Stochastic Processes

Lecture 3

CYS 2020/2021

- Definition of a stochastic process
- Definition of a Markov Chain
  - state space S
  - one-step transition probability
  - initial distribution
  - finite-dimensional distributions
  - some simple examples
- Transition probability matrices
- First Step Analysis
- Absorbing Markov Chains

Stochastic process

$X : \Omega \to \mathbb{R}$ random variable

$n = 2$ $\quad (X, Y) : \Omega \to \mathbb{R}^2$

$(X_1, X_2) : \Omega \to \mathbb{R}^2$

$\omega \longmapsto (X_1(\omega), X_2(\omega))$

male / female $\quad$ grade of the first exam

$n \in \mathbb{N}$ $\quad (X_1, \dots, X_n)$

$T = \{0, 1, 2, \dots, n, \dots\}$, $T = [0, +\infty)$

$n \in T$ $\quad X_n$

$X_0, X_1, X_2, \dots$ infinite sequence stochastic process

fix ω $\quad \boxed{\text{path}}$

$X = (X_0, X_1, \dots) : \Omega \times T \longrightarrow \mathbb{R}$

$\longrightarrow X_n(\omega) \in \mathbb{R}$

$X : \Omega \times T \longrightarrow \mathbb{R}$ $\qquad T = [0, +\infty)$

$X_t(\omega)$ random variable (fixed $t \in T$)

fix ω $\qquad \underbrace{t \longmapsto X_t(\omega)}$ real function

discrete time stochastic process

$X : \Omega \times \mathbb{N} \longrightarrow \mathbb{R}$

$X_0, X_1, X_2, \dots$

discrete time Markov chains

$X_n : \Omega \longrightarrow S$ discrete space

finite or (infinite) countable set

$S = \underbrace{\mathbb{N}}$

$$\mathbb{P}[X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i]$$

$$X_n : \Omega \to \underbrace{\mathbb{N}}_{\ni j} \qquad = \mathbb{P}[X_{n+1} = j \mid X_n = i]$$

$$\forall\, n \in \mathbb{N}, \quad i, j, i_0, \dots, i_{n-1} \in \mathbb{N}$$

One-step transition probability $P_{ij}^{n,n+1}$

$$P_{ij}^{n,n+1} := \mathbb{P}[X_{n+1} = j \mid X_n = i]$$

$$X_n = i \longrightarrow X_{n+1} = j$$

Stationary transition probabilities

$$P_{ij}^{n,n+1} := \boxed{P_{ij}} \qquad \forall\, n \in \mathbb{N}$$

(homogeneous in time)

$$P_{ij} := \underbrace{\mathbb{P}[X_1 = j \mid X_0 = i]} \qquad i, j \in \mathbb{N}$$

- $P_{ij} \geq 0 \qquad \forall i, j \in \mathbb{N}$

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

$$A \longmapsto \mathbb{P}[A \mid B]$$

is a probability

- $P_{i,j}$

$\boxed{\text{fixed } i}$

$$\sum_{j \in \mathbb{N}} P_{i,j} = \sum_{j \in \mathbb{N}} \mathbb{P}[X_1 = j \mid X_0 = i]$$

$\{X_1 = j\}$ are disjoint events $\forall j \in \mathbb{N}$

$$= \mathbb{P}[\underbrace{\bigcup_j \{X_1 = j\}}_{\Omega} \mid X_0 = i] = 1$$

$$\boxed{\begin{array}{ll} \bullet \; P_{ij} \geq 0 & \forall i, j \in \mathbb{N} \\ \bullet \; \sum_{j \in \mathbb{N}} P_{ij} = 1 & \forall i \in \mathbb{N} \end{array}}$$

$S = \{0, 1, 2\}$ $\quad P_{ij} \quad , i, j \in S$

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} + & P_{11} + & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \qquad P_{00} \geq 0 \qquad P_{10} + P_{11} + P_{12} = 1$$

transitions prob.

Stochastic Matrix

non-negative matrix

Fix i $\quad \sum_j P_{ij} = 1$

$S = \mathbb{N}$

$$\begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad P_{01} \geq 0$$

"matrix"

