Statistics of successful slots up to time n: (1)

Let the error process be Markov, two-state X_n
$0 =$ correct tx ; $1 =$ erroneous tx.

Define

$$\phi_{ij}(k,n) = P\left[k \text{ good slots in } 0,1,\dots,n-1, \text{ and } j \text{ in } n \mid i \text{ in } 0\right]$$

$$= P\left[\sum_{m=0}^{n-1} I\{X_m = 0\} = k, X_n = j \mid X_0 = i\right]$$

Condition on <u>last</u> transition:

$$\phi_{ij}(k,n) = \phi_{i0}(k-1,n-1)p_{0j} + \phi_{i1}(k,n-1)p_{1j}, \quad n>0$$

$$\phi_{ij}(k,0) = \begin{cases} 1 & \text{per } i=j, k=0, n=0 \\ 0 & \text{altrimenti} \end{cases} = \delta_{ij}\delta(n)\delta(k)$$

$$\phi_{ij}(k,n) = 0 \quad k<0 \text{ oppure } n<0 \text{ o } k>n.$$

Finally:

$$\phi_{ij}(k,n) = \phi_{i0}(k-1,n-1)p_{0j} + \phi_{i1}(k,n-1)p_{ij} + \delta_{ij}\delta(n)\delta(k)$$

In matrix form: $\phi(k,n) = \begin{pmatrix} \phi_{00}(k,n) & \phi_{01}(k,n) \\ \phi_{10}(k,n) & \phi_{11}(k,n) \end{pmatrix}$

$$\phi(k,n) = \phi(k-1,n-1)\begin{pmatrix} p_{00} & p_{01} \\ 0 & 0 \end{pmatrix} + \phi(k,n-1)\begin{pmatrix} 0 & 0 \\ p_{10} & p_{11} \end{pmatrix} + \delta(n)\delta(k) I$$

Let $P(0) = \begin{pmatrix} 0 & 0 \\ p_{10} & p_{11} \end{pmatrix}$ and $P(1) = \begin{pmatrix} p_{00} & p_{01} \\ 0 & 0 \end{pmatrix}$

Therefore:

$$\phi(k,n) = \phi(k-1,n-1)P(1) + \phi(k,n-1)P(0) + \delta(n)\delta(k)I, \quad n \geq 0$$

<u>note</u>: $P(1)$ contains "good" transitions (starting from 0)
$P(0)$ " "bad" " (" " 1)

→ $P(i)$ corresponds to i successes, $i=0,1$.

We can apply the same analysis to any finite-state Markov chain where we want to count transitions of a certain type.

<u>note</u>: if we condition on <u>first</u> transition:

$$\phi_{0j}(k,n) = p_{00}\phi_{0j}(k-1,n-1) + p_{01}\phi_{1j}(k-1,n-1) + \delta_{0j}\delta(n)\delta(k)$$

$$\phi_{1j}(k,n) = p_{10}\phi_{0j}(k,n-1) + p_{11}\phi_{1j}(k,n-1) + \delta_{1j}\delta(n)\delta(k)$$

In matrix form:

$$\phi(k,n) = P(1)\phi(k-1,n-1) + P(0)\phi(k,n-1) + \delta(n)\delta(k)I, \quad n \geq 0$$

* compared to the other equation, the order of the matrix products is reversed

* these matrix equations are convolutions.

How to solve these equations:

1) compute recursively
2) use transforms.

$$\varphi(s,z) = \sum_{k=0}^{+\infty} s^k \sum_{n=0}^{+\infty} z^n \phi(k,n)$$

$$\varphi(s,z) = \varphi(s,z) P(1) s z + \varphi(s,z) P(0) z + I$$

$$\varphi(s,z) = \left[I - P(1) s z - P(0) z\right]^{-1} = \left[I - z(P(1)s + P(0))\right]^{-1}$$

$$= \sum_{n=0}^{+\infty} \left[P(1)s + P(0)\right]^n z^n$$

$$\rightarrow \varphi(s,n) = \sum_{k=0}^{+\infty} s^k \phi(k,n)$$ = transform over k for a given n

$$= \left(P(1)s + P(0)\right)^n$$

which in principle could be inverted.

