# Stochastic Processes

Lecture 9

CYS 2020/2021

- Recurrence of the Random Walk on $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3$
- Basic Limit Theorem of MC
  - positive-recurrence
  - null-recurrence
- Invariants distribution
- Some Examples

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$$f_{ii}^{(n)} = \mathbb{P}[X_n = i, X_\nu \neq i, \nu = 1, 2, \dots, n-1 \mid X_0 = i], \quad f_{ii}^{(0)} = 0$$

$$f_{ii} = \sum_{n=0}^{+\infty} f_{ii}^{(n)} = \begin{cases} 1 & i \text{ recurrent} \\ < 1 & \underbrace{i \text{ transient}} \end{cases}$$

$i$ is recurrent $\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} = +\infty$

$i$ is transient $\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} < +\infty$

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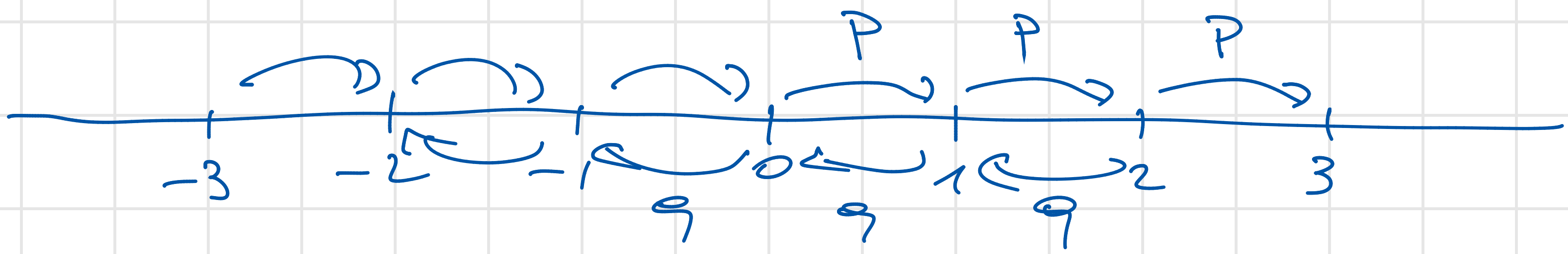
RW on $\mathbb{Z}$ is recurrent or transient?



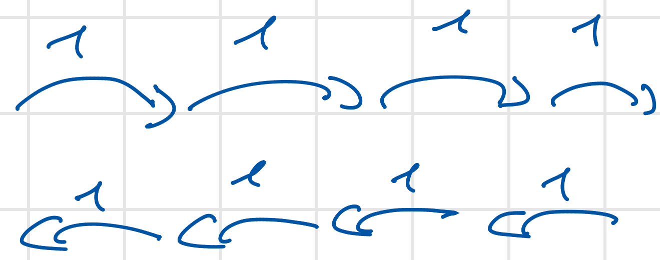

$p+q=1$

$\boxed{0<p<1}$

$0<q<1$

$p=1$

$p=0$




All the states communicate $\Rightarrow$ the MC is irreducible

$S=\mathbb{Z}$ integer numbers

$X_1, X_2, X_3, \dots$ iid r.v.'s s.t. $\boxed{X_i = \begin{cases} 1 & p \\ -1 & q \end{cases}}$

$S_0 = 0, \quad S_1 = X_1, \quad S_2 = X_1 + X_2, \quad \dots, \quad S_n = X_1 + \dots + X_n$

