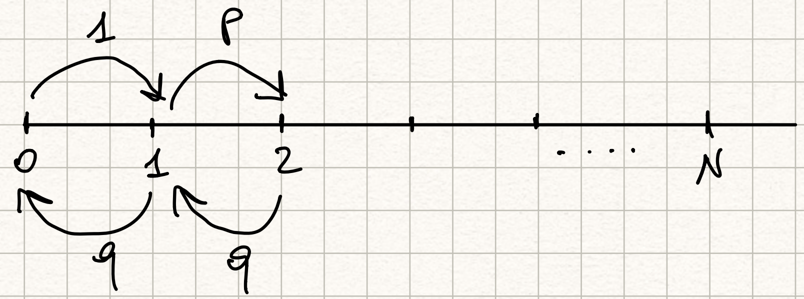
Q 3: consider a random walk over the non negative integers with the following transition probability: $P_{01} = 1$, $P_{i,i+1} = p$, $P_{i,i-1} = q$ $i > 0$ $p + q = 1$. Study it's behaviour and in particular characterize it's recurrence or transiency and derive the ready state distribution.



in the trivial case in which $p = 0$ the random walk is recurrent in the state 0 and 1 with period 2.

In the non - trivial case in wich $0 < p < 1$

if $p > 1/2 \longrightarrow$ RW goes to $+\infty$

if $p < 1/2 \longrightarrow$ RW goes to 0

Q 5: Prove that for a renewal process

$$E[S_{N(t)+1}] = E[X](m(t)+1)$$

we start by conditioning the time of the first renewal on $X_1 = x$

$$E[S_{N(t)+1} \mid X_1 = x] = \begin{cases} X_1 & \text{if } x > t \\ x + A(t-x) & \text{if } x \leq t \end{cases}$$

Invoking the law of total probability we obtain:

$$A(t) = E[S_{N(t)+1}] = E[X_1] + \int_0^t A(t-x)\,dF(x)$$

this is a renewal equation in the case of $E[X_1] = a(t)$

So we can rewrite $E[X_1] + \int_0^t A(t-x)\,dF(x)$

as

$$A(t) = a(t) + \int_0^t a(t-x)\,dM(x)$$

$$= E[X_1] \times [1 + M(t)]$$

Q9 prove that for a Markov chain the n-step transition probabilities $P_{ij}^{(n)}$ satisfy the relationship $P_{ij}^{(n)} = \sum P_{im}^{(k)} P_{mj}^{(n-k)}$ $k=0,1...n$

$$P_{ij}^{(n)} = \sum_{m=0}^{n} \mathbb{P}[X_n = j, X_k = m \mid X_0 = i]$$

$$= \sum_m \underbrace{\mathbb{P}[X_n = j \mid \cancel{X_0 = i}, X_k = m]}_{= P_{mj}^{(n-k)}} \cdot \underbrace{\mathbb{P}[X_k = m \mid X_0 = i]}_{= P_{im}^{(k)}}$$

forget for MC prop

$$= \sum_m P_{im}^{(k)} \cdot P_{mj}^{(n-k)}$$

Q7: Prove that a MC with finite number of states cannot have a null recurrent state:

null recurrent state means that

m_i for that state is equal to $+\infty$

and m_i being $\mathbb{E}[R_i \mid X_0 = i]$

so the expected time to return to stat i conditioned on the fact that we started from i.

so if $m_i = \infty$ and the state in recurrent we need infinite number of recurrent states in his class to satisfy this property $\longrightarrow$ but we have finite states for hypothesis so contradiction.

Q11: prove that in a Markov chain with a finite number of states:

a) there must be a recurrent state

b) it must not have a NULL recurrent state.

b) is answered in Q7.

2) a recurrent state is defined as a state with $f_{ii} = 1$ this means that is certain that starting from him we will return to him.

a transient state on the contrary has $f_{ii} < 1$ this means that even in an $+\infty$ number of step we are not guaranteed to return to him.

In a finite number of state if we don't have a recurrent state is trivial to guess that all transient states will have $f_{ii} = 1$ so becomming recurrent.
Thus we need at least a recurrent state to keep transient states transient.

