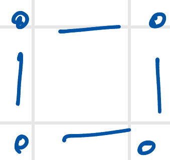
Stochastic Processes

Lecture 7

CYS 2020/2021

- Regular transition matrices — MATRICE TRANSIZIONE Regolare
- Limiting prob. distributions
- Theorem 1.1 (with proof)
- Some Examples
- Classification of the states
  - $i \leftrightarrow j$ communicate
  - class properties
  - periodicity 
  - recurrent and transient states

- Limiting probability distribution

stati
$$\underbrace{S = \{0, \dots, N\}}$$

$$\pi = (\pi_0, \dots, \pi_N), \quad \pi_j > 0 \quad \forall j \in S$$

$$\sum_{j \in S} \pi_j = 1 \rightarrow \boxed{\text{LA SOMMA} = 1}$$

matrice transizione

- $\underline{P}$ is a transition matrix $\boxed{\text{Regolare} \atop \text{regular}}$, i.e.

$$\exists\, k \in \mathbb{N} \text{ s.t. } \underline{P^k > 0} \quad \left(\underline{P^{(k)}_{ij} > 0 \quad \forall i, j \in S} \right)$$

Per ogni i, j

For a regular transition prob. matrix P we have

$$\boxed{\underbrace{\lim_{n \to \infty} P^{(n)}_{ij}}_{} = \underbrace{\pi_j}_{} > 0} \quad \text{for } j = 0, 1, \dots, N$$

$$\lim_{n \to \infty} \mathbb{P}[\underbrace{X_n = j}_{} \mid X_0 = i]$$

la % di trovare MC

in the long run, the prob. of finding the MC in state j is approximately π_j (no matter in which state the chain began at time 0.)

La prob. di trovare MC in stato J $\simeq \pi_j$

Esempio

Ex:

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 \le a \le 1, \quad 0 \le b \le 1$$

- $\boxed{a=0,\ b=0}$, $P = Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Is P regular? no! $P^n = P \quad \forall n$

$\boxed{P_{ij}^{(n)} \not\to \pi_j}$ non converge $\quad j = 0, 1$

- $\boxed{a=1,\ b=1}$ $\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P \cdot P$$

$$\begin{tikzcd} \boxed{0} \arrow[r, bend left, "1"] & \boxed{1} \arrow[l, bend left, "1"] \end{tikzcd}$$

Riga × colonna

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Id$$

$P^3 = P^2 \cdot P = Id \cdot P = P$, $P^4 = Id$ → sono tutti identici

$\underline{P^0 = Id}$ (by definition) $\quad \boxed{0!} = 1$

- $P^{2n} = Id \quad \forall n \in \mathbb{N}$ PARI even exponent
- $P^{2n+1} = P \quad \forall n \in \mathbb{N}$ DISPARI odd exponent

1 0 1 0 1 0 1 ...

! $P_{ij}^{(n)} \not\to \pi_j$ non converge to π_j ! $\begin{bmatrix} \boxed{0} & \\ & \end{bmatrix}$

$\boxed{0 < a, b < 1}$ allora $P^n \xrightarrow[n \to \infty]{} \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

• Se $0 < a, b < 1$

The limiting distribution is $\boxed{\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)}$

$$P = \begin{bmatrix} 0.33 & 0.67 \\ 0.75 & 0.25 \end{bmatrix}$$

$$P^{7} = \begin{bmatrix} 0.5271 & 0.4729 \\ 0.5294 & 0.4706 \end{bmatrix}$$

calcolato p7

$n = 7$

SAME

$$\pi = (0.5282, \ 0.4718)$$

• Example: Esempio

Son's Class

$$P = \begin{array}{c c} & \begin{array}{ccc} L & M & U \end{array} \\ \begin{array}{c} \text{Lower} \\ \text{Middle} \\ \text{Upper} \end{array} & \begin{bmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.65 \end{bmatrix} \end{array}$$

Father's class

FRACTION

What fraction of people are "middle class" in the long run?

