{:1}$  as in the Nyström method. As will be seen, this resultant approximation is in the form of  $AS_A A^T$ , and captures most of the information available in  $A$ . A comparison with some other possible approximations will be discussed in Section 3.4.

Recall that the definition of  $S_A$  in (10) relies on  $M_k$ , which in turn is the rank- $k$  approximation of  $M$  in (9). Hence, we first define an analogous matrix  $M_* \in \mathbb{R}^{m \times m}$  which is based on the submatrix  $A_{11}$ :

$$M_* = D_*^{-\frac{1}{2}} A_{11} D_*^{-\frac{1}{2}}, \quad (11)$$

where  $D_* = \text{diag}(A_{11}\mathbf{1})$ . Analogous to the definition of  $S_A$  in (10), we then define the following matrix  $S_* \in \mathbb{R}^{m \times m}$  based on the rank- $k$  approximation  $M_{*,k}$  of  $M_*$ :

$$S_* = D_*^{-\frac{1}{2}} M_{*,k}^{-1} D_*^{-\frac{1}{2}}. \quad (12)$$

Obviously,  $S_* = S_A$  when  $m = n$ . Finally, define, in a manner analogous to (9),

$$\hat{A} = A_{:1} S_* A_{:1}^T. \quad (13)$$

The following proposition shows that  $\hat{A}$  can exactly recover the degrees of the  $m$  sampled columns. The proof is in Appendix A.

**Proposition 1.** Assume that graph  $G$  has no more than  $k$  connected components. Then  $A_{:1} = \hat{A}_{:1}$ .

In other words, this approximation  $\hat{A}$  contains all the degree information about the selected columns that is in  $A$ . The assumption of Proposition 1 is reasonable since  $k$  is typically chosen not less than the number of connected components; otherwise we obtain  $k$  eigenvectors according to the 0 eigenvalue of  $L$  and are not unique [16]. Moreover,  $\hat{A}$  also has a close relationship with the Nyström approximation  $\hat{A}_{\text{nys}}$  of  $A$  in (2). Specifically, when  $k = m$ , we have

$$\hat{A} = A_{:1} D_*^{-\frac{1}{2}} M_*^{-1} D_*^{-\frac{1}{2}} A_{:1}^T = A_{:1} A_{11}^{-1} A_{:1}^T = \hat{A}_{\text{nys}},$$

on using (11). Thus,  $\hat{A}$  is similar to  $\hat{A}_{\text{nys}}$ , but with a further rank- $k$  approximation on  $M_*$ .

#### 3.2. Rank- $k$ Approximation of $M$

Let the eigen-decomposition of  $M_{*,k}$  be  $V \Lambda V^T$ , where  $V \in \mathbb{R}^{m \times k}$  and  $\Lambda \in \mathbb{R}^{k \times k}$ . On combining this with (12) and (13), we obtain  $\hat{A} = Q \Lambda Q^T$ , where

$$Q = A_{:1} D_*^{-\frac{1}{2}} V \Lambda^{-1}. \quad (14)$$

Since this  $\hat{A}$  contains all the degree information about the selected columns, we can approximate the degree matrix as

$$\hat{D} = \text{diag}(\hat{A}\mathbf{1}) = \text{diag}(Q \Lambda Q^T \mathbf{1}).$$

Subsequently, a rank- $k$  approximation of  $M$  can be obtained as

$$\hat{M}_k = \hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} = \hat{D}^{-\frac{1}{2}} Q \Lambda Q^T \hat{D}^{-\frac{1}{2}} = U \Lambda U^T, \quad (15)$$

where  $U = \hat{D}^{-\frac{1}{2}} Q$  contains its eigenvectors. Note that although the derivation of  $U$  above involves  $\hat{A}$ , it can be directly obtained from  $Q$  without first forming  $\hat{A}$ .