$$\boxed{\sum_{j=0}^{+\infty} P_{ij} = 1}$$

$$\sum \frac{1}{n} = +\infty, \quad \sum \frac{1}{n^2} < +\infty$$

Representation

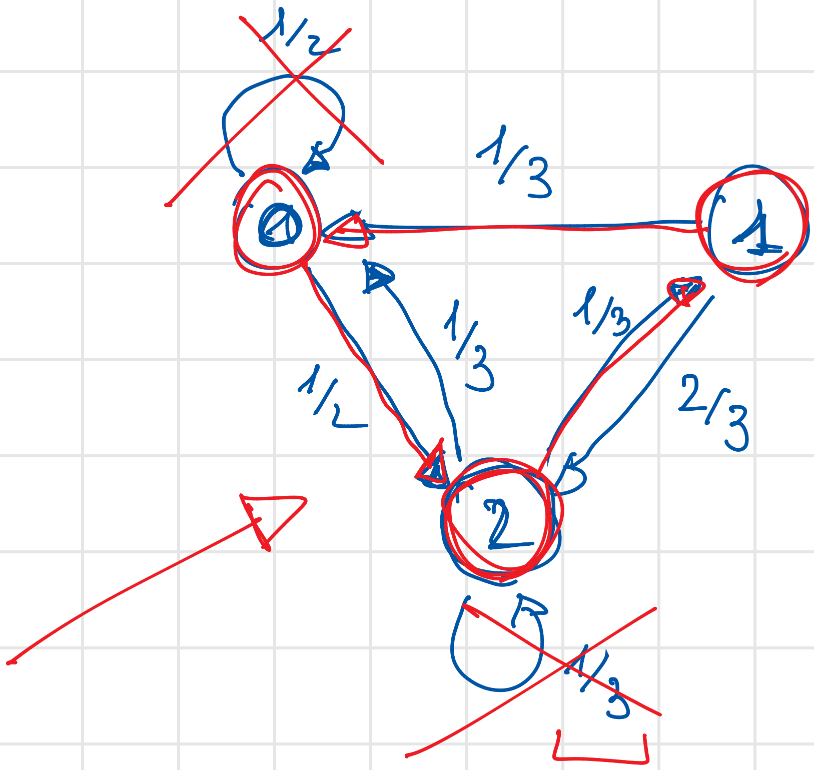
$$P_{ij} = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & \boxed{2} \end{array} \\ \hline \begin{array}{c} \boxed{0} \\ \boxed{1} \\ 2 \end{array} & \begin{bmatrix} \boxed{1/2} & 0 & \boxed{1/2} \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$\boxed{6}$

$S = \{0,1,2\}$

$\subseteq$

$\mathbb{N}$



Finite dimensional probabilities

$P[A \cap B] = P[A] \cdot P[B|A]$

$$P[X_0 = i] = p_i \qquad \forall i \in \mathbb{N}$$ initial distribution

$$P[\underbrace{X_0 = i_0, X_1 = i_1, \ldots,}_{A} \overbrace{\boxed{X_n = i_n}}^{B}]$$

$$= P[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}] \cdot$$

Markov property

$$P[X_n = i_n \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}]$$

$$= P[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}] \cdot P[X_n = i_n \mid X_{n-1} = i_{n-1}]$$

$$p_{i_{n-1}, i_n} = \mathbb{P}[X_n = i_n \mid X_{n-1} = i_{n-1}] \quad \dots$$

$$\mathbb{P}[X_0 = i_0, \dots, X_n = i_n] = p_{i_0} \, p_{i_0 i_1} \cdot p_{i_1 i_2} \cdots p_{i_{n-1}, i_n}$$

initial distribution

transition probabilities

Matrix description of transition probabilities

$$P = (p_{ij})_{i,j \in \mathbb{N}} \quad \text{one-step transition}$$

$$P^{(n)} = \left(p^{(n)}_{ij}\right)_{i,j \in \mathbb{N}}$$

$$p^{(n)}_{ij} = \mathbb{P}[X_n = j \mid X_0 = i] \qquad n \in \mathbb{N},\ i,j \in \mathbb{N}$$

n-step transition probabilities

Theorem 2.1 (pag 101)

The n-step transition probabilities of a MC satisfy

$$P_{ij}^{(n)} = \underbrace{\sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}}$$

where we define $P_{ij}^{(0)} = Id$

$$P^{(n)} = P \times P^{(n-1)} \qquad \sum_k a_{ik} b_{kj}$$

$$= P \times (P \times P^{(n-2)})$$

$$= P \times P \times (P \times P^{(n-3)})$$

$$\dots = P^n$$

n-th power of the matrix P

Proof: first-step analysis

$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B|A]$

$$P_{ij}^{(n)} = \mathbb{P}[X_n = j \mid X_0 = i]$$

$$= \sum_{k=0}^{\infty} \mathbb{P}[\underbrace{X_n = j}, \underbrace{X_1 = k} \mid \underline{X_0 = i}]$$

$$= \sum_{k=0}^{\infty} \underbrace{\mathbb{P}[\underbrace{X_1 = k} \mid X_0 = i]} \cdot \underbrace{\mathbb{P}[\underbrace{X_n = j} \mid \underbrace{X_1 = k}, \cancel{X_0 = i}]}$$

