Stochastic Processes

Lecture 8

$\boxed{22/7/2021}$

CTS 2020/2021

- Classification of the states (CLASSIFICAZIONE)
  - CLOSED closed classes of communication states (COMUNICAZIONE)
  - PERIODICITA' periodicity and aperiodicity
    - class property
    - examples
  - recurrent states RICORRENTI
  - transient states
    - RW on $\mathbb{Z}$ and $\mathbb{Z}^2$, $\mathbb{Z}^3$
- The Basic Limit theorem of the MC

- **Communication** $i \longleftrightarrow j$ $\exists n, m \in \mathbb{N}$

COMUNICAZIONE

$$P_{ij}^{(n)} > 0 \,, \quad P_{ji}^{(m)} > 0$$

$$i \to j \qquad j \to i$$

Communication is an equivalent relation

$$\begin{cases} i \leftrightarrow i & \text{comunicazione con se} \\ i \leftrightarrow j \Rightarrow j \leftrightarrow i & \text{comunicazione tra 2} \\ \underline{i \leftrightarrow j,\ j \leftrightarrow k} \Rightarrow \underline{i \leftrightarrow k} & \end{cases}$$

$i \xrightarrow{\text{comunica}} j$

$j \longrightarrow k$

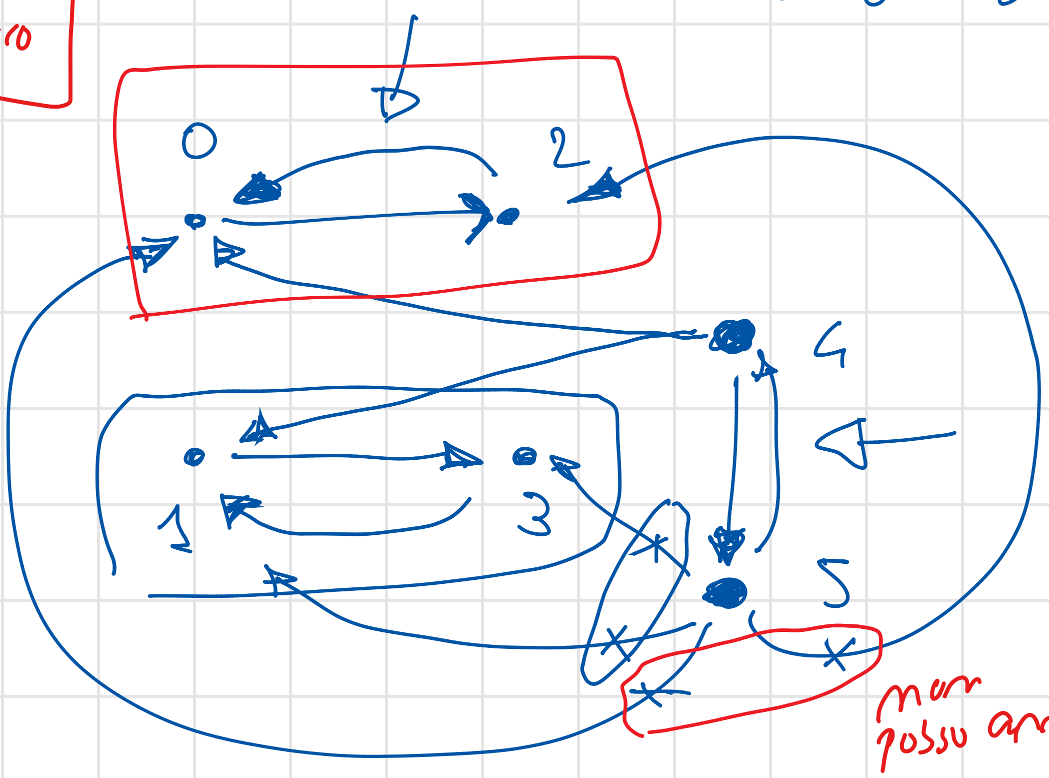
$\underline{\Rightarrow i \longrightarrow j}$

Example

$\boxed{\text{6 stati}}$

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \end{array}$$

$\boxed{\text{GRAFICO}}$



stati

$S = \{0, 1, 2, 3, 4, 5\}$

$\rightarrow C_0 = \{0, 2\}$ <u>closed</u>

$\rightarrow C_1 = \{1, 3\}$ closed

non posso andare

$\rightarrow C_2 = \{4, 5\}$ <u>not closed</u>

"transient states"