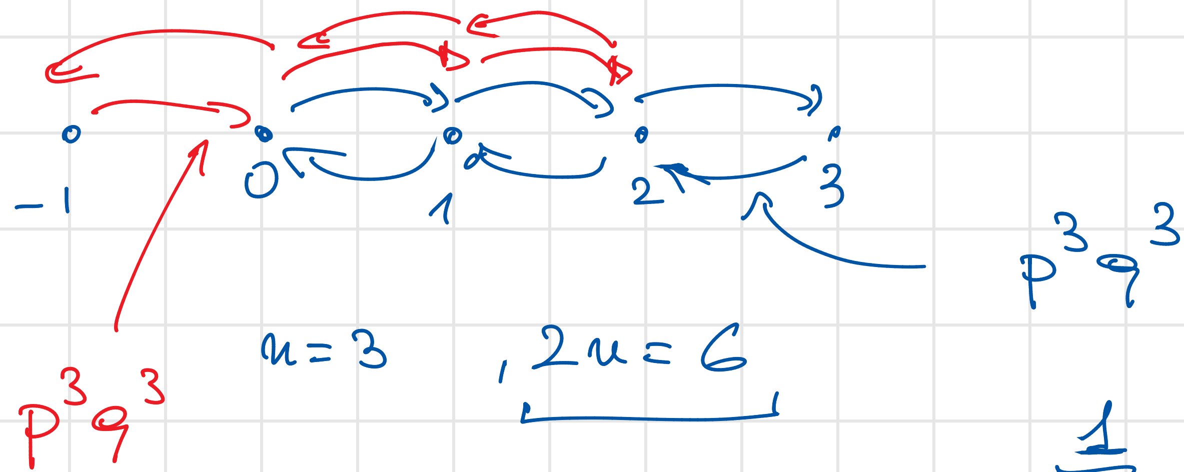
$= S_{n-1} + X_n$

0 is recurrent or transient ? $\sum_{n=1}^{\infty} P_{00}^{(n)} \begin{matrix} = +\infty \\ < +\infty \end{matrix}$

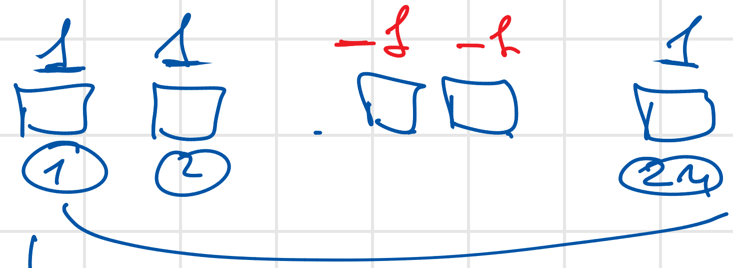
$$P_{00}^{(2n+1)} \equiv 0 \qquad n \in \mathbb{N}$$

$$\boxed{P_{00}^{(2n)} = \binom{2n}{n} p^n q^n}$$

$$P = \begin{vmatrix} \ddots & & & \\ q & 0 & p & \\ & q & 0 & p \\ & & q & 0 \quad p \\ & & & \ddots \end{vmatrix}$$




$n = 3$ , $\underline{2n = 6}$




$$\binom{2n}{n} = \frac{2n!}{n!\,(2n-n)!} = \underline{\frac{2n!}{n!\,n!}}$$

$$\underline{\sum_n \frac{2n!}{n!\,n!} p^n q^n} < \begin{matrix} = +\infty \\ < +\infty \end{matrix} \qquad ??$$

Stirling's formula $\quad n! \sim \underbrace{\boxed{n^n} \, \underline{e^{-n}} \cdot \sqrt{2\pi n}}$

$\sqrt{n}$

$$P_{00}^{(2n)} = \frac{2n!}{n!\,n!} \underbrace{p^n q^n} \approx (pq)^n \frac{2^{2n}}{\sqrt{\pi \cdot n}}$$

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \qquad (2n)! \sim \dots$$

$$n^n \qquad (2n)^{2n} = \underbrace{2^{2n}} \cdot \cancel{n^{2n}}$$

$$\cancel{n^n} \cdot \cancel{n^n}$$

$$\underbrace{P_{00}^{(2n)}} \sim \underbrace{\frac{(4 \cdot p \cdot q)^n}{\sqrt{\pi n}}}$$

