Stochastic Processes

Lecture 12

CYS 2020/2021

- Example of Poisson Processes
- Non-homogeneous Poisson Process
- The law of Rare Events
- Poisson point process
- Waiting times and sojourn times
- The Uniform distribution and Poisson Processes
- Combining / splitting Poisson Processes

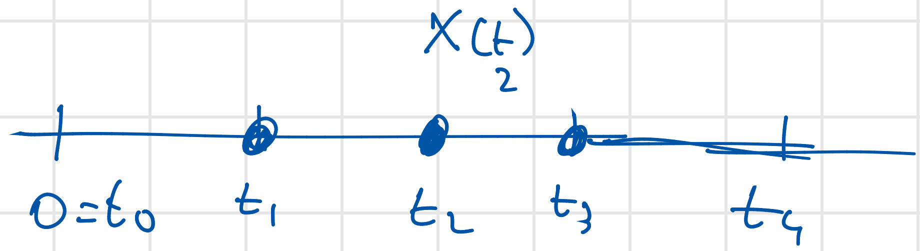
Poisson Process

$X(t)$, $t \geq 0$ $\quad$ $\lambda > 0$ intensity $\quad$ $\boxed{X(t) \in \mathbb{N},\ \forall t}$

$t_0 = 0 < t_1 < t_2 \dots < t_n$

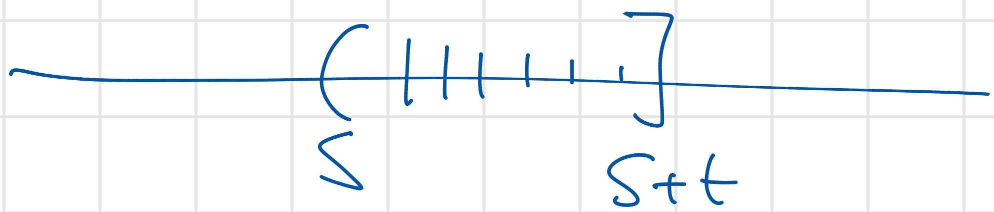
(i) $X(t_1) - X(t_0),\ X(t_2) - X(t_1),\ \dots,\ (X(t_n) - X(t_{n-1}))$

are indep. r.v.'s



(ii) $s \geq 0$, $t > 0$

$\Rightarrow$ $X(s+t) - X(s) \sim \text{Poisson}(\lambda t)$



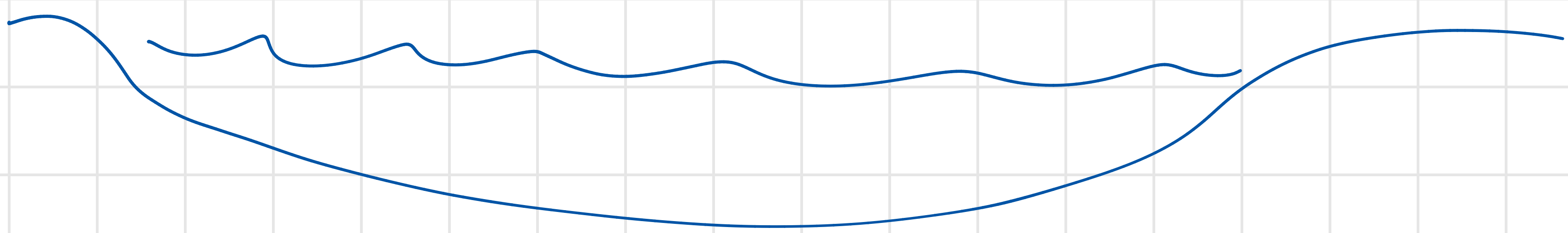
(iii) $X(0) = 0$

Counting process

$[0, t]$ $\quad$ $X(t) = \#$ occurrences in $[0, t]$

$t \in [0, +\infty)$

Exp. 1 undersea cable



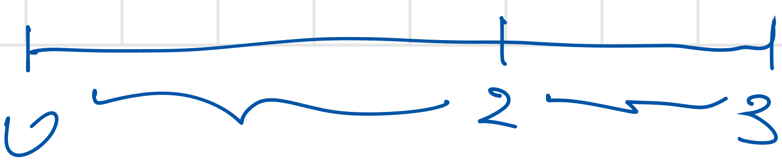
Defects occur along this cable according to a Poisson process of rate $\lambda = 0.1$ per mile. $X(t),\ t \geq 0$

