Using (24), it is not difficult to show that positive (null) recurrence is a class property (see Problem 11).

Let us suppose for the remainder of this section that the Markov renewal process is irreducible, positive recurrent and not lattice. Suppose further, that the embedded Markov chain $\{J_n, n \geq 0\}$ is also positive recurrent and aperiodic, and let $\{\pi_j, j = 0, 1, 2, ..\}$ denote the limiting probabilities for this Markov chain.

Since π_j equals the (long-run) proportion of transitions which are into state j, and since μ_j is the mean time spent in j per transition, it seems intuitive that the limiting probabilities should be proportional to $\pi_j \mu_j$. That is,

$$P_j = \frac{\pi_j \mu_j}{\sum_{i=0}^{\infty} \pi_i \mu_i} \qquad j = 0, 1, 2, \ldots \tag{25}$$

In order to prove (25), we shall need the following proposition which is useful in its own right.

Proposition 5.15 average first passage time from i to j

For all i, j

$$\mu_{ij} = \mu_i + \sum_{k \neq j} P_{ik} \mu_{kj} \tag{26}$$

PROOF. Let T_{ij} denote the time of the first transition into j. Then

$$\begin{aligned}
\mu_{ij} &= \sum_{k=0}^{\infty} E[T_{ij} \mid J_1 = k, J_0 = i] P_{ik} \\
&= \sum_{k \neq j} P_{ik}[\mu_{kj} + \eta_{ik}] + P_{ij}\eta_{ij} \\
&= \sum_{k=0}^{\infty} P_{ik}\eta_{ik} + \sum_{k \neq j} P_{ik}\mu_{kj} \\
&= \mu_i + \sum_{k \neq j} P_{ik}\mu_{kj}
\end{aligned}$$

We are now ready to prove the important

Theorem 5.16

For all states j,

$$\mu_{jj}\pi_j = \sum_{i=0}^{\infty} \pi_i \mu_i$$

and

$$P_j = \frac{\pi_j \mu_j}{\sum_{i=0}^{\infty} \pi_i \mu_i}$$

PROOF. Multiplying both sides of (26) by π_i and summing on i yields

$$\begin{aligned}
\sum_{i=0}^{\infty} \pi_i \mu_{ij} &= \sum_{i=0}^{\infty} \pi_i \mu_i + \sum_{i=0}^{\infty} \pi_i \sum_{k \neq j} P_{ik}\mu_{kj} \\
&= \sum_{i=0}^{\infty} \pi_i \mu_i + \sum_{k \neq j} \mu_{kj} \sum_{i=0}^{\infty} \pi_i P_{ik} \\
&= \sum_{i=0}^{\infty} \pi_i \mu_i + \sum_{k \neq j} \pi_k \mu_{kj}
\end{aligned}$$

or

$$\pi_j \mu_{jj} = \sum_{i=0}^{\infty} \pi_i \mu_i \tag{27}$$

where we have used the fact that $\pi_k = \sum_{i=0}^{\infty} \pi_i P_{ik}$ (see 4.9). From (27) it follows that

$$\frac{\pi_j \mu_j}{\sum_{i=0}^{\infty} \pi_i \mu_i} = \frac{\mu_j}{\mu_{jj}}$$

and the result follows from Proposition 5.14.

REMARK. Theorem 5.16 is extremely important, for it reduces the problem of calculating limiting probabilities of the Markov renewal process to one of calculating the stationary probabilities of the embedded Markov chain. It also shows that the limiting probabilities only depend on $Q_{ij}(\cdot)$ thru P_{ij} and μ_j.

EXAMPLE. The (M/M/1) Queue. Consider a single-server queueing system in which customers arrive at a Poisson rate λ, and service times are independent exponential random variables having mean $1/\mu$. Letting $n(t)$ denote the number of customers in the system at time t, then because of the lack of memory of the exponential it follows that the process $\{n(t), t \geq 0\}$ is semi-Markov with

$$P_{01} = 1 \qquad \mu_0 = \frac{1}{\lambda}$$

$$P_{i,i-1} = \frac{\mu}{\lambda + \mu} \qquad \mu_i = \frac{1}{\lambda + \mu} \qquad i = 1, 2, \ldots$$

$$P_{i,i+1} = \frac{\lambda}{\lambda + \mu} \qquad i = 1, 2, \ldots$$