$\frac{1^n}{\sqrt{\pi n}}$ $\quad p = q = 1/2$

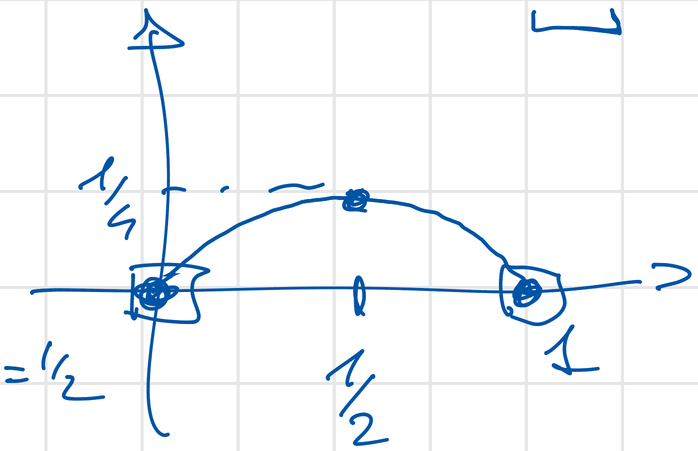
$\frac{(4pq)^n}{\sqrt{\pi n}}$ $\quad p \neq q$

$\underbrace{0} < pq \leq \underbrace{1/4}$

$$pq = \underbrace{p(1-p)}$$

$$0 < p < 1 \qquad p - p^2$$

$$1 - 2p = 0 \qquad p = 1/2$$




$$\boxed{p = q = 1/2} \qquad \sum \frac{1}{\sqrt{\pi n}} = +\infty \iff \sum P_{00}^{(2n)} = +\infty$$

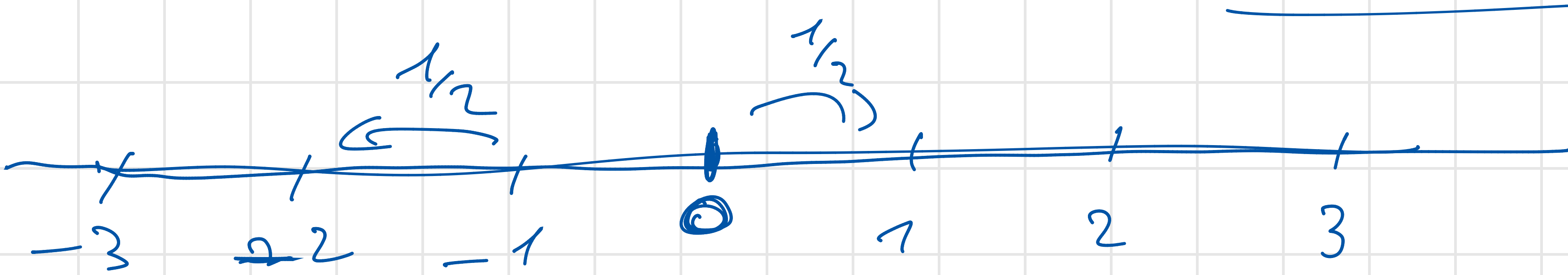
Symmetric RW

$\iff$ 0 <u>is recurrent</u>

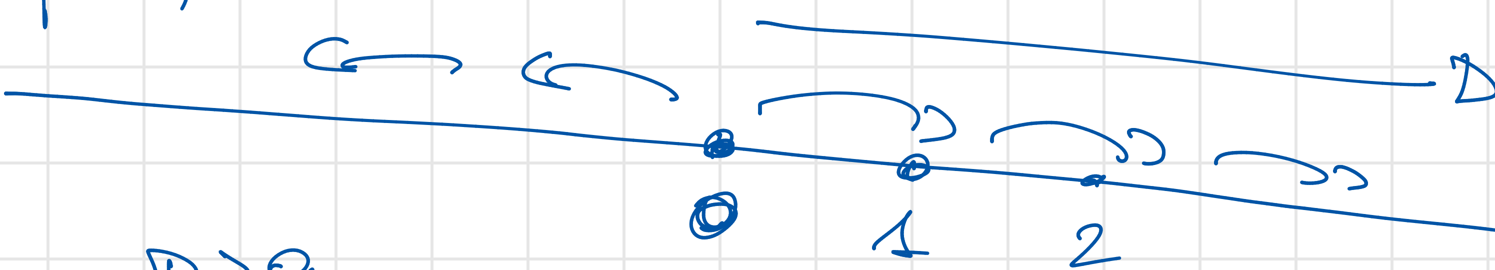
if $p \neq q$ $\quad \sum \frac{(4pq)^n}{\sqrt{\pi n}} < +\infty \iff \sum_{n=1}^{\infty} P_{00}^{(2n)} < +\infty$