(a) What is the probability that no defects appear in the first two mile of the cable?

$$\underbrace{X(2) - X(0)}_{\equiv\, 0} = X(2) \sim \text{Poisson}(0.1 \times 2) = \text{Poisson}(0.2)$$

$$\mathbb{P}[X(2) = \underset{k}{0}] = \frac{e^{-0.2}\, 0.2^{0}}{0!} = e^{-0.2} \simeq 0.8187$$

(b)



No defects between 2 and 3 given no defects in $(0, 2]$.

indep.

$$\mathbb{P}\left[\, X(3)-X(2)=0 \;\middle|\; \cancel{X(2)-X(0)=0} \,\right] = \circledast \qquad X(2)=0$$

$$= \mathbb{P}\left[\, X(3)-X(2)=0 \,\right] = \text{(ii)}$$

(i)

$$= \mathbb{P}\left[\, X(3)=0 \;\middle|\; \cancel{X(2)=0} \,\right]$$

$X(3)-X(2) \sim \text{Poisson}(\lambda \cdot t) =$

$\underbrace{t+s}$ $\underbrace{s}$ $\qquad 3-2=\underbrace{1}$

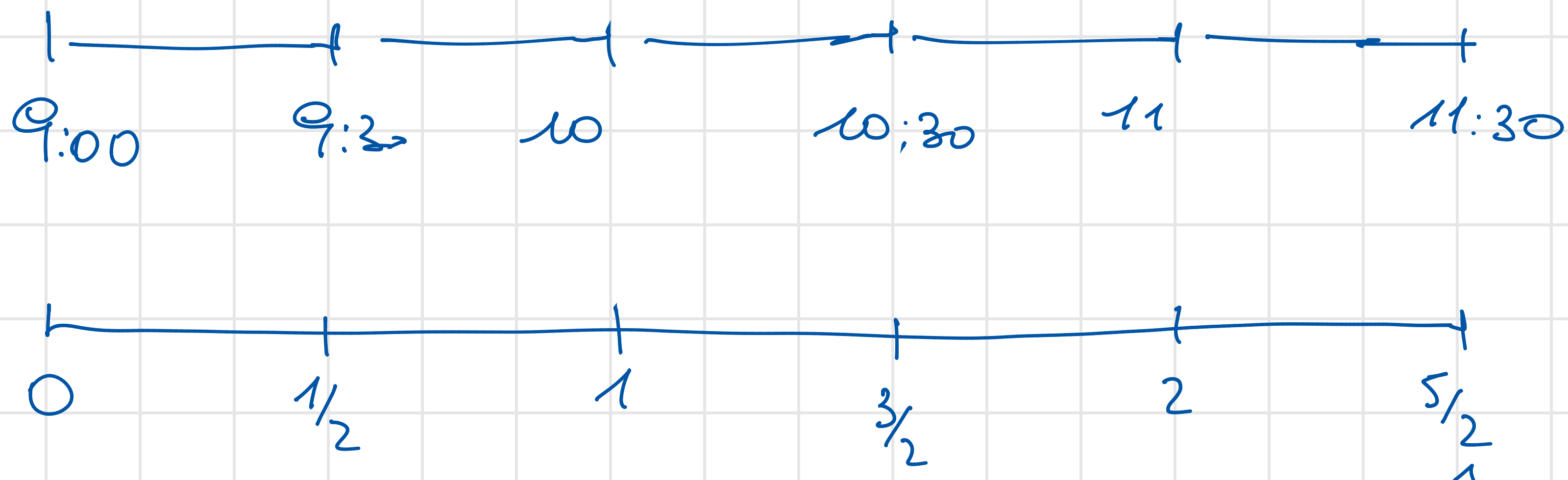
$t+s-s$

$X(2+1)-X(2)$ $\qquad = \text{Poisson}(0.1)$

$$\mathbb{P}\left[\, X(3)-X(2)=0 \,\right] = e^{-0.1} = 0.9048$$

Example Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour.