$$= \sum_{k=0}^{\infty} P_{ik} \, P_{kj}^{(n-1)}$$

(Corollary)

$$\mathbb{P}[X_n = k] = \sum_{j=0}^{\infty} p_j \, P_{jk}^{(n)}$$

$$X_0 \longrightarrow X_n$$

$$X_0 \sim p$$

$$p_j = \mathbb{P}[X_0 = j]$$

$$\sigma = \underbrace{(p_0, p_1, \dots, p_n, \dots)}$$

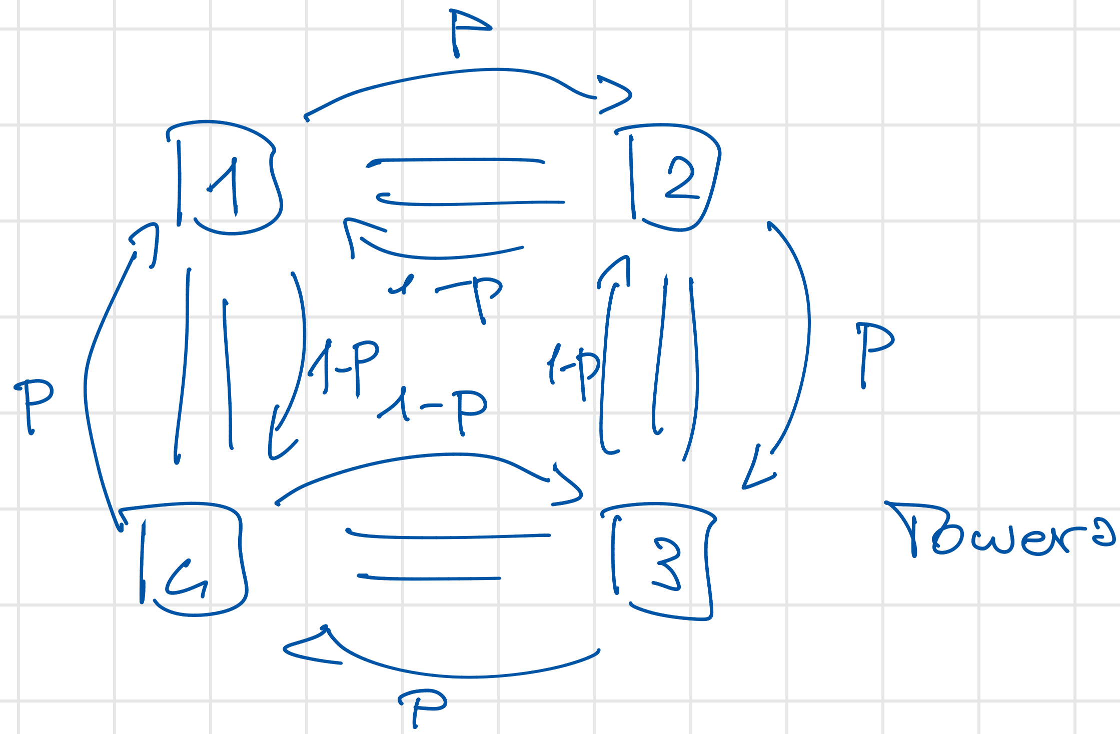
$$\underbrace{\sigma \cdot \boxed{P^n}} = \underbrace{\mathbb{P}[X_n = k]}$$

Remark: in general

A, B matrices

$$A \cdot B \neq B \cdot A$$

Ex: Castle



Sentinel

H	p
T	$1-p$

$n \in \mathbb{N}$ $\qquad X_n \in \{1, 2, 3, 4\}$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix} \end{array}$$

