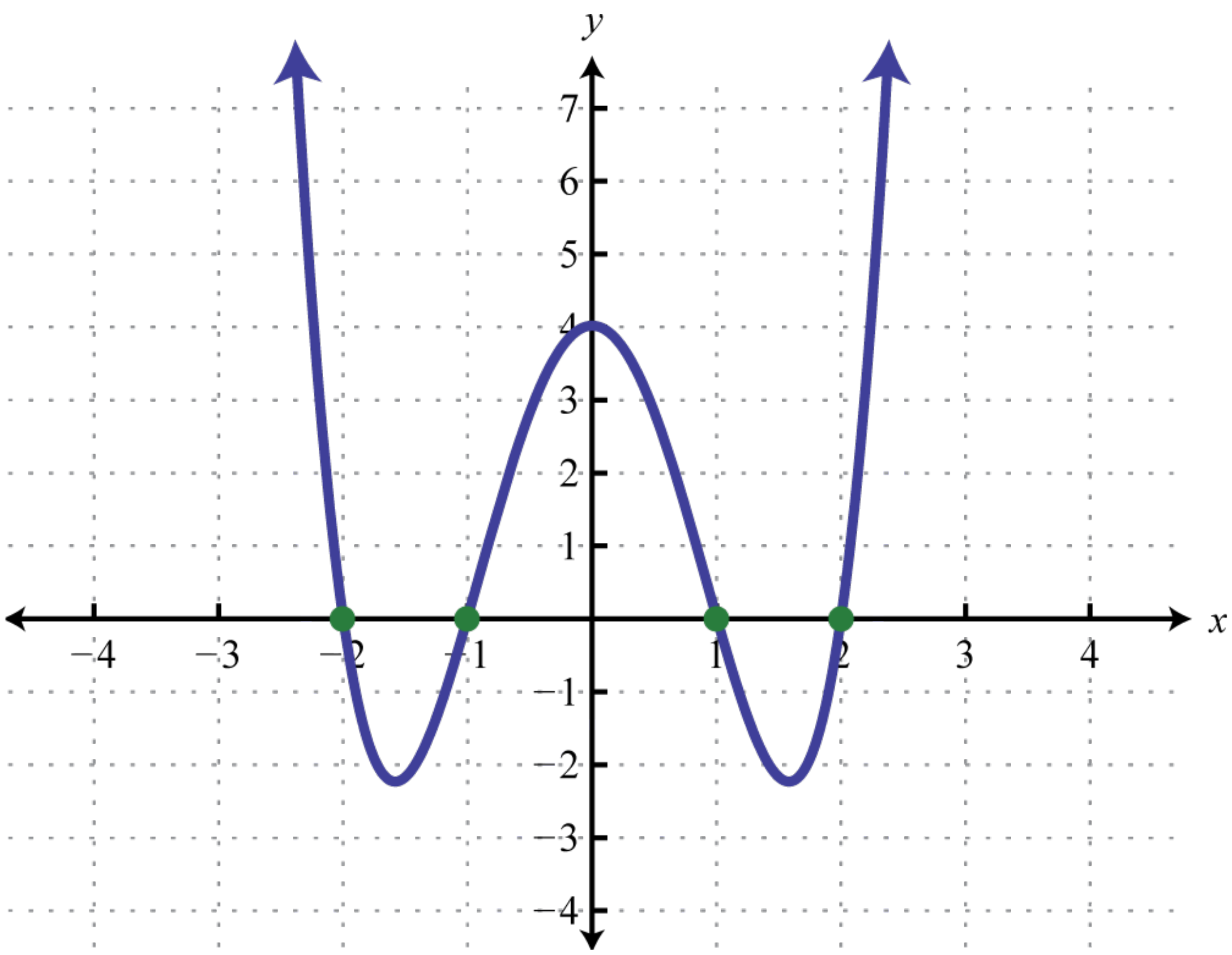
Functions

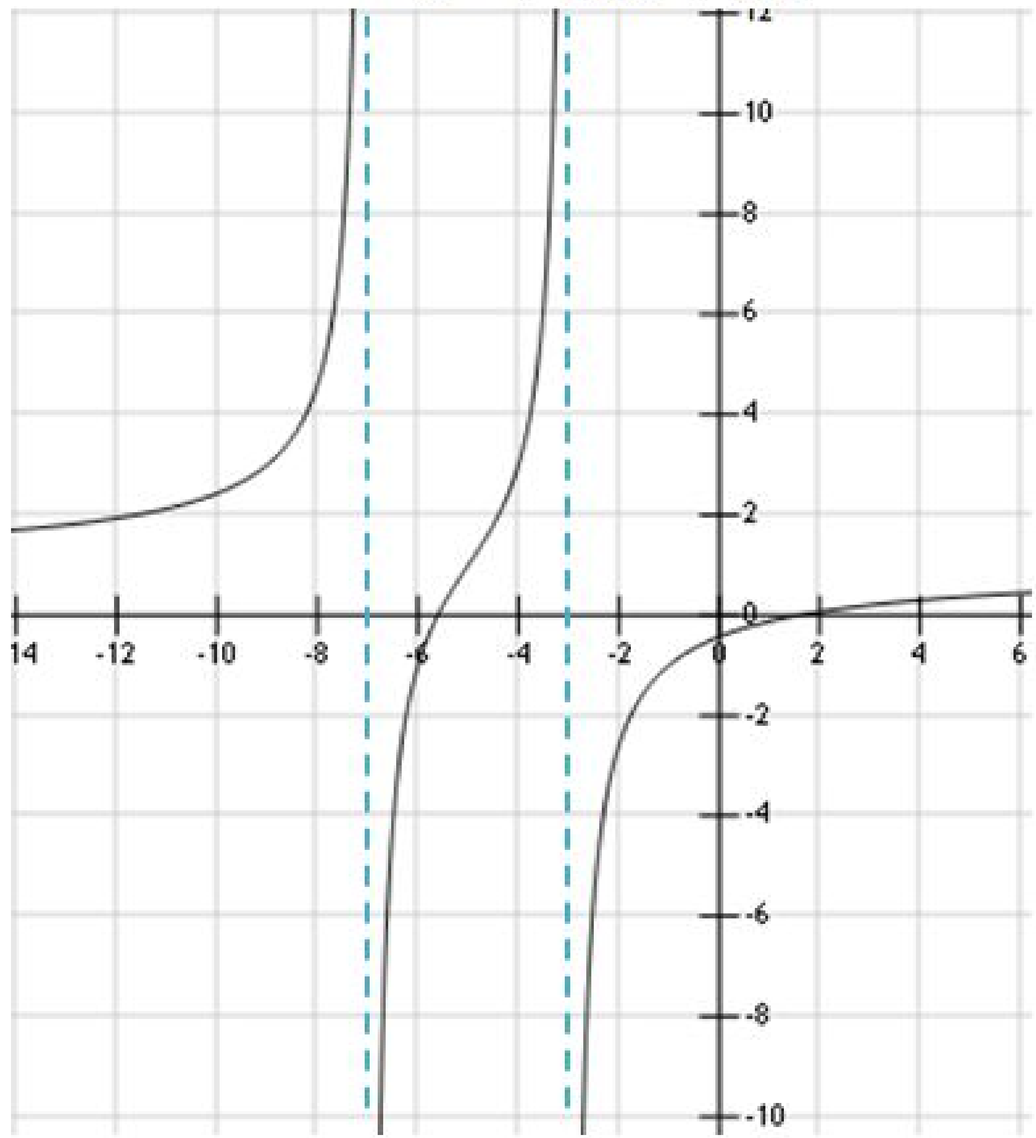
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



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Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3+3x^2+15x+7}{1-x^{137}}$$

- What does a function do?
  - Takes in as input a real number.
  - Outputs a real number.
  - … except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) numHeads++;

    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

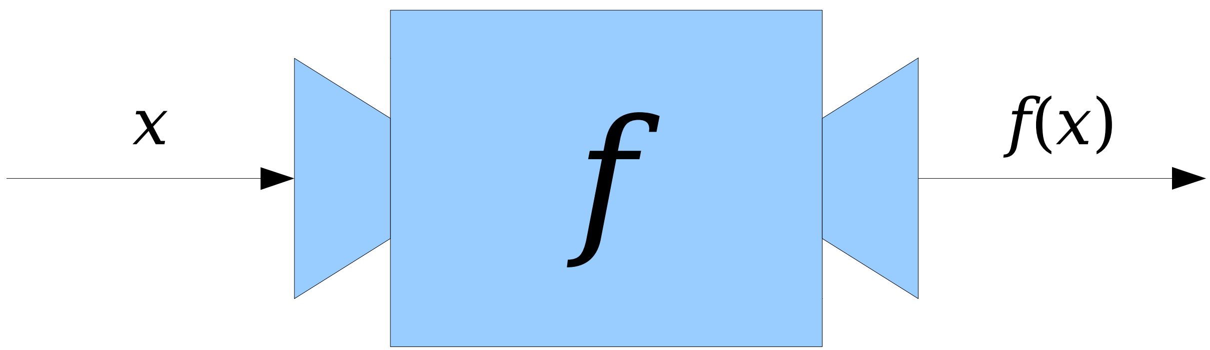
- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

Rough Idea of a Function:

A function is an object f that takes in one input and produces exactly one output.



(This is not a complete definition – we'll revisit this in a bit.)

High School versus CS Functions

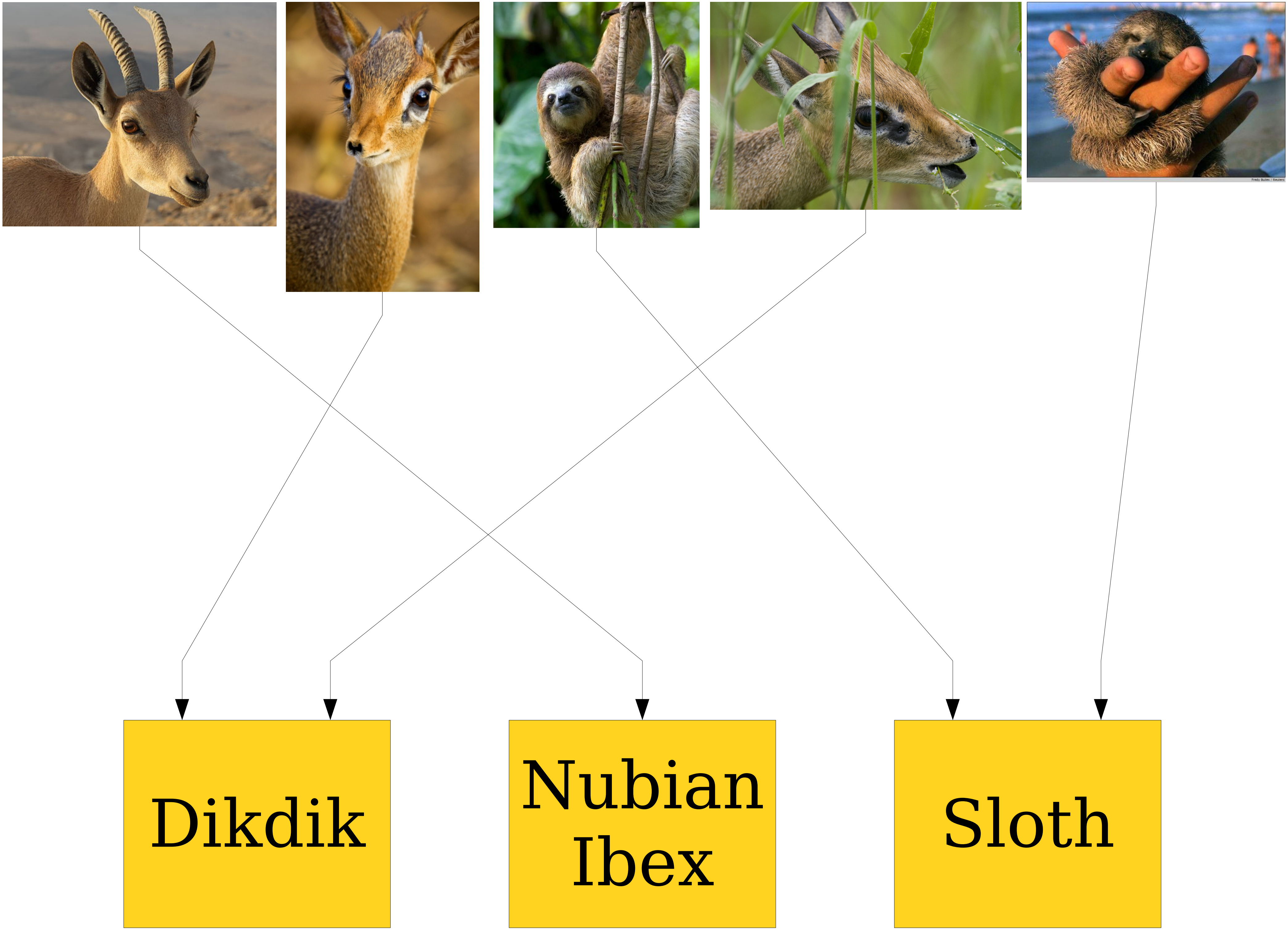
- In high school, functions usually were given by a rule:

$$f(x) = 4x + 15$$

- In CS, functions are usually given by code:

```
int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
        result *= i;
    }
    return result;
}
```

- What sorts of functions are we going to allow from a mathematical perspective?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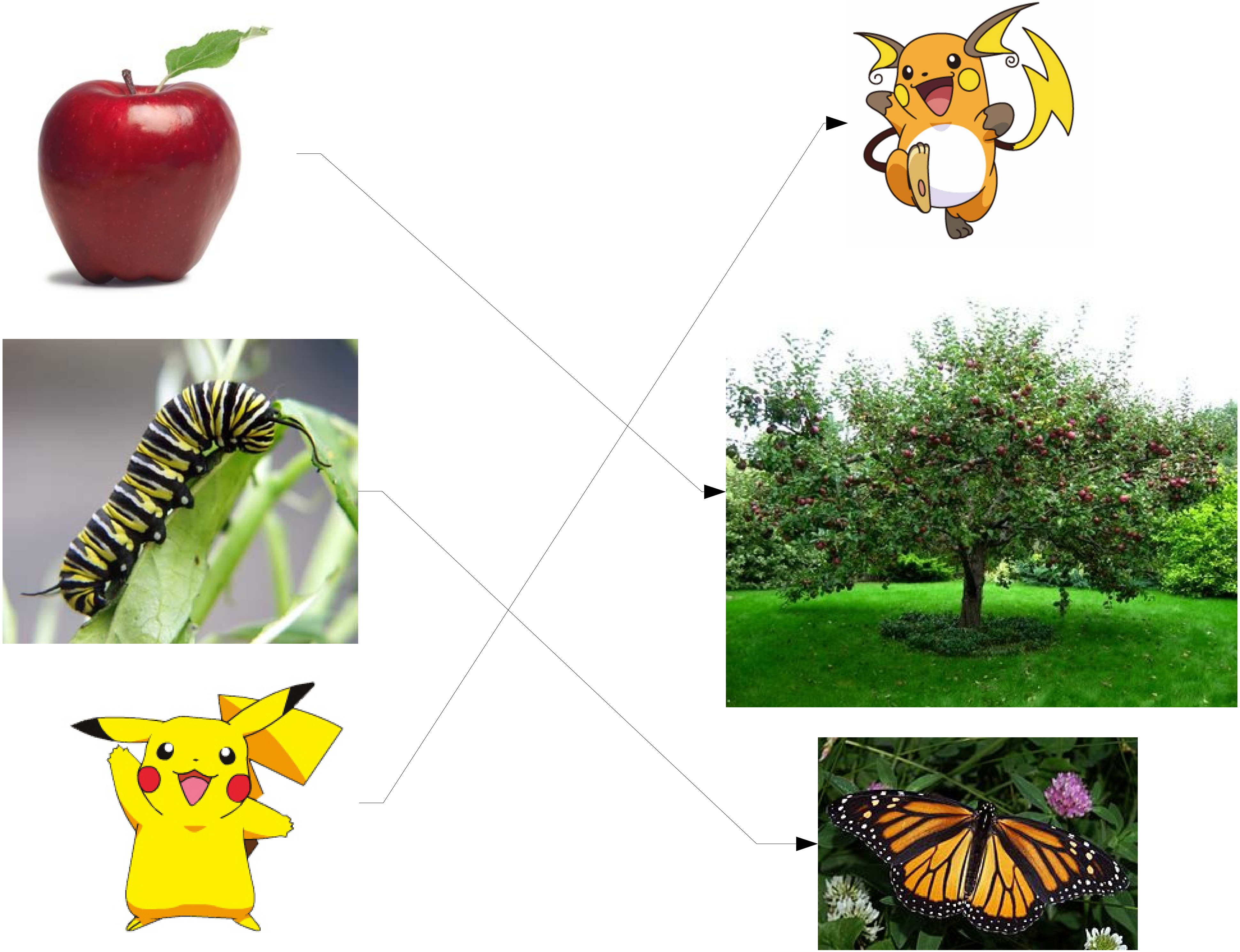
Nubian Ibex
Sloth



… but also …

$$f(x) = x^2 + 3x - 15$$

$$f(n)=\begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

Functions like these are called ***piecewise functions***.

To define a function, you will typically either

- draw a picture, or
- give a rule for determining the output.