9:00 A.M. 1 customer arrives by 9:30 A.M. and a total of 5 customers arrive by 11:30 AM



$$\boxed{\lambda = 4}$$

$$X(0) = 0$$

$\rightarrow \; X(t) \qquad (0, t]$

$\rightarrow \; X(t+s) - X(s) \qquad (s, t+s]$

$$\mathbb{P}\left[\underbrace{X(1/2) = 1, \; X(5/2) = 5}\right]$$

$$= \mathbb{P}\left[\underbrace{X(1/2) - \underset{0}{\underset{\parallel}{X(0)}} = 1, \; X(5/2) - X(1/2) = 4}\right]$$

(i) indep. of the increments of the P.P.

$$= \underbrace{\mathbb{P}[X(1/2) = 1]} \cdot \underbrace{\mathbb{P}[X(5/2) - X(1/2) = 4]}$$

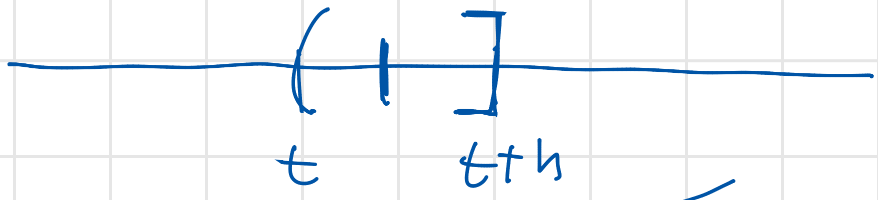
$$= \frac{e^{-2} \cdot 2^1}{1!} \cdot \frac{e^{-8} \cdot 8^4}{4!} \cong 0.0154\ldots$$

$X(1/2) \sim \text{Poisson}(\overset{2}{4 \cdot 1/2})$, $X(5/2) - X(1/2) \sim \text{Poisson}(\overset{8}{4 \cdot 2})$

$\boxed{1.3}$ Non homogeneous Poisson Process

$X(t+h)-X(t) \sim \text{Poisson}(\lambda \cdot h)$

$$\mathbb{P}[\,X(t+h) - X(t) = 1\,] =$$



$$= \frac{e^{(-)\lambda h} \cdot \boxed{\lambda h}}{1!}$$

$$= \lambda h \left(1 - \lambda h + \frac{\lambda^2 h^2}{2!} - \frac{\lambda^3 h^3}{3!} \cdots\right)$$

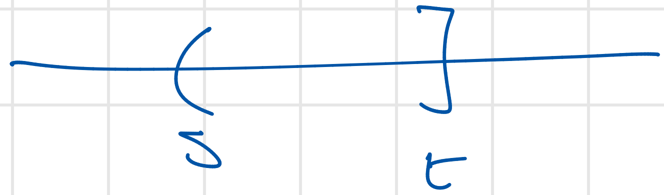
h^2 $\quad$ h^3

$$= \lambda h + o(h)$$

$\boxed{h \downarrow 0}$

does not depend on t.

