Stochastic Processes

Lecture 10

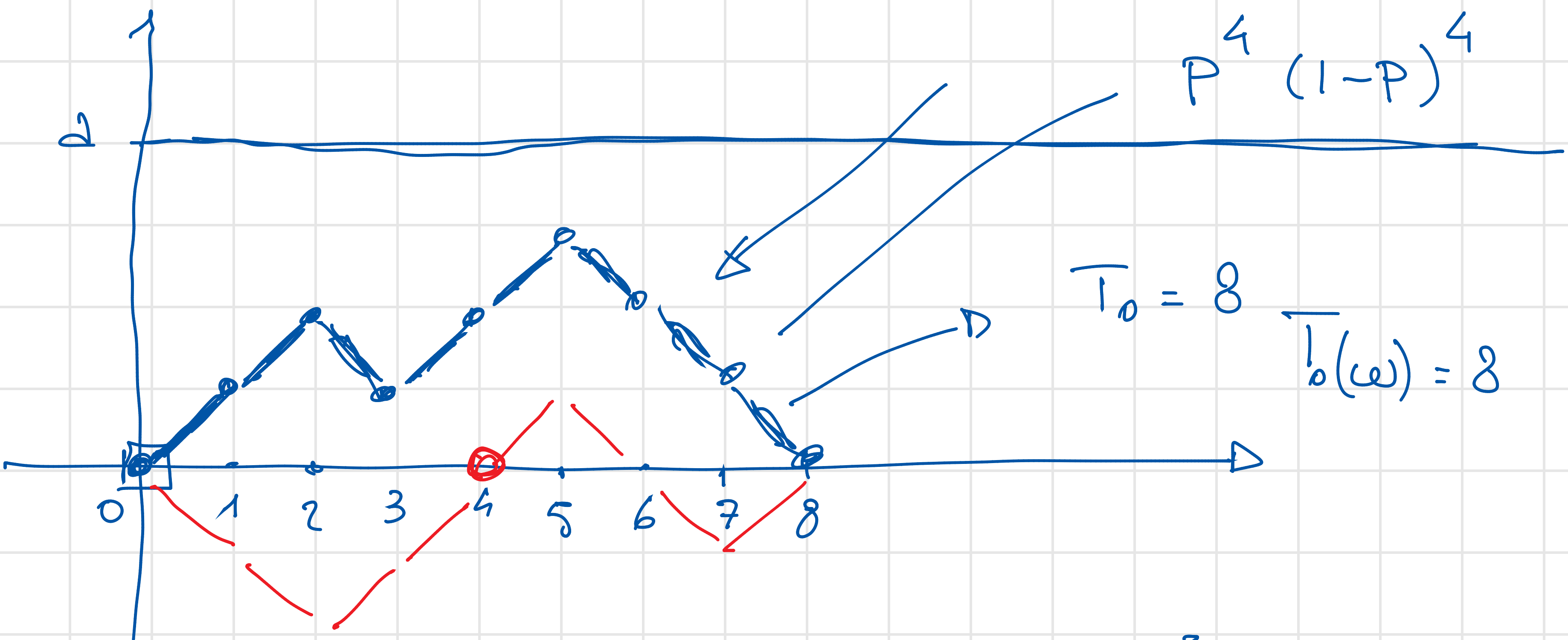
CYS 2020/2021

- Invariant distributions
  - Computation in the case of the RW with one reflecting barrier
  - Computation in the case of the success runs
  - Computation in the case of a RW on a graph
  - Detailed balance equation

Simple RW on $\mathbb{Z}$

$$\begin{cases} S_0 = 0 \\ S_n = X_1 + X_2 + \cdots + X_n \end{cases}$$

$X_n, n \geq 1$ iid $\quad X_1 = \begin{cases} 1 & p \\ -1 & q = 1-p \end{cases}$



$T_d = \inf \{ n \geq 1 : S_n = d \}$ r.v.

