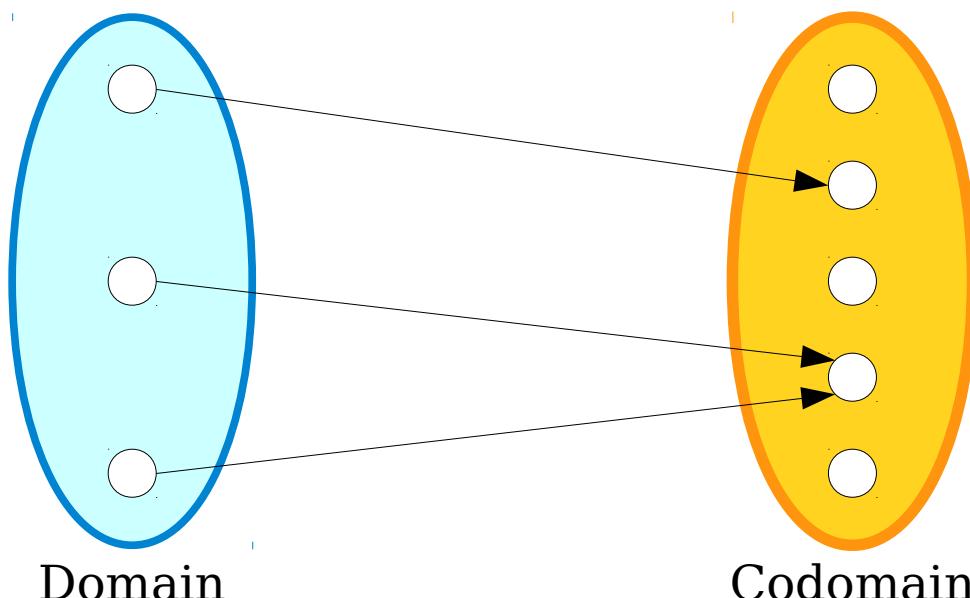


Cardinality

Recap from Last Time

Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.
- We write $f : A \rightarrow B$ to indicate that f is a function whose domain is A and whose codomain is B .



Function Composition

- Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary functions.
- The ***composition of f and g***, denoted $g \circ f$, is a function
 - whose domain is A ,
 - whose codomain is C , and
 - which is evaluated as $(g \circ f)(x) = g(f(x))$.

Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if each element of the codomain has at most one element of the domain that maps to it.
 - A function with this property is called an **injection**.
- Formally, $f : A \rightarrow B$ is an injection if this statement is true:
$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

(“*If the inputs are different, the outputs are different*”)
- Equivalently:
$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

(“*If the outputs are the same, the inputs are the same*”)
- **Theorem:** The composition of two injections is an injection.

Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if each element of the codomain is “covered” by at least one element of the domain.
 - A function with this property is called a **surjection**.
- Formally, $f : A \rightarrow B$ is a surjection if this statement is true:

$$\forall b \in B. \exists a \in A. f(a) = b$$

(“*For every possible output, there's at least one possible input that produces it*”)

- **Theorem:** The composition of two surjections is a surjection.

New Stuff!

Injections and Surjections

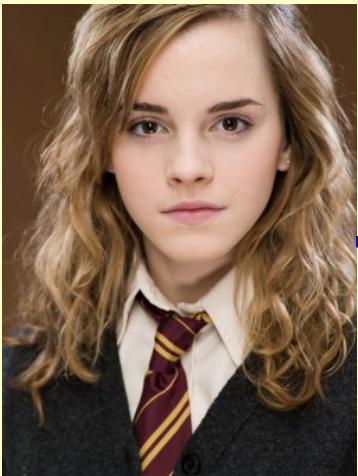
- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?



**Katniss
Everdeen**



Elsa



**Hermione
Granger**

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called **bijection**.
 - Such a function is a **bijection**.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- Bijections are sometimes called **one-to-one correspondences**.
 - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- **Yes!**
 - Since both f and g are injective, we know that $g \circ f$ is injective.
 - Since both f and g are surjective, we know that $g \circ f$ is surjective.
 - Therefore, $g \circ f$ is a bijection.