In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

One Challenge

$$f(x) = x^2 + 2x + 5$$

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$

$$f(x) = x^2 + 2x + 5$$

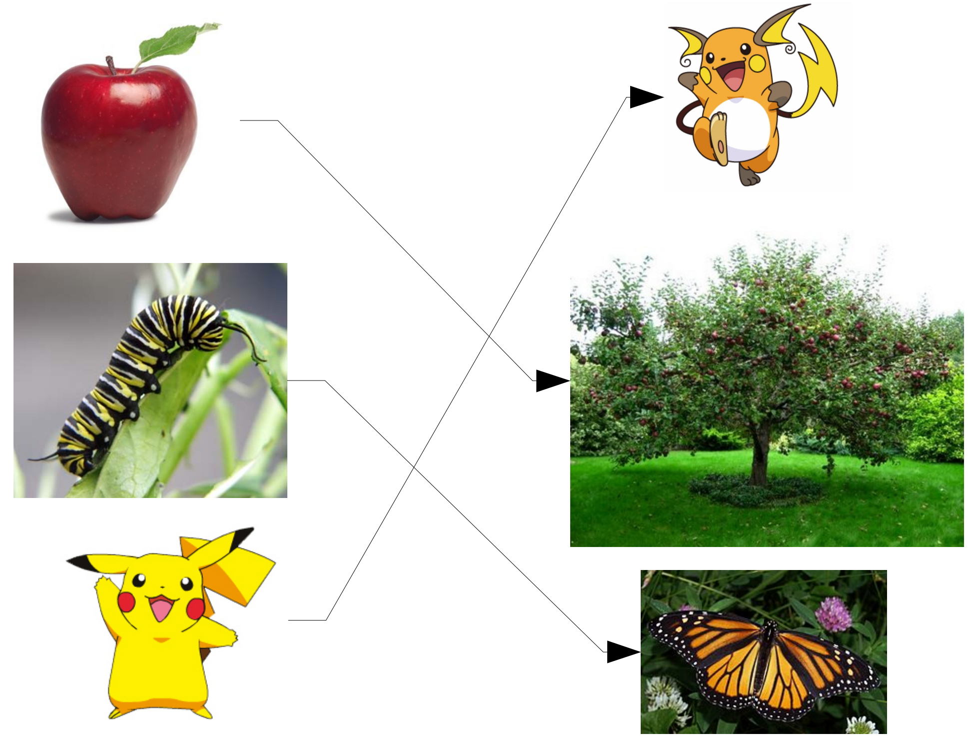
$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$
$$f(\ 0\ ) = 0^2 + 0 \cdot 2 + 5 = 5$$

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$

$$f(\ 0\ ) = 0^2 + 0 \cdot 2 + 5 = 5$$

$f($  $) = \ldots\ ?$



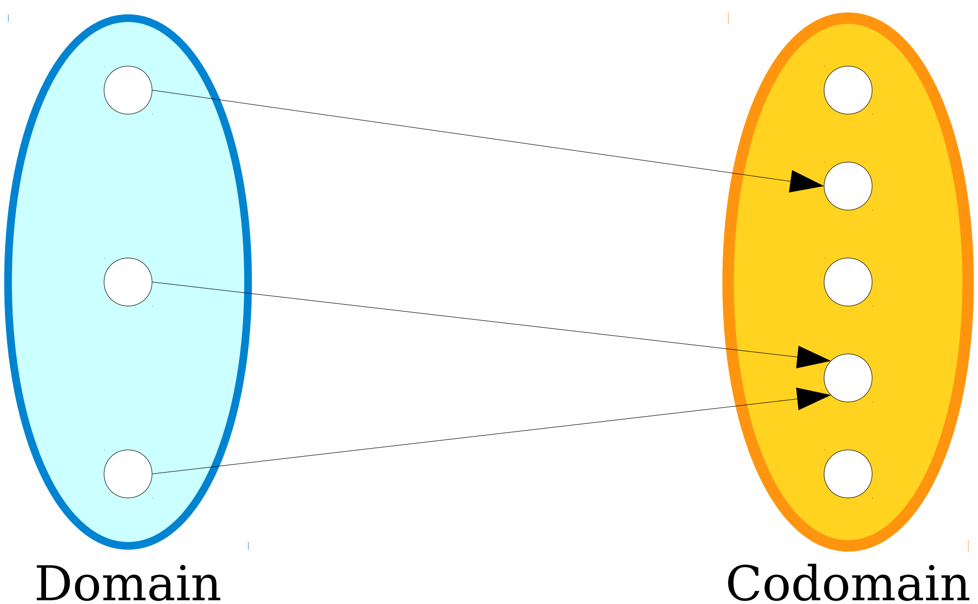
$f($  $) =$ 

$f(137) = \ldots?$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function f has two sets associated with it: its ***domain*** and its ***codomain***.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.



Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- This notation just says what the domain and codomain of the function is. It doesn't say how the function is evaluated.
- Think of it like a "function prototype" in C or C++. The notation $f : A \to B$ is like writing

  B f(A `argument);`

  We know that f takes in an A and returns a B, but we don't know exactly which B it's going to return for a given A.

Domains and Codomains

- A function f must be defined for every element of the domain.
  - For example, if $f : \mathbb{R} \to \mathbb{R}$, then the following function is ***not*** a valid choice for f:

$$f(x) = 1 / x$$

- The output of f on any element of its domain must be an element of the codomain.
  - For example, if $f : \mathbb{R} \to \mathbb{N}$, then the following function is ***not*** a valid choice for f:

$$f(x) = x$$

- However, a function f does not have to produce all possible values in its codomain.
  - For example, if $f : \mathbb{N} \to \mathbb{N}$, then the following function is a valid choice for f:

$$f(n) = n^2$$

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
  - $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
  - $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
  - $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$
- Notice that we're giving both a rule and the domain/codomain.

Defining Functions

Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.

This is the ceiling function – the smallest integer greater than or equal to x. For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

Examples:

- $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
- $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
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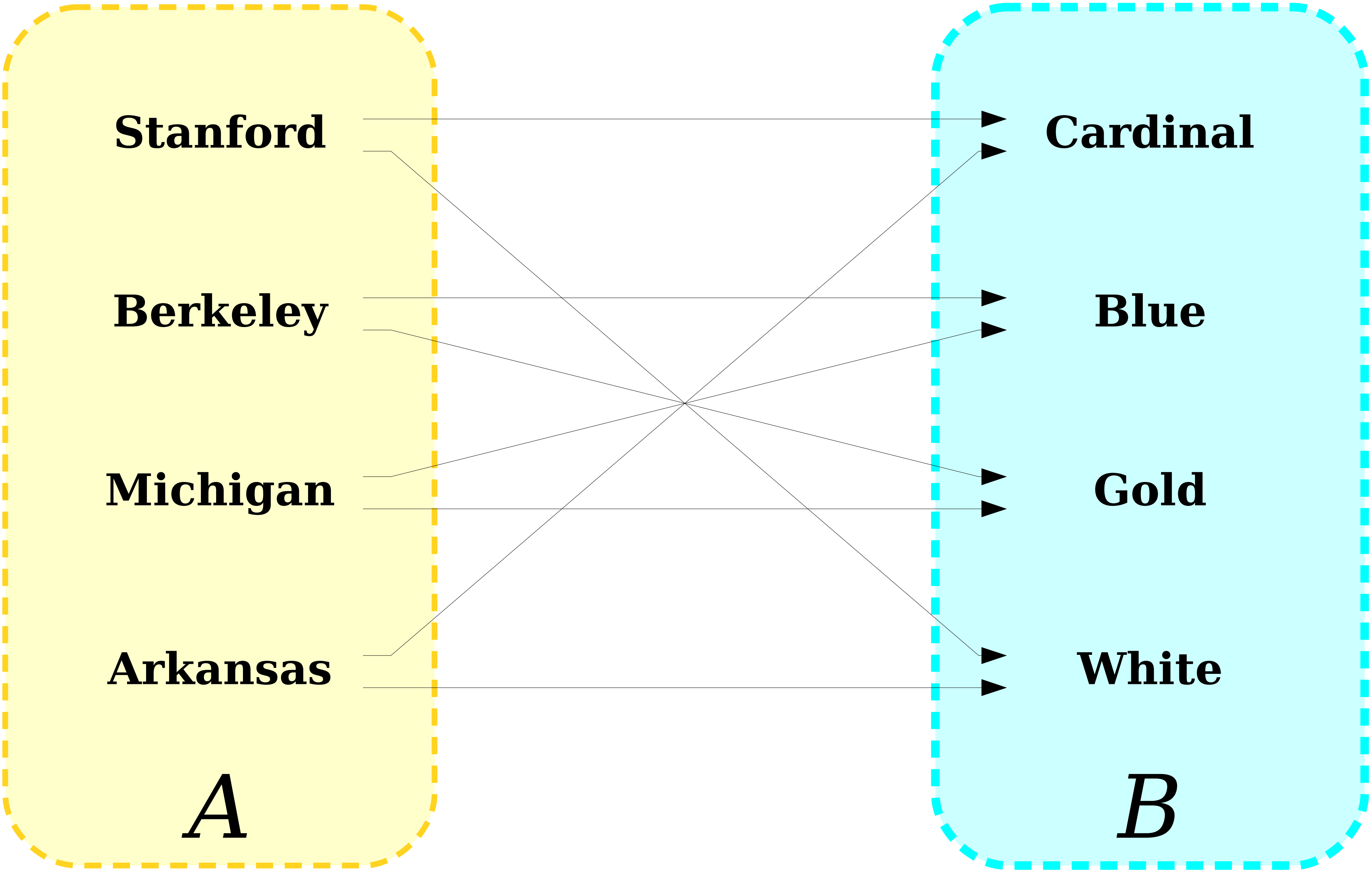
Notice that we're giving both a rule and the domain/codomain.

Is this a function from A to B?



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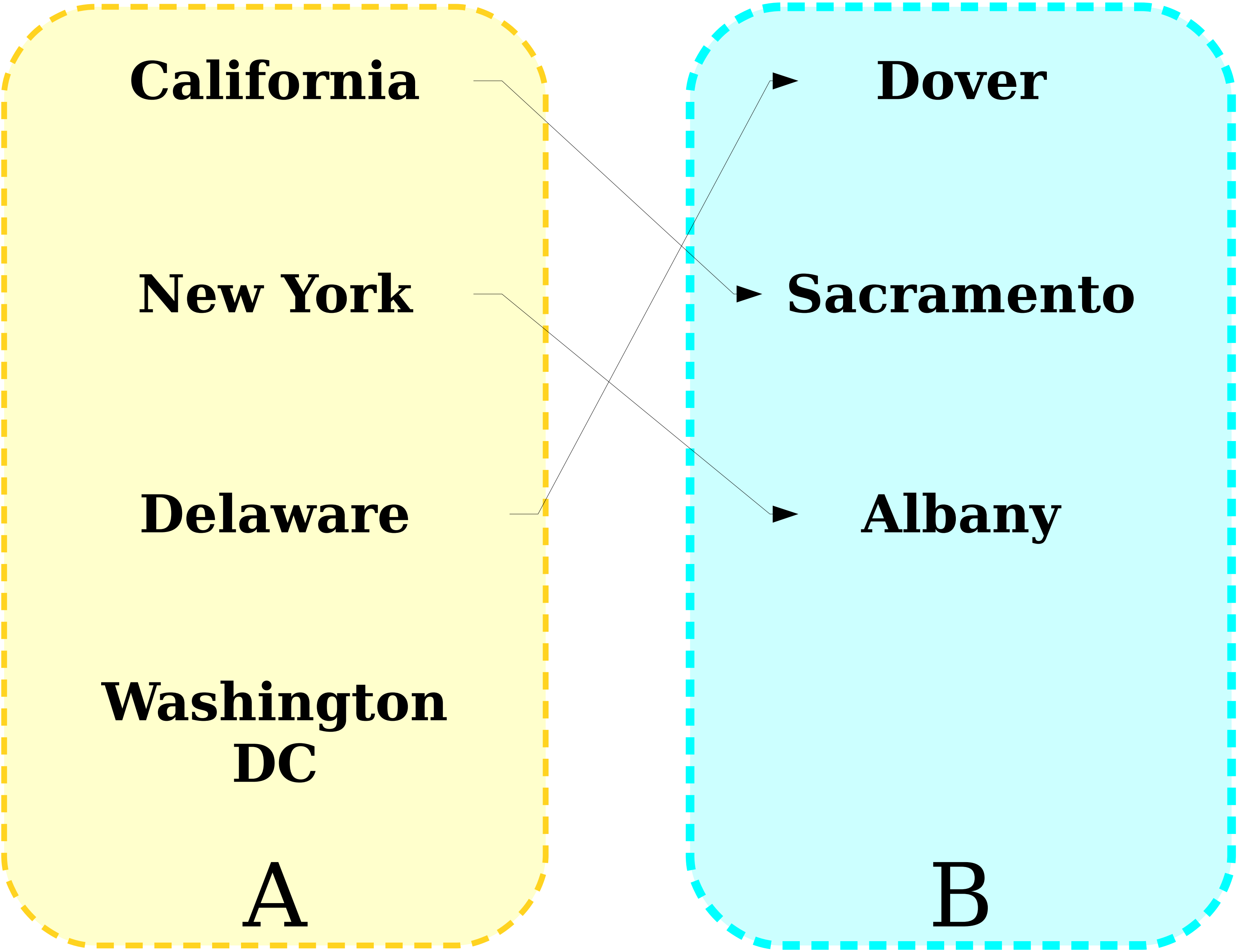
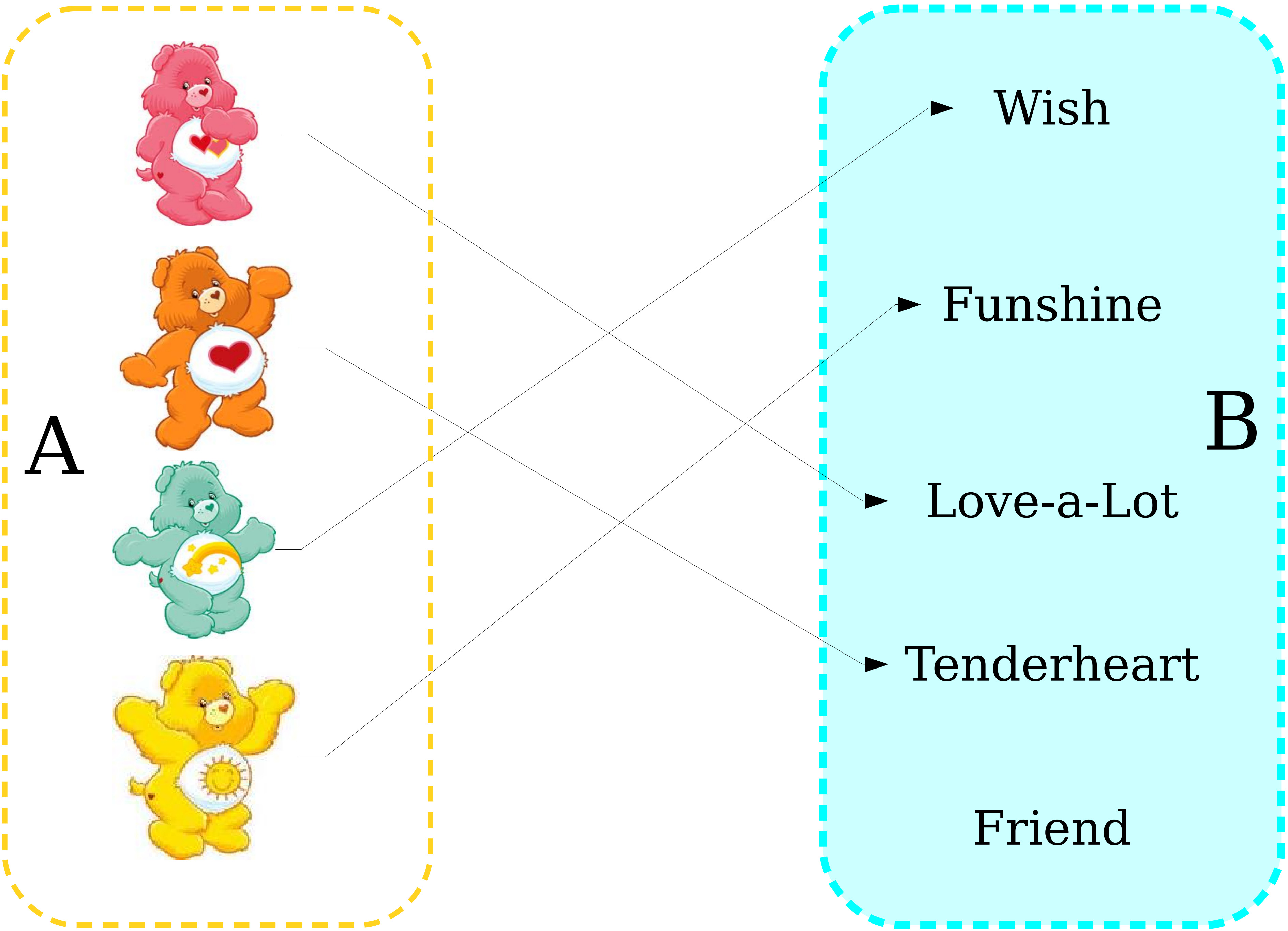
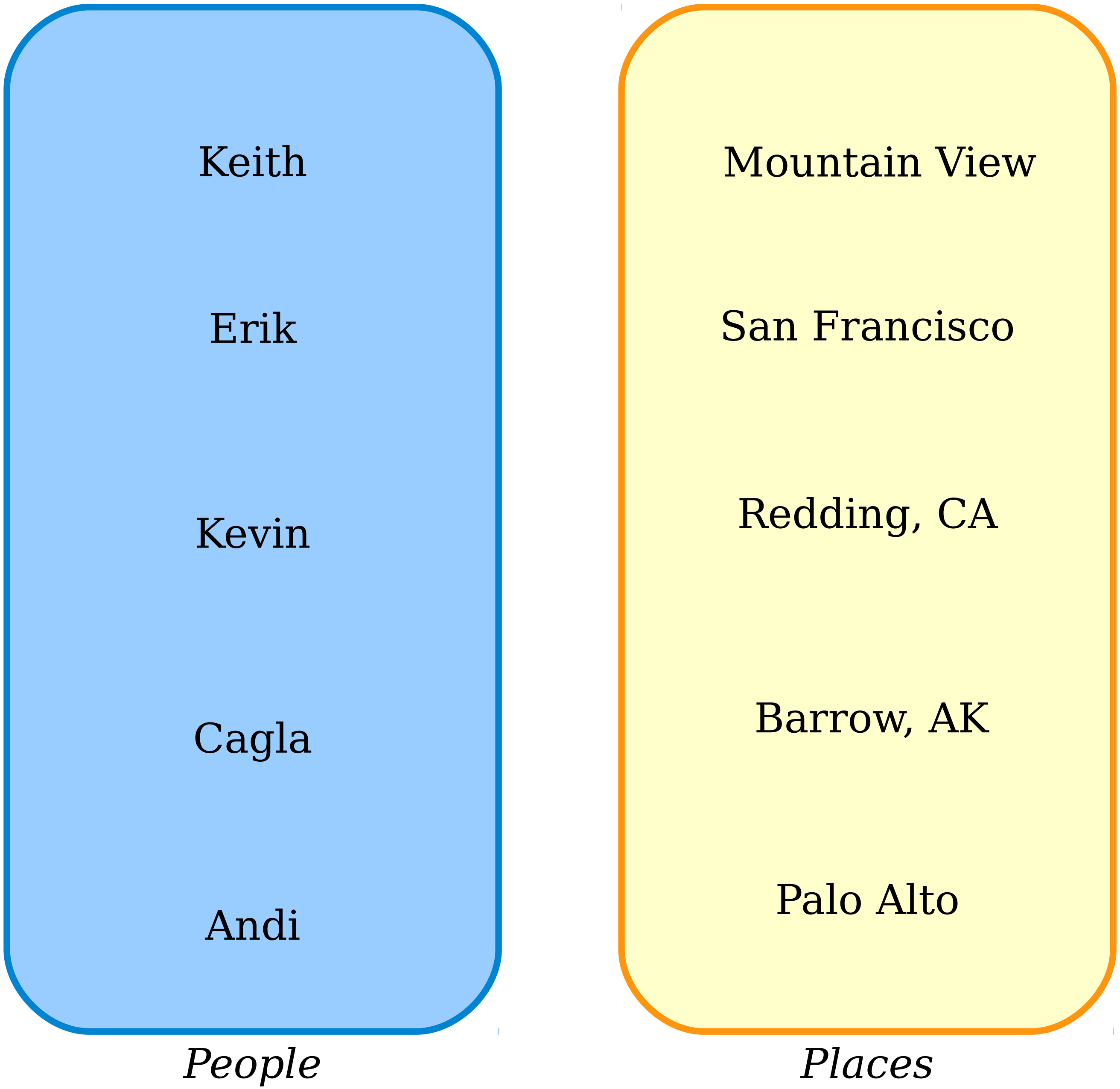


Is this a function from A to B?