$4pq < 1$

$\iff \boxed{0 \text{ is } \underline{\text{transient}}}$




$p \neq q$




$p > q$

$S_0 = 0$, $\quad S_n = X_1 + \dots + X_n$

$$\boxed{\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \xrightarrow[n\to\infty]{} E[X_1]}$$

0

$\omega$ $\quad \boxed{\frac{S_n(\omega)}{n}} \to \underbrace{E[X_1] > 0}$ $\quad X_i$ iid

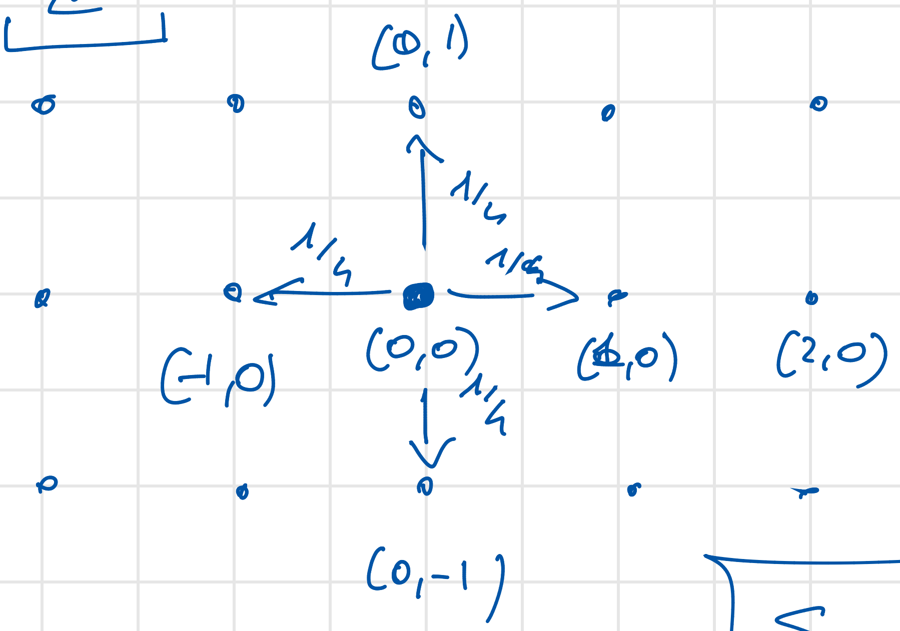
$$X_1 = \begin{cases} 1 & p \\ -1 & q = 1-p \end{cases}$$