When the data set is large, storage of the affinity matrix  $A$ , or even its submatrix  $A_{:1}$ , can become infeasible. Fortunately, note that, here,  $A_{:1}$  is only needed once in the computation of  $Q$  in (14). Thus, instead of storing the whole  $A_{:1}$  in memory, we can obtain  $Q$  row by row. For its  $i$ th row, we first compute the  $i$ th row of  $A_{:1}$  (which contains the affinities between pattern  $i$  and the sampled patterns in  $A_{:1}$ ), and then multiply it by  $D_*^{-\frac{1}{2}} V \Lambda^{-1}$ . Only the  $n \times k$  matrix  $Q$  needs to be stored. The whole algorithm is shown in Algorithm 2.

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**Algorithm 2** The proposed algorithm.

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**Input:** Data points  $X = \{x_1, \dots, x_n\}$  and  $m$  sampled points  $Z = \{z_1, \dots, z_m\}$  from  $X$ . Desired rank  $k$ .

**Output:** The approximated  $M_k$ :  $\hat{M}_k = U\Lambda U^T$ .

- 1: Form the affinity matrix  $A_{11} \in \mathbb{R}^{m \times m}$  using  $Z$ .
- 2:  $D_* \leftarrow \text{diag}(A_{11})$ .
- 3:  $M_* \leftarrow D_*^{-\frac{1}{2}} A_{11} D_*^{-\frac{1}{2}}$ .
- 4: Perform partial eigen-decomposition of  $M_*$ . Keep the  $k$  largest eigenvalues in  $\Lambda$  and the  $k$  leading eigenvectors in  $V$ .
- 5:  $B \leftarrow D_*^{-\frac{1}{2}} V \Lambda^{-1}$ .
- 6: **for**  $i = 1$  to  $n$  **do**
- 7:   Form the affinity vector  $a \in \mathbb{R}^{1 \times m}$  between  $x_i$  and points in  $Z$ .
- 8:   Form the  $i$ th row of  $Q$  as  $aB$ .
- 9: **end for**
- 10:  $\hat{D} \leftarrow \text{diag}(Q\Lambda Q^T)$ .
- 11:  $U \leftarrow \hat{D}^{-\frac{1}{2}} Q$ .

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### 3.3. Time and Space Complexities

Since only partial eigen-decomposition is used in Algorithm 2, it takes  $\mathcal{O}(nmk)$  time to compute  $k$  approximate eigenvectors. In contrast, as the procedure in [6] needs to compute  $A_{11}^{-1}$ , it will take  $\mathcal{O}(nmk + m^3)$  time instead.

As for the space complexity, when the matrix  $A_{11}$  is small, it can be stored in main memory and so takes  $\mathcal{O}(nm)$  space. However, when  $n$  is large,  $A_{11}$  may not be able to fit into memory. As discussed in the previous section, since the proposed algorithm only needs to access  $A_{11}$  once, we can compute it row by row. The space complexity is then reduced to  $\mathcal{O}(nk)$ . On the other hand, Algorithm 1 needs to use  $A_{11}$  more than once. To avoid computing  $A_{11}$  repeatedly, we will have to store it in memory, leading to  $\mathcal{O}(nm)$  space. A comparison of the time and space complexities between [6] and the proposed algorithm is shown in Table 1.

	un-orthogonalized		orthogonalized	
	Fowlkes <i>et al.</i> [6]	ours	Fowlkes <i>et al.</i> [6]	ours
time	$\mathcal{O}(nmk + m^3)$	$\mathcal{O}(nmk)$	$\mathcal{O}(nm^2 + m^3)$	$\mathcal{O}(nmk)$
space	$\mathcal{O}(nm)$	$\mathcal{O}(nk)$	$\mathcal{O}(nm)$	$\mathcal{O}(nk)$

Table 1. Time and space complexities for computing  $k$  approximate eigenvectors.