Q1 State and prove the elementary Renewal Theorem

A: Let $\{N(t) : t \geq 0\}$ be a Renewal (counting) process and $M(t) = E[N(t)]$

Then

$$\lim_{t \to +\infty} \frac{M(t)}{t} = \lim_{t \to +\infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}$$

, where μ is the mean interoccurrence time

<u>PROOF</u>: (<u>Note</u>: does not follow directly from strong LLN for renewal counting process, since almost sure convergence does not imply convergence of means

We start by showing that $M(t)$ is finite $\forall t > 0$

Let $S_n = X_1 + X_2 + \dots X_n = \sum_{i=0}^{n} X_i$, $n \geq 1$; waiting time until the n-th event, where X_i is the elapsed time from the $(i-1)$th until the i-th event

Let $F_n(t) = \mathbb{P}[S_n \leq t] = \mathbb{P}[N(t) \geq n]$

Using the convolution formula we have:

$$F_n(t) = \int_0^t F_{n-m}(t - \varepsilon)\, dF_m(\varepsilon)$$

to prove?

and

$$\int_0^t F_{n-m}(t - \varepsilon)\, dF_m(\varepsilon) \leq F_{n-m}(t) \cdot F_m(t) \quad , 1 \leq m \leq n-1$$

$\Rightarrow m(t) = E[N(t)] = \sum_{n=1}^{+\infty} P[S_n \le t] = \sum_{n=1}^{+\infty} F_n(t)$

$\Rightarrow m(t) = \sum_{n=1}^{+\infty} F_n(t) = \sum_{m=0}^{+\infty} \sum_{k=1}^{r} F_{mr+k}(t) \le \sum_{m=0}^{+\infty} \sum_{k=1}^{r} (F_r(t))^m F_k(t)$

$$= \underbrace{\left(\sum_{m=0}^{+\infty} (F_r(t))^m\right)}_{\text{geometrically}} \cdot \underbrace{\left(\sum_{k=1}^{r} F_k(t)\right)}_{\le r \text{ since } F_k(t)}$$

$\Rightarrow m(t)$ converges ✓ if $F_r(t) < 1$

There is a theorem which states: $\forall t, \exists r$ s.t. $F_r(t) < 1$

Proof: we can always find $t_0 > 0$ s.t. $F(t) < 1, \forall t < t_0$

$\Rightarrow$ We have:

$$1 - F_r(t) = \mathbb{P}[S_r > t] \ge (\mathbb{P}[X_i > t/r])^r = (1 - F(t/r))^r$$

which is > 0 if $\frac{t}{r} < t_0$ (by the ... $F(t) < 1 \; \forall t < t_0$)

$r > t/t_0$

So, $m(t)$ converges geometrically

Show $\frac{m(t)}{t} \to 1/\mu$ $t \to +\infty$?

Q2

Prove that in a Markov Chain the period is a class property

Proof:

Period is a class property means: $i \leftrightarrow j \Rightarrow d(i) = d(j)$

By hp ($i \leftrightarrow j \Rightarrow$ i and j communicate) $\exists\, m, n \in \mathbb{N}_{>0}$ s.t.

$$p_{ij}^{(m)} > 0 \quad \text{and} \quad p_{ji}^{(n)} > 0$$

$$\Rightarrow p_{jj}^{(m+n)} \geq p_{ji}^{(n)} \cdot p_{ij}^{(m)} > 0$$

Assume that $p_{ii}^{(s)} > 0$, $s \in \{ n \geq 1 : P_{ii}^{(n)} > 0 \}$, then

$$P_{jj}^{(n+s+m)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(s)} \cdot P_{ij}^{(n)} > 0$$

So $n+s+m \in \{ n \geq 1 : P_{jj}^{(n)} > 0 \}$,

Let $d(j)$ be g.c.d between $\{ n \geq 1,\ P_{jj}^{(n)} > 0 \} \Rightarrow d(j) | n+m$