Combining Functions

People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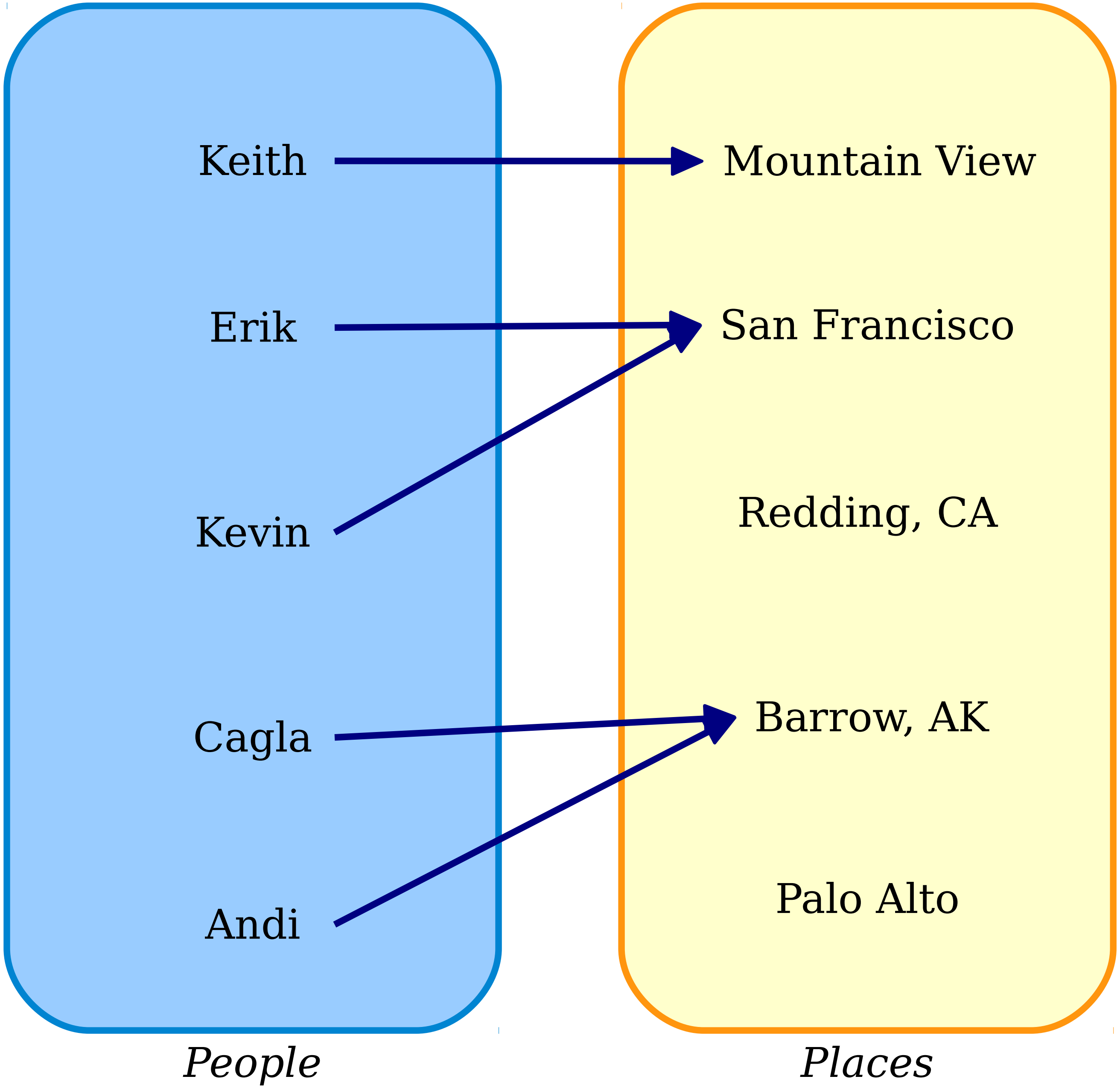
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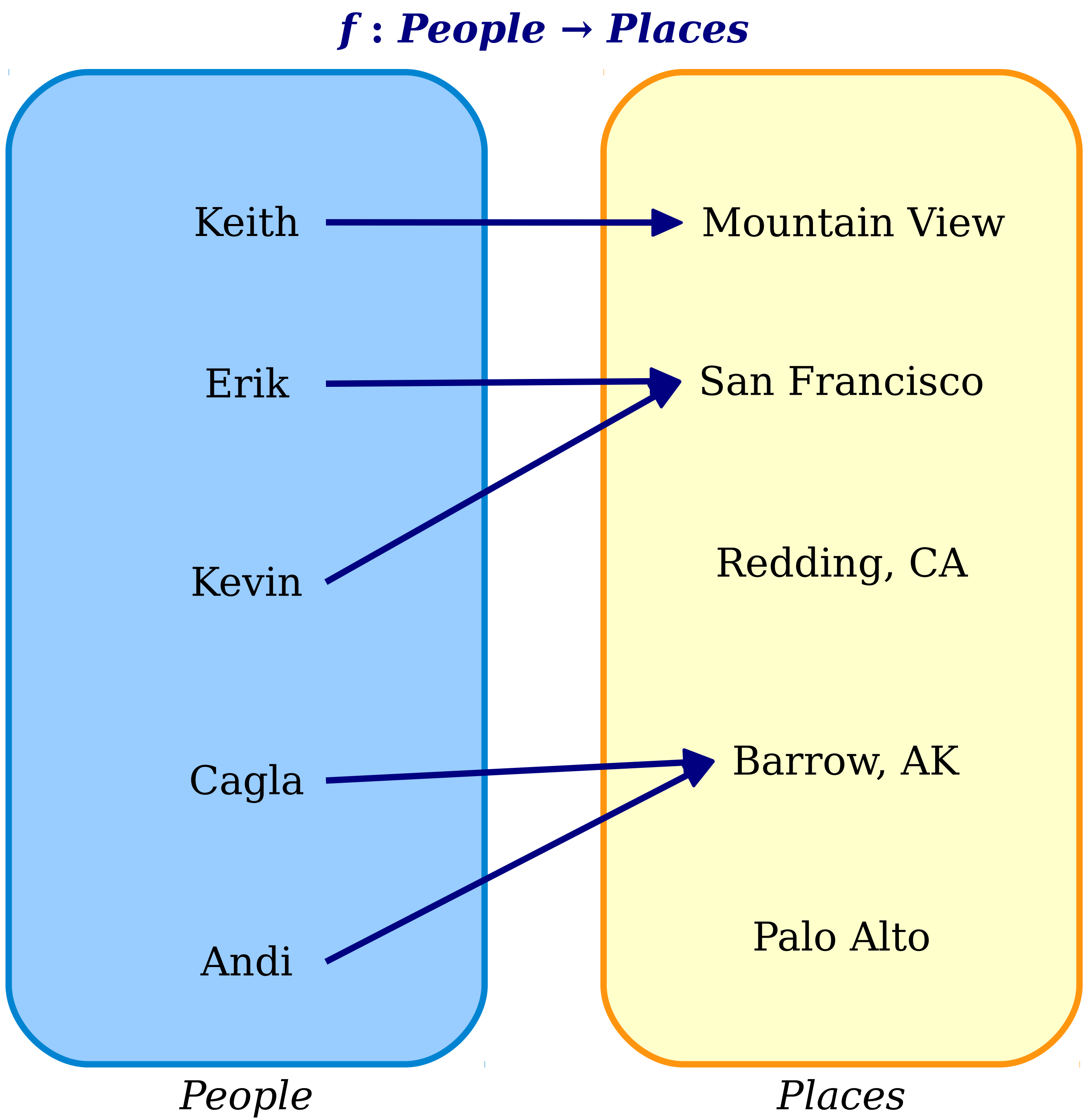
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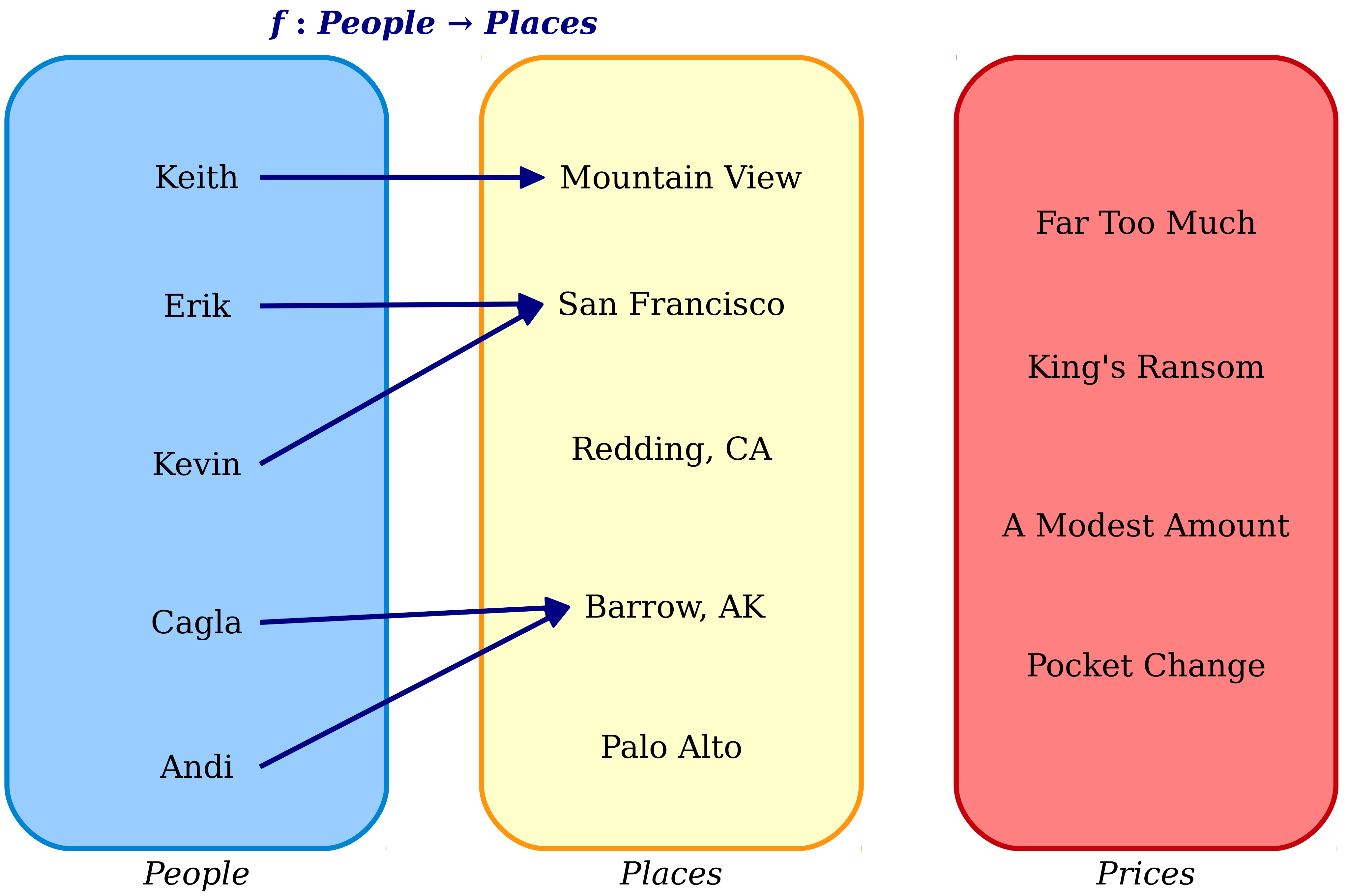
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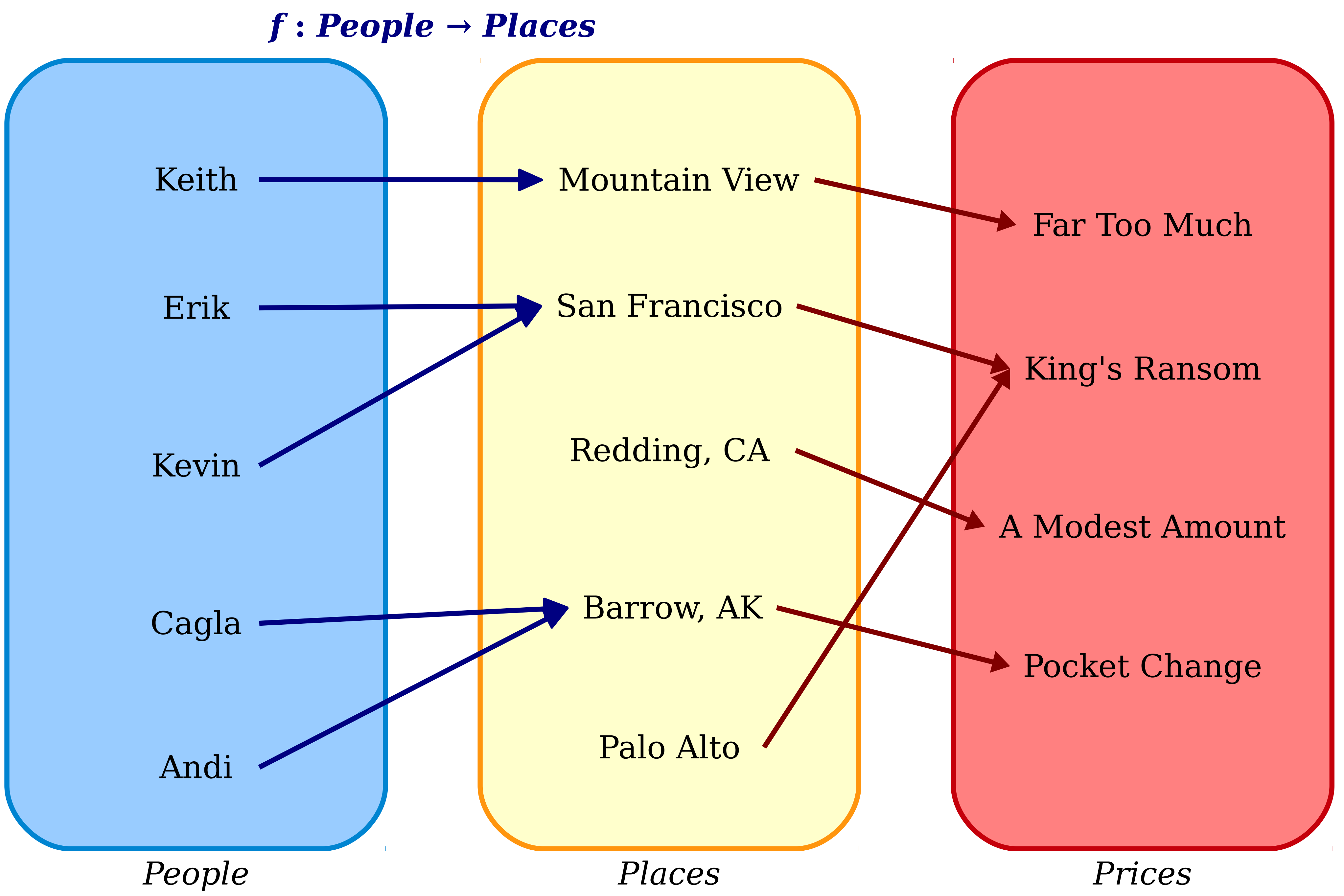

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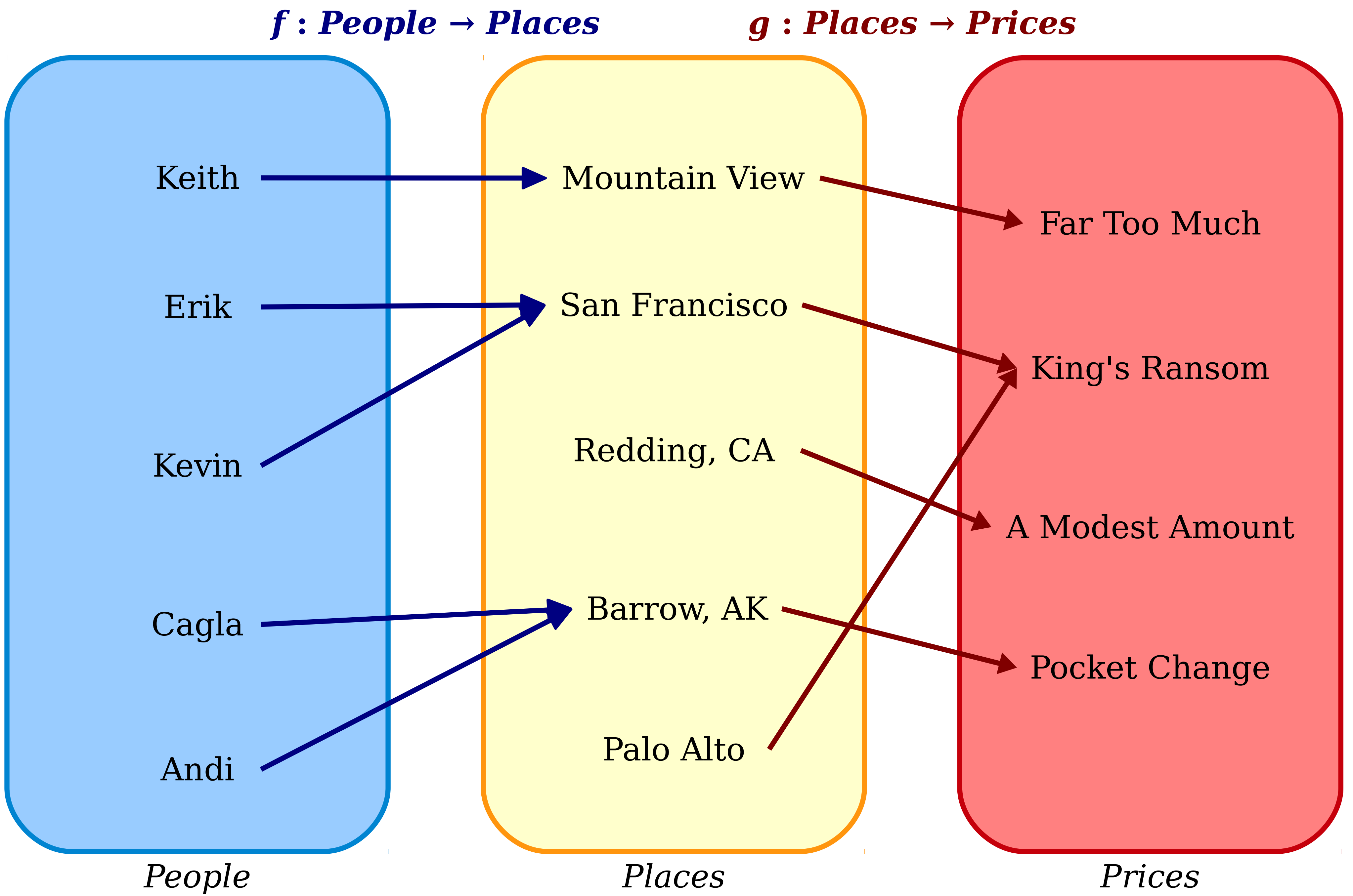