P is regular $\overset{\text{allora}}{\Rightarrow}$ $\exists$ limiting prob. π_J st.

(P è Regolare)

$$\boxed{P^{(n)}_{i,J} \to \pi_J} \qquad J \in \{L, M, U\}$$

$P^2, P^3, P^4, P^8 =$ $\begin{bmatrix} 0.0772 & 0.6250 & 0.2978 \\ \vdots & 0.6250 & \vdots \\ \vdots & 0.6250 & \vdots \end{bmatrix}$

Calcolando
$(p^2, p^3, p^4 \ldots p^8)$

Guess $\longrightarrow$ $\pi_M \cong 0.6250 = P^{(8)}_{M,M}$

S,M

U,M

è CIRCA $\cong$

$\underbrace{\pi}$, we need to evaluate $\lim_{n\to\infty} P^n$

In generale

PROPRIETÀ

Regular matrices:

$\underbrace{|S| = N}$; P is regular $\overset{\text{allora}}{\Rightarrow}$ $\boxed{P^{N^2+2N-2} > 0}$

il numero stati

P is not regular $\Leftarrow$ $P^{N^2+2N-2}_{i,J} = 0$ for some $i, J \in S$

Se fosse $== 0$

DIM

Sufficient condition to prove that P is regular

$$\boxed{(1) + (2) \Rightarrow P \text{ regular}}$$

(1) $\forall i, j \in S$ (S finite), there exists $\underline{n} = n(i,j)$

n dipende da (i, j)

s.t. $P^{(n)}_{i,j} > 0$

(2) $\exists\, i \in S$ s.t. $P_{ii} > 0$

Exercise: prove this sufficient condition!

TEOREMA 1.1

$\boxed{\text{TEOREMA}}$

TRANSITION - MAT

Thm 1.1 Let P be a regular trans. matrix on $\underbrace{\{0, 1, \ldots, N\}}_{N+1}$. Then the limiting prob. π is the unique (univoco) nonnegative solution (+ non negativo) of the equations

$$\begin{cases} \boxed{\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}} & j = 0, 1, \ldots, N \\ \sum_{k=0}^{N} \pi_k = 1 \end{cases}$$

$$\boxed{\underline{\pi} = \underline{\pi} \cdot P}$$

$\pi = (\pi_0, \ldots, \pi_N)$

Proof: essendo since P is regular, si ha

DIMOSTRAZIONE

$$\boxed{\lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j}$$

$\sum \pi_j = 1$

Ricordo: $P^n = \underbrace{P^{n-1}} \cdot P$

da $i \to k$

$k \to j$ (k è in mezzo)

$$\boxed{\underbrace{P_{i,j}^{(n)}} = \sum_{k=0}^{N} \underbrace{P_{i,k}^{(n-1)}} \cdot \underbrace{P_{k,j}}}$$

$j = 0, \dots, N$

passando al limite $\lim_{n\to\infty}$

$\lim_{n\to\infty}$

$\pi_j \qquad \boxed{\pi_k} \qquad \boxed{P_{k,j}}$

non dipende da n

Rimane costante

$$\pi_j = \lim_{n\to\infty} P_{i,j}^{(n)} = \lim_{n\to+\infty} \left(\sum_{k=0}^{N} \underbrace{P_{i,k}^{(n-1)} P_{k,j}} \right)$$

Sposto dentro Lim

$$= \sum_{k=0}^{N} \overbrace{\lim_{n\to\infty} \left(P_{i,k}^{(n-1)}\right.}^{\pi_k} \left. P_{k,j} \right) = \sum_{k=0}^{N} \pi_k \cdot P_{k,j}$$

$\boxed{|S| < +\infty \text{ finite number of states}}$

Warning: if $|S| = +\infty$ ($S = \mathbb{N}$)

we have to $\boxed{\text{add}}$ some conditions to be sure that we can switch lim with $\sum$.

