Stochastic Processes

Lecture 2

CYS 2020/2021

Probability Review (second part)

- Major discrete distributions

  Bernoulli - Binomial - Geometric - Poisson

- Major continuous distributions

  Normal - Exponential - Uniform - Gamma

- Conditional distributions:

  discrete case, continuous case

  random sums and their moments

$A, B \subseteq \Omega$ $\boxed{\mathbb{P}[B] > 0}$

$$\mathbb{P}[A \mid B] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

A and B are independent

$$\Leftrightarrow \mathbb{P}[A \mid B] = \mathbb{P}[A]$$

$$\Leftrightarrow \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \mathbb{P}[A]$$

$$\Leftrightarrow \boxed{\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]}$$

A, B, C they are indep. iff.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$
$$\mathbb{P}[B \cap C] = \mathbb{P}[B] \cdot \mathbb{P}[C]$$
$$\mathbb{P}[A \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[C]$$

$$\rightarrow \mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[B] \cdot \mathbb{P}[C]$$

Random Vectors

$$X = (X_1, \dots, X_n) : \Omega \longrightarrow \mathbb{R}^n$$

$n = 2$

$$(X_1, X_2)$$

Joint distribution function

$$F_{X_1,X_2}(x_1,x_2) = \mathbb{P}[\underbrace{X_1 \le x_1}_{\subseteq \, \Omega} \overset{\cap}{,} \underbrace{X_2 \le x_2}_{\subseteq \, \Omega}]$$

- discrete random vectors

$$p(x_1,x_2) := \mathbb{P}[X_1 = x_1, X_2 = x_2]$$

Joint prob. mass function

there exist a finite or countable set of values $(x_1,x_2) \in \mathbb{R}^2$ s.t.

- $p(x_1,x_2) > 0$
- $\sum_{x_1,x_2 \in \mathbb{R}^2} p(x_1,x_2) = 1$

- continuous random vectors $(n=2)$

$$X = (X_1, X_2)$$

$$F_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X,Y}(s,t)\, ds\, dt$$

$f_{X,Y}$ - joint distribution density

$$f_{X,Y} \geq 0 \quad , \quad \iint_{\mathbb{R}^2} f_{X,Y}(s,t)\, ds\, dt = 1$$

In both cases, from the joint distribution we can derive the marginal distributions of the single components of $X = (X_1, X_2, \dots, X_n)$

$X = (X_1, X_2)$

$$\underline{F_{X_1}(x_1)} = \lim_{x_2 \uparrow +\infty} F_{X_1,X_2}(x_1,x_2)$$

$$F_{X_2}(x_2) = \lim_{x_1 \uparrow +\infty} F_{X_1,X_2}(x_1,x_2)$$

joint mass function $\quad P_{X_1}(x_1) = \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2)$

joint density function (continuous r.v.)

$$f_{X_2}(x_2) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1$$

marginal density

X and Y are INDEPENDENT?

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$$

$$\boxed{\begin{aligned} F_{X,Y}(x,y) &= \mathbb{P}[\underbrace{X \le x}, \underbrace{Y \le y}] \\ &= \mathbb{P}[X \le x] \cdot \mathbb{P}[Y \le y] \end{aligned}}$$

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) \quad \forall x,y \in \mathbb{R}$$

discrete case this is equivalent to ask

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)}$$

while in the continuous case the indep. will be equivalent to

$$f_{X,Y}(x,y) = \underbrace{f_X(x) \cdot f_Y(y)}$$

Moments:

discrete r.v. the n-th moment

$$\mathbb{E}[X^n] := \sum_i \underbrace{x_i}^{n} \cdot \mathbb{P}[X=x_i]$$

expectation

cont. case

$$\mathbb{E}[X^n] := \int_{-\infty}^{+\infty} x^n f(x)\,dx$$

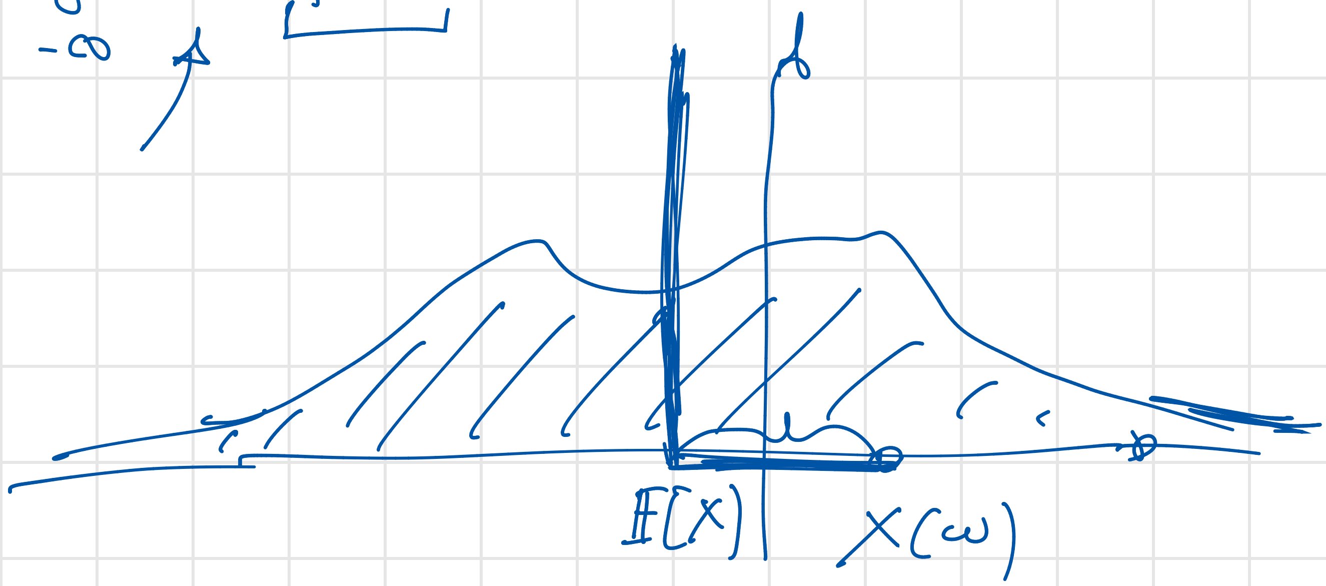
the infinite sum (the integral) converges absolutely.

$$\sum_i |x_i|^n \,\mathbb{P}[X=x_i] < +\infty \quad , \quad \int_{-\infty}^{+\infty} |x|^n f(x)\,dx < +\infty$$

$m=1$ The expectation (mean)

$$\mathbb{E}[X] = \sum_i x_i \, p(x_i)$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \, \underbrace{f(x)}_{} dx$$



<u>Variance</u> $\quad Var[X] = \mathbb{E}\left[\left(X - \mathbb{E}[X] \right)^2 \right]$

2nd - central moment

Discrete-case

- Bernoulli r.v.
- Binomial r.v.
- Geometric r.v.
- Poisson r.v.

Bernoulli $\boxed{p \in [0,1]}$

$X : \Omega \to \mathbb{R}$

$\{0,1\}$

$\mathbb{P}[X=0] = 1-p = p(0)$
$\mathbb{P}[X=1] = p = p(1)$

$X=1 \iff$ we win

$X=0 \iff$ we lose

$$\mathbb{E}[X] = \sum_i x_i \, \mathbb{P}[X=x_i] = 0 \cdot p(0) + 1 \cdot p(1)$$
$$= p$$

$$Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

always true $\equiv \underbrace{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$

$$\mathbb{E}[g(X)] = \begin{cases} \sum_i g(x_i) \cdot \mathbb{P}[X = x_i] \\ \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx \end{cases}$$

Bernoulli

$$\mathbb{E}[X^2] = 0^2 \cdot p(0) + 1^2 \cdot p(1)$$

$$= p$$

$$Var[X] = p - (p)^2 = p - p^2 = p(1-p)$$

- Binomial r.v.