Example: average number of good slots in $0, 1, 2, \ldots, n-1$, given initial state i:

$\varphi_{i0}(s,n) + \varphi_{i1}(s,n)$ is the generating function of the number of times state 0 is visited in $0, 1, 2, \ldots, n-1$, given that we start in i (note: we need to sum over j). The average number of visits is obtained by taking the first derivative for $s=1$.

$$\varphi'(1,n) = \left.\frac{d\varphi(s,n)}{ds}\right|_{s=1} =$$

$$= \sum_{k=0}^{n-1} \left(P(1)s + P(0)\right)^k P(1) \left(P(1)s + P(0)\right)^{n-1-k} \Big|_{s=1}$$

$$= \sum_{k=0}^{n-1} P^k P(1) P^{n-1-k}$$

What we want to compute is given by

$$\varphi'(1,n)\begin{pmatrix}1\\1\end{pmatrix} = \left(\begin{array}{l}\text{we sum the elements of each}\\ \text{row, i.e., over the final state } j\end{array}\right)$$

$$= \sum_{k=0}^{n-1} P^k P(1) P^{n-1-k}\begin{pmatrix}1\\1\end{pmatrix} = \sum_{k=0}^{n-1} P^k P(1)\begin{pmatrix}1\\1\end{pmatrix} =$$

$$= \sum_{k=0}^{n-1} P^k \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sum_{k=0}^{n-1} P_{00}^{(k)} \\ \sum_{k=0}^{n-1} P_{10}^{(k)}\end{pmatrix} \text{ as expected.}$$

<u>Remark</u>: we found the equation:

$$\phi(k,n) = \phi(k-1,n-1)P(1) + \phi(k,n-1)P(0) + \delta(n)\delta(k) I$$

$$n \geq 0$$

where we identified two types of transitions (good and bad, i.e., one or zero successes).

Suppose we associate an integer metric to each transition. Let $P(\ell)$ be the matrix that contains all elements of P that correspond to a "reward" of ℓ. We have:

$$\phi(k,n) = \sum_{\ell=0}^{+\infty} \phi(k-\ell,n-1)P(\ell) + I\,\delta(n)\,\delta(k)$$

with transform

$$\varphi(s,z) = \varphi(s,z)\psi(s)z + I$$

where $\psi(s) = \sum_{\ell=0}^{+\infty} P(\ell)s^{\ell}$

As before:

$$\varphi(s,z) = [I - \psi(s)z]^{-1}$$

$$\varphi(s,n) = [\psi(s)]^{n}.$$

note:

1. we can find average number of rewards in $0, 1, \dots, n-1$ as before (it was a particular case where $\varphi(s) = P(0) + P(1)s$)
2. each transition ij has a "label" $\varphi_{ij}(s)$
3. in $\varphi(s, z)$, s labels successes and z labels the number of transitions.
4. the label on each transition does not need to be a single term. That is, we can have

$$\varphi_{ij}(s) = \sum_{l=0}^{+\infty} P_{ij}(l)\, s^{l}$$

where $P_{ij}(l)$ is the probability of transition $i \to j$ <u>and</u> that the metric has the value l.

Properties of $\varphi_{ij}(s)$:

a) $\varphi_{ij}(1) = P_{ij}$; $\varphi(1) = P$

b) $\dfrac{\varphi_{ij}(s)}{P_{ij}}$ is the generating function of the distribution of the metric <u>given</u> the transition.

c) $\varphi'_{ij}(1) = \left.\dfrac{d\varphi_{ij}}{ds}\right|_{s=1} = P_{ij} \cdot$ average metric on ij

d) if the metric is the reward, $R_i = \sum_{j} \varphi'_{ij}(1)$
(average reward for a visit to state i)

