RW on $\mathbb{Z}$

Let $X_i, i \in \mathbb{N}$ be a sequence of iid random variables s.t.

$$X_i = \begin{cases} 1 & p \\ -1 & 1-p \end{cases}$$

where $p \in [0,1]$.

We define the simple RW on $\mathbb{Z}$ as

$$\begin{cases} S_0 = 0 \\ S_n = X_1 + \dots + X_n = S_{n-1} + X_n , & n \geq 1 \end{cases}$$

We have seen that S_n is a Markov Chain, irreducible when $0 < p < 1$, recurrent if $p = 1/2$ and transient if $p \neq 1/2$.

1. For any $i \in \mathbb{Z}$ and $n \in \mathbb{N} \setminus \{0\}$, we want to compute

$$\mathbb{P}[S_n = i].$$

Let us start by fixing some notation: if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$

$$\mathbb{P}[X_1 = \varepsilon_1, \dots, X_n = \varepsilon_n] = p^{N(\varepsilon)} (1-p)^{n - N(\varepsilon)}$$

where $N(\varepsilon) = \# \{ m : \varepsilon_m = 1 \}$

__Remark__ : Let $B_n(\varepsilon) = \sum_{i=1}^{n} \varepsilon_i$, then $N(\varepsilon) = \frac{n + S_n(\varepsilon)}{2}$

Indeed, if $\varepsilon_i \in \{-1,1\}$, $\frac{\varepsilon_i+1}{2} \in \{0,1\}$ and

$$N(\varepsilon) = \sum_{i=1}^{n} \frac{\varepsilon_i+1}{2} = \frac{S_n(\varepsilon)+n}{2}$$

Solution (1).

First of all $\mathbb{P}[\,|S_n| \le n\,] = 1 \Rightarrow \mathbb{P}[S_n = i] = 0$
$\forall\ |i| > n.$

Moreover, we have

if n is odd, then $\mathbb{P}[S_n \text{ is odd}] = 1$

if n is even, then $\mathbb{P}[S_n \text{ is even}] = 1$

Let us therefore assume that $i \in \{-n, -n+1, \dots, n-1, n\}$ and that i and n are both odd or even.

$$\mathbb{P}[S_n = i] = \mathbb{P}[(X_1, \dots, X_n) = \varepsilon,\ B_n(\varepsilon) = i]$$

$$= P_i \times |\{\varepsilon : B_n(\varepsilon) = i\}|$$

where $P_i = p^{N(\varepsilon)}(1-p)^{n-N(\varepsilon)} = p^{\frac{n+i}{2}} \cdot (1-p)^{\frac{n-i}{2}}$

(since $B_n(\varepsilon) = i$ and $N(\varepsilon) = \frac{B_n(\varepsilon)+n}{2}$).

We have to compute the cardinality of $\{E : B_n(E)=i\}$.

Since $B_n(E)=i \Rightarrow N(E)=\frac{n+i}{2}$, we can choose $\binom{n}{\frac{n+i}{2}}$ paths with this numbers of $+1$. Then

$$\mathbb{P}[S_n=i] = \binom{n}{\frac{n+i}{2}} p^{\frac{n+i}{2}} \cdot (1-p)^{\frac{n-i}{2}}$$

