Stochastic Processes

Lecture 6

CYS 2020/2021

- Examples of MC
- Success Runs
- Gambler's ruin. hitting time T, absorption prob.
  mean duration
- Another look at First Step Analysis (III.7)

Chapter IV

- Regular Transition Probability Matrices

Thm 1.1 (pag. 204-205)

Ex: Toss a coin

$$\begin{cases} H & p \\ T & 1-p=q \end{cases} \qquad 0<p<1$$

On average how many tosses are needed to obtain for the first time HTHT ?

• $\underbrace{HT}$ $\qquad \dfrac{1}{p(1-p)} = \boxed{\dfrac{1}{p\cdot q}} \qquad q = 1-p$

$X_n \in \{0,1,2\}$

... HT

... H

no seq.

HTHT $\qquad X_n \in \{0, 1, 2, 3, 4\}$

TT $\quad X_n = 0 \qquad \boxed{\dots H} \qquad \dots HT \qquad \dots HTH \qquad \dots HTHT$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & q & p & 0 & 0 & 0 \\ 1 & 0 & p & q & 0 & 0 \\ 2 & q & 0 & 0 & p & 0 \\ 3 & 0 & p & 0 & 0 & q \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

2 $\boxed{\dots HTT} \equiv 0$

$\boxed{3 \quad HTH}$

~~HTH~~ T $\longrightarrow 4$

H

$$P = \begin{array}{c} 0 \\ \\ \rightarrow \\ \\ 4 \end{array} \begin{bmatrix} q & p & 0 & 0 & 0 \\ 0 & p & q & 0 & 0 \\ q & 0 & 0 & p & 0 \\ 0 & \boxed{p} & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} w_0 = 1 + q w_0 + p \underbrace{w_1} \\ w_1 = 1 + p w_1 + q w_2 \\ \boxed{w_2 = 1 + q \underbrace{w_0} + p w_3} \\ w_3 = 1 + p w_1 + \cancel{q w_4} \\ w_4 = 0 \end{cases}$$

$\rightarrow \boxed{w_0 = \frac{1}{p} + w_1}$

$\rightarrow (1-p) w_1 = 1 + q w_2$

$q w_1 = 1 + q w_2$

$\boxed{w_1 = \frac{1}{q} + w_2}$

$\boxed{w_3 = 1 + p w_1}$

$\boxed{w_2 = 1 + q w_0 + p w_3}$

$$w_2 = 1 + q \cdot w_0 + p + p^2 \underbrace{w_1}$$

$$w_2 = 1 + q w_0 + p + p^2 \left(\frac{1}{q} + w_2 \right)$$

$$= 1 + q w_2 + p + p^2/q + p^2 w_2$$

$$\underbrace{(1-p^2)W_2} = 1 + p + p^2/q + q\underbrace{W_0}$$

$p+q=1$

$$W_2 = W_1 - 1/q = W_0 - \frac{1}{p} - \frac{1}{q} = W_0 - \left(\frac{p+q}{pq}\right)$$

$$W_1 = W_0 - 1/p$$

$$\boxed{W_2 = W_0 - \underbrace{\frac{1}{pq}}}$$

$$(1-p^2)W_2 = (\underset{q}{\underset{\parallel}{1-p}})(1+p)W_2 = \underbrace{q(1+p)}W_2$$