5. We can define multiple metrics, i.e., $\underline{s}$ could be a vector $(s_1, s_2, \dots)$

In this case, the average becomes:

$$P_{ij} \cdot \begin{array}{l}\text{average of the k-th} \\ \text{metric on } ij\end{array} = \left.\frac{\partial \psi_{ij}(s_1, s_2, \dots)}{\partial s_k}\right|_{\underline{s}=\underline{1}}$$

6. How to compute $\psi_{ij}(\underline{s})$:

a) let $\mathcal{E}(i,j)$ be the set of all events that correspond to a transition from state i to state j

b) let $P[A]$ the probability of event A

c) let $\underline{s}(A) = s_1^{l_1(A)} s_2^{l_2(A)} \cdots$ ~~[illegible]~~ where $l_k(A)$ is the value of the k-th metric that corresponds to the event A.

d) we have:

$$\psi_{ij}(\underline{s}) = \sum_{A \in \mathcal{E}(i,j)} P[A]\, \underline{s}(A)$$

note that $\psi_{ij}(\underline{1}) = \sum_{A \in \mathcal{E}(i,j)} P[A] = P_{ij}$

example: GBN with iid feedback errors (prob. δ).

table of possible events:

from	to	prob.	reward	time
G	G	$(1-\delta)\, p_{00}$	1	1
G	B	$(1-\delta)\, p_{01}$	1	1
G	G	$\delta\, p_{00}(m)$	0	m
G	B	$\delta\, p_{01}(m)$	0	m
B	G	$(1-\delta)\, p_{10}(m)$	0	m
B	B	$(1-\delta)\, p_{11}(m)$	0	m
B	G	$\delta\, p_{10}(m)$	0	m
B	B	$\delta\, p_{11}(m)$	0	m

$$\Phi(s_1, s_2) = \begin{pmatrix} (1-\delta) p_{00} s_1 s_2 + \delta p_{00}(m) s_2^m & (1-\delta) p_{01} s_1 s_2 + \delta p_{01}(m) s_2^m \\ p_{10}(m) s_2^m & p_{11}(m) s_2^m \end{pmatrix}$$

$$P = \psi(1,1) = \begin{pmatrix} (1-\delta)p_{00} + \delta p_{00}(m) & (1-\delta)p_{01} + \delta p_{01}(m) \\ p_{10}(m) & p_{11}(m) \end{pmatrix}$$

$$\pi_G = \frac{p_{10}(m)}{p_{10}(m) + (1-\delta)p_{01} + \delta p_{01}(m)} \; ; \; \pi_B = 1 - \pi_G$$

$P_{ij} R_{ij}$ are the elements of

$$\left.\frac{\partial \psi}{\partial s_1}\right|_{s_1 = s_2 = 1} = \begin{pmatrix} (1-\delta)p_{00} & (1-\delta)p_{01} \\ 0 & 0 \end{pmatrix}$$

$P_{ij} T_{ij}$ are the elements of

$$\left.\frac{\partial \psi}{\partial s_2}\right|_{s_1 = s_2 = 1} = \begin{pmatrix} (1-\delta)p_{00} + m\delta p_{00}(m) & (1-\delta)p_{01} + m\delta p_{01}(m) \\ m p_{10}(m) & m p_{11}(m) \end{pmatrix}$$

$$\underline{R} = \left.\frac{\partial \psi}{\partial s_1}\right|_{s_1 = s_2 = 1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\delta \\ 0 \end{pmatrix}$$

$$\underline{T} = \left.\frac{\partial \psi}{\partial s_2}\right|_{s_1 = s_2 = 1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\delta+m\delta \\ m \end{pmatrix}$$

$$\text{throughput} = \frac{\pi_G R_G + \pi_B R_B}{\pi_G T_G + \pi_B T_B} = \frac{(1-\delta)p_{10}(m)}{(1-\delta+m\delta)p_{10}(m) + m\left[(1-\delta)p_{01} + \delta p_{01}(m)\right]}$$