& $d(j) | n+m+s \Rightarrow d(j)$ divides $(n+m+s) - (n+m) = s$

$\Rightarrow d(j)$ divides $d(i)$

with similar computation we can prove that $d(i) | d(j)$

so $d(i) = d(j)$ $\Rightarrow$ i and j have the same period

Q4 - For a Poisson process of rate λ, prove that the interarrival times are i.i.d exp. with mean $\frac{1}{\lambda}$

A 4: interarrival times = sojourn time

Let's prove that $S_0 \sim \exp(\lambda)$

$$\mathbb{P}[S_0 > t] = \mathbb{P}[X(t) = 0] = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} \Rightarrow S_0 \sim \exp(\lambda t)$$

Since an exponent r.v. has mean $\frac{1}{\lambda}$, so is $\mathbb{E}[S_0]$

$\mathbb{P}[W_1 \leq s, W_2 > t]$ when W_i is waiting time and $\boxed{0 \leq s < t}$

$$\mathbb{P}[X(s) = 1, X(t) = 1] = \mathbb{P}[X(s) = 1, X(t) - X(s) = 0] =$$
$$\underset{\text{independent}}{=} \mathbb{P}[X(s) = 1] \cdot \mathbb{P}[X(t) - X(s) = 0]$$
$$= \frac{(\lambda s)^1 e^{-\lambda s}}{1!} \cdot \frac{\lambda(t-s)\, e^{-\lambda(t-s)}}{0!} = \lambda s e^{-\lambda t}$$

Instead, $\mathbb{P}[W_1 \leq s, W_2 \leq t] = \mathbb{P}[W_1 \leq s] - \mathbb{P}[W_1 \leq s, W_2 > t] = 1 - e^{-\lambda s} - \lambda s e^{-\lambda t}$

$= F_{W_1, W_2}(s, t) \Rightarrow$ density of r.v. (W_1, W_2) is:

$$f_{W_1, W_2}(s, t) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} F_{W_1, W_2}(s, t) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\left(1 - e^{-\lambda s} - \lambda s e^{-\lambda t}\right)$$
$$= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\left(-\lambda s e^{-\lambda t}\right)$$
$$= \lambda^2 e^{-\lambda t}, \quad 0 \leq s < t$$

Let us make this change of variables: $S_0 = W_1$, $S_1 = W_2 - W_1$

$\Rightarrow W_1 = S_0$, $W_2 = S_1 + S_0 \Rightarrow (S_0, S_1) \in (0, +\infty)^2$

and

$$f_{S_0,S_1}(u,r) = |\det J| \cdot f_{W_1,W_2}(u, u+r), \text{ where } J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, |\det J| = 1$$

$$f_{S_0,S_1}(u,r) = \lambda^2 e^{-\lambda(u+r)} = \lambda e^{-\lambda u} \cdot \lambda e^{-\lambda r} = f_{S_0}(u) \cdot f_{S_1}(r)$$

$\Rightarrow S_0 \perp S_1$ & $S_0 \sim \exp(\lambda)$, $S_1 \sim \exp(\lambda)$

$\Rightarrow \mathbb{E}[S_0] = \mathbb{E}[S_1] = \frac{1}{\lambda}$

the general case for $S_0 \dots S_n$ is similar

Q6: prove that if i and j of a MC communicate and i is recurrent, then j is also recurrent

A6: by hp, $i \leftrightarrow j \Rightarrow \exists\, n, m \geq 1$ s.t.

$$P_{ij}^{(n)} > 0\, ; \; P_{ji}^{(m)} > 0$$

Let $\alpha > 0$:

$$\sum_{\alpha=0}^{+\infty} P_{jj}^{(n+m+\alpha)} \geq \sum_{\alpha=0}^{+\infty} P_{ji}^{(m)} \cdot P_{ii}^{(\alpha)} \cdot P_{ij}^{(n)}$$

$$= P_{ji}^{(m)} \cdot P_{ij}^{(n)} \sum_{\alpha=0}^{+\infty} P_{ii}^{(\alpha)}$$