$\blacksquare$

Remark: we have seen in class, that

$$P_{00}^{(2n)} = \binom{2n}{n} p^n (1-p)^n$$

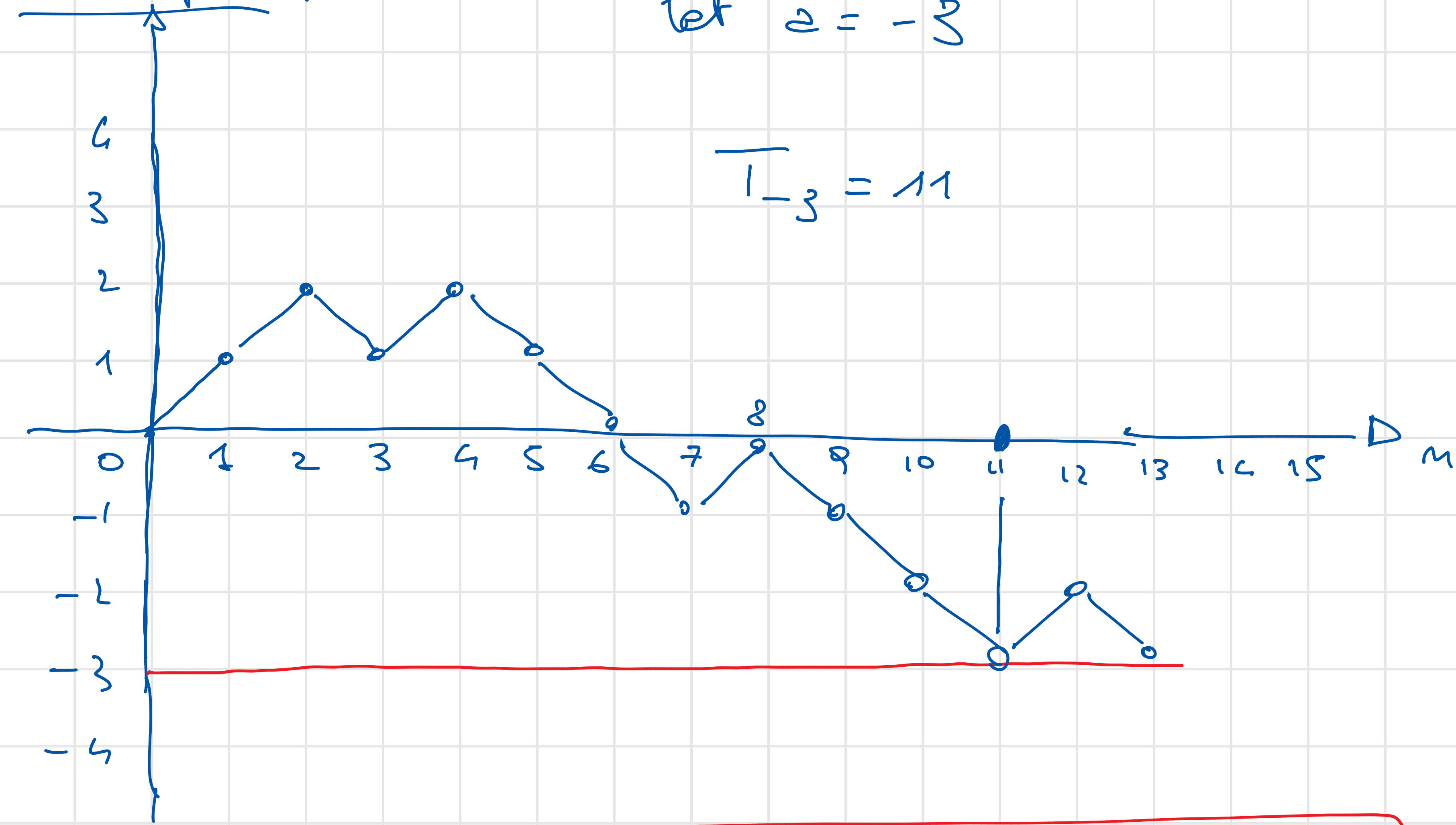
which is a special case of the previous result.

Let us now fix a barrier $a \in \mathbb{Z}$ and define the time of first passage to a as

$$T_a := \inf\{n \geq 1 : X_n = a\}$$

with the convention $\inf \emptyset = +\infty$.

Example :



Let $a = -3$

$T_{-3} = 11$

Let us compute $\forall\, i \in \mathbb{Z}$ and $n \in \mathbb{N} \setminus \{0\}$ **2**

$$\mathbb{P}[T_a = n]$$

Solution (2)

As before, note that $\mathbb{P}[T_a = n] = 0 \quad \forall\, |a| > n$,

and $\mathbb{P}[T_a = n] > 0 \iff |a| \leq n$ and a and n are both odd or even

Let us consider the case $a > 0$.

$$\mathbb{P}[T_a = n] = \mathbb{P}[S_n = a, T_a > n-1]$$

$$= \mathbb{P}[S_{n-1} = a-1, X_n = 1, T_a > n-1] =$$

Since $(X_n = 1)$ is independent of $(S_{n-1} = a-1) \cap (T_a > n-1)$

we get $= \mathbb{P}[X_n = 1] \cdot \mathbb{P}[S_{n-1} = a-1, T_a > n-1]$

$$= p \cdot \mathbb{P}[S_{n-1} = a-1, T_a > n-1]$$

Note that

$$\mathbb{P}[S_{n-1} = a-1, T_a > n-1] = \mathbb{P}[(X_1, \dots, X_{n-1}) = E : B_{n-1}(E) = a-1,$$
$$B_\ell(E) \le a-1 \quad \forall\, 0 \le \ell \le n-1]$$