Bernoulli : $A \in \Omega$

$\mathbb{P}[A] = p$

X Bernoulli r.v.

$X = \mathbb{1}_A$ indicator function

$$\mathbb{1}_A : \Omega \to \mathbb{R}$$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

(Prove)

$$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A] = p$$

$A_1, A_2, \dots, A_n \in \Omega$, independent and s.t. $\mathbb{P}[A_i] = p \quad \forall i = 1, \dots, n$.

Y = Binomial random variable of parameters p and n

Y counts the total number of events among $A_1, \dots, A_n$ that occur.

$$\{0, 1, 2, \dots, n\} \ni Y$$

$$\mathbb{P}[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$k \in \{0, 1, \dots, n\}$

$$\underbrace{A_1, A_2, \dots, A_n}$$ exactly k occurs.

<u>Binomial coeff.</u> $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1$$

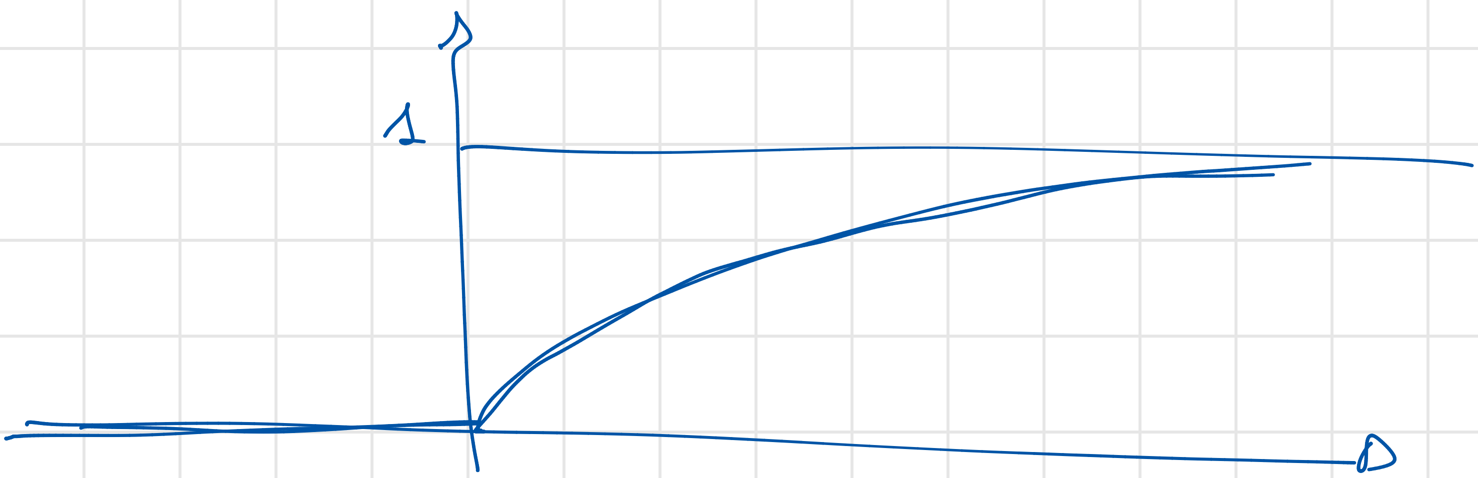
Continuous r.v.

- Normal

- Exponential $\qquad \lambda \in (0,+\infty)$

X is an exponential r.v. with parameter λ

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$



$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\boxed{F_X = \int f} \longleftrightarrow \boxed{f = \frac{d}{dx} F}$$

