Stochastic Processes

Lecture 11

CYS 2020/2021

- The Poisson Process

<u>Exercise</u>: $(X_n)_{n\geq 0}$ be a MC with π as its invariant distribution $\left(\pi_i = \sum_{j\in S} \pi_j P_{ji}, \ \forall i \in S\right)$.

Consider the process $Y_n := (X_n, X_{n-1})$, $(Y_n, \underline{n\geq 1})$

1. Prove that Y_n is a MC and determine its transition prob.

2. Prove that $\pi_{(i,j)} = \pi_j P_{ji}$ is an invariant distribution for (Y_n).

<u>Sol.</u> $Y_n \in S\times S$

$$\mathbb{P}\left[Y_{n+1} = (i', j') \mid Y_n = (i,j), \underline{\dots, Y_1 = (i_1, j_1)}\right]$$

$$\mathbb{P}[Y_{n+1} = (i', j') \mid Y_n = (i, j), \dots, Y_1 = (i_1, j_1)]$$

$$= \mathbb{P}[X_{n+1} = i', \underbrace{X_n = j' \mid X_n = i}, \cancel{X_{n-1} = j}, \cancel{X_{n-1} = i_{n-1}}, \dots$$

$$\dots \cancel{X_1 = i_1}, \cancel{X_0 = j_1}]$$

This expression makes sense only if (and is not zero)

$$j' = i, \quad j = i_{n-1}, \quad j_{n-1} = i_{n-2}, \dots, j_2 = i_1$$

By the Markov property of $(X_n, n \geq 0)$ we have

$$= \mathbb{P}[X_{n+1} = i', \underbrace{X_n = j'} \mid \underbrace{X_n = i}]$$

$$= \mathbb{P}[X_{n+1} = i' \mid X_n = i] = P_{ii'} \quad (\underline{j' = i})$$

$$\mathbb{P}[A \cap B \mid A] = \frac{\mathbb{P}[A \cap B \cap A]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \mathbb{P}[B \mid A]$$

Y_n is a MC and

$$\mathbb{P}[Y_{n+1} = (i', j') \mid Y_n = (i, j)]$$

$$= \begin{cases} P_{ii'} & \text{if } j' = i \\ 0 & \text{otherwise} \end{cases}$$

Remark: $P \in M(n \times n)$ $\quad |S| = n$

$\tilde{P}$ the transition matrix for Y_n

$$\tilde{P} \in M(n^2 \times n^2)$$

(2) $\boxed{\pi_{(i,J)} = \pi_J \, P_{Ji}}$ is invariant for $\tilde{P}$

$$\underbrace{\tilde{P}_{(i,J),(i',J')}} = \begin{cases} P_{ii'} & \text{if } J' = i \\ 0 & \text{otherwise} \end{cases}$$

We want to prove

(*) $$\pi_{(i,J)} = \sum_{(k,e)} \underbrace{\pi_{(k,e)}} \, \underbrace{P_{(k,e),(i,J)}}$$

Indeed,

$$\sum_{(k,\ell) \in S \times S} \pi_{(k,\ell)} \cdot P_{(k,\ell),(i,J)} = \sum_{\ell \in S} \underbrace{\pi_{(J,\ell)}} \cdot P_{J \cdot i}$$

$$= \underbrace{\sum_{\ell \in S} \underbrace{\pi_\ell \cdot P_{\ell J}}} \cdot P_{Ji} = \pi_J \cdot P_{Ji} = \pi_{(i,J)}$$

To prove that $\pi_{(i,J)}$ is a distribution, it remains to check

$$\sum_{(i,j)\in S\times S} \pi_{(i,j)} = \sum_{(i,j)\in S\times S} \pi_j P_{ji} =$$

$$= \sum_{j\in S} \pi_j \left(\sum_{i\in S} P_{ji} \right) =$$

$$= \sum_{j\in S} \pi_j \cdot 1 = 1$$

Thus $\pi_{(i,j)} = \pi_j P_{ji}$ is an invariant distribution for the MC $(Y_n, n\geq 1)$.

