$H = H_I(t) + H_N$

$$H_I(t)/\hbar = 2\pi \begin{pmatrix} \overline{E}_z & \frac{1}{2}E_x(t) & \frac{1}{2}E_x(t) & 0 & 0 & 0 \\ \frac{1}{2}E_x(t) & \frac{1}{2}\delta E_z & 0 & \frac{1}{2}E_x(t) & t_0 & t_0 \\ \frac{1}{2}E_x(t) & 0 & -\frac{1}{2}\delta E_z & \frac{1}{2}E_x(t) & -t_0 & -t_0 \\ 0 & \frac{1}{2}E_x(t) & \frac{1}{2}E_x(t) & -\overline{E}_z & 0 & 0 \\ 0 & t_0 & -t_0 & 0 & U-\epsilon & 0 \\ 0 & t_0 & -t_0 & 0 & 0 & U+\epsilon \end{pmatrix}$$

$$H_N/\hbar = 2\pi \begin{pmatrix} \frac{\beta_{E_{z_1}}+\beta_{E_{z_2}}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_{E_{z_1}}-\beta_{E_{z_2}}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta_{E_{z_1}}-\beta_{E_{z_2}}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{(\beta_{E_{z_1}}+\beta_{E_{z_2}})}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 2\pi \cdot \frac{1}{2}(\beta_{E_{z_1}}(Z \otimes I) + \beta_{E_{z_2}}(I \times Z)) = 2\pi \cdot \frac{1}{2}(\beta_{E_{z_1}} H_{E_{z_1}} + \beta_{E_{z_1}} H$$

$$< J_2 >= \frac{1}{2}\sum_{jk}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 C_{jk}(t_1,t_2)\mathrm{Tr}[(R_j(t_1)R_k(t_2))_{4\times 4}] - \frac{1}{16}\sum_{jk}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 C_{jk}(t_1,t_2)\mathrm{Tr}[R_{j,4\times 4}(t_1)]\mathrm{Tr}[R_{k,4\times 4}(t_2)],$$

where $R_j(t) = U_I^\dagger(t) H_{N_j} U_I(t)$, $C_{jk}(t_1,t_2) =< \beta_j(t_1)\beta_k(t_2) >$, and $j = E_{z_1}, E_{z_2}$.

The depahsing is quast-static so $C_{jk}(t_1,t_2) =< \beta_j \beta_k >$, and leave out the second term because H_N is traceless

$$\begin{aligned} < J_2 >= \frac{1}{2}\Bigg(&\beta_{E_{z_1}}\beta_{E_{z_1}}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}[(R_1(t_1)R_1(t_1))_{4\times 4}] + \beta_{E_{z_2}}\beta_{E_{z_2}}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}[(R_2(t_1)R_2(t_1))_{4\times 4}] \\ &+ \beta_{E_{z_1}}\beta_{E_{z_2}}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}[(R_1(t_1)R_2(t_1))_{4\times 4}] + \beta_{E_{z_1}}\beta_{E_{z_2}}\int_0^{t_f} dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}[(R_2(t_1)R_1(t_2))_{4\times 4}]\Bigg)' \end{aligned}$$

where $R_1(t) = U_I(t)(Z \otimes I)U_I(t)$, $R_2(t) = U_I(t)(I \otimes Z)U_I(t)$.