### 3.4. Comparison with Other Approximations

Other ways to obtain a rank- $k$  approximation of  $M$  are possible. A straightforward approximation can be obtained by first forming the rank- $k$  Nyström approximation  $\hat{A}^{(1)} = A_{11} A_{11,k}^{-1} A_{11}^T$  of  $A$ . Then, using (3), form a rank- $k$  approximation of  $M$  as  $\hat{M}_k^{(1)} = (\hat{D}^{(1)})^{\frac{1}{2}} \hat{A}^{(1)} (\hat{D}^{(1)})^{\frac{1}{2}}$ ,

where  $\hat{D}^{(1)} = \text{diag}(\hat{A}^{(1)} 1)$  is the approximate degree matrix. However, note that even when we sample all the  $n$  columns from  $A$  to form  $A_{11}$ ,  $\hat{M}_k^{(1)} \neq M_k$  in general and hence this cannot be expected to be a good approximation.

Another approximation, motivated from (10), is to use

$$S^{(2)} = D_{:1}^{-\frac{1}{2}} M_{11,k}^{-1} D_{:1}^{-\frac{1}{2}}, \quad (16)$$

instead of  $S_*$  in (12). Here,  $D_{:1} = \text{diag}(A_{:1})$ . It can be shown that the corresponding approximation of  $M$  becomes  $\hat{M}_k^{(2)} = \hat{M}_{:1} M_{11,k}^{-1} M_{:1}^T$ , which is similar to the Nyström approximation of  $M$  in (6) except for using a rank- $k$  approximation on  $M_{11}$ . However, with this approximation, we will then need to access  $A_{:1}$  twice, once to form  $D_{:1}$  and another to calculate  $Q$  in (14). Subsequently, to avoid computing  $A_{:1}$  twice, we will need  $A_{:1}$  to be stored, which takes  $\mathcal{O}(nm)$  space and can be infeasible when  $n$  is large.

Moreover, unlike  $S_*$  proposed in Section 3.1, these two approximations may not recover the degrees of  $A_{:1}$  (as in Proposition 1). Empirical results in Section 4.2 confirm that these two alternatives are less accurate.

### 3.5. Orthogonalizing the Approximate Eigenvectors

Note that the approximate eigenvectors  $U$  obtained in (15) may not be orthogonal. In this section, we propose an inexpensive orthogonalization procedure which can transform any decomposition of the form  $T = U\Lambda U^T$  to  $T = \tilde{U}\tilde{\Lambda}\tilde{U}^T$ , such that  $\tilde{U}$  is orthogonal (i.e.,  $\tilde{U}^T \tilde{U} = I$ ). Our procedure is more general than Algorithm 1 in that it does not require  $T$  or  $\Lambda$  to be psd.

The orthogonalization procedure is shown in Algorithm 3. Note that  $P = U^T U$  is always psd and so it is safe to use  $\Sigma^{\frac{1}{2}}$  and  $\Sigma^{-\frac{1}{2}}$ . The following proposition shows the correctness of Algorithm 3.

**Proposition 2.** *In Algorithm 3,  $U\Lambda U^T = \tilde{U}\tilde{\Lambda}\tilde{U}^T$  and  $\tilde{V}^T \tilde{V} = I$ .*

Proof is in Appendix B. It can be easily seen that the time complexity of Algorithm 3 is  $\mathcal{O}(nk^2)$  and its space complexity is  $\mathcal{O}(nk)$ . Thus, unlike the orthogonalization procedure in [6], this does not increase the time complexity of the proposed algorithm (Table 1).

## 4. Experiments

### 4.1. Spectral Clustering

In this section, we perform spectral clustering experiments on a number of data sets with various sizes (Table 2). The USPS, MNIST and MNIST8M<sup>1</sup> data sets contain handwritten digit images, with the raw pixels as features. We select the digits 0–4 for MNIST and MNIST8M. The RCV1<sup>2</sup>

<sup>1</sup><http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>

<sup>2</sup><http://alumni.cs.ucsb.edu/~wychen/sc>

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**Algorithm 3** Procedure to orthogonalize  $U$ , where  $T = U\Lambda U^T$ .