$$= q(1+p)W_0 - \frac{1+p}{p}$$

$$\underbrace{q}(1+p)W_0 - \frac{1+p}{p} = 1 + p + \frac{p^2}{q} + qW_0$$

$$qp \cdot W_0 = \underbrace{\boxed{1} + p + p^2/q + \frac{1+p}{p}}$$

$\boxed{p+q=1}$

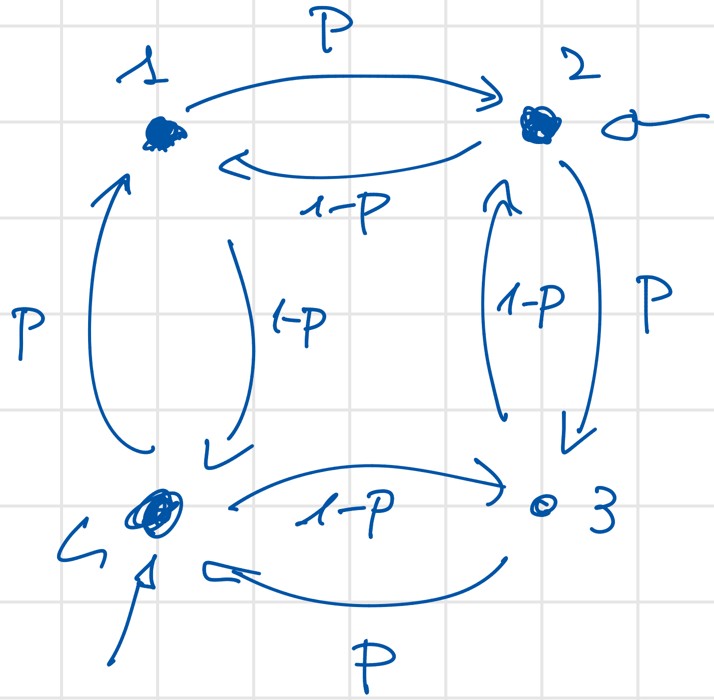
$$\boxed{W_0 = \frac{1}{qp} + \frac{1}{q^2p^2}}$$

HTHT

$$\boxed{W_0 = \frac{1}{qp} + \frac{1}{q^2p^2} + \cdots + \frac{1}{q^np^n}}$$

$\underbrace{HTHT \ldots HT}_{n\text{-times}}$

2. Example   Castle with the sentinel



RW on a Graph

$$p = 1-p = \frac{1}{2}$$

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & & \end{array} \\ 1 & \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \end{array}$$

$$P, P^2, P^3, P^4, P^5, \dots$$
$$0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \dots$$

$$P^3 = P^2 \cdot P = P \cdot P^2 = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \equiv P$$

$$\boxed{A \cdot B \neq B \cdot A}$$

the product is not commutative !!

$$\forall n \in \mathbb{N} \quad \underbrace{P^{2n} = P^2 , \quad P^{2n+1} = P}$$

Success Runs

$X_n \in \mathbb{N}$

- finite $\{0, 1, \dots, N\}$, $\{1, 2, \dots, N\}$
- infinite $\mathbb{N}$, $\mathbb{Z}$

$$
P = \begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & p_0 & q_0 & 0 & 0 & 0 & \cdots \\
1 & p_1 & r_1 & q_1 & 0 & 0 & \cdots \\
2 & p_2 & 0 & r_2 & q_2 & 0 & \cdots \\
3 & p_3 & 0 & 0 & \cancel{r_3} & q_3 & \cdots \\
4 & p_4 & 0 & 0 & & & \ddots \\
5 & p_5 & 0 & 0 & & & \\
\vdots & & & & & & \\
n & & & & & & \\
\vdots & & & & & &
\end{array}
$$

β $\qquad \alpha \qquad k = 3$

(1) Repeated trials each of which admits two possible outcomes, success S, failure F $\quad (H, T)$

At any trial $\quad S \quad \alpha$

$\qquad\qquad\quad F \quad 1 - \alpha = \beta$

$$\boxed{X_n = k}$$

$\to X_{n+1} = 0$ $\quad \beta$

$\to X_n = k+1$ $\quad \alpha$

$F, S, S, S, \dots, S$

n success in a row!

Succ. Runs indep. trials

$$
P = \begin{bmatrix}
\beta & \alpha & 0 & 0 & \cdots \\
\beta & 0 & \alpha & & \\
\beta & 0 & 0 & \alpha & \\
\beta & 0 & 0 & 0 & \alpha \\
\vdots & & & & & \ddots
\end{bmatrix}
$$

upper diagonal all of α's

column of β's

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(2) $\underbrace{\text{light bulbs}}$ lifetime

$$
\boxed{\mathbb{P}[\xi = k] = \underbrace{a_k}_{} > 0 \qquad k = 1, 2, \ldots \qquad \sum_{k=1}^{\infty} a_k = 1}
$$

Each bulb will be replaced by a new one when it burns out.

