Theo 3.2 (5.5): Interarrival times S_i are iid $\exp(\lambda)$

Proof: seen before.

Theo 3.1 (5.4): waiting time W_n is gamma with n, λ

Proof: $W_n = \sum_{i=0}^{n-1} S_i$ where S_i $i = 0, \dots, n-1$ are iid $\exp(\lambda)$.

Theo 4.1 (5.7): Let W_i be the event times of $PP(\lambda)$. Conditional on $X(t) = n$, the joint pdf is

$$f_{W_1 \dots W_n | X(t)=n}(w_1, \dots, w_n) = n! \, t^{-n} \quad 0 < w_1 < \dots < w_n \le t$$

Proof: Let the w_i's be all different, and pick Δw_i such that all intervals $[w_i, w_i + \Delta w_i]$ are disjoint.

$$P[w_i \le W_i \le w_i + \Delta w_i, i = 1, \dots, n \mid X(t) = n] =$$

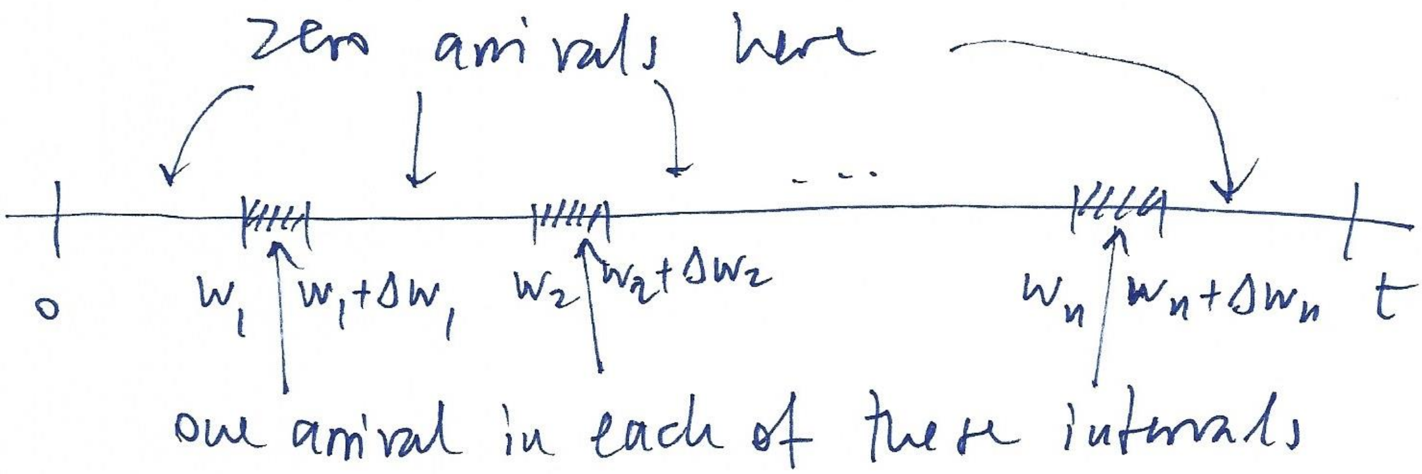
$$= P\left[\begin{array}{l}\text{one arrival in } [w_i, w_i + \Delta w_i], i = 1, \dots, n, \text{ and} \\ \text{zero arrivals everywhere else in } [0, t)\end{array} \middle| X(t) = n\right]$$

$$= \frac{\lambda \Delta w_1 e^{-\lambda \Delta w_1} \dots \lambda \Delta w_n e^{-\lambda \Delta w_n} e^{-\lambda (t - \sum_{i=1}^{n} \Delta w_i)}}{e^{-\lambda t} (\lambda t)^n / n!} =$$

$$= n! \, t^{-n} \Delta w_1 \dots \Delta w_n =$$

$$= f_{W_1 \dots W_n | X(t)=n}(w_1, \dots w_n) \Delta w_1 \dots \Delta w_n + o(\Delta w_1 \dots \Delta w_n)$$

Divide by $\Delta w_1 \dots \Delta w_n$ and let $\Delta w_i \to 0$ $i = 1 \dots n$



<u>remember</u>: arrivals in disjoint intervals are independent.

Thm 3.3 (5.6): $X(t)$ PP(λ), $0<u<t$, $0\le k\le n$

$$P[X(u)=k \mid X(t)=n] = \binom{n}{k}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}.$$

Proof: Since given $X(t)=n$ the n events are iid $\sim U[0,t]$, the prob. that each falls in $[0,u]$ is u/t and therefore $X(u)$ is Binomial$(n, u/t)$

Thm: $X_1(t), X_2(t)$ indep PP with λ_1, λ_2.

$$P[X_1(t)=k \mid X_1(t)+X_2(t)=n] = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

Proof: $\cancel{P[X_1(t), X_2(t) = n-k]}$ $\dfrac{P[X_1(t)=k,\ X_1(t)+X_2(t)=n]}{P[X_1(t)+X_2(t)=n]} =$

$$= \frac{P[X_1(t)=k,\ X_2(t)=n-k]}{P[X_1(t)+X_2(t)=n]} =$$ independent processes

$$= \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}\cdot\frac{n!}{e^{-(\lambda_1 t+\lambda_2 t)}(\lambda_1 t+\lambda_2 t)^n} =$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

More generally : $0 < s < t, \; 0 \le k \le n$

$$P\left[X_1(s) = k \mid X_1(t) + X_2(t) = n\right] =$$

$$= \frac{P\left[(X_1(s) = k),\ (X_1(t) - X_1(s) + X_2(t) = n-k)\right]}{P\left[X_1(t) + X_2(t) = n\right]} =$$

independent events

$$= \frac{\dfrac{e^{-\lambda_1 s}(\lambda_1 s)^k}{k!} \cdot \dfrac{e^{-(\lambda_1(t-s)+\lambda_2 t)}\left(\lambda_1(t-s)+\lambda_2 t\right)^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)t}\left((\lambda_1+\lambda_2)t\right)^n / n!} =$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \left(\frac{\lambda_1 s}{(\lambda_1+\lambda_2)t}\right)^k \left(\frac{\lambda_1(t-s)+\lambda_2 t}{(\lambda_1+\lambda_2)t}\right)^{n-k}$$