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**Input:**  $U \in \mathbb{R}^{n \times k}$ ,  $\Lambda \in \mathbb{R}^{k \times k}$ .

**Output:** orthogonalized  $\tilde{U}$  and  $\tilde{\Lambda}$ .

- 1:  $P \leftarrow U^T U$ .
  - 2: eigen-decomposition:  $P = V\Sigma V^T$ .
  - 3:  $B \leftarrow \Sigma^{\frac{1}{2}} V^T \Lambda V \Sigma^{\frac{1}{2}}$ .
  - 4: eigen-decomposition:  $B = \tilde{V} \tilde{\Lambda} \tilde{V}^T$ . Reorder the eigenvalues in  $\tilde{\Lambda}$  and according eigenvalues if necessary.
  - 5:  $\tilde{U} \leftarrow U V \Sigma^{-\frac{1}{2}} \tilde{V}$ .
- 

is an extensively used text data set. It contains categorized news from Reuters, and each news article is represented by a sparse log-transformed TF-IDF feature vector [9]. After removing news categories with fewer than 1,000 articles, we obtain 55 categories that are used in the experiment. The ImageNet<sup>3</sup> is an image database containing images organized according to the WordNet hierarchy. Each image is represented as a bag of words based on the SIFT features. We select the 100 largest categories from the training set. Each of these contains an average of 2,000 images.

For the ImageNet data set, we use the histogram intersection kernel [14], which has been commonly used on SIFT features, to construct the affinity matrix. For the other data sets, we use the Gaussian kernel  $\exp(-\mu\|x - y\|^2)$ , where  $\mu$  is set to the inverse of the average distance among all data pairs (*i.e.*  $\mu^{-1} = \frac{1}{n^2} \sum_{ij} \|x_i - x_j\|^2$ ).

size	data	#samples	dim	#clusters
small	USPS	9,298	256	10
	MNIST	35,735	784	5
medium	RCV1	160,633	47,236	55
	image-net	191,050	1,000	100
large	MNIST8M	4,130,460	784	5

Table 2. Data sets used in the spectral clustering experiments.

#### 4.1.1 Setup

The following spectral clustering methods will be compared:

1. Normalized cut algorithm [12] (denoted `ncut`).
2.  $k$ -means-based approximate spectral clustering [18] (denoted `kasp`).
3. Standard Nyström-based spectral clustering [6] (denoted `psc`).
4. The proposed algorithm (denoted `ours`).

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<sup>3</sup><http://www.image-net.org/challenges/LSVRC/2010/download-public>

The `kasp` is implemented<sup>4</sup> in R, while the others are in Matlab. The implementation for `psc` is from the highly optimized code used in [2]<sup>5</sup>. The `kasp` and `psc` implementations can directly output the clustering results. For `ncut` and `ours`, after dropping the leading eigenvector of  $M$ , we perform  $k$ -means clustering using the remaining  $c$  leading eigenvectors, and with each data row normalized to unit norm. As the  $c+1$  leading eigenvectors are required, we set  $k=c+1$  for `ours`.

Recall that both `psc` and `ours` are based on column sampling. The number of sampled columns is fixed at  $m=1,000$ , and these columns are selected by uniform sampling without replacement.

The following measures are used for performance evaluation: (1) accuracy; (2) normalized mutual information (NMI) [13], whose value is always between zero and one, and the higher the better; and (3) time. Moreover, we follow the common practice of permuting the obtained clusters to best match the ground truth. To reduce statistical variability, results for all the methods are averaged over 10 repetitions. All experiments, including the image segmentation in Sec. 4.2, are performed on a machine with Intel Xeon X5540 CPU and 16GB memory.