Since i is recurrent by hypotesis, then

$$\sum_{\alpha=0}^{+\infty} P_{ii}^{(\alpha)} = +\infty \Rightarrow \sum_{\alpha=0}^{+\infty} P_{jj}^{(\alpha)} = +\infty$$

j is recurrent

prob. because

$$\sum_{\alpha=0}^{+\infty} P_{jj}^{(n+m+\alpha)} \geq \sum_{\alpha=0}^{+\infty} P_{ii}^{(\alpha)} = +\infty$$

Q8: prove that for a PP $X(t)$ the statistic of $X(S)$ conditioned on $X(t)$, $s<t$, $k \leq m$, is binomial and provide the expression of $\mathbb{P}[X(s)=k \mid X(t)=m]$

$$\mathbb{P}[X(s)=k \mid X(t)=m] = \frac{\mathbb{P}[X(s)=k, X(t)=m]}{\mathbb{P}[X(t)=m]}$$

$$= \frac{\mathbb{P}[X(s)=k, X(t)-X(s)=m-k]}{\mathbb{P}[X(t)=m]} =$$

$$= \mathbb{P}[X(s)=k] \cdot \mathbb{P}[X(t)-X(s)=m-k] \cdot \frac{1}{\mathbb{P}[X(t)=m]}$$

$$= \frac{(\lambda s)^k e^{-\lambda s}}{k!} \cdot \frac{(\lambda(t-s))^{m-k} \cdot e^{-\lambda(t-s)}}{(m-k)!} \frac{m!}{(\lambda t)^m e^{-\lambda t}} = \binom{m}{k} \frac{(\lambda s)^k (\lambda(t-s))^{m-k}}{(\lambda t)^m}$$

$$= \binom{m}{k} \frac{s^k (t-s)^{m-k}}{\underbrace{t^m}_{\to \boxed{t^k \cdot t^{m-k}}}} = \boxed{\binom{m}{k} \left(\frac{s}{t}\right)^k \cdot \left(1 - \frac{s}{t}\right)^{m-k}}$$

$$\Rightarrow \mathbb{P}[X(s)=k \mid X(t)=m] = \mathrm{Bin}(m, s/t)$$

Q10: For a Renewal process state precisely and prove what is the value of $\lim_{t \to +\infty} \frac{N(t)}{t}$

Note: $N(\infty) = \infty$ with prob. 1:

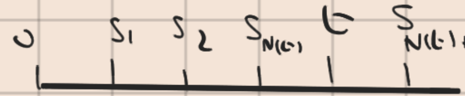
$$\mathbb{P}[N(\infty) < \infty] = \mathbb{P}[X_n = \infty \text{ for some } n] =$$
$$= \mathbb{P}\left[\bigcup_{n=1}^{+\infty} \{X_n = +\infty\}\right] \le \sum_{n=1}^{+\infty} \mathbb{P}[X_n = +\infty] = 0$$

cannot have finite renewals in an infinite time

$\Downarrow$

$N(t) \to +\infty$ as $t \to +\infty$

$$\lim_{t \to +\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{with prob. 1}$$



Proof: since $S_{N(t)} \le t < S_{N(t)+1}$ $\to$

$$\Rightarrow \frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

$\frac{S_{N(t)}}{N(t)}$ is the average of the first $N(t)$ interarrival times

then by the strong LLN: $\frac{S_{N(t)}}{N(t)} \to \mu$ as $N(t) \to +\infty$

but also $\frac{S_{N(t)}}{N(t)} \to \mu$ as $t \to +\infty$

$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \to$ then we obtain

$$\frac{S_{N(t)+1}}{N(t)} \to \mu \quad \text{as } t \to +\infty$$

By the combined lemma we get also that

$$\frac{t}{N(t)} \to \mu \quad \text{as } t \to \infty$$

or

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

Q12 : same as Q8

Q14: Give a definition of recurrent state and prove that a state is recurrent $\iff$