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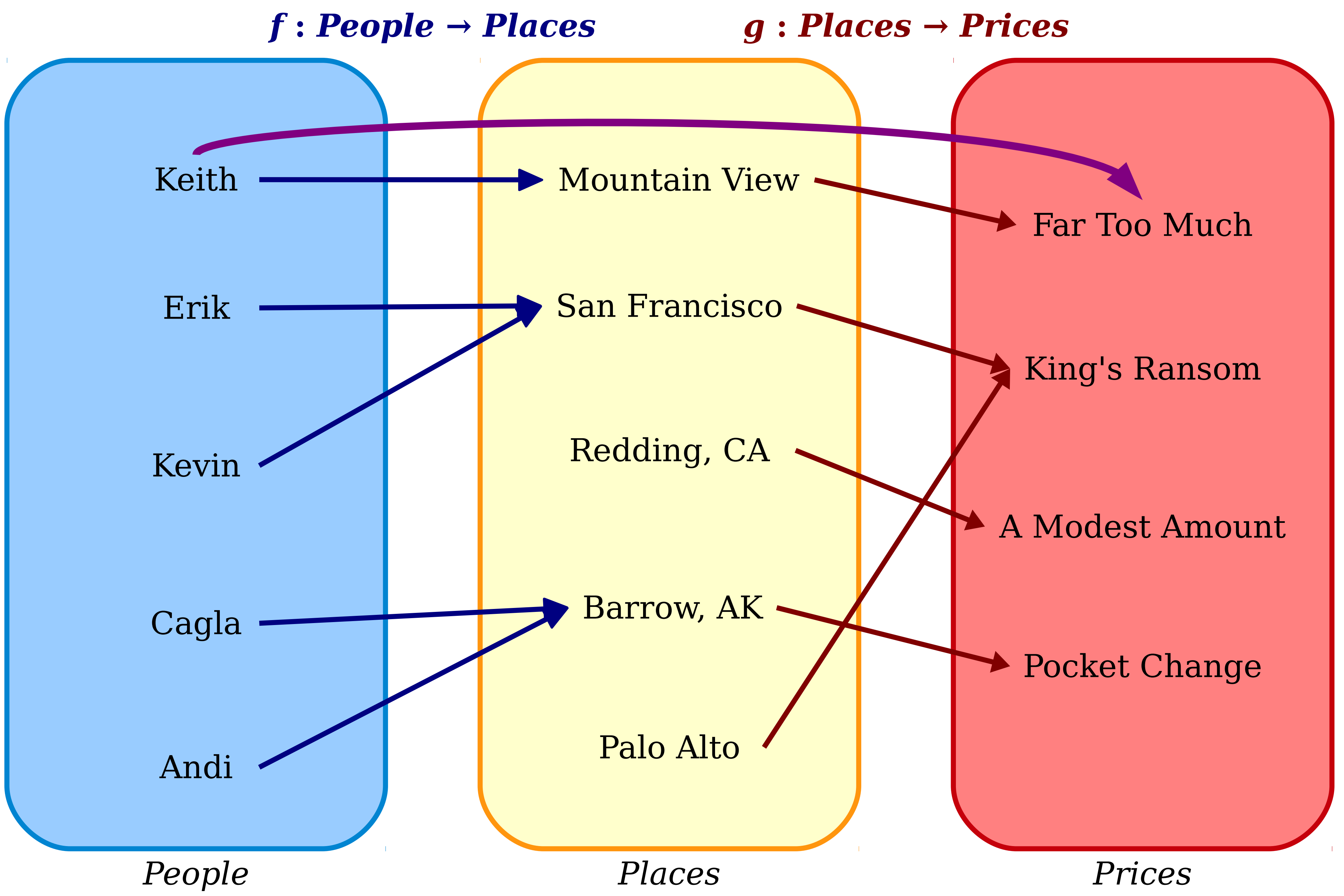

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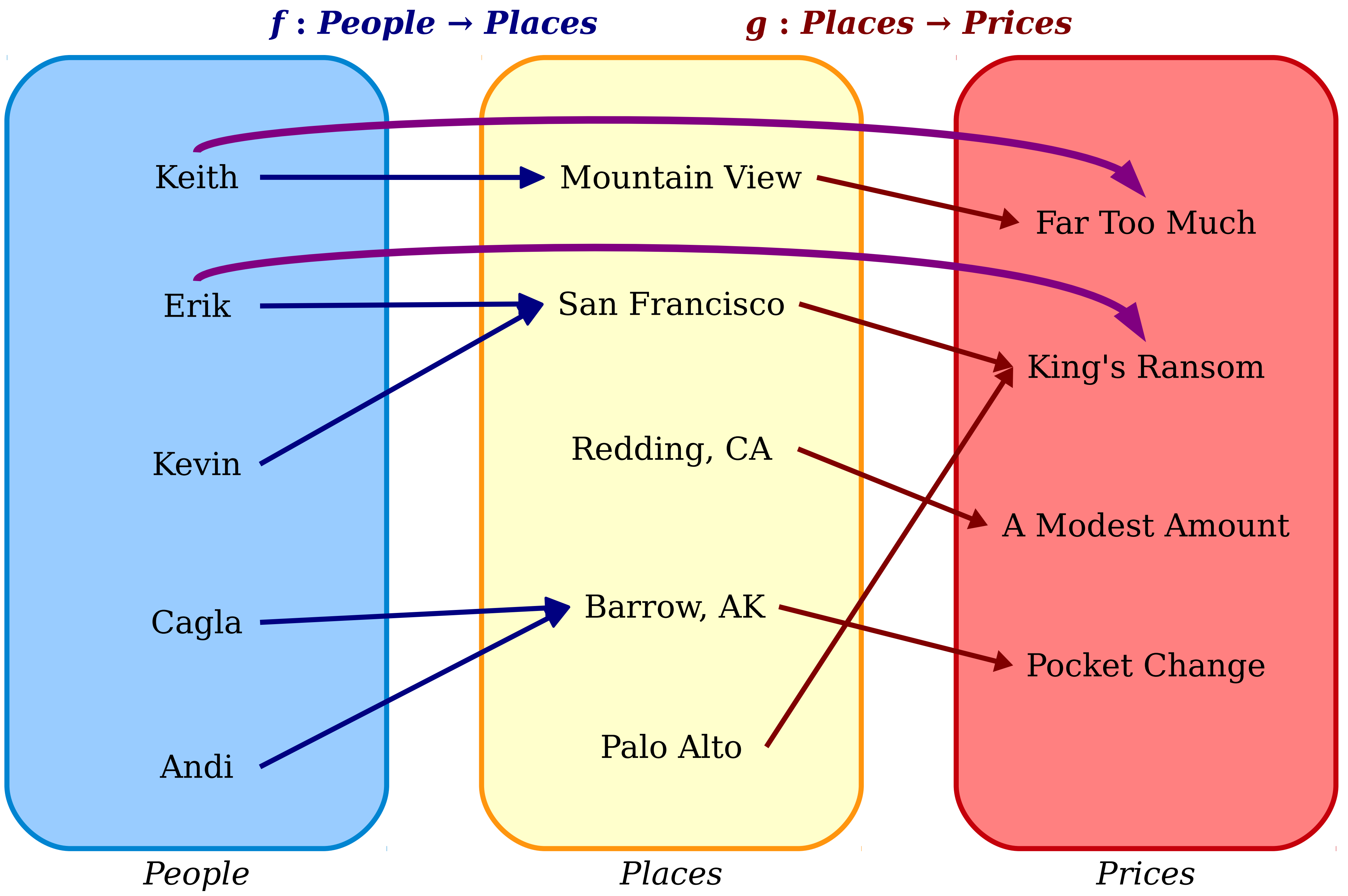
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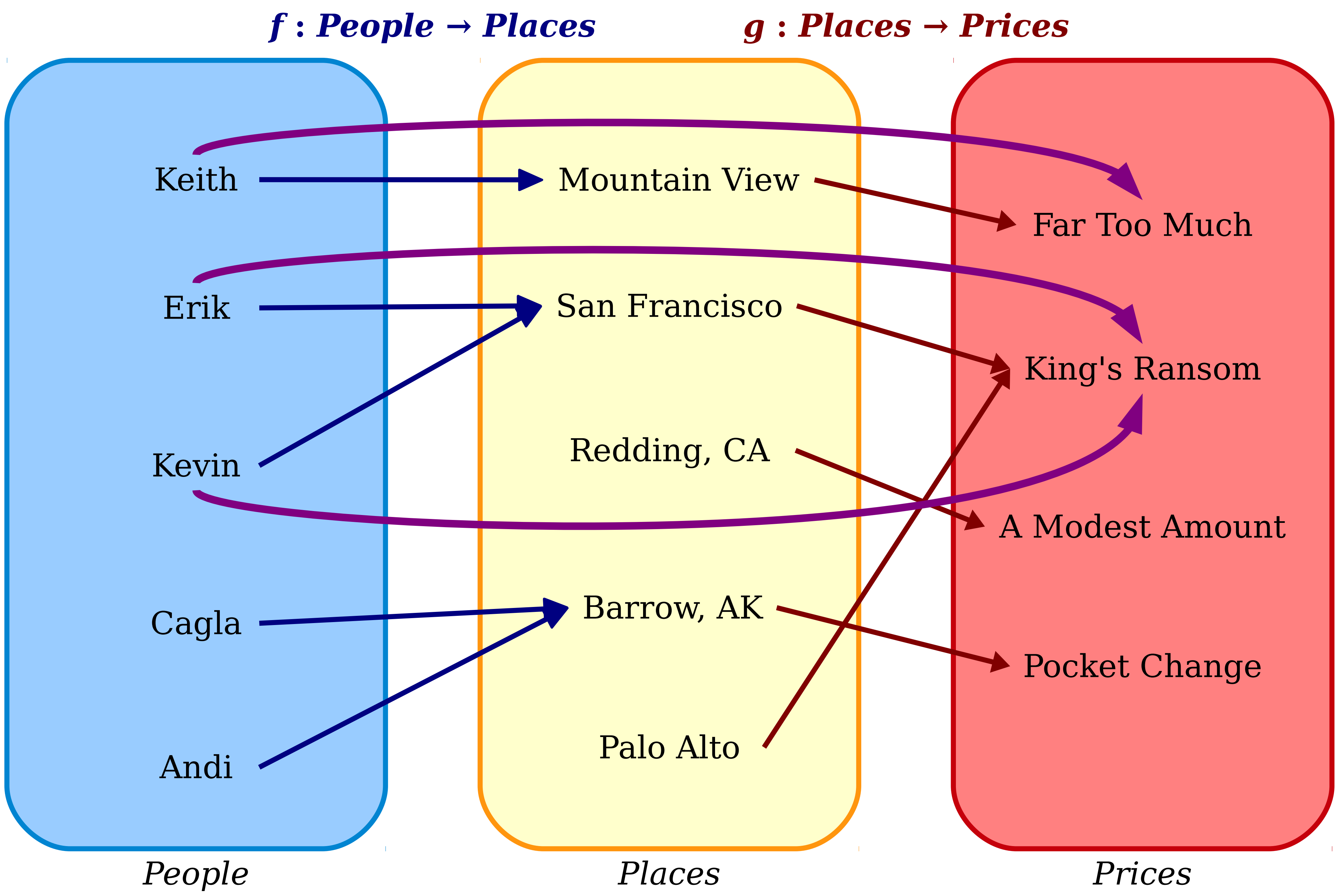
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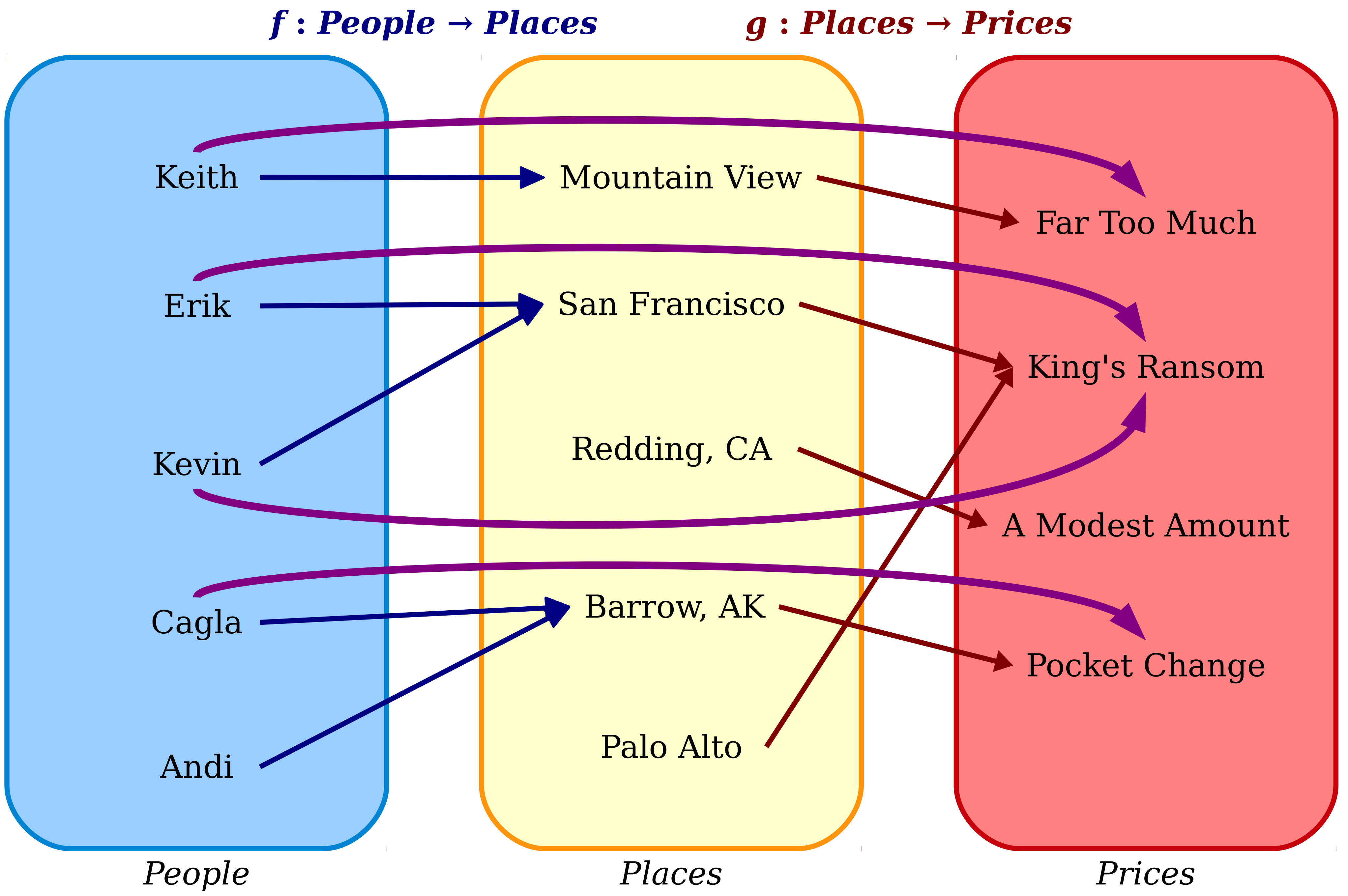

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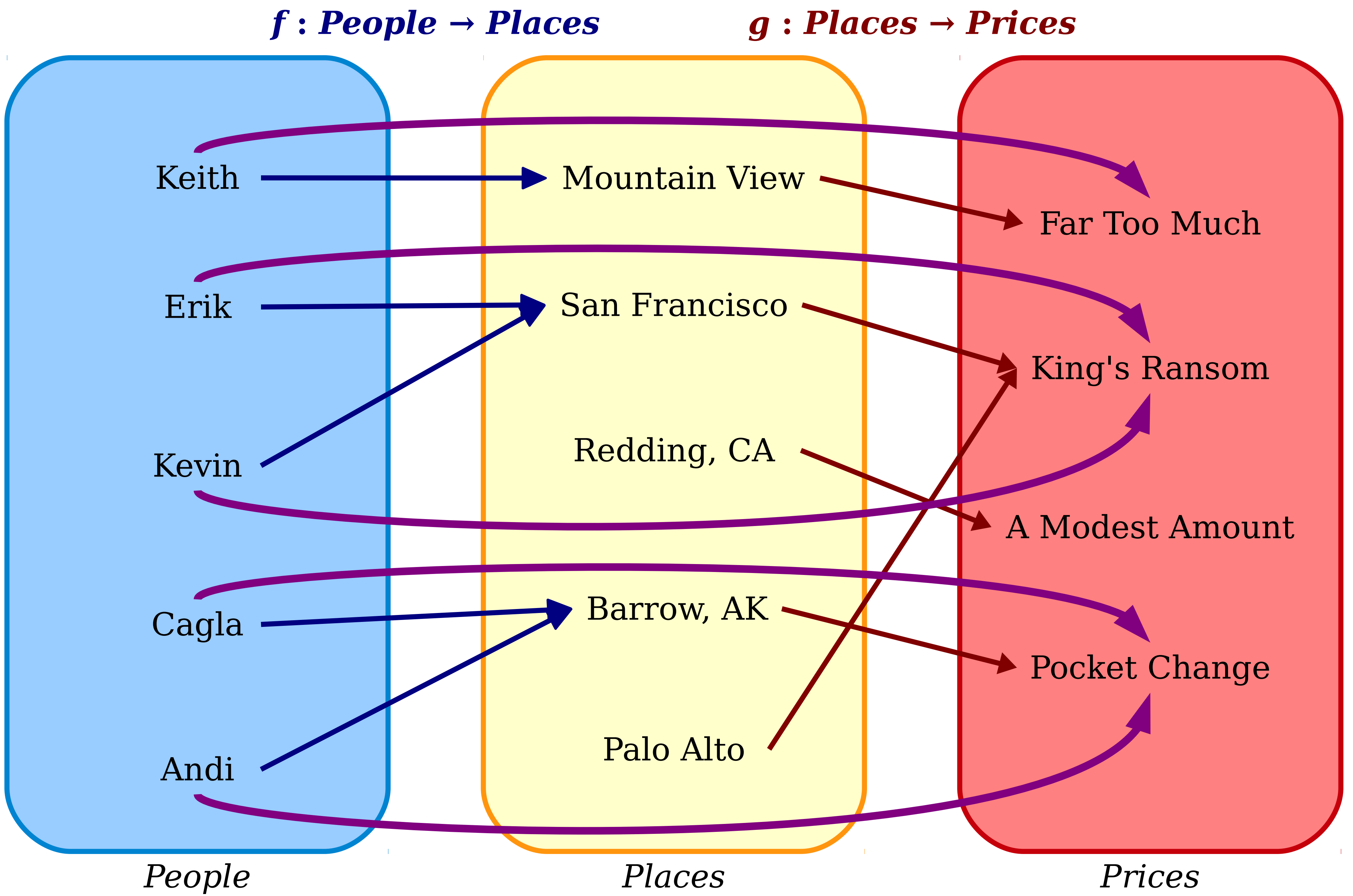