$$X(t) - X(s)$$

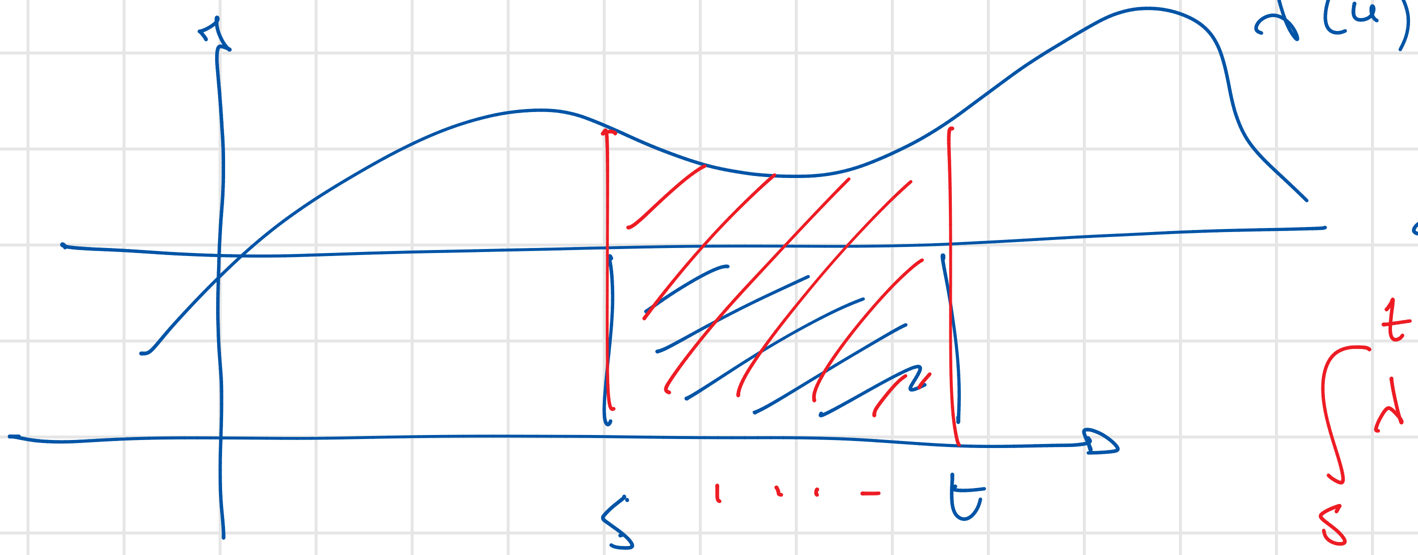


$$\sim \text{Poisson}\left(\lambda\,(t-s)\right)$$

$\lambda(u) \quad u \in (0, +\infty)$

$$\text{Poisson}\left(\int_s^t \lambda(u)\,du\right)$$

$\lambda(u) \equiv \lambda$ $\qquad \lambda\,(t-s)$



$\int_s^t \lambda(u)\, du$

$\boxed{\text{Example pag. 272}}$

Section 2 Second construction of the PP

Poisson point process of intensity λ

Poisson counting process

Law of Rare Events

large number N of indep. Bernoulli trials where the prob. of success is p

$X_{N,p}$ = total number of successes $\sim Bin(N,p)$

$$\mathbb{P}[X_{N,p}=k] = \frac{N!}{k!\,(N-k)!}\cdot p^k(1-p)^{N-k}$$

$k=0,\dots,N$

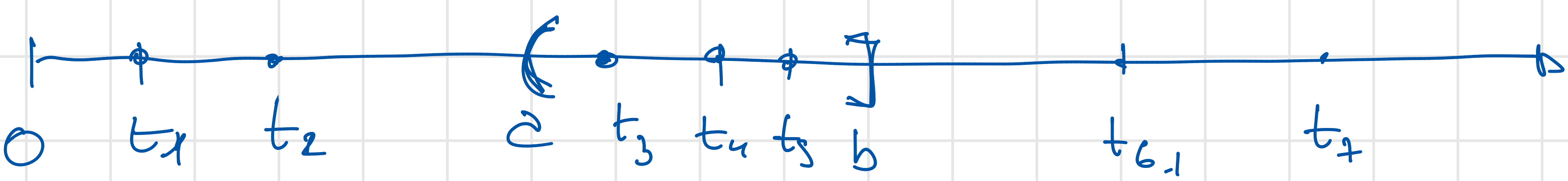
$N \longrightarrow +\infty \qquad p \longrightarrow 0$

$$\boxed{Np = \mu > 0}$$

$$\underbrace{\mathbb{P}[X_{N,p}=k]} \longrightarrow \underbrace{\frac{e^{-\mu}\,\mu^k}{k!}} \qquad k=0,1,2,\dots$$

Poisson(μ)

$t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6$



$N((a,b])$ = number of events that occur during the interval $(a,b]$

(1) $t_0 = 0 < t_1 < t_2 < t_3 \quad \dots < t_m$

$m = 2, 3, \dots$

the random variables

$$N((t_0, t_1]),\ N((t_1, t_2]),\ \dots,\ N((t_{m-1}, t_m])$$

are independent.

(2) $\forall$ time t and positive number h

the distribution of $N((t, t+h])$

depends only on the length h and not on t,

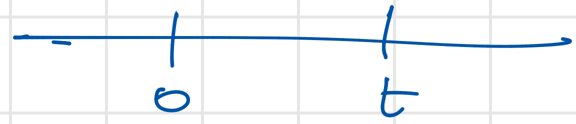
(3) $\mathbb{P}[N((t, t+h]) \geq 1] = \boxed{\lambda h} + o(h) \quad h \downarrow 0$

for a given constant $\lambda > 0$.