#### 4.1.2 Results on Small and Medium-Sized Data Sets

Table 3 shows the results on the small data sets. The variances of the time measurements are very small and so are not reported. Moreover, only the orthogonalized versions of `psc` and `ours` are shown. Those for the un-orthogonalized versions are similar. As can be seen, the three methods `sc`, `psc` and `ours`, yield very similar accuracies and NMI values, while `kasp` is much inferior. This may be due to that a fixed kernel parameter ( $\mu$ ) has been used in our experiment, while the `kasp` experiments in [18] search  $\mu$  over a range of possible values. As for the time, `ours` is faster than the others. Recall that the `kasp` implementation is in R while the others are in Matlab. Hence, the time comparison for `kasp` is for reference only.

On the medium-sized data sets, `ncut` fails to run because of insufficient memory. As for `kasp`, its implementation does not allow use of the sparse data format and histogram intersection kernel. Hence, it is also dropped from the comparison. Results comparing the orthogonalized and un-orthogonalized versions of `psc` and `ours` are shown in Table 4. As can be seen, the use of orthogonalized eigenvectors improves the clustering results. However, this nearly doubles the time for `psc`, while that for `ours` is only increased by a very small fraction.

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<sup>4</sup><http://www.stat.berkeley.edu/~dhyan/fasp.html>.

<sup>5</sup><http://alumni.cs.ucsb.edu/~wychen/sc.html>

data	method	accuracy (%)	NMI	time (sec)	
				eig	total
USPS	ncut	<b>70.32 ± 0.13</b>	<b>0.658 ± 0.001</b>	4.2	8.5
	kasp	45.63 ± 3.83	0.395 ± 0.013	-	8 min
	psc	68.14 ± 0.11	0.634 ± 0.001	28.9	30.0
	ours	70.00 ± 0.61	0.654 ± 0.009	<b>2.9</b>	<b>3.5</b>
MNIST	ncut	<b>81.37 ± 0.15</b>	<b>0.636 ± 0.001</b>	51.4	289.4
	kasp	59.08 ± 6.25	0.439 ± 0.045	-	40 min
	psc	80.62 ± 0.18	0.619 ± 0.001	39.1	46.2
	ours	80.88 ± 1.42	0.624 ± 0.015	<b>2.5</b>	<b>6.7</b>

Table 3. Clustering results on the small data sets. Here, “eig” measures the time spent on computing the eigenvectors of the normalized Laplacian.

data	method	accuracy (%)	NMI	time (sec)	
				eig	total
RCV1	psc (orthogonalized)	<b>25.05 ± 0.60</b>	0.255 ± 0.007	86.7	121.5
	ours (orthogonalized)	24.72 ± 0.85	0.253 ± 0.008	<b>5.8</b>	<b>27.4</b>
	psc (un-orthogonalized)	22.98 ± 0.95	0.239 ± 0.009	30.7	64.2
	ours (un-orthogonalized)	22.54 ± 1.23	0.236 ± 0.011	<b>4.9</b>	<b>26.5</b>
ImageNet	psc (orthogonalized)	<b>10.56 ± 0.08</b>	<b>0.196 ± 0.002</b>	83.9	179.2
	ours (orthogonalized)	<b>10.56 ± 0.10</b>	<b>0.196 ± 0.002</b>	<b>5.8</b>	<b>85.8</b>
	psc (un-orthogonalized)	9.42 ± 0.31	0.169 ± 0.003	28.2	114.0
	ours (un-orthogonalized)	9.18 ± 0.11	0.163 ± 0.002	<b>5.4</b>	<b>85.3</b>

Table 4. Clustering results on the medium-sized data sets.

#### 4.1.3 Results on Large Data Sets

The proposed method is particularly useful on large data sets. Using the MNIST8M data set, Figure 1 shows how the time for computing the orthogonalized approximate eigenvectors scales with number of samples  $n$ . As can be seen, ours grows much slower than psc. Moreover, psc suffers from insufficient memory when  $n > 10^6$ . These agree with the complexity results discussed in Table 1. Detailed clustering results on a subset of  $10^6$  samples and on the full set are shown in Table 5.

