Stochastic Processes — Lecture 1

CYS 2020/2021

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- Short review of Probability theory
  - Probability spaces $(\Omega, 2^{\Omega}, \mathbb{P})$
  - Conditional probability
  - Random variables
  - Moments and Expected values
  - Random vectors

A deterministic model predicts a single outcome from a given set of circumstances.

A stochastic model predicts a set of possible outcomes weighted by their likelihoods, or probabilities.

Probability space

$(\Omega, 2^{\Omega}, \mathbb{P})$ $\quad\quad \Omega$ = sample space

Ω = set of all the possible outcomes

- Toss a coin $\quad \Omega = \{T, H\}$
- Roll a die $\quad \Omega = \{1, 2, 3, 4, 5, 6\}$
- Roll 2 dice $\quad \Omega = \{ \underbrace{(i, j)}_{\text{RED BLUE}} : i, j \in \{1, \dots, 6\} \}$

(false) $\underline{\mathbb{P} : \Omega \longrightarrow [0,1]}$

what is the prob. to have an even number

$E = \{\text{even number}\}$ "event"

$= \{2, 4, 6\} \subseteq \Omega$

The events will be (in general) all the

subsets of Ω

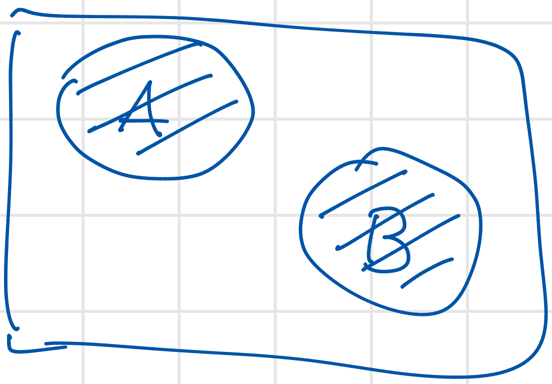
$$2^{\Omega} = \text{power set of } \Omega = \{A : A \subseteq \Omega\}$$

$$\mathbb{P} : 2^{\Omega} \longrightarrow [0,1]$$

1. $\mathbb{P}[\Omega] = 1$

2. $A_i \in 2^{\Omega}$ s.t. $A_i \cap A_j = \emptyset \quad \forall i \neq j$

$$\mathbb{P}\Big[\bigcup_i A_i\Big] = \sum_i \mathbb{P}[A_i]$$



Ω

Venn diagram

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$\rightarrow \boxed{A \cap B}$

$\rightarrow \boxed{A \cup B}$

$\rightarrow \boxed{A^c}$

$A \setminus B$

the event A occurs

if the outcome

of my experiment

$(\omega \in \Omega)$ $\omega \in A$

Properties of a probability

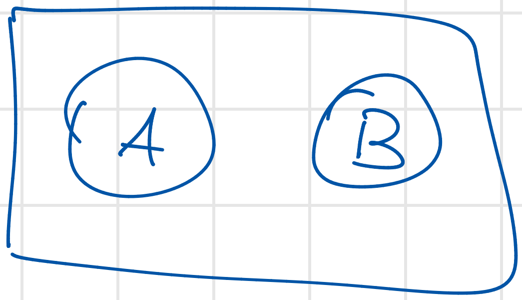
- $\mathbb{P}[\emptyset] = 0$ $\quad\quad$ $\emptyset$ = impossible event

  Ω = certain event

- $\boxed{A \subseteq B} \Rightarrow \mathbb{P}[A] \le \mathbb{P}[B]$

  <u>monotonicity of the probability</u>

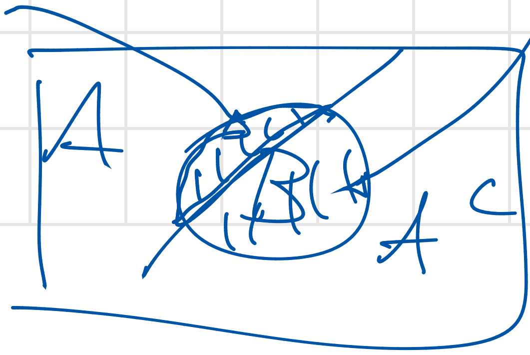
- $A \cap B = \emptyset$



$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$

- $A, B \in 2^{\Omega}$

$\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[A^c \cap B]$



$A_1, A_2, \dots$ family of events s.t.