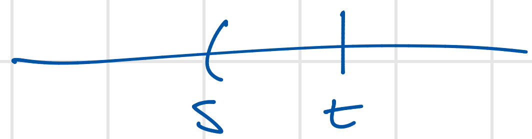
(4) $\mathbb{P}[N((t, t+h]) \geq 2] = \underline{\underline{o(h)}} \quad h \downarrow 0$

$$\Rightarrow \boxed{\underbrace{\mathbb{P}[N((0, t]) = k]}_{\text{POISSON}} = \frac{(\lambda t)^k \cdot e^{-\lambda t}}{k!}, \quad k = 0, 1, 2}$$

$X(t)$, $t \geq 0$



$N((s,t])$, $s \geq 0$, $t > s$



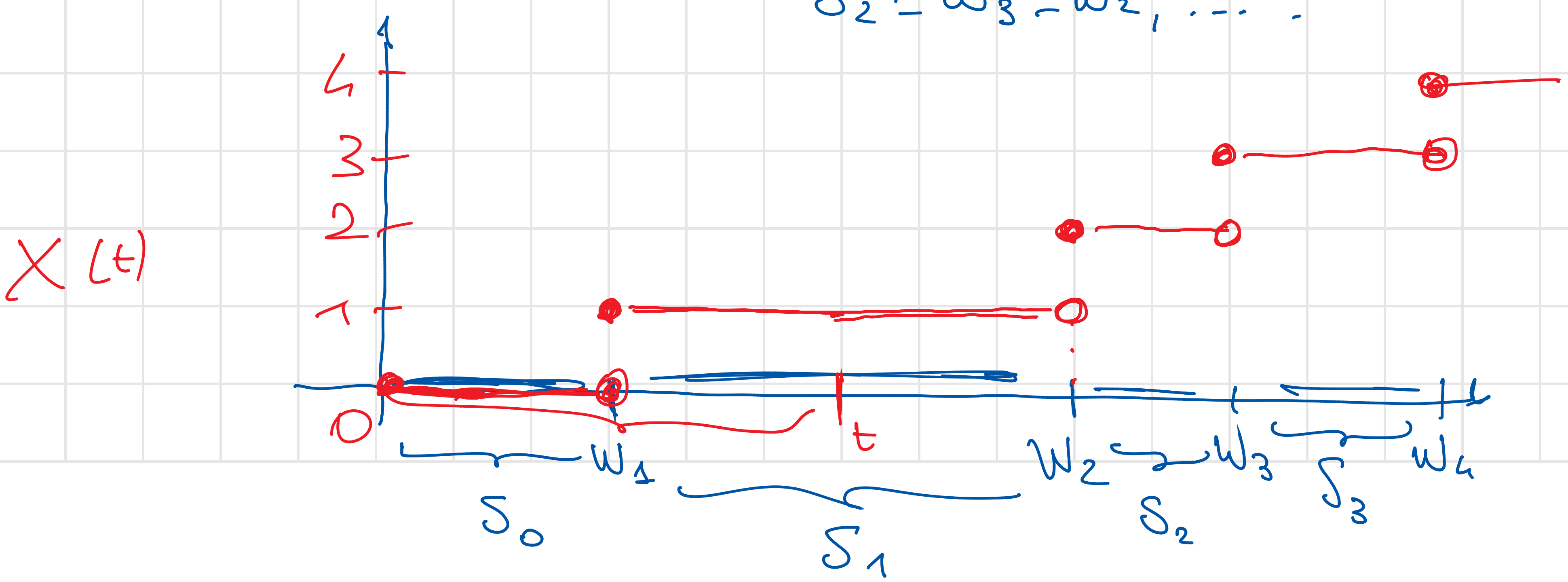
↳ Poisson point process of intensity λ.

- $X(t) = N((0,t])$
- $N((s,t]) = X(t) - X(s)$

Section 3 Distributions Associated with the PP

- Waiting times W_1, W_2, W_3, W_4, W_5
- Sojourn times $S_0 = W_1$, $S_1 = W_2 - W_1$,

  $S_2 = W_3 - W_2$, ...