$\inf \emptyset = +\infty$ $\boxed{q = 1-p}$

$\mathbb{P}[T_d = n] = ?$ $\forall n$

$d > 0$, $d < 0$, $\boxed{d = 0}$

$\mathbb{P}[T_0 = n] = 0$ n is odd

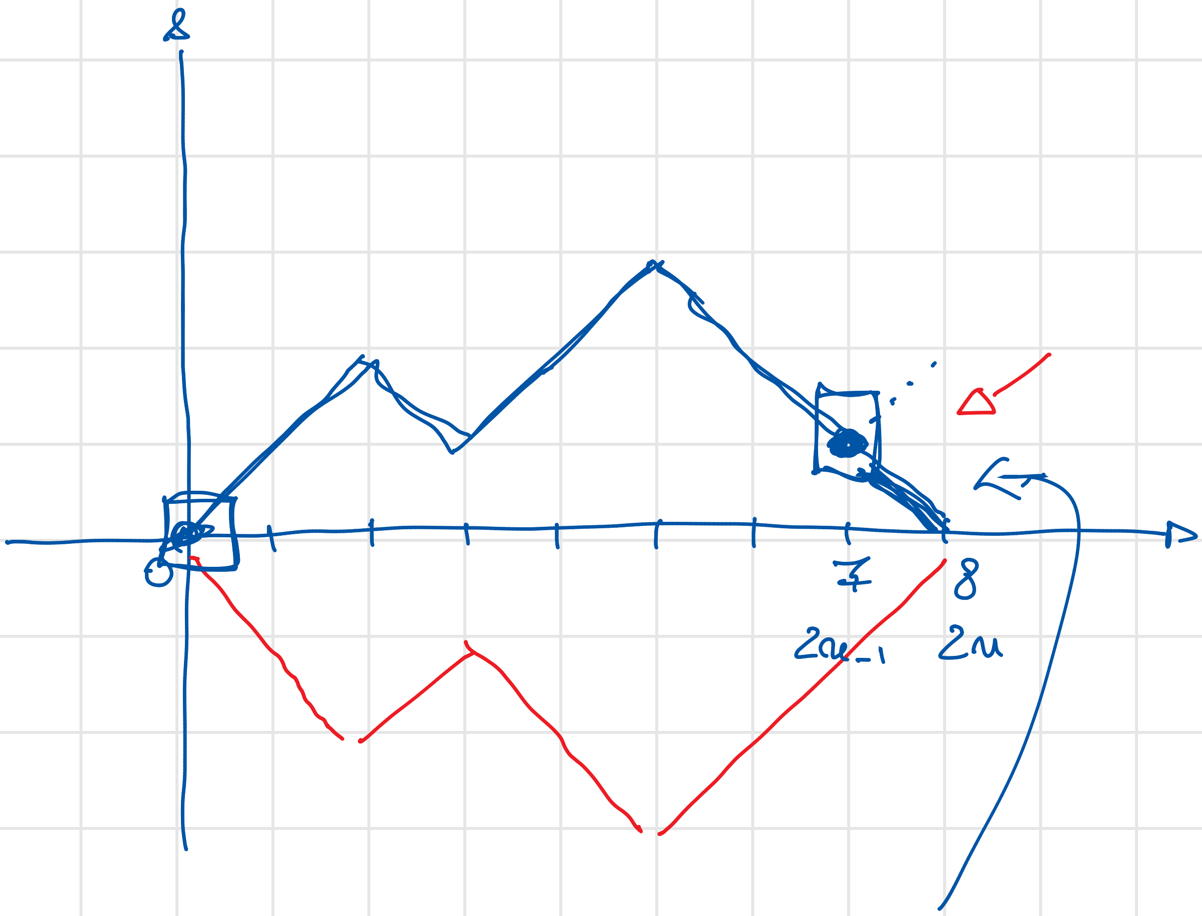
$\boxed{\mathbb{P}[T_0 = 2n] = ?}$

$[T_0 = 2n]$

$\underbrace{p^n q^n}$

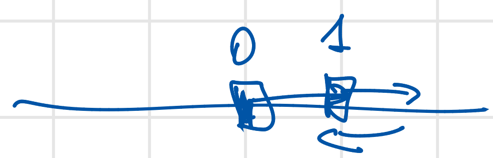
$\underbrace{\mathbb{P}[S_{2n} = 0]} = \binom{2n}{n} \cdot p^n \cdot q^n$