I said that a class is <u>closed</u> if $i \in C$ and

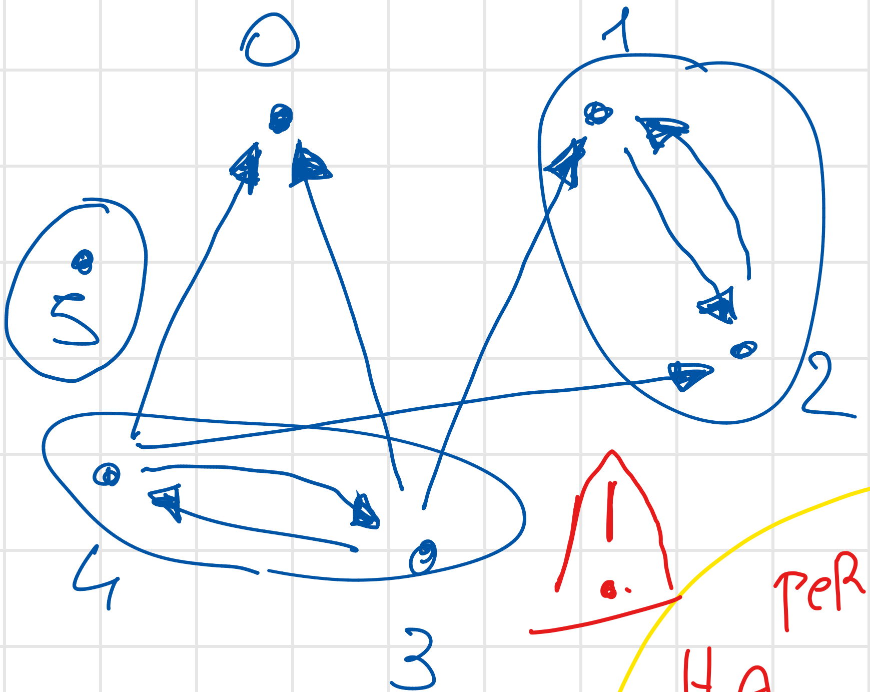
$$\exists n : P_{ij}^{(n)} > 0 \Rightarrow j \in C \quad (i \in C,\ i \to j \Rightarrow j \in C)$$

Ex. 2

$$P = \begin{array}{c|c} & \begin{matrix} 0 & \;1\; & \;2\; & \;3\; & \;4\; & \;5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 \\ 1/3 & 0 & 1/6 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ui

3



GRAPh. Ci

$C_0 = \{5\}$ closed

$C_1 = \{0\}$ closed

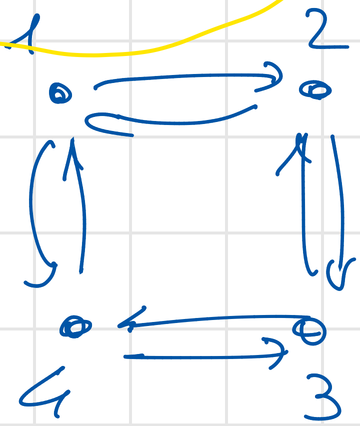
$C_2 = \{1, 2\}$ closed

$C_3 = \{3, 4\}$ not closed

perché HA FRecce IN USCITA

PERIODICITA'

- Periodicity



$$P^2 = \begin{bmatrix} \boxed{*} & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{bmatrix}$$

$$P^{2n} = P^2$$

$$P^{2n-1} = P$$

$$P = \begin{array}{c|c} & \begin{matrix} \quad & 2 & \quad & 3 \end{matrix} \\ 1 & \begin{bmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \end{bmatrix} \end{array}$$

Definition: we define the <u>period</u> of state i, written $d(i)$, to be the greatest common divisor (g.c.d) of all the integers $n \geq 1$ for which $P_{ii}^{(n)} > 0$.