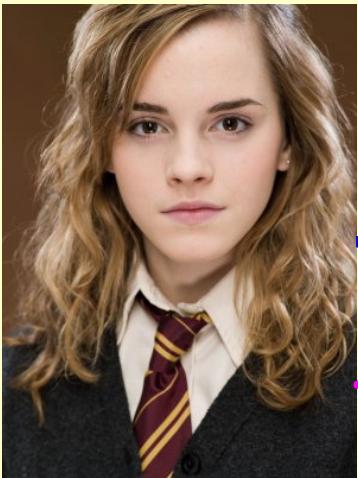
Inverse Functions



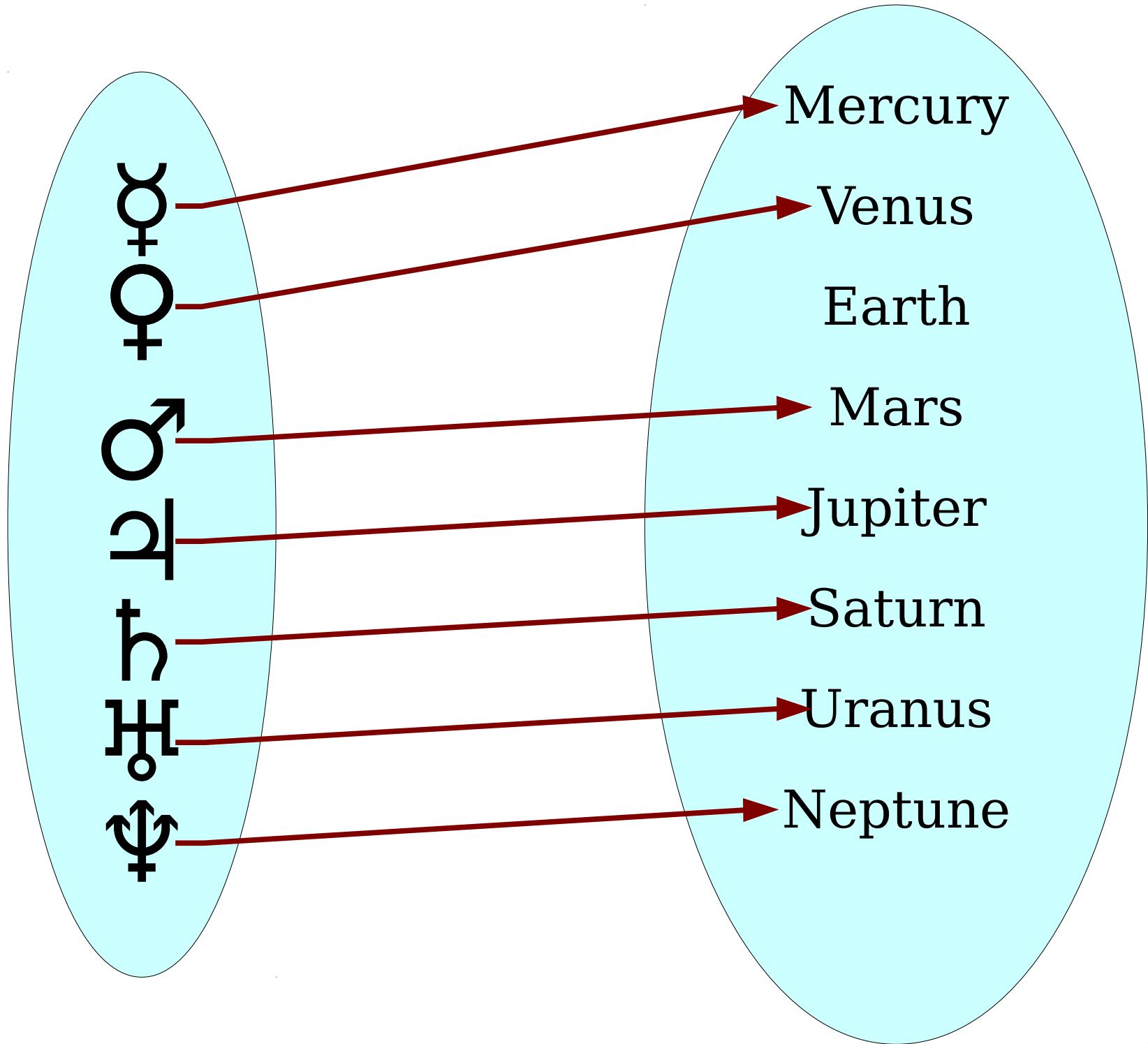
**Katniss
Everdeen**

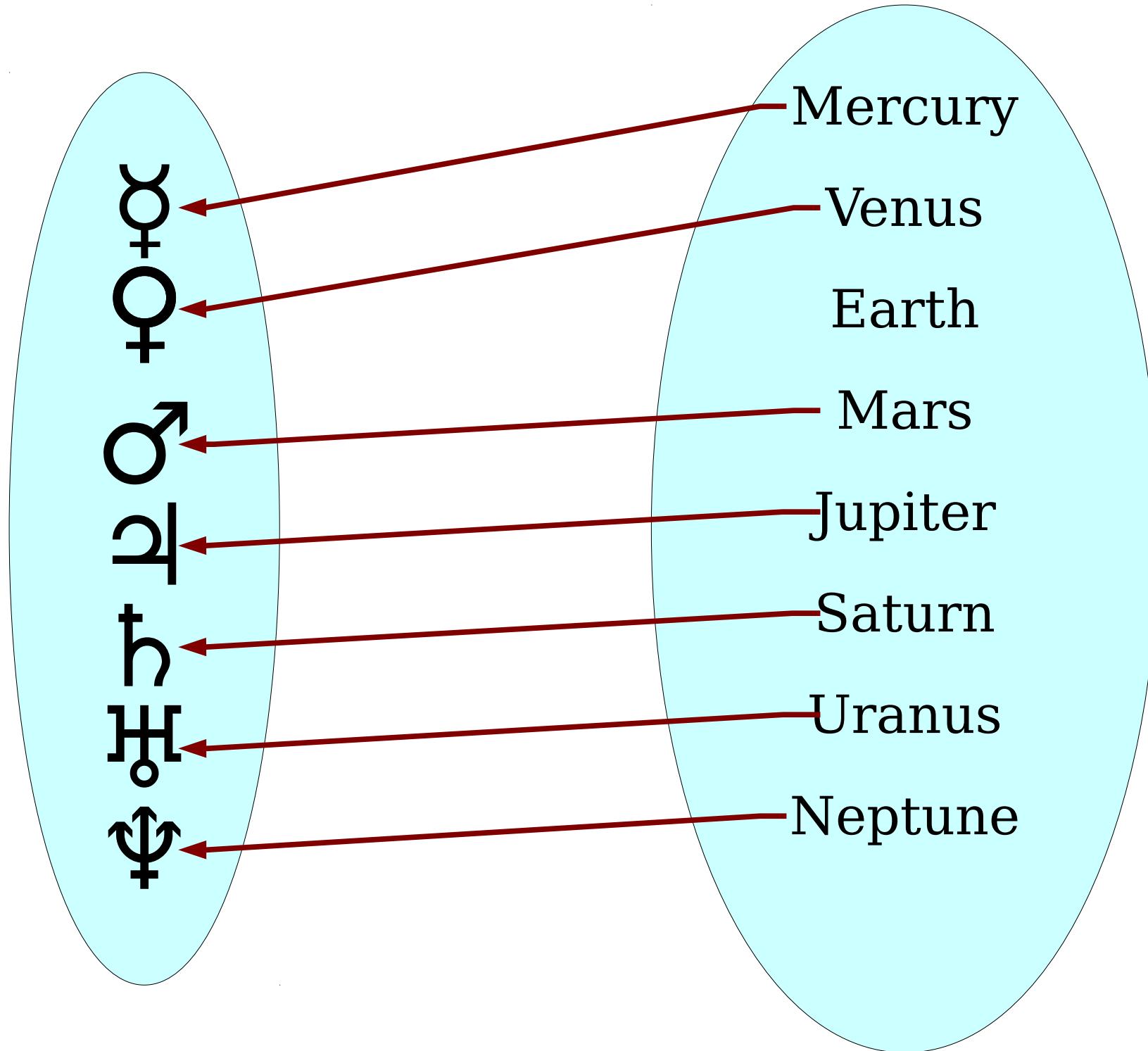


Elsa



**Hermione
Granger**





Mercury