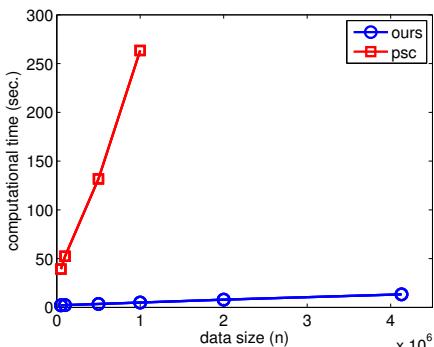


Figure 1. Time for computing the approximate eigenvectors versus data set size.

#### 4.2. Image Segmentation

In this section, we evaluate our method on image segmentation. The approach in [7] is adopted to compute the affinity matrix. It uses a multi-layer graph  $G^* = (V^*, E^*, W^*)$ , where the nodes  $V^* = V \cup \{V^{(l)}\}_{l=1}^L$  are a set of pixels  $V$  and the sets of regions  $\{V^{(l)}\}_{l=1}^L$ ,  $E^*$  is the edge set and  $W^*$  is the weights on  $E^*$ . The node subset  $V^{(l)}$  at the  $l$ th layer corresponds to the over-segmentation result generated by the mean shift algorithm [3]. Different layers have different mean shift parameters. An undirected edge  $e_{ij} \in E^*$  exists if (1) two pixels  $i, j$  are within a neighborhood of radius  $\alpha$ ; or (2) two regions  $i, j \in V^{(l)}$  share a common boundary at the same layer; or (3) a pixel  $i \in V$  is included in its corresponding region  $j \in V^{(l)}$ . This edge  $e_{ij}$  then has the following weight  $w_{ij} \in W^*$ :

$$w_{ij} = \begin{cases} \exp(-\theta_g \|g_i - g_j\|), & i, j \in V \text{ & } \|x_i - x_j\| \leq \alpha \\ \exp(-\theta_g \|\bar{g}_i - \bar{g}_j\|), & i, j \in V^{(l)} \\ \gamma, & i \in V, j \in V^{(l)}, \end{cases} \quad (17)$$

where  $x_i$ , and  $g_i$  are the position and color of pixel  $i$ , respectively,  $\bar{g}_i$  is the mean color of inner pixels of region  $i$ , and  $\theta_g$  is a constant that controls the strength of the weight. Note that in (17), the boundary cue is implicitly incorporated in the third case and  $\gamma$  controls the trade-off between color and boundary cues. To perform segmentation, we ap-

#samples used	method	accuracy(%)	NMI	time (sec.)	
				eig	total
1,000,000	psc	78.68 ± 0.18	0.555 ± 0.002	245.0	391.8
1,000,000	ours	79.07 ± 0.62	0.564 ± 0.002	<b>4.9</b>	<b>101.5</b>
4,130,460	ours	77.91 ± 0.78	0.551 ± 0.002	<b>13.4</b>	<b>407.0</b>

Table 5. Clustering results on the MNIST8M data set.

ply the proposed method on  $W^*$ . We use grid sampling with step size  $\alpha$  on images. Region nodes are not involved in the sampling step so as to avoid zero rows in the resulting matrix.

We first compare the qualities of the three approximation schemes:  $S_*$  defined in (12),  $A_{11,k}^{-1}$ , and  $S^{(2)}$  defined in (16), by using the Berkeley image database. It contains 300 images with ground truth segmentation results. As in [7], we construct a three-layer graph  $G^*$  with the same setting for mean-shift parameters and  $\theta_g$ . The neighborhood radius  $\alpha$  is set to 3 pixels. Following [7], we also manually chose the number of segments for each image. The inter-layer edge weight  $\gamma$  in (17) is empirically chosen as 3. For quantitative evaluation, the following four measures are used (as in [7]): Probabilistic Rand Index (PRI), Variation of Information (VoI), Global Consistency Error (GCE), and Boundary Displacement Error (BDE). For PRI, the larger the better, while for the other three, the smaller the better.