gcd

integers

$$\{ n \geq 1 \,;\, P_{ii}^{(n)} > 0 \}$$

Example: 1 $\quad P_{11}^{(2)} > 0,\ P_{11}^{(6)} > 0,\ P_{11}^{(6)} > 0, \ldots$

$$\boxed{d(1) = 2}$$

1 has period equal to 2

Remark: $P_{ii}^{(n)}$ is not $\neq$ $\left(P_{ii}\right)^n$

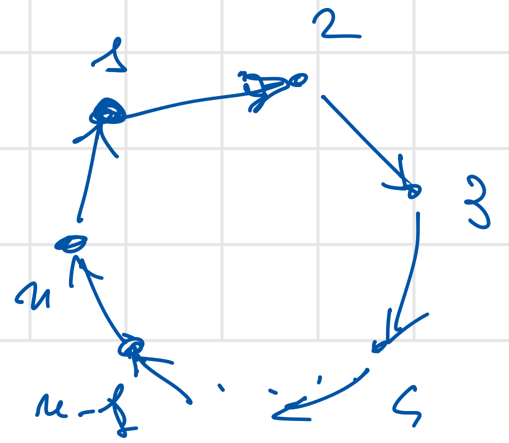
↑ is the i,i entry of the matrix P^n

$$\lim_{n \to \infty} P_{ii}^{(n)} \ \nexists$$

$P_{11}^{(2N)} > 0,\ P_{11}^{(N)} > 0$

Ex:

$$P = \overbrace{\begin{bmatrix} 0 & \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 \end{bmatrix}}^{N}$$



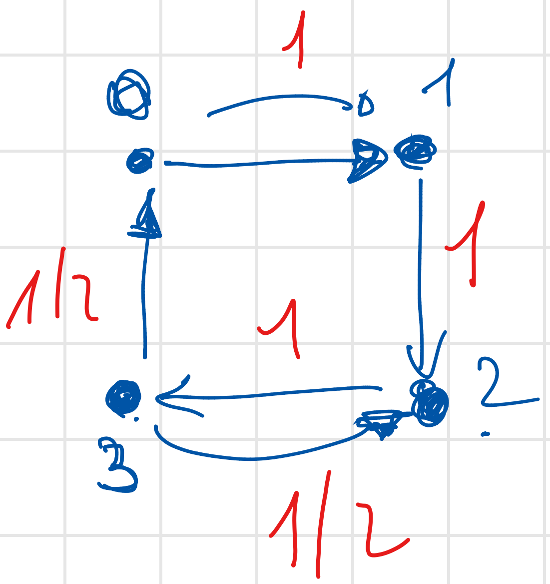
$$\boxed{d(1) = N}$$

What is the period 1?

Ex:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \boxed{0} & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1/2 & 0 & 1/2 & 0 \end{array}$$

5 passi bisogna "consumarli"



state 0:

Impossibile rimanere a 0 con 5 passi

$P_{00} = 0$

$\boxed{P_{00}^{(2)} = 0}$

Rimanere in 0 dopo 2 step

$P_{00}^{(3)} = 0$

$\boxed{P_{00}^{(4)} = 1/2}$ Yes

$P_{00}^{(5)} = 0$

$$\boxed{P_{00}^{(6)} = \underset{P_{32}}{\tfrac{1}{2}} \cdot \underset{P_{23}}{1} \cdot \underset{P_{30}}{\tfrac{1}{2}}}$$

Soluzioni

$$\{n : P_{00}^{(n)} > 0\} = \{\underbrace{4, 6, 8, 10, \dots}\} \quad \gcd \quad \boxed{d(0) = 2}$$

~~Gambler's ruin~~ . $S = \{0, 1, \dots, N-1, N\}$

Dimostrato tua mamma.

$$\cancel{\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ q & 0 & p & & \vdots \\ & q & 0 & p & \\ & & & \ddots & \\ & & q & 0 & p \\ 0 & & \cdots & 0 & 1 \end{bmatrix}}$$

$\cancel{C_0 = \{0\}}$

$\cancel{C_1 = \{N\}}$

$\cancel{C_2 = \{1, 2, \dots, N-1\}}$

~~Prove that $d(i)$, for $i \in C_2$, is 2.~~

Three properties 3 PROPRIETÀ

Remark: If $P_{ii}^{(n)} = 0 \quad \forall n \geq 1$ $\quad (P_{ii}^{(0)} = 1)$

we define $d(i) = 0$.

1. If $i \leftrightarrow j$ then $d(i) = d(j)$ [periodo di i = p. di j]

Periodicity is a class property!