UNICITÀ

Uniqueness : $x_0, x_1, x_2, \dots$

$$\begin{cases} x_j = \underbrace{\sum_{k=0}^{N} x_k P_{k,j}} & \forall j \in \{0,\dots,N\} \\ \sum_{k=0}^{N} x_k = 1 \end{cases}$$

we want to prove that $\boxed{x_j = \pi_j \quad \forall j}$

$$x_\ell = \underbrace{\sum_{j=0}^{N} x_j P_{j,\ell}} = \sum_{j=0}^{N} \Big(\sum_{k=0}^{N} \underbrace{x_k} P_{k,j} \Big) P_{j,\ell}$$

" Σ " → Fubini's theorem

switch summation $= \sum_{k=0}^{N} x_k \underbrace{\sum_{j=0}^{N} P_{k,j} P_{j,\ell}} = \sum_{k=0}^{N} x_k P^{(2)}_{k,\ell}$

$$\Rightarrow \boxed{x_\ell = \sum_{k=0}^{N} x_k \cdot P^{(2)}_{k,\ell}} \qquad \ell = 0,\dots,N$$

Repeat this n-times ← Ripeto n-volte

$$\underbrace{x_\ell} = \underbrace{\sum_{k=0}^{N} x_k P^{(n)}_{k,\ell}} \qquad \boxed{\ell = 0,\dots,N}$$

$n \in \mathbb{N}$

passaggio al limite $\lim_{n\to\infty}$

$$\boxed{x_\ell} = \sum_{k=0}^{N} x_k \underbrace{\lim_{n\to\infty} P^{(n)}_{k,\ell}} = \underbrace{\sum_{k=0}^{N} x_k}_{=1} \cdot \pi_\ell = \boxed{\pi_\ell}$$

<u>Remark</u> if P is regular, it admits a limiting distribution π (<u>we do not prove this</u>)

and we can compute π by solving

$$\begin{cases} \pi = \pi P \\ \sum_j \pi_j = 1 \end{cases}$$

linear system of equations

Esempio: Social class di Prima — 3 classi

<u>Ex</u>: <u>Social class matrix</u>

$S = \{0, 1, 2\}$

$$P = \begin{bmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{bmatrix}$$

regular

$P > 0$

$(\pi_0, \pi_1, \pi_2) P$

$\boxed{\pi \cdot P = \pi}$

RISOLVO

$$\begin{cases} \pi_0 \cdot 0.40 + \pi_1 \cdot 0.05 + \pi_2 \cdot 0.05 = \pi_0 \\ \pi_0 \cdot 0.50 + \pi_1 \cdot 0.70 + \pi_2 \cdot 0.50 = \pi_1 \\ \pi_0 \cdot 0.10 + \pi_1 \cdot 0.25 + \pi_2 \cdot 0.65 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$\leftarrow$ infine la $\sum \pi_j = 1$

$\pi_0 = \frac{5}{65} = \frac{1}{13}$, $\pi_1 = \frac{5}{8}$, $\pi_2 = 1 - \pi_0 - \pi_1 = \frac{31}{104}$

$= 0.6250$

Special case : DOUBLE stochastic MATRICES

Doubly Stochastic Matrices

$P = (P_{ij})$ $\quad P_{i,j} \geq 0 \quad \forall i, j$

$\sum_j P_{ij} = 1 \quad \forall i$ (trans. matrix)

$\rightarrow \sum_i P_{ij} = 1 \quad \forall j$

(d.s.M.) + Regular + unique Dist.