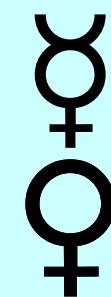
Venus

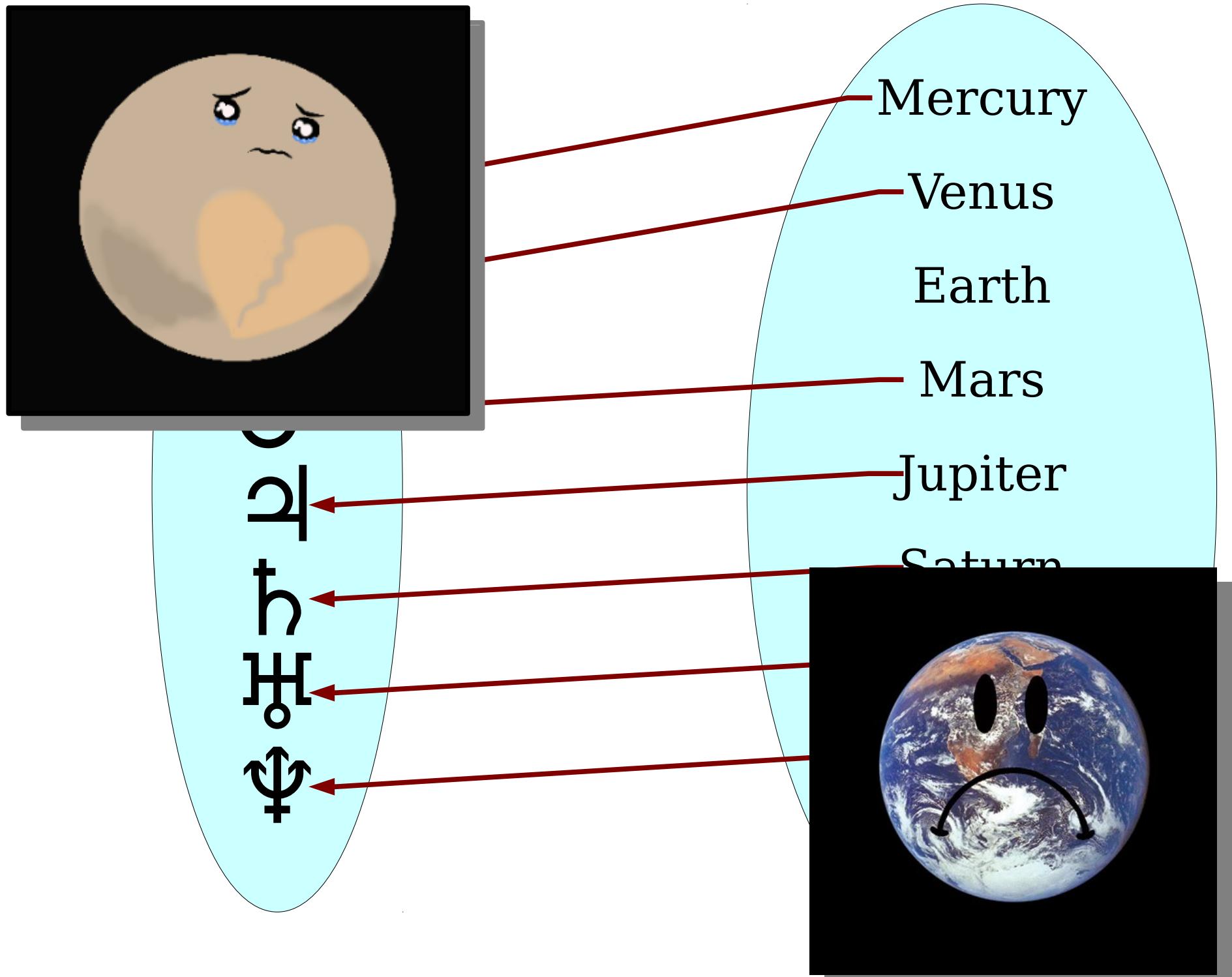
Earth

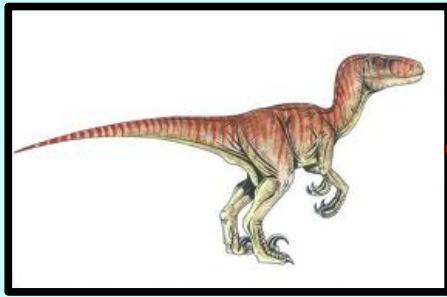
Mars

Jupiter

Saturn



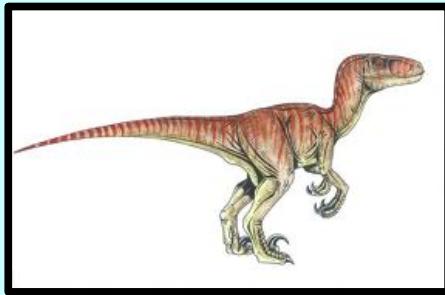
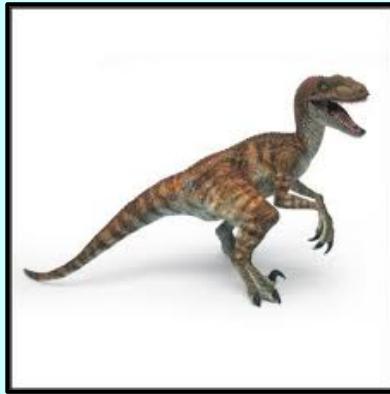




