Graphs

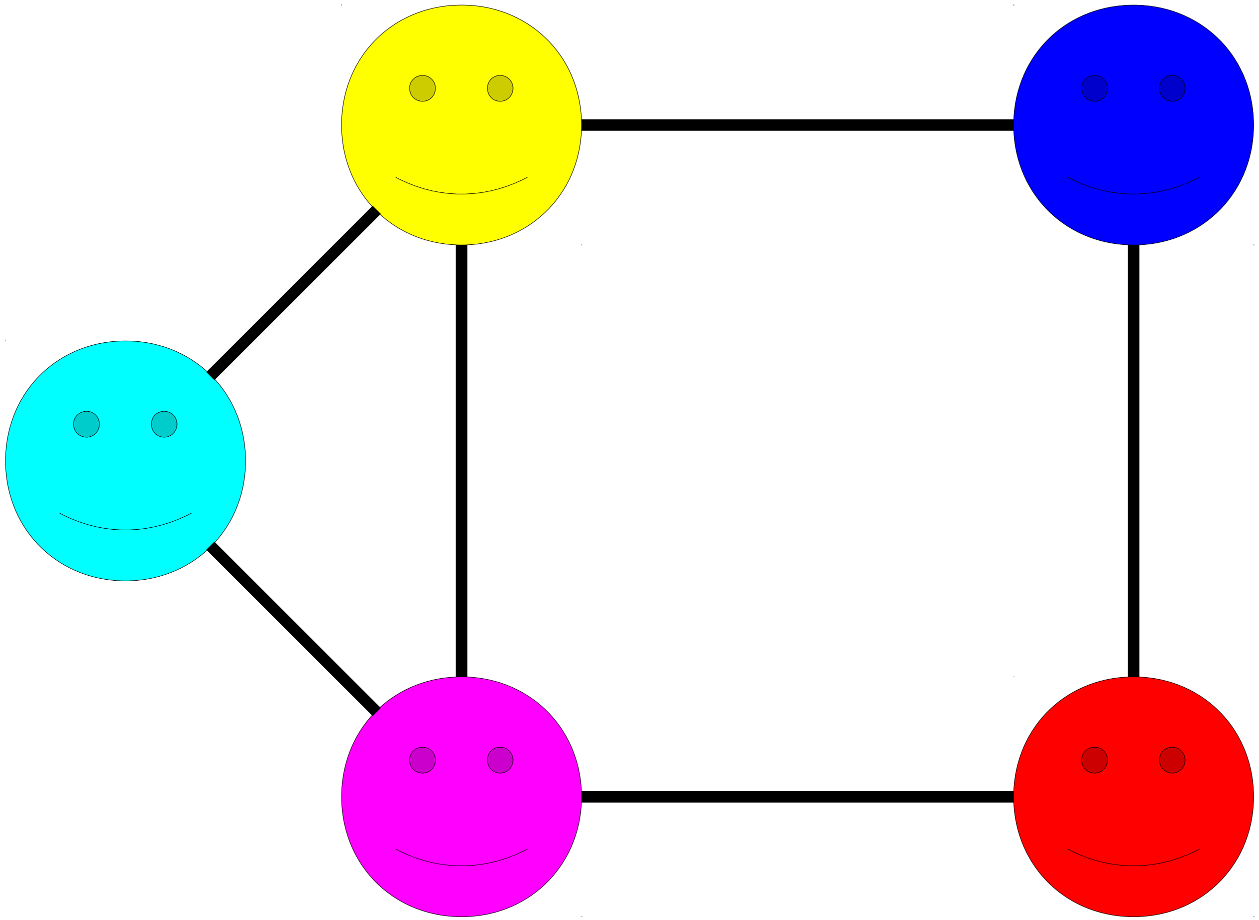
Part Two

Outline for Today

- **Planar Graphs**
  - A special class of graph with numerous applications.
- **Graph Coloring**
  - A common operation on graphs with surprising properties.
- **The Pigeonhole Principle**
  - Revisiting a core idea in counting.
- **Ramsey Theory**
  - Is large-scale disorder possible?
- **The Hadwiger-Nelson Problem (ITA)**
  - Looking at infinitely large graphs.

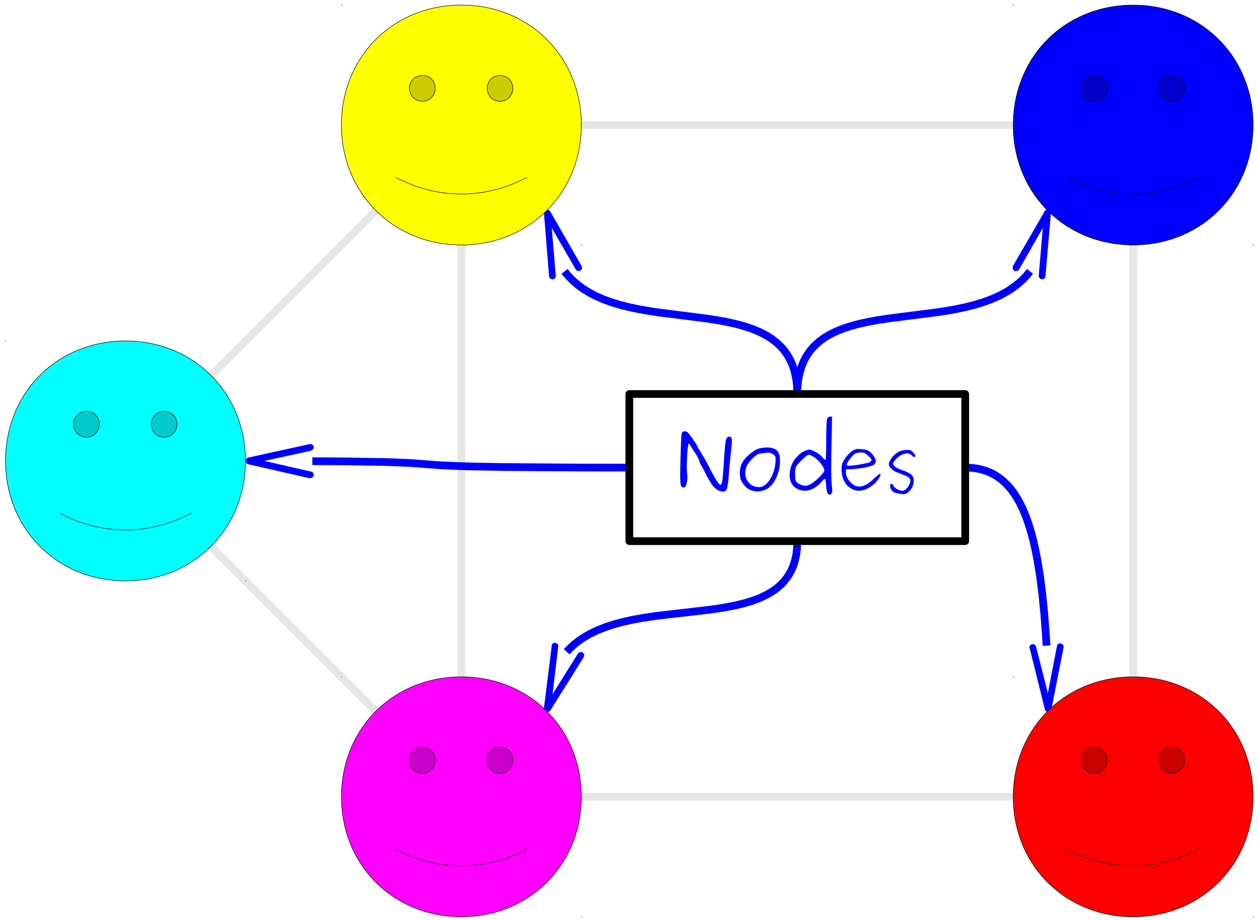
Recap from Last Time

A ***graph*** is a mathematical structure for representing relationships.



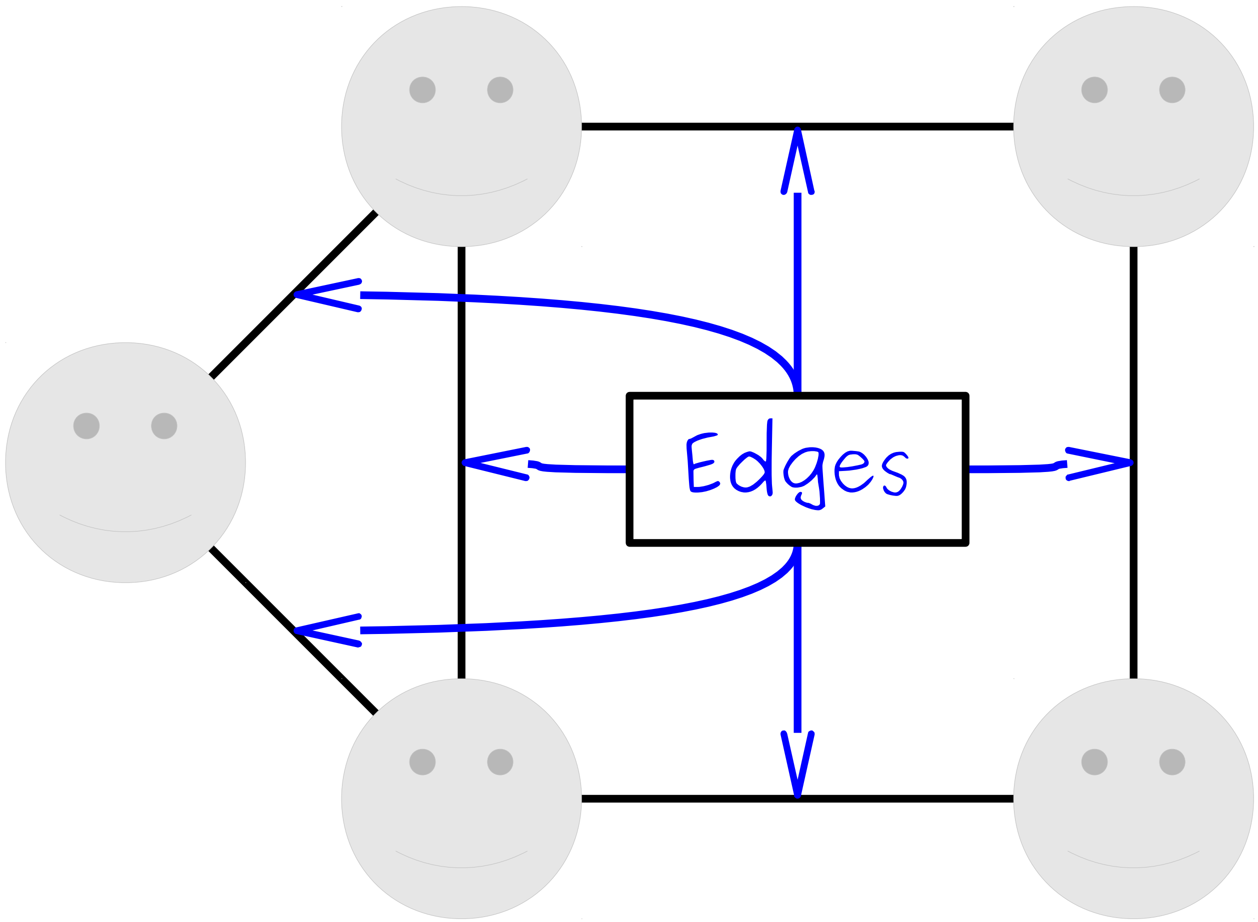
A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

A ***graph*** is a mathematical structure for representing relationships.



A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

A ***graph*** is a mathematical structure for representing relationships.



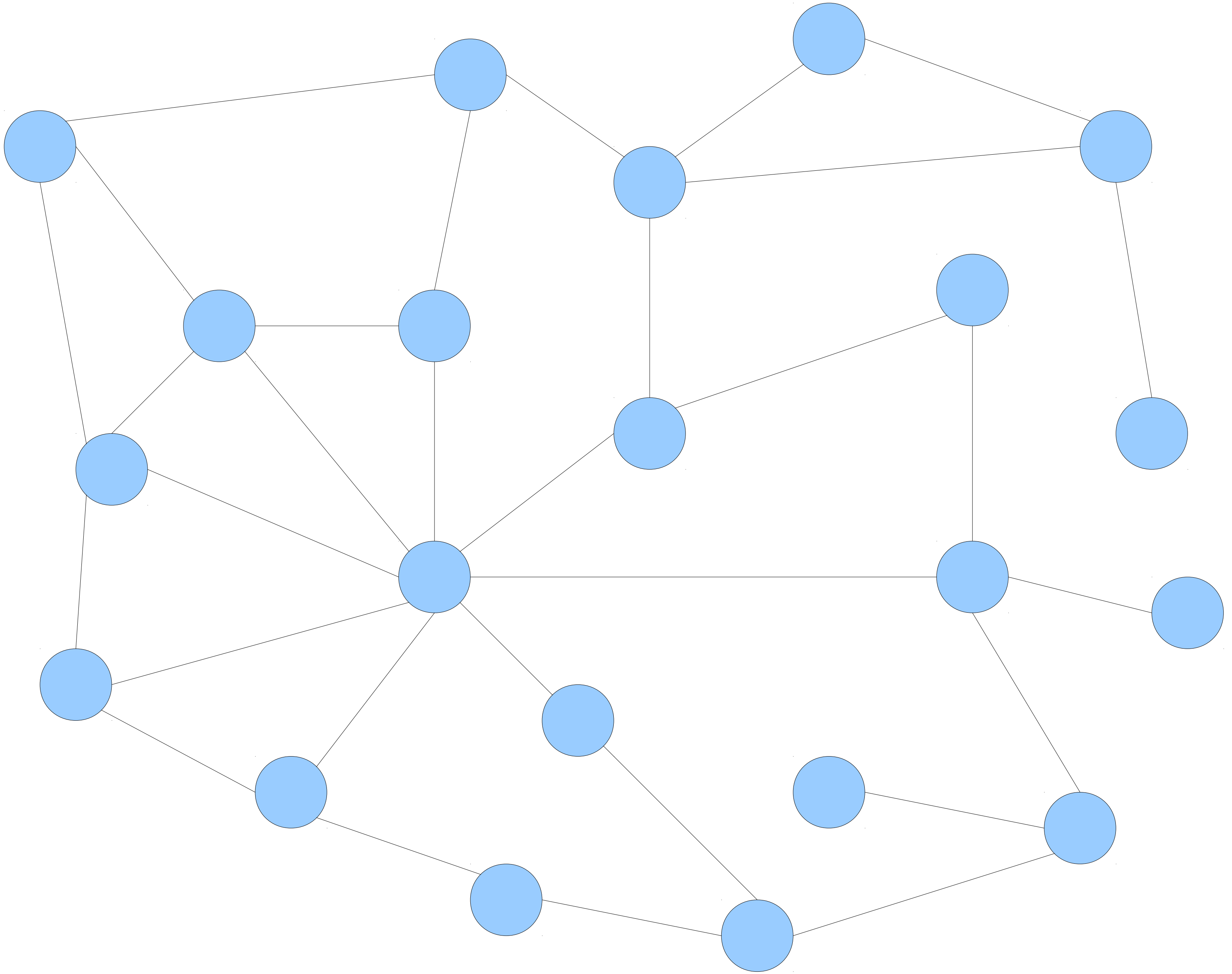
A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

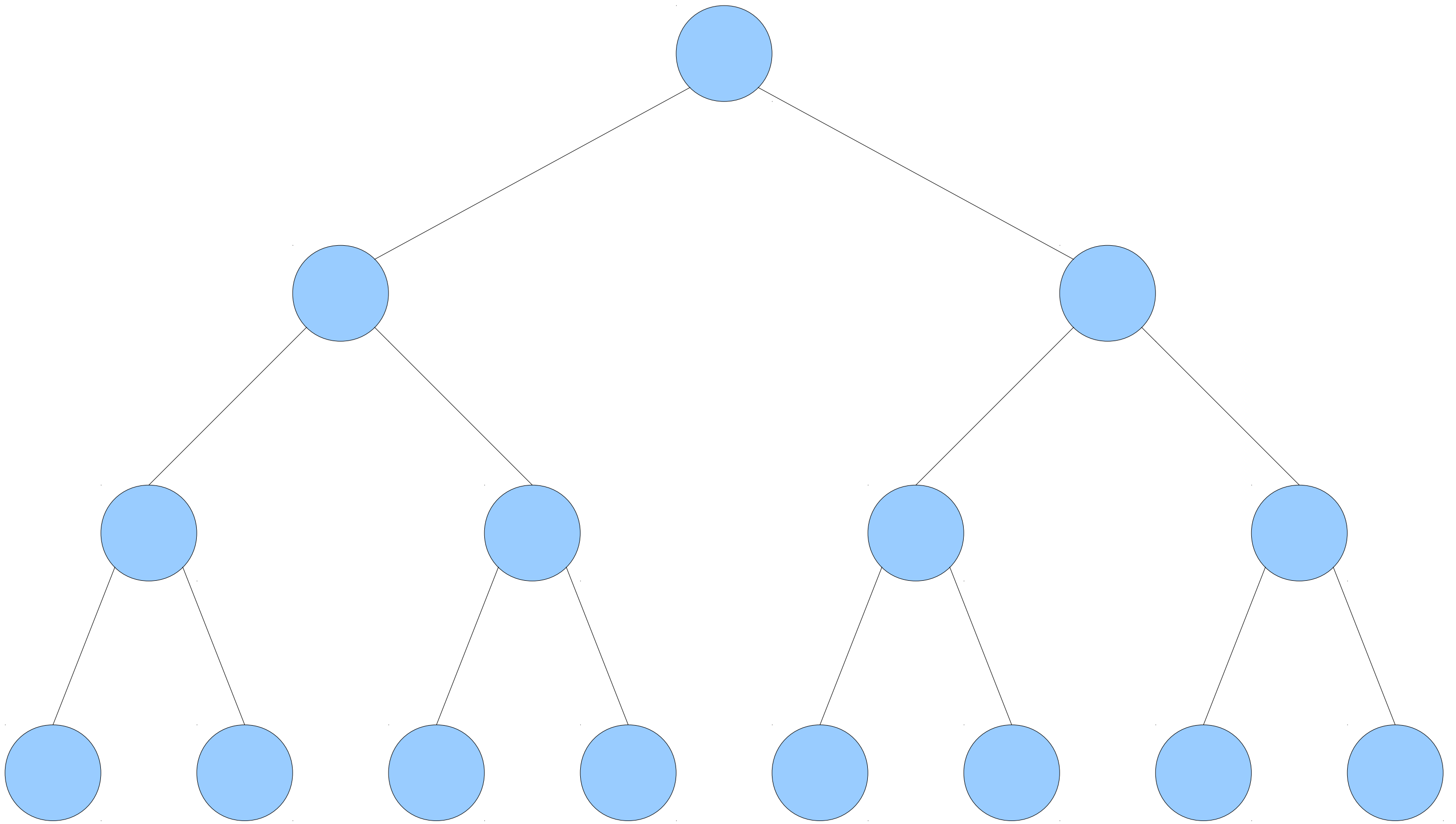
Adjacency and Connectivity

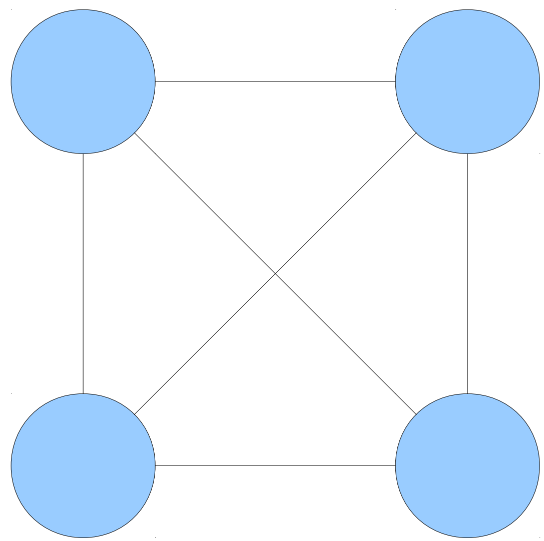
- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***connected*** if there's a path between them.
  - A path is a series of one or more nodes where consecutive nodes are adjacent.

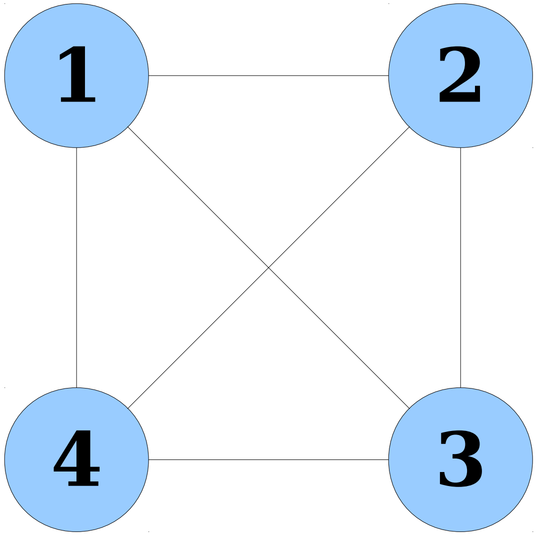
New Stuff!