partition of Ω

$$\bigcup_i A_i = \Omega \,, \quad A_i \cap A_j = \emptyset \quad \forall i \neq j$$

$$\boxed{\mathbb{P}[B] = \sum_j \mathbb{P}[A_j \cap B]}$$

Example:

2 red balls
3 black balls

$R_1 = \{$ the first ball drawn is red $\}$

$R_2 = \{$ the second " " " " $\}$

without replacement.

$$\mathbb{P}[R_1] = \frac{2}{5}$$

$$\boxed{\begin{aligned} \mathbb{P}[R_2] &= \\ &= \mathbb{P}[R_2 \cap R_1] + \\ &\quad \mathbb{P}[R_2 \cap R_1^c] \end{aligned}}$$

Conditional probability

$A, B \in \Omega$, $\quad \mathbb{P}[B] > 0$

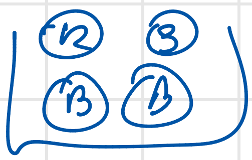
$$\underbrace{\mathbb{P}[A \mid B]}_{\text{conditional probability of } A \text{ given } B} := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

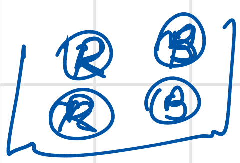
The probability that we assign to A once we know that B occurs.

"information"

Example box

$$\mathbb{P}[R_2 \cap R_1] = \underbrace{\mathbb{P}[R_2 \mid R_1]}_{1/4} \cdot \underbrace{\mathbb{P}[R_1]}_{2/5}$$





$$\mathbb{P}[R_2 \cap R_1^c] = \underbrace{\mathbb{P}[R_2 \mid R_1^c]}_{2/4} \cdot \mathbb{P}[R_1^c] \quad \text{"}3/5$$

$$\mathbb{P}[R_2] = \frac{2}{5}\cdot\frac{1}{4} + \frac{3}{5}\cdot\frac{2}{4} = \frac{2+6}{5\cdot 4} = \frac{8}{5\cdot 4} = \frac{2}{5}$$

$$\mathbb{P}[R_2] = \mathbb{P}[R_1] \quad \text{(surprise)}$$

Random Variables

Random function

$$\boxed{X : \Omega \longrightarrow \begin{matrix}\mathbb{R}\\ \mathbb{R}^n\end{matrix}}$$

$$\boxed{X \in \mathbb{R}}\ , \quad X \in \mathbb{R}^n$$

$$\Omega = \{T, H\}$$

Events:

$$\{ X \leq x \} \qquad x \in \mathbb{R}$$

prob. space

$$X : \Omega \to \mathbb{R}$$

$$\{ X \leq x \} = \underbrace{\{ \omega \in \Omega : X(\omega) \leq x \}}_{\subseteq \ \underbrace{\Omega}}$$

$$\{ X \in A \} \qquad A \subseteq \mathbb{R}$$

$$\{ \omega : X(\omega) \in A \} \subseteq \Omega$$

$$\boxed{\mathbb{P}[X \leq x]} \ , \ \boxed{\mathbb{P}[X \in A]} \qquad A \subseteq \mathbb{R}$$

$$x \longmapsto \mathbb{P}[X \leq x] = F(x)$$

$x \in \mathbb{R}$

$\boxed{\text{distribution function}}$

Discrete random variables

Continuous " "

Discrete RV

$X : \Omega \longrightarrow \mathbb{R}$

$\exists$ finite or at most countable set of values $x_1, x_2, \dots \in \mathbb{R}$ s.t.