Front Door

Balcony
Window

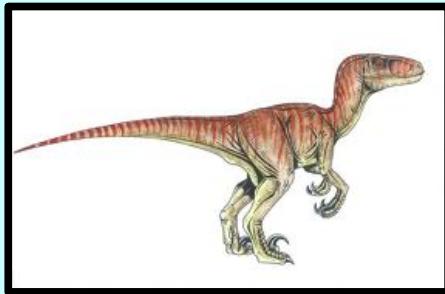
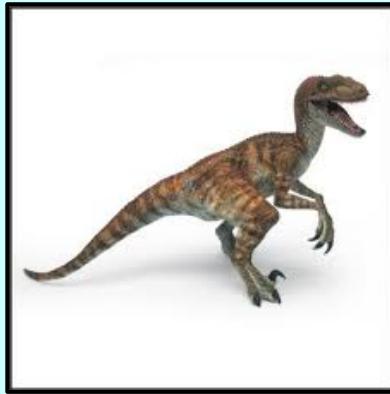
Bedroom
Window



Front Door

Balcony
Window

Bedroom
Window



Front Door

Balcony
Window

Bedroom
Window

Inverse Functions

- In some cases, it's possible to "turn a function around."
- Let $f : A \rightarrow B$ be a function. A function $f^{-1} : B \rightarrow A$ is called the **inverse of f** if the following is true:
$$\forall a \in A. \forall b \in B. (f(a) = b \leftrightarrow f^{-1}(b) = a)$$
- In other words, if f maps a to b , then f^{-1} maps b back to a and vice-versa.
- Not all functions have inverses (we just saw a few examples of functions with no inverse).
- If f is a function that has an inverse, then we say that f is **invertible**.

Inverse Functions

- **Theorem:** Let $f : A \rightarrow B$. Then f is invertible if and only if f is a bijection.
- To prove this result, we need to prove that
 - if $f : A \rightarrow B$ is invertible, then f is a bijection, and
 - if $f : A \rightarrow B$ is a bijection, then f is invertible.
- These proofs are in the course reader. Feel free to check them out if you'd like!

Where We Are

- We now know
 - what an injection, surjection, and bijection are;
 - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
 - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters.
Well, there's a good reason...

Cardinality Revisited

Cardinality

- Recall (*from our first lecture!*) that the **cardinality** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
 - $|\{1, 2, 3\}| = 3$
 - $|\{100, 200\}| = 2$
- For infinite sets, we introduced **infinite cardinals** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

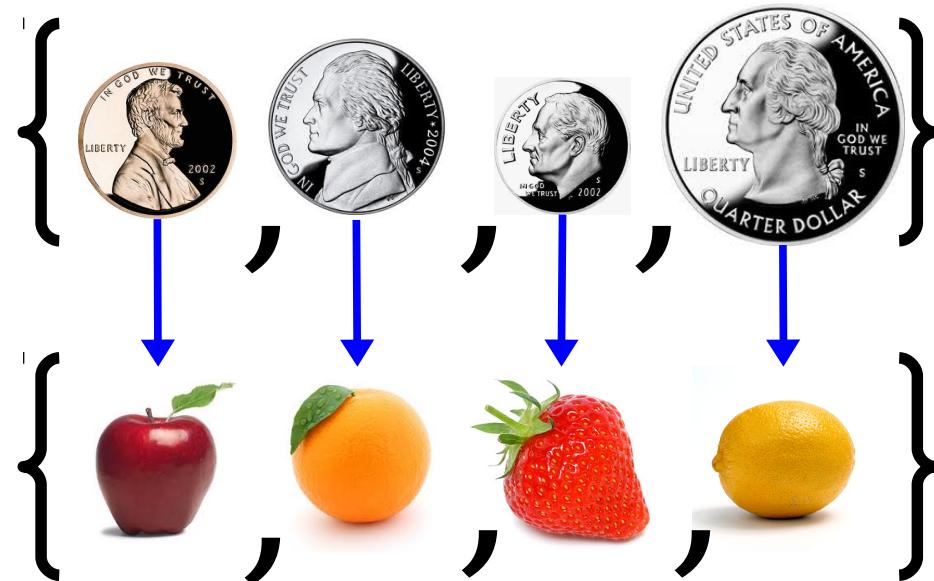
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
 - What is 4 ? What is \aleph_0 ?
 - (Take Math 161 for an answer!)
- **Idea:** Define cardinality as a *relation* between two sets rather than an absolute quantity.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

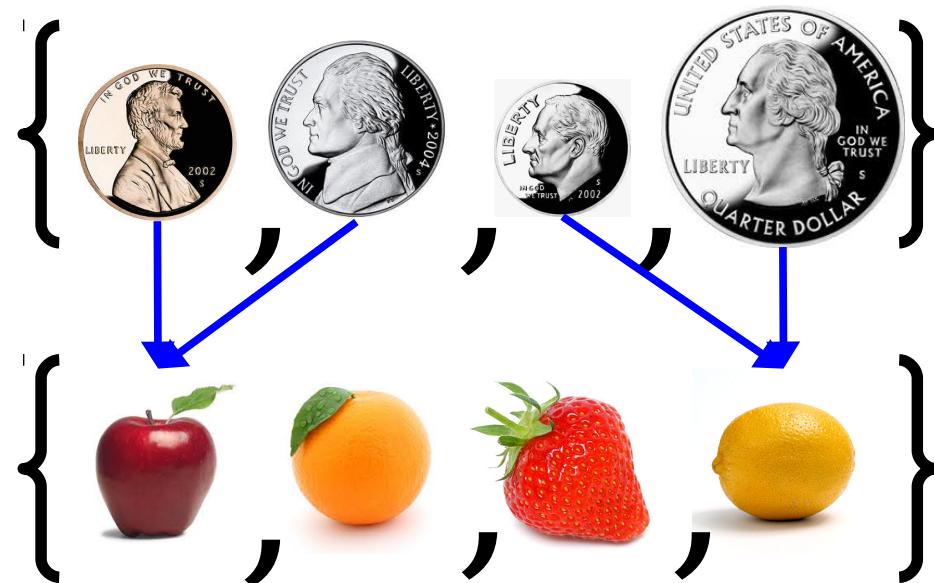
$|S| = |T|$ if there exists a **bijection** $f: S \rightarrow T$



Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a **bijection** $f: S \rightarrow T$



Properties of Cardinality

- For any sets A , B , and C , the following are true:
 - $|A| = |A|$.
 - Define $f : A \rightarrow A$ as $f(x) = x$.
 - **If $|A| = |B|$, then $|B| = |A|$.**
 - If $f : A \rightarrow B$ is a bijection, then $f^{-1} : B \rightarrow A$ is a bijection.
 - **If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.**
 - If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections, then $g \circ f : A \rightarrow C$ is a bijection.