Thm 3.1 The waiting time W_n has a gamma distribution whose prob. density function is

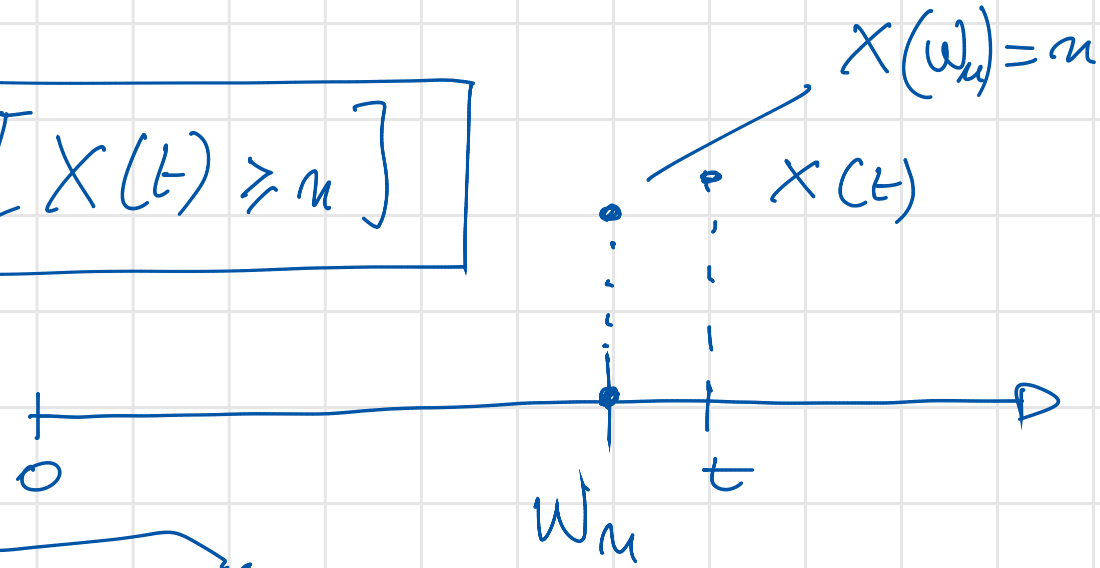
$$f_{W_n}(t) = \frac{\lambda^n \, t^{n-1}}{(n-1)!} \cdot e^{-\lambda t}, \quad n = 1, 2, \dots$$

$\boxed{\Gamma(n)}$

$\boxed{\Gamma(\alpha)}$

$\boxed{t \geq 0}$

Proof: $\boxed{[W_n \leq t] = [X(t) \geq n]}$



$$\underbrace{F_{W_n}(t)} = P[W_n \leq t] = P[\underbrace{X(t)} \geq n]$$

$$= \sum_{k=n}^{+\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad n$$

$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$f_{W_n}(t) = \frac{d}{dt} F_{W_n}(t) =$$

$$\vdots$$

$$= \frac{\lambda^n \; t^{n-1}}{(n-1)!} e^{-\lambda t} \qquad n = 1, 2, 3, \ldots \quad t \geq 0$$

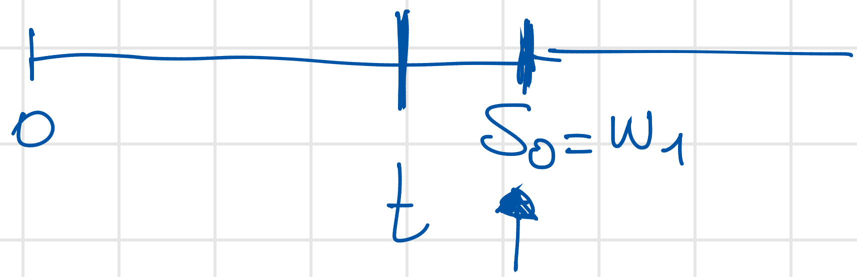
W_n Gamma r.v. $\blacksquare$

<u>Thm 3.2</u> The sojourn times $S_0, S_1, S_2, \ldots$ are <u>independent</u> random variables, each having the exponential prob. density given by

$$f_{S_n}(s) = \lambda e^{-\lambda s}, \quad s \geq 0$$

Proof: $S_0 \sim \mathrm{Exp}(\lambda)$?

