Absorption probabilities

Define the following probabilities, for $k \in C$, with C a recurrent class, and $i \in T$, T a transient class:

$$\pi_{ik}^{n}(C) = P\left[X_n = k,\ X_m \notin C,\ m = 1, 2, \ldots, n-1 \,\middle|\, X_0 = i\right] =$$

$$= P\left[\begin{matrix}\text{entering class } C \text{ in state } k \\ \text{at time } n \text{ for the first time}\end{matrix} \,\middle|\, X_0 = i\right]$$

$$\pi_{i}^{n}(C) = \sum_{k \in C} \pi_{ik}^{n}(C) = P\left[\begin{matrix}\text{entering class } C \\ \text{at time } n \\ \text{for the first time}\end{matrix} \,\middle|\, X_0 = i\right]$$

$$\pi_i(C) = \sum_{n=1}^{+\infty} \pi_i^n(C) = P\left[\begin{matrix}\text{absorption in} \\ \text{class } C\end{matrix} \,\middle|\, X_0 = i\right]$$

Theorem 3.1 (KT p. 91)

Let $j \in C$ (aperiodic recurrent class) and $i \in T$.

We have:

$$\lim_{n \to \infty} P_{ij}^{(n)} = \pi_i(C) \lim_{n \to \infty} P_{jj}^{(n)} = \pi_i(C) \cdot \pi_j$$

Note: the limit depends on both i and j

(The same result applies in the periodic case for $\lim_{n \to \infty} \sum_{m=0}^{n-1} P_{ij}^{(m)}$)

j	i	$\lim_{n\to\infty} P_{ij}^{(n)}$
transient	any	0
recurrent $\in C$	recurrent $\in C_1 \neq C$	0
recurrent $\in C$	recurrent $\in C$	$\pi_j = \frac{1}{m_j}$
recurrent $\in C$	transient	$\pi_i(C)\,\pi_j = \frac{\pi_i(C)}{m_j}$

C, C_1 recurrent classes

$$\pi_i(C) = P[X_T \in C \mid X_0 = i] = \lim_{n\to\infty} \sum_{k \in C} P_{ik}^{(n)}$$

if j (and therefore C) is periodic, the same results apply for $\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} P_{ij}^{(m)}$

(*) In a MC with a finite number of states, there must be at least one positive recurrent state.
("the chain must be somewhere at infinity")

<u>Proof</u>: Assume no pos. rec. states. Then:

$$\sum_{j=1}^{N} P_{ij}^{(n)} = 1 \qquad N = \#\text{ of states} < \infty$$

$$1 = \lim_{n\to\infty} \sum_{j=1}^{N} P_{ij}^{(n)} \overset{\text{finite sum}}{=} \sum_{j=1}^{N} \lim_{n\to\infty} P_{ij}^{(n)} \underset{\text{all states transient or null recurrent}}{=} 0$$

Therefore, our initial assumption must be wrong.
($\Rightarrow$ a chain with all transient states must be infinite) Q.E.D.

(*) In a MC with a finite number of states, there ~~must~~ cannot be any null recurrent states.

<u>Proof</u>: Suppose there is one, which will then belong to a finite null rec. class.
Since a recurrent class is a MC by itself, this is not possible from the previous result.
Q.E.D.

($\Rightarrow$ null recurrent states are only allowed in infinite chains) exercise: Study the limiting behavior of P^n for the following two matrices:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/6 & 1/3 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e., find $\lim_{n\to\infty} P^{2n}$ and $\lim_{n\to\infty} P^{2n+1}$ and compare.

Comment the result.

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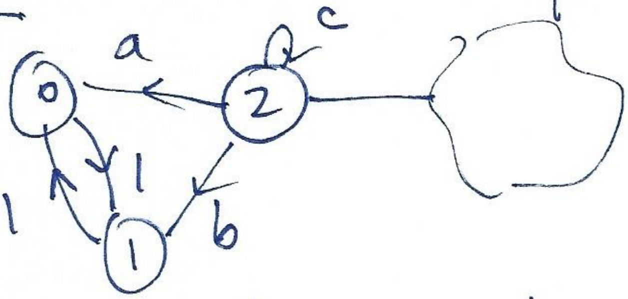
exercise: Consider the following chain:



Compute $P[X_{2n}=0 \mid X_0=2]$ and $P[X_{2n+1}=0 \mid X_0=2]$ and find under which condition on a, b, c

$$\lim_{n\to\infty} P[X_{2n}=0 \mid X_0=2] = \lim_{n\to\infty} P[X_{2n+1}=0 \mid X_0=2]$$

Comment the results.

X_n a Markov Chain, S a set of states.

$$Y_i(n) = P[X_j \in S,\ j = 1, 2, \dots, n \mid X_0 = i],\ i \in S$$

$$= P\left[\begin{array}{c}\text{chain remains in } S \\ \text{up to time } n\end{array} \Big| X_0 = i\right]$$

We have $Y_i(1) = \sum_{j \in S} P_{ij}$; $Y_i(n) = \sum_{j \in S} P_{ij} Y_j(n-1)$ $n > 1$.

<u>Lemma</u>: $Y_i(n)$ is non-increasing in n, $\forall i \in S$

<u>Proof</u>: By induction: $Y_i(2) = \sum_{j \in S} P_{ij} Y_j(1) \le \sum_{j \in S} P_{ij} = Y_i(1)$

Assume $Y_i(n) \le Y_i(n-1)$. Then

$$Y_i(n+1) = \sum_{j \in S} P_{ij} Y_j(n) \le \sum_{j \in S} P_{ij} Y_j(n-1) = Y_i(n)$$

$\Rightarrow$ true $\forall n$. Q.E.D.

Let Z_i be a solution of

$$Z_i = \sum_{j \in S} P_{ij} Z_j,\quad |Z_i| \le 1,\quad i \in S.$$

<u>Lemma</u>: $|Z_i| \le Y_i$.

<u>Proof</u>: By induction: $|Z_i| \le \sum_{j \in S} P_{ij} |Z_j| \le \sum_{j \in S} P_{ij} = Y_i(1)$

Suppose $|Z_i| \le Y_i(n)$ for some n. Then,

$$|Z_i| \le \sum_{j \in S} P_{ij} |Z_j| \le \sum_{j \in S} P_{ij} Y_j(n) = Y_i(n+1)$$

$\Rightarrow |Z_i| \le Y_i(n)$ $\forall n$, also as $n \to \infty$ Q.E.D.

Lemma 4.13:

An irreducible MC with states 0, 1, 2, ... is recurrent $\iff$ $f_{i0} = 1 \quad \forall i \neq 0$.

Proof: 1) We assume $f_{i0} = 1 \; \forall i \neq 0$. Then:

$$f_{00} = P_{00} + \sum_{i \neq 0} P_{0i} f_{i0} = 1$$

2) We assume $\exists \, i \neq 0$ s.t. $f_{i0} < 1$. Since the MC is irreducible, $\exists \, m$ s.t. $P_{0i}^{(m)} > 0$. Define $n = \min \{ m > 0 : P_{0i}^{(m)} > 0 \}$. The path from 0 to i of minimum length n cannot go through 0.

Then, an event for which the MC does not return to 0 is that it goes to i in n steps and from then never comes back. Such event has probability $P_{0i}^{(n)} \cdot (1 - f_{i0})$ and

$$1 - f_{00} \geq P_{0i}^{(n)} (1 - f_{i0}) > 0 \implies f_{00} < 1.$$

Therefore, the condition is necessary.