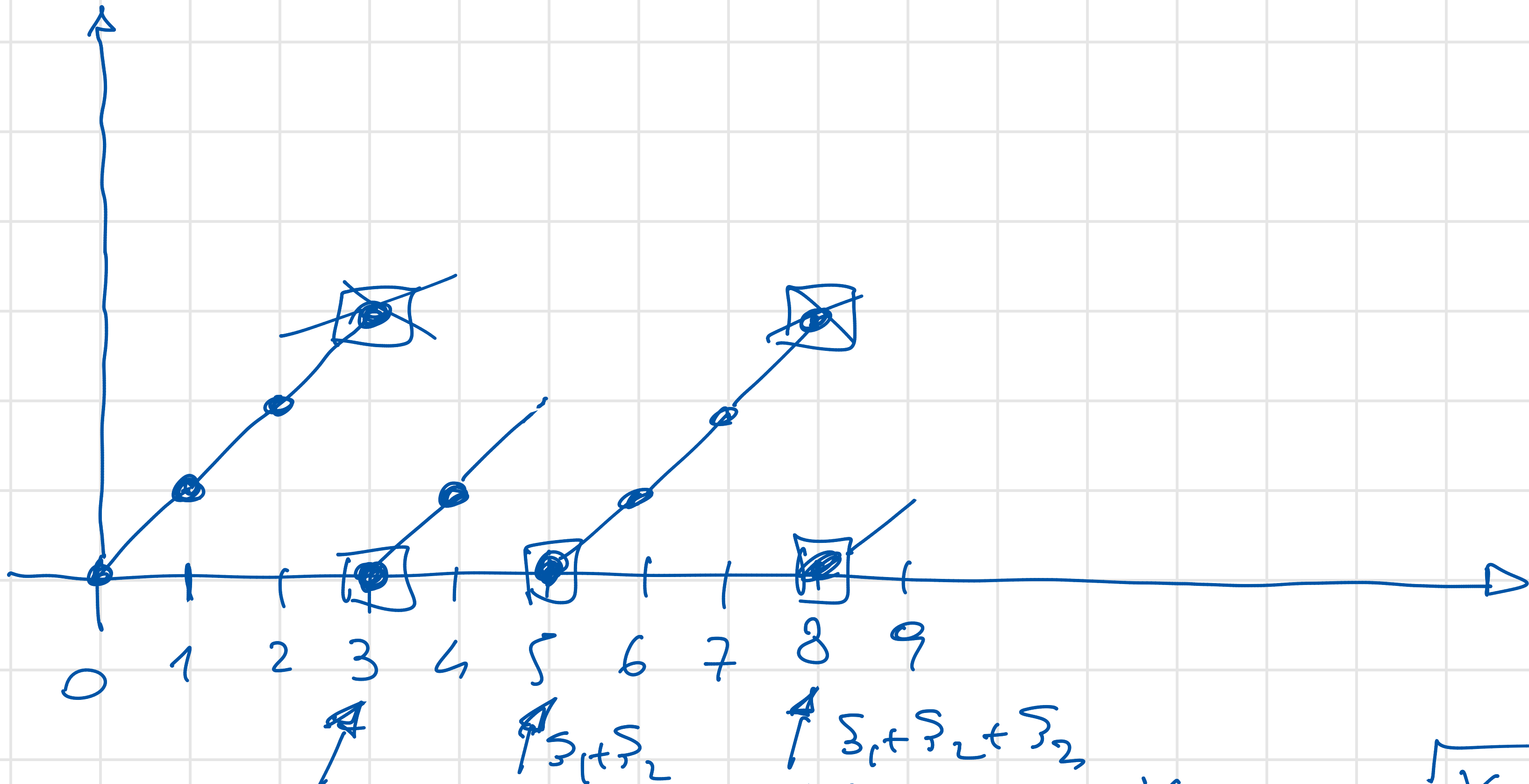
ξ_1 first bulb

ξ_2 second bulb

$\boxed{\xi_1 + \xi_2}$

$\underbrace{\xi_1 + \cdots + \xi_n}$ $\qquad$ $\xi_1, \ldots, \xi_n$ independent

X_n = the age of the bulb in service at time n.