$$E[X_1] = 1 \cdot p + (-1)(1-p) = p - 1 + p = \underbrace{2p - 1}$$

$p > 1/2$ $\quad (p > q)$ $\quad E[X_1] = 2p - 1 > 0$ $\qquad p = 1/2$

$p < 1/2$

RW on $\underline{\mathbb{Z}^2}$




$\boxed{\text{Symmetric Case}}$

$S_0 = (0,0)$

$S_n = X_1 + \cdots + X_n$

$\mathbb{Z}$

$\left(\frac{1}{\sqrt{2\pi}\sqrt{n}}\right)$

$P^{(2n)}_{(0,0)(0,0)} \sim \frac{2}{C \cdot n} \Rightarrow$ (0,0) is recurrent

$\mathbb{Z}^2$

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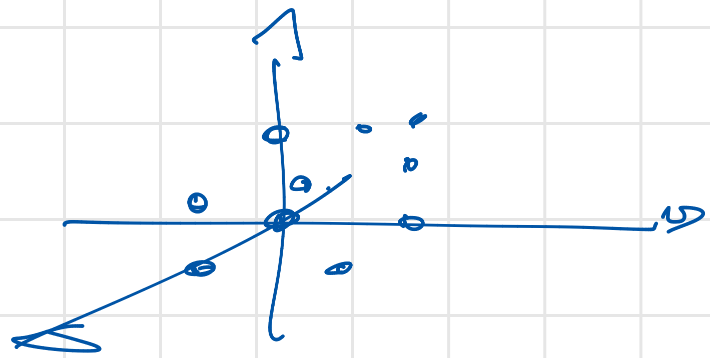
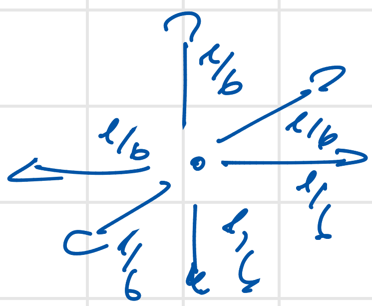
$\mathbb{Z}^3$






$P^{(2n)}_{(0,0,0)(0,0,0)} \sim C \cdot \left(\frac{6}{n}\right)^{3/2} \sim \frac{1}{n^{3/2}}$ (0,0,0) is transient

$X_0 = 0, \quad X_1 = 1, \quad X_2 = 2, \ldots$

$+1 \qquad +1 \qquad +1$

$S = \mathbb{Z}$

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**Exercise:** Consider the RW on $\mathbb{Z}$

$S_0 = 0, \quad S_1 = X_1, \quad \ldots, \quad S_n = X_1 + \cdots + X_n$

Prove that

$$\mathbb{P}[S_n = i] = \begin{cases} 0 & |i| > n \\ 0 & |i| \le n \text{, } n \text{ is even and } i \text{ is odd or viceversa} \\ \binom{n}{\frac{n+i}{2}} p^{\frac{n+i}{2}} (1-p)^{\frac{n-i}{2}} & |i| \le n \text{, } i \text{ and } n \text{ are odd (are even)} \end{cases}$$

(the mass prob. function of $S_n$)

**Exercise 2:** $a \in \mathbb{Z}$ $\quad T_a := \inf\{n \geq 1 : X_n = a\}$

$\boxed{\mathbb{P}[T_a = n]}$

(*Hint:* divide the case $a = 0$ and $a \neq 0$)

# The Basic Limit Theorem of MC

$$f_{ii}^{(n)} = \mathbb{P}\left[X_n = i, X_\nu \neq i, \nu = 1, \dots, n-1 \mid X_0 = i\right]$$

probability of first return at time n.

$$R_i := \min\{n \geq 1 : X_n = i\} \in \mathbb{N}$$

$$f_{ii}^{(n)} = \mathbb{P}\left[R_i = n \mid X_0 = i\right]$$

$$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} \begin{cases} = 1 & i \text{ is recurrent} \\ < 1 & i \text{ is transient} \end{cases}$$

If $f_{ii} = 1 \Rightarrow R_i$ is a R.V. $\sum_{n=1}^{\infty} \mathbb{P}[R_i = n \mid X_0 = i] = 1$

$$m_i = \mathbb{E}[R_i \mid X_0 = i]$$

conditional expectation

# Theorem 4.1 The basic limit of MC

(a) Consider a recurrent irreducible aperiodic MC.

Let $P_{ii}^{(n)}$ be the prob. of entering state i at the n-th transition ($n \in \mathbb{N}$) given that $X_0 = i$. ($P_{ii}^{(0)} = 1$)
Let $f_{ii}^{(n)}$ be the prob. of first returning to state i at the n-th transition ($n \in \mathbb{N}$), where $f_{ii}^{(0)} = 0$.
Then

$$\boxed{\lim_{n \to +\infty} P_{ii}^{(n)} = \boxed{\frac{1}{m_i}}} \qquad \left( m_i = \sum_{n=0}^{\infty} n \cdot f_{ii}^{(n)} \right)$$

(b) Under the same conditions as in (a),

$$\lim_{n \to \infty} P_{ji}^{(n)} = \lim_{n \to \infty} P_{ii}^{(n)} \quad \text{for all states } j$$