Let $\Sigma := \{E : B_{n-1}(E) = a-1, B_\ell(E) \le a-1, \forall\, 0 \le \ell \le n-1\}$

When $E \in \Sigma$, $\mathbb{P}[(X_1, \dots, X_{n-1}) = E] = p^{\frac{n+a}{2}-1} (1-p)^{\frac{n-a}{2}}$

since $N(E) = \dfrac{n-1 + B_{n-1}(E)}{2} = \dfrac{n-1+a-1}{2} = \dfrac{n+a}{2} - 1$

and $n-1 - N(E) = n-1 - \dfrac{n+a}{2} + 1 = \dfrac{n-a}{2}$.

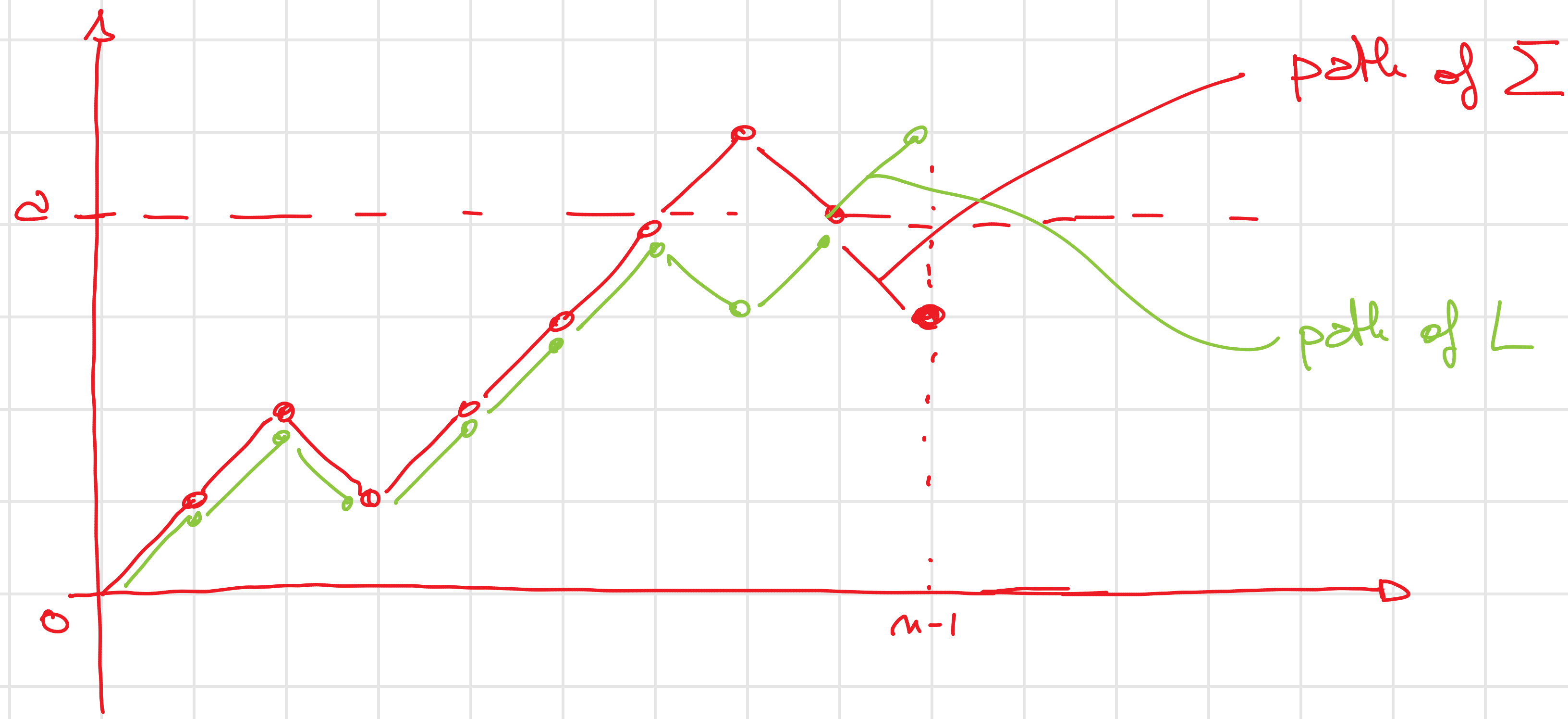
So $\mathbb{P}[T_a = n] = p \cdot p^{\frac{n+a}{2}-1} \cdot (1-p)^{\frac{n-a}{2}} \cdot |\Sigma|$

