Proof of basic limit theo. for M.C. (Ross 173)

for a rec, irred, aper, M.C. we have
$$\lim_{n\to\infty} P_{jj}^{(n)} = \lim_{n\to\infty} P_{ij}^{(n)} = \pi_j = \frac{1}{\sum_{n=1}^{+\infty} n f_{jj}^{(n)}} = \frac{1}{m_j}.$$

Proof: let $N_j(t)$ be the number of visits to j by time t.
Since j is recurrent, if ~~$X_0 = j$~~ the chain starts in state j, $N_j(t)$ is a renewal process. In this case, the distribution of the inter-event times is arithmetic with span ~~d~~ $=1$.
From renewal theorem we have:
$$\lim_{t\to\infty} [M(t+h) - M(t)] = \frac{h}{\mu} \quad \text{if } h \text{ is multiple of the span.}$$

In this case $\mu = m_j$
$$P_{jj}^{(n)} = M(n) - M(n-1) \to \frac{1}{m_j}$$
$N_j(n) = N_j(n-1) + \underbrace{\text{visit at } n}_{I[X_n = j]}$

If the chain is periodic, we have
$$P_{jj}^{(nd)} = M(nd) - M(nd-d) \to \frac{d}{m_j}.$$
$N_j(nd) - N_j(nd-d) = \sum_{i=nd-d+1}^{nd} I[X_i = j]$ all zero except for $i = nd$

If the initial state is $i \neq j$, $N_j(t)$ is a delayed renewal process, and the same results apply regardless of i.

4.3 Limit Theorems

It is easy to show that if state j is transient, then

$$\sum_{n=1}^{\infty} P_{ij}^{n} < \infty \qquad \text{for all } i,$$

meaning that, starting in i, the expected number of transitions into state j is finite. As a consequence it follows that for j transient $P_{ij}^{n} \to 0$ as $n \to \infty$.

Let μ_{jj} denote the expected number of transitions needed to return to state j. That is,

$$\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^{n} & \text{if } j \text{ is recurrent.} \end{cases}$$

By interpreting transitions into state j as being renewals, we obtain the following theorem from Propositions 3.3.1, 3.3.4, and 3.4.1 of Chapter 3.

THEOREM 4.3.1

If i and j communicate, then:

(i) $P\left\{\lim_{t\to\infty} N_j(t)/t = 1/\mu_{jj} \mid X_0 = i\right\} = 1.$

(ii) $\lim_{n\to\infty} \sum_{k=1}^{n} P_{ij}^{k}/n = 1/\mu_{jj}.$

(iii) *If j is aperiodic, then* $\lim_{n\to\infty} P_{ij}^{n} = 1/\mu_{jj}.$

(iv) *If j has period d, then* $\lim_{n\to\infty} P_{jj}^{nd} = d/\mu_{jj}.$

If state j is recurrent, then we say that it is *positive* recurrent if $\mu_{jj} < \infty$ and *null* recurrent if $\mu_{jj} = \infty$. If we let

$$\pi_j = \lim_{n\to\infty} P_{jj}^{nd(j)},$$

it follows that a recurrent state j is positive recurrent if $\pi_j > 0$ and null recurrent if $\pi_j = 0$. The proof of the following proposition is left as an exercise.