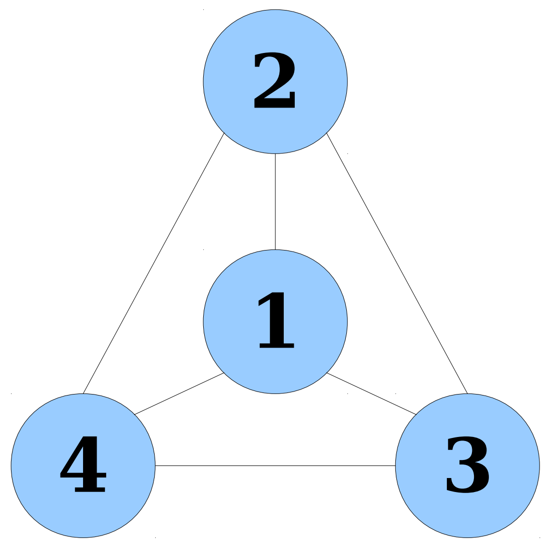
Planar Graphs

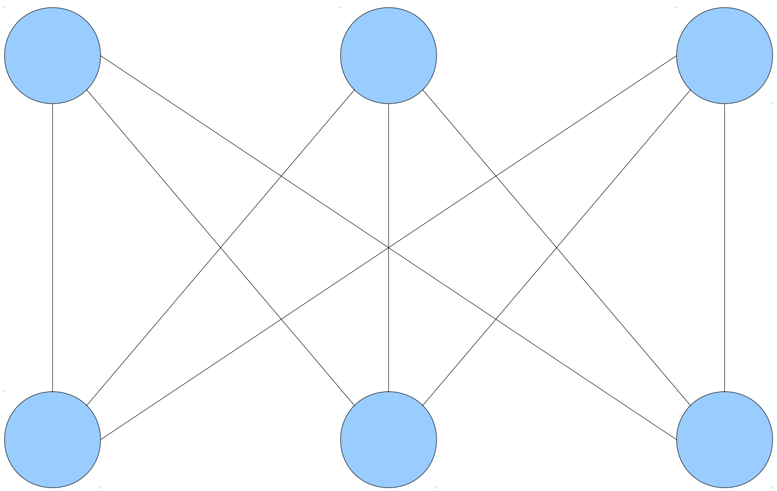


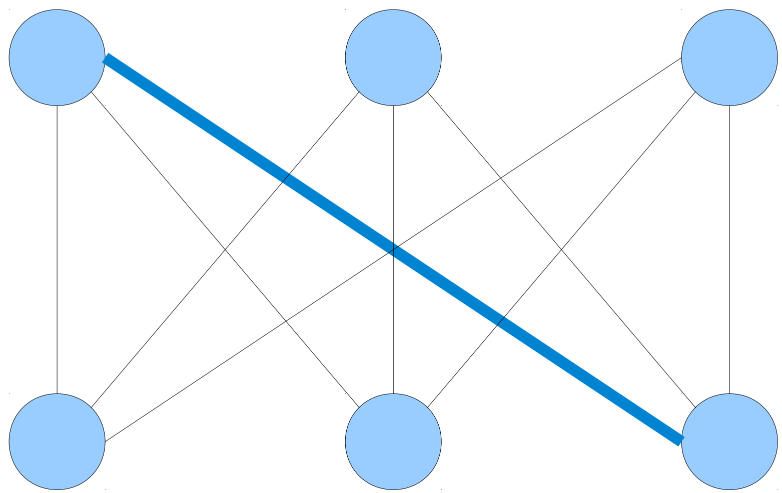


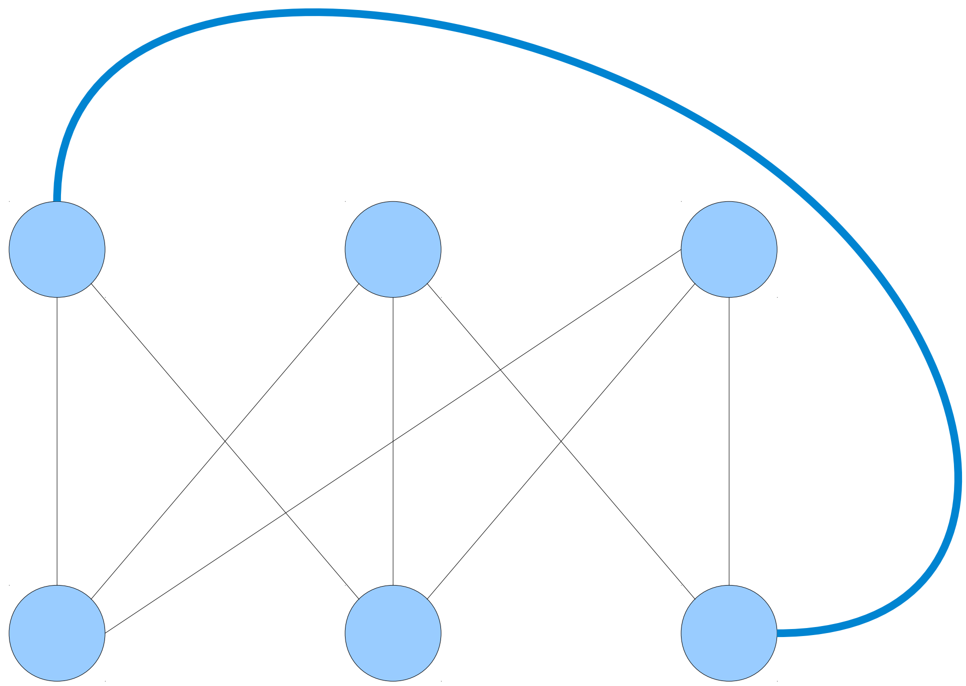


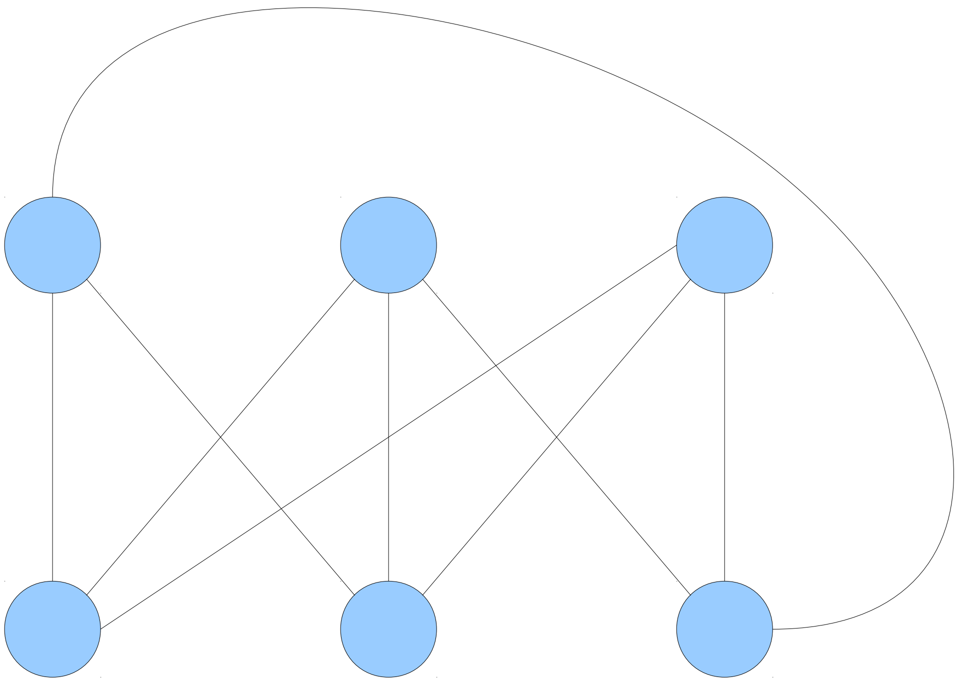

1
2
4
3

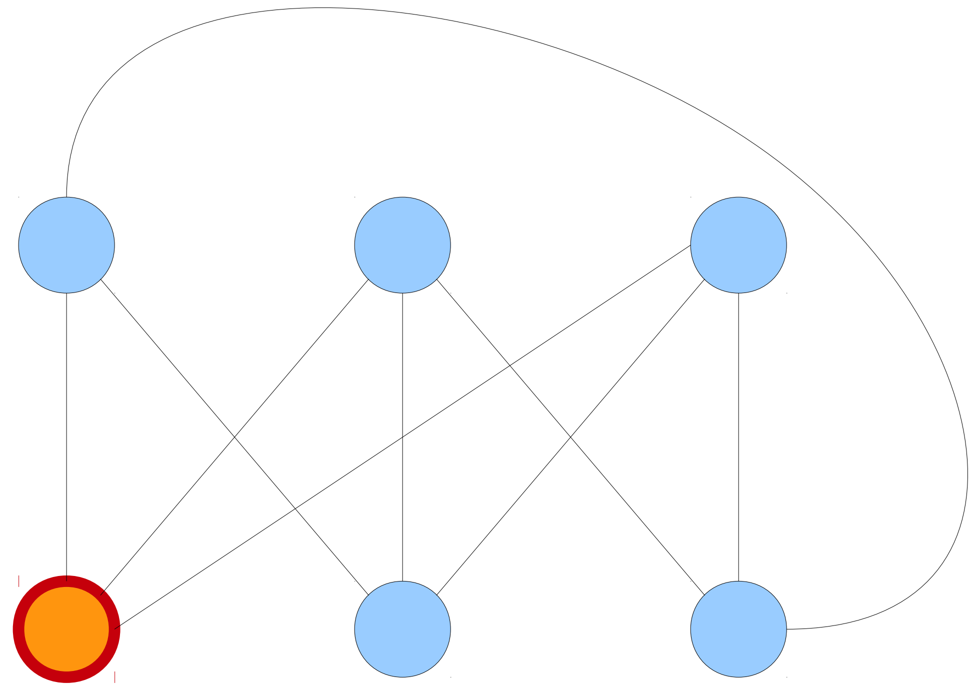

2
1
4
3

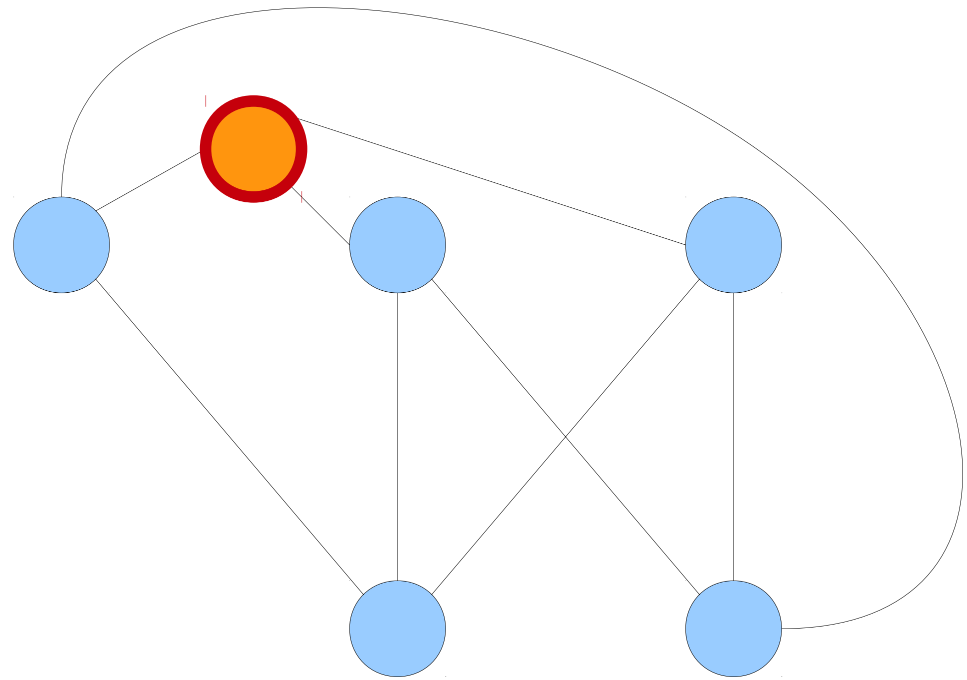


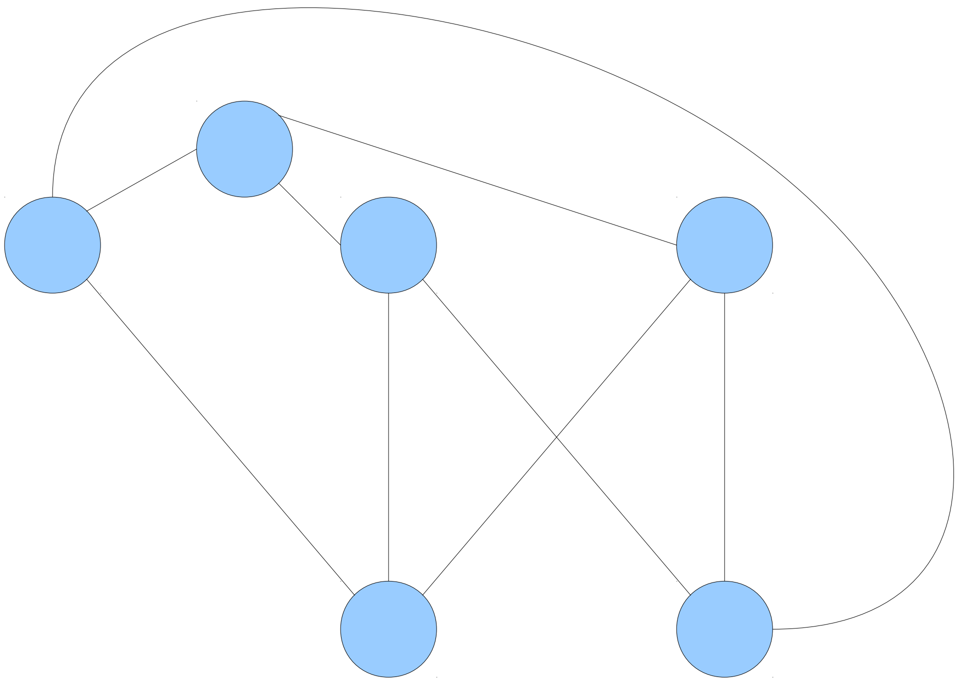


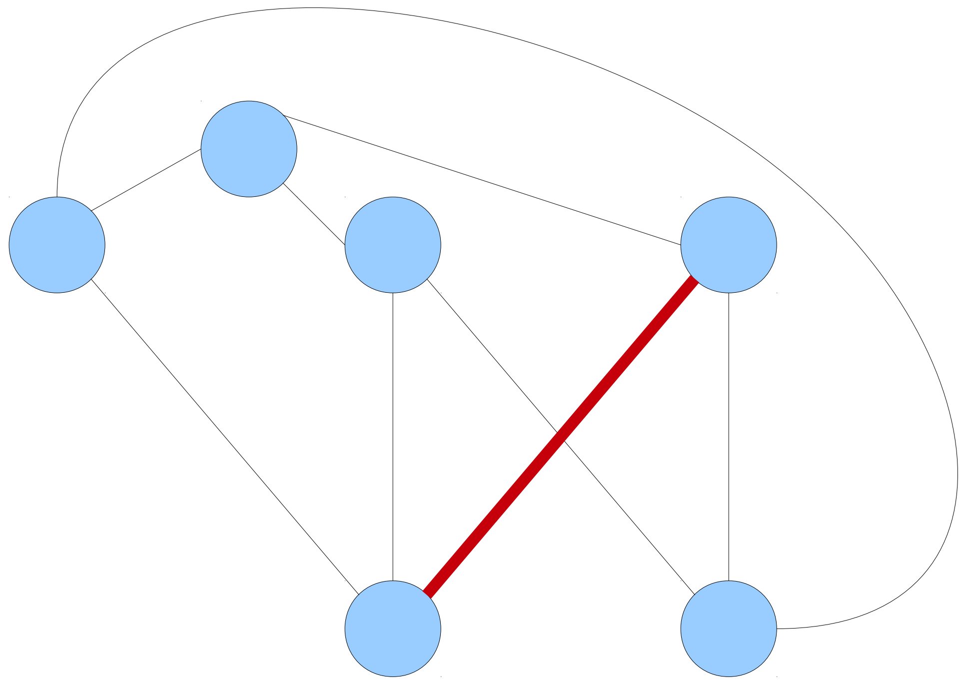


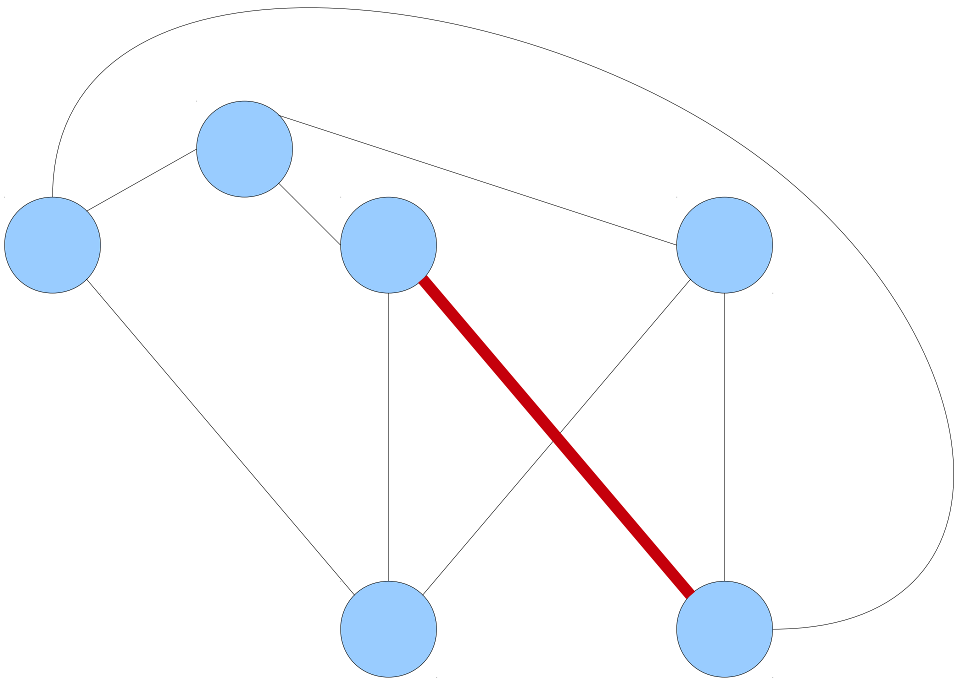


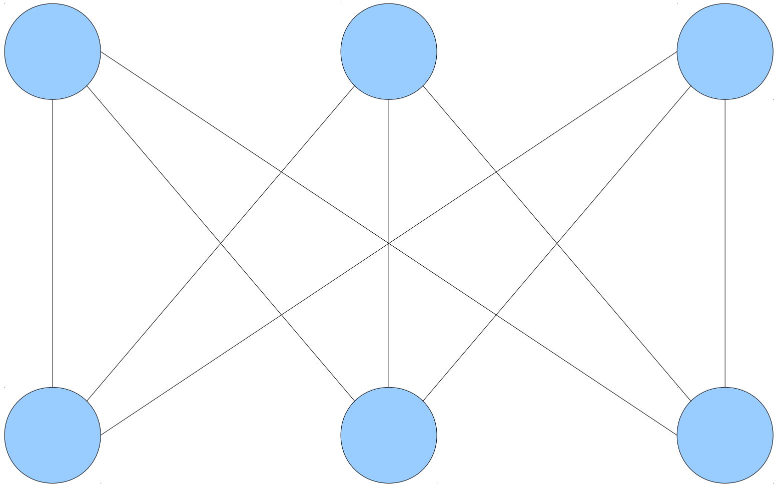










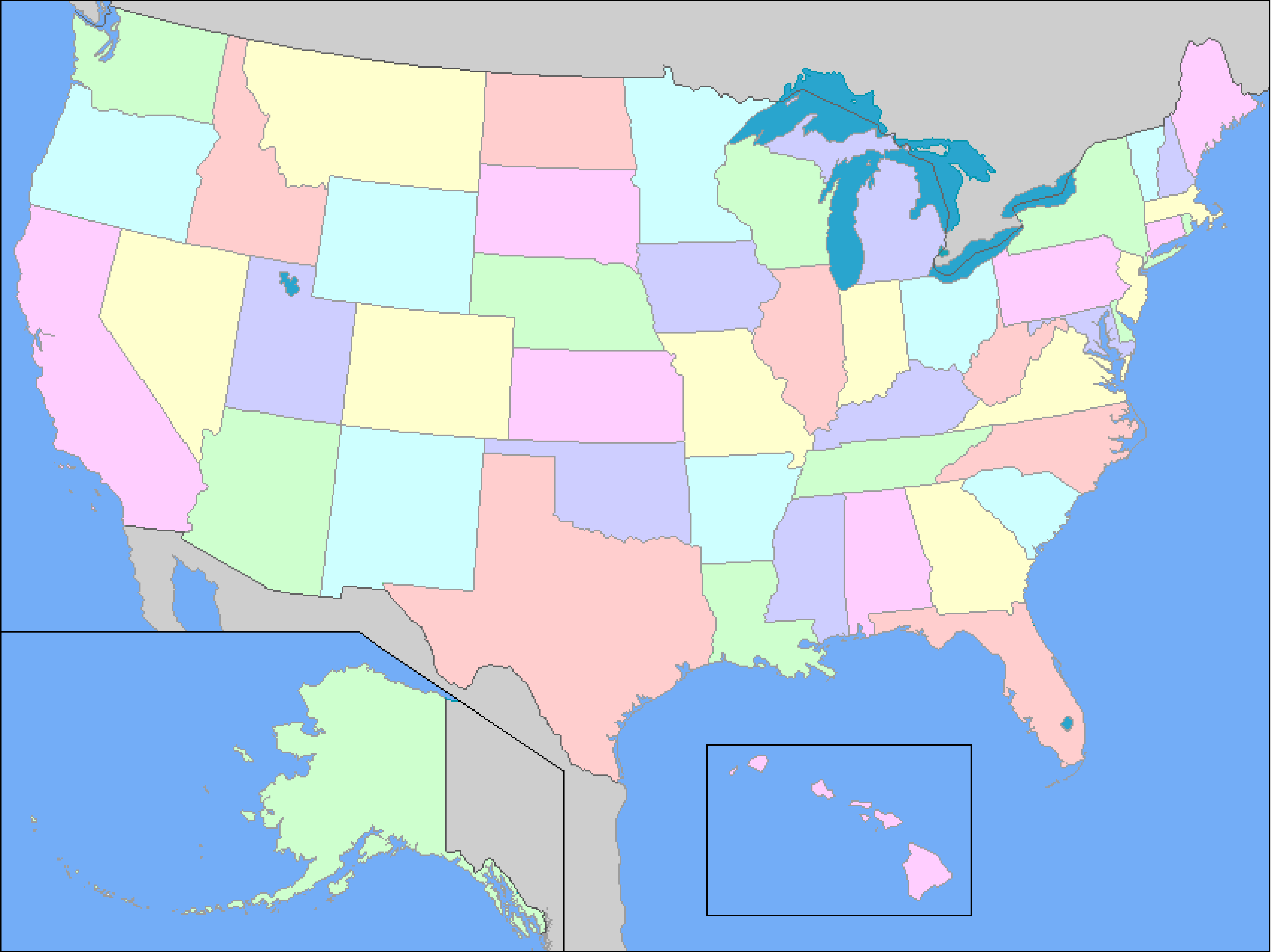


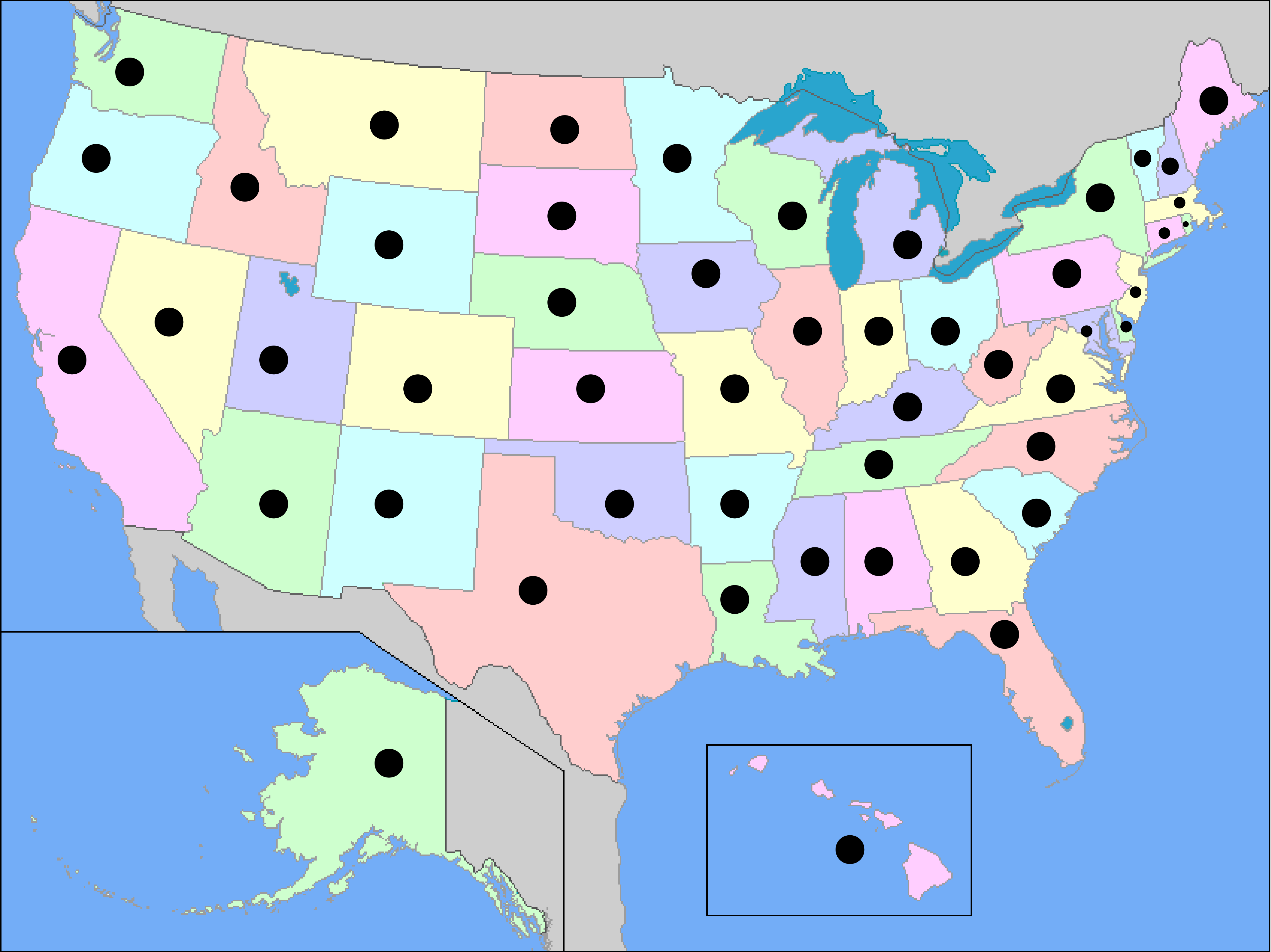
This graph is called the ***utility graph***. There is no way to draw it in the plane without edges crossing.

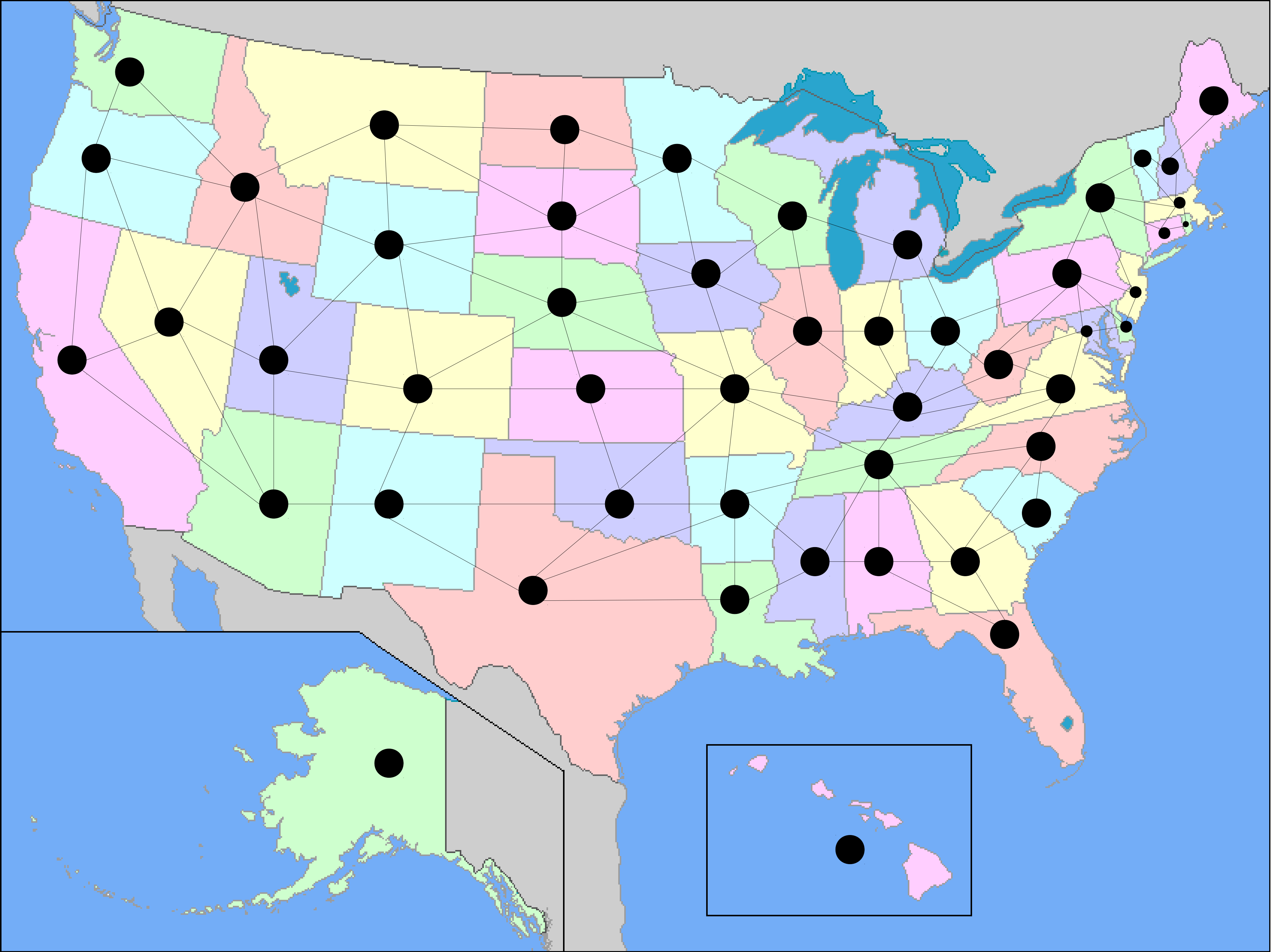
A graph is called a ***planar graph*** if there is some way to draw it in a 2D plane without any of the edges crossing.

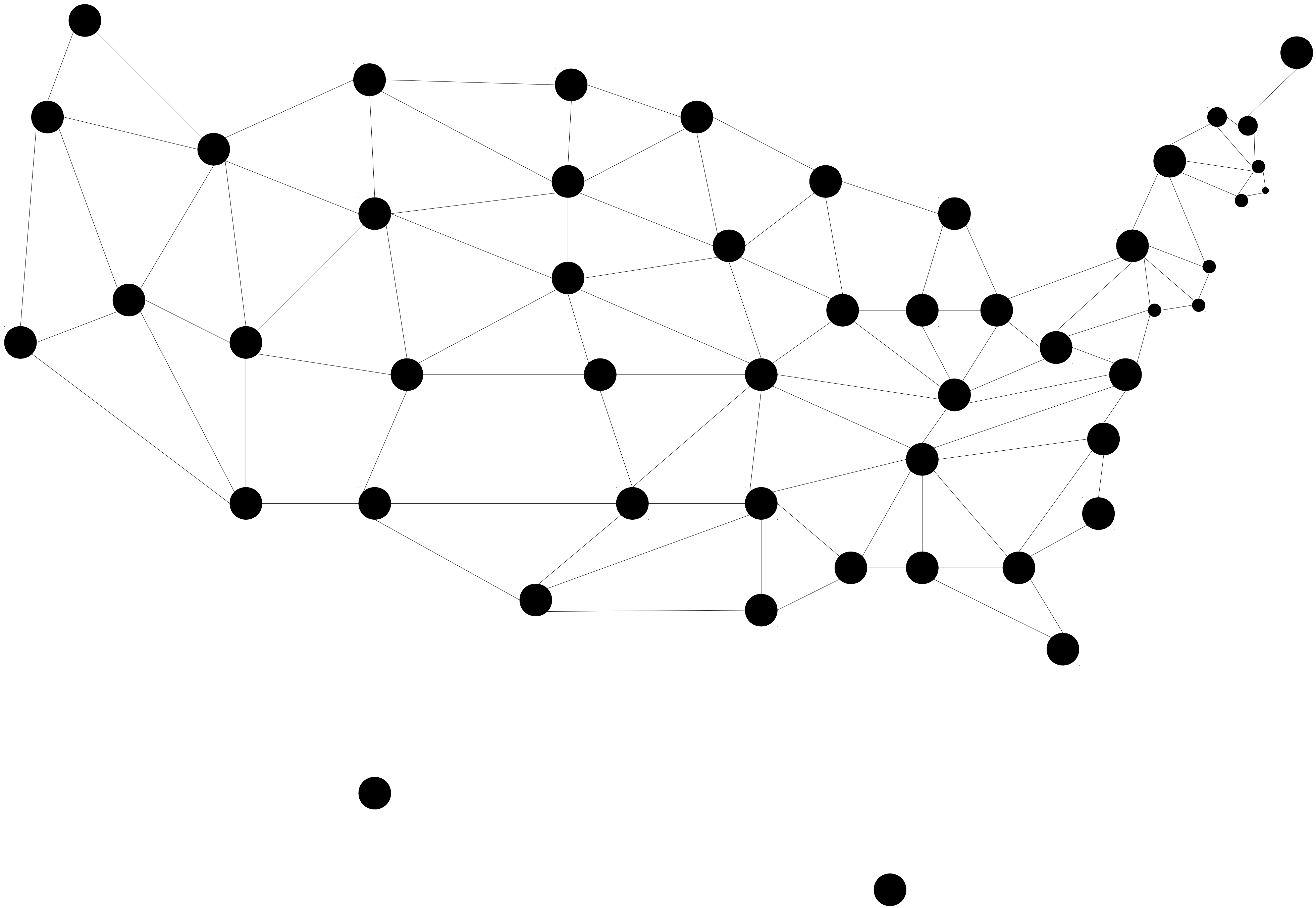
A Fun (And Strangely Addicting) Game:

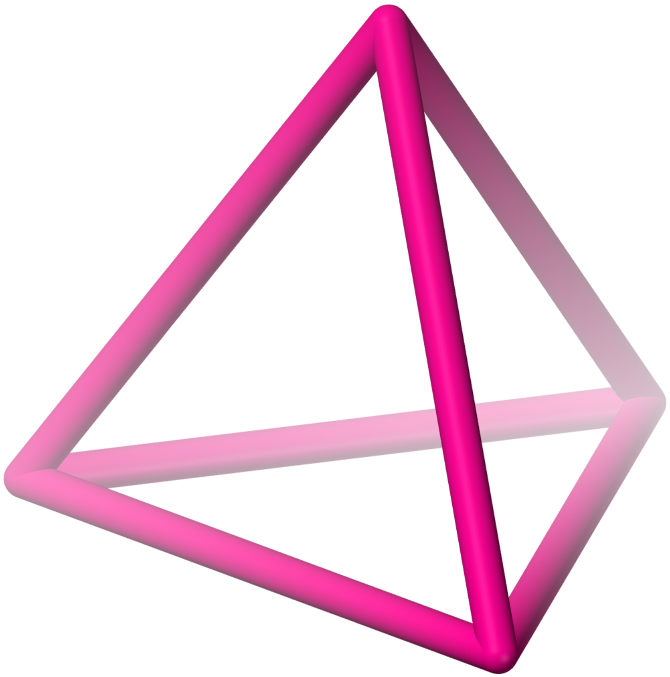
http://planarity.net/

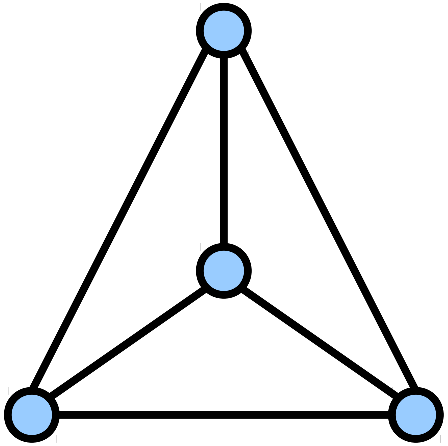


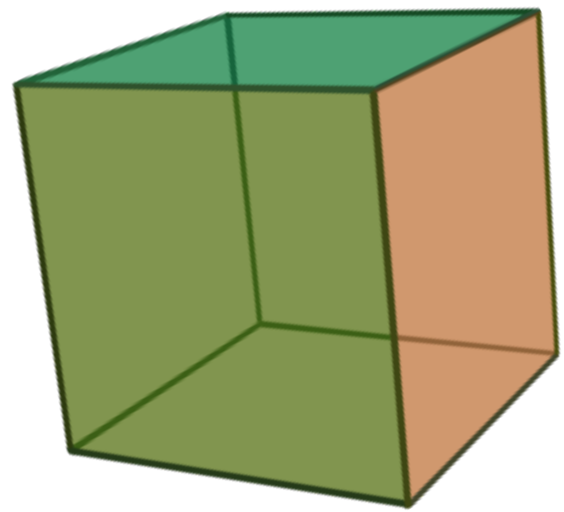


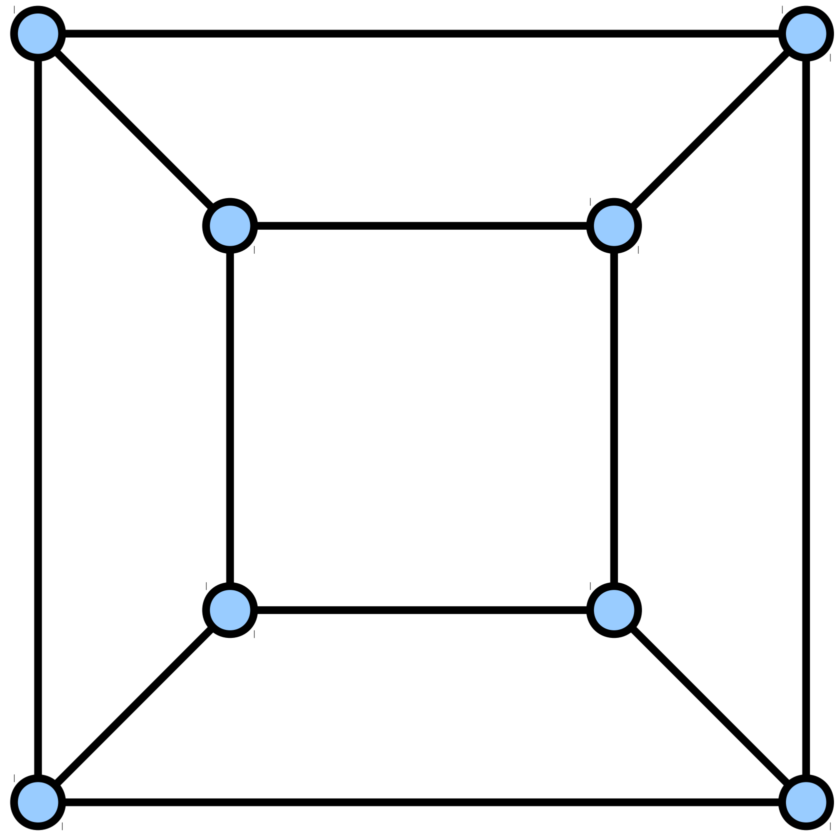


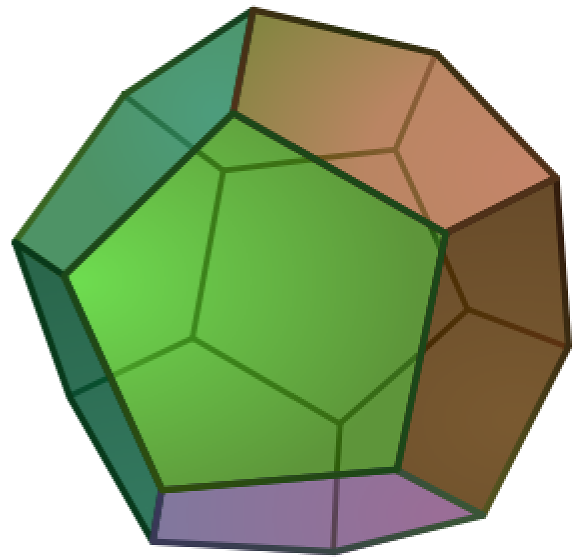


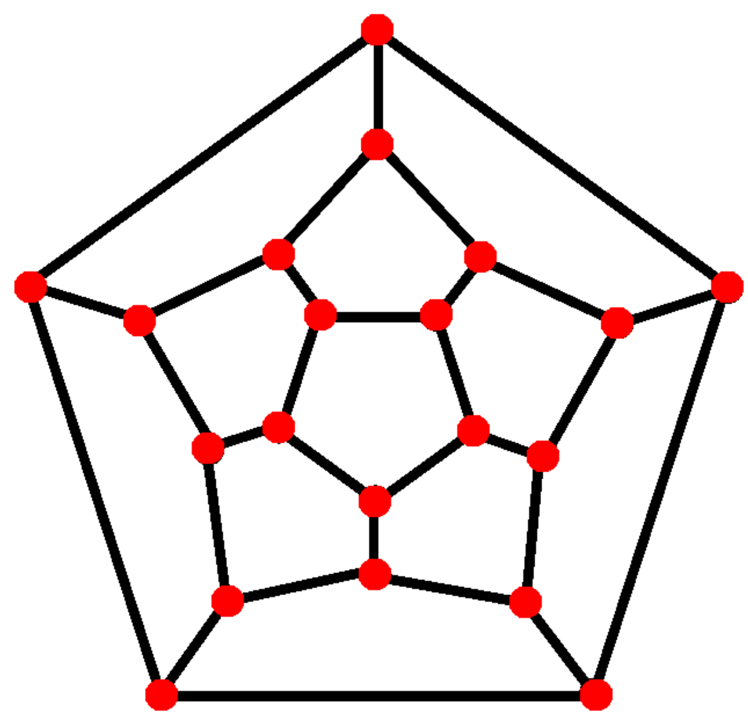




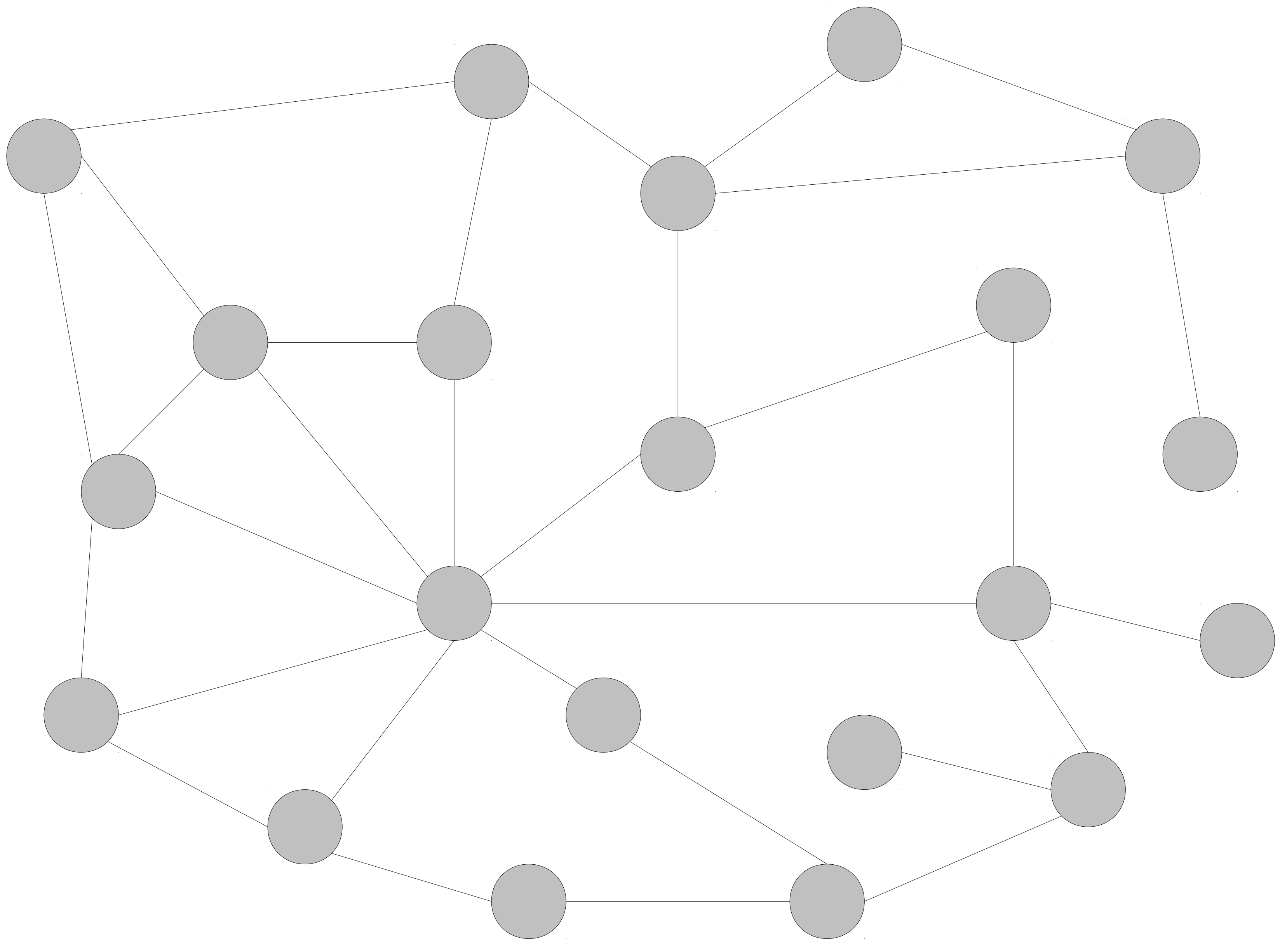




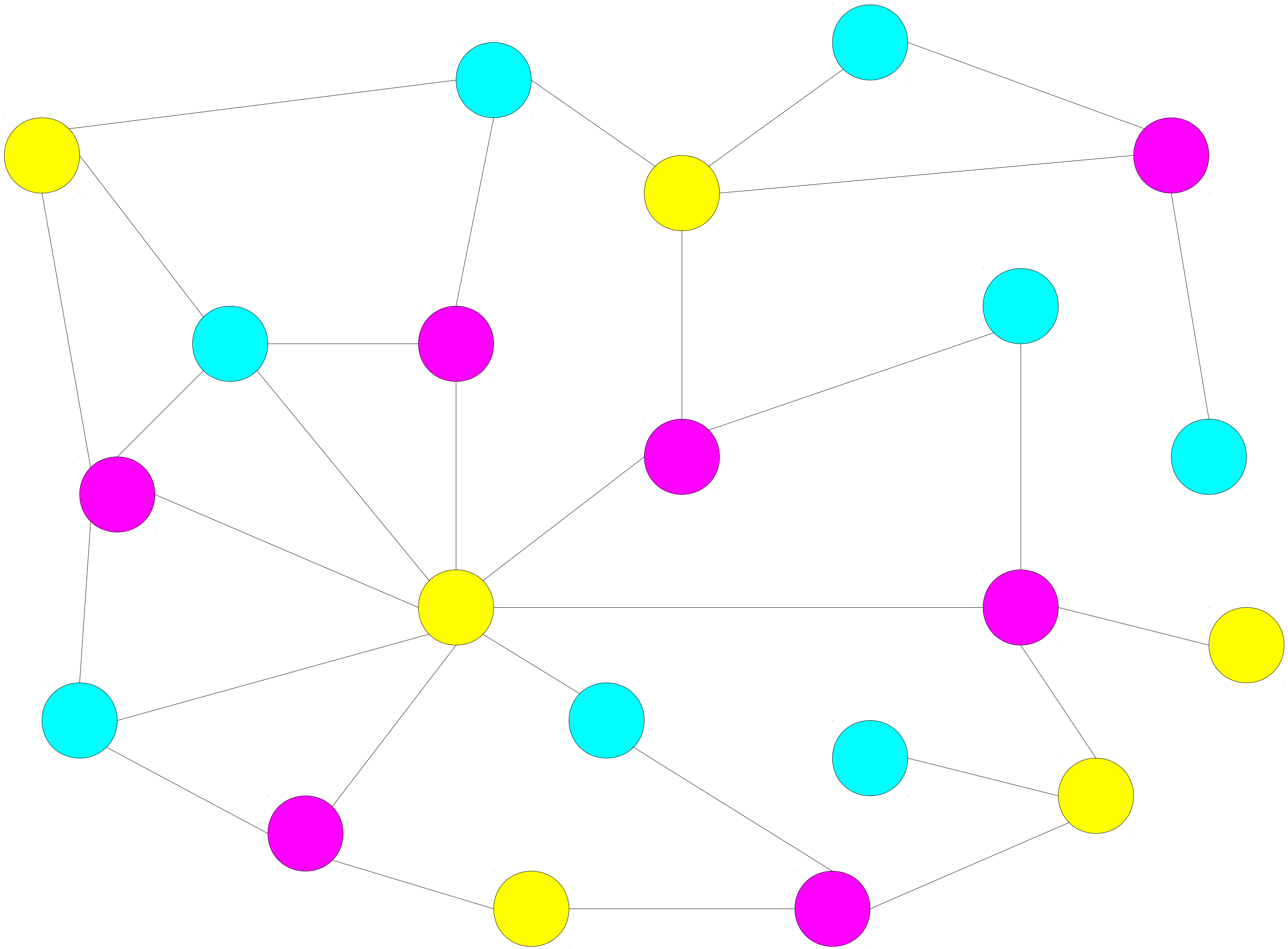


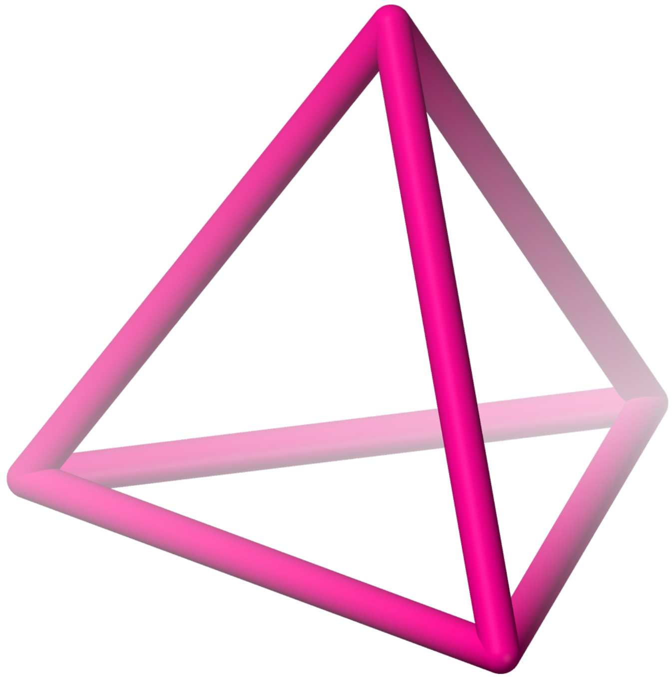


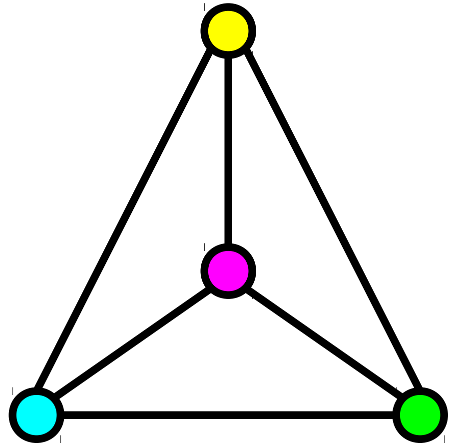
Graph Coloring

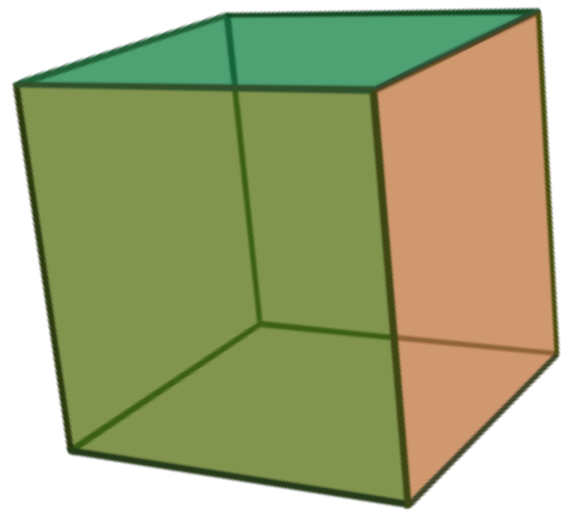


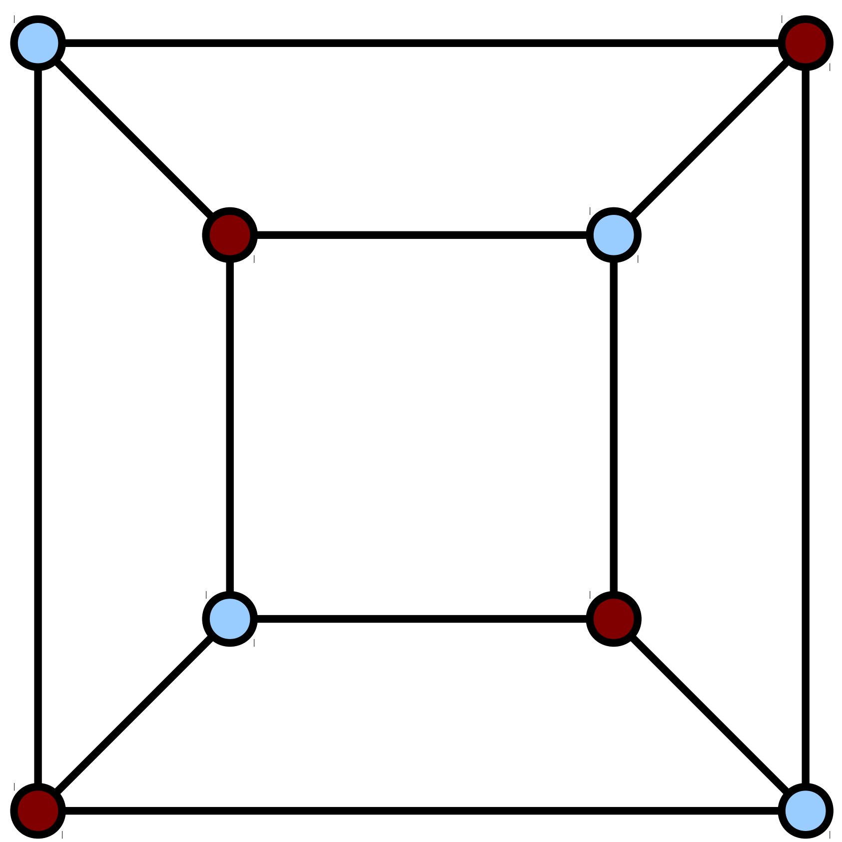
Graph Coloring











Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.

Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k-coloring*** of a graph $G = (V, E)$ is a function

$$\mathbf{f : V \to \{1, 2, \ldots, k\}}$$

such that

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$

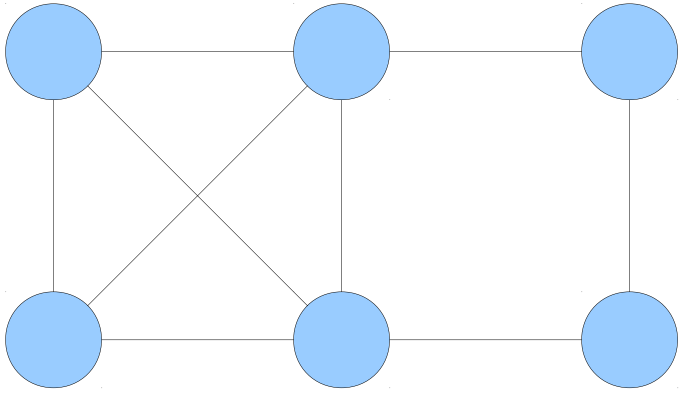
Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k-coloring*** of a graph $G = (V, E)$ is a function

$$\mathbf{f} : V \to \{1, 2, \ldots, k\}$$

  such that

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$



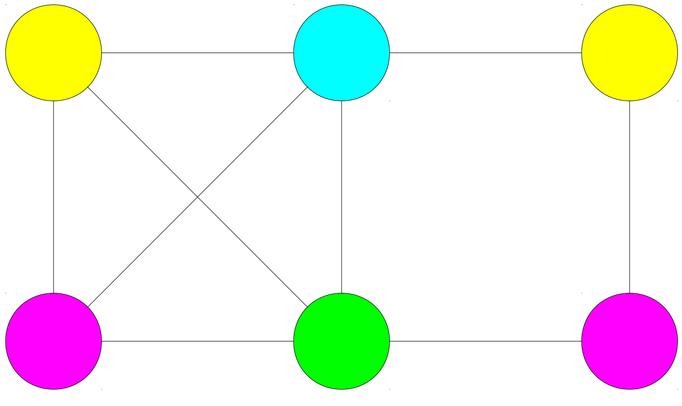
Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k-coloring*** of a graph $G = (V, E)$ is a function

$$\mathbf{f : \mathit{V} \to \{1, 2, \dots, \mathit{k}\}}$$

such that

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$



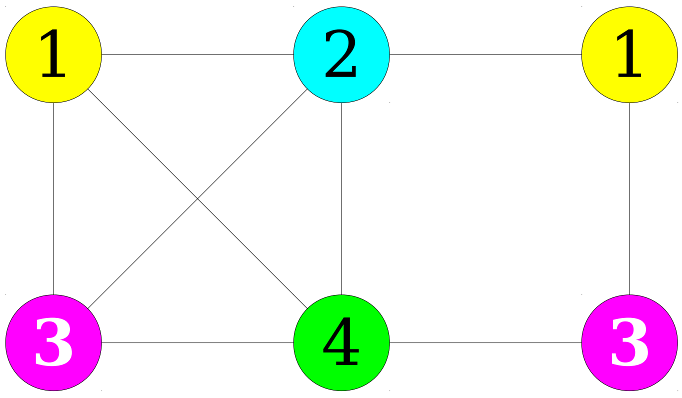
Graph Coloring

- Intuitively, a ***k*-coloring** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k*-coloring** of a graph $G = (V, E)$ is a function

$$\mathbf{f} : V \to \{1, 2, \ldots, k\}$$

such that

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$



Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k-coloring*** of a graph $G = (V, E)$ is a function

$$\mathbf{f} : V \to \{1, 2, \ldots, k\}$$

such that

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$

Although this is the formal definition of a k-coloring, you rarely see it used in proofs. It's more common to just talk about assigning colors to nodes. However, this definition is super useful if you want to write programs to reason about graph colorings!

Graph Coloring

- Intuitively, a ***k-coloring*** of a graph $G = (V, E)$ is a way to color each node in V one of k different colors such that no two adjacent nodes in V are the same color.
- Formally, a ***k-coloring*** of a graph $G = (V, E)$ is a function

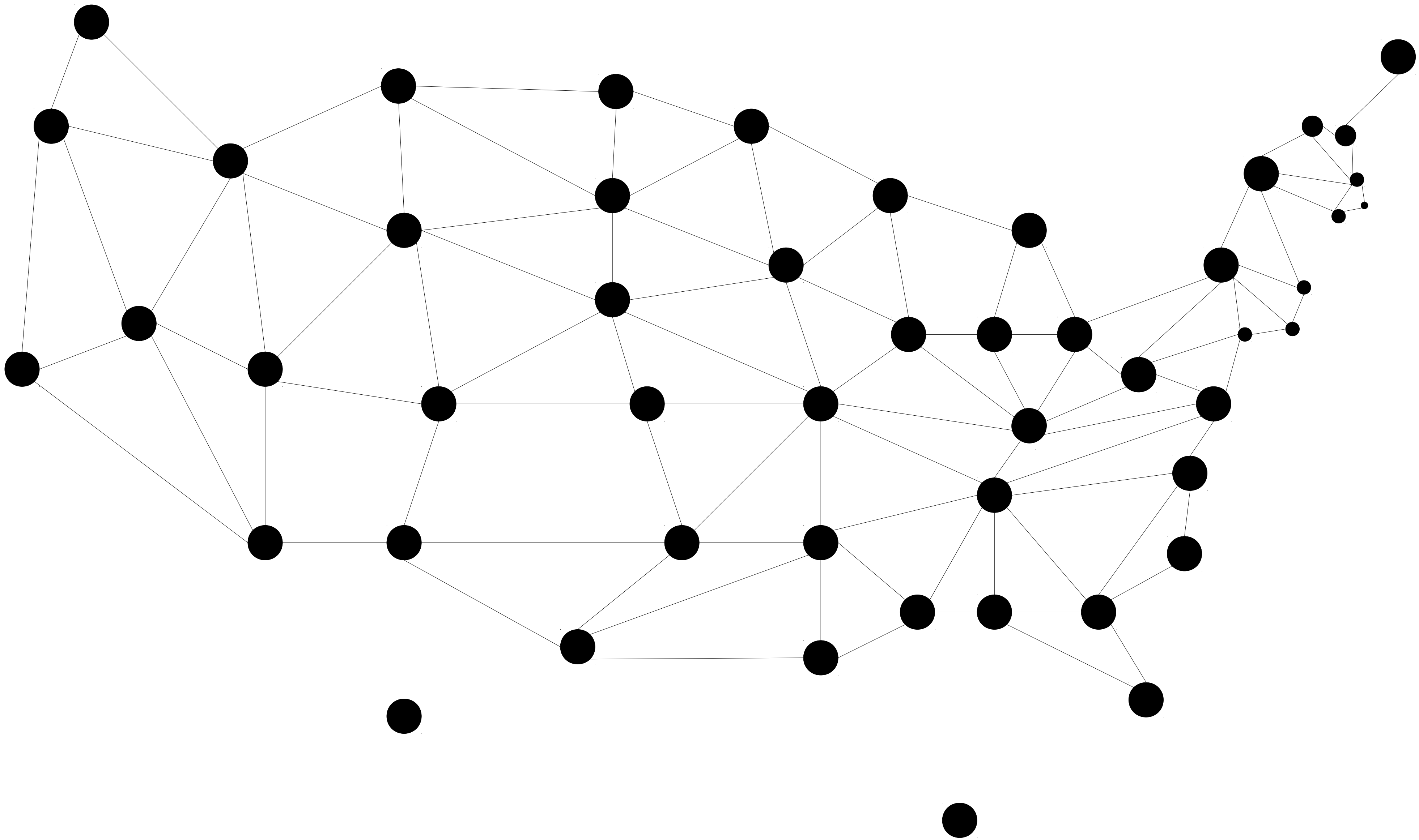
$$\mathbf{f : V \to \{1, 2, \ldots, k\}}$$

  such that

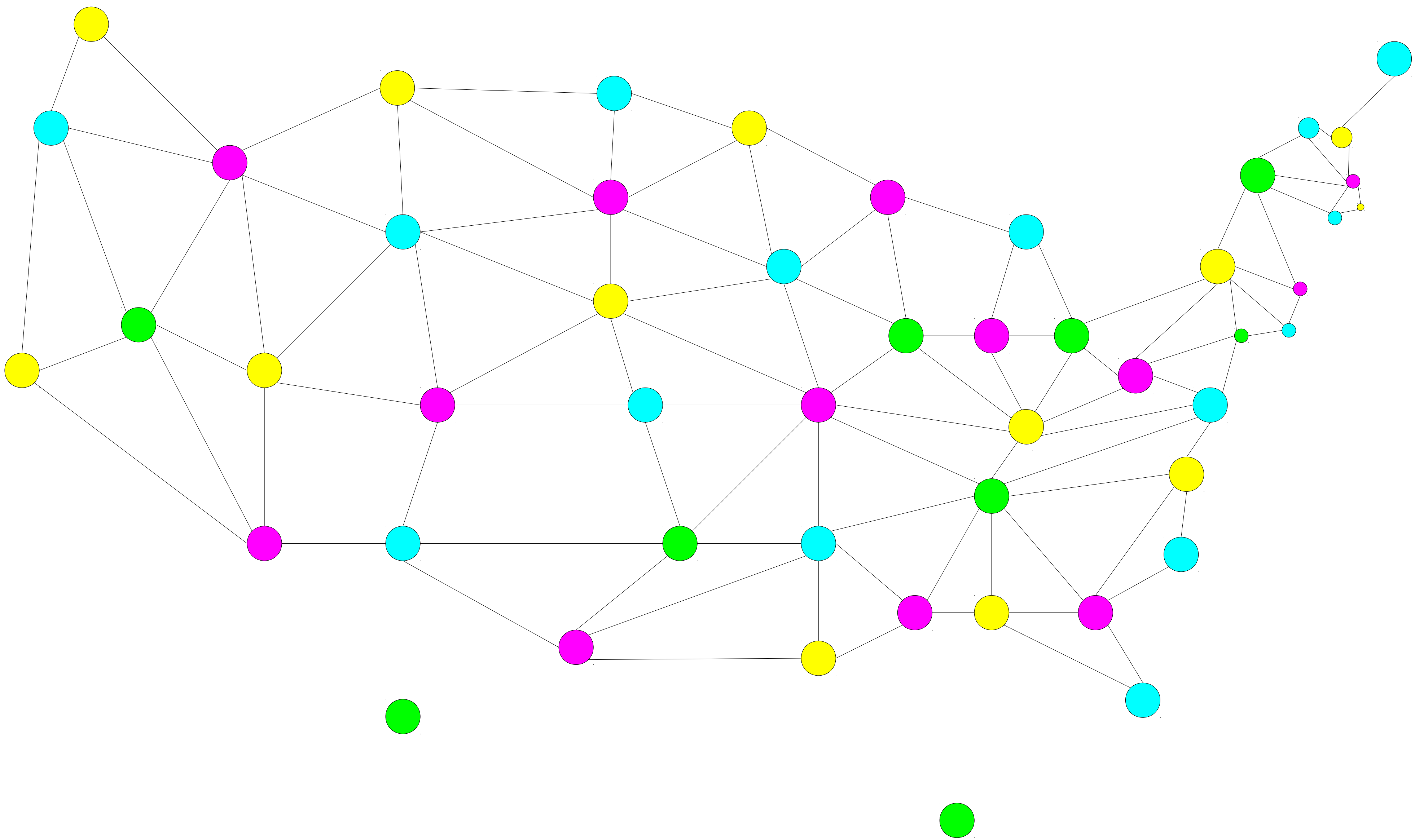
$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to f(u) \neq f(v))$$

- A graph G is called ***k-colorable*** if a k-coloring of G exists.
- The smallest k for which G is k-colorable is its ***chromatic number***.
  - The chromatic number of a graph G is denoted $\chi(G)$, from the Greek *χρώμα*, meaning “color.”

Graph Coloring



Graph Coloring



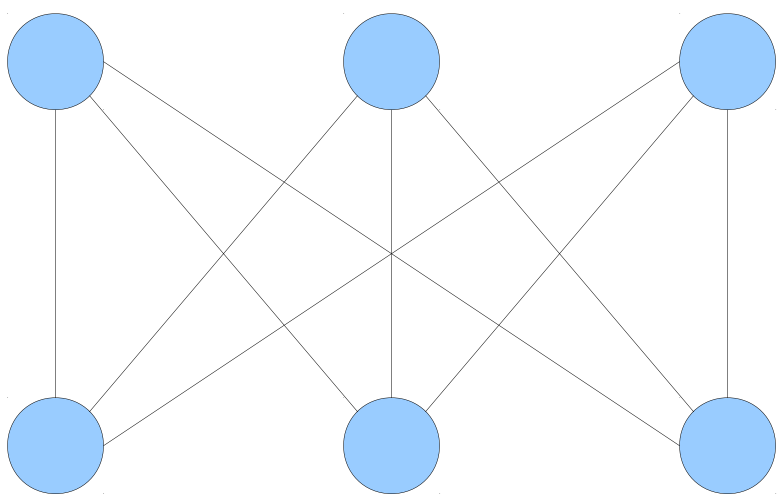
Theorem (Four-Color Theorem): Every planar graph is 4-colorable.

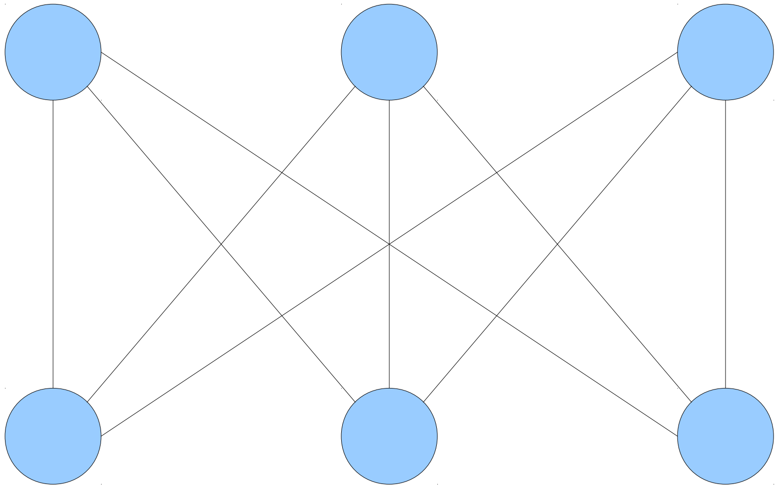
- **1850s:** Four-Color Conjecture posed.
- **1879:** Kempe proves the Four-Color Theorem.
- **1890:** Heawood finds a flaw in Kempe's proof.
- **1976:** Appel and Haken design a computer program that proves the Four-Color Theorem. The program checked 1,936 specific cases that are “minimal counterexamples;” any counterexample to the theorem must contain one of the 1,936 specific cases.
- **1980s:** Doubts rise about the validity of the proof due to errors in the software.
- **1989:** Appel and Haken revise their proof and show it is indeed correct. They publish a book including a 400-page appendix of all the cases to check.
- **1996:** Roberts, Sanders, Seymour, and Thomas reduce the number of cases to check down to 633.
- **2005:** Werner and Gonthier repeat the proof using an established automatic theorem prover (Coq), improving confidence in the truth of the theorem.

Philosophical Question: Is a theorem true if no human has ever read the proof?

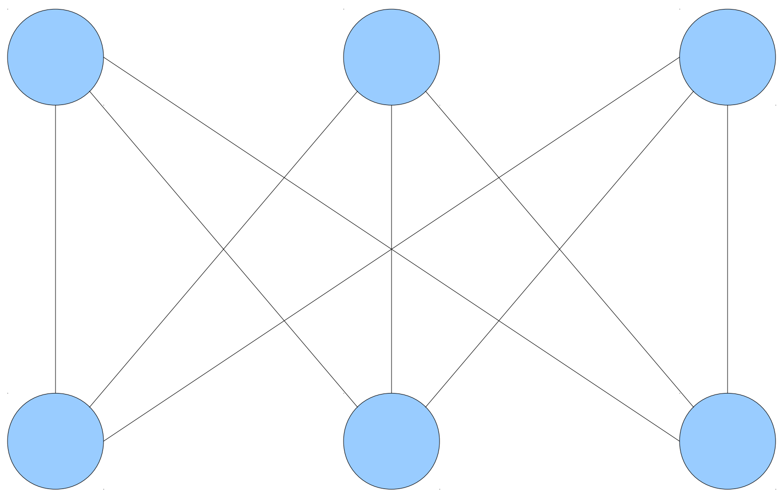
Mathematics and Vision

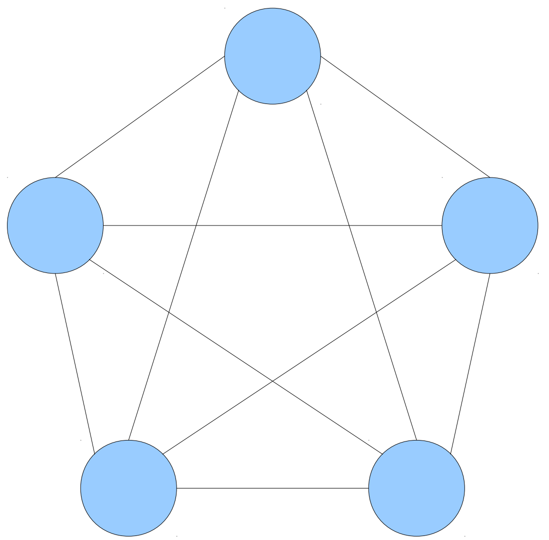
- Earlier, when we discussed connected components, we spent a lot of time pinning down a formal definition of a connected component.
- When talking about planar graphs, we're still relying on a visual definition.
- ***Question:*** Is there a way to rigorously define planar graphs without relying on visual intuitions?



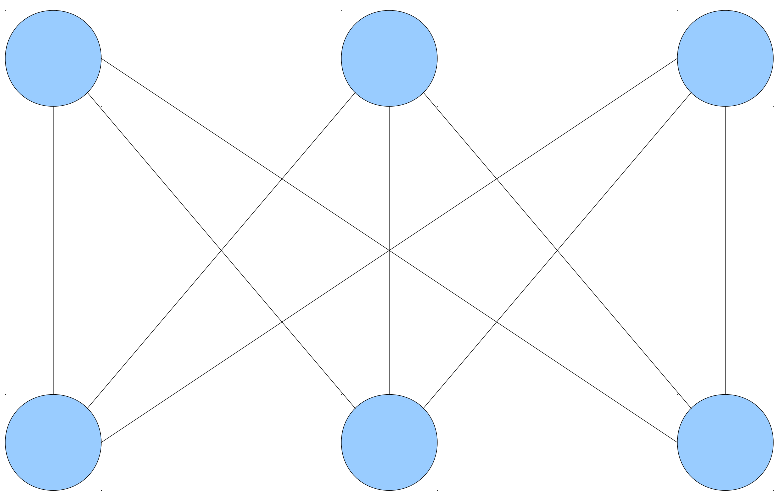


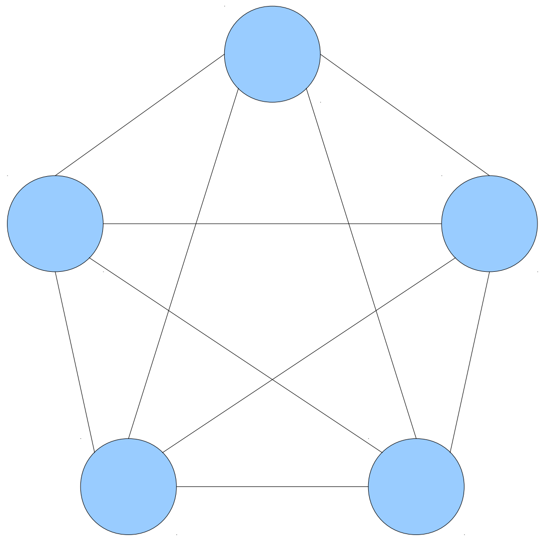
This is the ***utility graph*** we saw earlier. It's also called $K_{3, 3}$.





This is the ***utility graph*** we saw earlier. It's also called $K_{3,\,3}$.



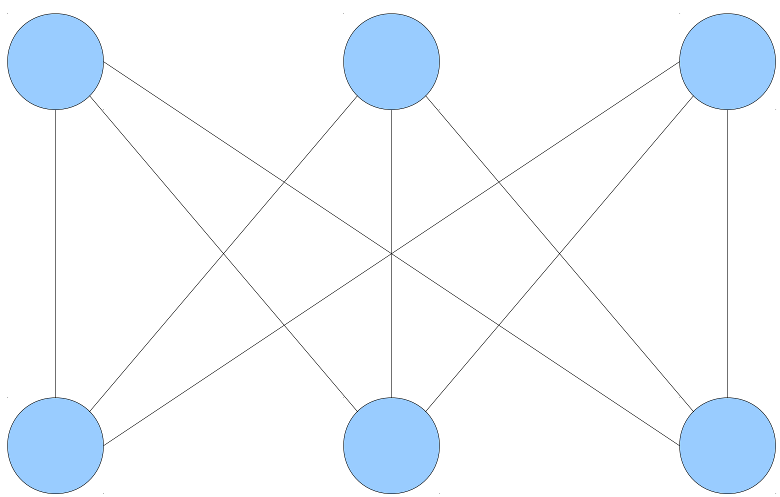


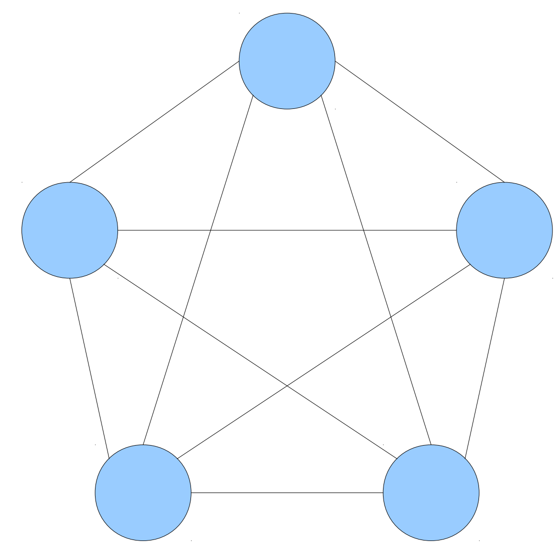
This is the ***utility graph*** we saw earlier. It's also called $K_{3,\,3}$.

This graph is called a ***5-clique***. You often see it called K_5.

Theorem: Neither of these graphs are planar.

Fun challenge: Prove that K_5 is not planar.

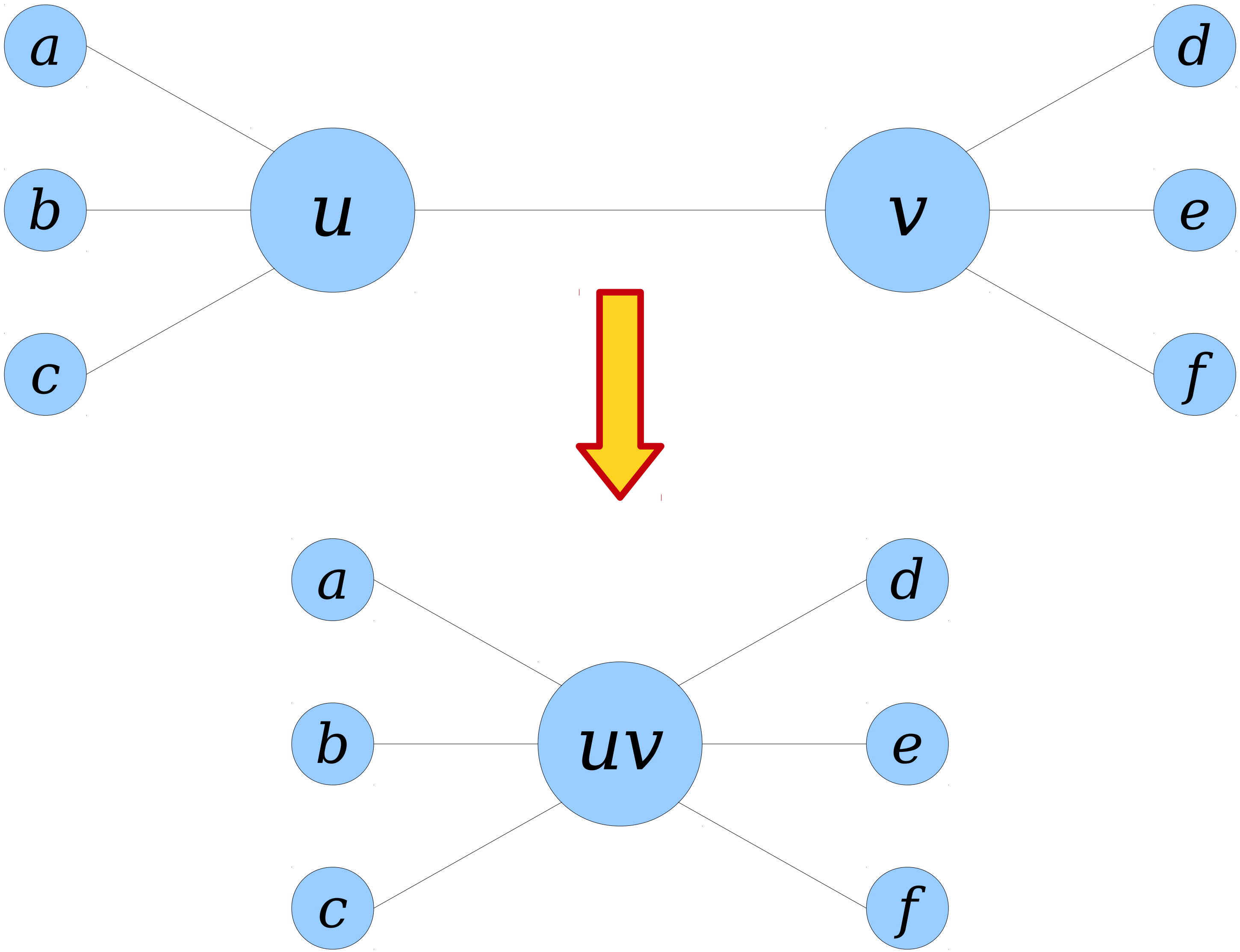




This is the ***utility graph*** we saw earlier. It's also called $K_{3,\,3}$.

This graph is called a ***5-clique***. You often see it called K_5.

Edge Contraction



Theorem (Wagner's Theorem): A graph G is planar if and only if is impossible to turn G into $K_{3,3}$ or K_5 by deleting nodes and edges from G or contracting edges of G.

Proof: Haha no.

Algorithms on Graphs

- Results like the previous one give a way to rigorously define certain types of graphs without appealing to any visual intution.
  - Cool term: this is a type of ***forbidden minor characterization*** of a family of graphs.
- There is a *ton* of research about the computational difficulty of designing good graph algorithms.
- Curious to learn more? Take CS161, CS267, or CS367!

Time-Out for Announcements!

Problem Sets

- Problem Set Three was due at the start of today's lecture.
  - Want to use late days? Feel free to turn it in by Monday at 3:00PM.
- Problem Set Four goes out now.
  - The checkpoint problem is due on Monday.
  - Remaining problems are due on Friday.
- Explore strict orders, Hasse diagrams, graphs, and the pigeonhole principle!

Event Tonight

- Jad Abumrad, host of *Radiolab*, will be performing at the Stanford Storytelling Project tonight in Memorial Auditorium.
  - If you've never heard of this show, you should go listen to it. It's phenomenal.
- There are a few tickets left at the door. The event opens to students at 7:00PM.

Your Questions

"Keith, will you go to Flo Formal with me tonight? (I'll reveal my identity when you say yes)"

"PS: The person (me) asking you to Flo Formal is in SLE. Your interest is piqued."

I'm flattered, but I think I'll have to decline because of the "half your age plus seven" rule. ☺

PS: To everyone in this class that's crushing on someone – *go ask them out already!*

“When is it appropriate to negotiate an internship salary?”

It can't hurt to try! Figure it's good practice for negotiating a full-time salary even if it doesn't work out.

Pro tip – literally – if you're applying for full-time jobs, you should **absolutely** negotiate your salary. There's tons of evidence explaining why this is so important. Email me about this if you're curious.

“How does one fall in love with math if one is not good at it?”

Okay – you have <u>not</u> been doing math long enough to say that you're not good at it. Trust me on this one.

I'm going to post a YouTube link on the course website later today. Take a look at it and let me know your thoughts. ☺

Back to CS103!

The Pigeonhole Principle and Graphs

The ***pigeonhole principle*** is the following:

If m objects are placed into n bins, where $m > n$, then some bin contains at least two objects.

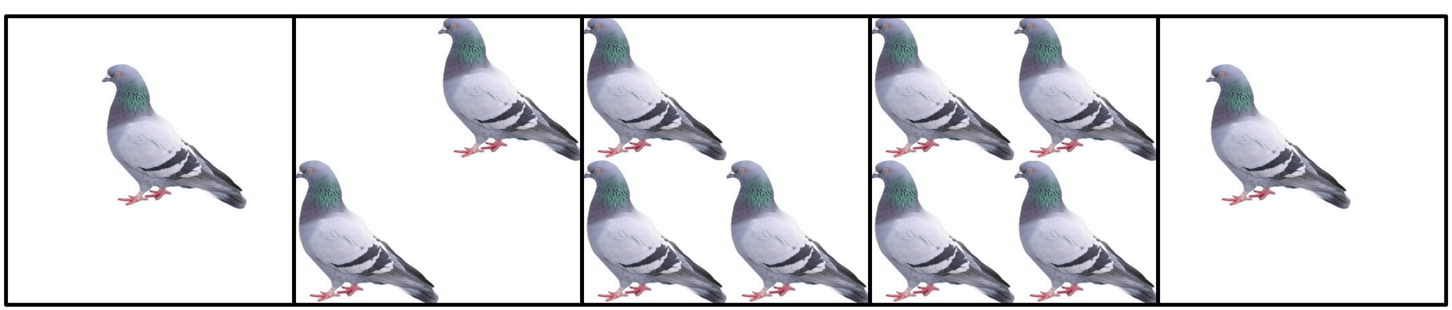
(We sketched a proof in Lecture #02)

The Generalized Pigeonhole Principle

The ***generalized pigeonhole principle*** is the following:

If m objects are placed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects and some bin contains at most $\lfloor m/n \rfloor$ objects.

(Here, $\lceil x \rceil$ is the ***ceiling function*** and denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ is the ***floor function*** and denotes the largest integer less than or equal to x.)



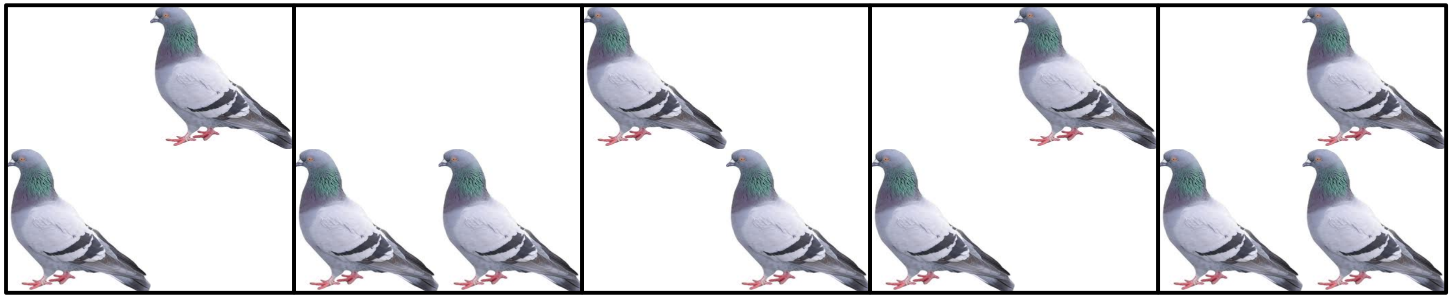
$m = 11$
$n = 5$

$\lceil m / n \rceil = 3$
$\lfloor m / n \rfloor = 2$

The ***generalized pigeonhole principle*** is the following:

If m objects are placed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects and some bin contains at most $\lfloor m/n \rfloor$ objects.

(Here, $\lceil x \rceil$ is the ***ceiling function*** and denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ is the ***floor function*** and denotes the largest integer less than or equal to x.)



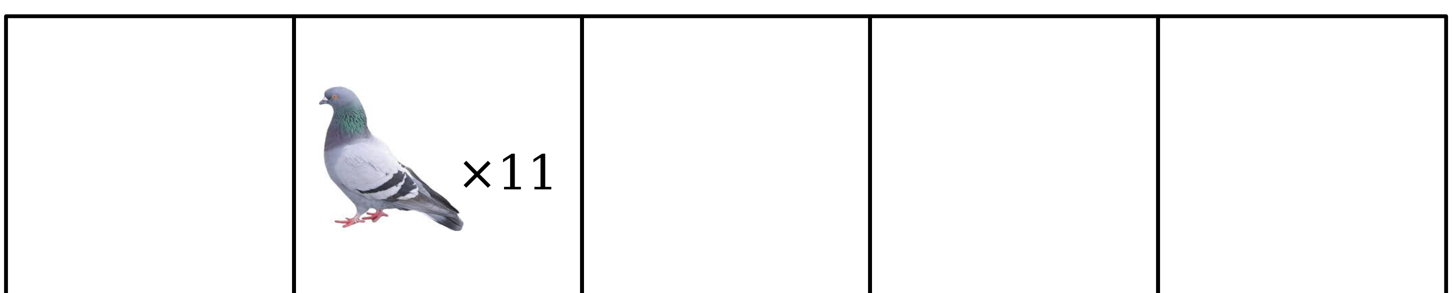
$m = 11$
$n = 5$

$\lceil m / n \rceil = 3$
$\lfloor m / n \rfloor = 2$

The ***generalized pigeonhole principle*** is the following:

If m objects are placed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects and some bin contains at most $\lfloor m/n \rfloor$ objects.

(Here, $\lceil x \rceil$ is the ***ceiling function*** and denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ is the ***floor function*** and denotes the largest integer less than or equal to x.)



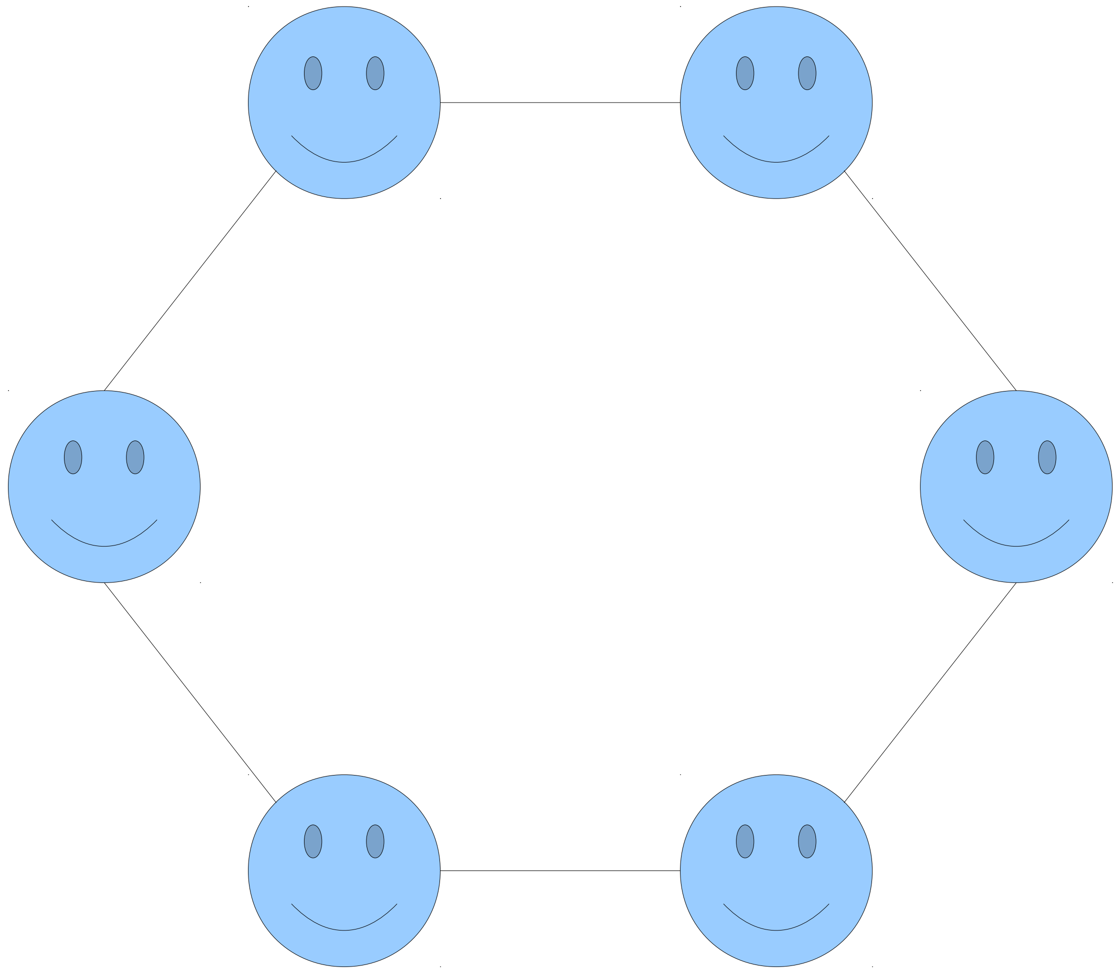
$m = 11$
$n = 5$

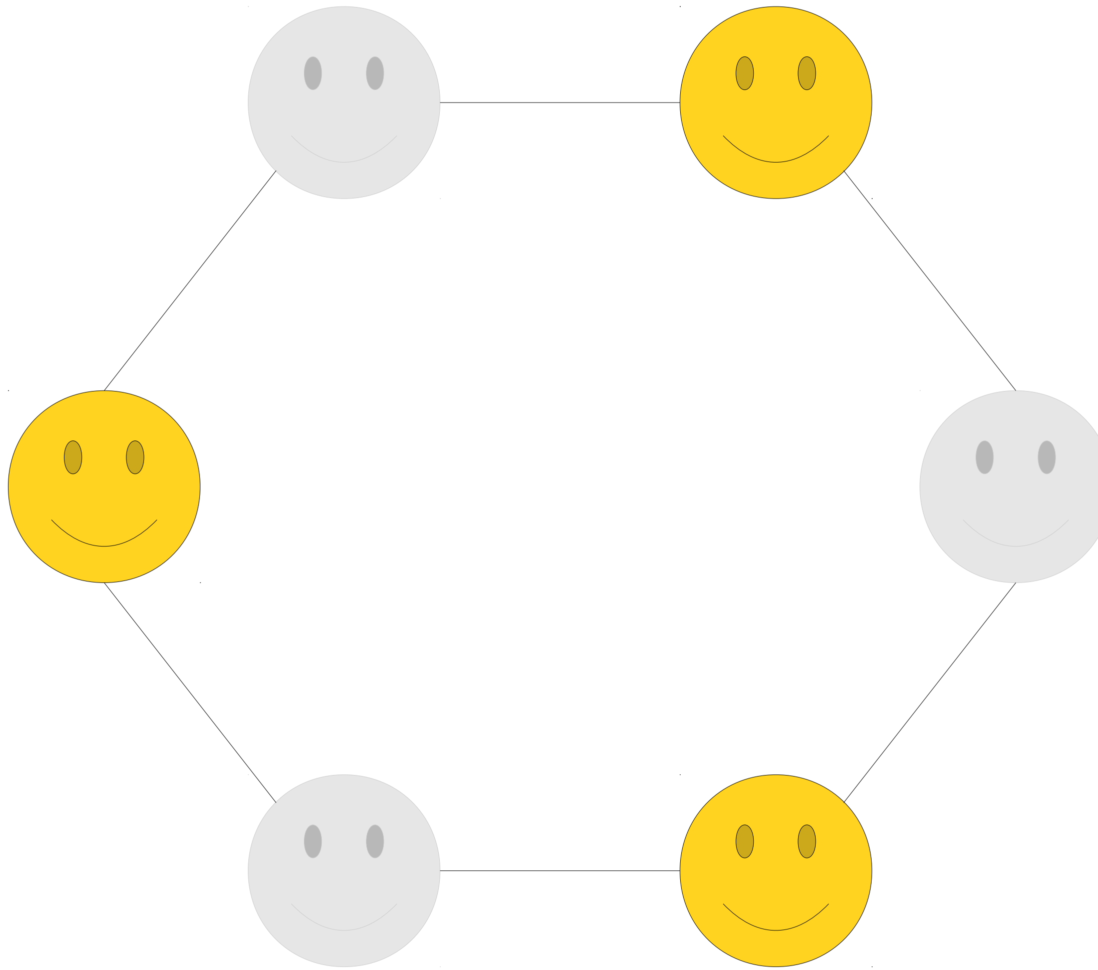
$\lceil m / n \rceil = 3$
$\lfloor m / n \rfloor = 2$