Exercise: $p = 1/2$ and compute P^n

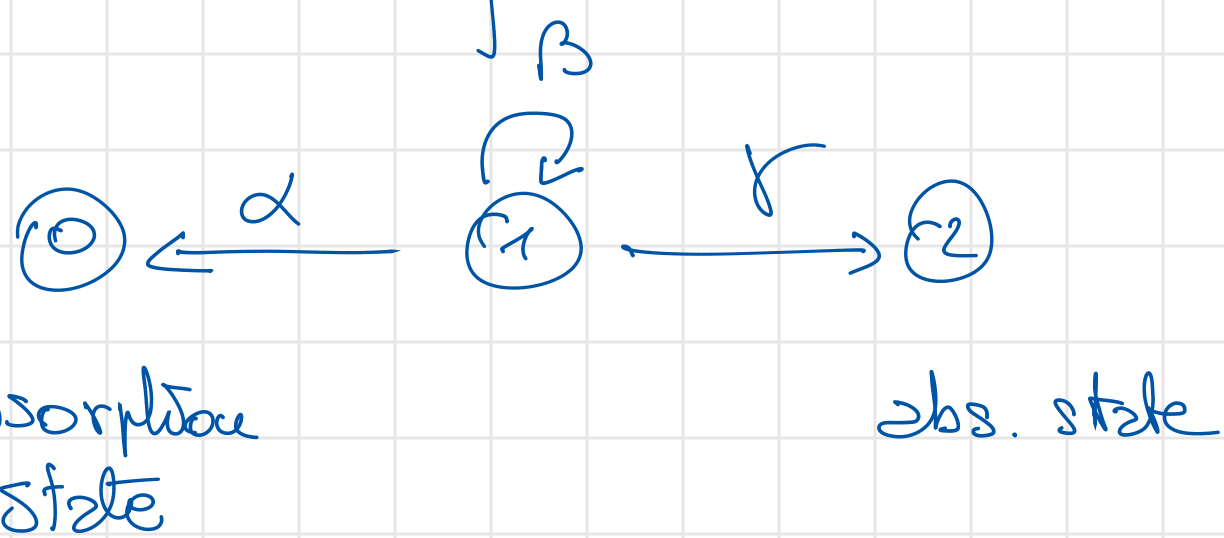
$\forall n \in \mathbb{N}$

Section 4 First Step Analysis

$S = \{0, 1, 2\}$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\beta = 1 - \alpha - \gamma$



If I start from 1, what is the prob. that I goes (forever) to 0? to 2?

$$T = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 2\}$$

$\in \mathbb{N}$ time

$$u = \mathbb{P}[X_T = 0 \mid X_0 = 1]$$

$$v = \mathbb{E}[T \mid X_0 = 1]$$

$$\boxed{\begin{aligned} &\underbrace{\mathbb{P}[X_T = 0 \mid X_1 = 0]}\ = 1 \\ &\mathbb{P}[X_T = 0 \mid X_1 = 2] = 0 \\ &\mathbb{P}[X_T = 0 \mid X_1 = 1] = u \end{aligned}}$$

$$u = \mathbb{P}[X_T = 0 \mid X_0 = 1] =$$

$$= \sum_{k=0}^{2} \underbrace{\mathbb{P}[X_T = 0 \mid X_1 = k}, \cancel{X_0 = 1}] \cdot \mathbb{P}[X_1 = k \mid X_0 = 1]$$

$$= 1 \cdot \alpha + u \cdot \beta + 0 \cdot \gamma$$

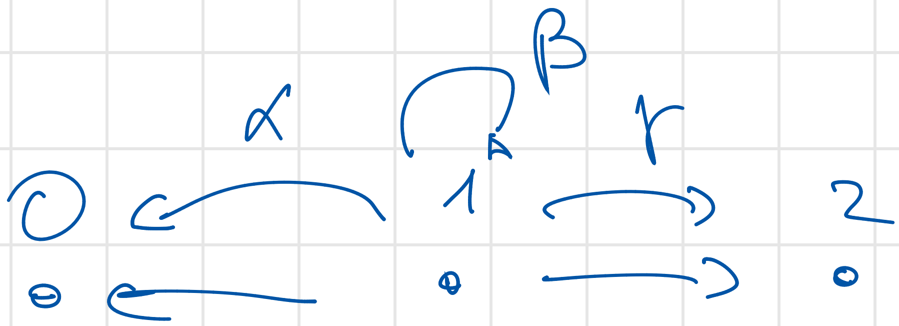
$$u = \alpha + \beta \cdot u \qquad \boxed{u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha + \gamma}}$$

Exercise

$$w = \mathbb{P}[X_T = 2 \mid X_0 = 1] = 1 - u = \frac{\gamma}{\alpha + \gamma}$$

$\nu = \mathbb{E}[T \mid X_0 = 1]$ expected absorption time

$X_1 = k$



$$\nu = \underbrace{1} + \alpha \cdot 0 + \gamma \cdot 0 + \beta \cdot \nu$$

$$\nu = \mathbb{E}[T \mid X_0 = 1]$$

$$\nu = \mathbb{E}[T \mid X_1 = 1]$$

$$\nu = 1 + \beta \nu \qquad \nu = \frac{1}{1-\beta} = \frac{1}{\alpha+\gamma}$$

(since $\alpha+\beta+\gamma = 1$
$\Rightarrow 1-\beta = \alpha+\gamma$)

pag. 118 - 119 - 120

$\boxed{\text{pag. 121}}$ . . . .