$X(t) = 0$



$$\mathbb{P}[S_0 > t] = \mathbb{P}[X(t) = 0] = \frac{e^{-\lambda t} \cdot (\lambda t)^0}{0!}$$

$X(t) \sim \mathrm{Poisson}(\lambda t)$ $\quad = e^{-\lambda t}$

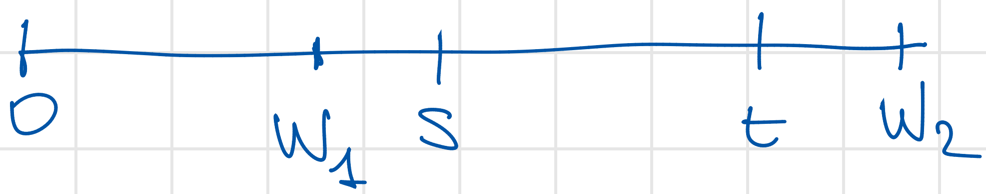
$\boxed{t \geq 0}$

$$\mathbb{P}[S_0 > t] = e^{-\lambda t} \implies S_0 \sim \mathrm{Exp}(\lambda)$$

$1 - \mathbb{P}[S_0 \leq t]$

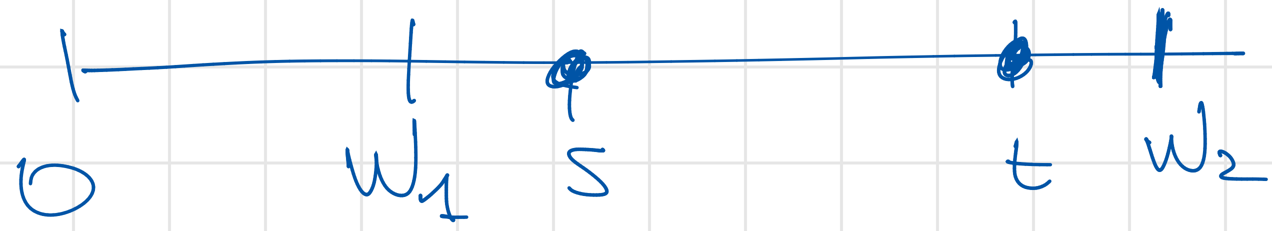
$F_{S_0}(t) = 1 - \underbrace{e^{-\lambda t}}$

$0 \leq s < t$



$$\mathbb{P}[\underset{S_0}{\underset{\parallel}{W_1}} \leq s, \underset{S_1 - S_0}{\underset{\parallel}{W_2}} > t] = \mathbb{P}[S_0 \leq s, S_1 - S_0 > t]$$

$$= \mathbb{P}[W_1 \le s,\ W_2 > t] = \mathbb{P}[X(s) \ge 1,\ X(t) < 2]$$



$$= \mathbb{P}[X(s) = 1,\ X(t) = 1]$$

$$= \mathbb{P}[\underbrace{X(s)=1}, \underbrace{X(t)-X(s)=0}] = \cdots$$

$$= \mathbb{P}[X(s)=1] \cdot \mathbb{P}[X(t)-X(s)=0]$$

$$= \lambda s e^{-\lambda s} \cdot e^{-\lambda(t-s)} = \boxed{\lambda s e^{-\lambda t}}$$

$$\mathbb{P}[W_1 \le s, W_2 > t]$$

So

$$\mathbb{P}[W_1 \le s,\ W_2 \le t] = \boxed{\mathbb{P}[W_1 \le s]} - \mathbb{P}[W_1 \le s, W_2 > t]$$

$$\frac{\partial}{\partial s} \cdot \frac{\partial}{\partial t}$$

$$f_{W_1, W_2}(s,t)$$

(Completed after the Lecture)

$$\mathbb{P}[W_1 \le s, W_2 \le t] = \mathbb{P}[W_1 \le s] - \mathbb{P}[W_1 \le s, W_2 > t]$$