n (+1) among the 2n steps



$n=4 \qquad T_{2n} = T_8$

$2 \; p^4 q^4$



$(0,0) \longrightarrow (8,0)$ without touching 0 before the 8^{th} step

$(0,0) \longrightarrow (7,1)$

$A = \{$ all paths that join $(0,0)$ with $(2n-1,1)\}$

$B = \{$ " " " " " " " without touching 0 $\}$

$A = B \cup C \cup D$ disjoint $\Rightarrow |A| = |B| + |C| + |D|$

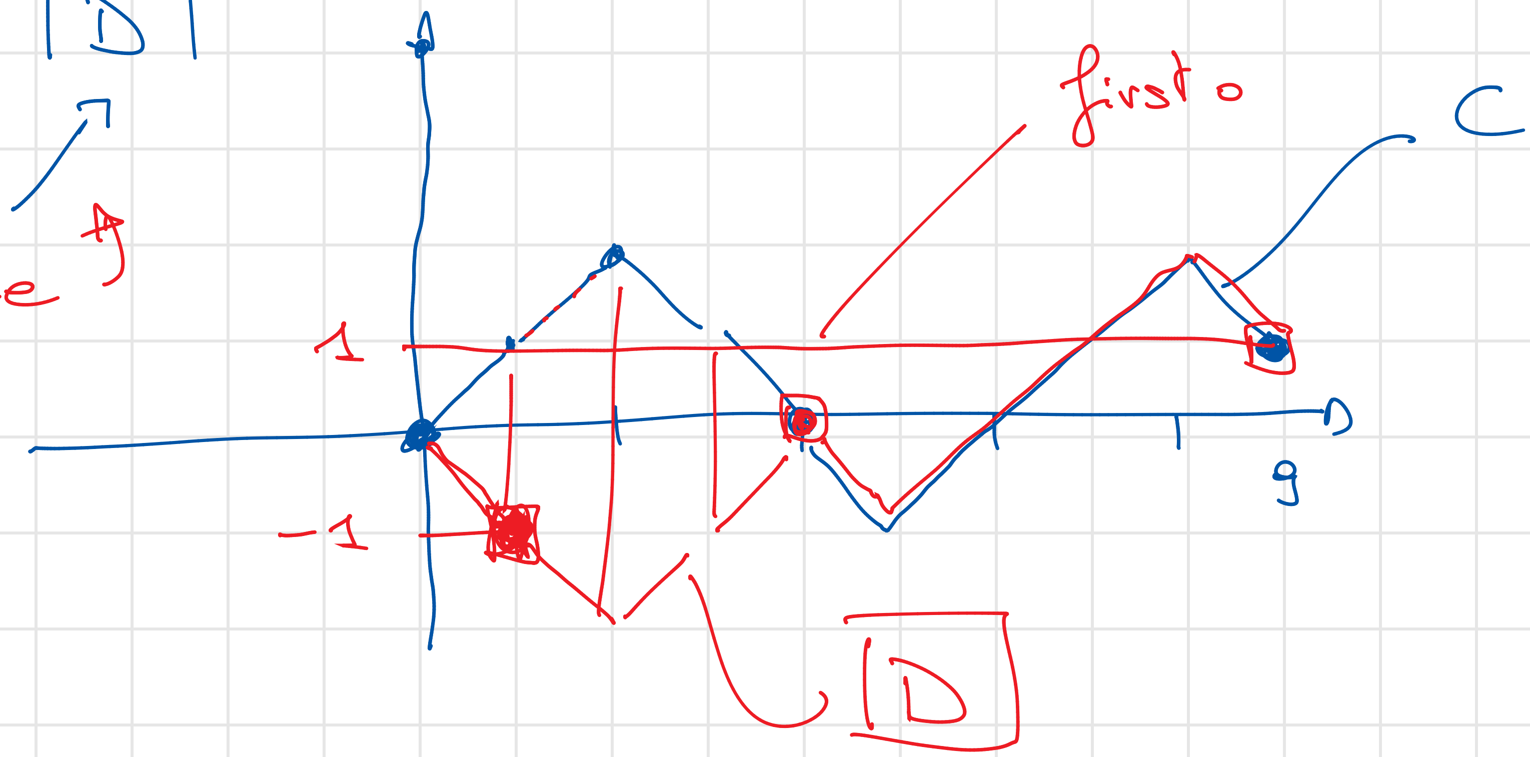
$|A| = \binom{2n-1}{n}$

$C = \{$ all the paths in $A \setminus B$ s.t. $\underbrace{X_1 = 1}\}$

$D = \{$ all the paths in $A \setminus B$ s.t. $\underbrace{X_1 = -1}\}$

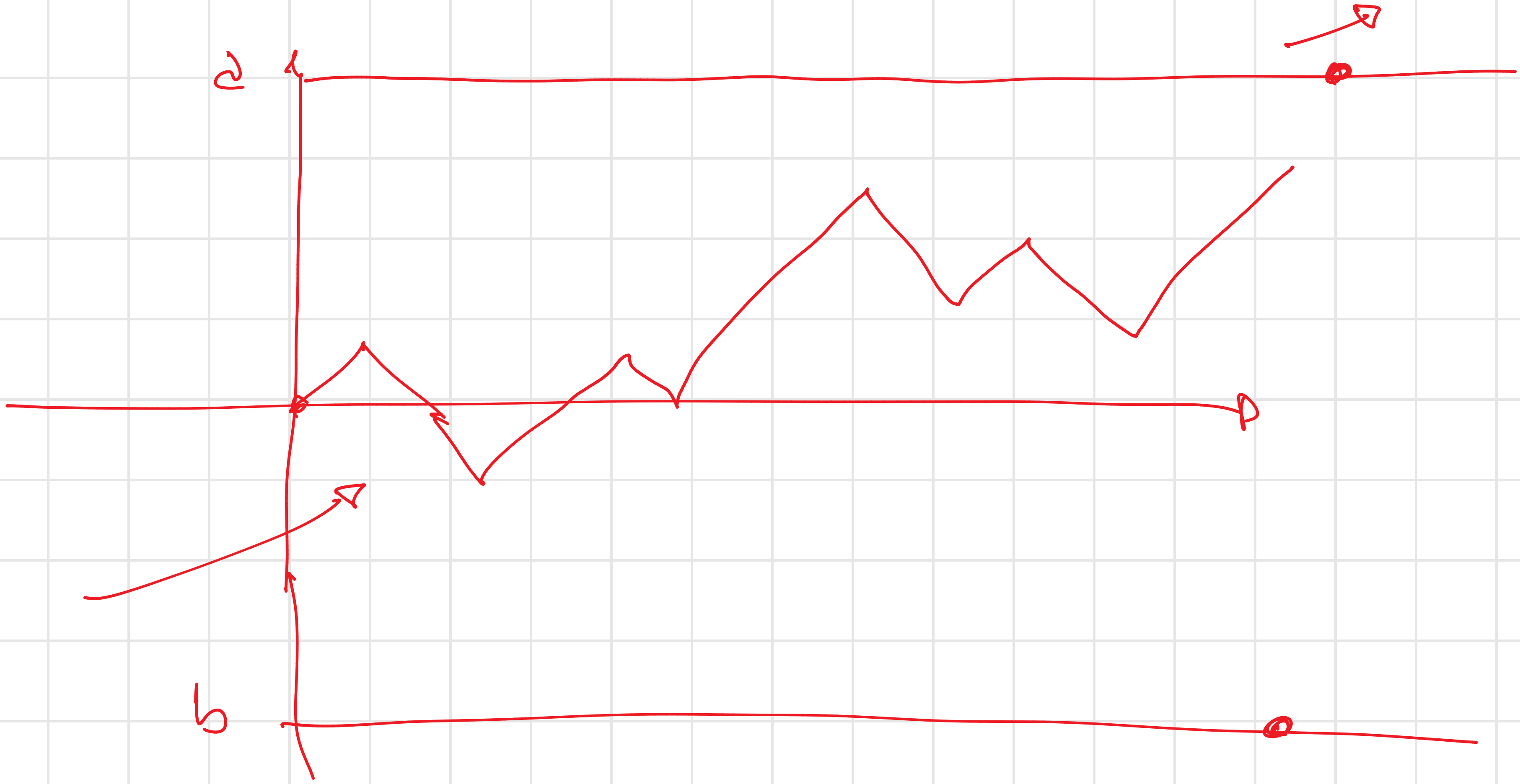
$|C| = |D|$

true



$D = \{ \text{paths from } (1,-1) \text{ to } (2n-1, 1) \}$