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

We say that a state i is $\boxed{\text{recurrent}}$ if $f_{ii} = 1$

$\hookrightarrow$ prob. of returning to i

$$i \text{ recurrent} \iff \sum_{n=1}^{\infty} P_{ii}^{(n)} = +\infty$$

Proof: i recurrent $\Rightarrow f_{ii} = 1$

M counts the total number of visits to state i

$$M = \sum_{n=1}^{\infty} \mathbb{1}\{X_n = i\}$$

$$\mathbb{E}[M \mid X_0 = i] = \frac{f_{ii}}{1 - f_{ii}} = +\infty \qquad \to f_{ii} = 1$$

so

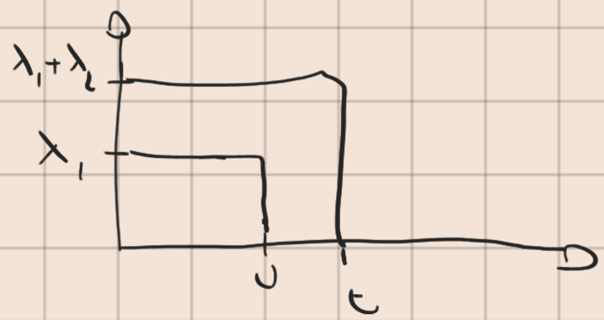
$$\boxed{+\infty} = \mathbb{E}[M \mid X_0 = i] = \mathbb{E}\left[\sum_{n=1}^{+\infty} \mathbb{1}\{X_n = i\} \,\Big|\, X_0 = i\right]$$

$$= \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}\{X_n = i\} \mid X_0 = i]$$

since $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$

$$= \sum_{n=1}^{\infty} \mathbb{P}[X_n = i \mid X_0 = i] = \boxed{\sum_{n=1}^{\infty} P_{ii}^{(n)}}$$

Q16: For two PP $X_1(t)$ and $X_2(t)$ with rates λ_1, λ_2 respectively, derive the expression of $\mathbb{P}[X_1(u) = k \mid X_1(t) + X_2(t) = n]$ for $0 < u < t$, $0 \le k \le n$



$$X(t) = X_1(t) + X_2(t)$$

$$\mathbb{P}[X_1(u) = k \mid X(t) = n] = \frac{\mathbb{P}[X_1(u) = k,\ X_1(t) - X_1(u) + X_2(t) = n - k]}{\mathbb{P}[X(t) = n]}$$

$$= \mathbb{P}[X_1(u) = k] \cdot \mathbb{P}[X_1(t) - X_1(u) + X_2(t) = n - k] \cdot \frac{1}{\mathbb{P}[X(t) = n]}$$

$$\overset{\text{indep.}}{=} \frac{(\lambda_1 u)^k e^{-\lambda_1 u}}{k!} \cdot \frac{(\lambda_1(t-u)+\lambda_2 t)^{n-k} e^{-(\lambda_1(t-u)+\lambda_2 t)}}{(n-k)!} \cdot \frac{n!}{(t(\lambda_1+\lambda_2))^n e^{-t(\lambda_1+\lambda_2)}} =$$

$$= \boxed{\binom{n}{k} \left(\frac{\lambda_1 \cdot u}{t(\lambda_1+\lambda_2)}\right)^k \cdot \left(\frac{\lambda_1(t-u)+\lambda_2 t}{t(\lambda_1+\lambda_2)}\right)^{n-k}}$$

Q15: Prove that for a Renewal process $M(t) < \infty$ $\forall t$ finite

We start by introducing a definition

$$\boxed{F_n(t)} = \int_0^t F_{n-m}(t-\xi)\, dF(\xi) \leq \boxed{F_{n-m}(t) F_m(t)} \quad 1 \leq m \leq n-1$$

$\leq$

Then

$$M(t) = \sum_{j=1}^{+\infty} F_j(t) = \sum_{m=0}^{+\infty} \sum_{k=1}^{r} F_{mr+k}(t)$$

$$\leq \sum_{m=0}^{+\infty} \sum_{k=1}^{r} (F_r(t))^m F_k(t) =$$

$$= \sum_{m=0}^{+\infty} (F_r(t)^m) \underbrace{\left(\sum_{k=1}^{r} \overbrace{F_k(t)}^{\to \text{ prob. so } \leq 1}\right)}_{\leq r}$$

converges geometrically if $F_r(t) < 1$ $\to$ There is a Th that states that $\forall t, \exists r$ s.t. $F_r(t) < 1$