$$= 1 - e^{-\lambda s} - \lambda s e^{-\lambda t}$$

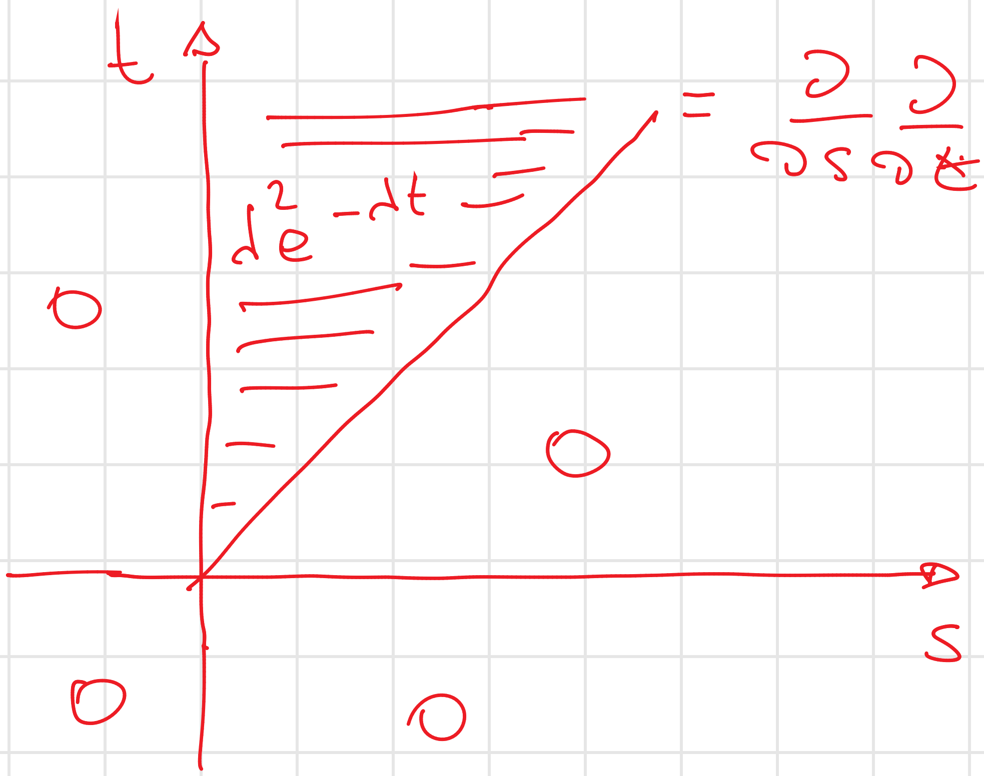
Then the density of the random vector (W_1, W_2) is

$$f_{W_1,W_2}(s,t) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} F_{W_1,W_2}(s,t)$$

$$= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\left(1 - e^{-\lambda s} - \lambda s e^{-\lambda t}\right) =$$

$$= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\left(-\lambda s e^{-\lambda t}\right) = \lambda^2 e^{-\lambda t}$$

$0 \le s < t$



Make the following change of variables:

$S_0 = W_1$, $S_1 = W_2 - W_1$

$\Rightarrow$ $W_1 = S_0$, $W_2 = S_1 + S_0$; we get $(S_0, S_1) \in (0, +\infty)^2$

and Jacobian of the transf.

$$f_{S_0,S_1}(u,v) = \underbrace{|\det J|}_{=1} \cdot f_{W_1,W_2}(u, u+v)$$

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad |\det J| = 1$$

$$f_{S_0,S_1}(u,v) = \lambda^2 e^{-\lambda(u+v)}$$

$$= \lambda e^{-\lambda u} \cdot \lambda e^{-\lambda v}$$

$$= f_{S_0}(u) \cdot f_{S_1}(v)$$

$\Rightarrow$ $S_0 \sim Exp(\lambda)$, $S_1 \sim Exp(\lambda)$

and S_0 and S_1 are indep.

The proof for the general case

$S_0, \dots, S_n$

is longer, but follows a similar path ∎

Prove the following result:

<u>Thm 3.3</u> Let $\{X(t), t \geq 0\}$ be a PP of rate $\lambda > 0$.

Then for $0 < u < t$ and $0 \leq k \leq n$

$$\mathbb{P}[X(u) = k \mid X(t) = n] = \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$$

i.e. the conditional probability that k events occur up to time u, given that a total number n of events has occurred up to time n, is that of a Binomial r.v. of parameters $\frac{u}{t}$ and n.