$\underline{\xi_1 = 3}$ $\qquad$ $X_0 = 0$, $X_1 = 1$, $X_2 = 2$, $\boxed{X_3 = 0}$

$\xi_2 = 2$ $\qquad$ $X_4 = 1$, $\boxed{X_5 = 0}$, $X_6 = 1$, ...

$\xi_3 = 3$

$\boxed{X_n = 0 \text{ at the time of a failure}}$

$k \in \mathbb{N}$

$\boxed{r_k \equiv 0}$ $\qquad$ $p_k : k \longrightarrow 0$

$q_k = 1 - p_k : k \longrightarrow k+1$

$k \longrightarrow 0$

$$\mathbb{P}[X_{n+1} = 0 \mid X_n = k] = P_k$$

$$\mathbb{P}[\xi = k] = a_k > 0 \qquad \forall k \geq 1$$

$$\mathbb{P}[X_{n+1} = 0 \mid X_n = k] = \frac{\mathbb{P}[X_{n+1} = 0, X_n = k]}{\mathbb{P}[X_n = k]}$$

$$\mathbb{P}[X_n = k] = \mathbb{P}[\xi \geq k+1]$$

$$\mathbb{P}[X_{n+1} = 0, X_n = k] = \mathbb{P}[\xi = k+1]$$

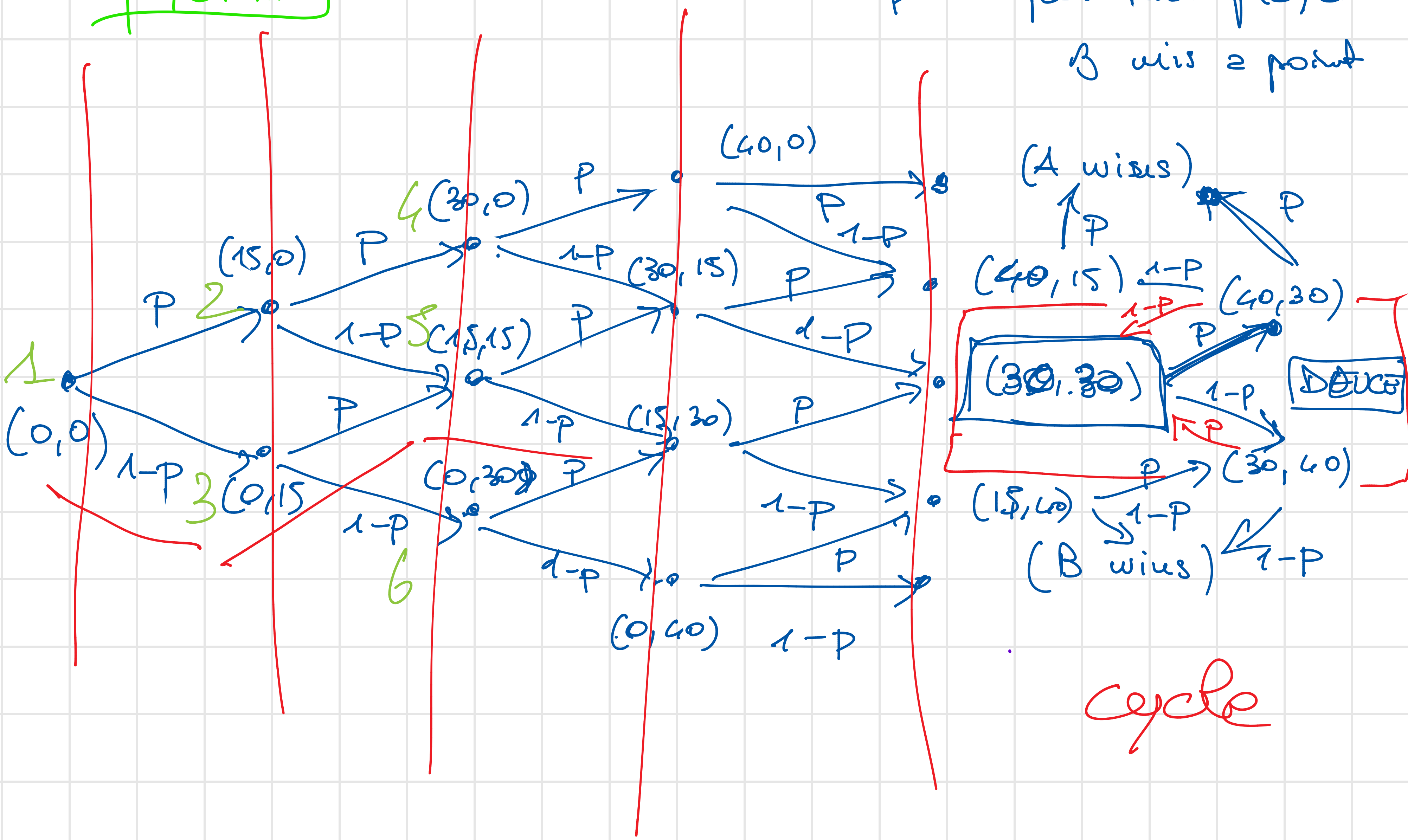
$$= \frac{\mathbb{P}[\xi = k+1]}{\mathbb{P}[\xi \geq k+1]} = \boxed{\frac{a_{k+1}}{a_{k+1} + a_{k+2} + \cdots}}$$