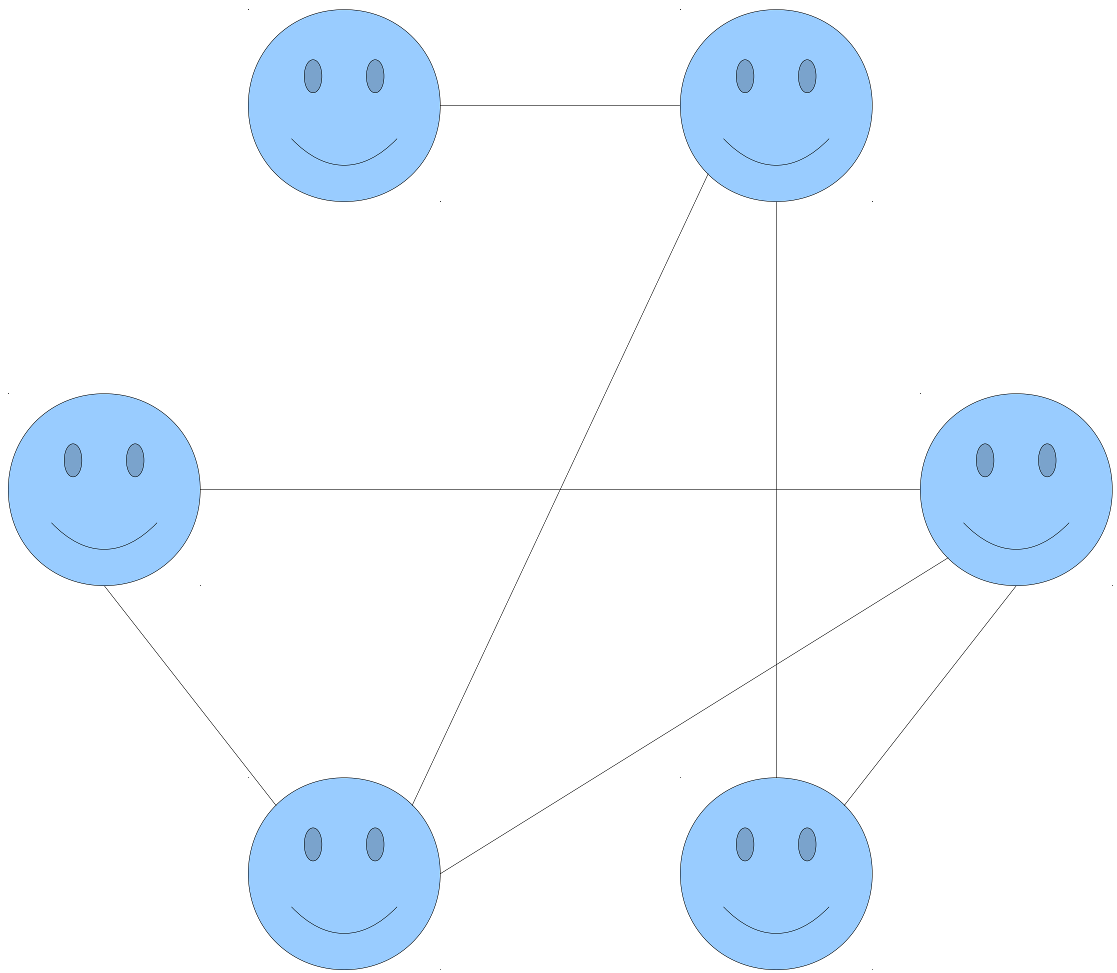
An Application: Friends and Strangers

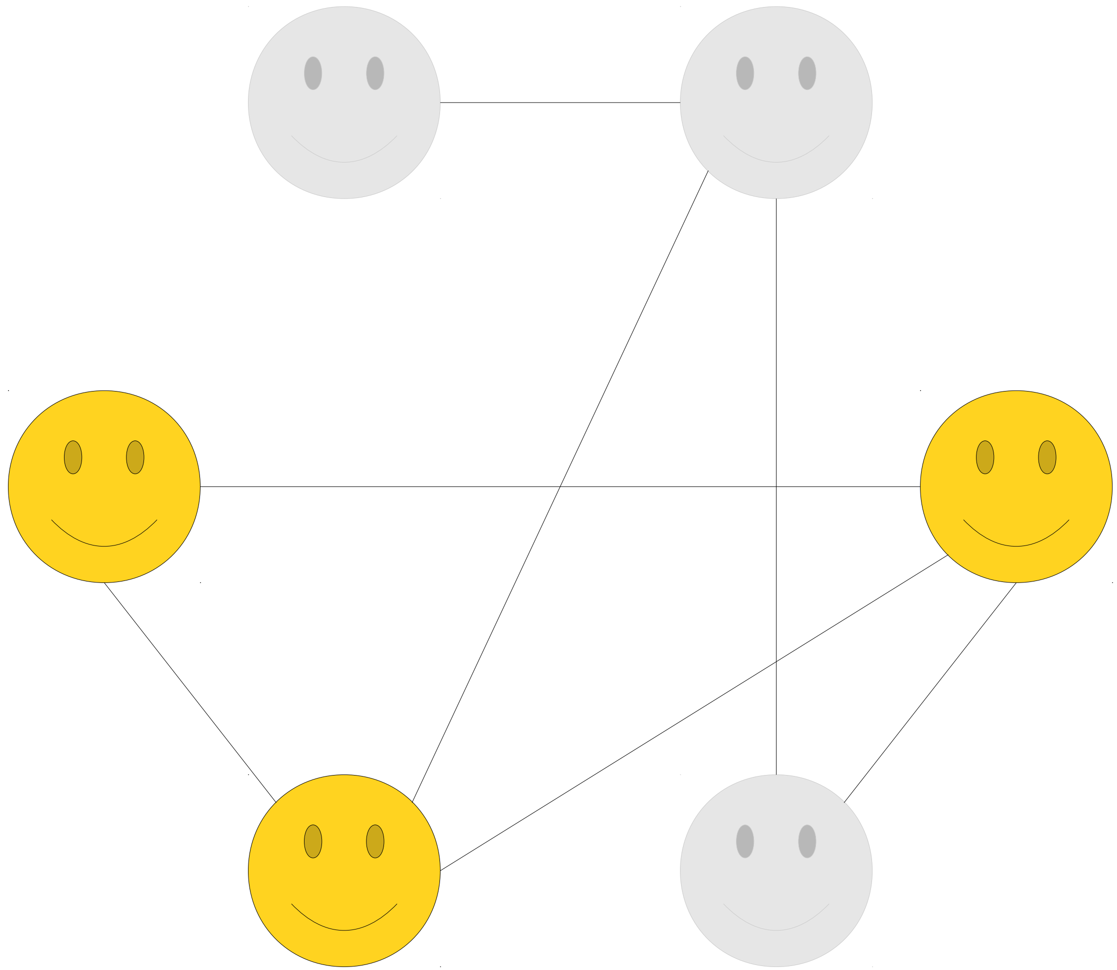
Friends and Strangers

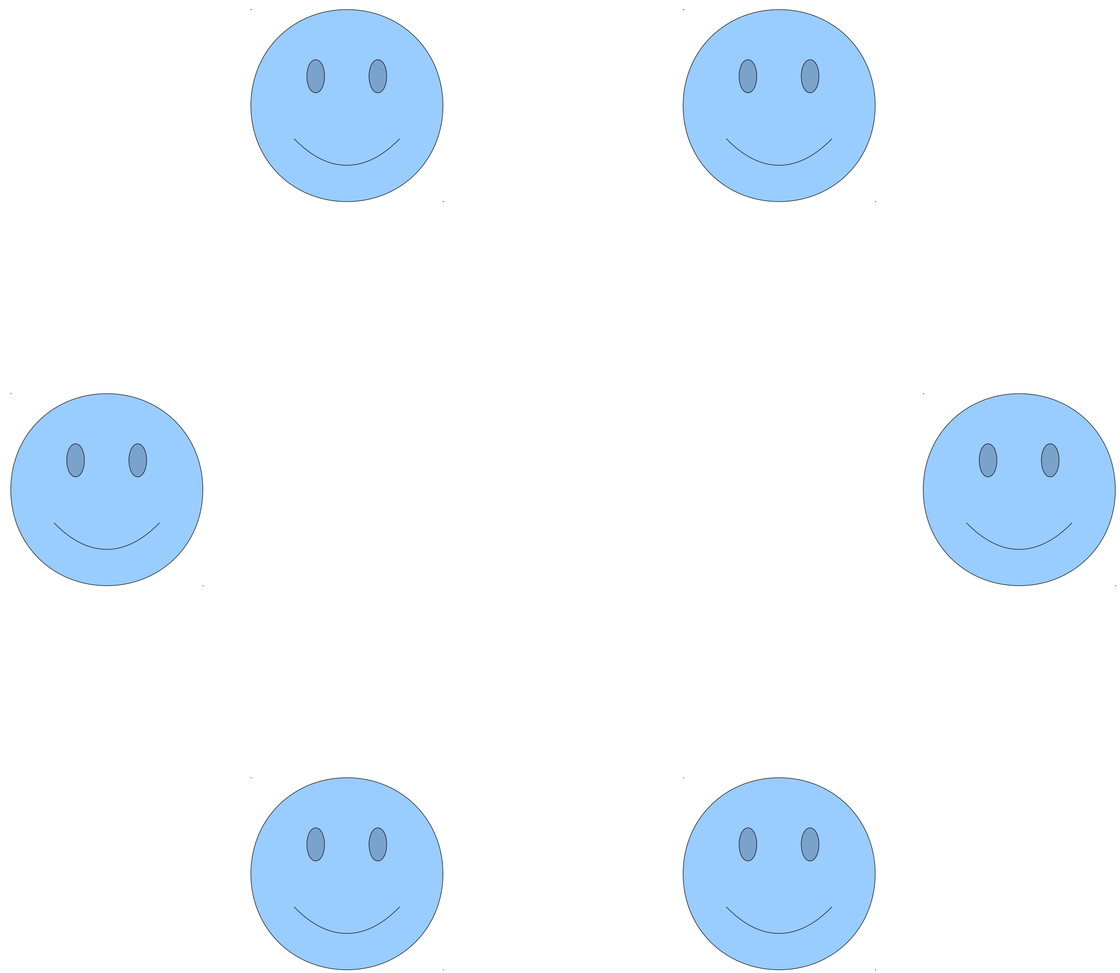
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people where no one knows anyone else).

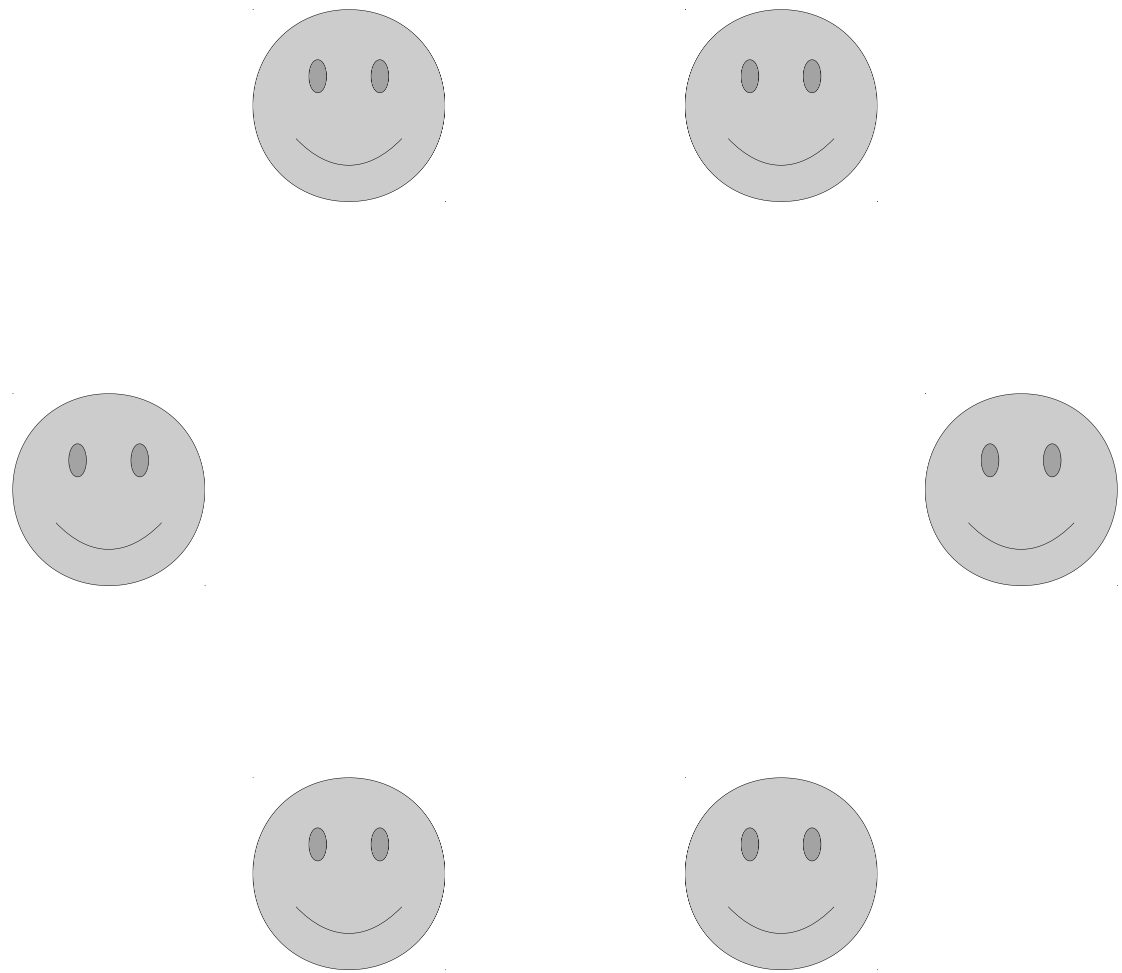


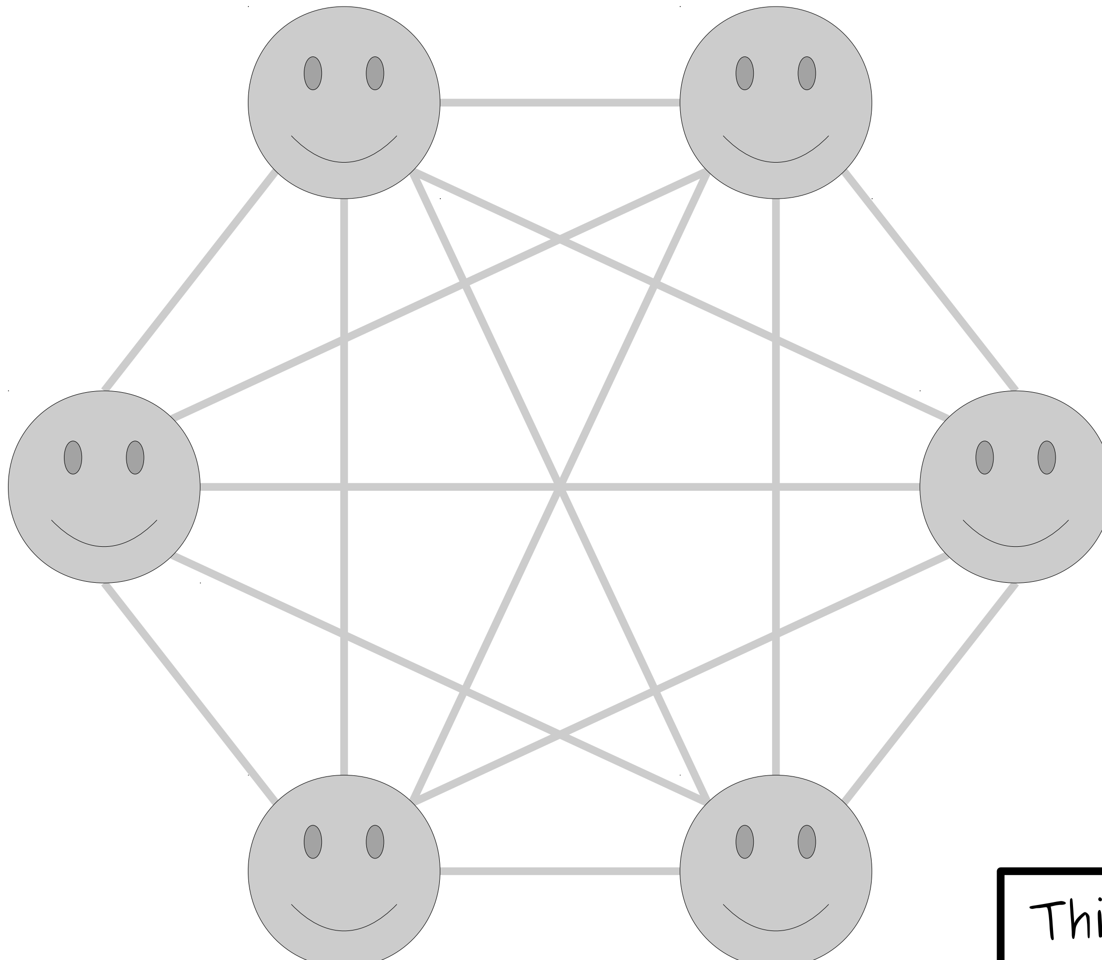




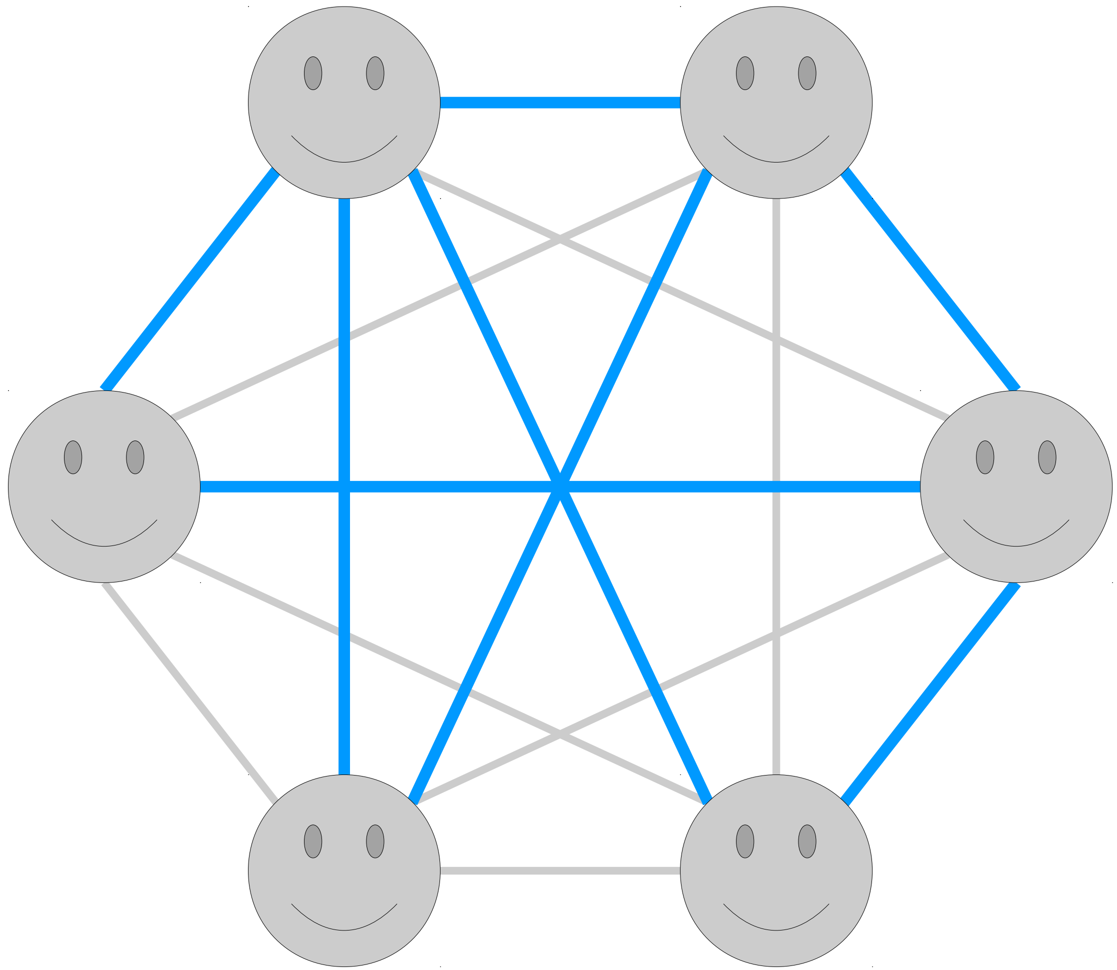


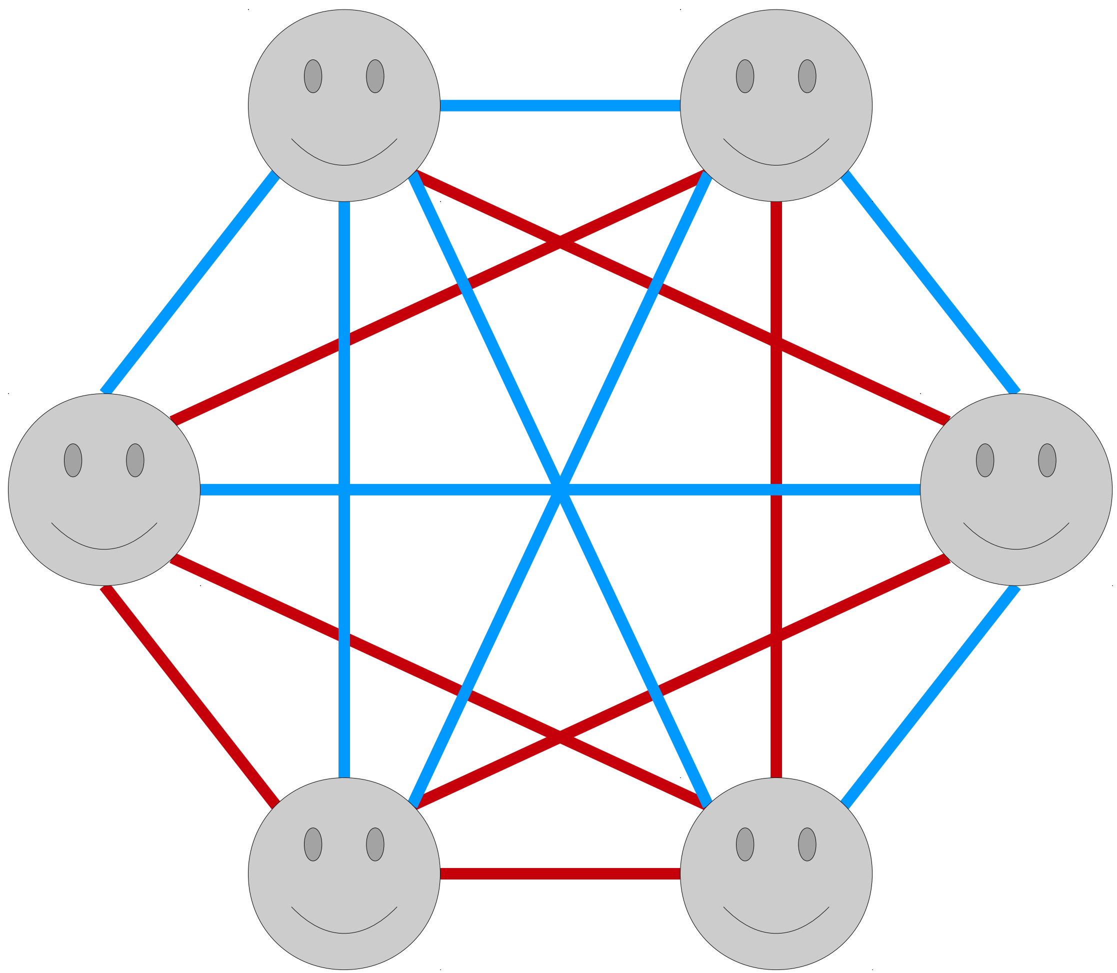


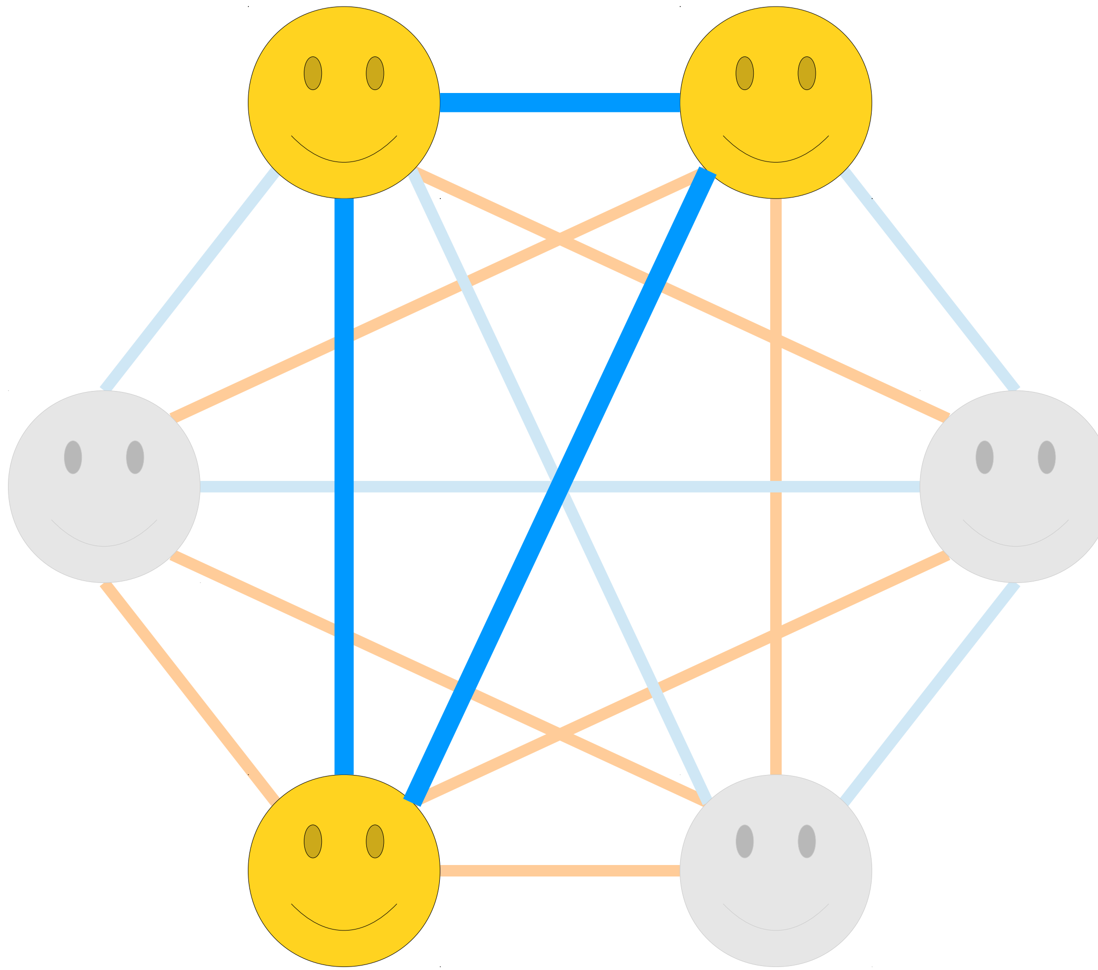


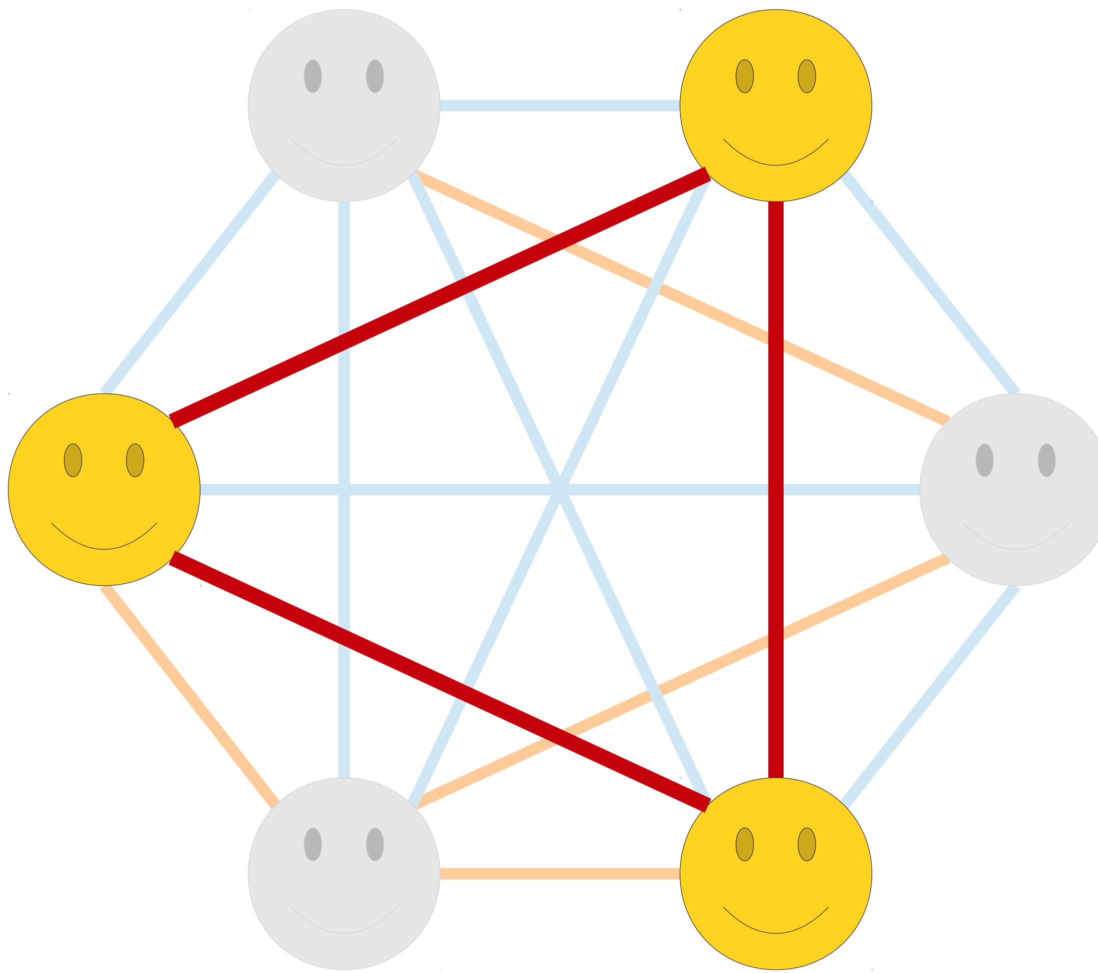


This graph is called a *6-clique*, by the way.