$|D|$ is easy . . . . $\mathbb{P}[T_0 = 2n] = \dfrac{\mathbb{P}[S_{2n} = 0]}{2n-1}$



Chpt. 4 , Section 4

$\boxed{\text{Thm } 4.2}$: In a positive recurrent, aperiodic class with states $j = 0, 1, 2, \ldots$

$$\boxed{\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j} = \sum_{i=0}^{\infty} \pi_i P_{ij},$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

and the π's are uniquely determined by the set of equations

$$\boxed{\pi_i \geq 0, \quad \sum_{i=0}^{\infty} \pi_i = 1 \quad \text{and} \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad j = 0, 1, 2, \ldots}$$

By Thm 4.1 $\boxed{\pi_i = \frac{1}{m_i}}$ $\boxed{m_i = E[R_i \mid X_0 = i]}$

$m_i < +\infty$ (positive rec.)

$\pi_i > 0$

(invariant)

$\pi = (\pi_0, \pi_1, \ldots)$ this distribution is called a stationary probability distribution for the MC if $\boxed{\mathbb{P}[X_0 = i] = \pi_i}$

$$\mathbb{P}[X_1 = i] \overset{\text{by Markov property}}{=} \sum_{k=0}^{\infty} \mathbb{P}[X_0 = k] \cdot \mathbb{P}[X_1 = i \mid X_0 = k]$$

$$= \sum_{k=0}^{\infty} \pi_k P_{ki} = \pi_i$$

$\forall i = 0, 1, 2, \dots$

and in general $\boxed{\mathbb{P}[X_n = i] = \pi_i}$ $i = 0, 1, 2, \dots$ $\forall n \geq 0$

Remark : $Y_n = (X_n, X_{n+1})$ $(X_n, n \geq 0)$ is a MC

Prove that $Y_n, n \geq 0$ is a Markov Chain

if $\mathbb{P}[X_0 = i] = \pi_i$, $i = 0, 1, 2, \dots$ invariant distr.