The Poisson Processes

One dimension

Paths $X(t)$, $t \in [0, +\infty)$

continuous time stochastic process

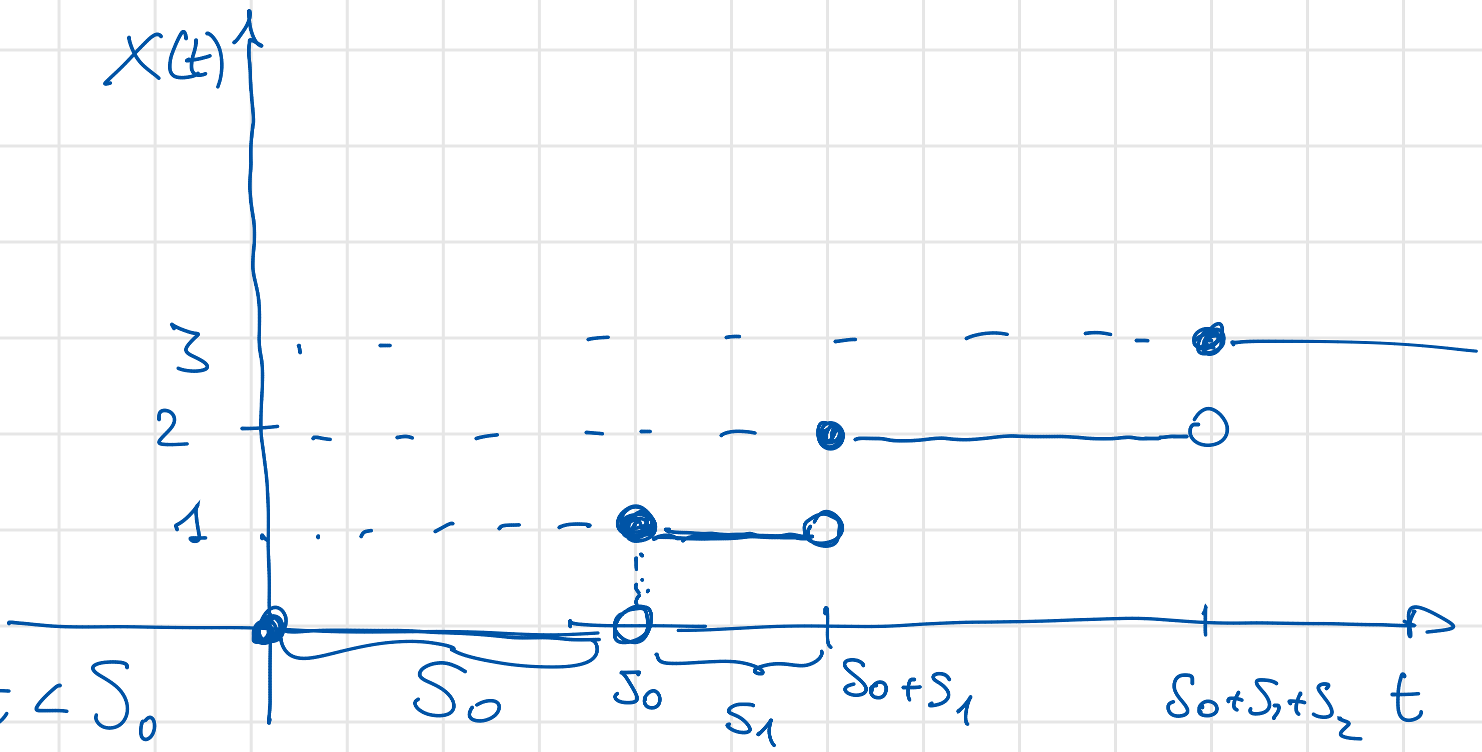
$\lambda \in (0, +\infty)$ rate of the process.