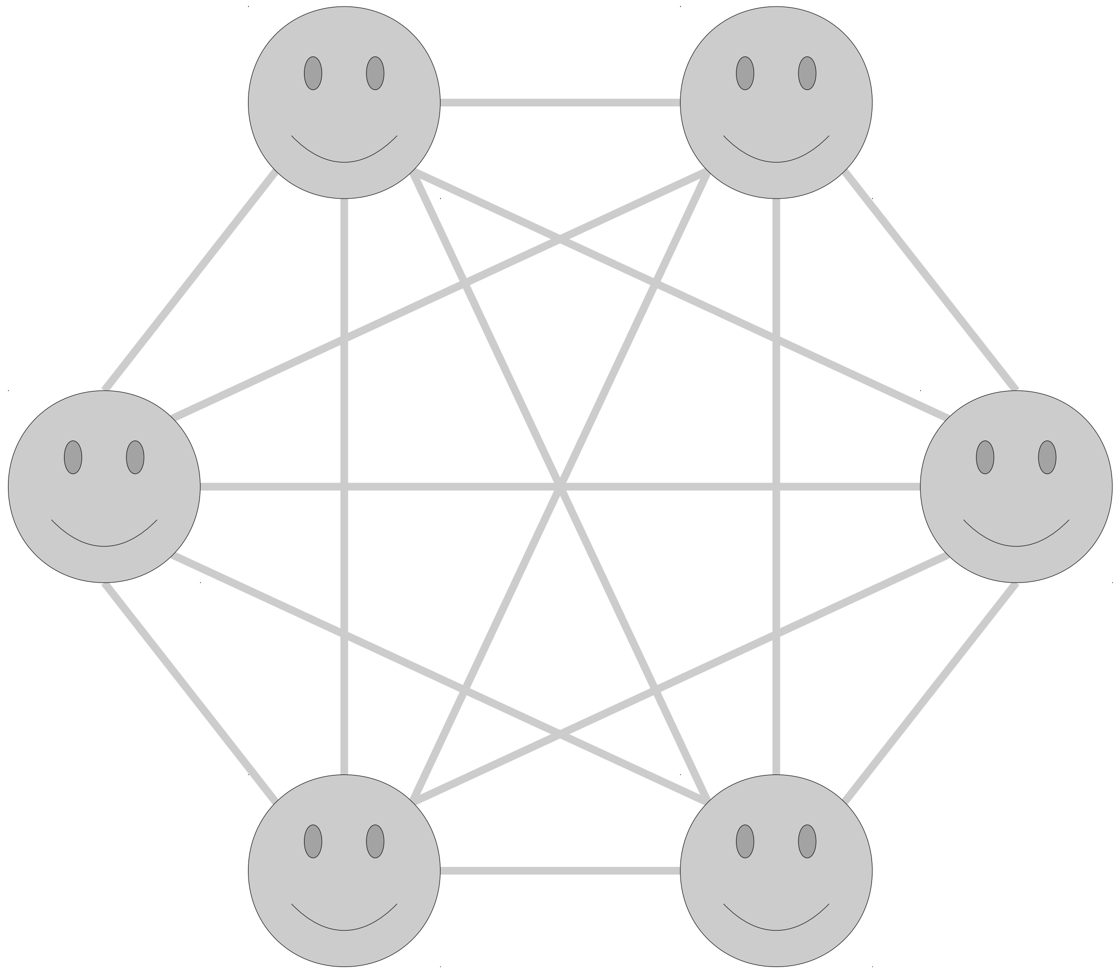


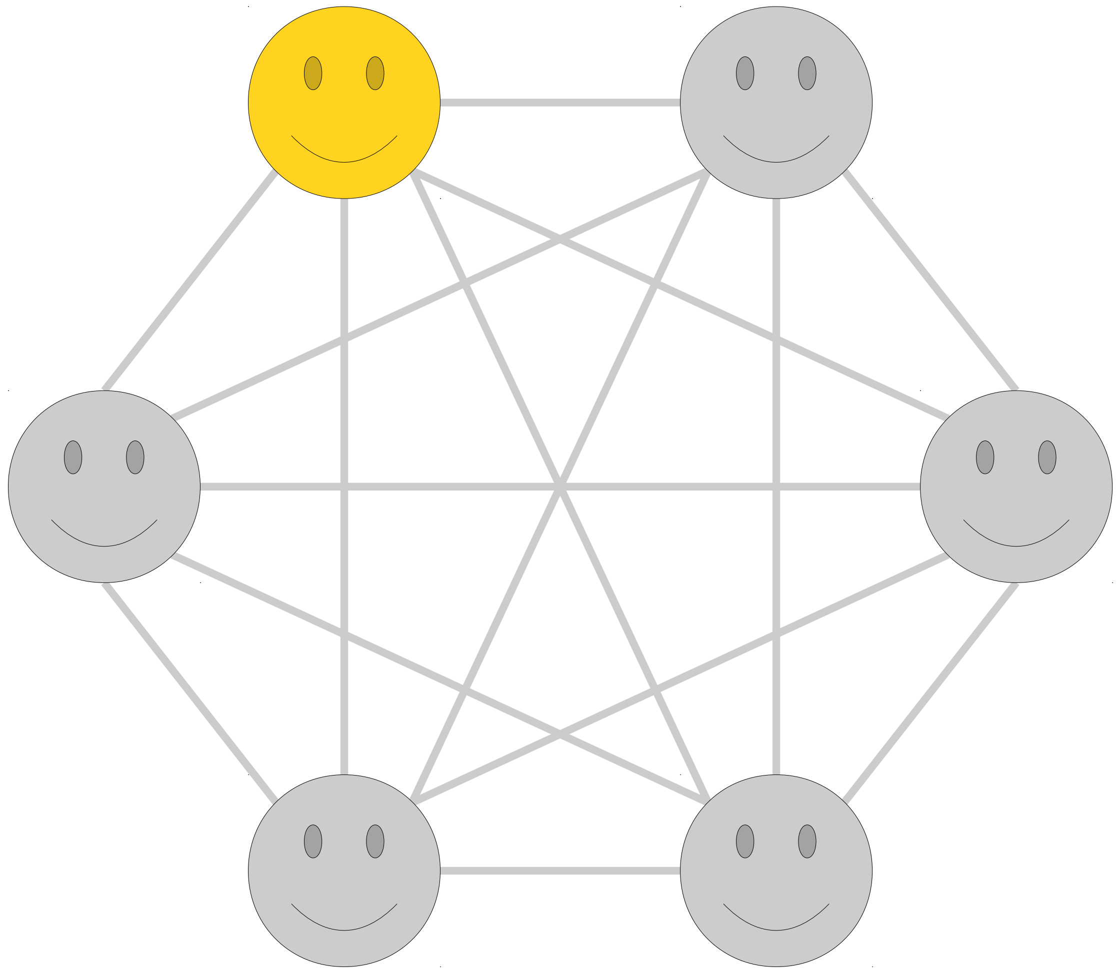


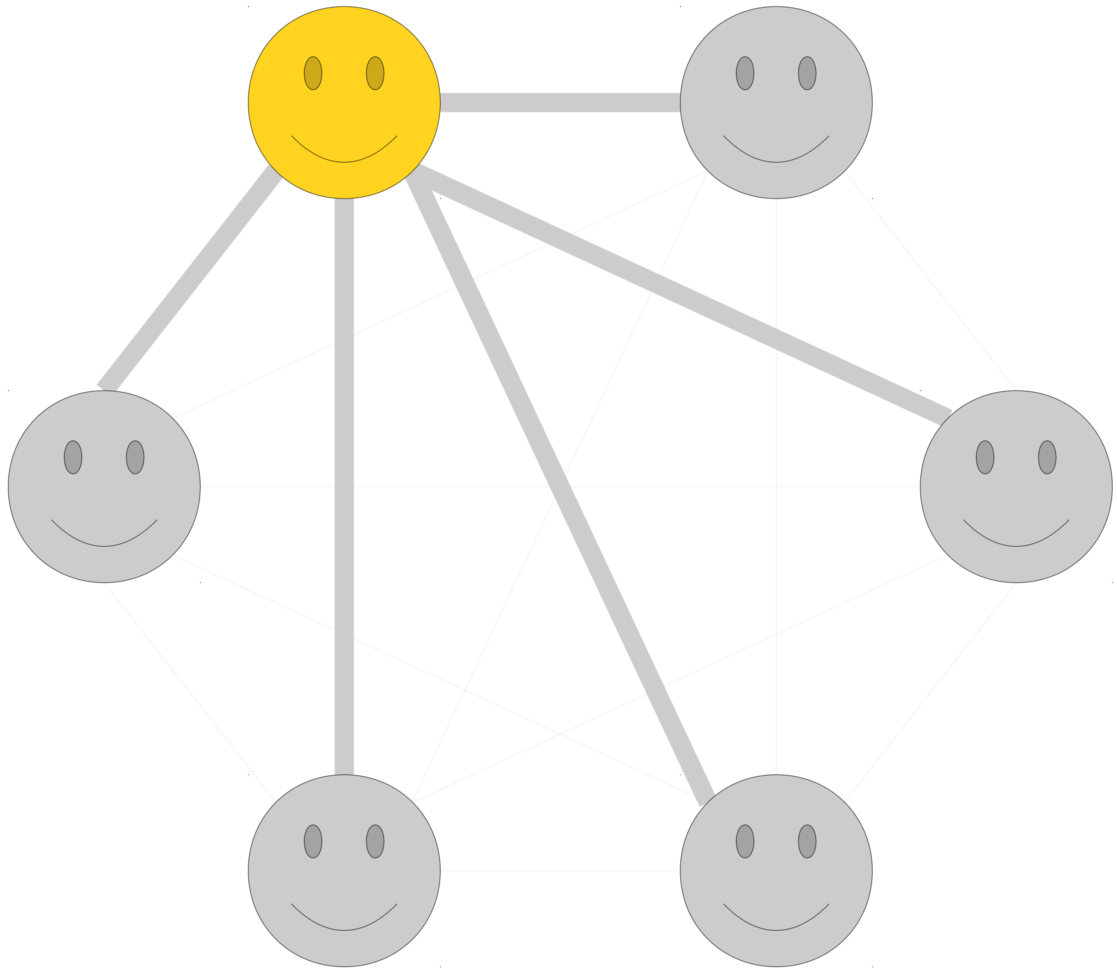


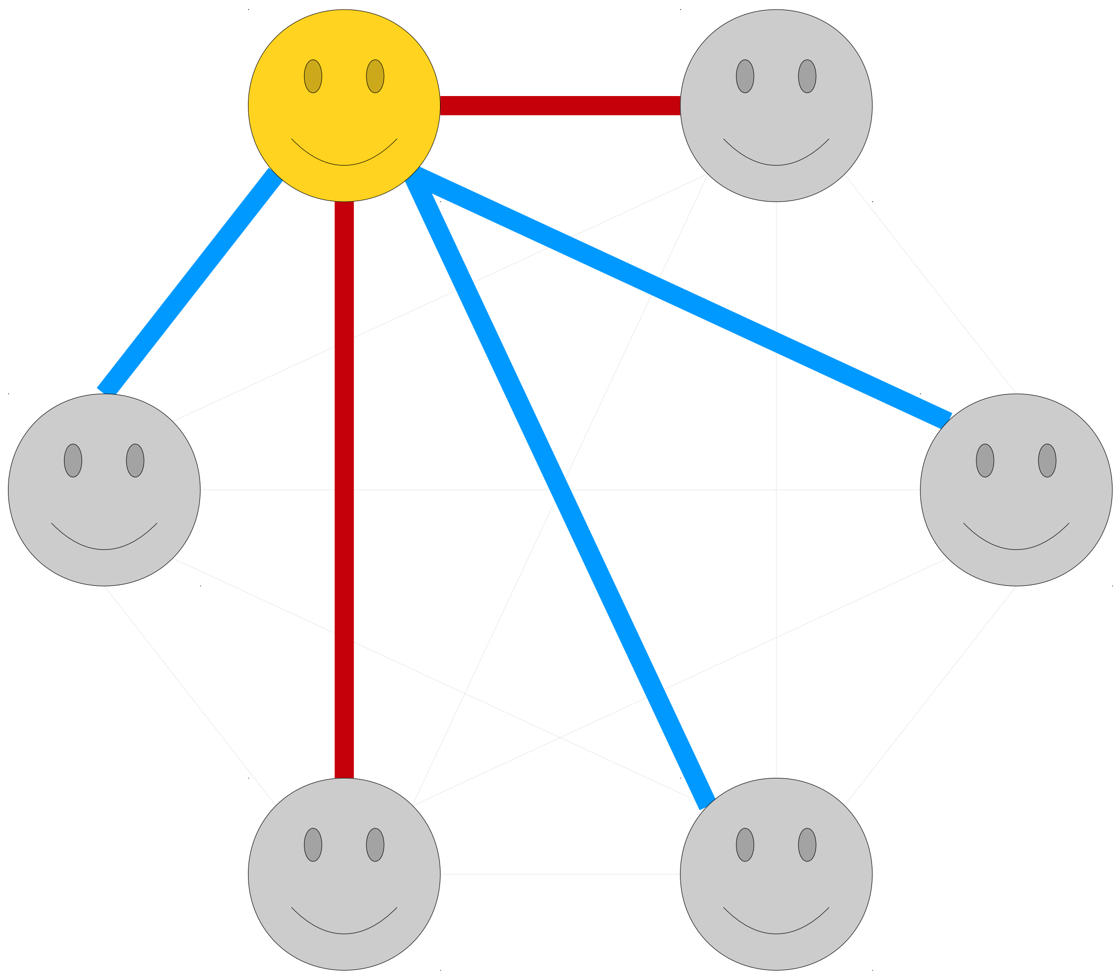
Friends and Strangers Restated

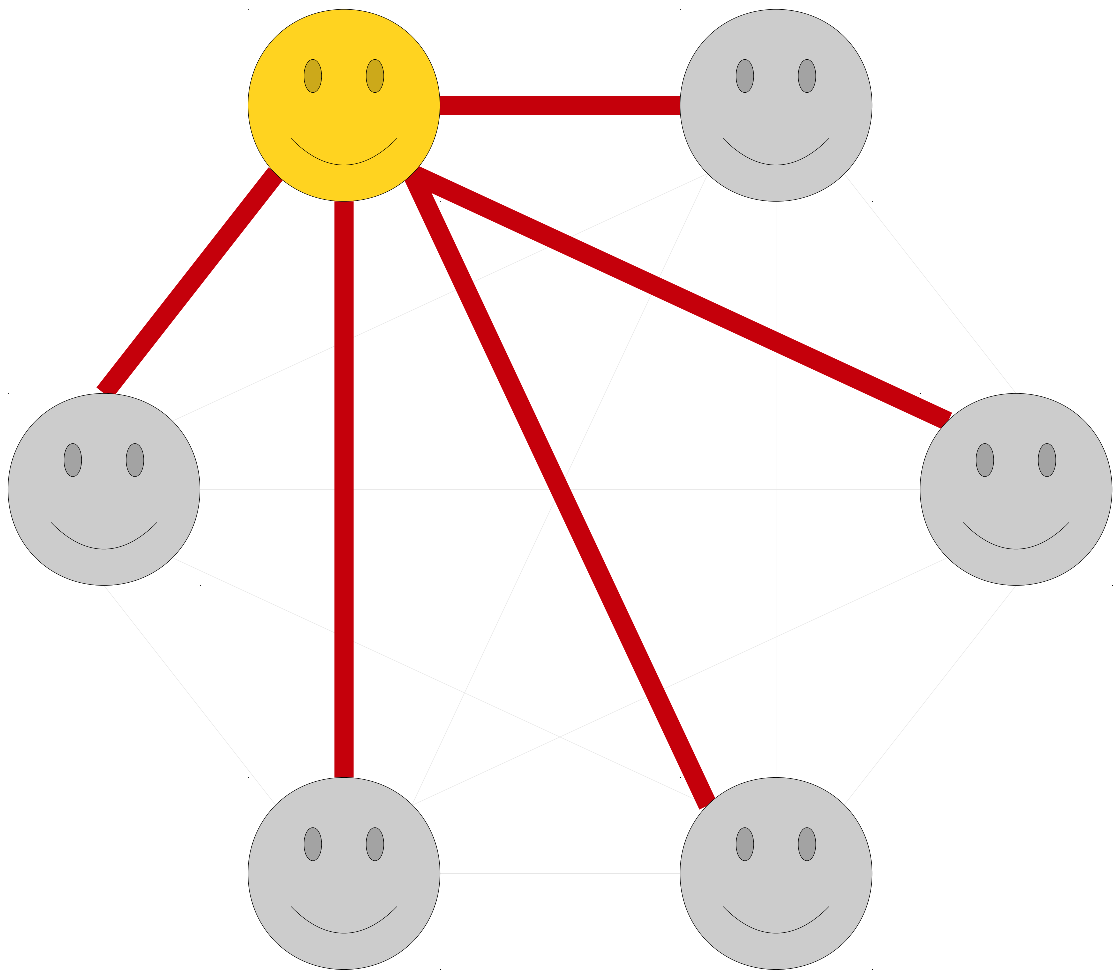
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:
- ***Theorem:*** Consider a 6-clique where every edge is colored red or blue. The the graph contains a red triangle, a blue triangle, or both.
- How can we prove this?

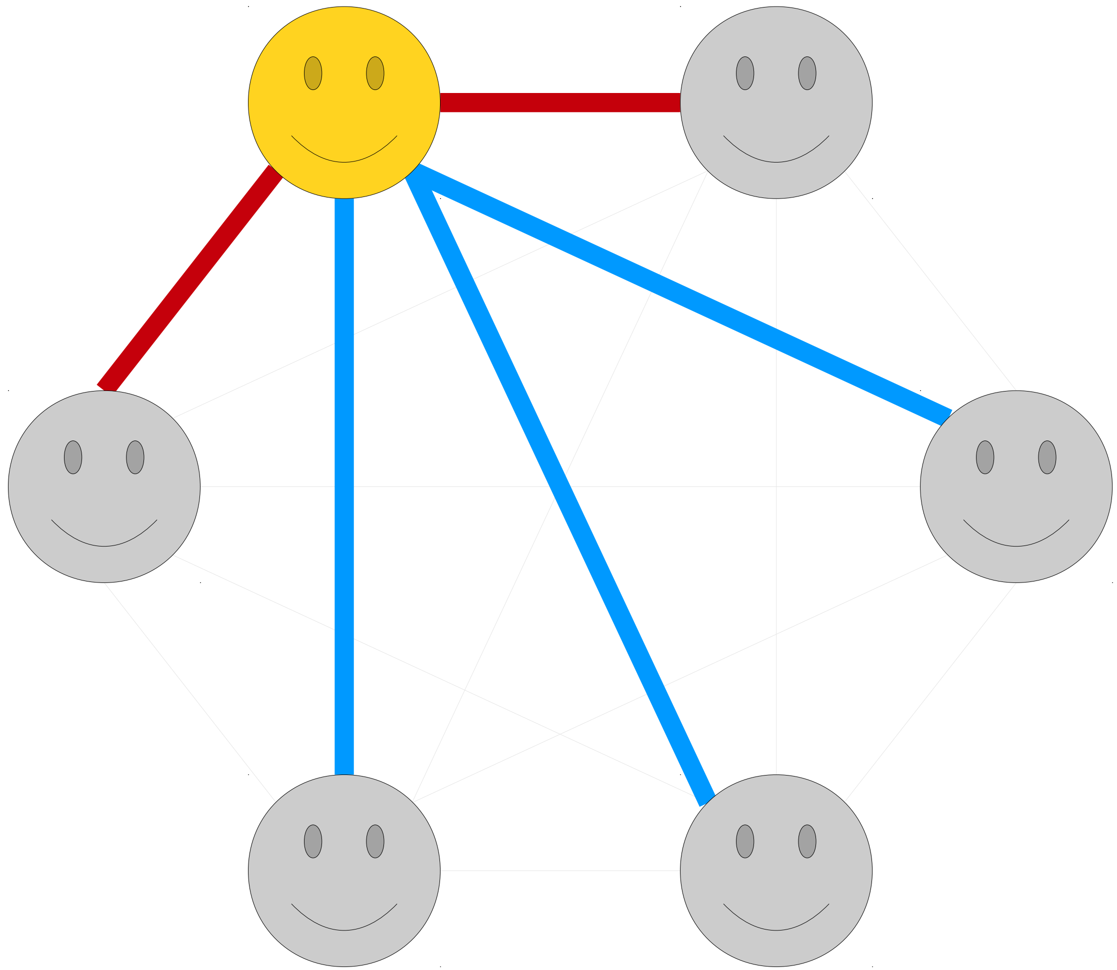


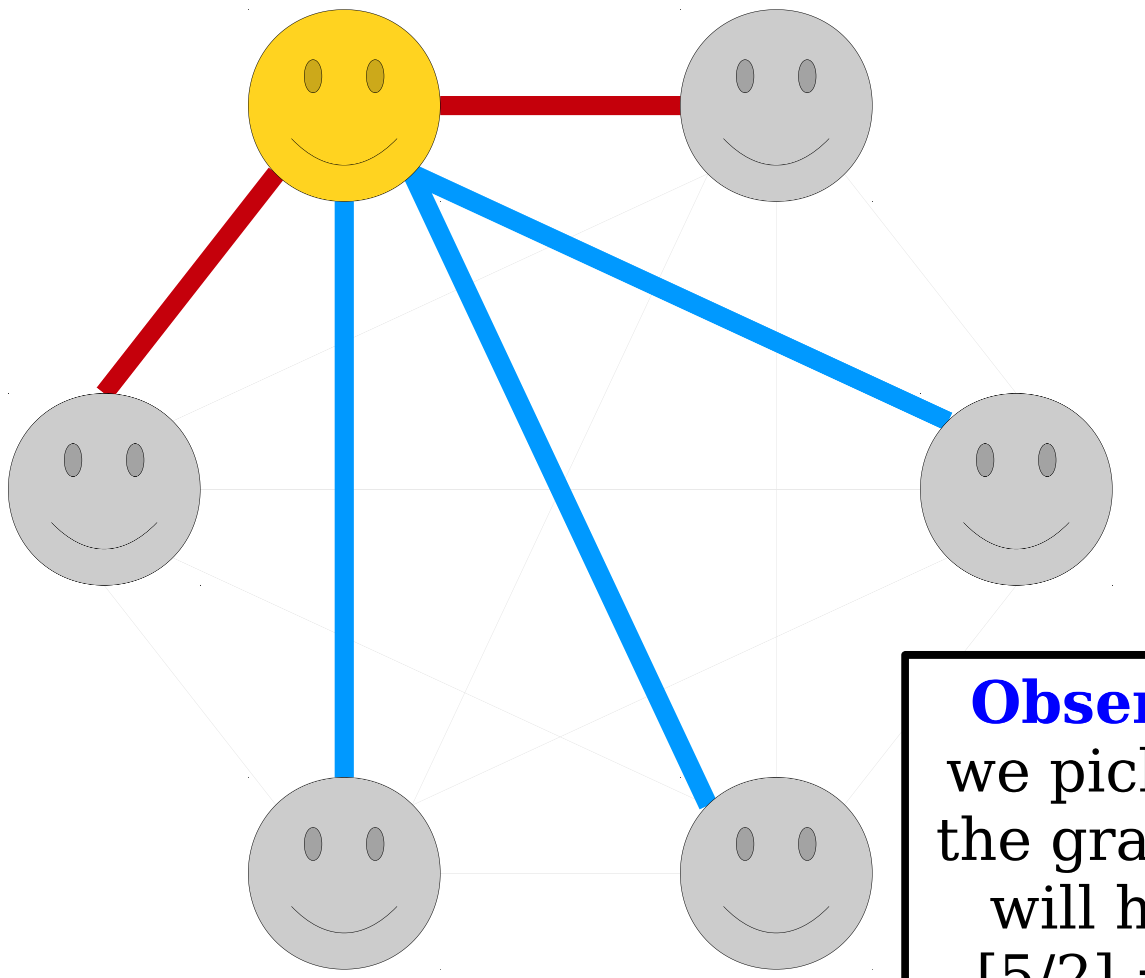




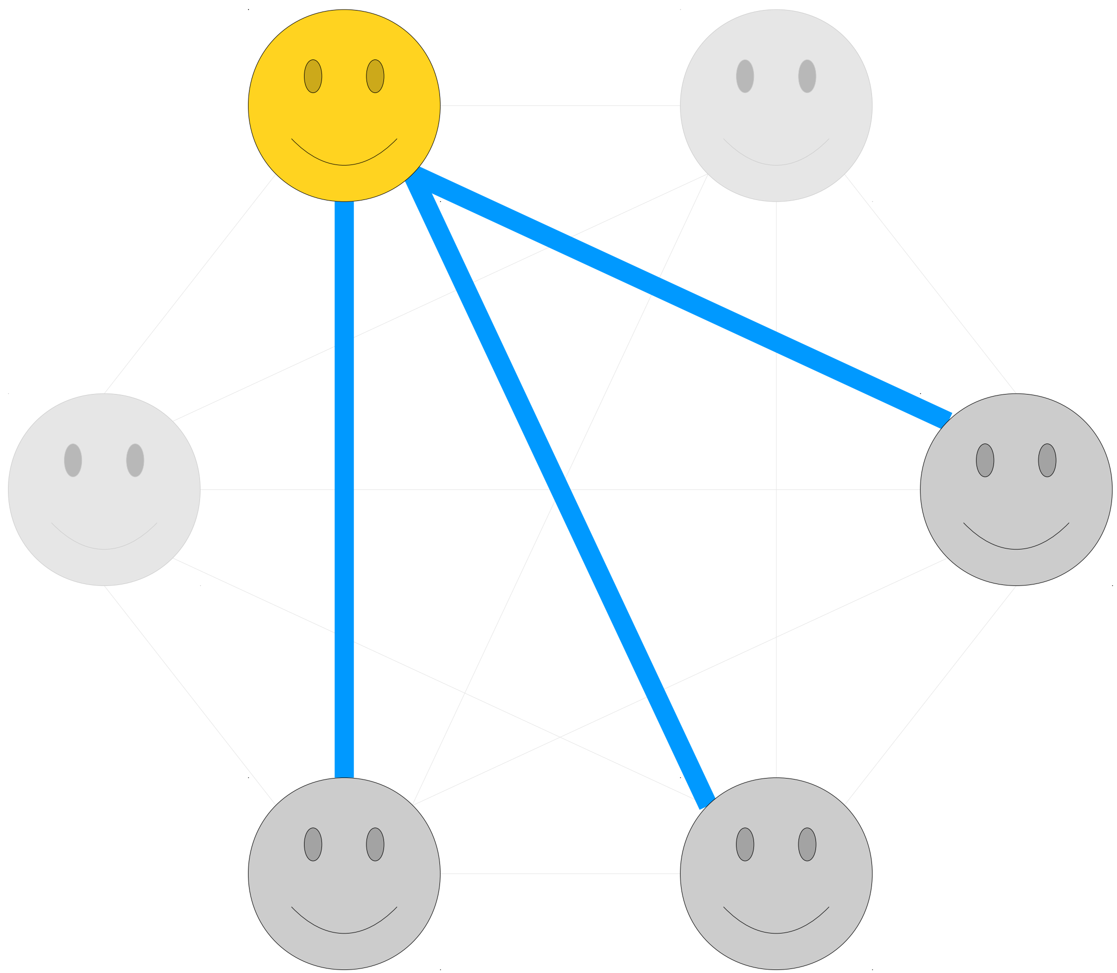


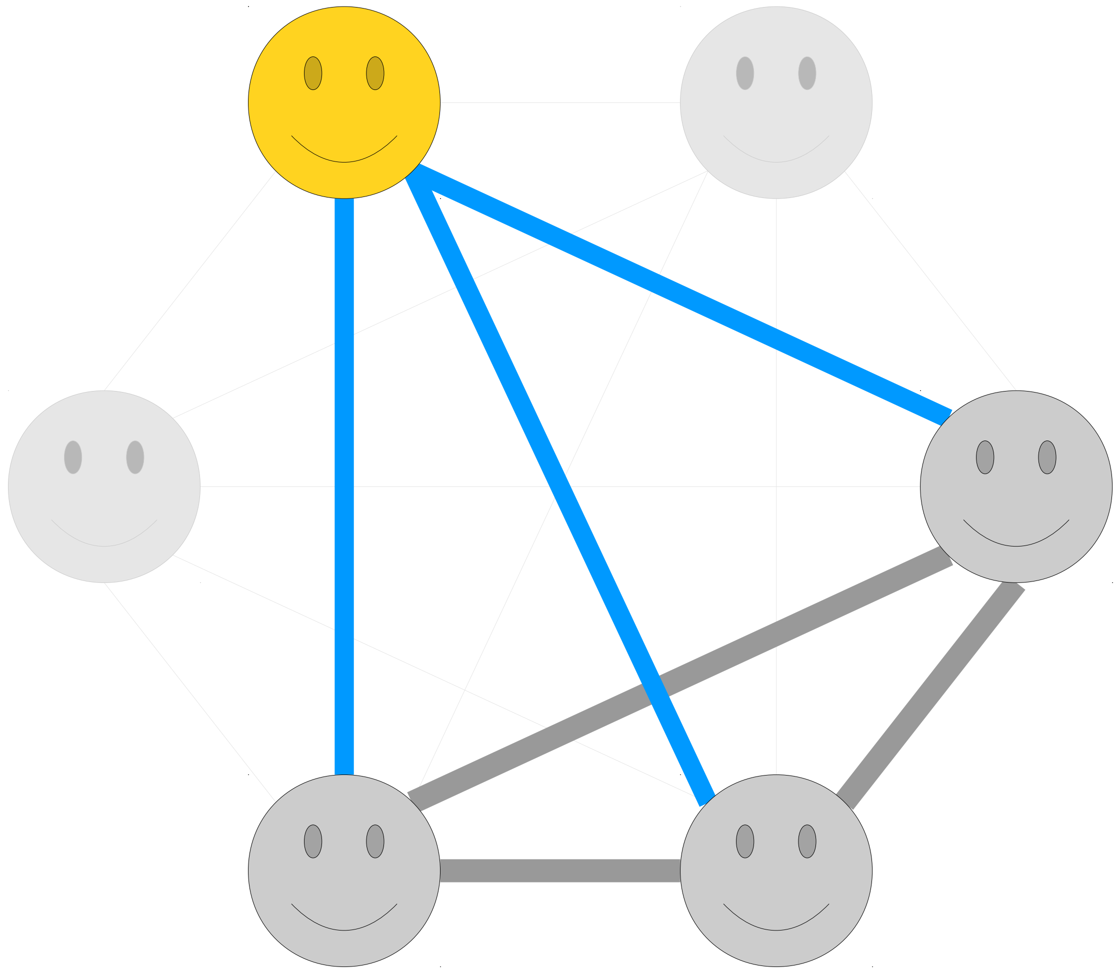


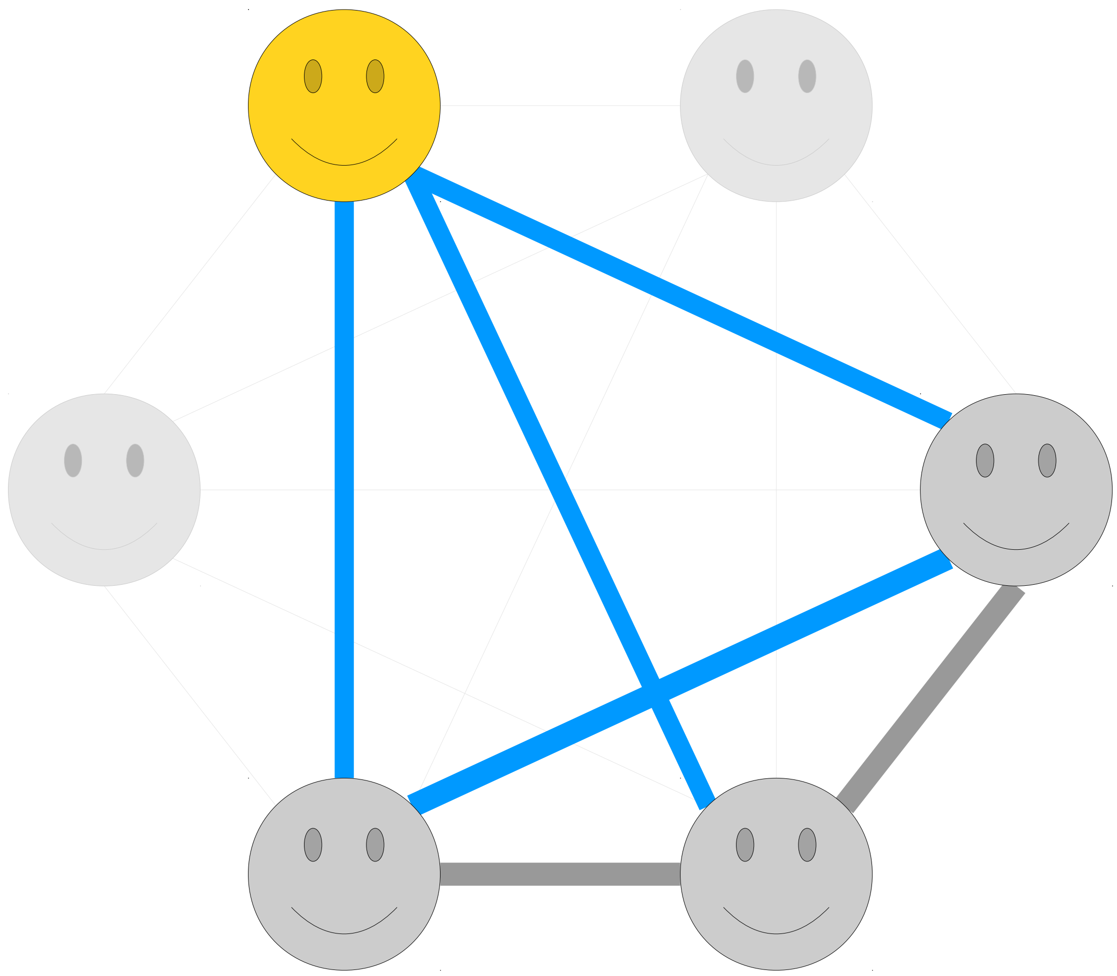


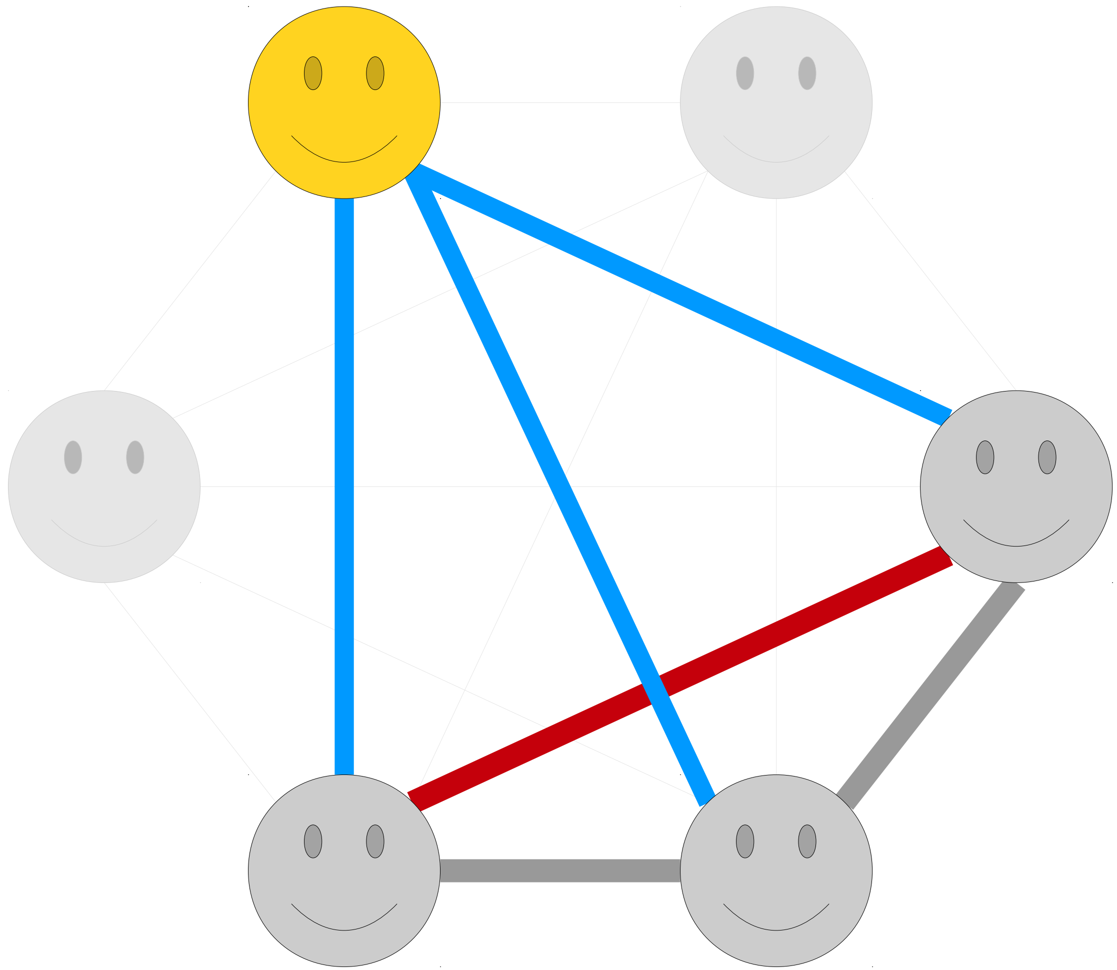


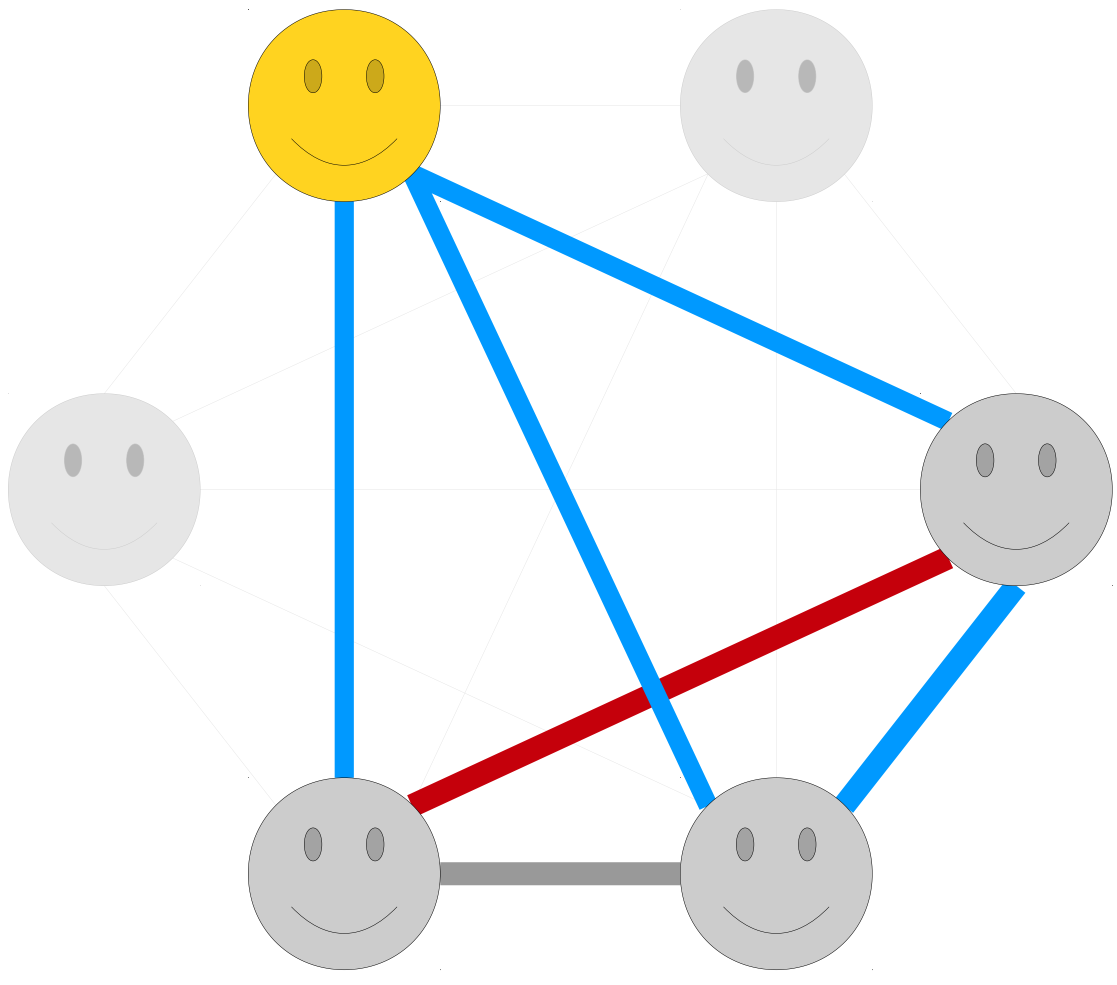
Observation 1: If we pick any node in the graph, that node will have at least ⌈5/2⌉ = 3 edges of the same color incident to it.

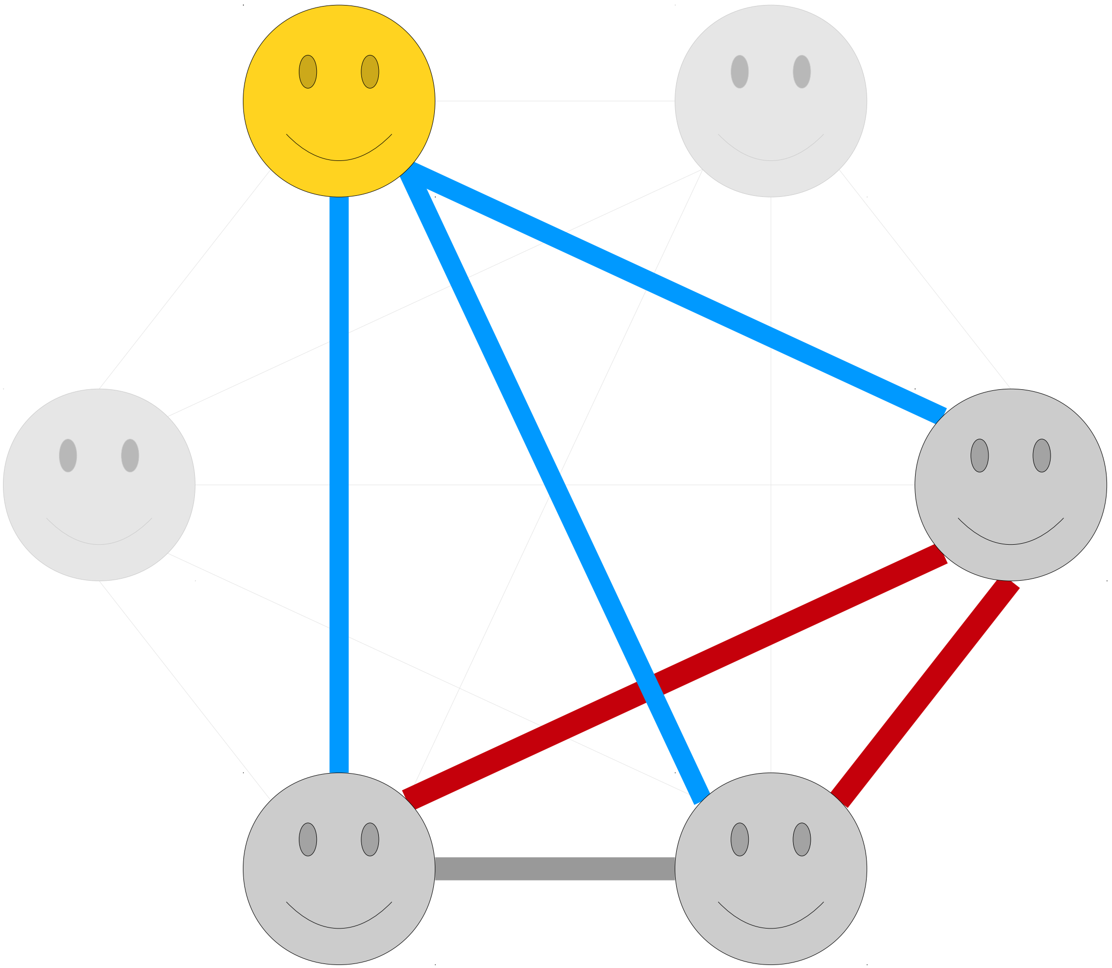


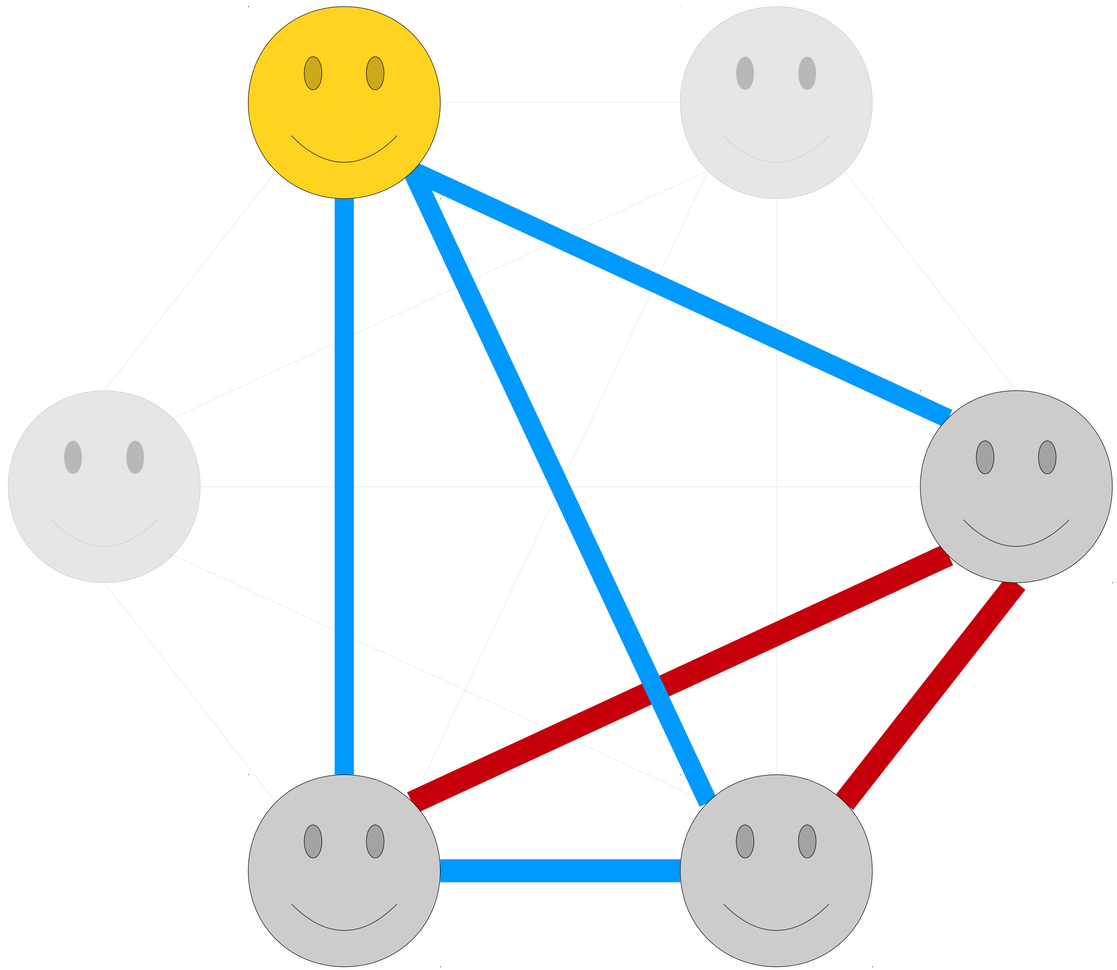


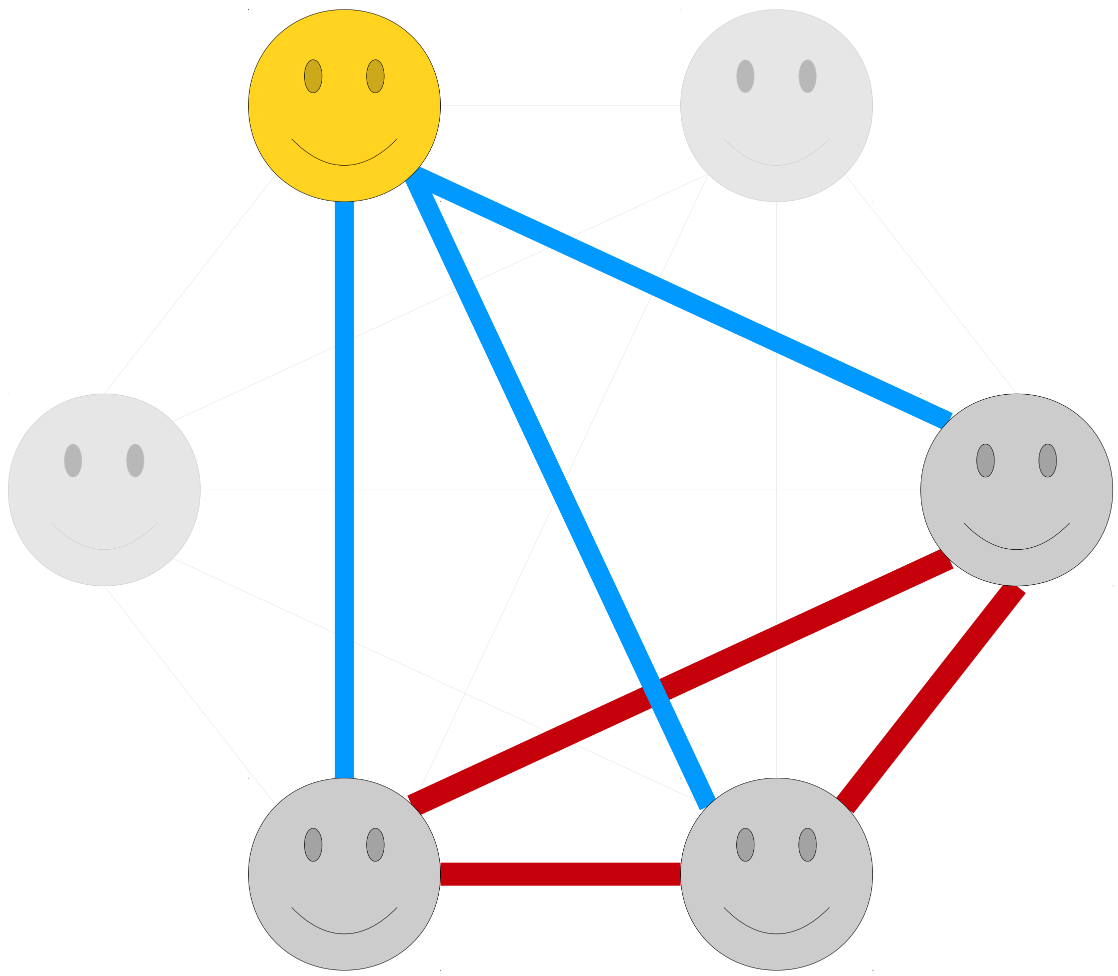












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Consider any node x in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5 / 2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2.