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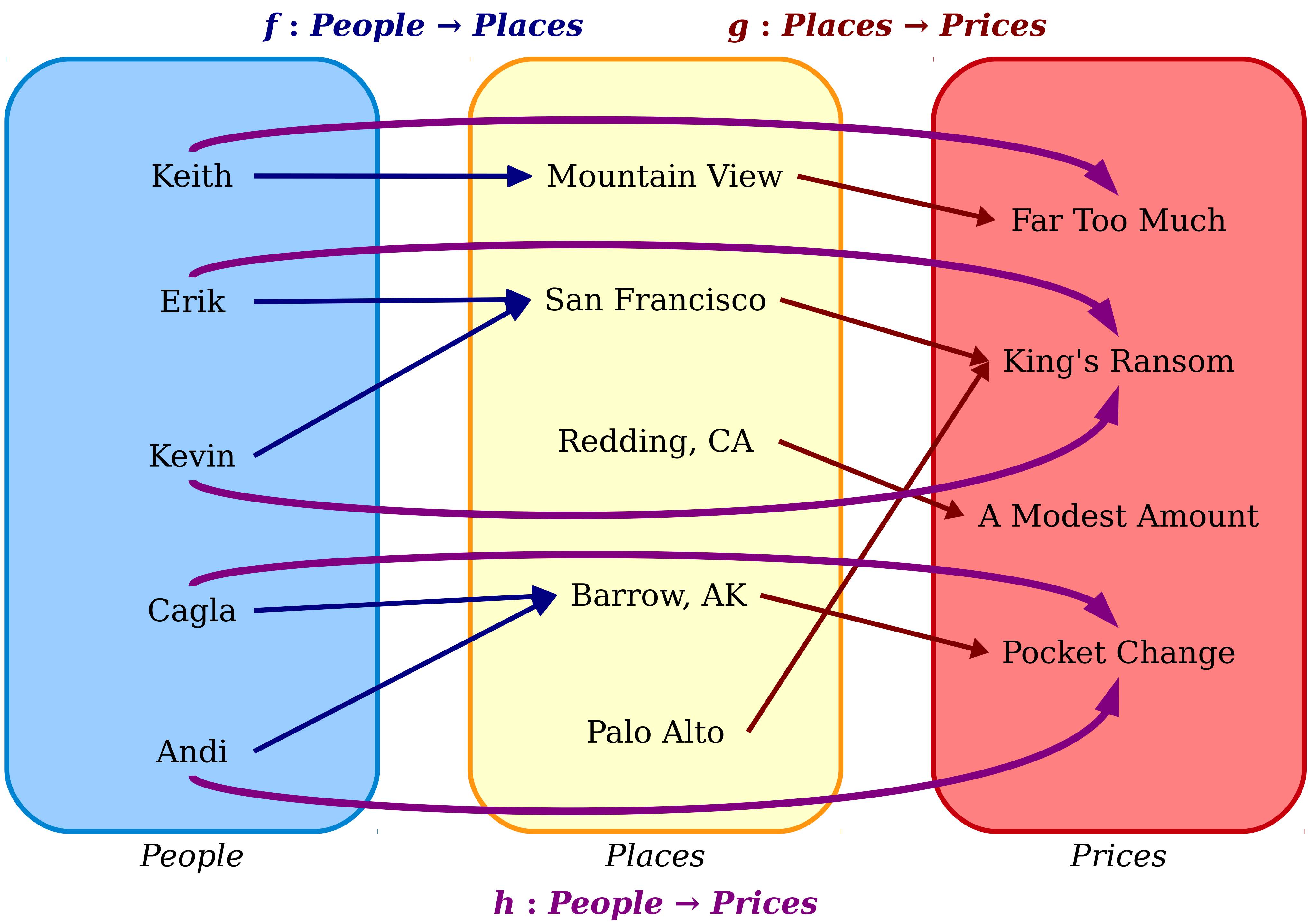

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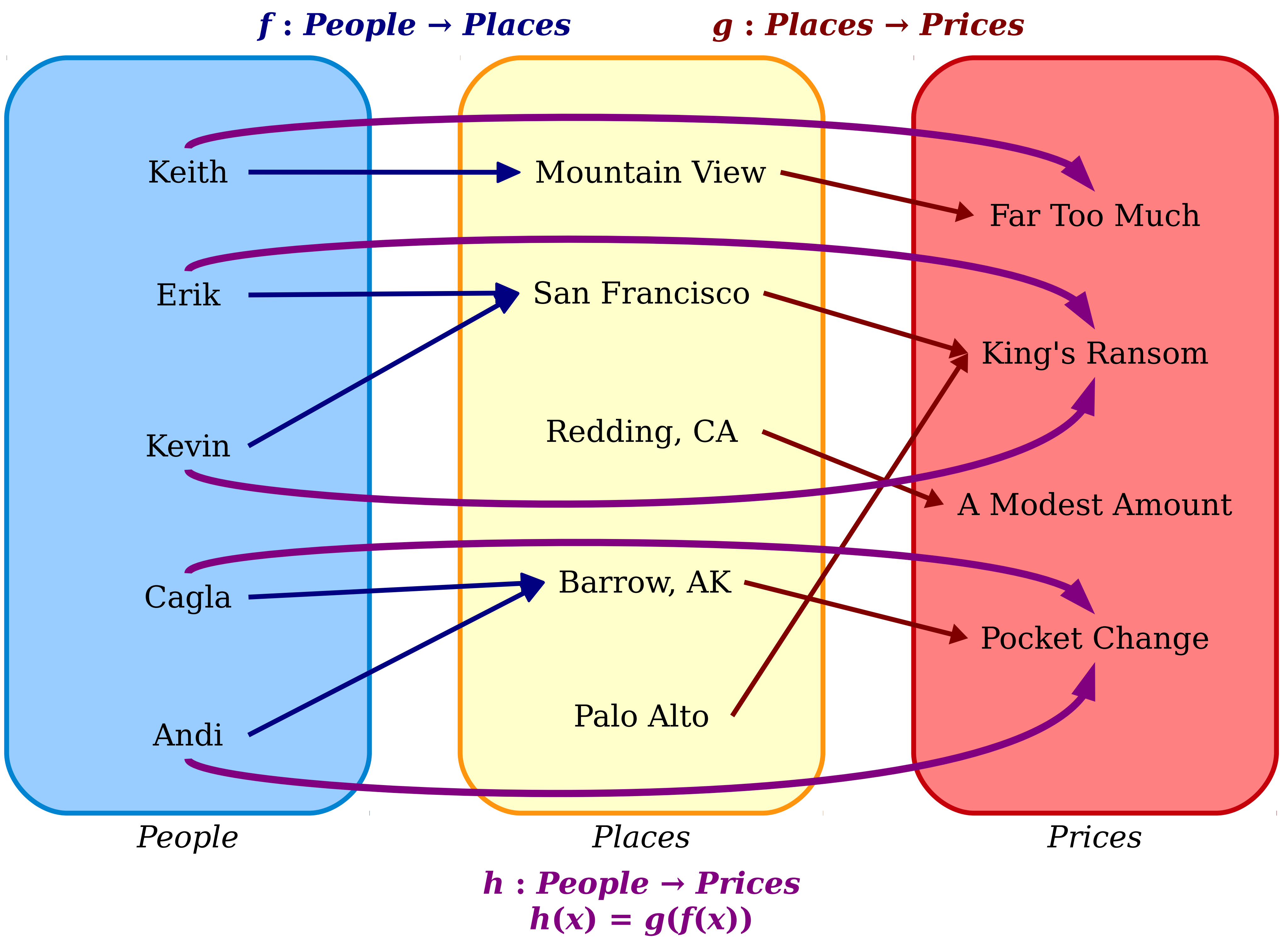
h : People → Prices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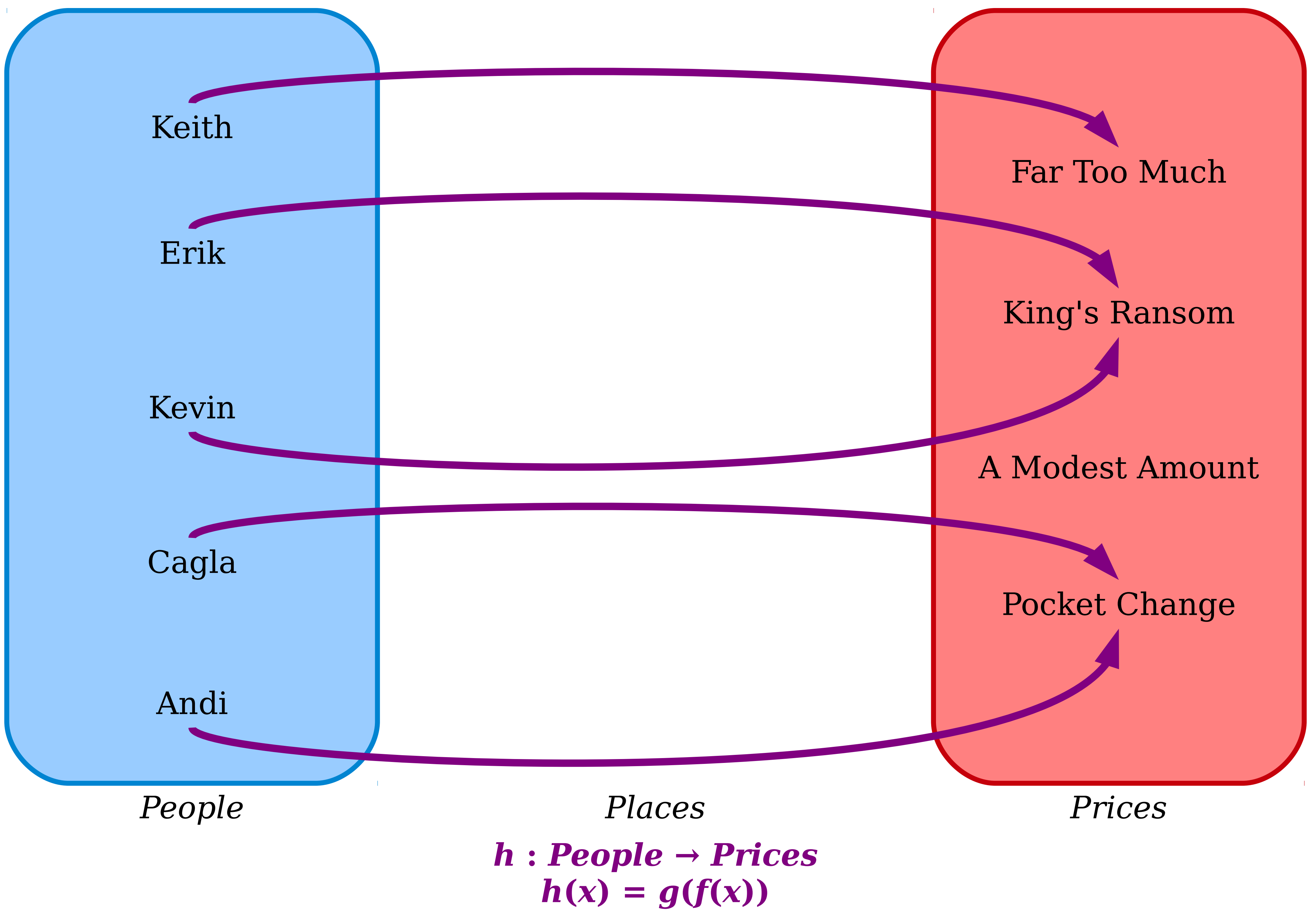
h : People → Prices
h(x) = g(f(x))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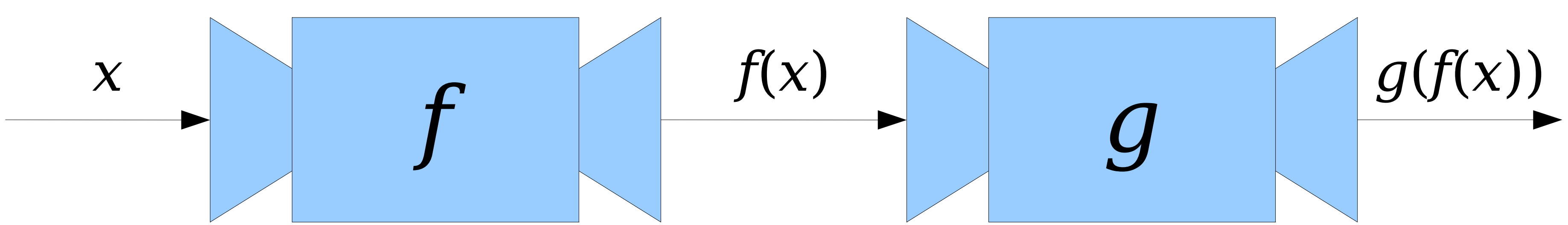
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

Function Composition

- Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = n^2$.
- What is $g \circ f$?

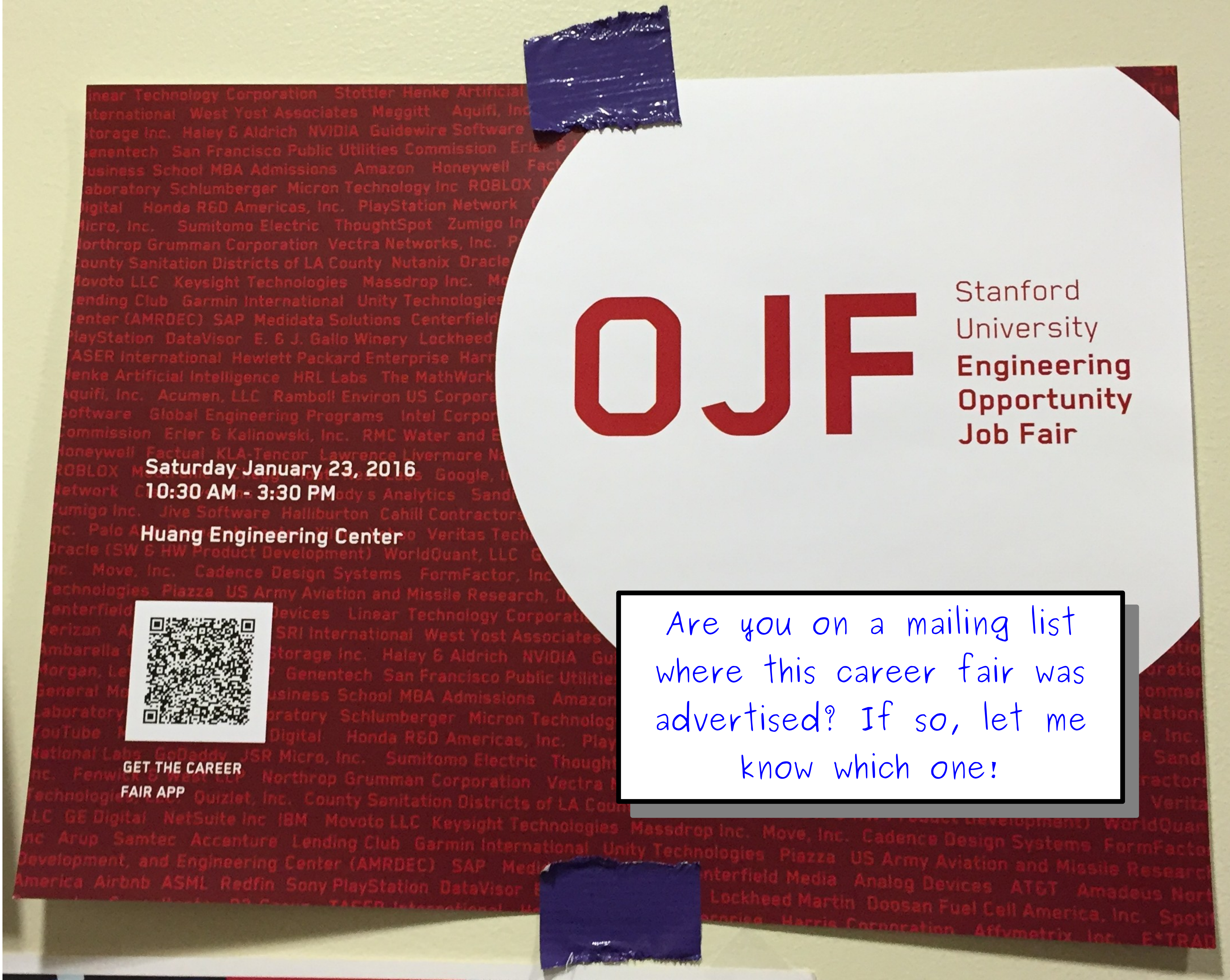
$$\begin{aligned}(g \circ f)(n) &= g(f(n)) \\ &= g(2n+1) \\ &= (2n+1)^2 = 4n^2 + 4n + 1\end{aligned}$$

- What is $f \circ g$?

$$\begin{aligned}(f \circ g)(n) &= f(g(n)) \\ &= f(n^2) \\ &= 2n^2 + 1\end{aligned}$$

- In general, if they exist, the functions $g \circ f$ and $f \circ g$ are usually not the same function. ***Order matters in function composition!***

Time-Out for Announcements!



Are you on a mailing list where this career fair was advertised? If so, let me know which one!

WiCS Casual CS Dinner

- Stanford WiCS (Women in Computer Science) is holding a Casual CS Dinner tonight at 6:00PM at the Women's Community Center.
- All are welcome. Highly recommended!

oSTEM Mixer

- Stanford's chapter of oSTEM (Out in STEM) is holding a mixer event tonight at 6:00PM at the LGBT-CRC.
  - If I'm not mistaken, that's literally right above the Casual CS Dinner!
- Interested in attending? RSVP using this link.