If P is dsm and is regular, it admits a unique limiting distribution $(|S| = N)$

which is $\boxed{\pi = \left(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right)}$ uniform distribution on S

$\pi = \pi \cdot P$

$\pi_i = \sum_{j=1}^{N} \pi_j P_{ji} \overset{?}{=} \frac{1}{N} = \sum_{j=1}^{N} \frac{1}{N} P_{ji}$

$= \frac{1}{N} \sum_{j=1}^{N} P_{ji} = 1/N$

pag. 206 $\quad Y_n$ sum of n independent rolls of a fair die

in the long run what is the prob. that Y_n is multiple of 7 $\quad \boxed{\frac{1}{7}}$

UNA ALTRA INTERPRETAZIONE

A second interpretation of the Limiting Distribution

$a_0, a_1, a_2, \dots$ real numbers $\quad a_n \to a \in \mathbb{R}$

$$\boxed{\underbrace{\frac{1}{n} \sum_{k=0}^{n-1} a_k}_{} \xrightarrow[n \to \infty]{} a}$$

(exercise)! fca mamma

indicated function

$$\frac{1}{n} \sum_{k=0}^{n-1} \underbrace{\mathbb{1}_{\{X_k = j\}}}_{}$$

fraction

(random) fraction of time spent in state j spent

take expectation

$$\mathbb{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = j\}} \middle| X_0 = i\right] = \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{X_k = j\}} \middle| X_0 = i\right]$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}[X_k = j \mid X_0 = i]$$

mean fraction of visits to state j starting from i

$$= \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{(k)} = \mathbb{E}\left[\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = j\}} \,\middle|\, X_0 = i \right]$$

$$\lim_{n\to\infty} \mathbb{E}[\quad] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_{ij}^{(k)}$$

$$= \pi_j$$

π_j is the long run mean fraction of time that the process spends in state j

$\boxed{0.65}$

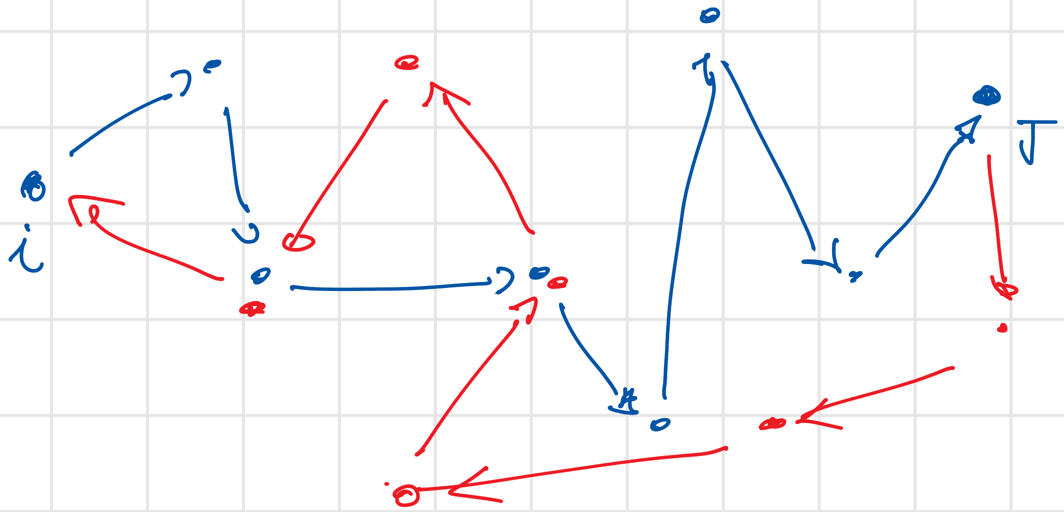
2. Examples $\boxed{(2.1)}$ Chapter IV

3. $\boxed{\text{The classification of States}}$ (classificazione stati)

- accessible: j is accessible from i if $\boxed{P_{ij}^{(n)} > 0}$ for some integer $\boxed{n \in \mathbb{N}}$

$$\boxed{i \longrightarrow j} \quad P_{ij} > 0$$

$$\boxed{P^{(2)}_{1,3} > 0}$$



- if i is accessible from j and j is accessible from i, we say that i and j communicate

$$\boxed{i \longleftrightarrow j} \longrightarrow \begin{cases} i \longrightarrow j \\ j \longrightarrow i \end{cases} \Longrightarrow \text{COMMUNICATE}$$