$$\mathbb{P}[Y_n = (i, j)] = \mathbb{P}[X_n = i, X_{n+1} = j]$$

$$= \underbrace{\pi_i P_{ij}}$$ $\boxed{\forall n \geq 0}$

Ex: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ irreducible, periodic (2) MC

$$P^2 = Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{cases} P^{2n} = Id & \forall n \\ P^{2n-1} = P & \forall n \end{cases}$$

$$\lim_{n\to\infty} P^n_{ii} \nexists$$

$$(\pi_0, \pi_1) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\pi = \pi P}$$

$$\begin{cases} \pi_0 = \pi_1 \\ \pi_1 = \pi_0 \\ \pi_0 + \pi_1 = 1 \end{cases} \to \boxed{\pi_0 = \pi_1 = 1/2}$$

$(1/2, 1/2)$ is the unique invariant distribution to the MC

Invariant or Stationary distribution

Definition: A distribution π and a transition matrix P are in detailed balance if

$$\boxed{\pi_i P_{ij} = \pi_j P_{ji} \qquad \forall\, i,j \in S}$$

Remark: we have to check only the cases $i \neq j$

Proposition: If P and π are in detailed balance, then π is invariant for P.

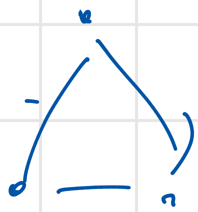
Proof:

$$(\pi \cdot P)_i = \sum_{j\in S} \pi_j P_{ji} \overset{DB}{=} \sum_{j\in S} \pi_i P_{ij}$$

$$= \pi_i \underbrace{\sum_{j\in S} P_{ij}}_{=1} = \pi_i \qquad \blacksquare$$

Ex 1:

$$\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

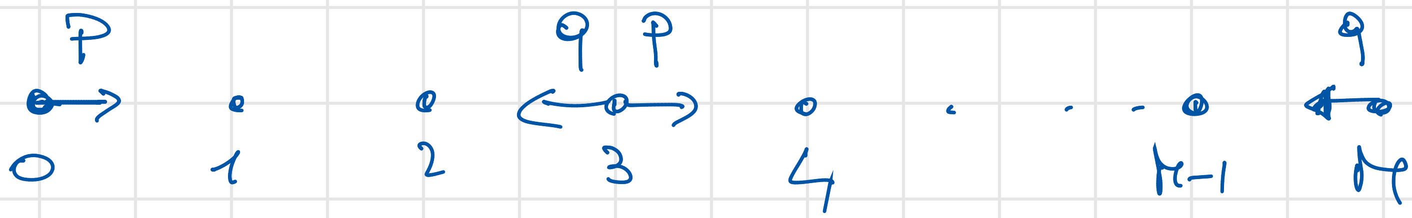


doubly stochastic

$$\boxed{\pi_0 = \pi_1 = \pi_2 = 1/3}$$

$(1/3, 1/3, 1/3)$

Ex 2:



$S = \{0, 1, \dots, M\}$ RW on S

$0 < p = 1 - q < 1$

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ M-1 \\ M \end{array} \begin{bmatrix} q & p & 0 & \cdot & \cdot & - & - & \cdot & 0 \\ q & 0 & p & \cdot & - & - & - & \cdot & 0 \\ 0 & q & 0 & p & \cdot & \cdot & - & & \cdot \\ & & & & & & & & \\ & & & & & & & & \\ \cdot & - & - & - & - & q & 0 & p & \\ 0 & - & - & - & - & 0 & q & p & \end{bmatrix}$$

Find the invariant distribution!

Detailed balance equation

$$\boxed{\pi_i \cdot P_{i,i+1} = \pi_{i+1} P_{i+1,i} \qquad \forall\, i = 0, 1, \dots, M-1}$$

$$P_{i,i+1} \equiv p \quad , \quad P_{i+1,i} \equiv q$$

$$\pi_{i+1} = \frac{p}{q} \cdot \pi_i = \frac{p^2}{q^2} \pi_{i-1} = \dots$$

$$\boxed{\pi_i = \left(\frac{p}{q}\right)^i \pi_0 , \quad i = 0, \dots, M}$$

one solution of $\pi = \pi P$

$$\sum_{i=0}^{M} \pi_i = 1$$

$$\pi_0 + \left(\frac{p}{q}\right) \pi_0 + \dots + \left(\frac{p}{q}\right)^M \pi_0 = 1$$