Problem Set One

- Problem Set One has been graded. Feedback is now available online through GradeScope, and solutions are now available in hardcopy.
- Please read Handout #15 (“Reviewing Graded Work”) for our advice about what to do next. In particular:
  - Make sure you understand every piece of feedback you received.
  - Ask questions about feedback you don't fully understand.
  - Read over our solution sets, especially the “why we asked this question” section, to make sure that you understand what skills we were trying to help you build.
- Late PS1 submissions will be returned by tomorrow afternoon at 3:00PM.

Problem Set Two

- The checkpoint assignments for PS2 have been graded.
- Please be sure to read over the checkpoint solutions set – there's a lot of information in there!
- The remaining problems are due on Friday. Please stop by office hours with questions, and continue to ask questions on Piazza!

Problem Set One: A Common Mistake

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$.

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$.

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

Since $y - x = kr$, we see that $x - y = -kr$.

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Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

Since $y - x = kr$, we see that $x - y = -kr$. Earlier we noted that $x - y = kq$, so collectively we see that $-kr = kq$.

An Incorrect Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

Since $y - x = kr$, we see that $x - y = -kr$. Earlier we noted that $x - y = kq$, so collectively we see that $-kr = kq$. Therefore, we see that $r = -q$.

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Since $y - x = kr$, we see that $x - y = -kr$. Earlier we noted that $x - y = kq$, so collectively we see that $-kr = kq$. Therefore, we see that $r = -q$. ■

We're assuming what we're trying to prove!

A Better Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

A Better Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

Since $x - y = kq$, we see that $y - x = -kq$

A Better Proof

Theorem: For any integers x, y, and k, if $x \equiv_k y$, then $y \equiv_k x$.

Proof: Consider any arbitrary integers x, y, and k where $x \equiv_k y$. This means that there is an integer q where $x - y = kq$. We need to prove that $y \equiv_k x$, meaning that we need to prove that there is an integer r where $y - x = kr$.

Since $x - y = kq$, we see that $y - x = -kq = k(-q)$.

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Notice that we start with our initial assumptions and use them to derive the required result.

General Advice

- ***Be careful not to assume what you're trying to prove.***
- In a proof, we recommend using the phrases “we need to show that” or “we need to prove that” to clearly indicate your goals.
- If you later find yourself relying on a statement marked “we need to prove that,” chances are you've made an error in your proof.

Your Questions

“What was the best piece of advice you ever received? Both math-related and not.”

For math advice: “All models are wrong. Some are useful.” (source)

General life advice: “if you can’t make your opponent’s point for them, you don’t truly grasp the issue.” (source)

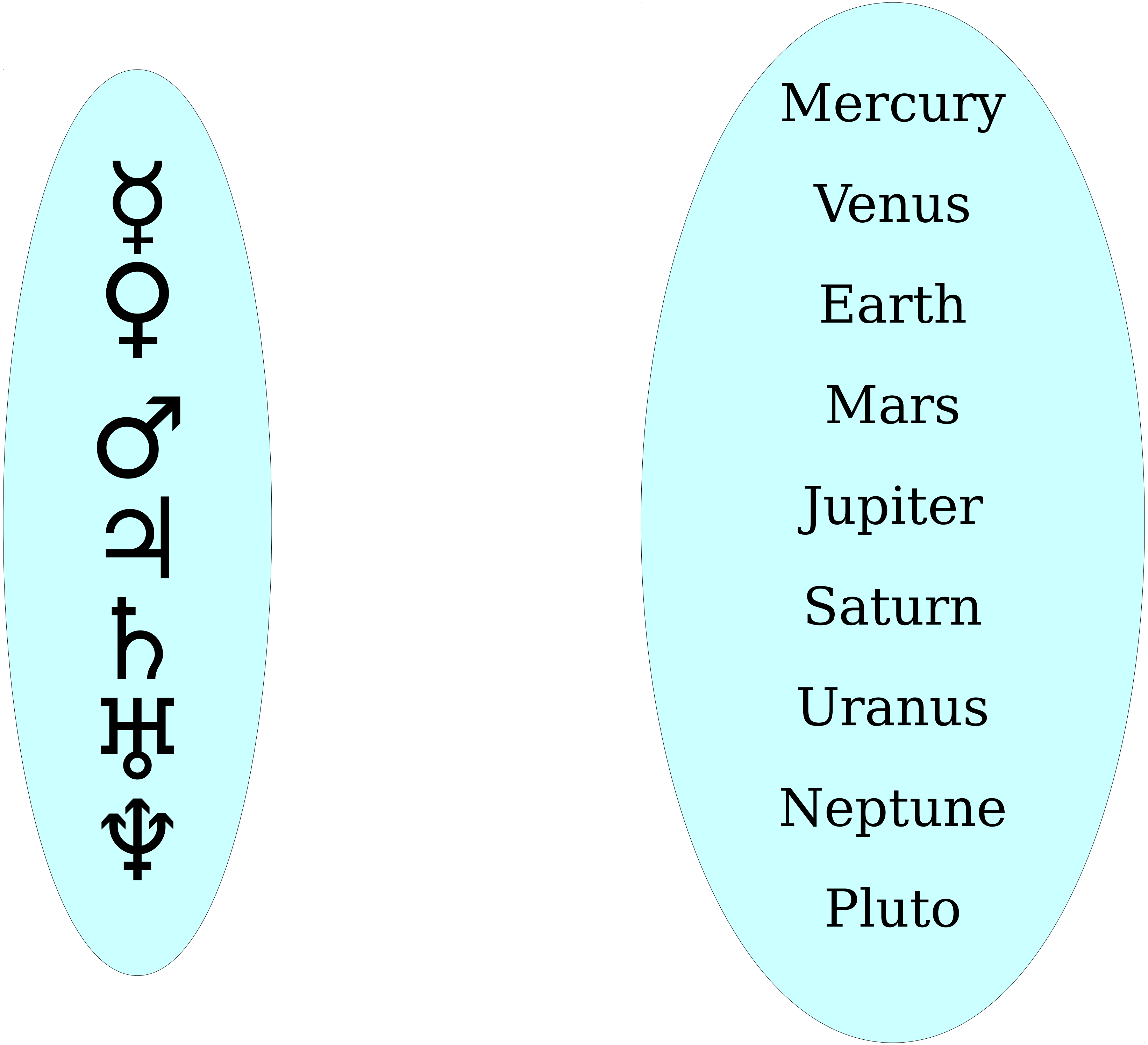
"You asked us to ask you again about research versus industry over the summer. :-)"

We're lucky to be in a spot where undergrads can get internships and where recruiting is strong. However, I think it's important that everyone get some research experience as well. Keep in mind that post-graduation, unless you directly go for a Ph.D, it's really, *really* hard to get into research. It's worth exploring it to see if it's something you're interested in before you graduate, even if you decide not to continue on in academia.

If you're an undergraduate in CS, I strongly recommend looking into the CURIS summer research program. It's a great way to get exposed to research, make connections in the CS department, and get a feel for what academia is like.

Back to CS103!

Special Types of Functions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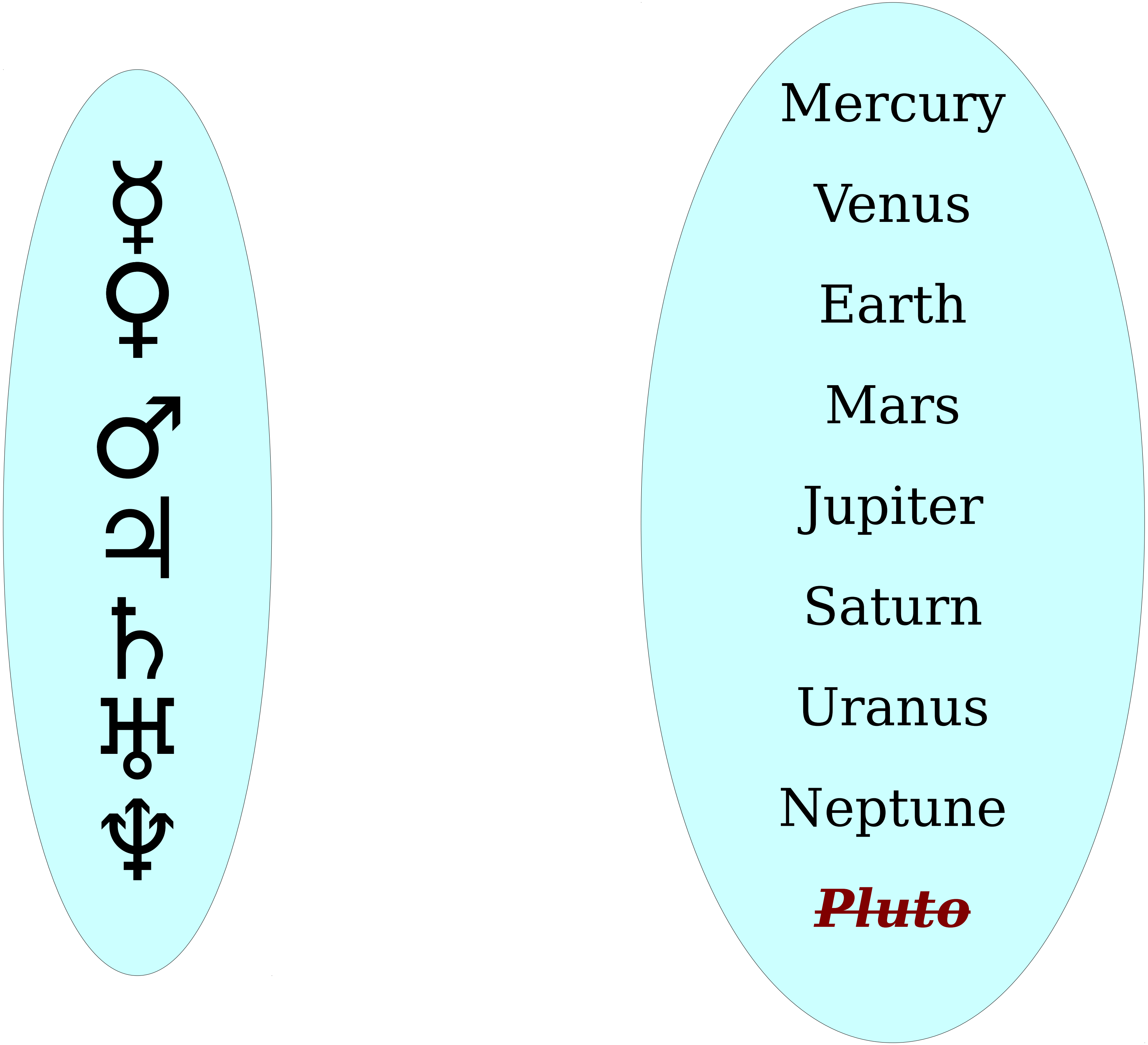
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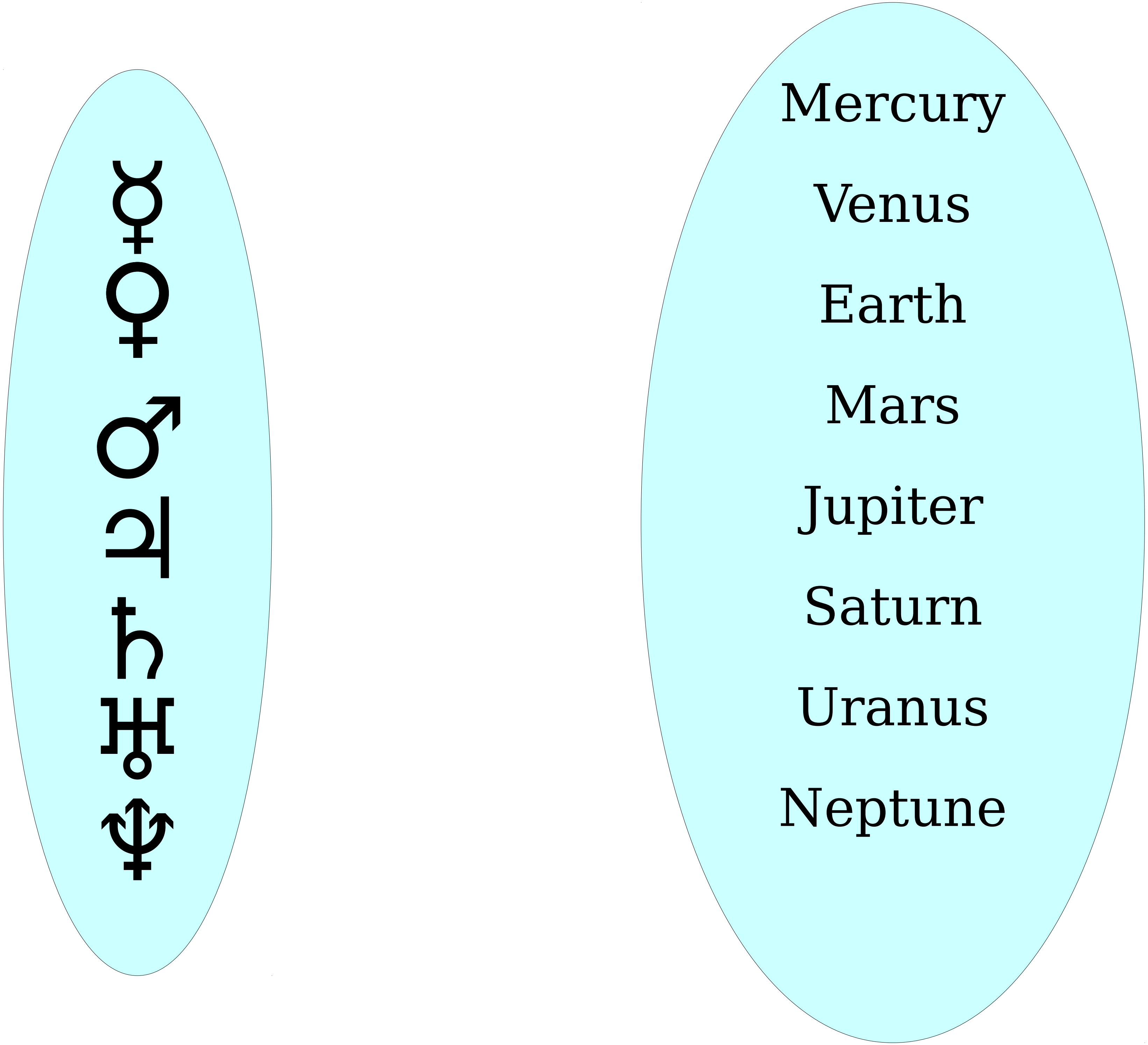
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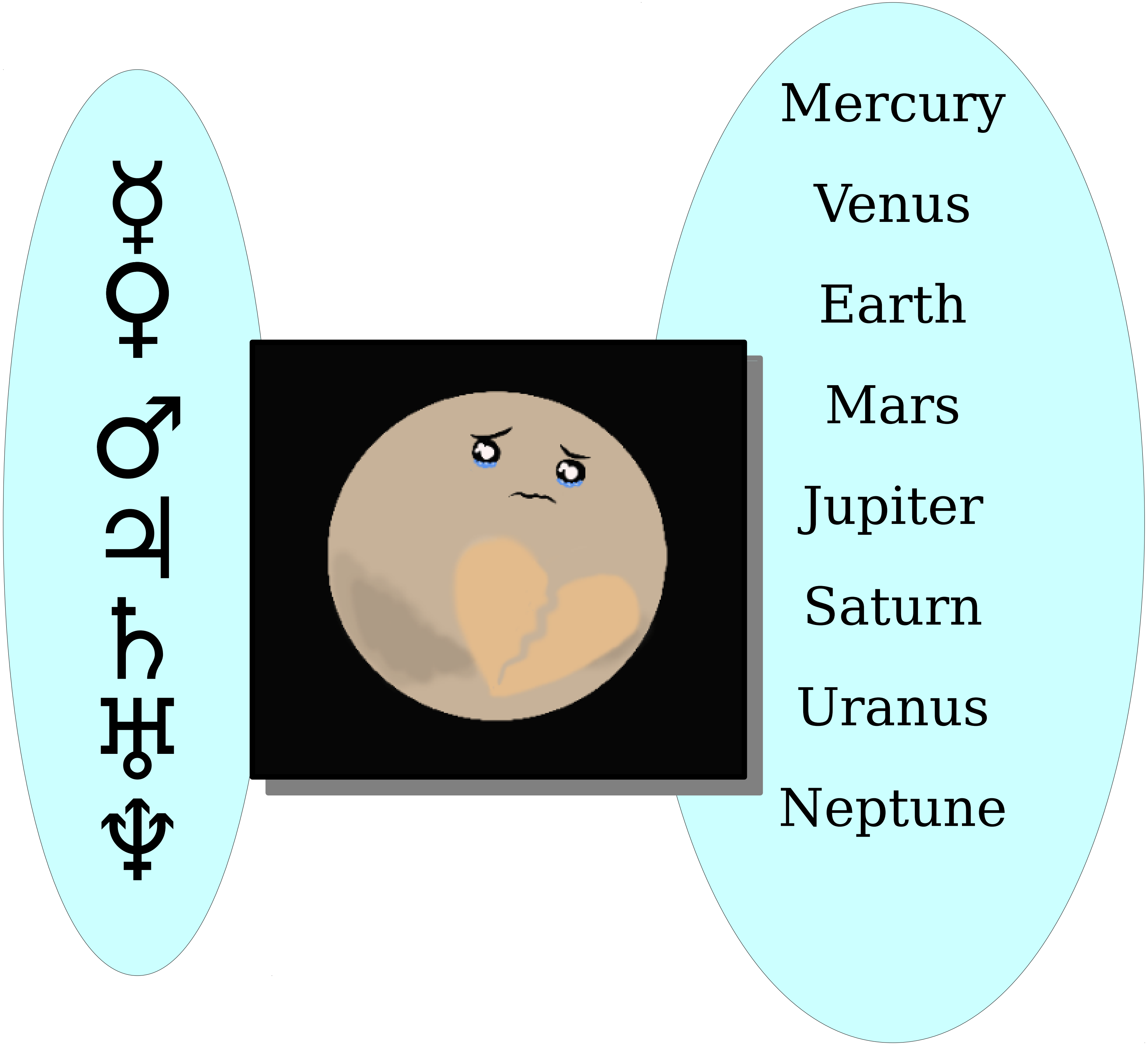
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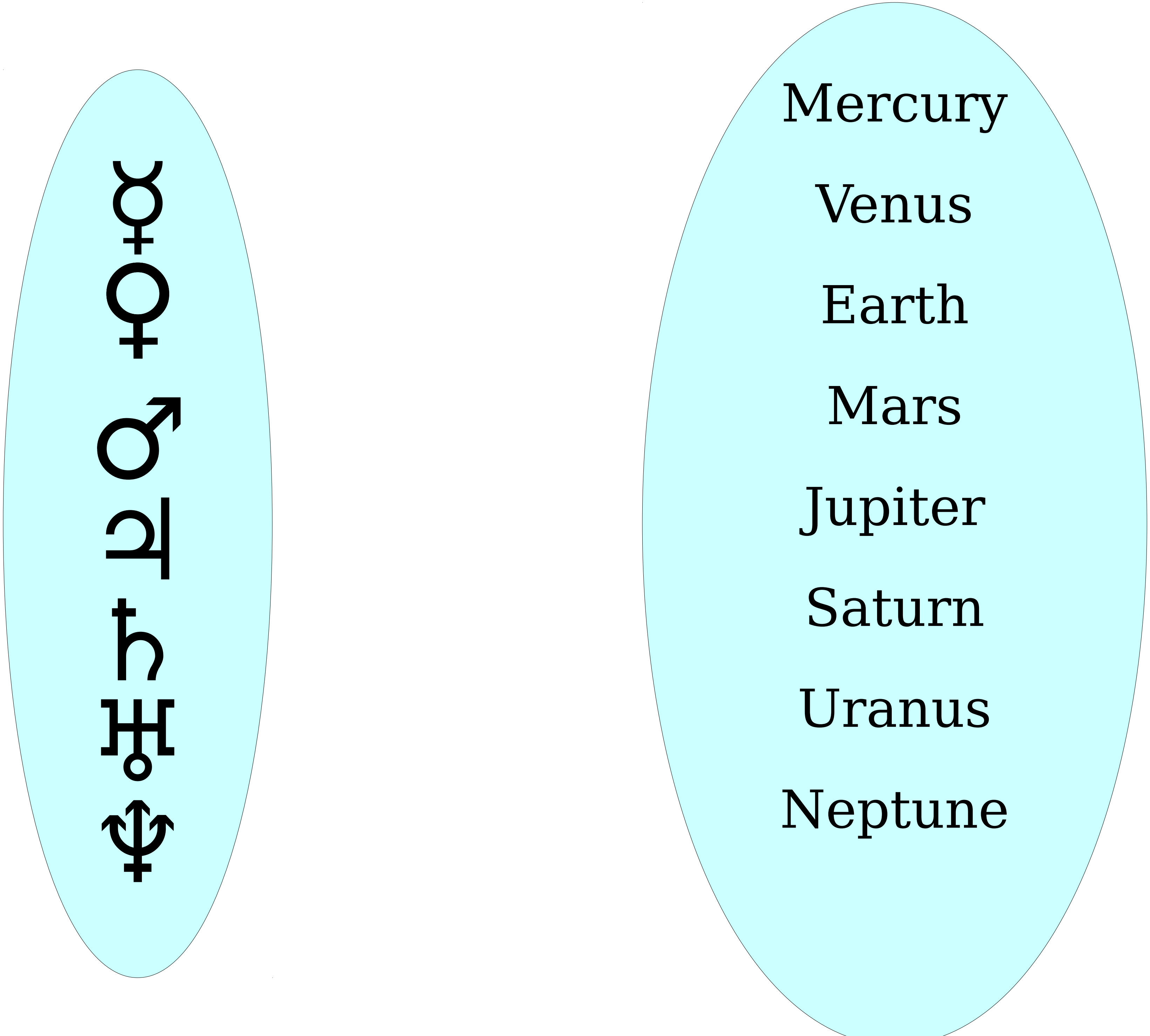
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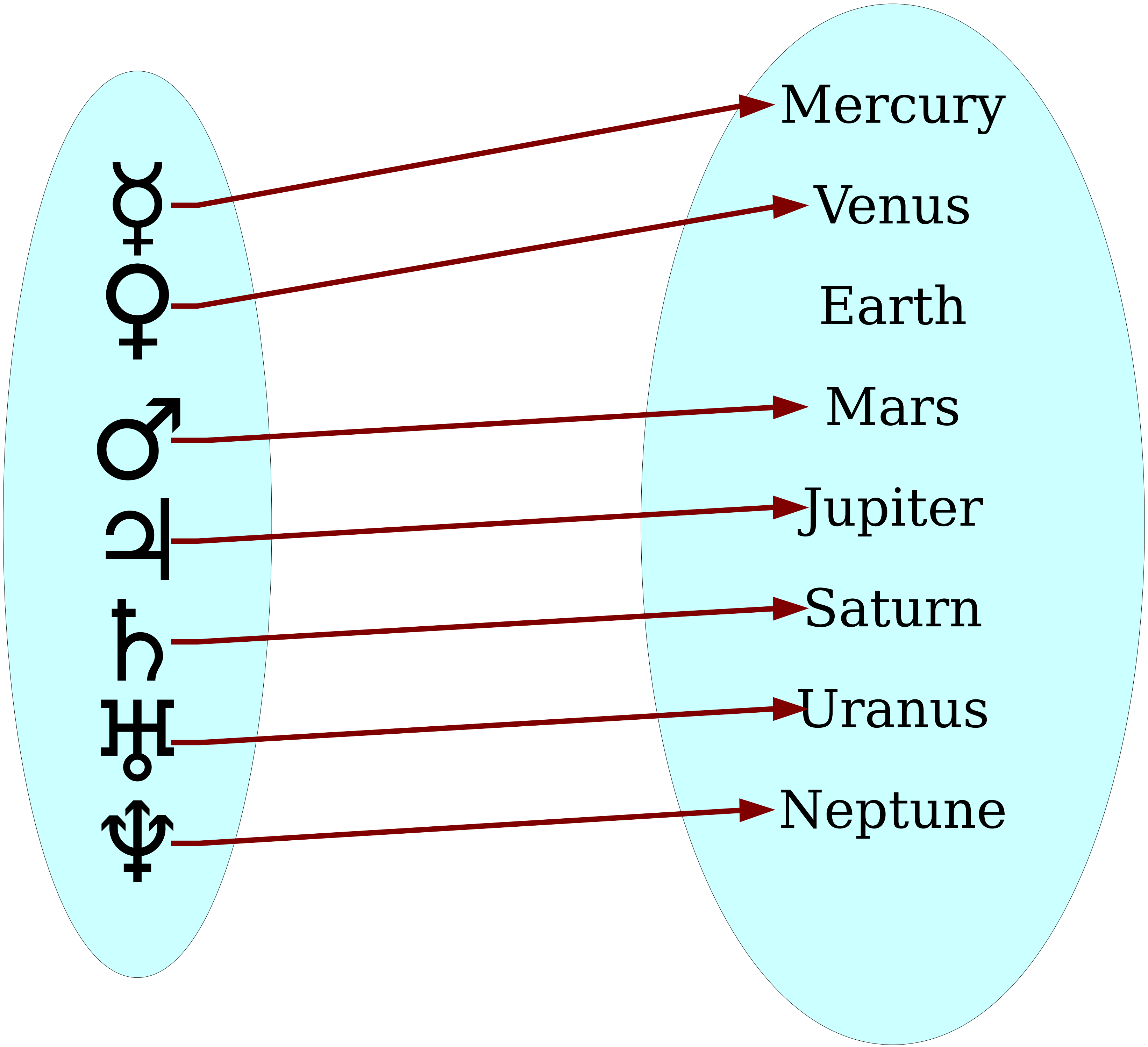