Let r, s, and t be three of the nodes connected to node x by an edge of color c_1. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1, then one of those edges plus the two edges connecting back to node x form a triangle of color c_1. Otherwise, all three of those edges are of color c_2, and they form a triangle of color c_2. Overall, this gives a red triangle or a blue triangle, as required. ■

What This Means

- The proof we just did was along the following lines:

  If you choose a sufficiently large object, you are guaranteed to find a large subobject of type A or a large subobject of type B.

- Intuitively, it's not possible to find gigantic objects that have absolutely no patterns or structure in them – there is no way to avoid having some interesting structure.
- There are numerous theorems of this sort. The mathematical field of ***Ramsey theory*** explicitly studies problems of this type.
- You'll see two examples of this on the upcoming problem set.

On Triangles

- Triangles in graphs have all sorts of crazy properties.
- From an algorithmic perspective, many problems on graphs are, in a sense, no harder than finding triangles in graphs.
- Curious? Take CS267!

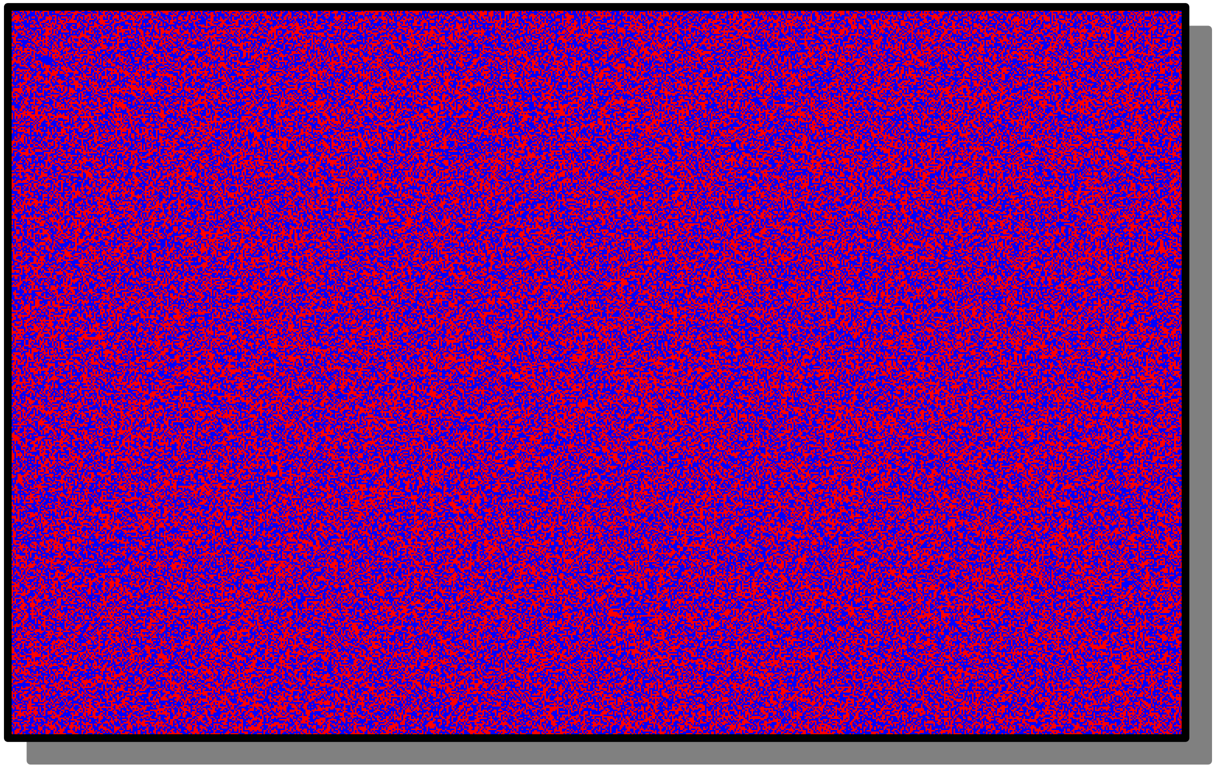
Coda: ***The Hadwiger-Nelson Problem***

Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

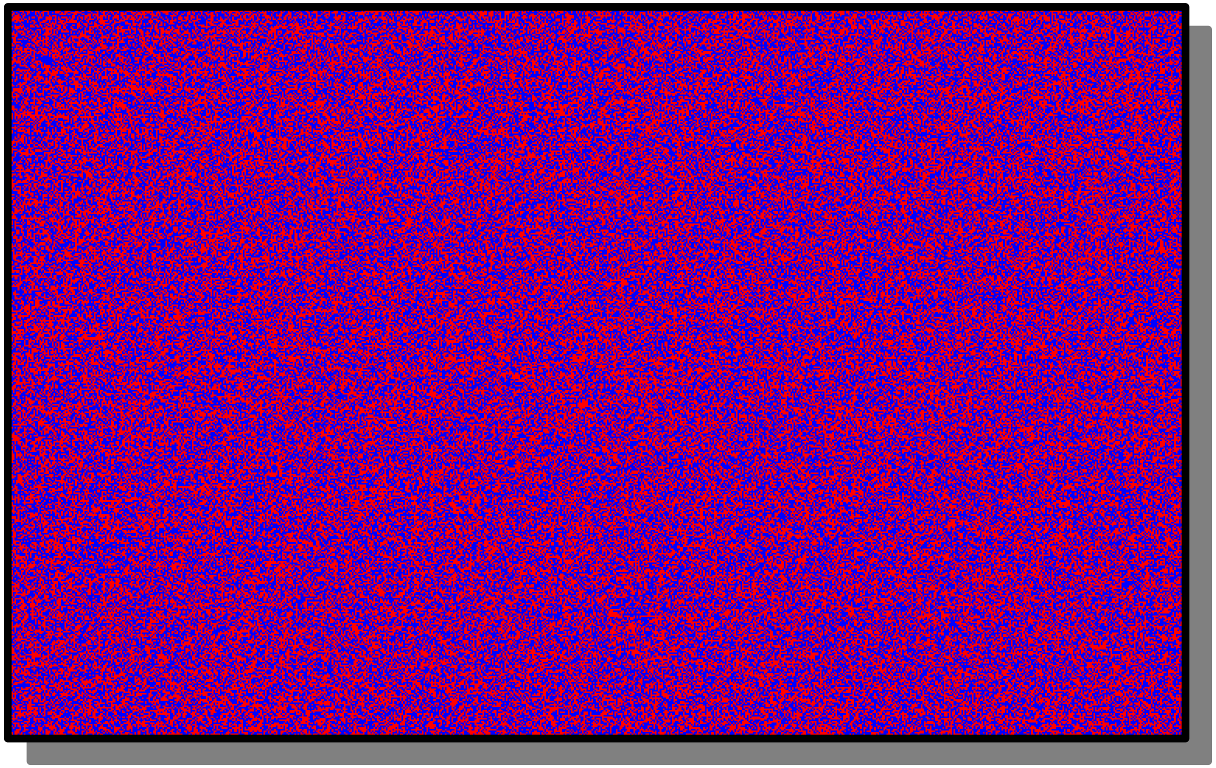
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



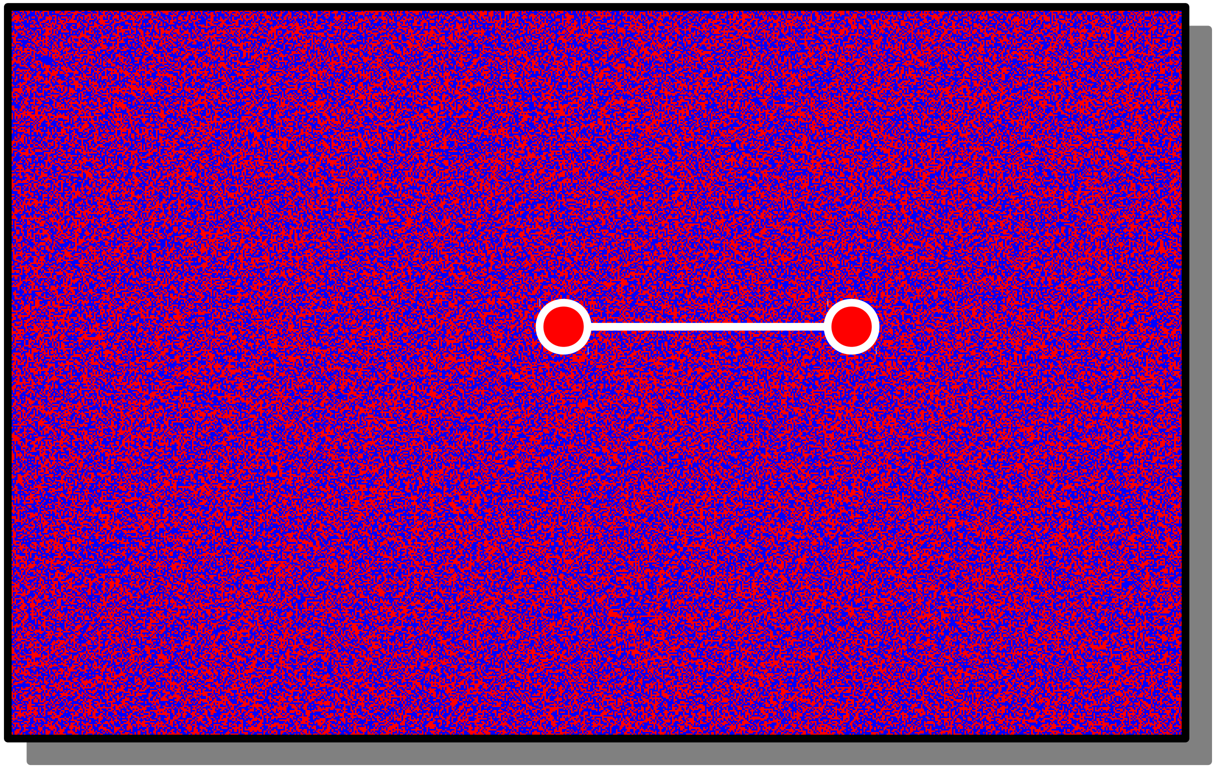
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



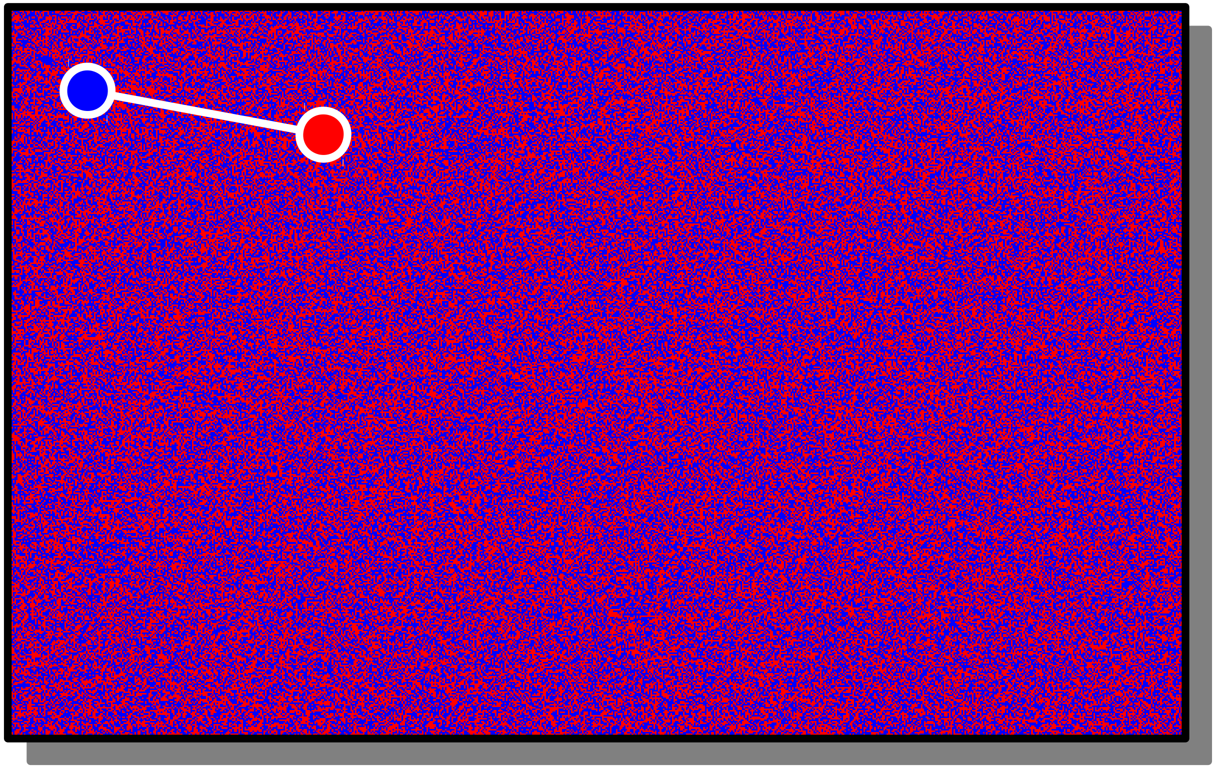
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



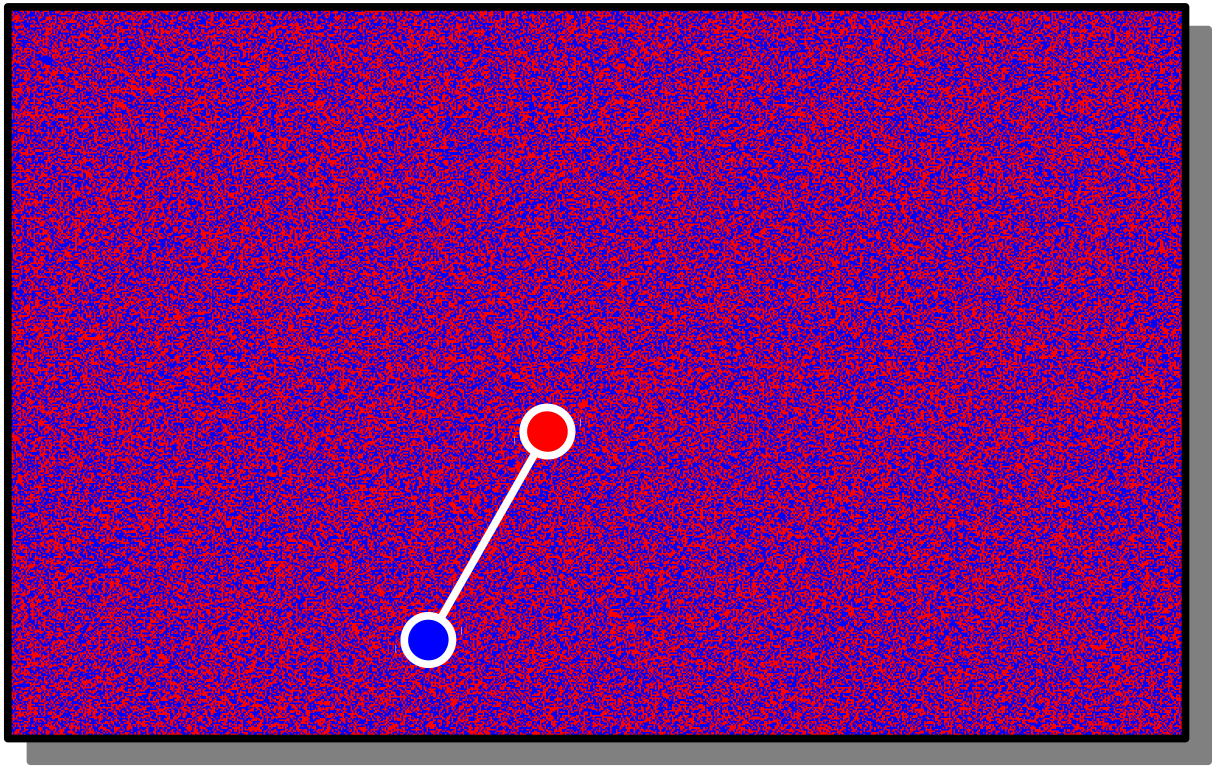
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



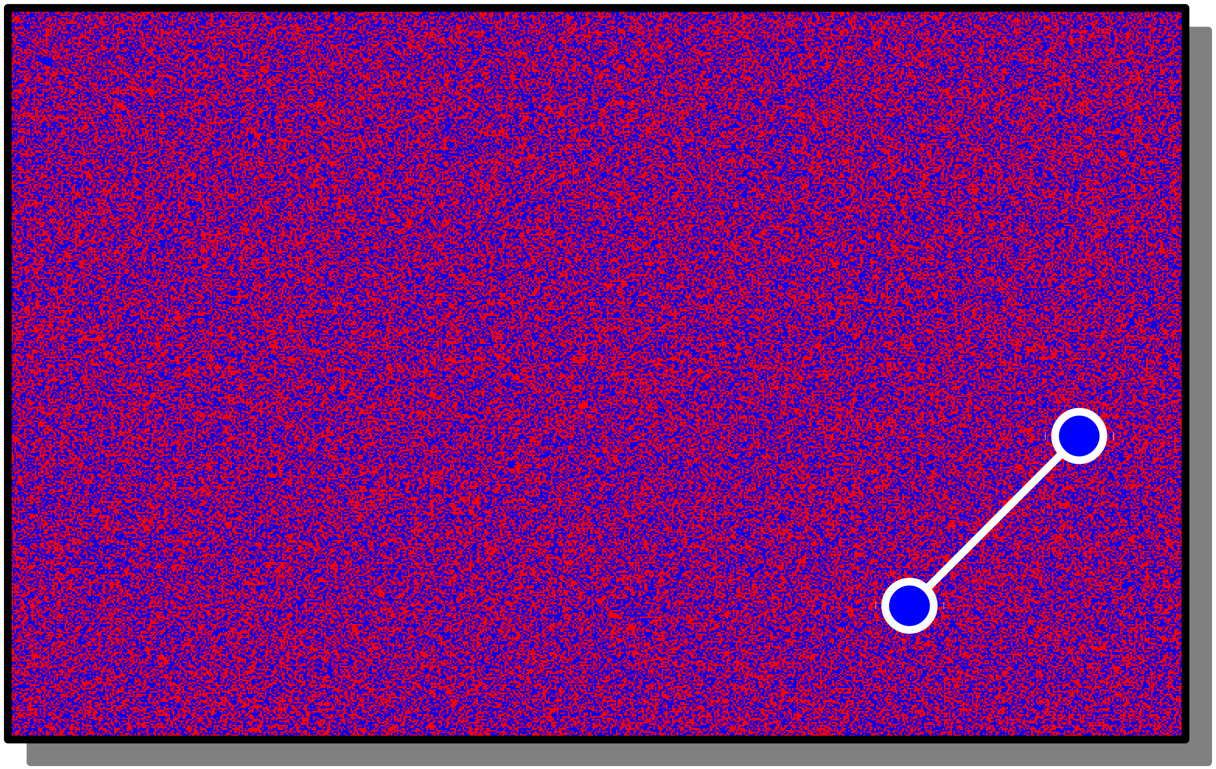
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

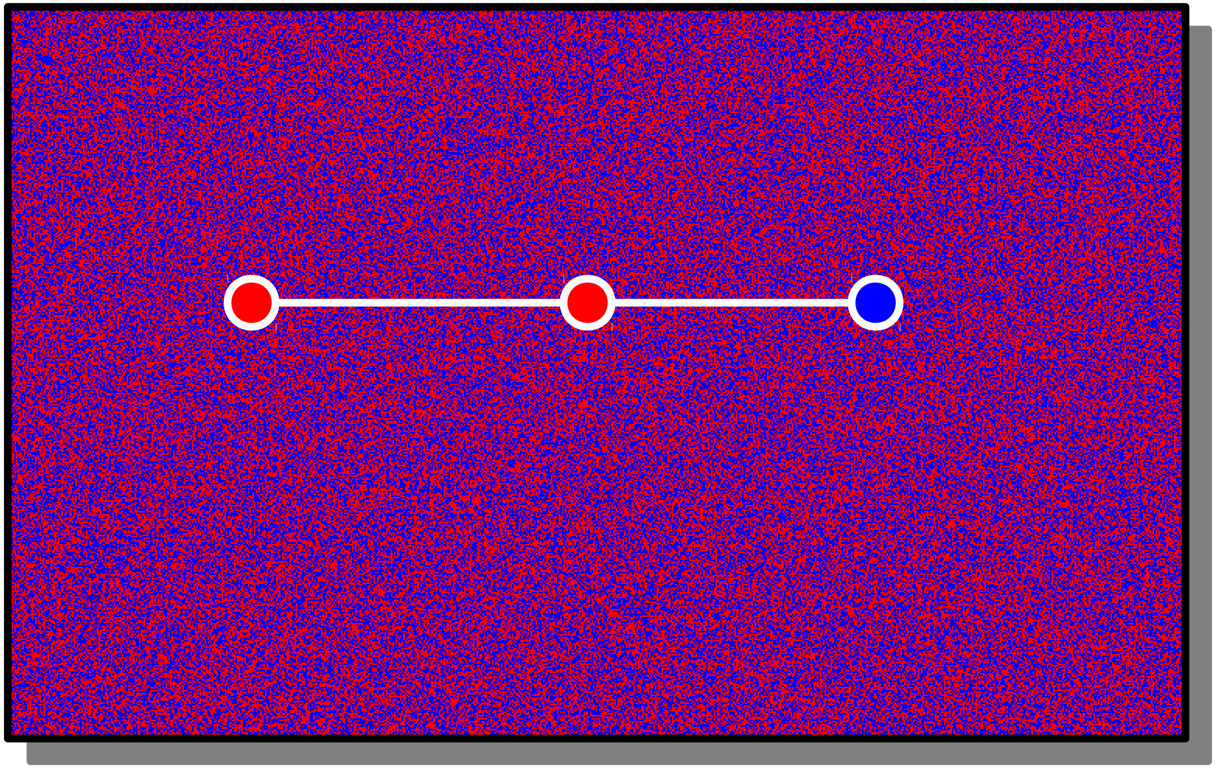
> Suppose we pick *k* points in the plane. What is the minimum choice of *k* we can pick before we're guaranteed to get at least two points of the same color?

Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

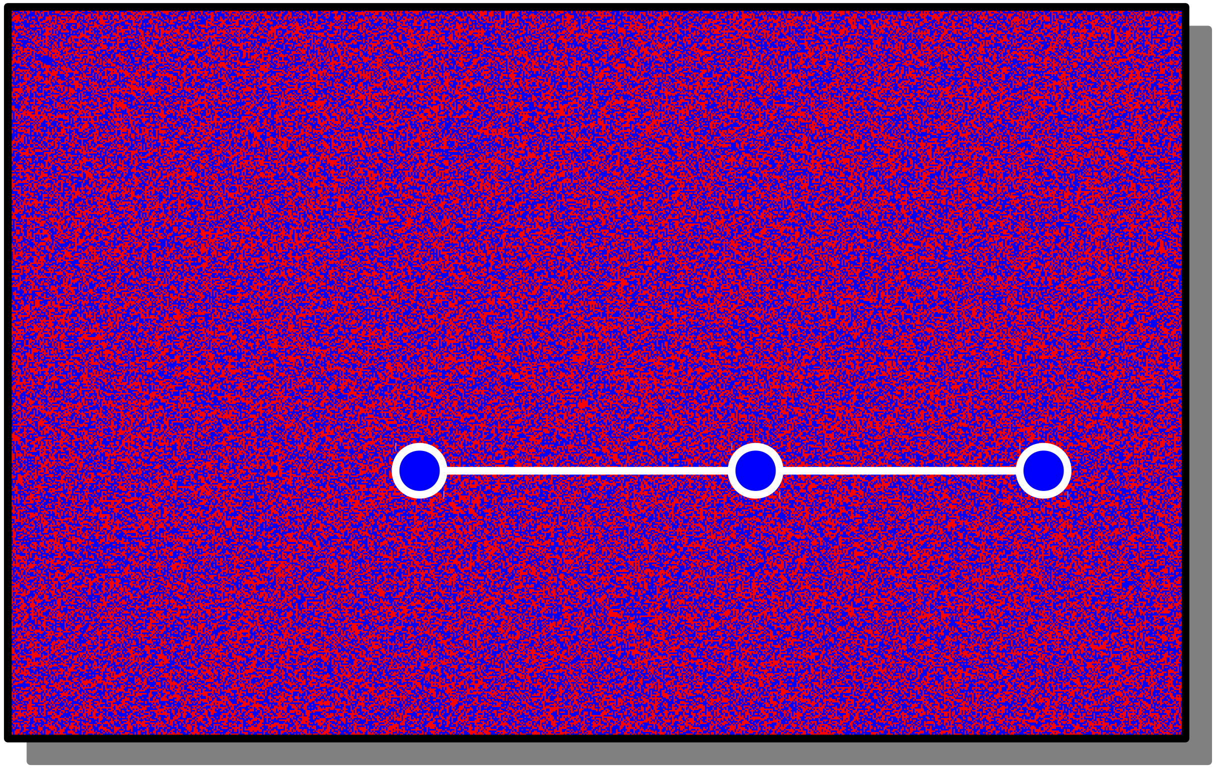
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



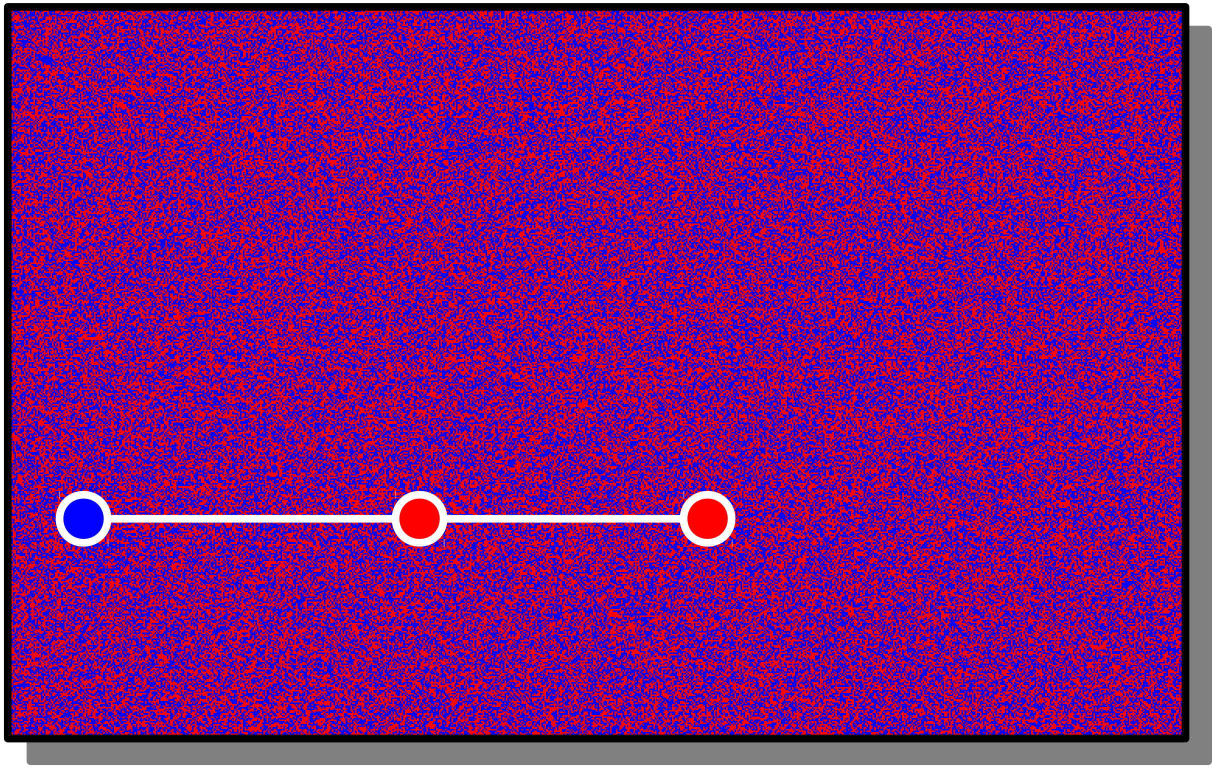
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



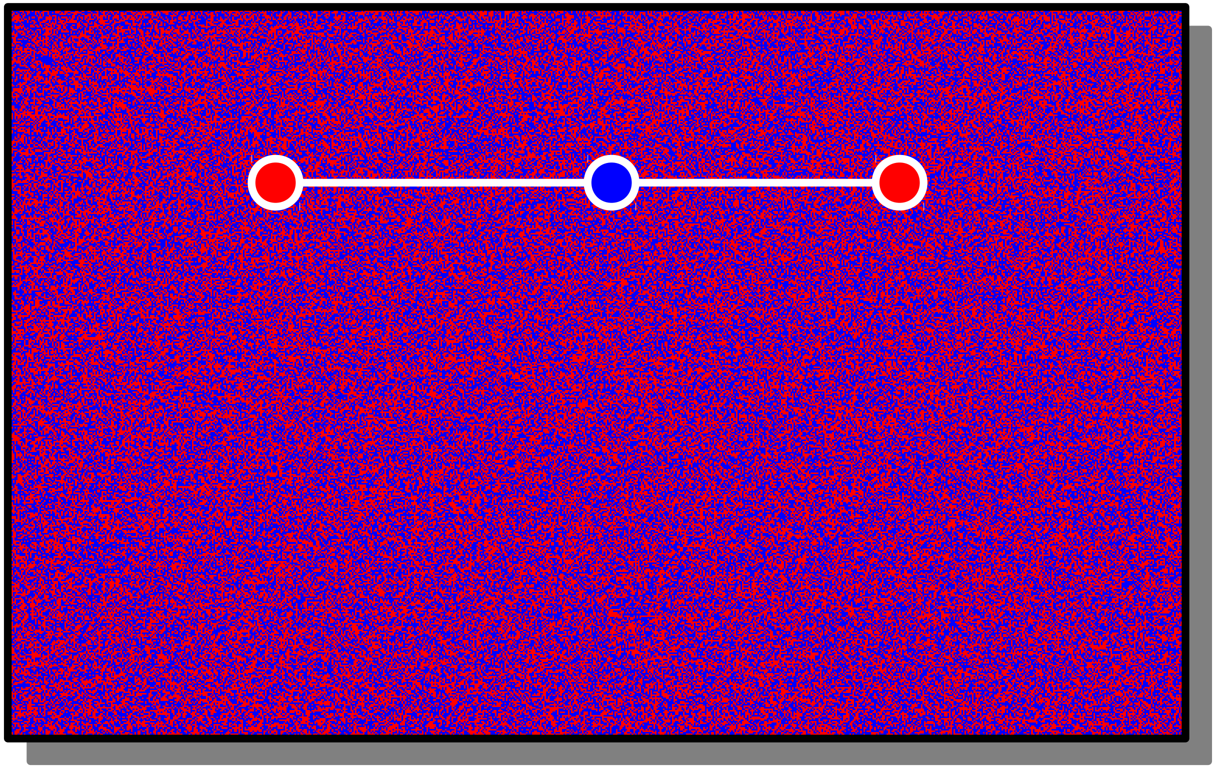
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



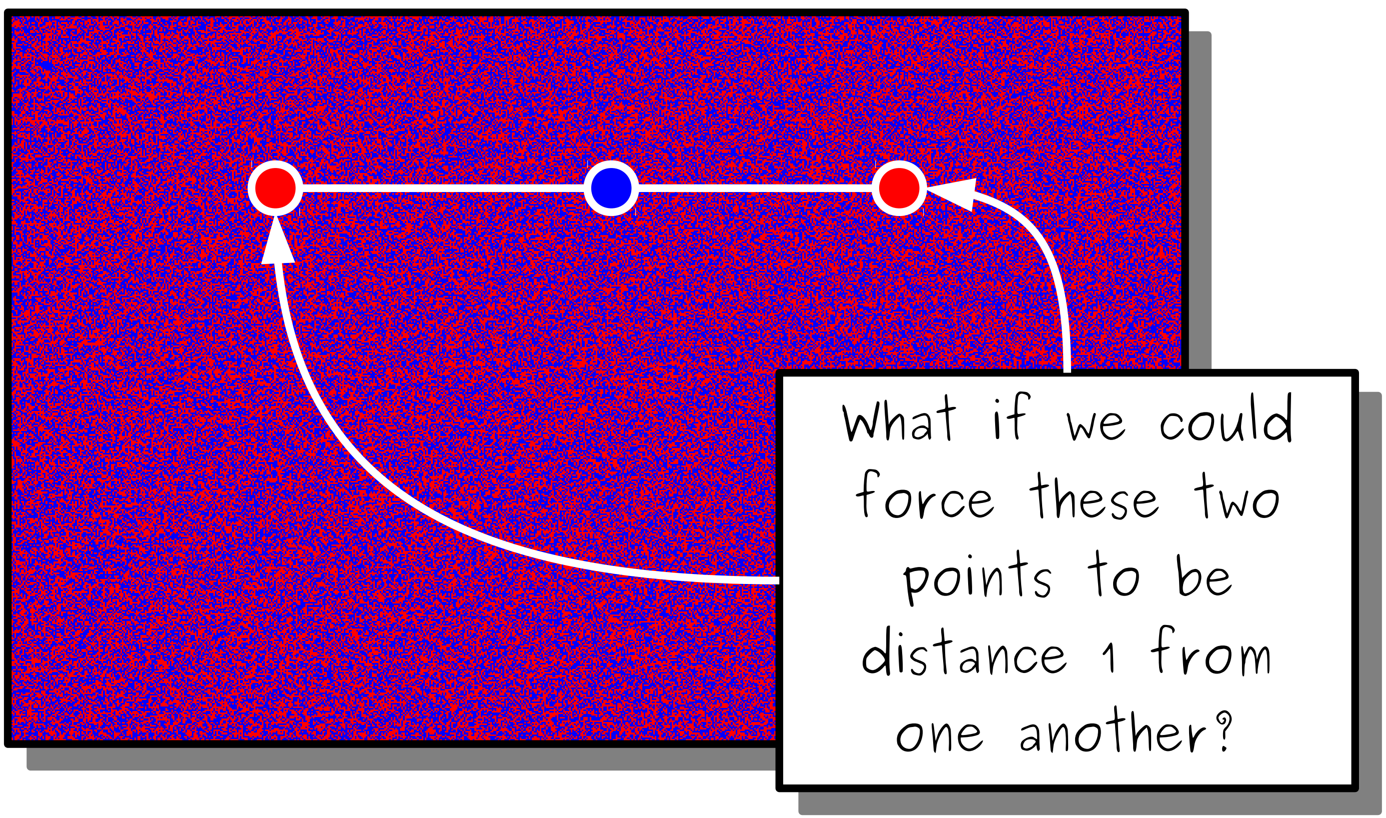
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