Fun with Cardinality

Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

\mathbb{Z} ... -3 -2 -1 0 1 2 3 4 ...

Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

\mathbb{Z}

... -3 -2 -1 0 1 2 3 4 ...

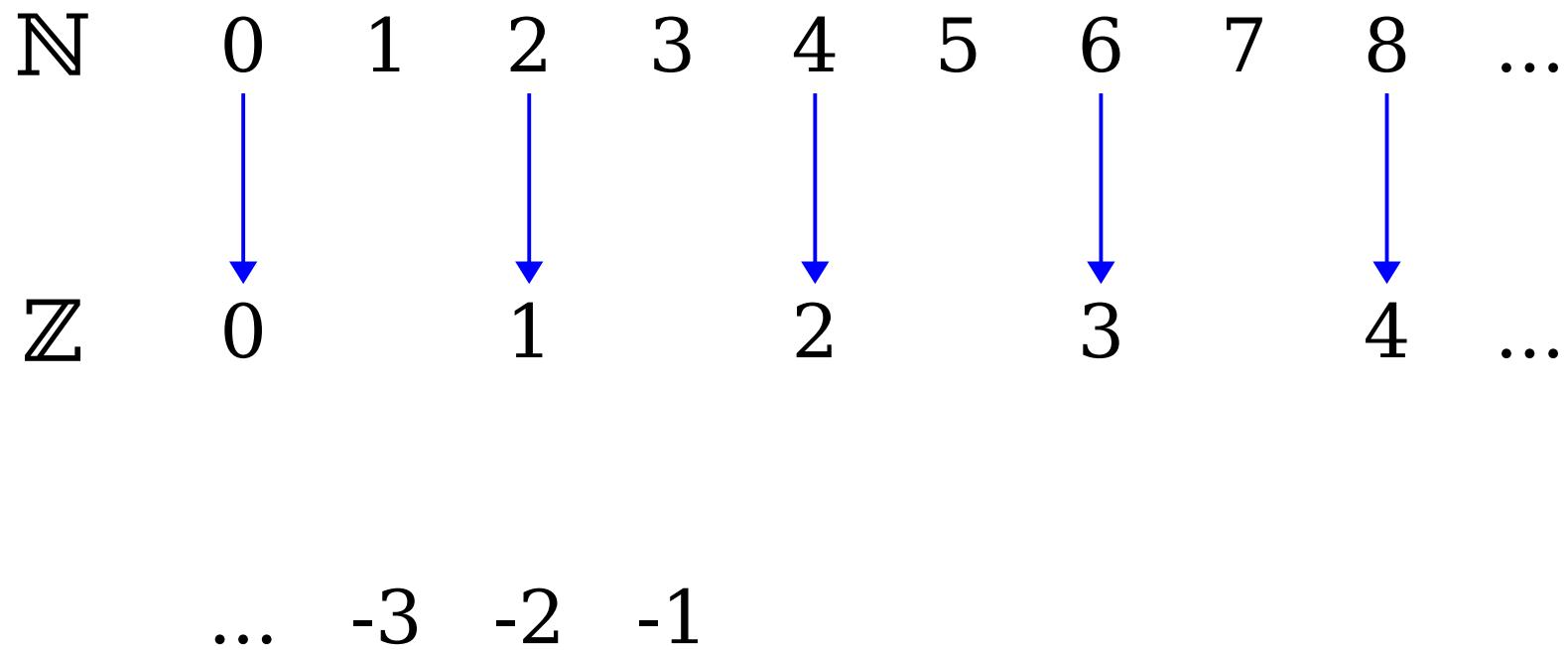
Infinite Cardinalities

\mathbb{N} 0 1 2 3 4 5 6 7 8 ...

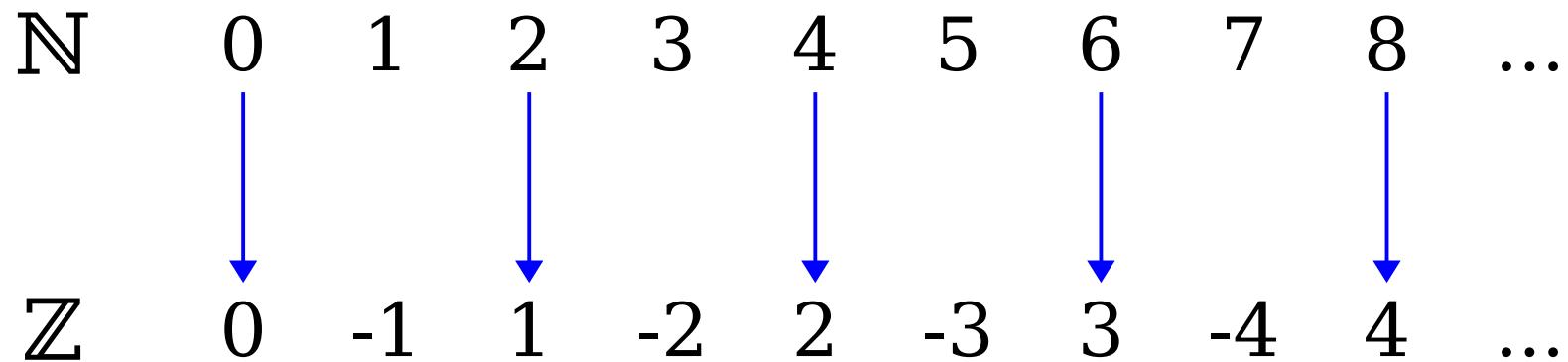
\mathbb{Z} 0 1 2 3 4 ...