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Two cases. $\boxed{m_i < +\infty}$ or $m_i = +\infty$

$$m_i = \mathbb{E}[R_i \mid X_0 = i]$$

$$\lim_{n \to \infty} P_{ii}^{(n)} > 0 \qquad \boxed{\pi_i > 0} \implies \boxed{\pi_j > 0 \quad \forall j \in S}$$

If $i$ is recurrent and $\boxed{m_i < +\infty}$, we call

$i$ positive recurrent

$P_{ii}^{(n)} \longrightarrow \pi_i > 0$ and $\pi_i = \frac{1}{m_i} > 0$

$\boxed{|S| < +\infty \text{, Regular}}$

$\Uparrow \not\Downarrow$

$|S| = +\infty$ $\quad$ $\boxed{\text{irreducible, aperiodic, positive recurrent}}$

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If $i$ is recurrent and $\boxed{m_i = +\infty}$, we call

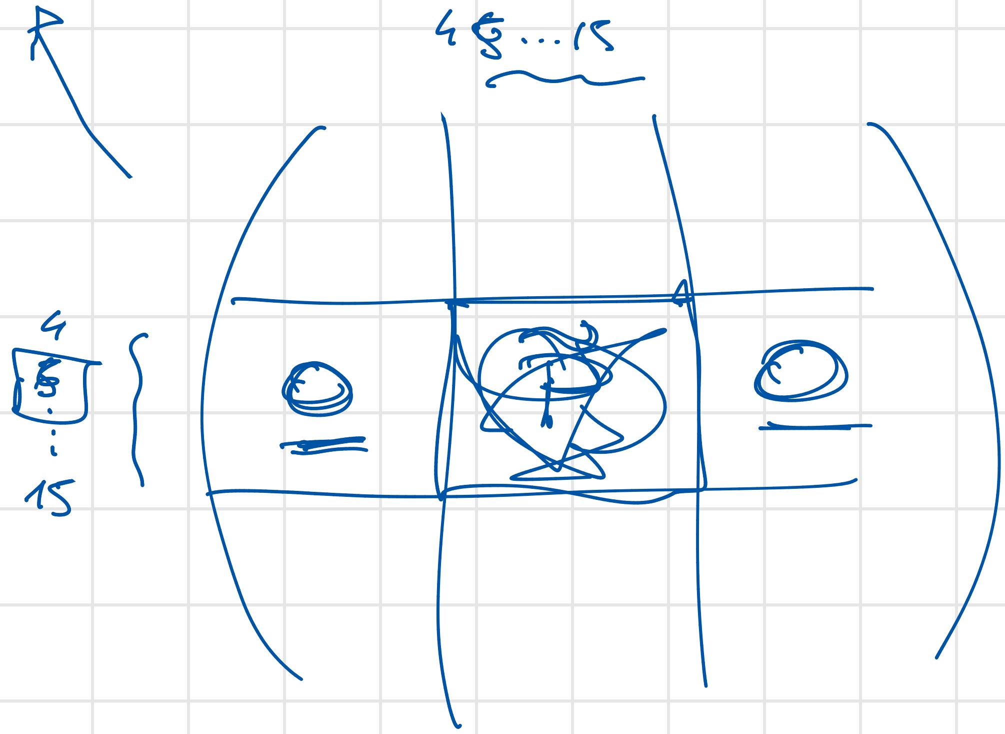
$i$ <u>null recurrent</u> state

$$\lim_{n \to \infty} P_{ii}^{(n)} = 0$$

<u>Question</u>: for the RW on $\mathbb{Z}$, $0$ is positive recurrent or null recurrent?