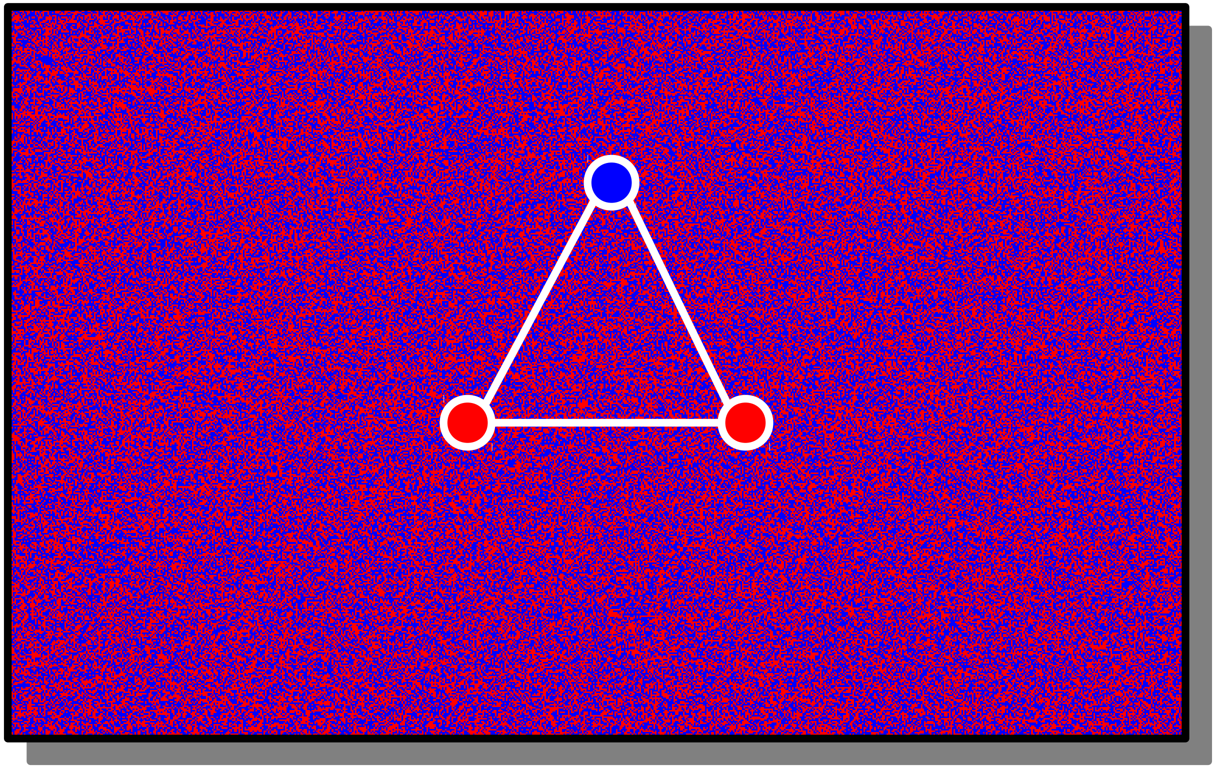
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



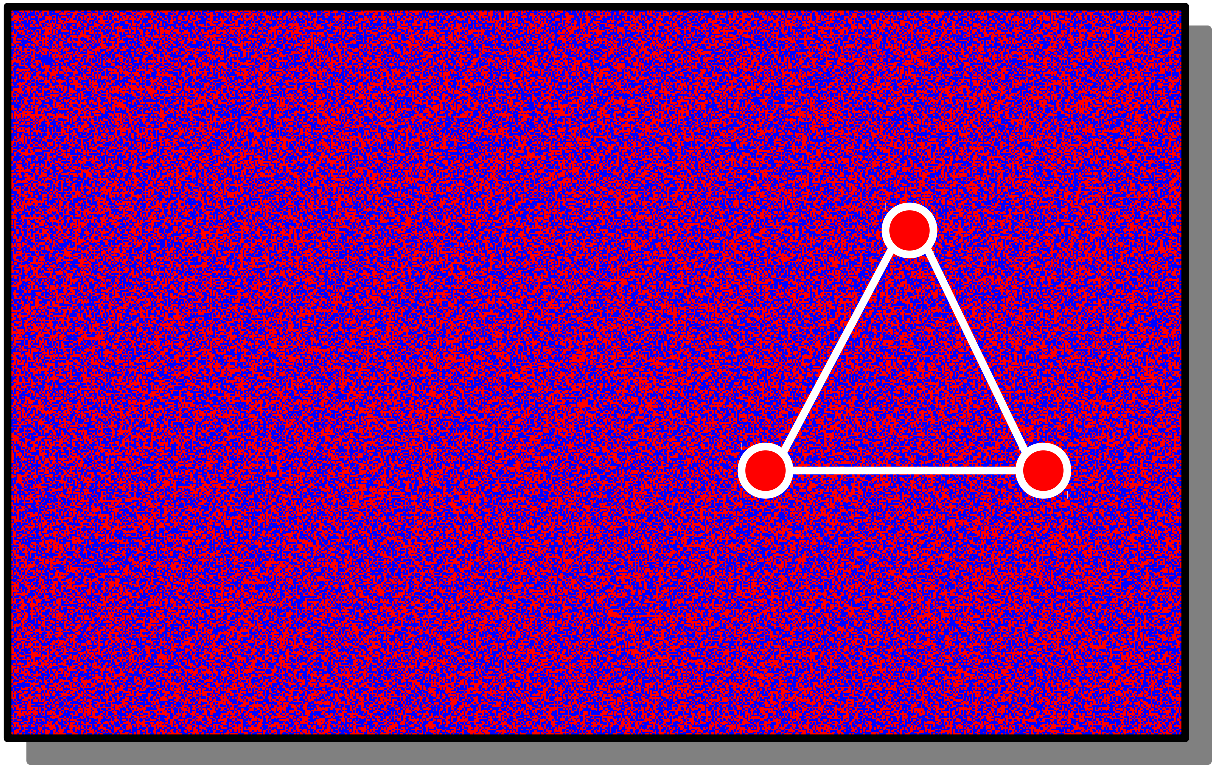
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



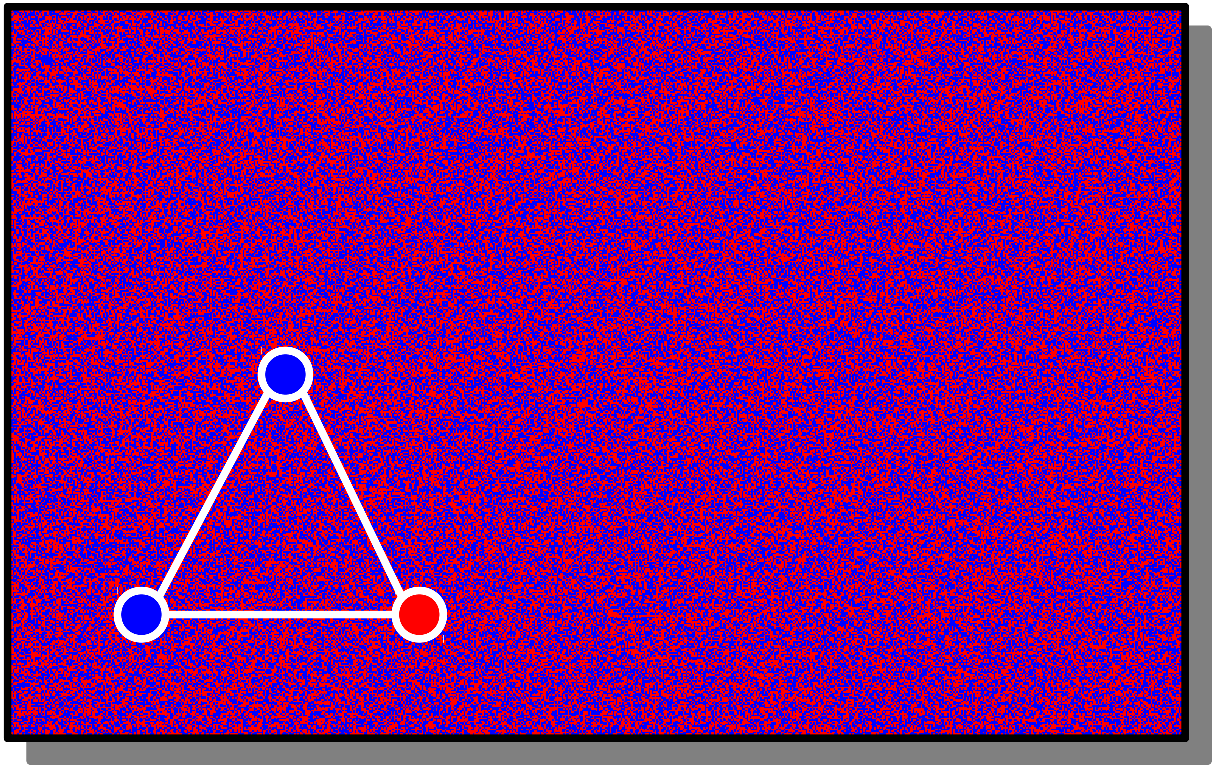
Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.



Coloring the Plane

Theorem: Suppose that every point in the Cartesian plane is colored either red or blue. Regardless of how those points are colored, there will be a pair of points at distance 1 from each other that are the same color.

Proof: Consider any equilateral triangle whose sides are length 1. Put this triangle anywhere in the plane. Because the triangle has three vertices and each point in the plane is only one of two different colors, by the pigeonhole principle at least two of the vertices must have the same color. These vertices are at distance 1 from each other, as required. ■

The Hadwiger-Nelson Problem

Infinite Graphs

- Consider the Cartesian plane, $\mathbb{R}^2$.
  - Each point in the plane has a coordinate (x, y), where x and y are real numbers. The set of all such points is $\mathbb{R}^2$.
- Imagine we make the following graph:
  - The nodes are the points in the plane.
  - There's an edge between any two points that are exactly one unit apart from each other.
- ***Question:*** What is the chromatic number of this graph?
- In other words, how many colors are necessary to color all the points in the plane so that no two points at distance 1 are the same color?
- This problem is called the ***Hadwiger-Nelson problem***.

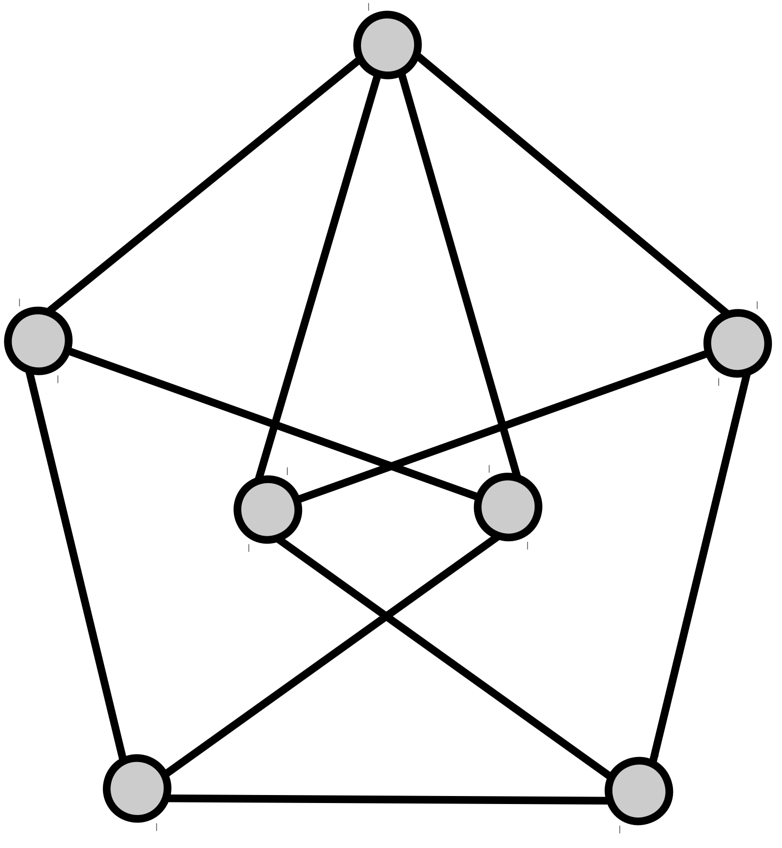
The Hadwiger-Nelson Problem

- ***Theorem:*** The answer to the Hadwiger-Nelson problem is not 2.
- ***Proof:*** We just proved that no matter how you color the points in the real plane with two colors, there will always be two points at are at distance 1. ■

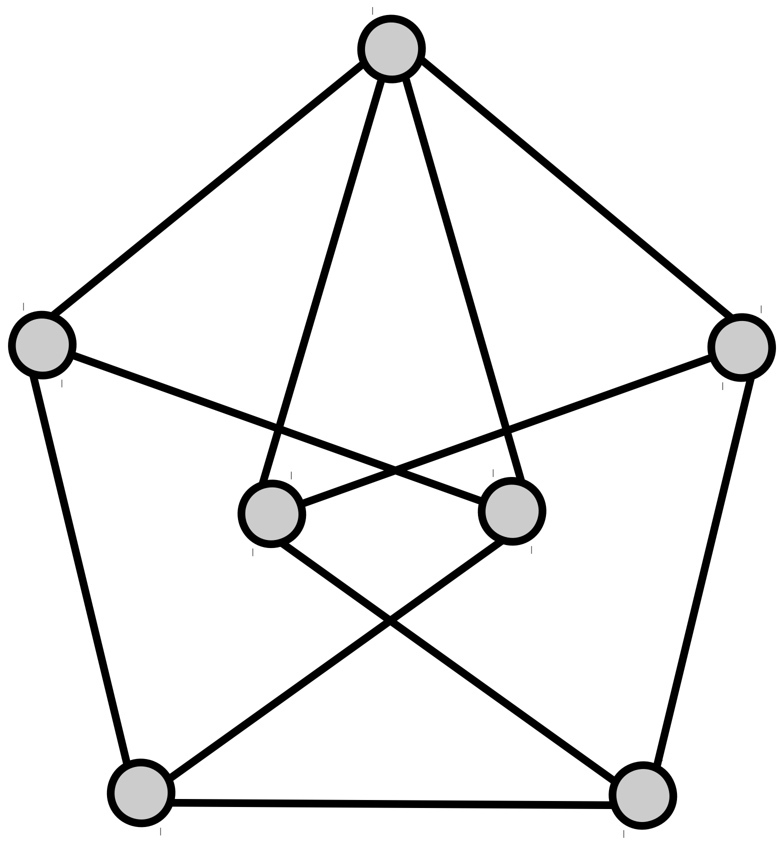
Theorem (Moser): The answer to the Hadwiger-Nelson problem is not three.

In other words, if you color all points in the Cartesian plane one of three different colors, there will always be a pair of points that are the same color and one unit apart.

The Moser Spindle

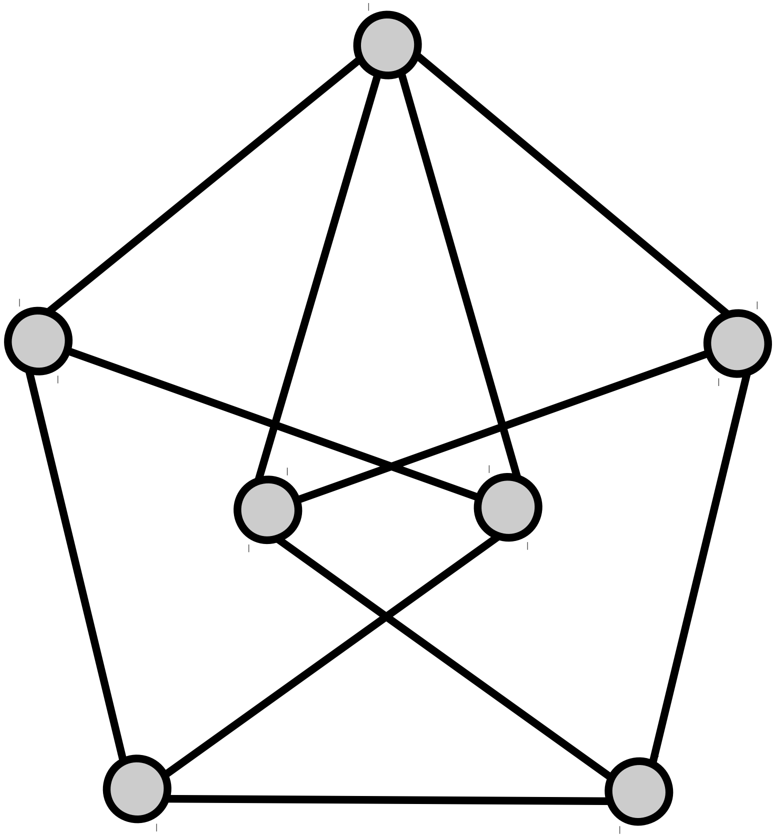


The Moser Spindle



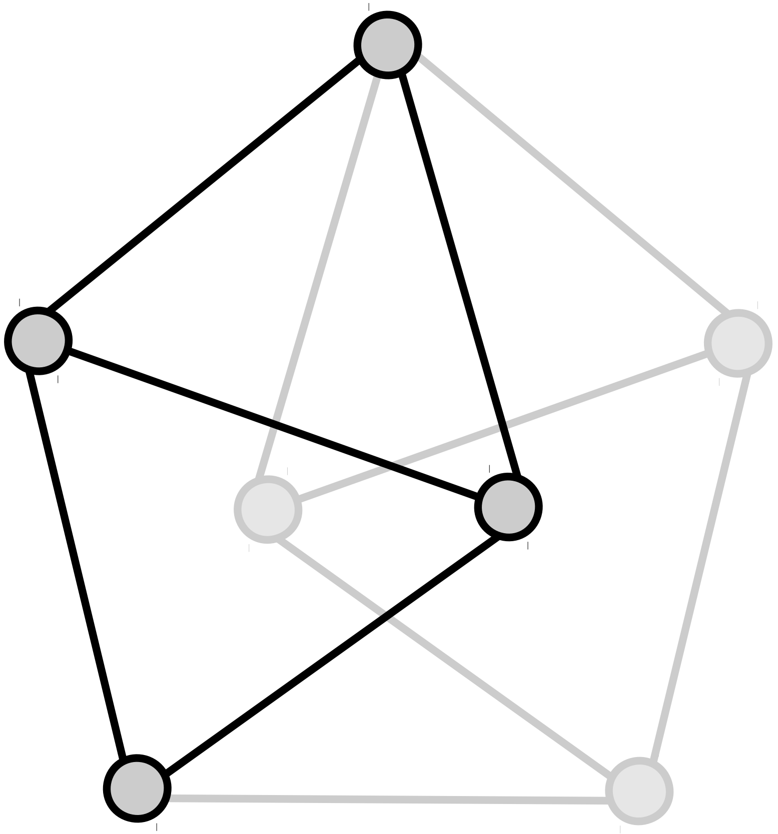
This graph can be drawn so that each of these edges has length exactly 1.

The Moser Spindle



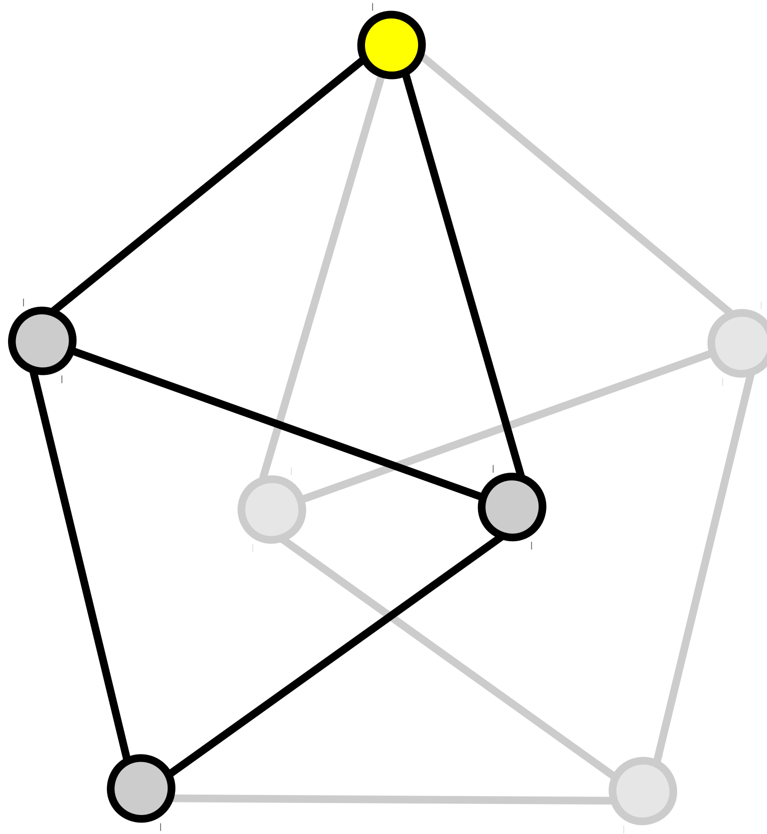
Claim: This graph is not 3-colorable.

The Moser Spindle



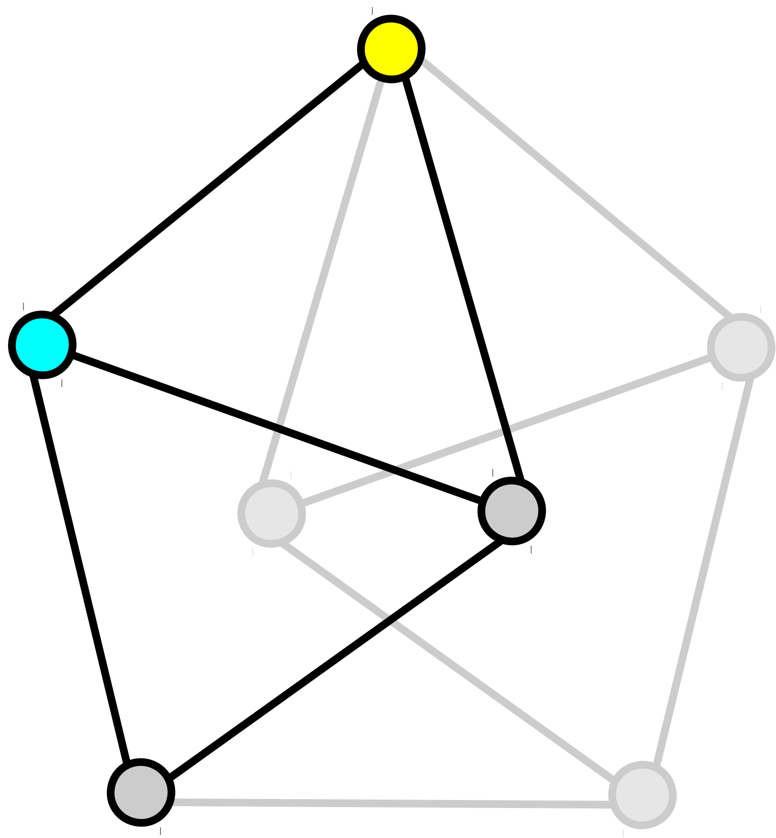
Claim: This graph is not 3-colorable.

The Moser Spindle



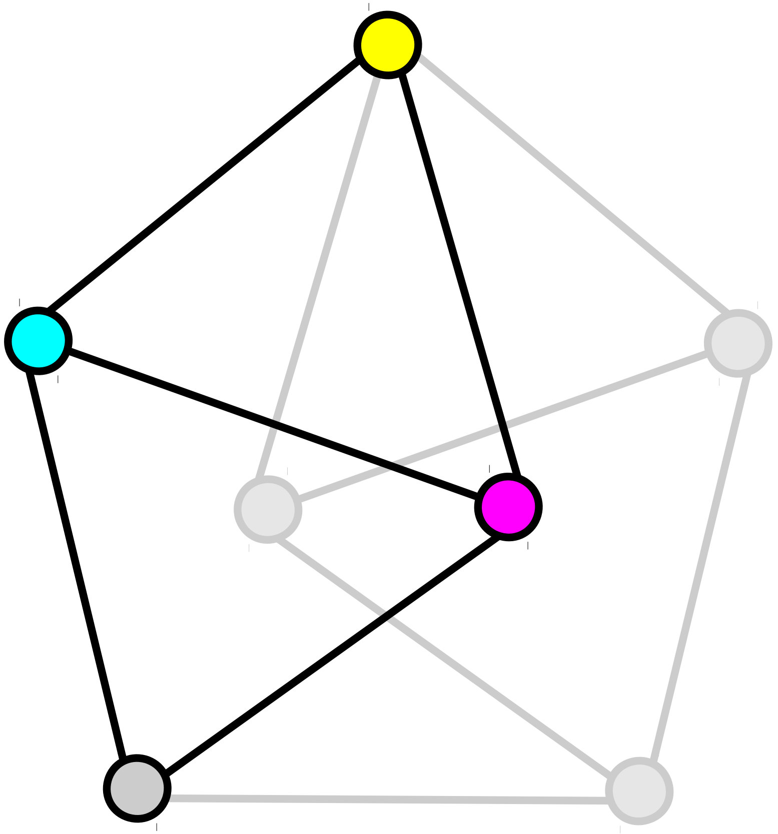
Claim: This graph is not 3-colorable.

The Moser Spindle



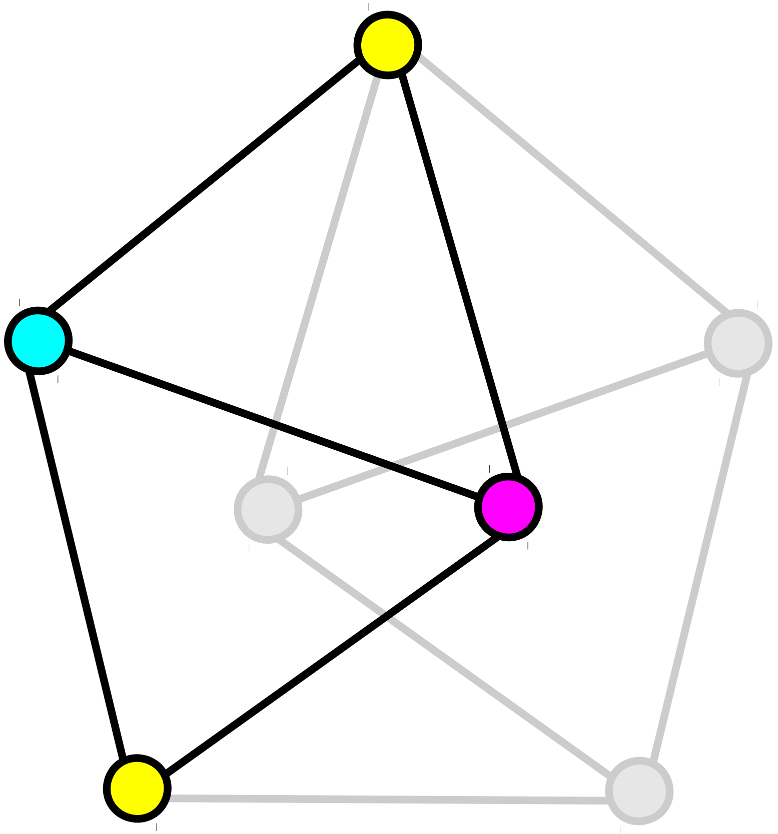
Claim: This graph is not 3-colorable.

The Moser Spindle



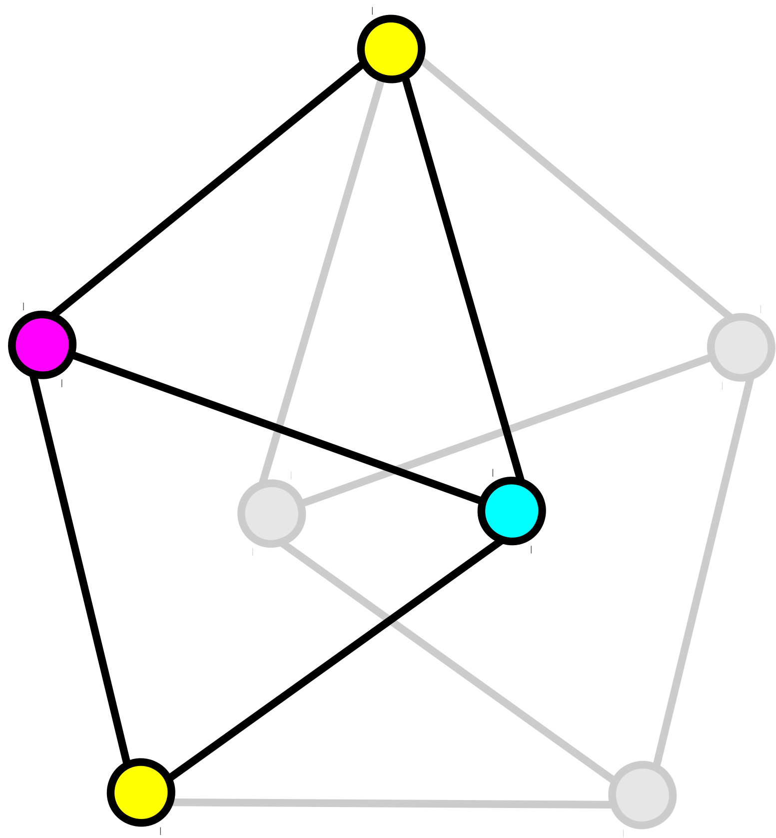
Claim: This graph is not 3-colorable.

The Moser Spindle



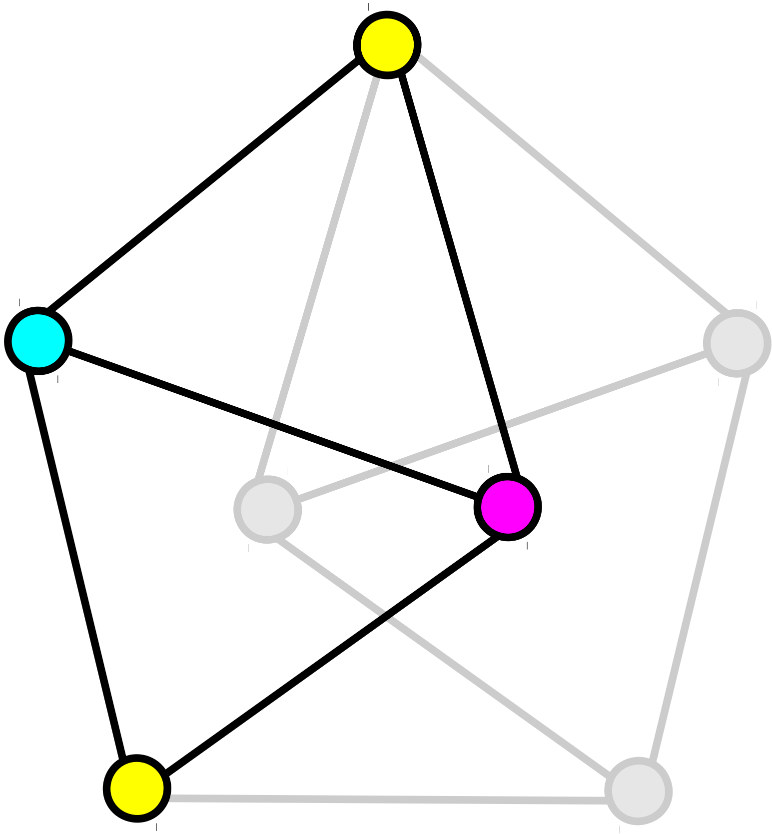
Claim: This graph is not 3-colorable.

The Moser Spindle



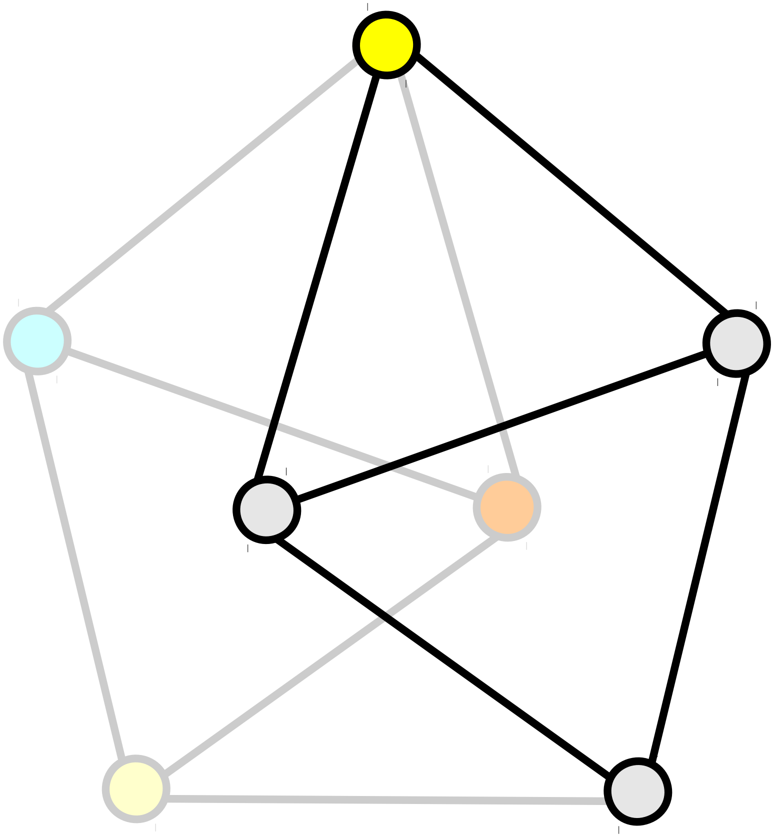
Claim: This graph is not 3-colorable.

The Moser Spindle



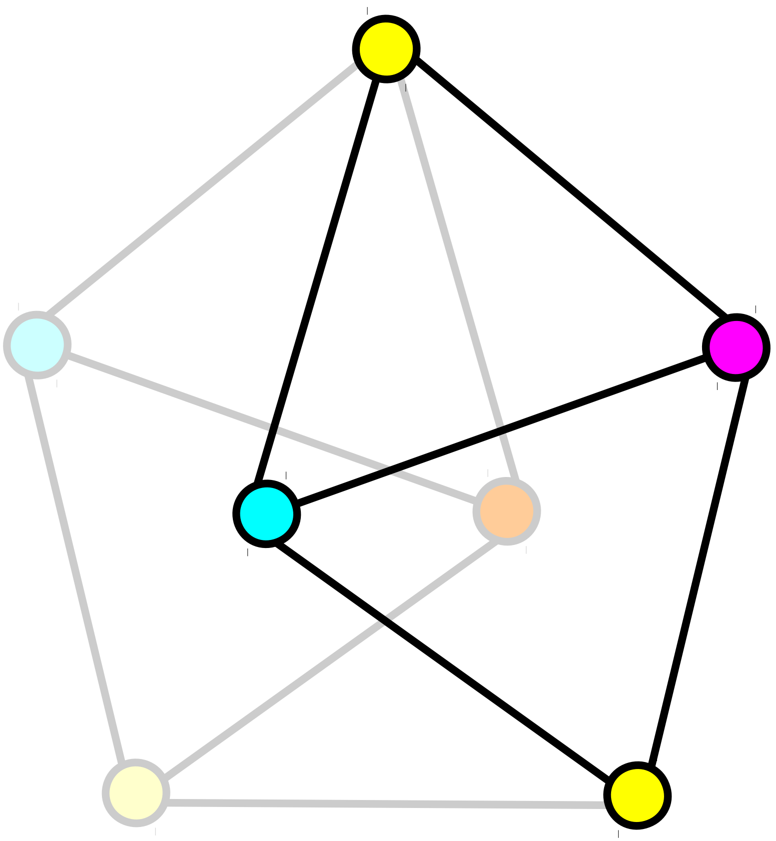
Claim: This graph is not 3-colorable.

The Moser Spindle



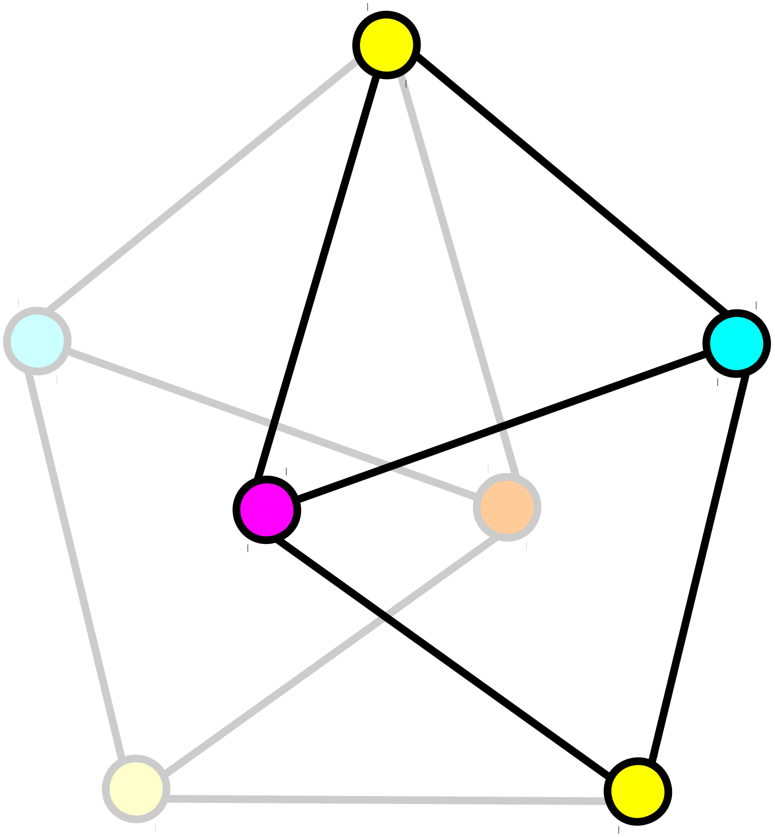
Claim: This graph is not 3-colorable.

The Moser Spindle



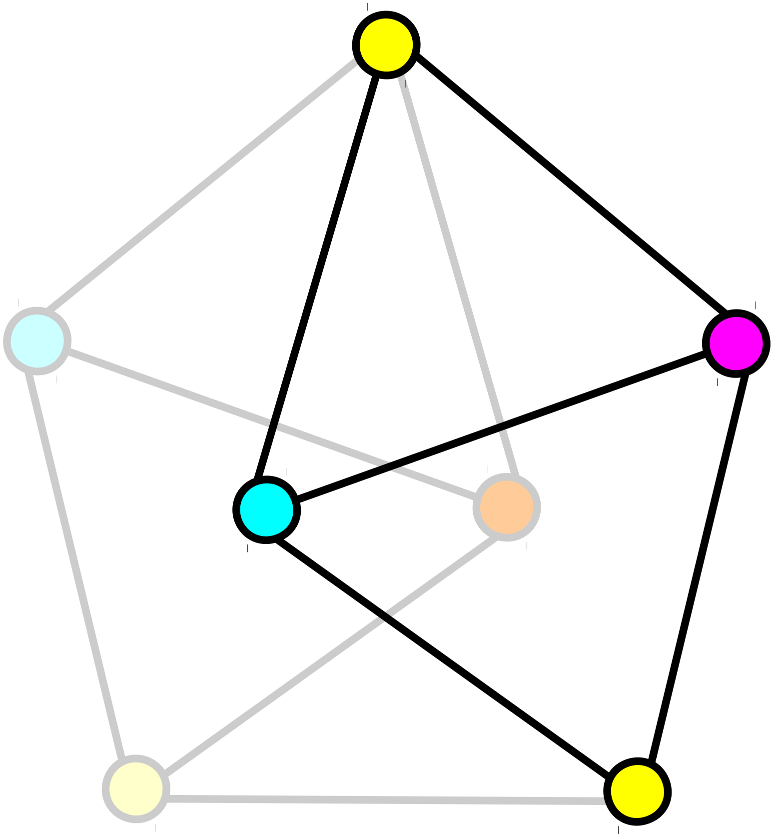
Claim: This graph is not 3-colorable.