... -3 -2 -1

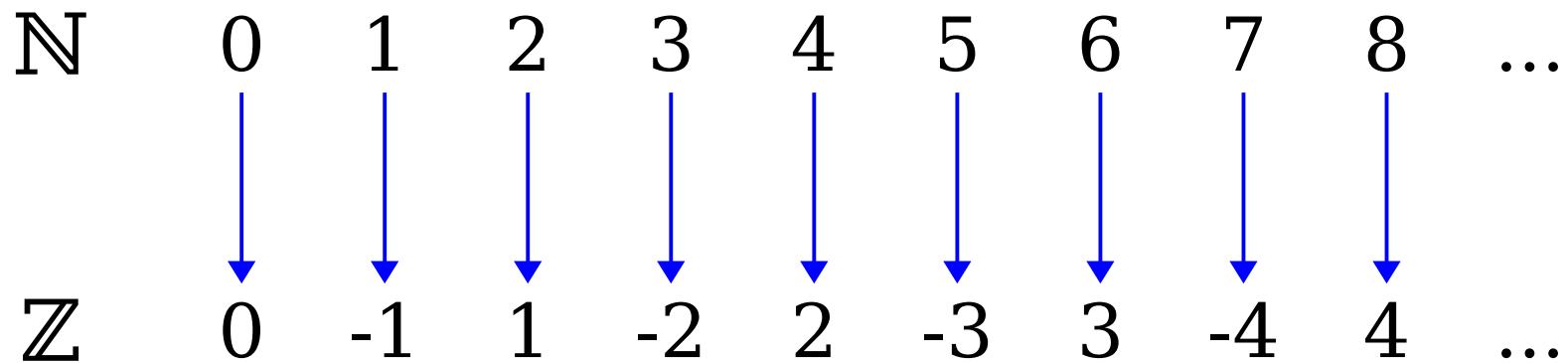
Infinite Cardinalities



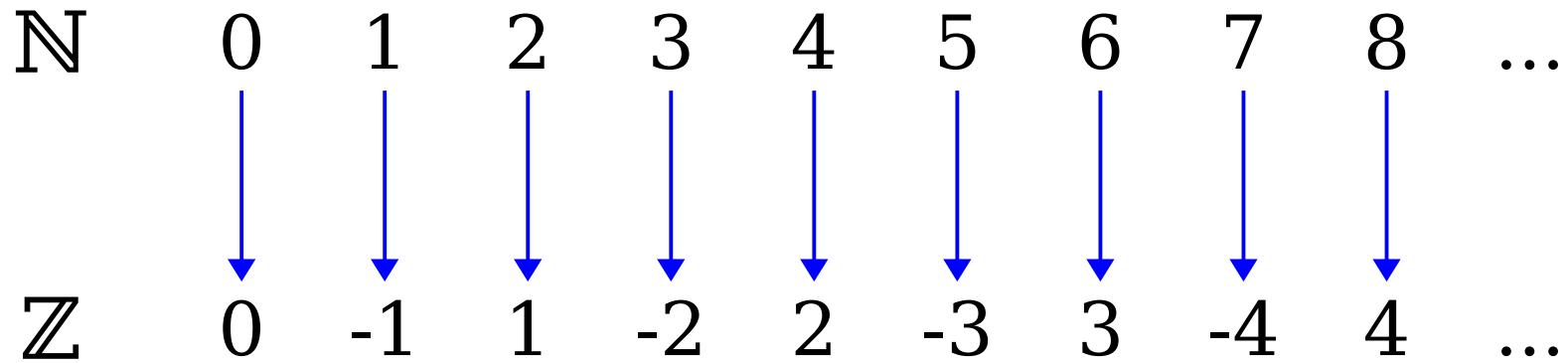
Infinite Cardinalities



Infinite Cardinalities

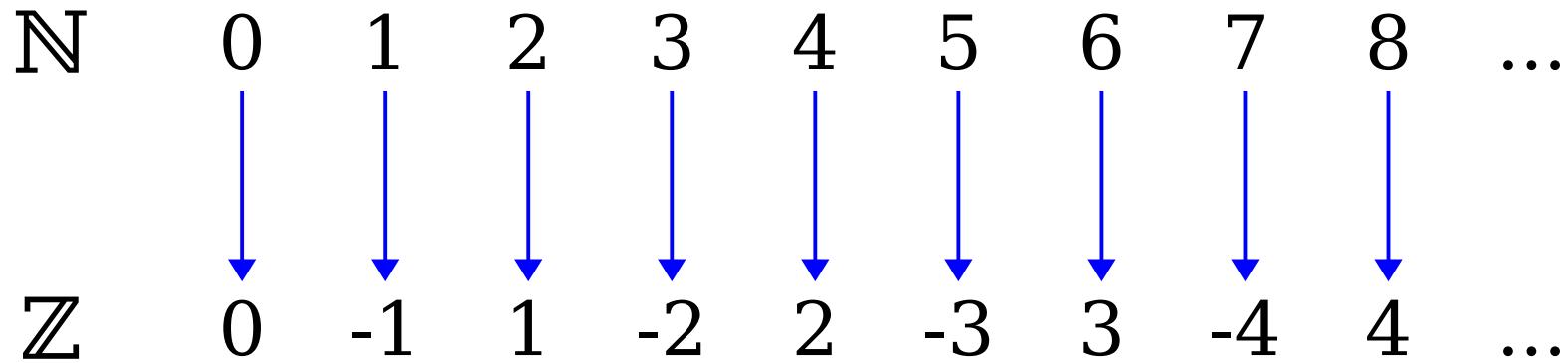


Infinite Cardinalities



Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as follows:

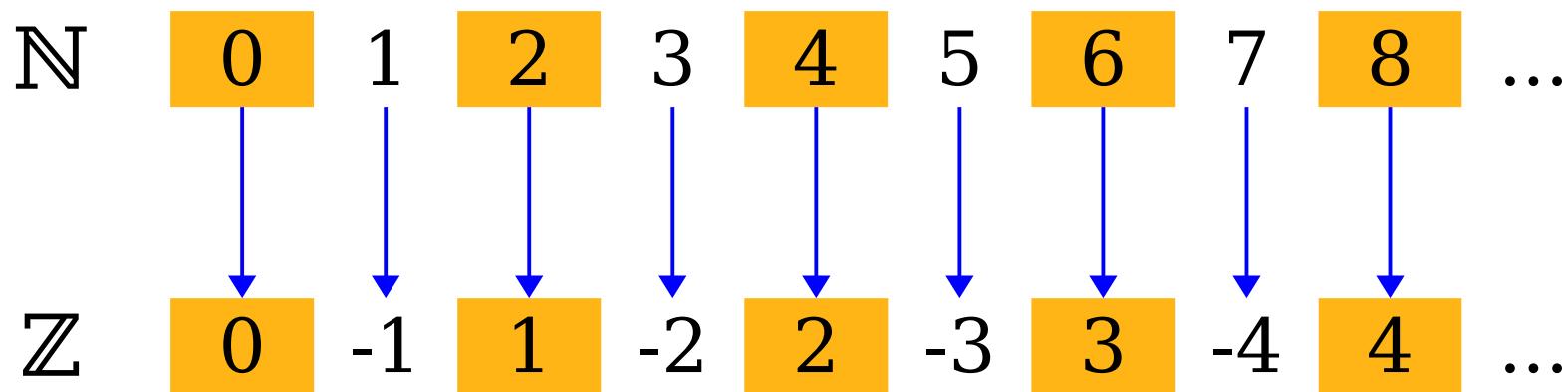
Infinite Cardinalities



Define $f : \mathbb{N} \rightarrow \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \text{...} & \text{if } n \text{ is even} \\ \text{...} & \text{if } n \text{ is odd} \end{cases}$$

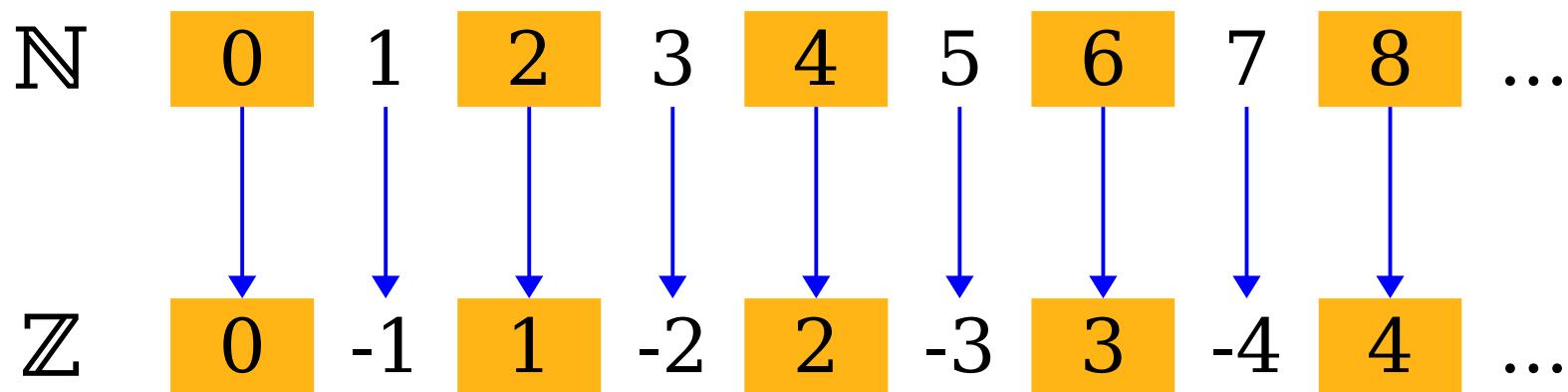
Infinite Cardinalities



Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \text{ } & \\ \text{ } & \end{cases}$$