Remark: if two states i and j do not communicate

DO NOT

then either $P^{(n)}_{ij} = 0 \quad \forall n \geq 0$

or $P^{(n)}_{ji} = 0 \quad \forall n > 0$

$$\begin{array}{c|l} 0 & \\ 1 & \\ 2 & \end{array} \begin{bmatrix} 1 & 0 & \cdots & & \\ q & 0 & p & \cdots & \\ 0 & q & 0 & p & \cdots \end{bmatrix}$$

or both relations are true. Gambler's ruin

since $P^{(n)}_{0i} = 0 \quad \forall n, \ i > 0 \quad \Longleftarrow$ 0 do not communicate to any other state

The concept of communication is an equivalence relation: $\rightarrow$ Relation equivalent

(i) $i \leftrightarrow i$ (reflexivity)

Con se stesso $\leftarrow$ $P_{ii}^{(n)} > 0$ for some n? $n = 0, 1, 2, \dots$

$$P^0 = Id$$

$$P_{ii}^{(0)} = 1 \quad \forall i$$

simmetrico

(ii) if $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetry)

(iii) if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity)

k is accessible from i (the same ~~is~~ acc. from k)

$$i \longrightarrow k$$

$$i \rightarrow j \Rightarrow \exists\, n : P_{ij}^{(n)} > 0$$

$$j \rightarrow k \Rightarrow \exists\, m : P_{jk}^{(m)} > 0$$

$$\underbrace{P_{ik}^{(n+m)}} = \underbrace{\sum_{r\in S} P_{ir}^{(n)} \cdot P_{rk}^{(m)}}$$

$$\left(\exists n : P_{ij}^{(n)} > 0 \,,\quad \exists m : P_{jk}^{(m)} > 0 \right) \Leftarrow$$

$r = j$

$$\geq P_{ij}^{(n)} \cdot P_{jk}^{(m)} > 0$$

$\Rightarrow$ k is accessible from i.

Since communication is a equivalence relation we can partition the state space into equivalence classes.

$$S = C_0 \cup C_1 \cup C_2 \dots$$

$C_0, C_1, \dots$ are equivalence classes $\Leftarrow$ classe

$i, j \in C_0$ then $i \Leftrightarrow j$

If a MC has a unique equivalence class, i.e. all the states communicate, then the MC is called <u>irreducible</u>. (irreducibile)

- The seafood YES
- The Gambler's ruin NO

$$\begin{array}{c} 0 \\ \\ \\ N \end{array} \begin{pmatrix} \boxed{1} & & \\ & \boxed{} & \\ & & \boxed{1} \end{pmatrix}$$

$\boxed{0 < p, q < 1}$

$S = \{0, 1, \dots, N-1, N\}$

$\{1, \dots, N-1\}$ is a <u>class</u>

CLOSED

$\boxed{C_0 = \{0\},\ C_1 = \{N\}}$

$\boxed{1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow \dots \longleftrightarrow N-1}$

$$\begin{array}{c|cccccc} & & 1 & 2 & & & \\ 0 & 1 & 0 & 0 & \cdot & - & - \\ 1 & q & 0 & p & & & \\ 2 & & q & 0 & p & & \\ & & & q & 0 & & \\ N & & & & & & 1 \end{array}$$

$\longrightarrow \boxed{C_2 = \{1, \dots, N-1\}}$ NOT CLOSED

$1 \xrightarrow[q]{} 0$, but $0 \notin C_2$

C_0 and C_1 are <u>closed</u>, while C_2 is not <u>closed</u>

<u>C is closed</u> if $i \in C$, $i \longrightarrow j \Longrightarrow j \in C$ $\leftarrow$ (C comprende i e j)