The Moser Spindle



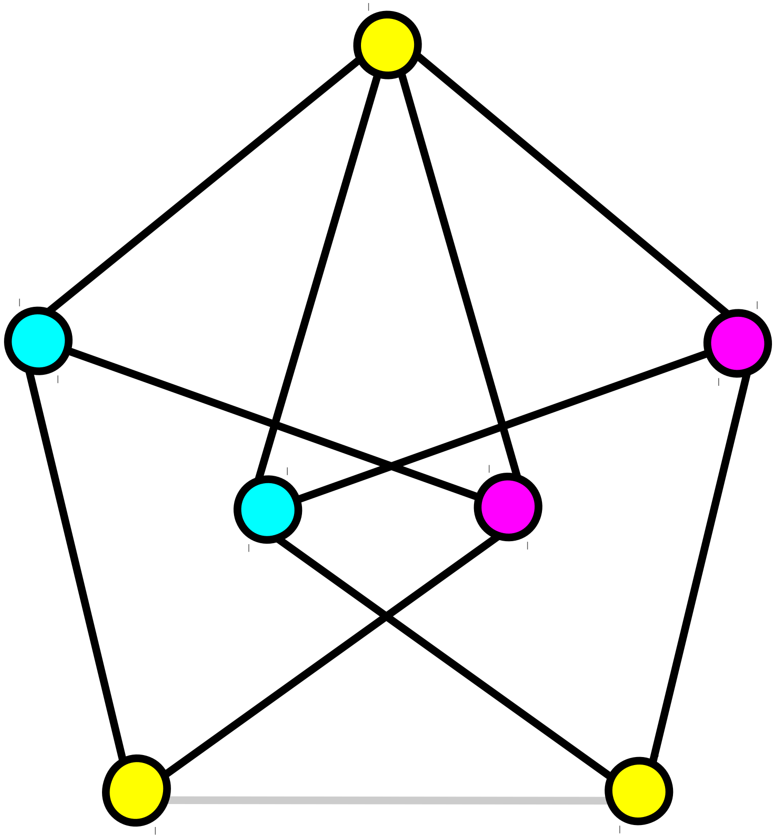
Claim: This graph is not 3-colorable.

The Moser Spindle



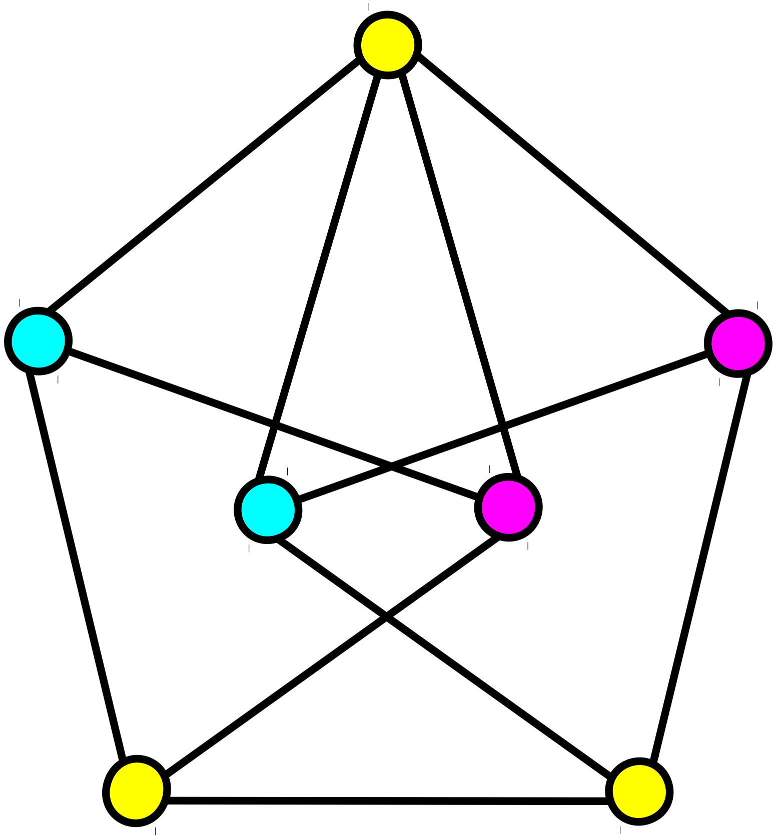
Claim: This graph is not 3-colorable.

The Moser Spindle



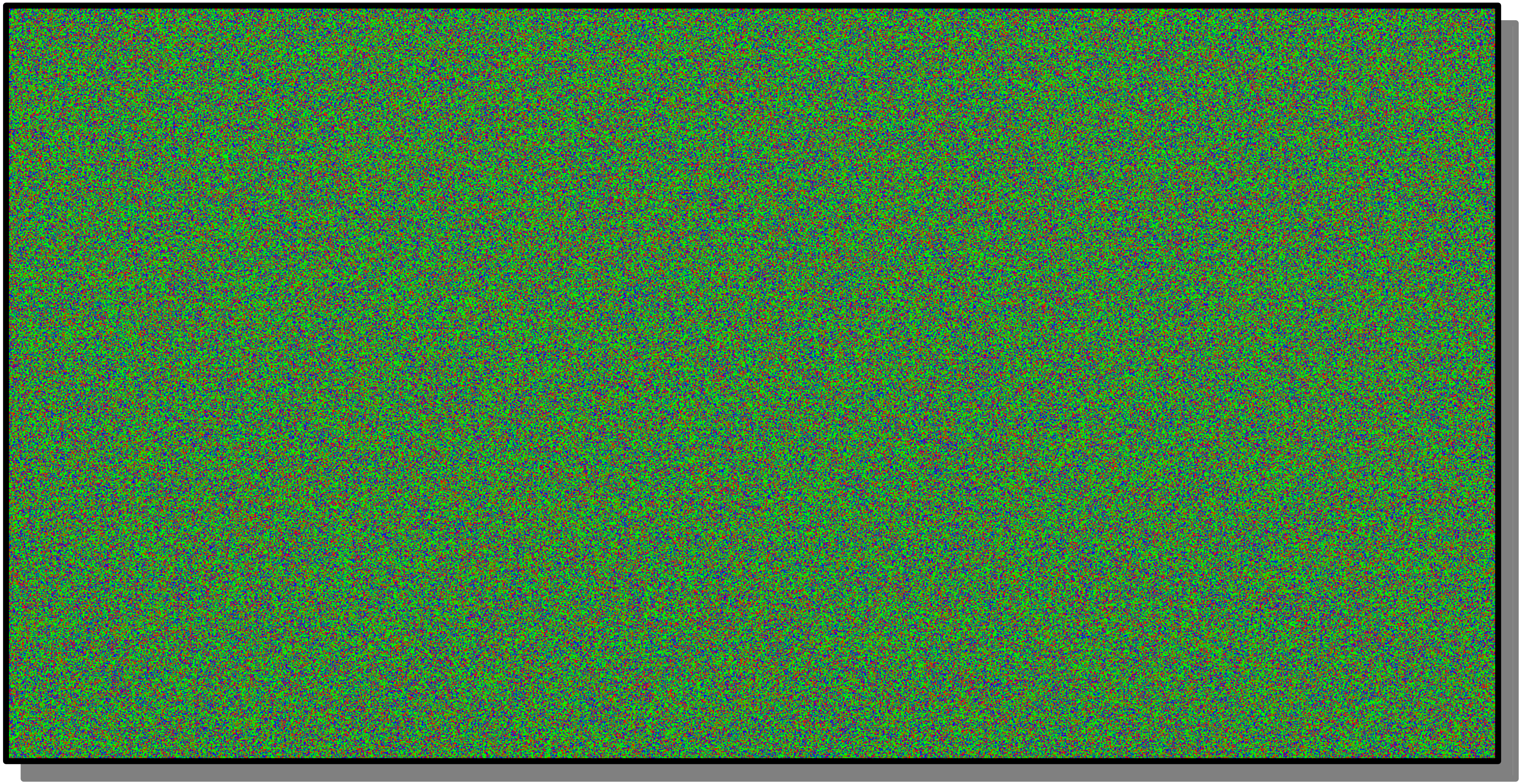
Claim: This graph is not 3-colorable.

The Moser Spindle

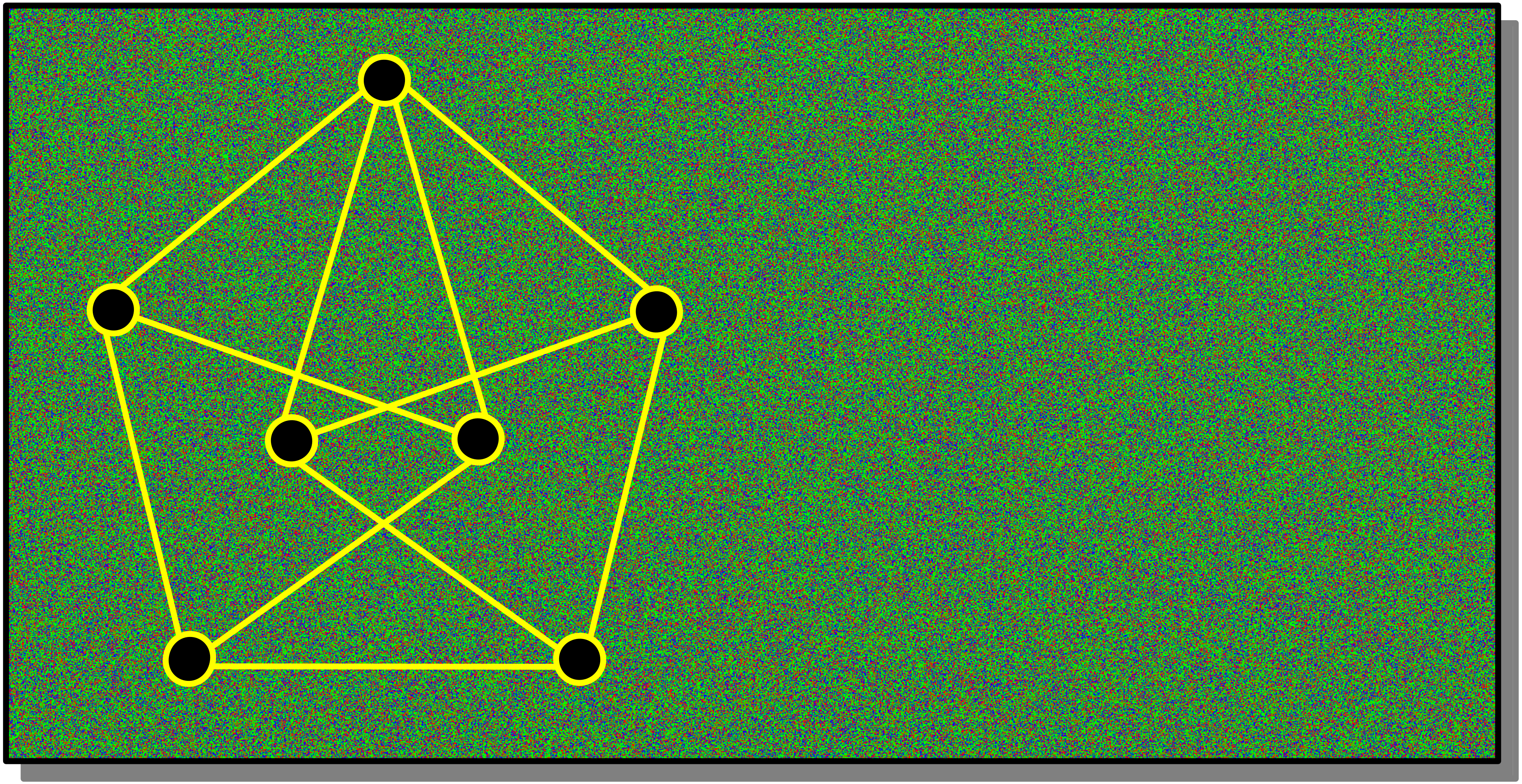


Claim: This graph is not 3-colorable.

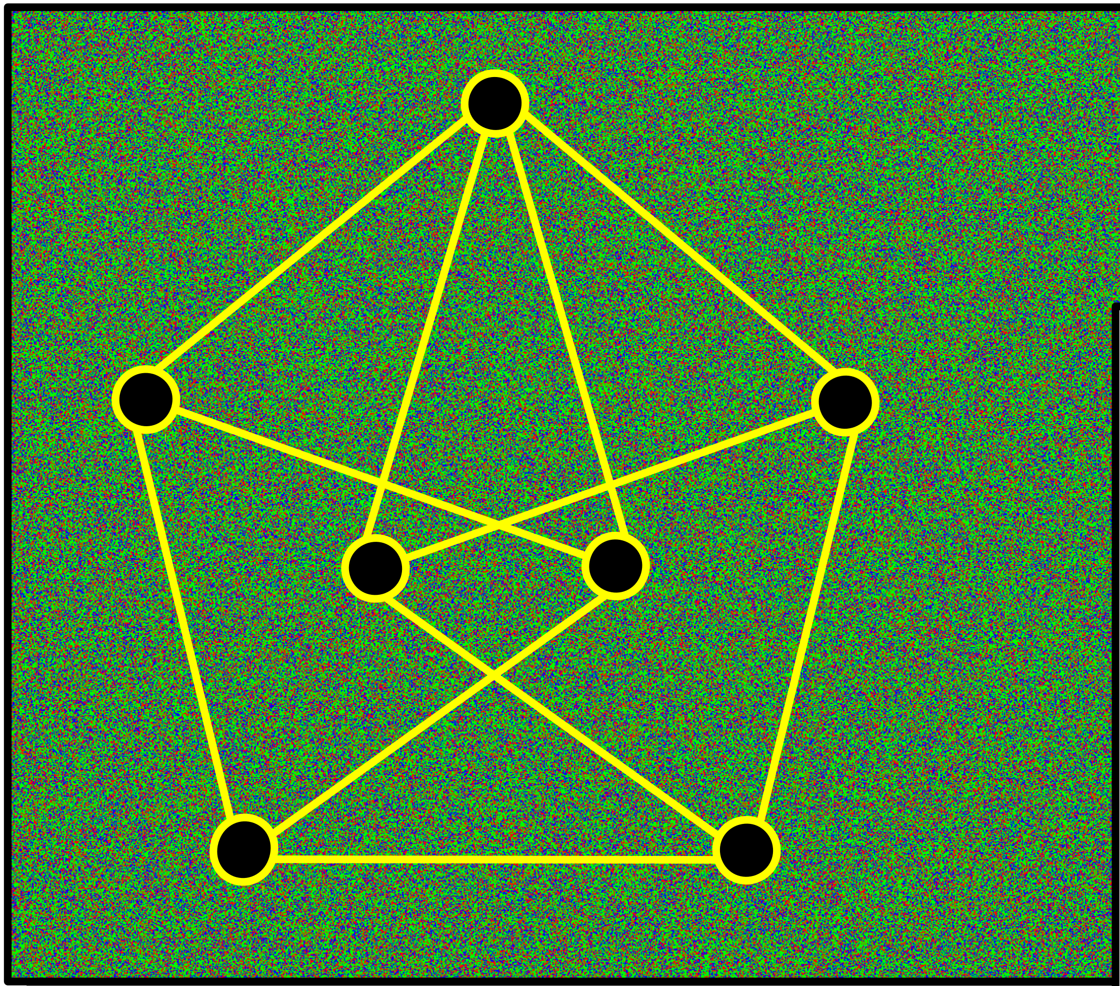
The Moser Spindle



The Moser Spindle

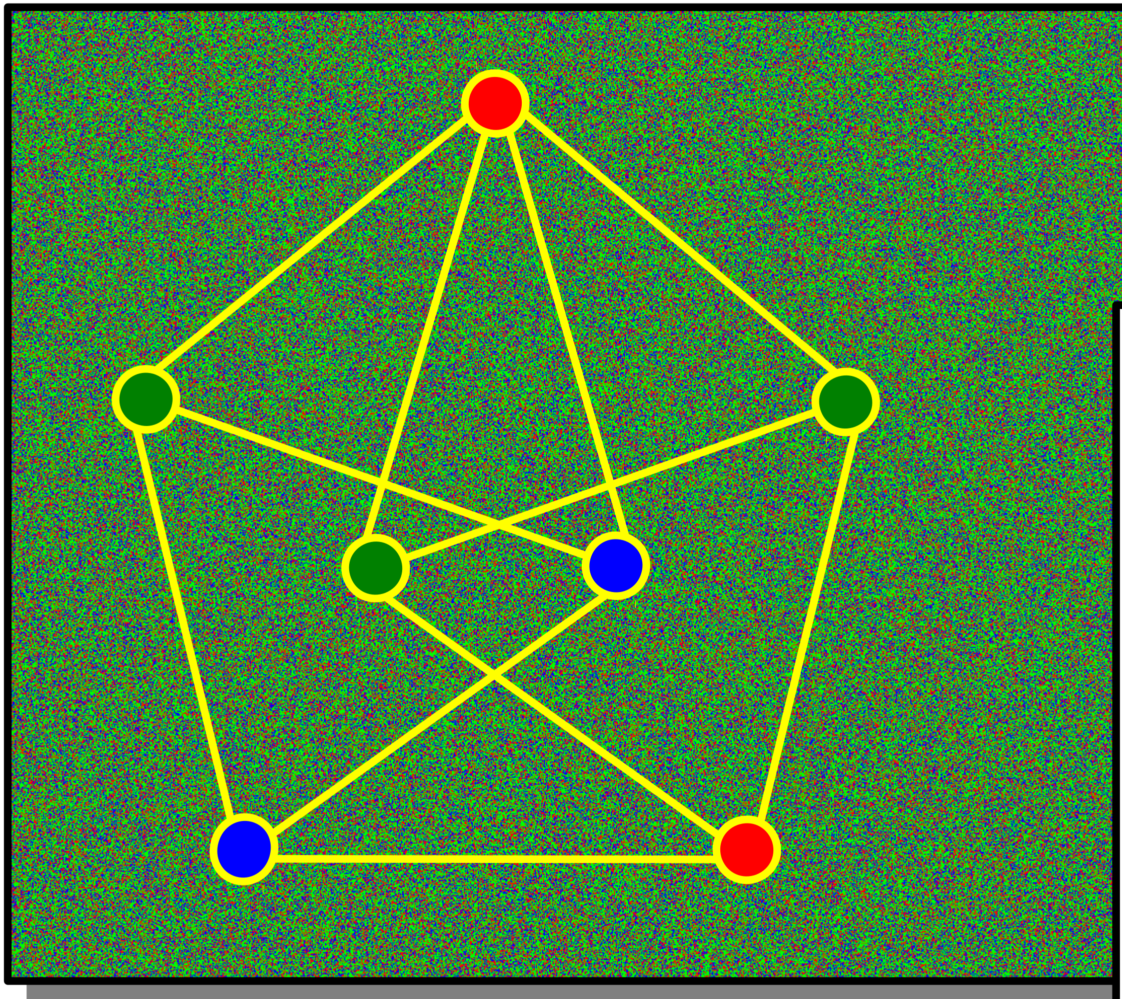


The Moser Spindle



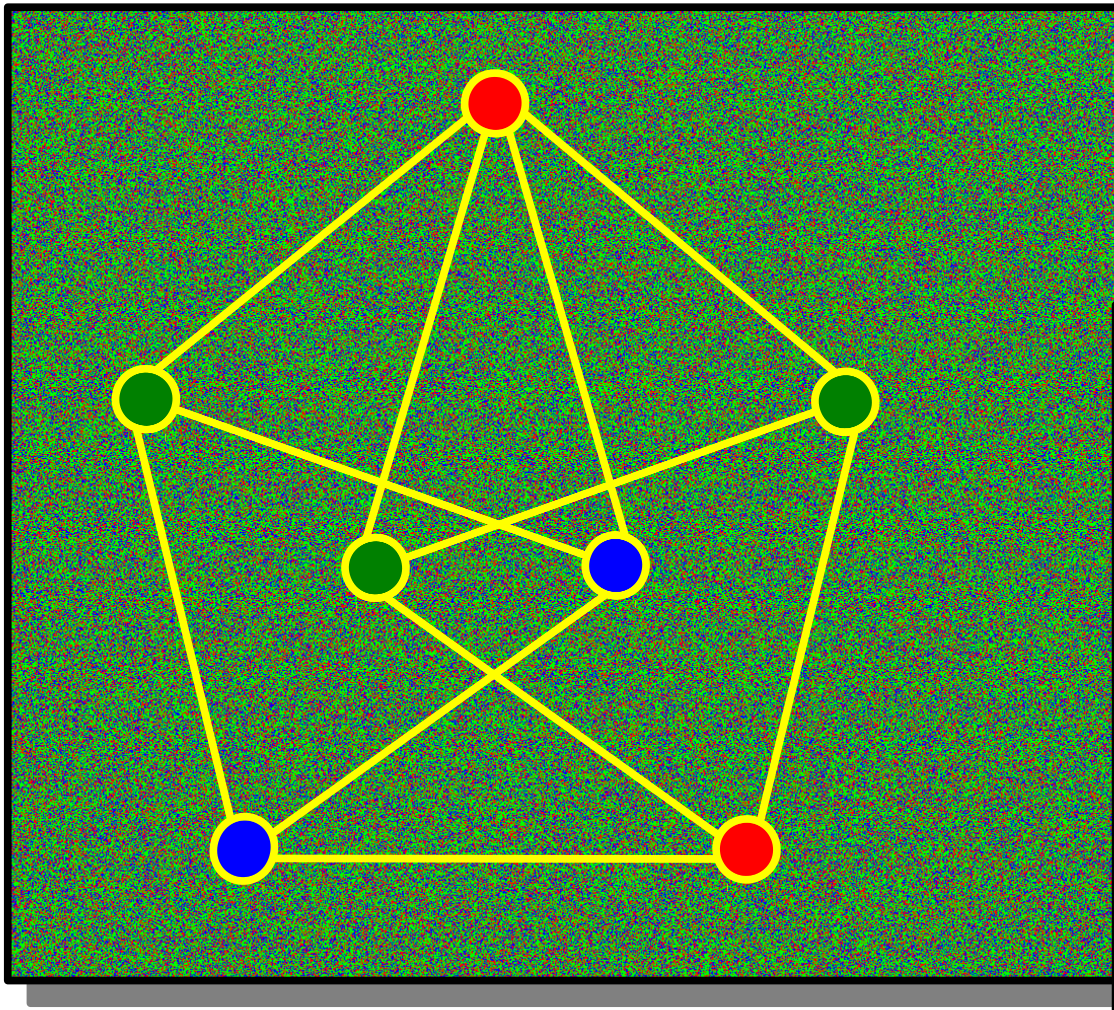
If we drop the Moser spindle into a 3-colored plane, each vertex will be given some color.

The Moser Spindle



If we drop the Moser spindle into a 3-colored plane, each vertex will be given some color.

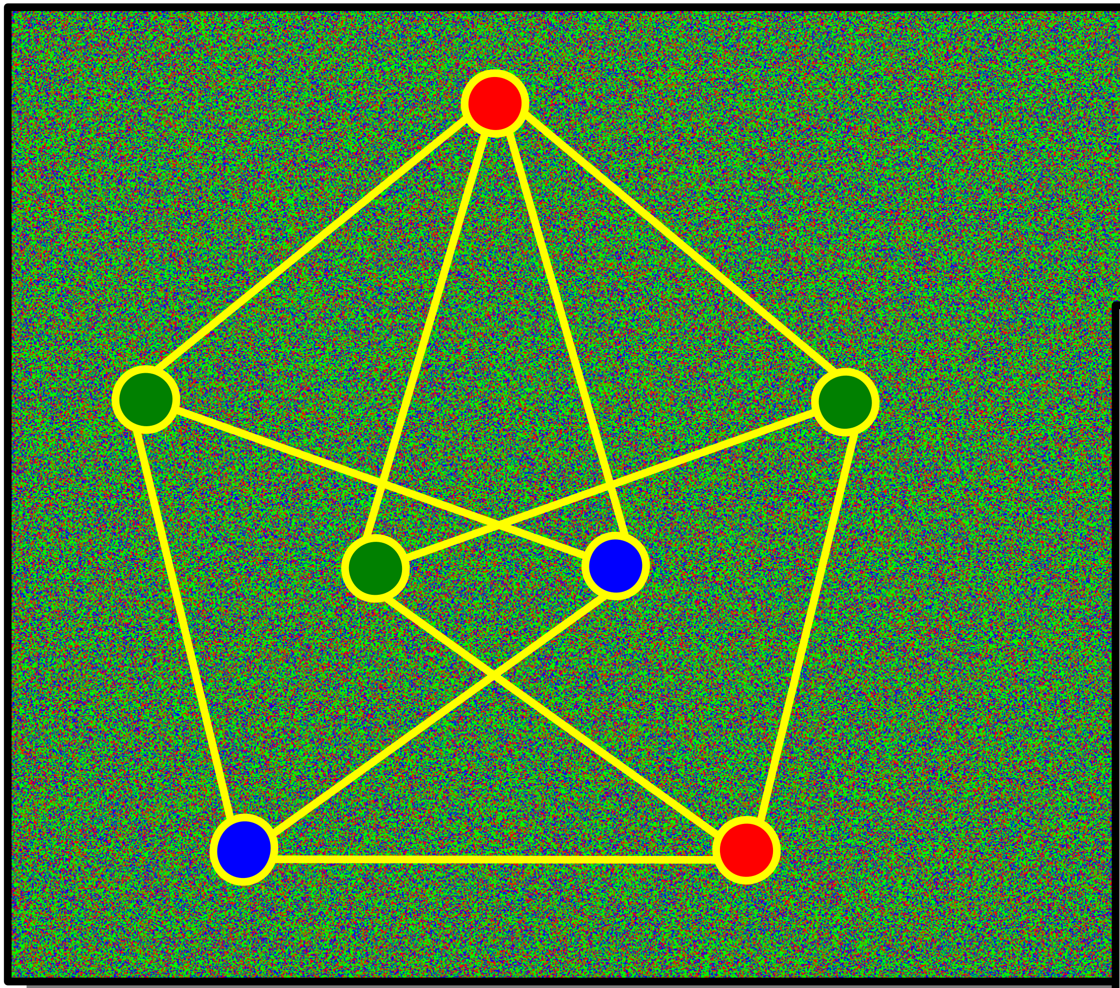
The Moser Spindle



If we drop the Moser spindle into a 3-colored plane, each vertex will be given some color.

We know that the Moser spindle isn't 3-colorable.

The Moser Spindle

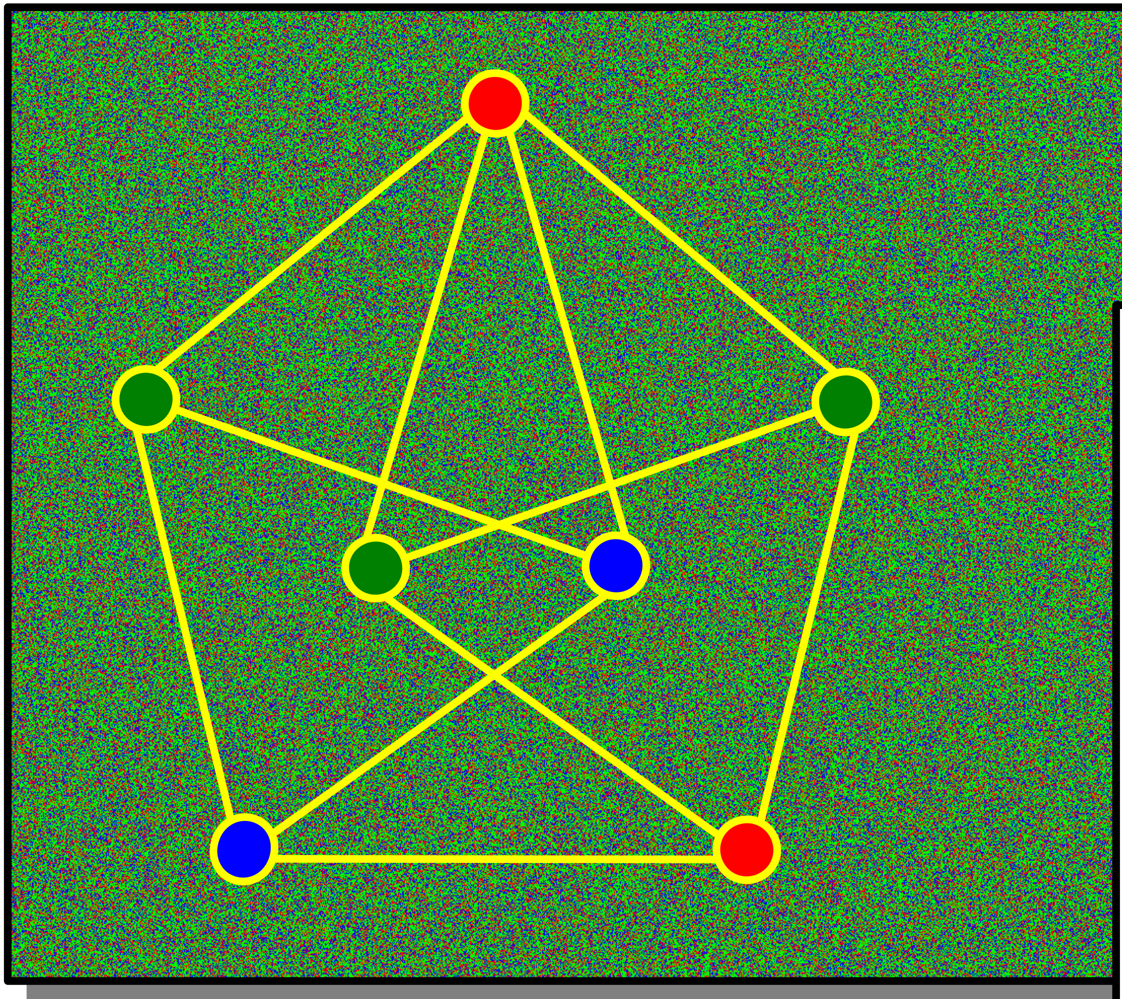


If we drop the Moser spindle into a 3-colored plane, each vertex will be given some color.

We know that the Moser spindle isn't 3-colorable.

Therefore, at least two adjacent vertices must be the same color.

The Moser Spindle



If we drop the Moser spindle into a 3-colored plane, each vertex will be given some color.

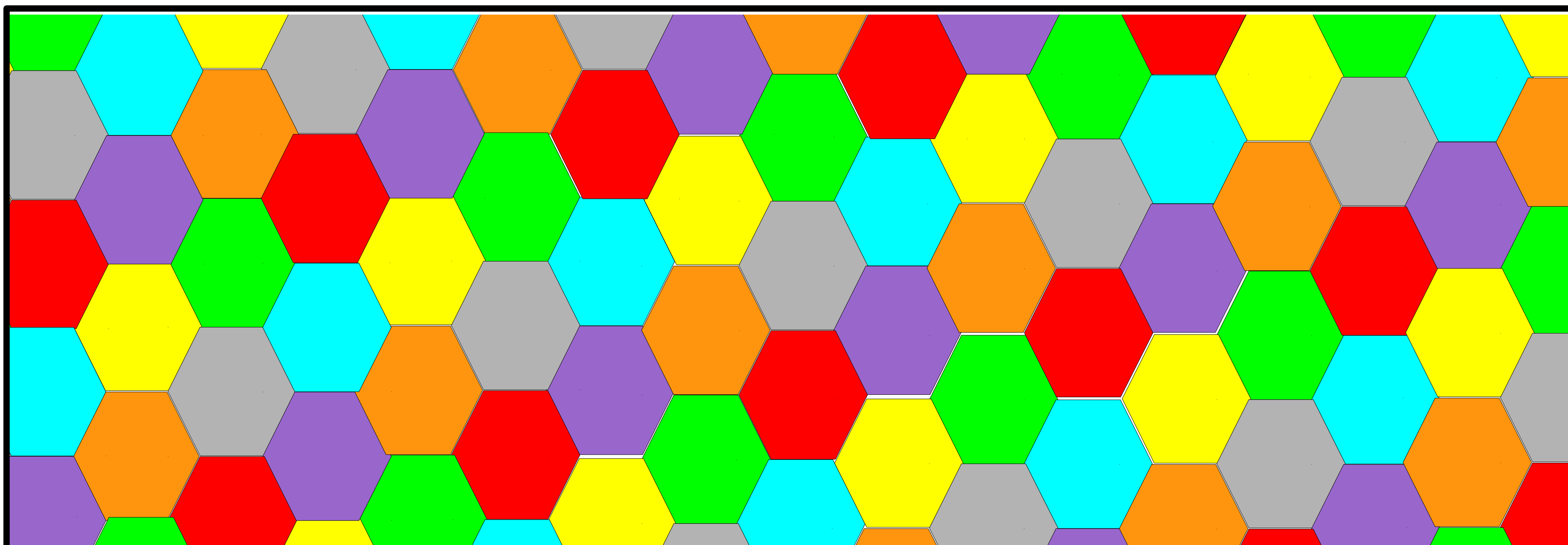
We know that the Moser spindle isn't 3-colorable.

Therefore, at least two adjacent vertices must be the same color.

Therefore, there must be two points in the plane at distance one that are the same color!

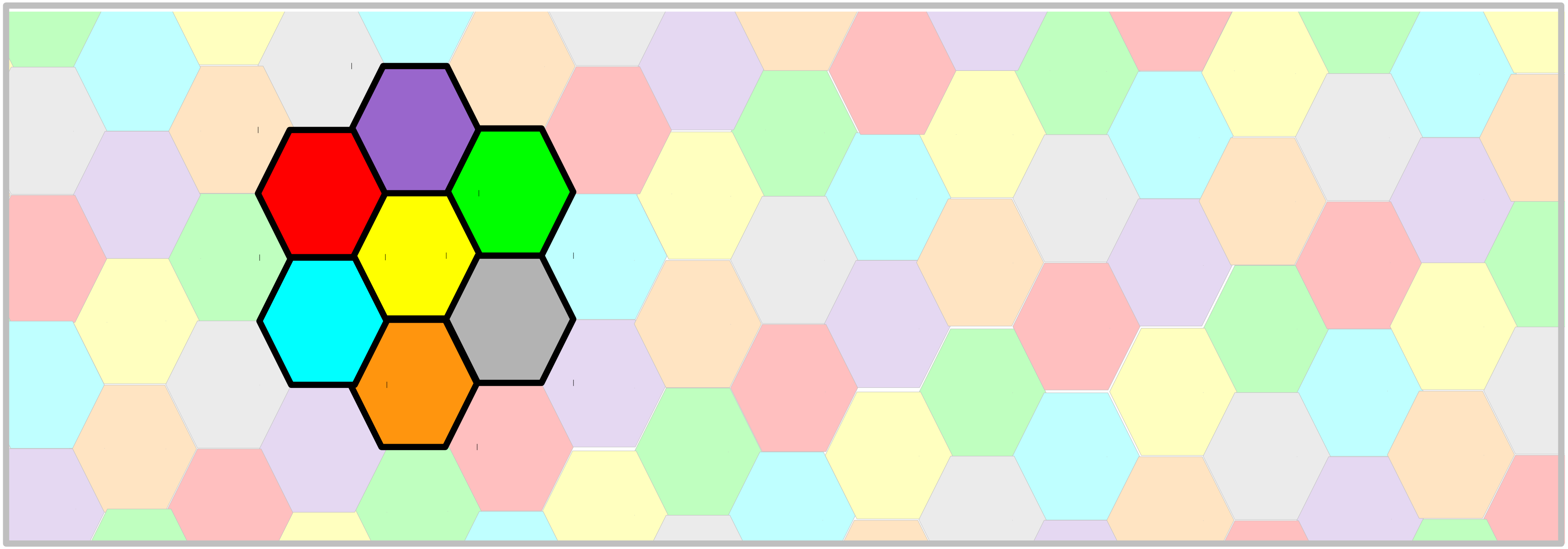
What We Know

- We know that the solution to Hadwiger-Nelson is at least four, since if you drop the Moser spindle into a plane with 3 colors, you'll find at least two of the nodes at distance 1 are the same color.
- It turns out that it's possible to color the plane using seven colors so that no two points at distance 1 are the same color by tiling the plane with hexagons of diameter ¾.



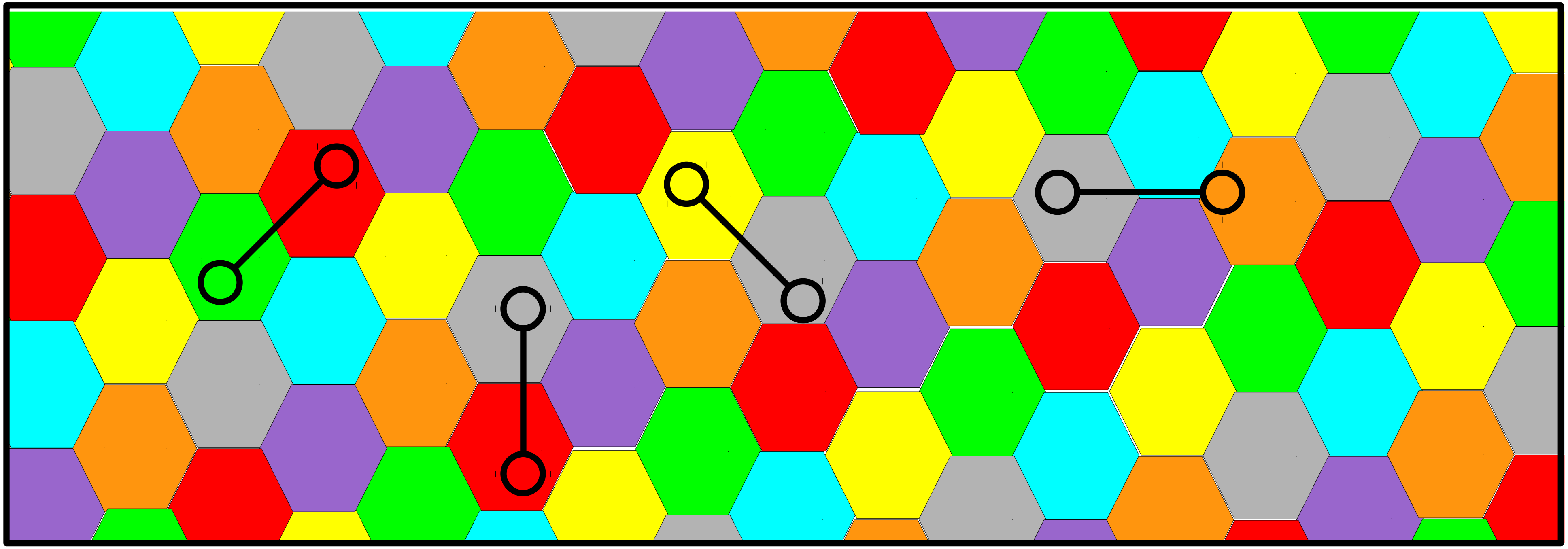
What We Know

- We know that the solution to Hadwiger-Nelson is at least four, since if you drop the Moser spindle into a plane with 3 colors, you'll find at least two of the nodes at distance 1 are the same color.
- It turns out that it's possible to color the plane using seven colors so that no two points at distance 1 are the same color by tiling the plane with hexagons of diameter ¾.



What We Know

- We know that the solution to Hadwiger-Nelson is at least four, since if you drop the Moser spindle into a plane with 3 colors, you'll find at least two of the nodes at distance 1 are the same color.
- It turns out that it's possible to color the plane using seven colors so that no two points at distance 1 are the same color by tiling the plane with hexagons of diameter ¾.



The Hadwiger-Nelson Problem

- We know that the answer to Hadwiger-Nelson problem can't be 1, 2, or 3.
- We also know the solution to the Hadwiger-Nelson problem must be at most 7.
- This means that the answer to the Hadwiger-Nelson problem must be 4, 5, 6, or 7.
- ***Amazing fact:*** No one knows which of these numbers is correct!
- ***This is an open problem in mathematics!***