$X(\omega, t)$

random time

$X(0) = 0$

$\underline{S_0 \sim Exp(\lambda)}$



$X(t) = 0 \quad t < S_0$

$X(S_0) = 1$

$S_1 \sim Exp(\lambda)$, $S_0 \perp\!\!\!\perp S_1$

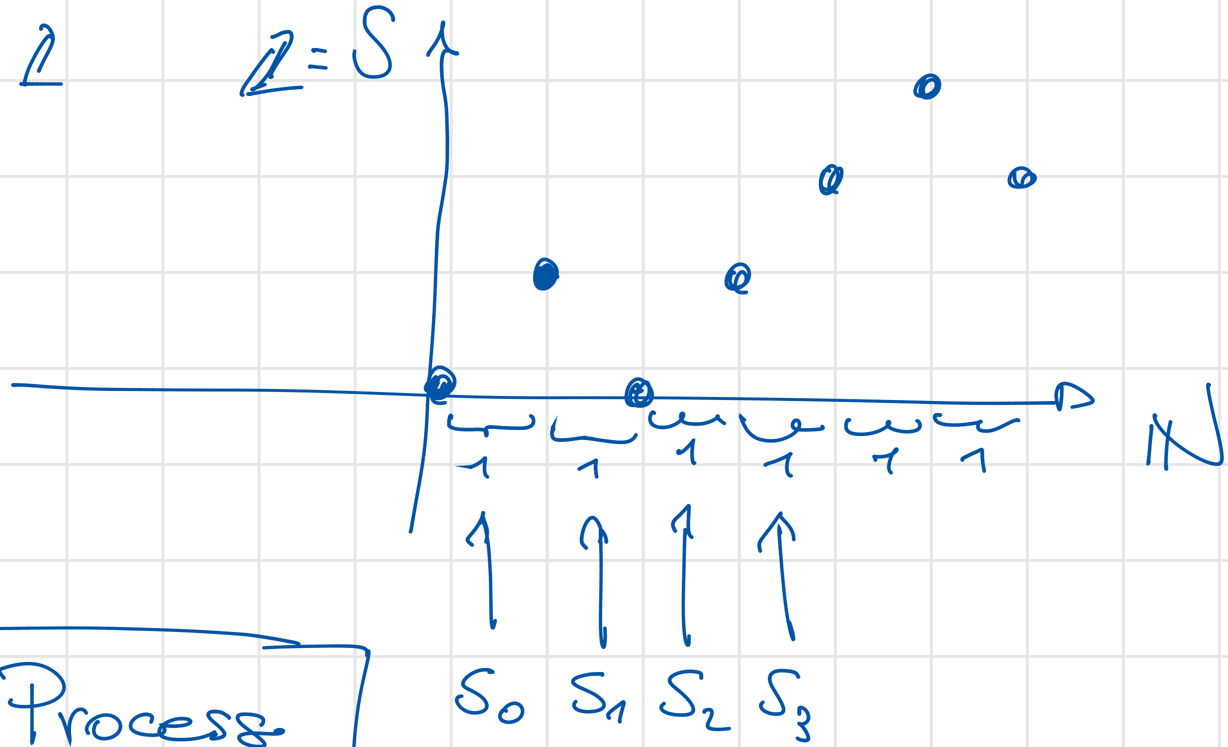
independent

indep. S_1, S_2, S_3

$S_2 \sim Exp(\lambda)$

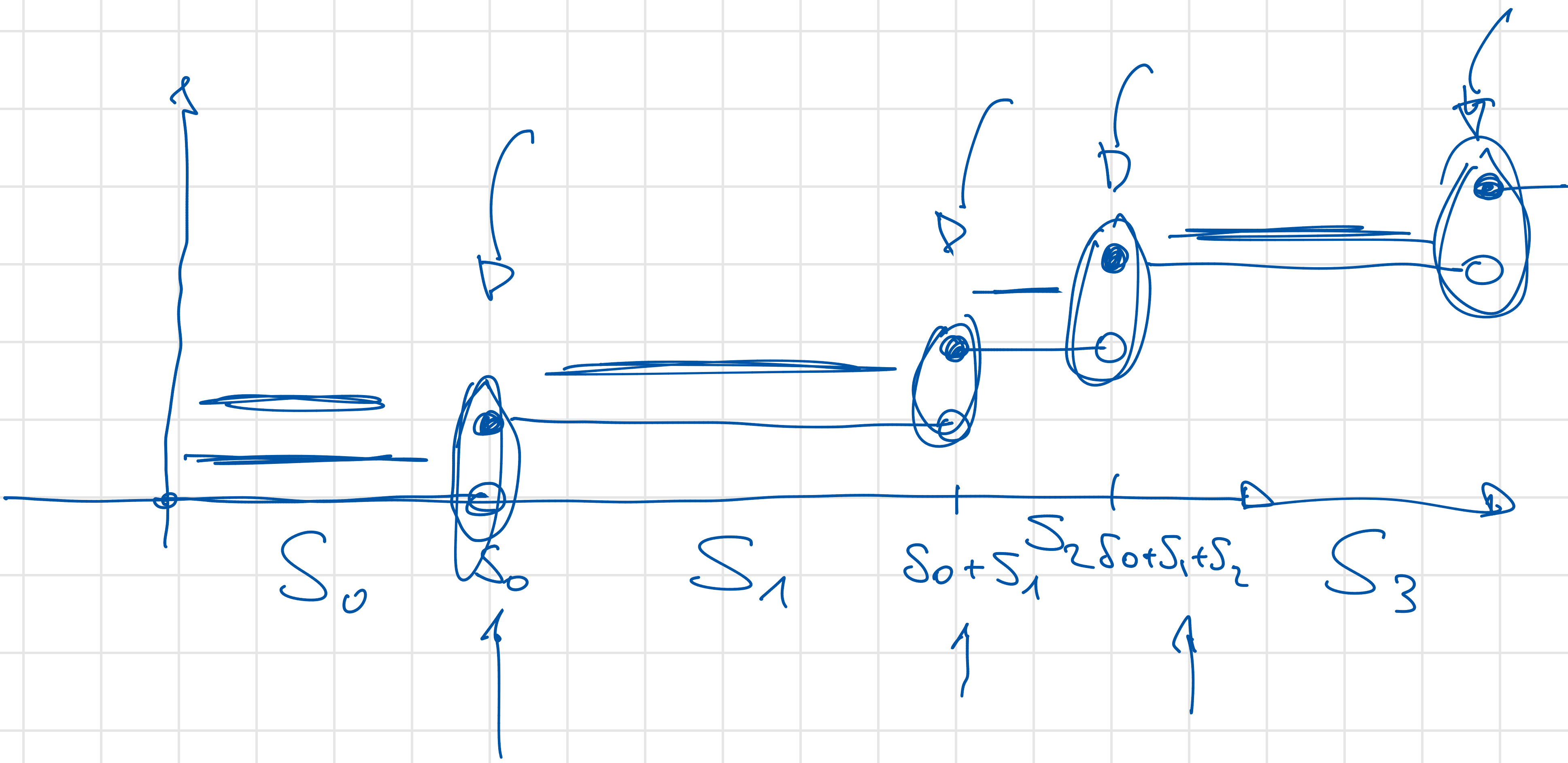
Ex. RW in $\mathbb{Z}$

$\mathbb{Z} = S$



Poisson Process

Fix λ $\qquad S_i \sim \mathrm{Exp}(\lambda)$ r.v.



$$\boxed{i \xrightarrow{1} i+1 \xrightarrow{1} i+2 \xrightarrow{1} i+3}$$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 0 & 0 & 1 \\ 1 & & 1 \quad 1 \quad \ddots \quad 1 \\ & & \qquad\qquad 0 \end{array}$$

Renewal

$\boxed{1}$ definition of the Poisson Process

1.1 Poisson Distribution

$X \sim \text{Poisson}(\mu) \qquad \mu > 0$

$X \in \mathbb{N}$

$$\mathbb{P}[X=k] = p_k = e^{-\mu}\frac{\mu^k}{k!}, \quad k \in \mathbb{N}$$

Exponential Serie : $\boxed{\displaystyle\sum_{k=0}^{+\infty} \frac{\mu^k}{k!} = e^{\mu}}$

$$\Rightarrow \sum_{k=0}^{+\infty} p_k = e^{-\mu} \sum_{k=0}^{+\infty} \frac{\mu^k}{k!} = e^{-\mu} \cdot e^{\mu} = 1$$