$$a_i = \mathbb{P}[X = x_i]$$

- $a_i > 0 \quad \forall i = 1, 2, \dots$
- $\sum_i a_i = 1$

$\boxed{\text{probability mass function}}$

$$p(x_i) = \mathbb{P}[X = x_i] = a_i \qquad i = 1, 2, \dots$$

$$\boxed{\mathbb{P}[X \in A] = \sum_{x_i \in A} p(x_i)}$$

$A \subseteq \mathbb{R}$

$$F(x) = \mathbb{P}[X \le x] = \sum_{x_i \le x} p(x_i)$$

Some properties of distribution function:

$$\mathbb{P}[X > x] = 1 - F(x)$$

$$F(x) = \mathbb{P}[X \le x]$$

$$\{X > x\} \cap \{X \le x\} = \emptyset$$

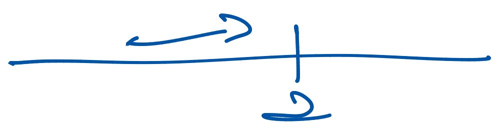
$$\{X > x\} \cup \{X \le x\} = \Omega$$

$$1 = \mathbb{P}[\Omega] = \mathbb{P}[\{X > x\} \cup \{X \le x\}] = \mathbb{P}[X > x] + \mathbb{P}[X \le x]$$

$$\mathbb{P}[a < X \le b] = F(b) - F(a)$$

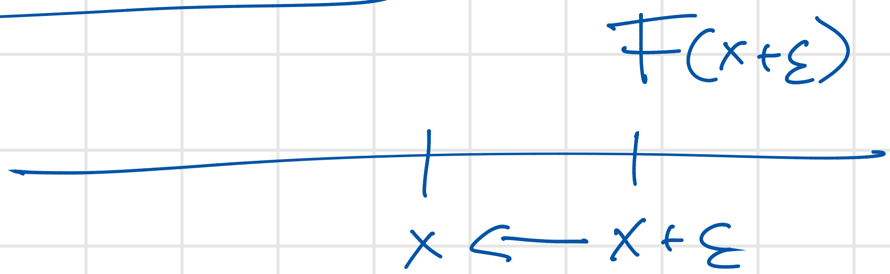
$$\mathbb{P}[a \le X \le b] = F(b) - F(a-)$$