$E[X] = 1/\lambda$, $Var[X] = 1/\lambda^2$

Memoryless property

$s, t > 0$

$$\mathbb{P}[X > s + t \mid \underbrace{X > t}] = \mathbb{P}[X > s]$$

$$\frac{\mathbb{P}[\underbrace{X > s+t}, \underbrace{X > t}]}{\mathbb{P}[X > t]} = \frac{\mathbb{P}[X > s+t]}{\mathbb{P}[X > t]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$(X > s+t) \subseteq (X > t)$$

$$= e^{-\lambda s - \cancel{\lambda t} + \cancel{\lambda t}}$$

$$= e^{-\lambda s} = \underbrace{\mathbb{P}[X > s]}$$

$\boxed{x > 0}$

$$F(x) = 1 - e^{-\lambda x} = \mathbb{P}[X \leq x]$$

$$\boxed{\mathbb{P}[X > x] = 1 - \mathbb{P}[X \leq x] = e^{-\lambda x}}$$

The exponential distribution is the only random variable that satisfies the memoryless property !!

$$\boxed{\mathbb{P}[X > s+t] = \mathbb{P}[X > s] \cdot \mathbb{P}[X > t]}$$

Euler equation

$$g(s+t) = g(s) \cdot g(t) \qquad s,t > 0$$

$$g(s) = \boxed{e^{-\lambda s}}$$

- Conditional probability and conditional expectation.

$$\boxed{\mathbb{P}[A|B] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]}}$$

(X,Y) joint discrete r. vector

$$P_{X,Y}(x,y) = \mathbb{P}[X=x, Y=y]$$

joint mass function

$$P_{X|Y}(x|y) = \frac{\mathbb{P}[X=x, Y=y]}{\mathbb{P}[Y=y]} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

conditional probability mass function if $P_Y(y) \neq 0$

fixed y s.t. $\underbrace{P_Y(y) > 0}$

$x \longmapsto P_{X|Y}(x|y)$ is a prob. mass function

$$\mathbb{E}[g(X) \mid Y = y] := \sum_{x} g(x)\, P_{X|Y}(x|y)$$

$$\mathbb{E}[g(X)] = \sum_{y} \mathbb{E}[g(X) \mid Y = y] \cdot P_Y(y)$$

$$\| $$

$$\boxed{\mathbb{E}\big[\, \mathbb{E}[g(X) \mid Y] \,\big]}$$

$$h(y) = \mathbb{E}[g(X) \mid Y = y]$$

$$\mathbb{E}[g(X) \mid Y] := h(Y)$$

Example X binomial r.v. with parameter p and N $\quad X \sim \mathrm{Bin}(p, N)$

$$N \sim \mathrm{Bin}(q, M)$$

$$\Rightarrow \underbrace{P_{X|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}}$$

$k = 0, 1, \dots, n$

$n \in 0, \dots, M$

$$\Rightarrow \underbrace{P_N(n) = \binom{M}{n} q^n (1-q)^{M-n}}$$

$$\mathbb{P}[X=k] = \sum_n \mathbb{P}[X=k, N=n] = \underbrace{\sum_{n \in \mathbb{N}} P_{X|N}(x|n) P_N(n)}$$

marginal distribution

$$\underbrace{P_{X|N}(x|n)} = \frac{P_{X,N}(x,n)}{\underbrace{P_N(n)}}$$

$$\boxed{P_{X,N}(x,n) = P_{X|N}(x|n) \cdot P_N(n)}$$

pag. 58-59

$$= \sum_{n=0}^{M} P_{X|N}(x|n) \cdot P_N(n)$$

$$= \sum_{n=0}^{M} \binom{n}{k} p^k (1-p)^{n-k} \cdot \binom{M}{n} q^n (1-q)^{M-n}$$

$$= \cdots\cdots = \binom{M}{k} (pq)^k (1-pq)^{M-k}$$

$k=0,\dots,M$

$$X \sim Bn(pq, M)$$

$$\underbrace{\mathbb{P}[X_1=x_1, X_2=x_2]} = \mathbb{P}[X_2=x_2 \mid X_1=x_1] \cdot \mathbb{P}[X_1=x_1]$$

$$1 \to 2 \to 3 \to 4$$

$$\mathbb{P}[X_1=x_1, X_2=x_2, X_3=x_3] = \mathbb{P}[X_1=x_1] \cdot \mathbb{P}[X_2=x_2 \mid X_1=1]$$

$$\cdot\ \mathbb{P}[X_3=x_3 \mid X_1=x_1, X_2=x_2]$$