If $\Sigma' = \{E : B_{n-1}(E) = a-1,\ B_l(E) \geq a \text{ for some } l \leq n-1\}$

we get $\Sigma \cup \Sigma' = \{E : B_{n-1}(E) = a-1\}$ and $\Sigma \cap \Sigma' = \emptyset$.

Then $|\Sigma| + |\Sigma'| = \binom{n-1}{\frac{n+a}{2}-1}$

$$\Rightarrow \quad |\Sigma| = \binom{n-1}{\frac{n+a}{2}-1} - |\Sigma'|$$



If $L = \{E : B_m(E) \geq a \text{ for some } 1 \leq l \leq n-1,\ B_{n-1}(E) = a+1\}$

we have that $|L| = |\Sigma'|$. Indeed, there is a one to one correspondence of the path in L and Σ' if we swap the $+1$ and the -1 after the first time the path reaches a.

Why is this property useful? Since $L = \{E : B_{n-1}(E) = a+1\}$

(in this case the property "$B_m(E) \geq a$ for some $1 \leq l \leq n-1$" is clearly true)

Then, $|L| = \binom{n-1}{\frac{n+a}{2}}$ and we get

$$\mathbb{P}[T_a = n] = p^{\frac{n+a}{2}} \cdot (1-p)^{\frac{n-a}{2}} \cdot \left[\binom{n-1}{\frac{n+a}{2}-1} - \binom{n-1}{\frac{n+a}{2}} \right]$$

$$= p^{\frac{n+a}{2}} (1-p)^{\frac{n-a}{2}} \cdot \left[\frac{(n-1)!}{\left(\frac{n+a}{2}-1\right)! \left(n - \frac{n+a}{2}\right)!} - \frac{(n-1)!}{\left(\frac{n+a}{2}\right)! \left(n - \frac{n+a}{2} - 1\right)!} \right]$$

$$= p^{\frac{n+a}{2}} (1-p)^{\frac{n-a}{2}} \cdot \frac{(n-1)!}{\left(\frac{n+a}{2}\right)! \left(n - \frac{n+a}{2}\right)!} \cdot \Bigg[\underbrace{\frac{n+a}{2} - \left(n - \frac{n+a}{2}\right)}_{a} \Bigg]$$

$$= \frac{a}{n} \cdot \binom{n}{\frac{n+a}{2}} \cdot p^{\frac{n+a}{2}} (1-p)^{\frac{n-a}{2}}$$