Table 6 shows the results. As can be seen,  $S_*$  outperforms the other two on all four measures. For benchmarking, we also show the results reported in [7], which are comparable to ours.

method	PRI	VoI	GCE	BDE
ours ( $S_*$ )	<b>0.8205</b>	1.9517	0.1998	<b>12.09</b>
ours ( $A_{11,k}$ )	0.7814	2.4519	0.2718	13.61
ours ( $S^{(2)}$ )	0.7598	2.6852	0.3093	14.38
Kim <i>et al.</i> [7]	0.8146	<b>1.8545</b>	<b>0.1809</b>	12.21

Table 6. Comparison of different  $S$ 's and the method in [7] on the Berkeley image database.

To further demonstrate the scalability of the proposed method, we perform segmentation on several very large images downloaded from the web. We manually select the parameter  $\alpha$  and  $\gamma$  for each image. Results are shown in Figure 2. The first row is the original image and the second row shows the segmentation results and the corresponding computation time. As can be seen, on the  $4752 \times 3168$  image, computing the eigenvectors only takes 23 seconds.

## 5. Conclusion

In this paper, we proposed a time- and space-efficient spectral clustering algorithm. We directly estimate the rank- $k$  approximation of  $M = I - L$  ( $L$  is the normalized Lapla-

cian) to avoid storing the sampled affinity matrix, which allows the proposed algorithm scaling to very large scale data sets. A general procedure is also proposed to orthogonalize the approximated eigenvectors suitable for both psd and not psd affinity matrices. We performed extensive experiments to show the accuracy and time efficiency of our algorithms and demonstrated the scalability on spectral clustering with several millions data points and image segmentation with 10 million pixels scale.

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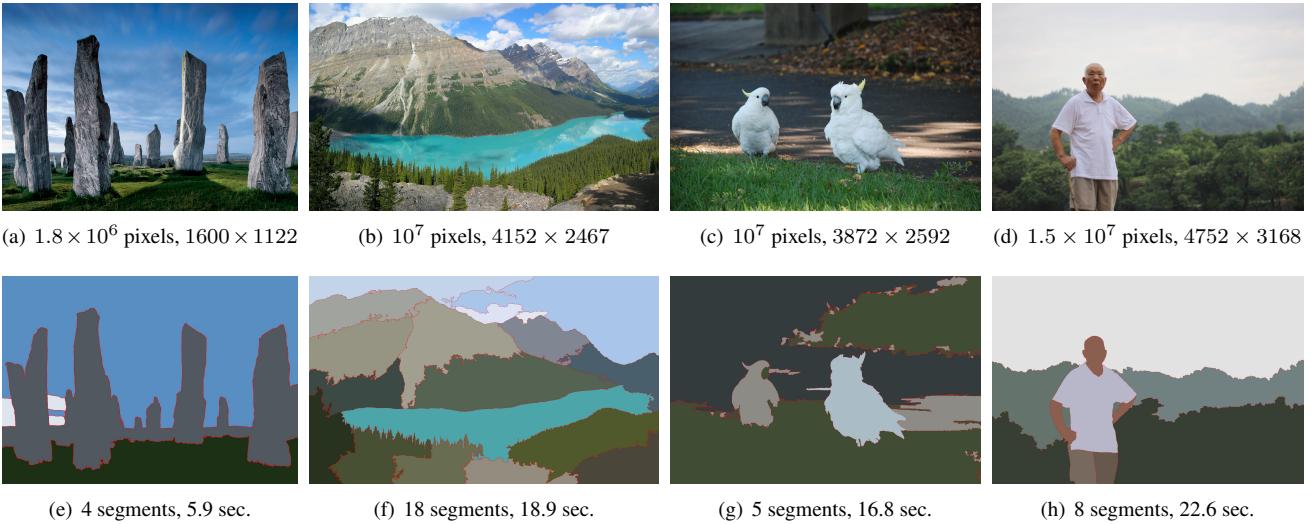


Figure 2. Image segmentation on very large images. The first row shows the image, its size, and the number of pixels. The second row shows the segmentation result and the time used on computing the eigenvectors.