$$\left(\sum_{s=1}^{\infty} a_{k+s} = \sum_{s=1}^{\infty} \mathbb{P}[\xi = k+s] \right.$$

Ex: Game in tennis

Tenis

$\left\langle \begin{array}{ll} p & \text{prob. that player A wins a point} \\ 1-p & \text{prob that player B wins a point} \end{array} \right.$



- What is the prob. that player A wins the game?
- On average, how long it takes to complete a game?

how many points have to be played ...

$$P = \left[\begin{array}{c|c} Q & R \\ \hline 0 \;\cdots\; 0 & 1 \;\; 0 \\ 0 \;\cdots\; 0 & 0 \;\; 1 \end{array}\right]$$

transient

win A

win B

$$\begin{array}{c|cccc} & 1 & 2 & 3 & \cdots \\ \hline 1 & 0 & p & 1-p & 0 \;\cdots \\ 2 & & & & \\ 3 & & & & \end{array}$$

$(I-Q)^{-1}$

6. Functionals of RW and SR

Random Walker

$$P = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ \\ N \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & & & & N \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ q & 0 & p & & \\ 0 & q & 0 & p & \\ & & \ddots & \ddots & p \\ 0 & \cdots & & 0 & 1 \end{bmatrix} \end{array}$$

Gambler's ruin

Hitting Time T

$\underbrace{A}_{\text{wins } p}, B$

$$T := \min\{n \geq 0 : X_n = 0 \text{ or } X_n = N\}$$

$$\boxed{u_k = \mathbb{P}[X_T = 0 \mid X_0 = k]} \qquad k = 0, \dots, N$$

(1) $\boxed{u_0 = 1, \quad u_N = 0}$

a second order difference equation

$\boxed{k = 1, \dots, N-1}$ (2) $\boxed{u_k = p\, u_{k+1} + q\, u_{k-1}}$

$N-1$ equations

$p + q = 1$

$$(p+q) u_k = p\, u_{k+1} + q\, u_{k-1}$$

$$p\, u_k + q\, u_k = p\, u_{k+1} + q\, u_{k-1}$$

$$p(u_{k+1} - u_k) = q\,(u_k - u_{k-1})$$

$\boxed{x_k = u_k - u_{k-1}}$

$$p\, x_{k+1} = q \cdot x_k \longrightarrow \boxed{x_{k+1} = \frac{q}{p} \cdot x_k}$$

$$x_2 = \frac{q}{p} x_1, \quad x_3 = \frac{q}{p} x_2 = \frac{q}{p} \cdot \frac{q}{p} x_1 = \frac{q^2}{p^2} x_1$$

$$\dots \quad x_k = \left(\frac{q}{p}\right)^{k-1} \cdot x_1 \quad \dots \quad x_N = \left(\frac{q}{p}\right)^{N-1} x_1$$

$$\boxed{u_0 = 1, \; u_N = 0}$$

$$u_k = \begin{cases} \boxed{\dfrac{N-k}{N}} & p = q = \frac{1}{2} \\[2ex] \boxed{\dfrac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}} & p \neq q \end{cases}$$