$$= \frac{a}{n} \cdot \mathbb{P}[S_n = a]$$

In a similar way we get for $a<0$

$$\mathbb{P}[T_a = n] = \frac{-a}{n} \, \mathbb{P}[S_n = a]$$

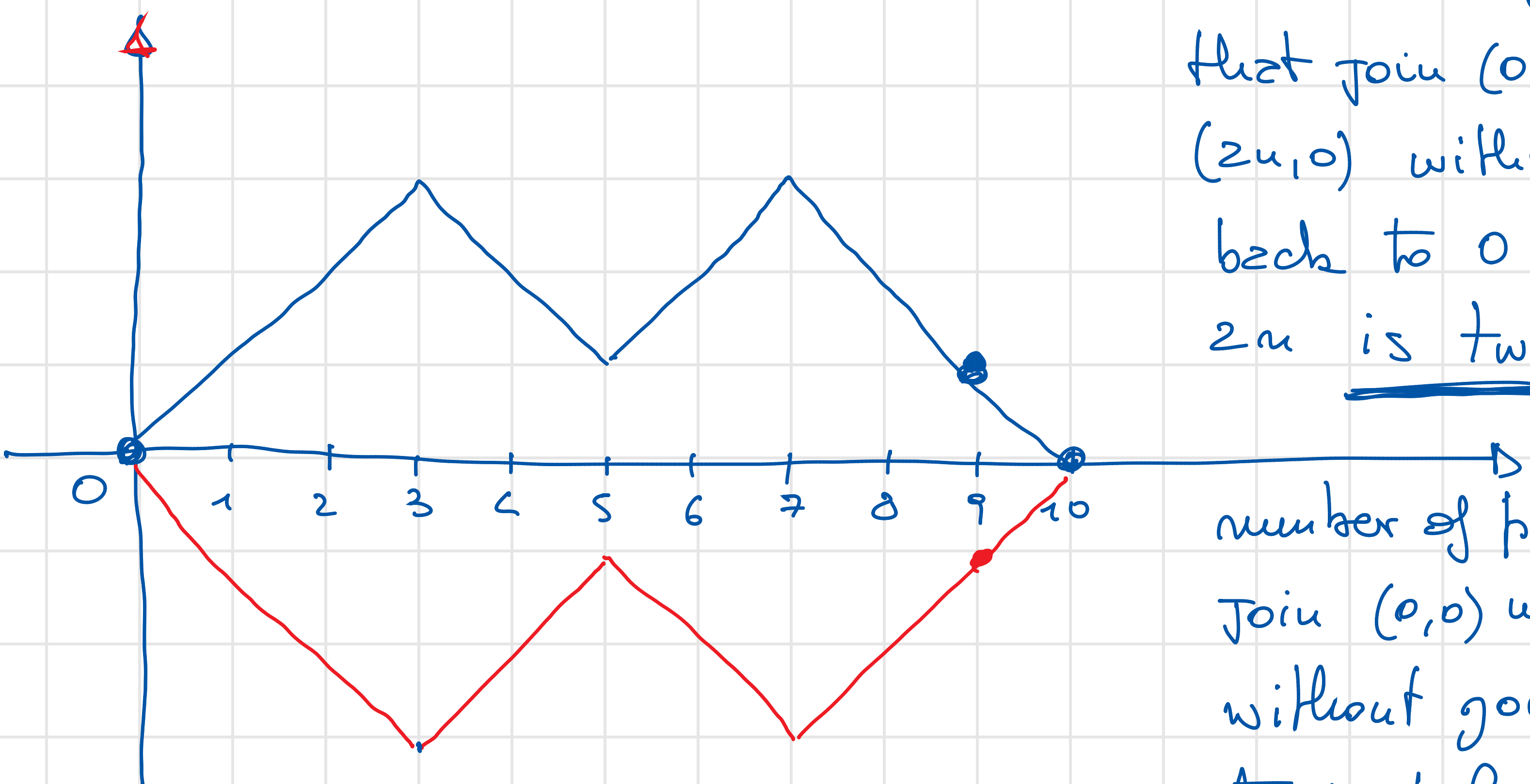
and the general result for $a \neq 0$ reads

$$\mathbb{P}[T_a = n] = \frac{|a|}{n} \binom{n}{\frac{n+a}{2}} p^{\frac{n+a}{2}} (1-p)^{\frac{n-a}{2}}$$

$$= \frac{|a|}{n} \mathbb{P}[S_n = a]$$

Let us now compute $\mathbb{P}[T_0 = 2n]$.