## Theorem 4.2

In a positive recurrent, aperiodic class with states $j = 0, 1, \dots$




$$\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j \quad \left(= \frac{1}{m_j}\right)$$

$$\boxed{\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}}, \quad \sum_{i=0}^{\infty} \pi_i = 1$$

and $\pi$'s are uniquely determined by the set of equations

$$\pi_i \geq 0, \quad \boxed{\sum_{i=0}^{\infty} \pi_i = 1} \quad \text{and} \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \text{for } j = 0, 1, \dots$$

$(\pi_i)_{i=0,\dots,\infty}$ stationary (invariant) prob. distribution of MC

RW on $\mathbb{Z}$ $\quad$ $\underline{0}$ is <u>null recurrent</u>

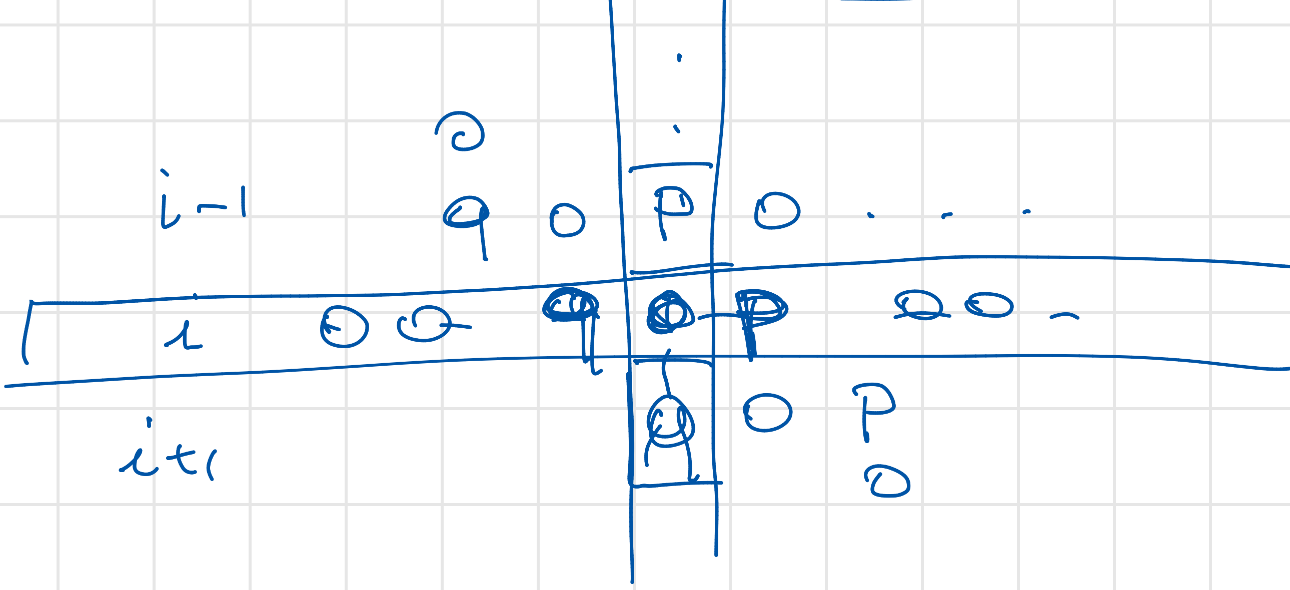
$$\boxed{\mu_{\underline{0}} = +\infty}$$

$i \to i+1 \quad p$

$i \to i-1 \quad q$

$i-1 \to i \quad p$

$i-1 \to i-2 \quad q$



$i+1$

$$\to \quad \boxed{\pi_j = \sum_{i=-\infty}^{\infty} \pi_i P_{ij}}$$

$$\boxed{\pi_j = p\,\pi_{j-1} + q\,\pi_{j+1}}$$

$p = q = 1/2 \qquad \pi_j = \frac{1}{2}\pi_{j-1} + \frac{1}{2}\pi_{j+1} \qquad \forall j \in \mathbb{Z}$

$\boxed{\pi_j \equiv 1} \qquad 1 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1$ YES !

$\boxed{\pi_j \equiv 3} \qquad \sum_{i\in\mathbb{Z}} \pi_i = +\infty$

$\boxed{\pi_j \equiv 0} \qquad \boxed{\sum_{i\in\mathbb{Z}} \pi_i \equiv 0} \qquad$ 0 cannot be <u>positive recurrent</u>