$$\lim_{\varepsilon \downarrow 0} F(a - \varepsilon) = F(a-)$$



F is right continuous

$$\boxed{F(x) = \lim_{\varepsilon \downarrow 0} F(x+\varepsilon)} =: F(x+)$$



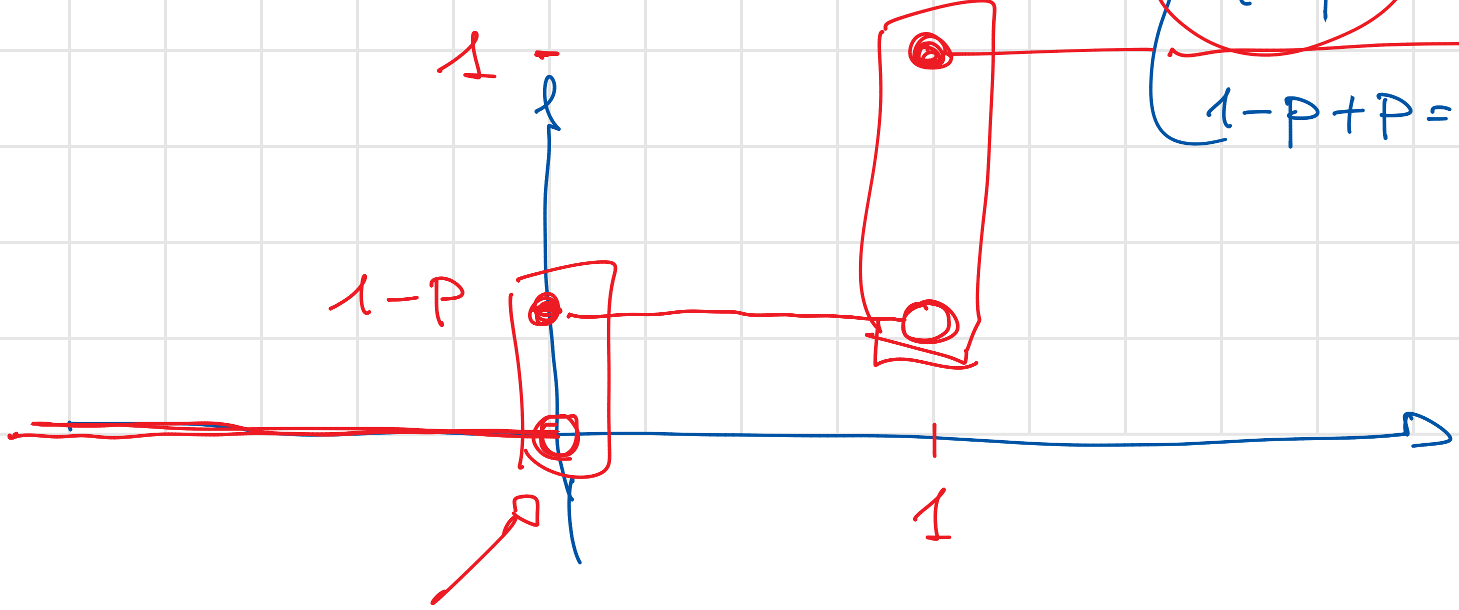
$$\boxed{F(x-) \le F(x)}$$

<u>Example</u> : $X : \Omega \to \mathbb{R}$

$X = \underline{0}$ $1-p$ $p(0) = 1-p$

$X = \underline{1}$ p $p(1) = p$

$$F(x) = \mathbb{P}[X \le x] = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1-p+p = 1 & x \ge 1 \end{cases}$$



Continuous r.v.

$$\mathbb{P}[X=x] = \underbrace{F(x) - F(x-)}$$

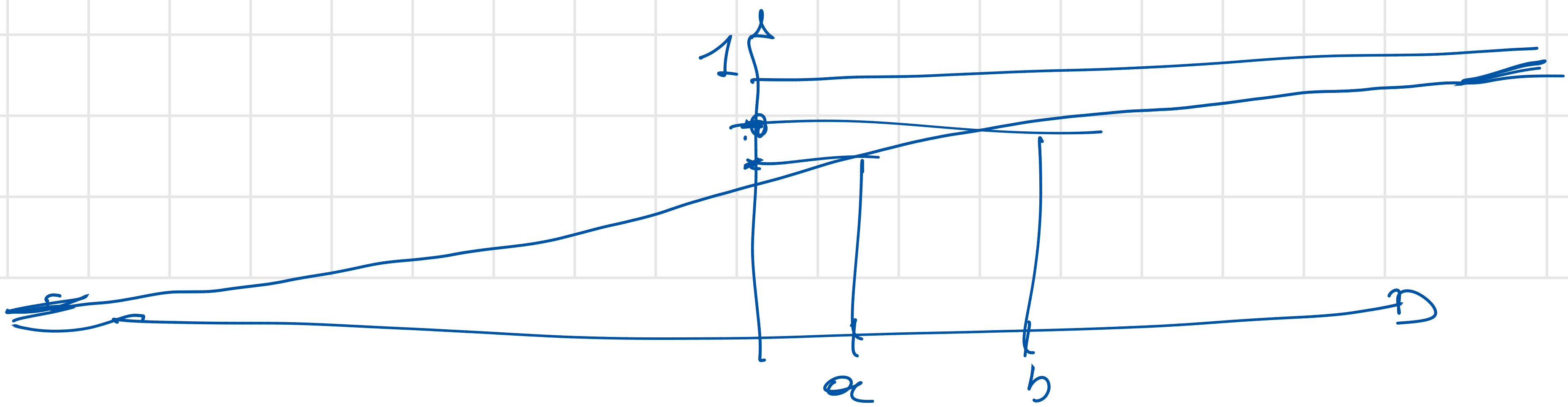
Jump of the function F in x

X is cont. $\iff \mathbb{P}[X=x]=0 \quad \forall x \in \mathbb{R}$

$\iff F$ is a continuous function

It is concentrated on the intervals

$$\mathbb{P}[X \in (a,b]] = F(b) - F(a)$$



There exists a function

$$f : \mathbb{R} \to \mathbb{R}_+$$

that we call density of the r.v. X

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

(absolutely) continuous function

2.3 , 2.4

$$\{X = x\}$$