$$\mathbb{E}[X] = \sum_{k=0}^{+\infty} k \cdot p_k = \sum_{k=0}^{+\infty} k \cdot e^{-\mu} \frac{\mu^k}{k!}$$

$$= \sum_{k=1}^{+\infty} \cancel{k}\, e^{-\mu} \frac{\mu^k}{\cancel{k!}\,(k-1)!} = \sum_{k=1}^{+\infty} e^{-\mu} \frac{\mu^k}{(k-1)!}$$

$$= \sum_{k=1}^{+\infty} e^{-\mu} \frac{\mu^k}{(k-1)!} = e^{-\mu}\mu \underbrace{\sum_{k=1}^{+\infty}\frac{\mu^{k-1}}{(k-1)!}}_{s=k-1}$$

$$= e^{-\mu}\mu \cdot \underbrace{\sum_{s=0}^{+\infty}\frac{\mu^s}{s!}}_{e^{\mu}} = e^{-\mu}\cdot\mu e^{\mu} = \mu$$

$$X \sim \text{Poisson}(\mu) \Rightarrow \boxed{\mathbb{E}[X]=\mu}$$

$$\text{Var}[X] = \mathbb{E}[(X-\mathbb{E}[X])^2] = \underbrace{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

$$\mathbb{E}[X(X-1)] = \mathbb{E}[g(X)] = \sum_{k=0}^{+\infty} g(k)\cdot p_k$$

$$= \sum_{k=0}^{+\infty} k(k-1)\cdot \frac{e^{-\mu}\mu^k}{k!} = \sum_{k=2}^{+\infty} \cancel{k(k-1)}\, \frac{e^{-\mu}\mu^k}{\cancel{k!}}$$

$$= \sum_{k=2}^{+\infty}\frac{e^{-\mu}\mu^k}{(k-2)!} = \mu^2 e^{-\mu}\underbrace{\sum_{k=2}^{+\infty}\frac{\mu^{(k-2)}}{(k-2)!}}_{=e^{\mu}}$$

$$= \mu^2$$

$$\mathbb{E}[X(X-1)] = \mu^2$$

$$\parallel$$

$$\mathbb{E}[X^2 - X] = \mathbb{E}[X^2] - \mathbb{E}[X]$$

$$\Rightarrow \quad \mathbb{E}[X^2] = \mu^2 + \mathbb{E}[X] = \mu^2 + \mu$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mu^2 + \mu - (\mu)^2 = \mu$$

$X \sim \mathrm{Poisson}(\lambda)$ $\qquad$ $\mathbb{E}[X] = \mathrm{Var}[X] = \mu$.

Thm 1: Let X and Y be independent r.v. having Poisson distribution with parameters λ and μ. Then the sum $X+Y$ has a Poisson distribution with parameter $\lambda + \mu$.

Remark: $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, $X \perp\!\!\!\perp Y$

$$\Rightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \quad (\text{Normal})$$

Proof: $X+Y \in \mathbb{N}$ discrete r.v.

$\forall n \in \mathbb{N}$

$$p_n := \mathbb{P}[\underbrace{X+Y=n}_{A}] = \sum_{k=0}^{\infty} \mathbb{P}[\underbrace{X+Y=n}_{A}, \underbrace{X=k}_{B_k}]$$

$$= \sum_{k=0}^{+\infty} \mathbb{P}[Y=n-k, X=k]$$

$n-k \geq 0 \Rightarrow k \leq n$

$$= \sum_{k=0}^{n} \mathbb{P}[Y=n-k, X=k]$$

$$\overset{\text{indep}}{=} \sum_{k=0}^{n} \mathbb{P}[Y=n-k] \cdot \mathbb{P}[X=k]$$

$$\overset{\text{Poisson}}{=} \sum_{k=0}^{n} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{\mu^{n-k} e^{-\mu}}{(n-k)!}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \cdot \sum_{k=0}^{n} \frac{n! \cdot \lambda^k \mu^{n-k}}{k!\,(n-k)!}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \cdot \underbrace{\sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k}}_{= (\lambda+\mu)^n} = \frac{e^{-(\lambda+\mu)} (\lambda+\mu)^n}{n!}.$$

$\blacksquare$

$N \sim \text{Poisson}(\mu)$

I cancel any of these 1's with probability $1-p$, independently.

$$N = \underbrace{\cancel{1} + \dots + 1 + \dots + \cancel{1}}_{N\text{-times}}$$

$$M = \underbrace{\overset{X_1 \sim \text{Bin}(1,p)}{1} + \overset{X_2 \sim \text{Bin}(1,p)}{0} + \overset{X_3}{0} \dots + 1 + 1 + \dots + 0 + \overset{X_N \sim \text{Bin}(1,p)}{1}}_{N}$$

What is the distribution of M?