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## A. Appendix: Proof of Proposition 3

**Lemma 3.** Assume that graph  $G$  has no more than  $k$  connected components, then  $\text{diag}(D^{\frac{1}{2}}M_kD^{\frac{1}{2}}1) = D$ .

*Proof.* Let  $L = I - M$  be the normalized Laplacian matrix of  $W$ . Then  $L$  is psd and has exactly  $c$  zero eigenvalues, where  $c$  is the number of connected components of  $G$  [16]. Therefore, the  $c$  largest eigenvalues of  $M$  are equal to 1. As  $M_k$  keeps the  $k$  largest eigenvalues of  $M$  and  $k \geq c$ ,  $M_k$  also keeps the eigenspace which is spanned by the eigenvectors with eigenvalue 1. As  $M(D^{\frac{1}{2}}1) = D^{-\frac{1}{2}}W1 = D^{\frac{1}{2}}1$ , so  $D^{\frac{1}{2}}1$  is an eigenvector of  $M$  with eigenvalue 1. Thus, it is also an eigenvector of  $M_k$ . Hence,  $M_k(D^{\frac{1}{2}}1) = D^{\frac{1}{2}}1$  and the result follows.  $\square$

*Proof.* (Proposition 1) Let  $D_* = \text{diag}(A_{11}1)$  and  $M_* =$

$$D_*^{-\frac{1}{2}}A_{11}D_*^{-\frac{1}{2}}. \text{ Note that,}$$

$$\begin{aligned} & D_*^{-\frac{1}{2}}M_{*,k}^{-1}D_*^{-\frac{1}{2}}A_{11}1 \\ &= A_{11}^{-1}D_*^{\frac{1}{2}}(D_*^{-\frac{1}{2}}A_{11}D_*^{-\frac{1}{2}})M_{*,k}^{-1}(D_*^{-\frac{1}{2}}A_{11}D_*^{-\frac{1}{2}})D_*^{\frac{1}{2}}1 \\ &= A_{11}^{-1}D_*^{\frac{1}{2}}M_{*,k}M_{*,k}^{-1}M_{*,k}D_*^{\frac{1}{2}}1 \\ &= A_{11}^{-1}D_*^{\frac{1}{2}}M_{*,k}D_*^{\frac{1}{2}}1 = A_{11}^{-1}A_{11}1 = 1, \end{aligned}$$

where the second last equation is from Lemma 3. Therefore,  $\hat{A}_{:,1} = A_{:,1}D_*^{-\frac{1}{2}}M_{*,k}^{-1}D_*^{-\frac{1}{2}}A_{11}1 = A_{:,1}$ .  $\square$

## B. Proof of Proposition 2

*Proof.*

$$\begin{aligned} U\Lambda U^T &= (UV\Sigma^{-\frac{1}{2}})(\Sigma^{\frac{1}{2}}V^T\Lambda V\Sigma^{\frac{1}{2}})(UV\Sigma^{-\frac{1}{2}})^T \\ &= (UV\Sigma^{-\frac{1}{2}})\tilde{V}\tilde{\Lambda}\tilde{V}^T(UV\Sigma^{-\frac{1}{2}})^T = \tilde{U}\tilde{\Lambda}\tilde{U}^T. \end{aligned}$$

Moreover,  $\tilde{U}$  is orthogonal as

$$\begin{aligned} \tilde{U}^T\tilde{U} &= \tilde{V}^T\Sigma^{-\frac{1}{2}}V^T(U^TU)V\Sigma^{-\frac{1}{2}}\tilde{V} \\ &= \tilde{V}^T\Sigma^{-\frac{1}{2}}V^T(V\Sigma V^T)V\Sigma^{-\frac{1}{2}}\tilde{V} \\ &= \tilde{V}^T\tilde{V} = I. \end{aligned}$$

$\square$