2. If state i has period $d(i)$, then there exists an integer N (depending on i) s.t. $\forall$ integers $n \geq N$

$$P_{ii}^{(n\, d(i))} > 0$$

3. If $P_{ji}^{(m)} > 0$, then $P_{ji}^{(m + n\, d(i))} > 0$

$\forall$ n positive sufficiently large integers.

Let us prove property [1] fine.

dimostrazioni MERDA

Proof: $i \longleftrightarrow j$ communicate!

$\exists m, n$ two positive integers s.t.

$$P_{ij}^{(m)} > 0 \quad \text{and} \quad P_{ji}^{(n)} > 0$$

by assumption

Then

$$\Rightarrow P_{jj}^{(n+m)} \geq P_{ji}^{(n)} \cdot P_{ij}^{(m)} > 0$$

Assume $\boxed{P_{ii}^{(s)} > 0}$ $\left(s \in \{ n \geq 1 : P_{ii}^{(n)} > 0 \} \right)$

$$\Rightarrow P_{jj}^{(n+s+m)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(s)} \cdot P_{ij}^{(n)} > 0$$

$$n+s+m \in \{ n \geq 1 : P_{jj}^{(n)} > 0 \} \ni n+m$$

$d(j)$ the gcd of $\Rightarrow d(j)$ divides $n+m$

and $d(j)$ divides $n+m+s$

$\Rightarrow d(j)$ divides $n+m+s - n+m = S$

$\Rightarrow d(j)$ divides $d(i)$

With a similar computation we have that $d(i)$ divides $d(j)$ $\Rightarrow$ $d(i) = d(j)$ $\Rightarrow$ i and j have the same period. ∎

<u>Definition</u>: A Markov Chain in which each state has period 1 is called <u>aperiodic</u>

<u>Proposition</u>: if $|S| < +\infty$ $(S = \{0, 1, \dots, N\})$

a MC is <u>irreducible</u> and <u>aperiodic</u>

$\Updownarrow$

is regular

Recurrent and Transient States

Fix $i \in S$ $\qquad$ $\{X_n, n \geq 0\}$ is a MC with P
transition matrix

$\forall \underbrace{n \geq 1}$

$$f_{ii}^{(n)} = \mathbb{P}[X_n = i, X_\nu \neq i, \nu = 1, 2, .., n-1 \mid X_0 = i]$$

$f_{ii}^{(n)}$ is the prob. that starting from i, the first return to state i occurs at the n-th transition

$$f_{ii}^{(1)} = \mathbb{P}[X_1 = i \mid X_0 = i] = P_{ii}$$

$f_{ii}^{(n)}$

$$\boxed{P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)} \quad , n \geq 1}$$

with $f_{ii}^{(0)} = 0$, $\forall i$.

$X_0 = i$, $X_n = i$

$X_0 = i$, $X_n = i$ and

$E_k = \{$ first return to state i is at k-th transition $\}$

$E_1, E_2, \dots, E_n$ are disjoint events.

$$\mathbb{P}[E_k] = \mathbb{P}[\text{ first return is at } k\text{th transition } | X_0 = i]$$

$$\cdot\, \mathbb{P}[X_n = i \mid X_k = i] \qquad \boxed{1 \le k \le n}$$

$$= f_{ii}^{(k)} \cdot P_{ii}^{(n-k)}$$

$$P_{ii}^{(n)} = \mathbb{P}[X_n = i \mid X_0 = i] = \sum_{k=1}^{n} \mathbb{P}[E_k]$$

$$= \sum_{k=1}^{n} f_{ii}^{(k)} \cdot P_{ii}^{(n-k)} = \sum_{k=0}^{n} f_{ii}^{(k)} \cdot P_{ii}^{(n-k)}$$

since $f_{ii}^{(0)} = 0$

$$\underbrace{f_{ii}} = \sum_{n=0}^{\infty} f_{ii}^{(n)} = \lim_{N \to \infty} \sum_{n=0}^{N} f_{ii}^{(n)}$$

starting from state i the MC returns to state i at some time

<u>Definition</u> We say that the state i is <u>recurrent</u> if $f_{ii} = 1$.

On the converse, a state i is <u>transient</u> if $f_{ii} < 1$.

$f_{ii} = 1$ $\quad$ i is recurrent

- $f_{ii} < 1$ $\quad$ i is transient

By the Markov property, $f_{ii} < 1$ is the prob. that I return once to state i.