<u>Thm 2</u>: Let N be a Poisson r.v. with parameter μ and, conditioned on N, let M be a Binomial distribution with parameters $\underline{N}$ and $\underline{p}$. Then the unconditional distribution of M is Poisson with parameter $\underline{\mu \cdot p}$

<u>Proof</u>:

$$\boxed{\begin{array}{l} M \mid N=n \sim \text{Bin}(n,p) \quad \text{r.v.} \\ N \sim \text{Poisson}(\mu) \quad \text{r.v.} \end{array}}$$

$$\mathbb{P}[\underline{M=k}] = \sum_{n=0}^{\infty} \underline{\mathbb{P}[M=k \mid N=n]} \cdot \underline{\mathbb{P}[N=n]}$$

$$\mathbb{P}[M=k] = \sum_{\underbrace{n=0}} ^{\infty} \underbrace{\mathbb{P}[M=k \mid N=n]} \cdot \underbrace{\mathbb{P}[N=n]}_{>0}$$

$n \in \mathbb{N}$

$$M \mid N=n \sim \mathrm{Bin}(n, p) \in \{0, 1, \dots, n\}$$

is $\neq 0 \iff \boxed{k \leq n}$

$$= \sum_{\boxed{n=k}}^{+\infty} \underbrace{\binom{n}{k} p^k (1-p)^{n-k}} \cdot \frac{\mu^n e^{-\mu}}{n!}$$

$$= e^{-\mu} p^k \sum_{n=k}^{+\infty} \frac{\cancel{n!}}{k!(n-k)!} \cdot \frac{(1-p)^{\boxed{n-k}} \boxed{\mu^n}}{\cancel{n!}}$$

$$= \frac{e^{-\mu} p^k}{k!} \cdot \mu^k \underbrace{\sum_{n=k}^{+\infty} \frac{(1-p)^{n-k} \mu^{n-k}}{(n-k)!}}$$

$$\sum_{n=k}^{+\infty} \frac{((1-p)\mu)^{n-k}}{(n-k)!} = e^{(1-p)\cdot\mu}$$

$$= \frac{e^{-\mu} (p\cdot\mu)^k}{k!} \cdot e^{(1-p)\mu} = \frac{e^{-p\mu} \cdot (p\mu)^k}{k!}$$

$\Rightarrow \quad M \sim \text{Poisson}(p \cdot \mu)$

$\square$

Remark: $N \sim \text{Poisson}(\mu) \rightsquigarrow \boxed{M \sim \text{Poisson}(p\cdot\mu)}$

$p \in [0,1]$

The Poisson Process:

Definition: A Poisson process of intensity λ, $\lambda > 0$, is an integer-valued stochastic process $\{X(t) : t \geq 0\}$ for which:

(i) for any time points $t_0 = 0 < t_1 < t_2 \cdots < t_n$, the increments

$$\underbrace{X(t_1) - X(t_0)}, \underbrace{X(t_2) - X(t_1)}, \ldots, \underbrace{X(t_n) - X(t_{n-1})}$$

are independent random variables

(ii) for $s \geq 0$ and $t > 0$, the random variable

$X(s+t) - X(t)$ has a Poisson distribution

$$\mathbb{P}\left[X(s+t) - X(t) = k\right] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k \in \mathbb{N}$$

(Poisson (λt) r.v.)

(iii) $X(0) = 0$

$$\boxed{\begin{aligned} \mathbb{E}[X(t)] &= \lambda t \\ \mathrm{Var}[X(t)] &= \lambda t \end{aligned}}$$