$N \longrightarrow +\infty$ What happens?

$$\underbrace{u_k}_{} \xrightarrow[N \to +\infty]{} \begin{cases} \boxed{1} & \boxed{p = q} \quad p \leq q \\[2ex] \underbrace{\left(\frac{q}{p}\right)^k}_{} & p > q \end{cases}$$

$\boxed{p > q} \Rightarrow \dfrac{q}{p} < 1 \Rightarrow \left(\dfrac{q}{p}\right)^N \xrightarrow[N \to \infty]{} 0$

$p < q \Rightarrow q/p > 1 \quad \left(\frac{q}{p}\right)^N \longrightarrow +\infty$

Give a look the paragraph 7 of Chapter III

$$P = \begin{bmatrix} \boxed{Q} & R \\ O & \boxed{I} \end{bmatrix}$$

pag. 172 $\boxed{W = (I - Q)^{-1}}$ fundamental matrix

Chapter IV Long Run Behavior of a MC

RUN BEHAVIOUR

P Tennis Game 17 states (STATI)

- $P \in M(17 \times 17)$ $\quad P^5$
- $P^n = \left(P_{ij}^{(n)}\right)_{i,j \in \{1,2,\dots,17\}}$

$$P^n \xrightarrow[n \to +\infty]{} \boxed{?}$$

$$\left[\alpha \cdot P^n\right] = \left(\mathbb{P}[X_n = k]\right)$$

$\mathbb{P}[X_n = 0]$

$\mathbb{P}[X_0 = k] = \alpha_k$ $\quad \alpha' = (\alpha_0, \alpha_1, \dots)$

1. Regular Transition Probability Matrices

$\longrightarrow S = \{0, \dots, N\}$ finite state space

P transition probability, it is regular

if $\exists\, k \in \mathbb{N}$ s.t. $P^k > 0$ positive matrix

i.e. $P^{(k)}_{i,j} > 0 \quad \forall\, i,j \in \{0, \dots, N\}$

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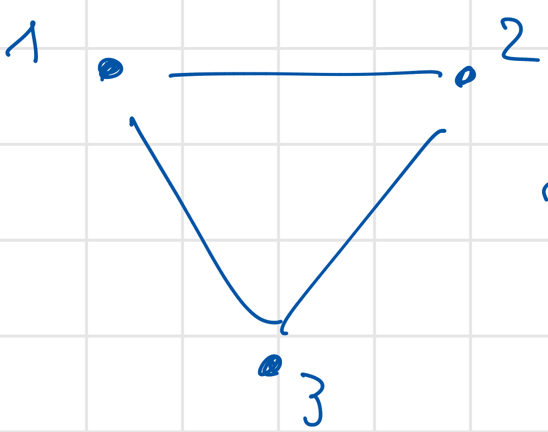
1 Sentinel $P^{2n} = P^2$, $P^{2n+1} = P$, P is not regular

4 towers

$$P^{(n)}_{0,0} = \begin{cases} 1/2 & n = 2k \text{ (even)} \\ 0 & n = 2k+1 \text{ (odd)} \end{cases}$$

2

3 towers



$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$\boxed{P^2 > 0}$

$\boxed{P^n > 0}$ $\forall n \geq 2$

Limiting prob. distribution

$\underbrace{\pi = (\pi_0, \pi_1, \dots, \pi_N)}$

$\underbrace{\pi_j > 0 \quad j = 0, \dots, N \quad , \quad \sum_{j=0}^{N} \pi_j = 1}$

For a regular transition matrix we can prove

$$\boxed{\lim_{n \to +\infty} P_{ij}^{(n)} = \underbrace{\pi_j} \quad \text{for } j = 0, 1, \dots, N}$$

$$P^n \longrightarrow \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Theorem 1.1