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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following definition is equivalent and tends to be more useful in proofs:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- Intuitively, in an injection, every element of the codomain has at most one element of the domain mapping to it.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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What does it mean for the function f to be injective?

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$$\forall n_0 \in \mathbb{N}.\ \forall n_1 \in \mathbb{N}.\ (\ f(n_0) = f(n_1) \to n_0 = n_1\ )$$

$$\forall n_0 \in \mathbb{N}.\ \forall n_1 \in \mathbb{N}.\ (\ n_0 \neq n_1 \to f(n_0) \neq f(n_1)\ )$$

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Therefore, we'll pick arbitrary $n_0, n_1 \in \mathbb{N}$ where $f(n_0) = f(n_1)$, then prove that $n_0 = n_1$.

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Since $f(n_0) = f(n_1)$, we see that

$$2n_0 + 7 = 2n_1 + 7.$$

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so $n_0 = n_1$, as required.

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Therefore, we need to find $x_0, x_1 \in \mathbb{Z}$ such that $x_0 \neq x_1$, but $f(x_0) = f(x_1)$.
Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_0 and x_1 such that $x_0 \neq x_1$, but $f(x_0) = f(x_1)$.

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Let $x_0 = -1$ and $x_1 = +1$.

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and

$$f(x_1) = f(1) = 1^4 = 1,$$

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Injections and Composition

Injections and Composition

- ***Theorem:*** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.
- Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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Proof:

What's the high-level structure of this proof?

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$$\forall f.\ \forall g.\ (Inj(f) \land Inj(g) \to Inj(g \circ f))$$

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof:

What's the high-level structure of this proof?

$$\forall f.\ \forall g.\ (Inj(f) \land Inj(g) \to Inj(g \circ f))$$

Therefore, we'll choose two arbitrary injective functions $f : A \to B$ and $g : B \to C$ and prove that $g \circ f$ is injective.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.

What does it mean for $g \circ f : A \to C$ to be injective?

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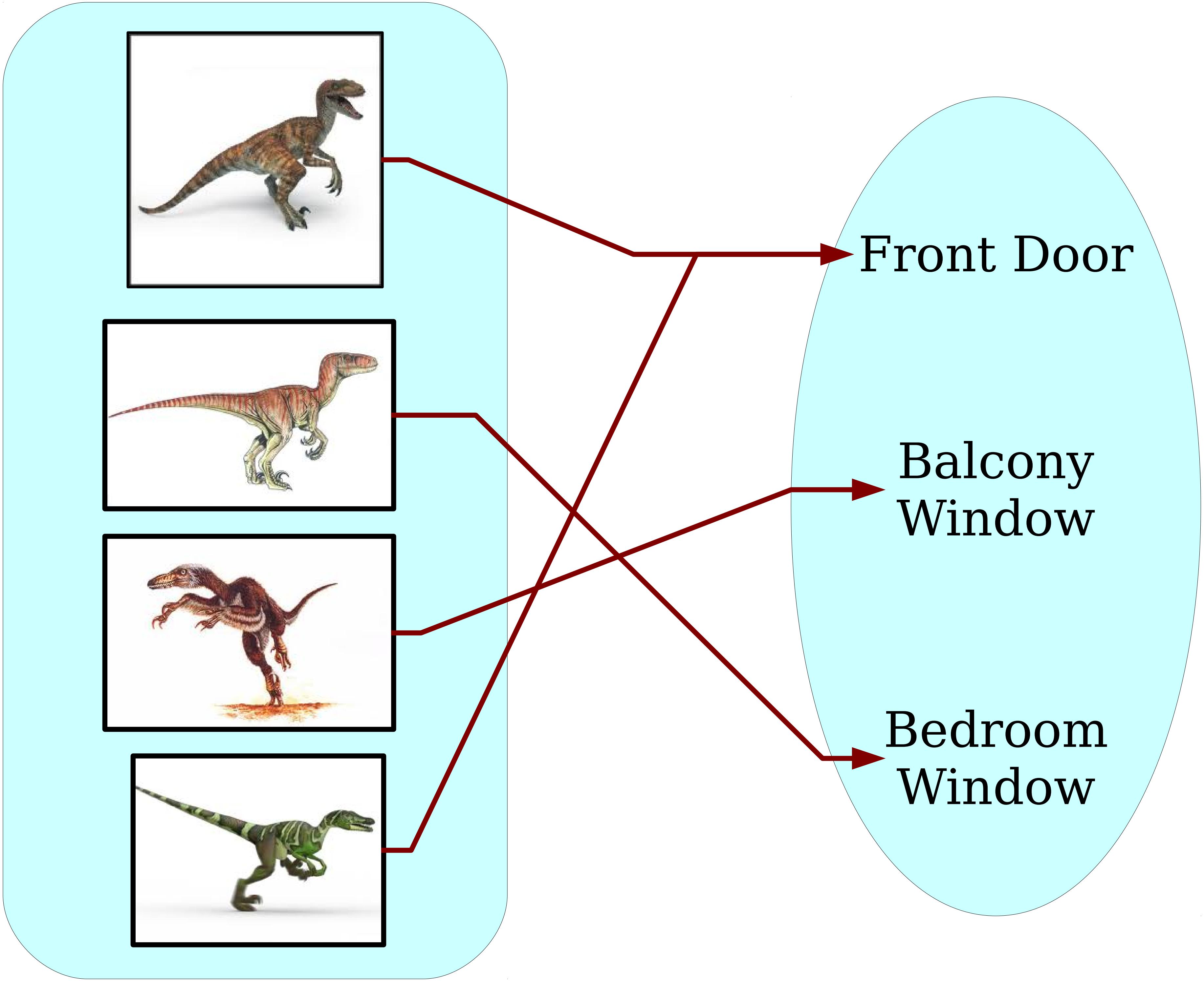
Good exercise: Repeat this proof using the other definition of injectivity.

Another Class of Functions


Front Door
Balcony Window
Bedroom Window

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

- A function with this property is called a ***surjection***.
- Intuitively, every element in the codomain of a surjection has at least one element of the domain mapping to it.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

Proof:

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Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Composing Surjections

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Therefore, we'll choose two arbitrary surjective functions $f : A \to B$ and $g : B \to C$ and prove that $g \circ f$ is surjective.

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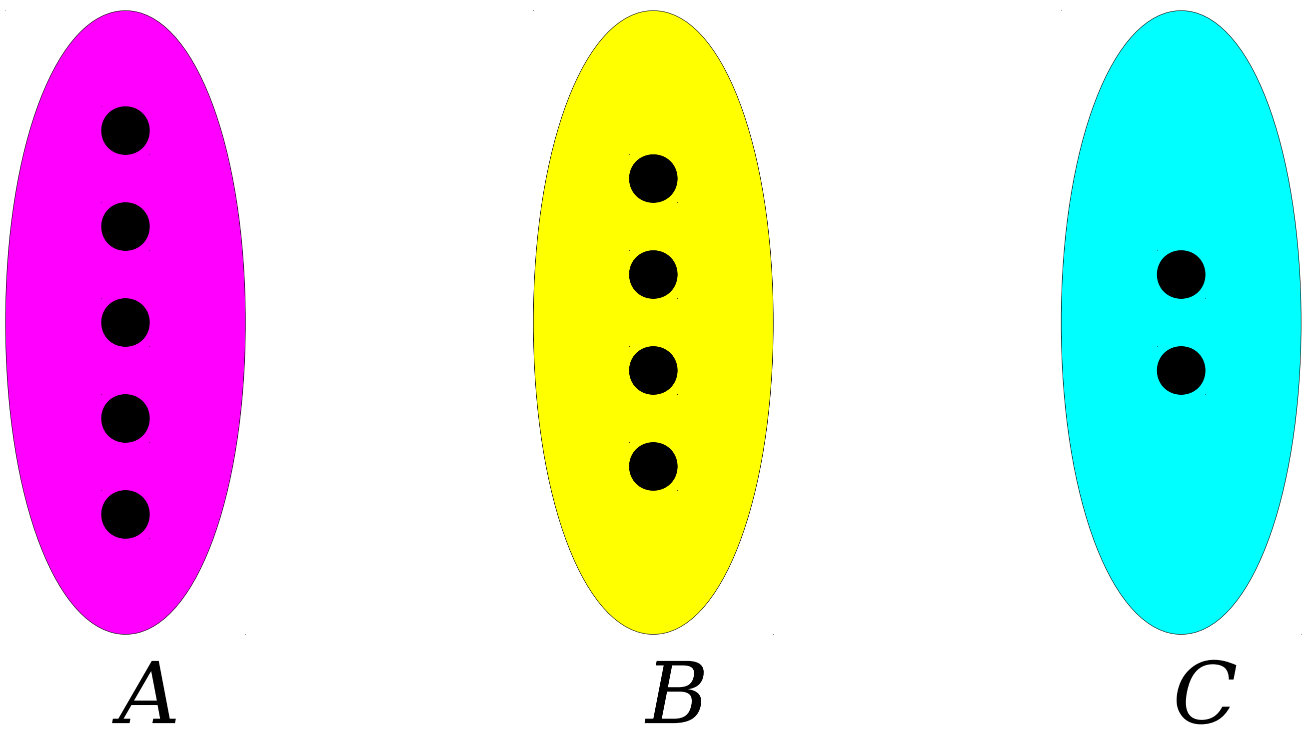

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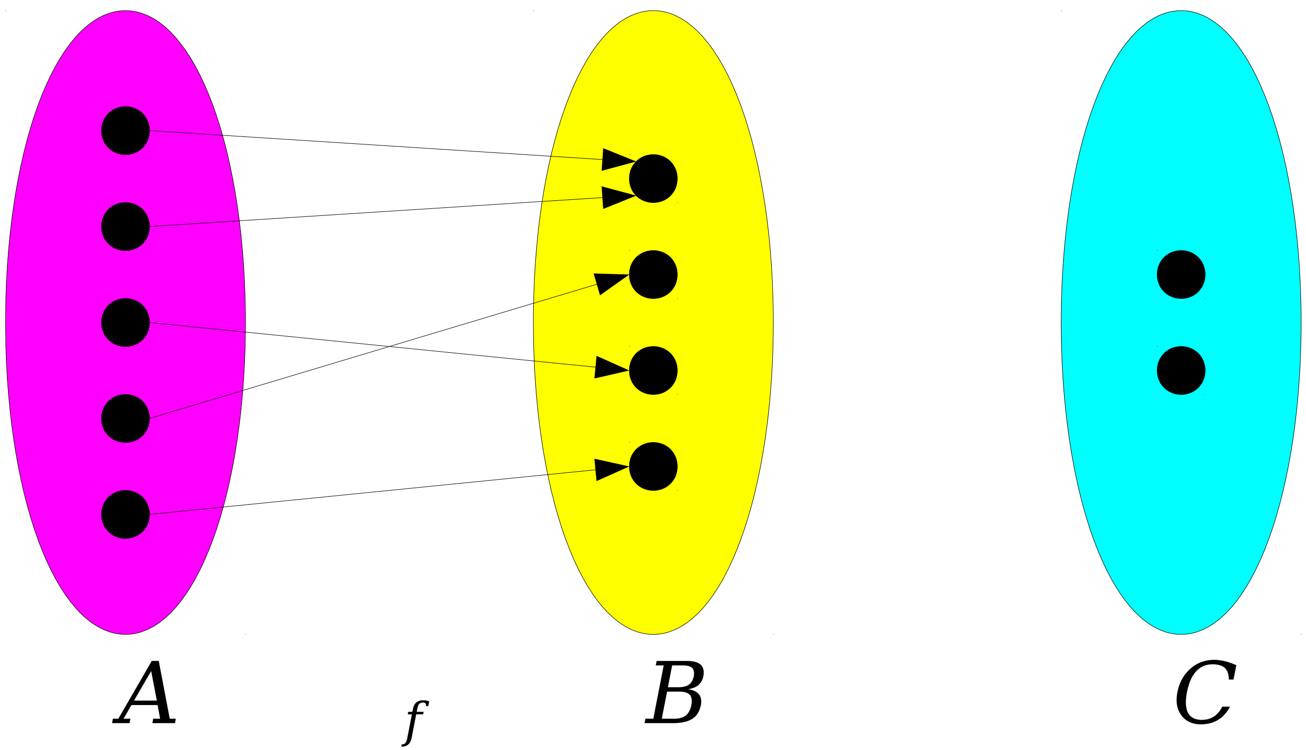