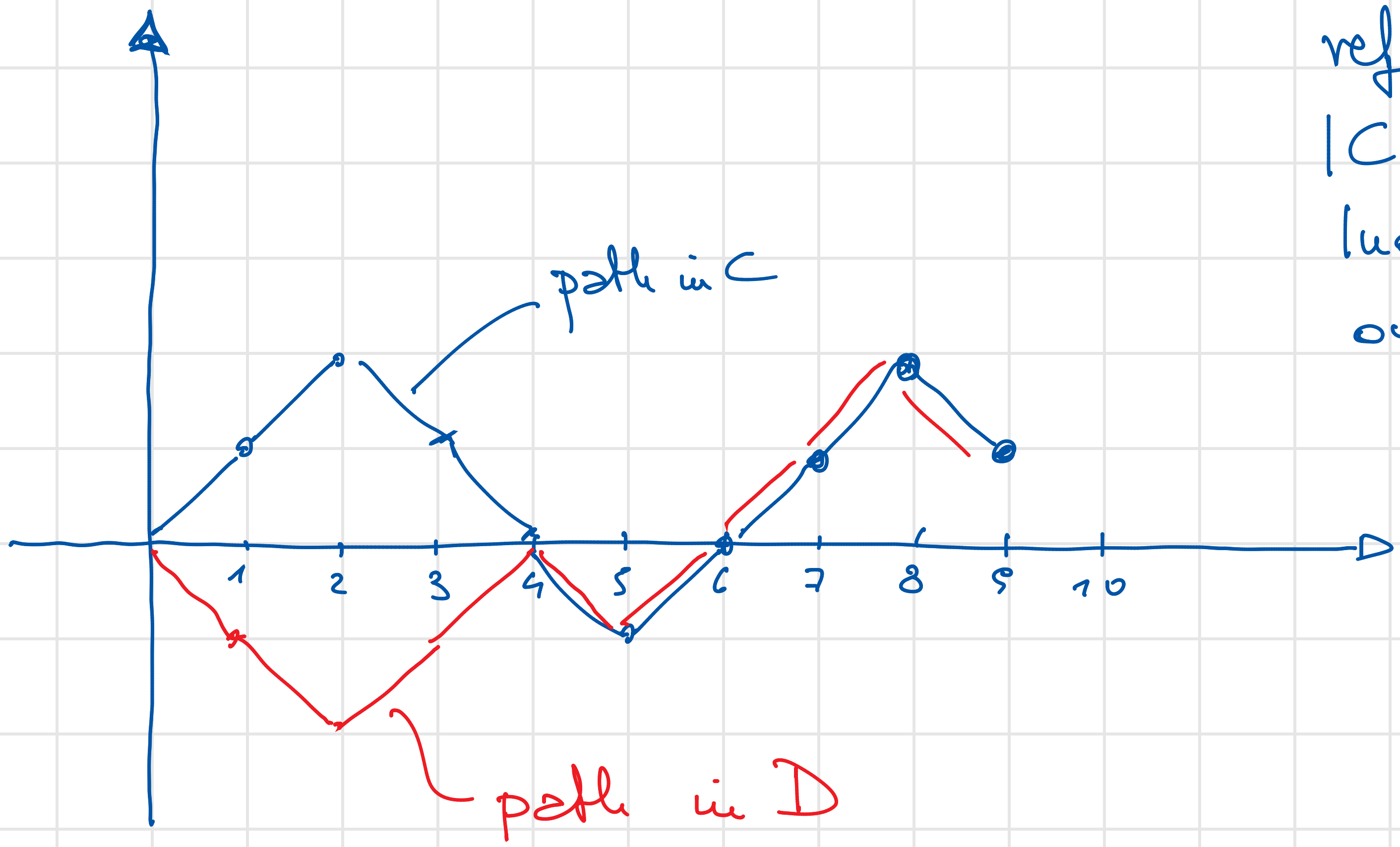
In this case the computation is slightly different: indeed



the number of the paths that join $(0,0)$ with $(2n,0)$ without going back to 0 before $2n$ is twice the number of paths that join $(0,0)$ with $(2n-1,1)$ without going back to 0 before $2n-1$ (reflecting property)

Let $B = \{$paths from $(0,0)$ to $(2n-1,1)$ without touching $0\}$ and $A = \{$paths from $(0,0)$ to $(2n-1,1)\}$. We have $A = B \cup C \cup D$

where $C = \{$ paths not in B s.t. $X_1 = 1\}$ and B, C, D are disjoint.
$D = \{$ paths not in B s.t. $X_1 = -1\}$



Now, by the reflection property, $|C| = |D|$

Indeed, we have a one to one relation of the paths in C and D by changing the $+1$ and -1 before the first return to 0.

As before, $D = \{$ paths not in B s.t. $X_1 = 1\}$

$D' = \{$ all the paths that join $(1,-1)$ with $(2n-1, 1)\}$

have the same cardinality, and

$$|D'| = \binom{2n-2}{n-2}$$ (all the possible positions of the $n-2$ "-1" in $2n-2$ slots)

Then $|A| = \binom{2n-1}{1}$, $|C| = |D| = |D'| = \binom{2n-2}{n-2}$

$$\Rightarrow |B| = \binom{2n-1}{1} - 2\binom{2n-2}{n-2} = \frac{(2n-2)!}{n!\,(n-1)!}$$

So we get

$$\mathbb{P}[T_0 = 2n] = 2 \cdot |B| \cdot p^n (1-p)^n$$

$$= 2 \frac{(2n-2)!}{n!\,(n-1)!} \cdot p^n (1-p)^n$$

$$= \frac{\binom{2n}{n} p^n (1-p)^n}{2n-1} = \frac{\mathbb{P}[S_{2n} = 0]}{2n-1}$$

$\blacksquare$