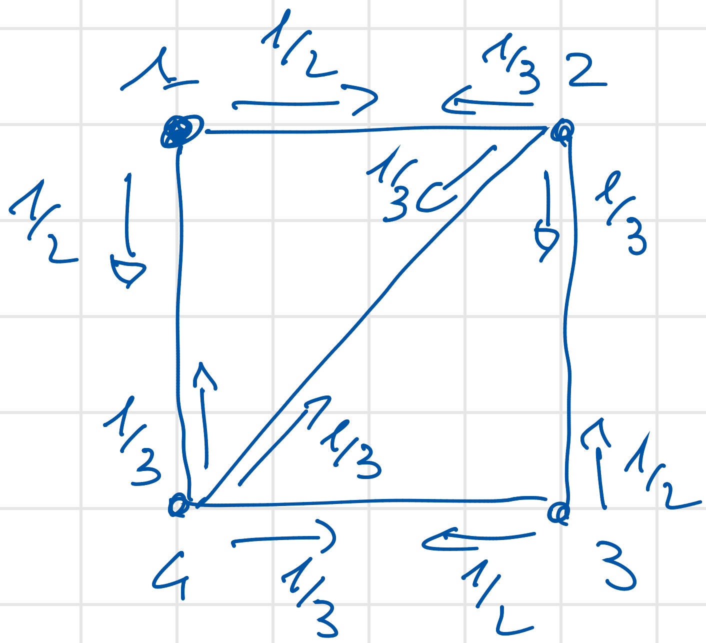
$$\pi_0 \left(1 + p/q + \dots + \left(\frac{p}{q}\right)^M\right) = 1$$

$$\pi_0 = \frac{1}{1 + p/q + \dots \left(p/q\right)^M} = \frac{1 - p/q}{1 - \left(p/q\right)^{M+1}}$$

Example page 248 + Example page 249

$\boxed{\text{Lemma 4.1}}$ ← read!

Random Walk on a Graph



4 nodes
5 edges

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

$$\boxed{\pi = \pi P}$$

invariant distribution

$$\pi_i P_{ij} = \pi_j P_{ji}$$

v_i = # of neighbors nodes to i

$$P_{ij} = \begin{cases} 0 & i,j \text{ are not neighb.} \\ 1/v_i & i,j \text{ are neighb.} \end{cases}$$