$(f_{ii})^2$ the prob. to return twice

$(f_{ii})^n$ the prob. to return n times to state i (starting from i)

M = r.v. counts the number of times that the process returns to i

→ M is a Geometric r.v. $\left(\mathbb{P}[\cdot \mid X_0 = i] \right)$ p. 240

$$\mathbb{P}[M \geq k \mid X_0 = i] = (f_{ii})^k \qquad k = 1, 2, 3, \dots$$

with parameter $1 - f_{ii}$

so $$\boxed{\mathbb{E}[M \mid X_0 = i] = \frac{f_{ii}}{1 - f_{ii}}}$$

Remark

$X \sim Geo(p)$ $\quad$ $\mathbb{E}[X] = 1/p$ $\quad$ $X \in \{1, 2, 3, \dots\}$, $\quad$ $\mathbb{P}[X = k] = (1-p)^{k-1} \cdot p$

$\underline{M \in \{0, 1, 2, \dots\}}$

$$\mathbb{P}[M = k \mid X_0 = i] =$$

$$= \mathbb{P}[M \geq k \mid X_0 = i] - \mathbb{P}[M \geq k+1 \mid X_0 = i]$$

$$= (f_{ii})^k - (f_{ii})^{k+1} = (f_{ii})^k (1 - f_{ii})$$

$M \sim \widetilde{Geo}(p)$, $\underline{p = 1 - f_{ii}}$, $\boxed{\mathbb{E}[M] = \frac{1-p}{p}}$

$X \sim Geo(p) \Rightarrow M = X - 1 \Rightarrow \mathbb{E}[M] = \mathbb{E}[X] - 1$

$\mathbb{E}[X] = \frac{1}{1 - f_{ii}}$, $\mathbb{E}[M] = \frac{1}{1 - f_{ii}} - 1 = \frac{1 - 1 + f_{ii}}{1 - f_{ii}}$

$$= \frac{f_{ii}}{1 - f_{ii}}$$

Theorem 3.1 A state i is recurrent if and only if

$$\rightarrow \quad \sum_{n=1}^{\infty} P_{ii}^{(n)} = +\infty$$

Equivalently, the state i is transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < +\infty$

Proof: i transient $\Rightarrow$ $\boxed{f_{ii} < 1}$ and M

counts the total number of visits to state i.

$$M = \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}}$$

$$\mathbb{E}[M \mid X_0 = i] = \frac{f_{ii}}{1 - f_{ii}} < +\infty$$

$$+\infty > \mathbb{E}[M \mid X_0 = i] = \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}} \mid X_0 = i\right]$$

switch

$$= \sum_{n=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{X_n = i\}} \mid X_0 = i\right]$$

Fubini theorem

$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$

$$= \sum_{n=1}^{\infty} \mathbb{P}[X_n = i \mid X_0 = i] = \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

$(\Leftarrow)$ $\sum_{n=1}^{\infty} P_{ii}^{(n)} < +\infty \Rightarrow f_{ii} < 1 \Rightarrow i$ transient

$\blacksquare$

Corollary: if $i \leftrightarrow j$ communicate and i is recurrent (transient), then j is recurrent (transient)

[i.e. recurrency and transience are class properties]

Proof: $i \leftrightarrow j$ $\exists\, n, m \geq 1$ s.t.

$$P_{ij}^{(n)} > 0 \quad , \quad P_{ji}^{(m)} > 0$$

Let $\nu > 0$

$$\underbrace{\sum_{\nu=0}^{\infty} P_{jj}^{(\underbrace{m+n}+\nu)}} \geq \sum_{\nu=0}^{\infty} \underbrace{P_{ji}^{(m)}} P_{ii}^{(\nu)} \underbrace{P_{ij}^{(n)}} =$$

$$= P_{ji}^{(m)} \cdot P_{ij}^{(n)} \cdot \sum_{\nu=0}^{\infty} P_{ii}^{(\nu)}$$

Hence, if $\sum_{\nu=0}^{\infty} P_{ii}^{(\nu)} = +\infty$ (i - recurrent)

$\Rightarrow \sum_{\nu=0}^{\infty} P_{jj}^{(\nu)} = +\infty \Rightarrow j$ is recurrent