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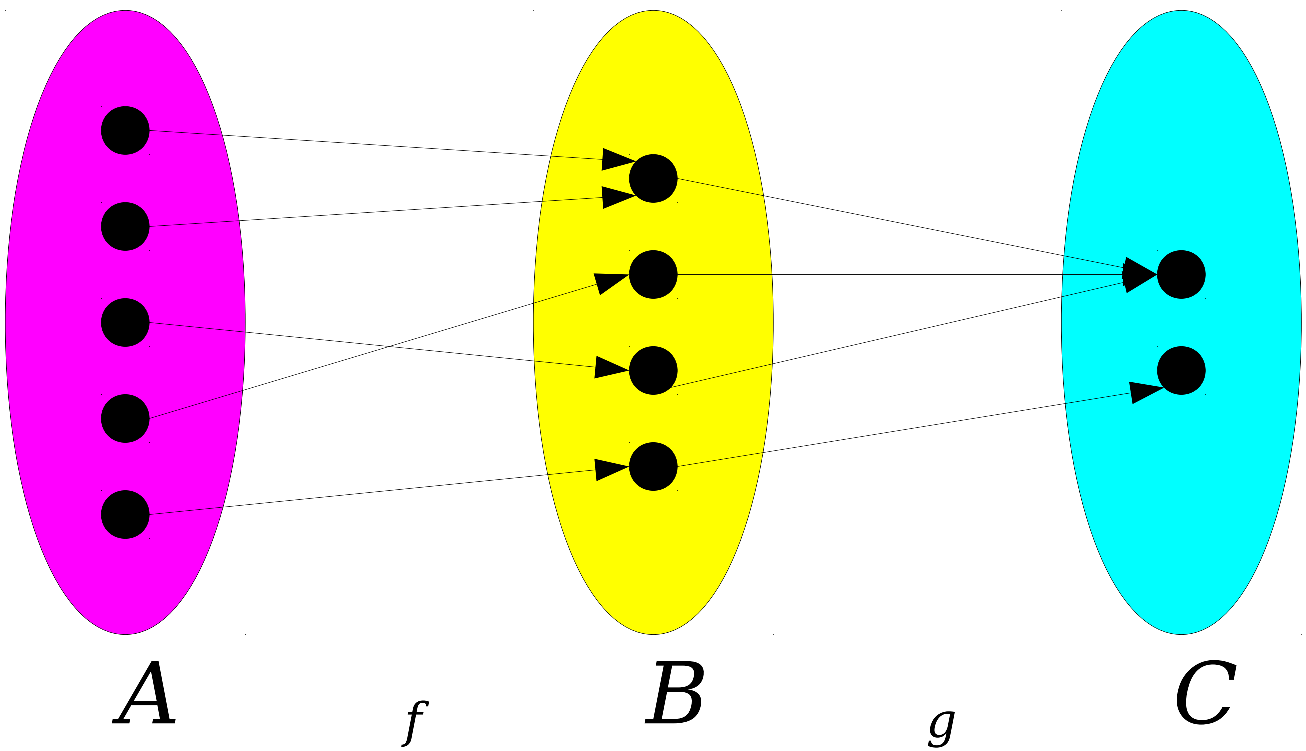

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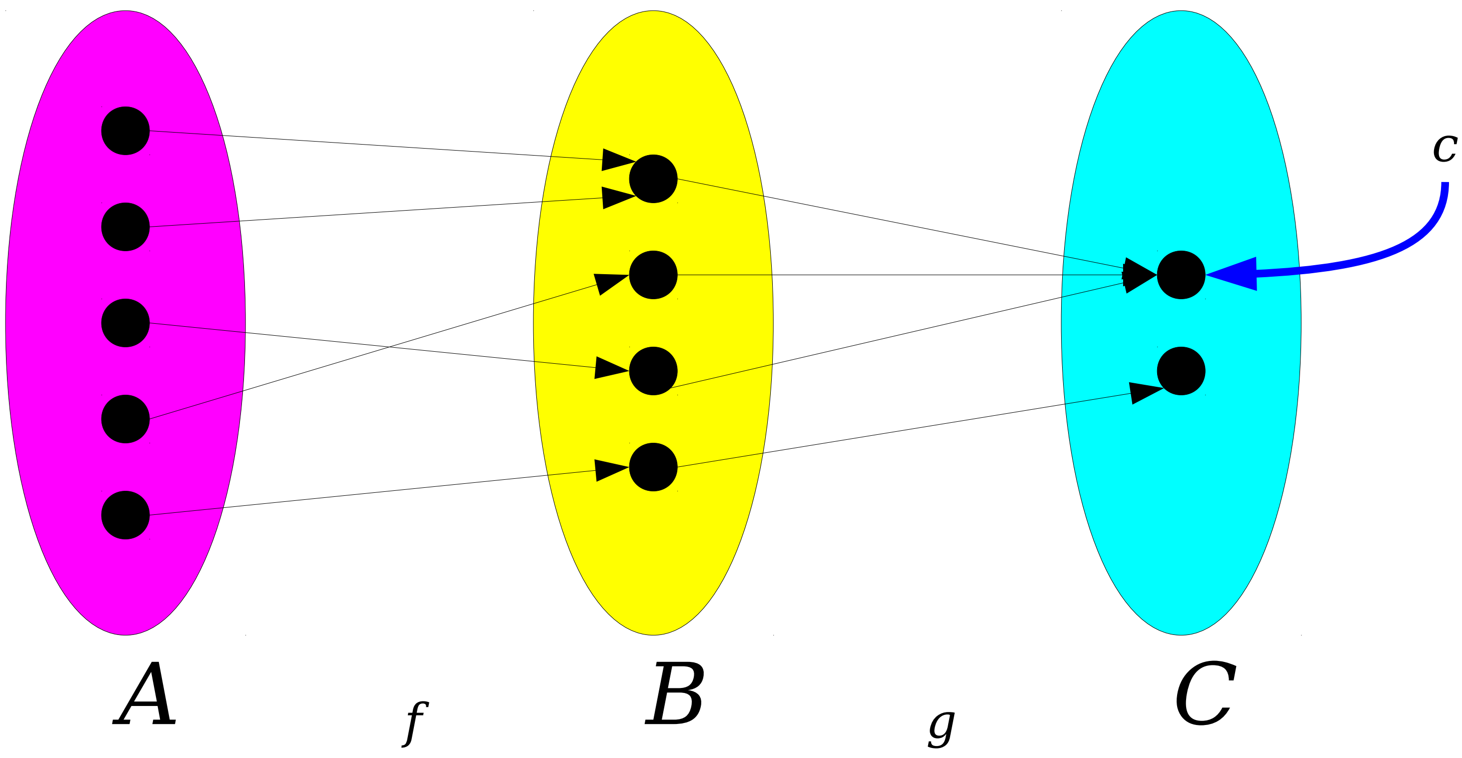

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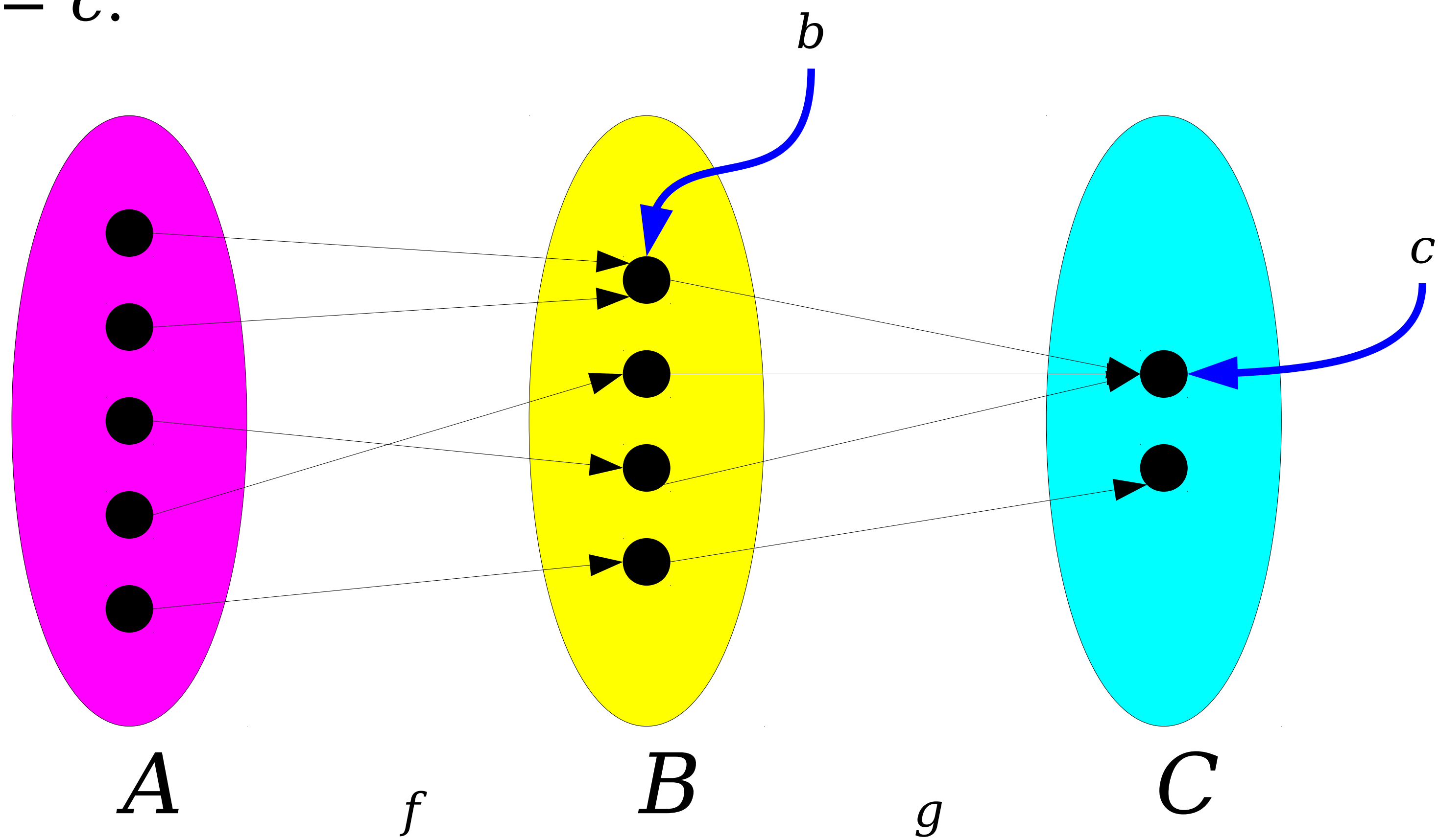

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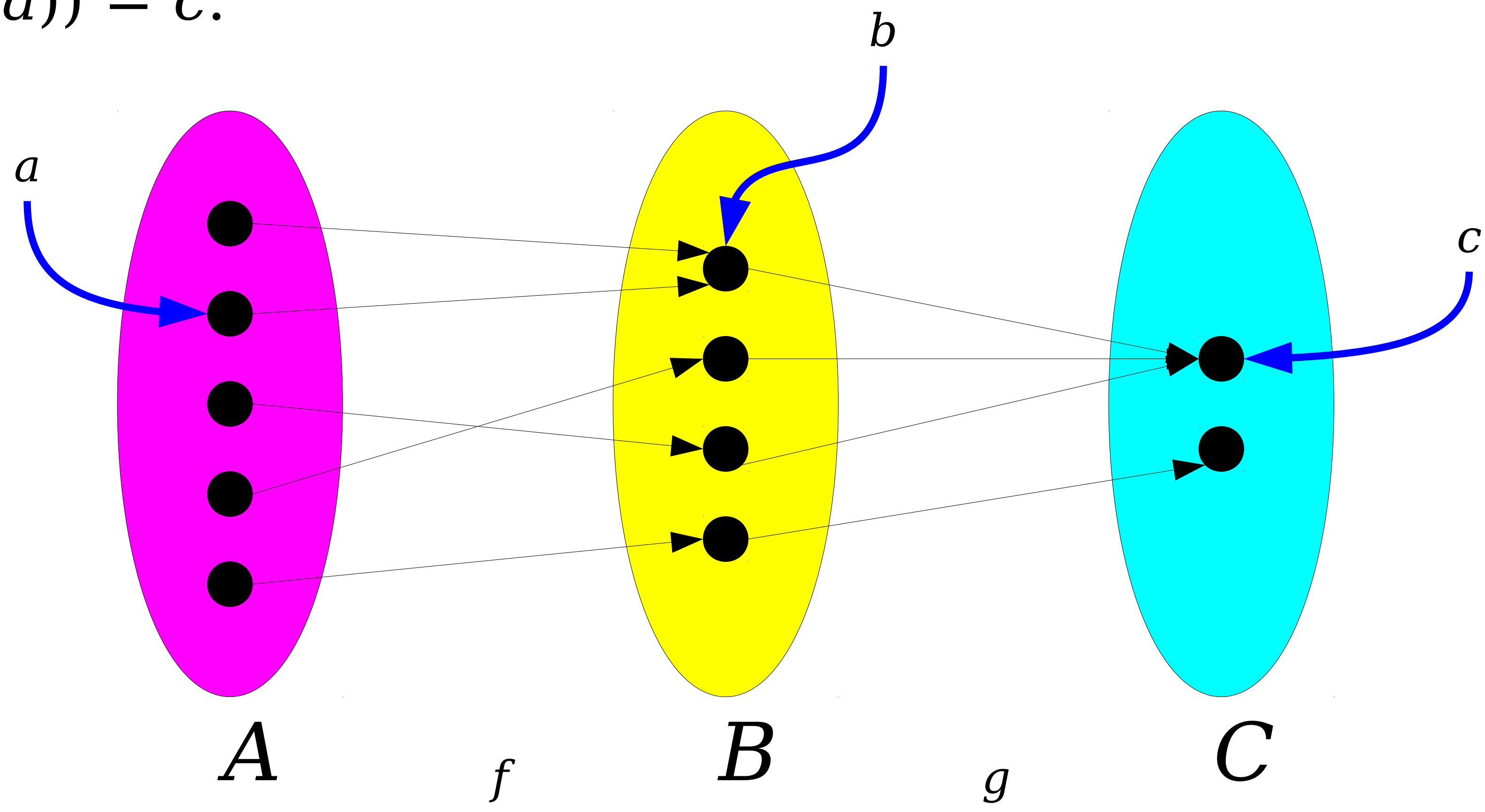

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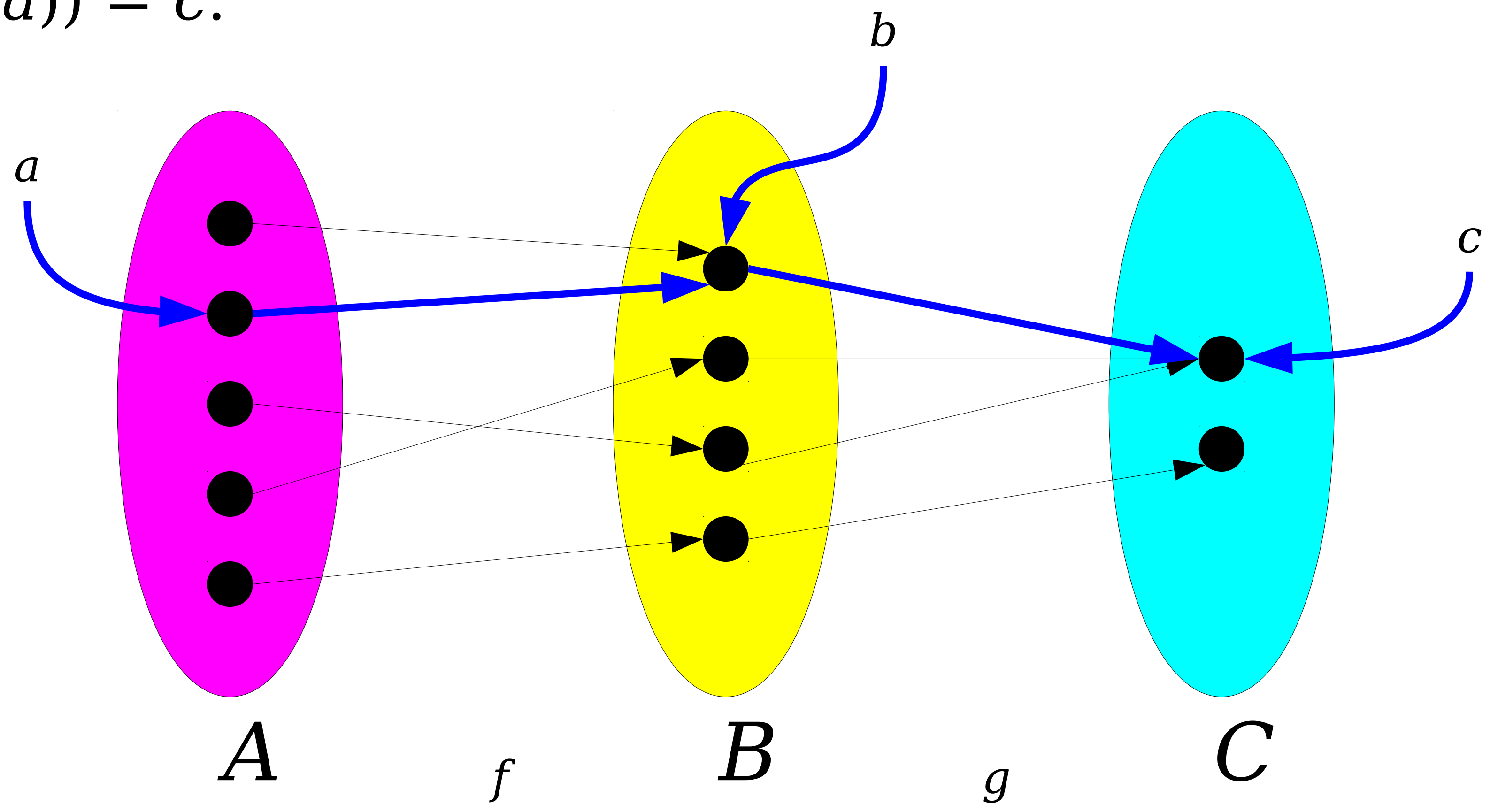

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Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.

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