$\pi_i P_{ij} = \pi_j P_{ji}$ if and only if

i and j are neighbors

$$\boxed{\pi_i \cdot \frac{1}{\sigma_i} = \pi_j \cdot \frac{1}{\sigma_j}}$$

$$\boxed{\sigma = \sum_{i \in S} \sigma_i}$$

$$\boxed{\pi_i = \frac{\sigma_i}{\sigma}}$$

$\pi_i > 0$, $\sum_i \pi_i = 1$

ok

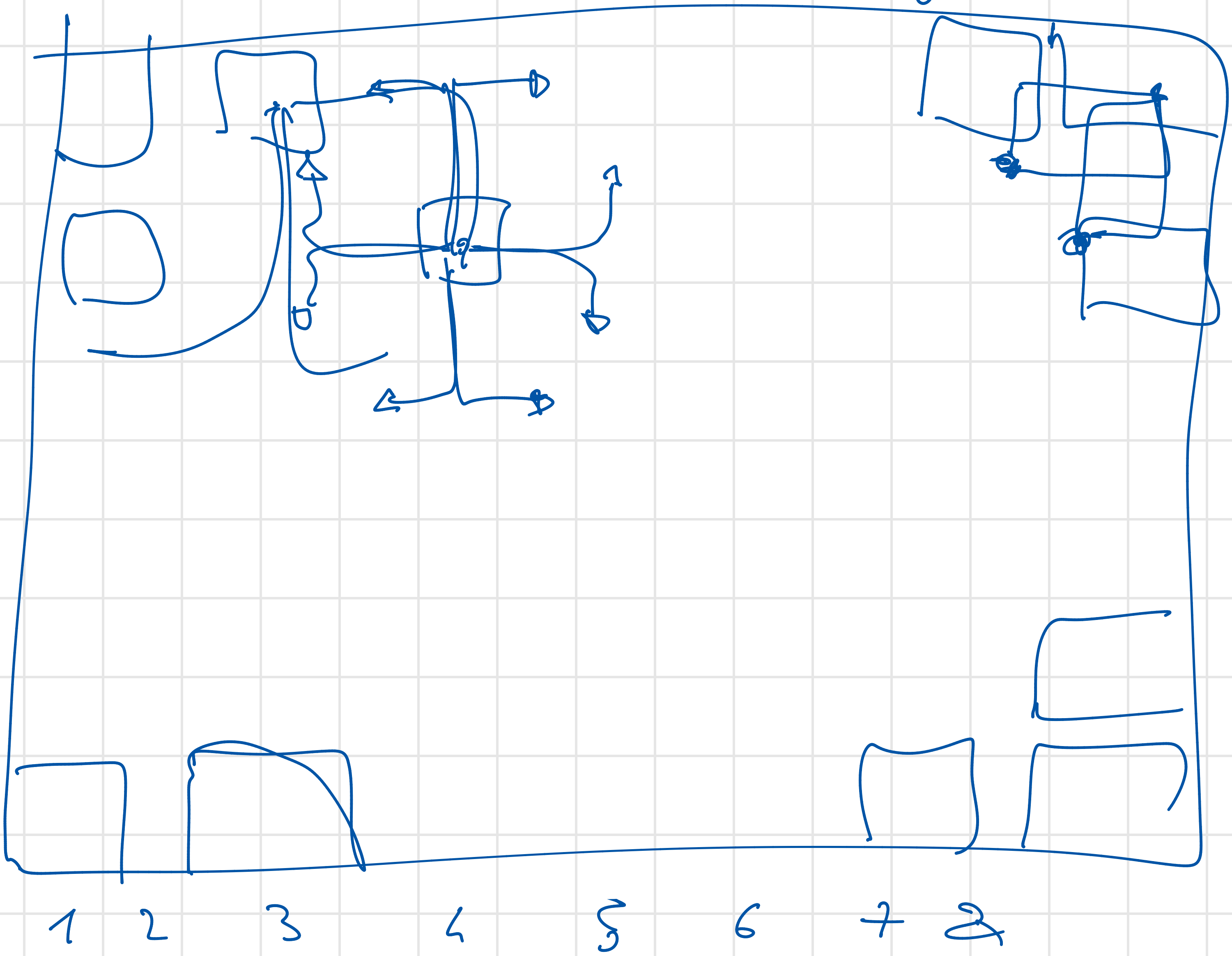
$$\frac{\sigma_i}{\sigma} \cdot \frac{1}{\sigma_i} = \frac{1}{\sigma} = \frac{\sigma_j}{\sigma} \cdot \frac{1}{\sigma_j} = \frac{1}{\sigma}$$

If the Graph is connected $\Rightarrow$ $\pi_i = \frac{\sigma_i}{\sigma}$, $i \in S$

is the unique invariant distribution

Exercise

Chess Knight moves randomly



$\boxed{168}$

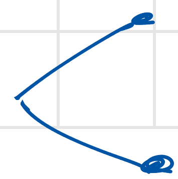
(i, j)

On average how many moves of this random Knight are needed starting from $(1,1)$ to go back to $(1,1)$.

$\boxed{\frac{d_i}{d}}$

RW on a Graph

$(1,1)$



$\boxed{d_i}$