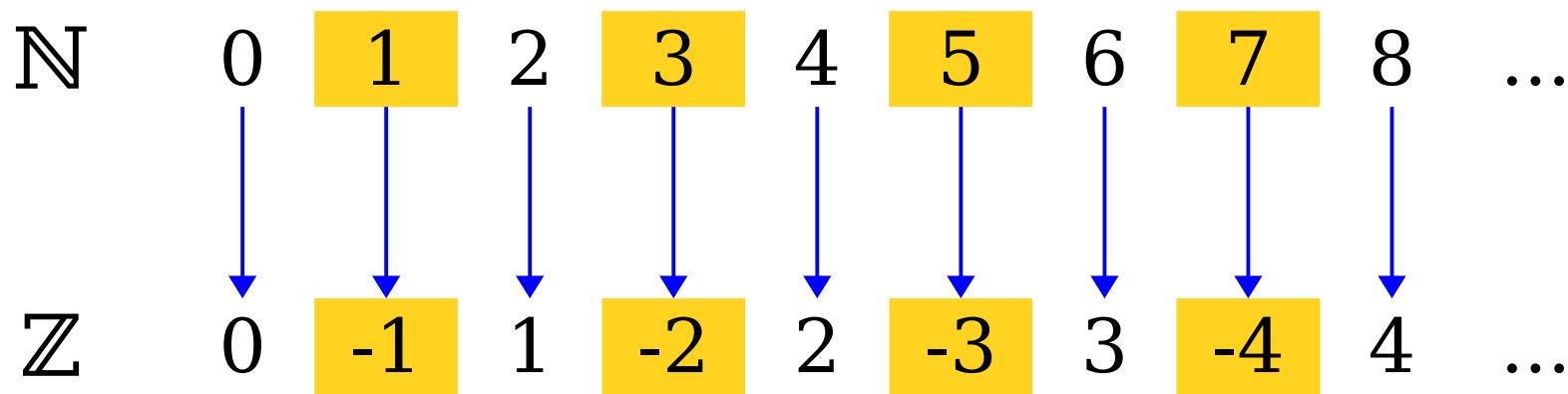
Infinite Cardinalities



Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \end{cases}$$

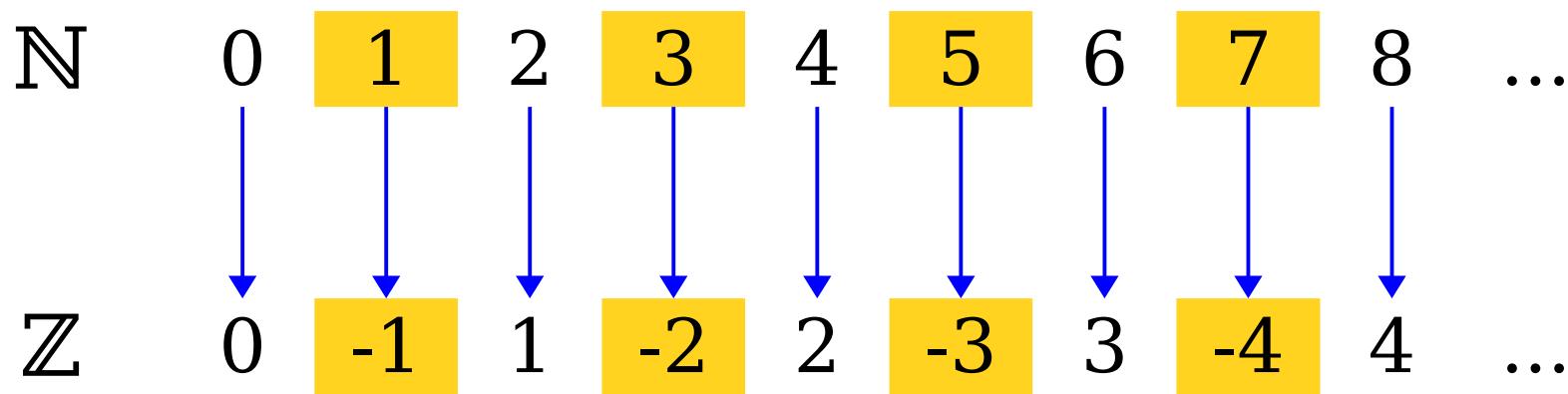
Infinite Cardinalities



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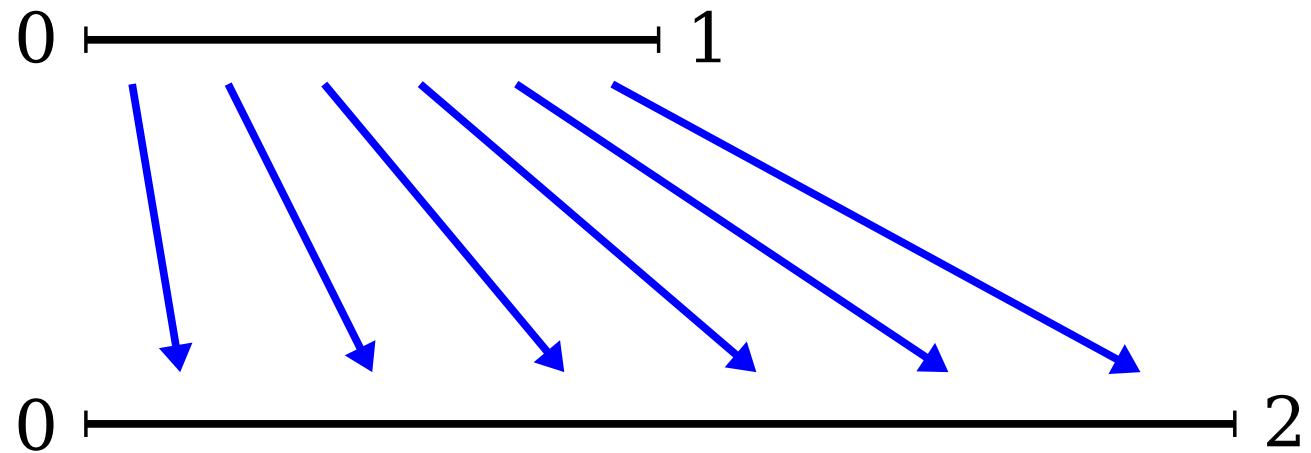
Infinite Cardinalities



Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{otherwise} \end{cases}$$

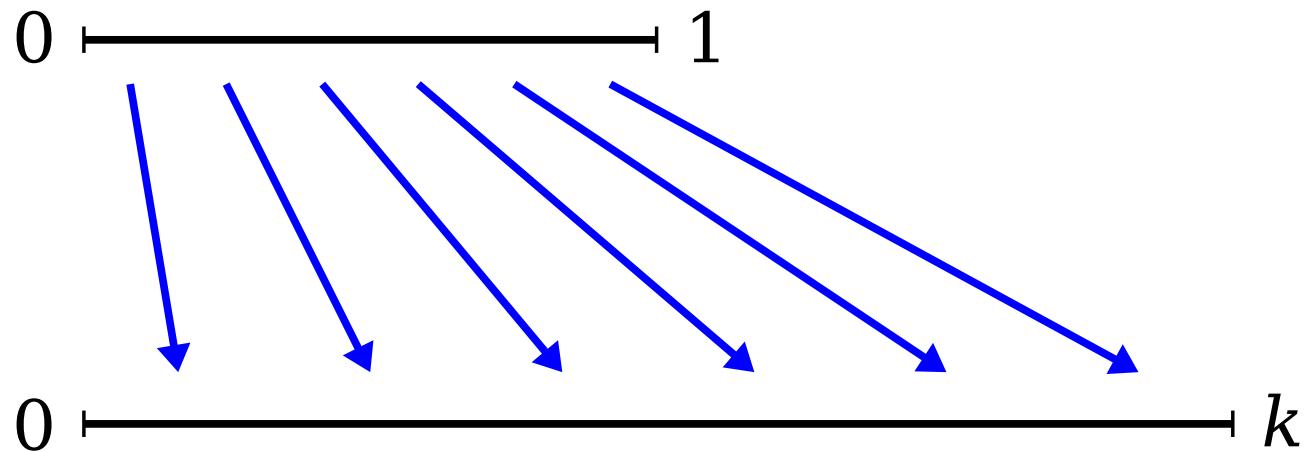
Home on the Range



$$\begin{aligned}f : [0, 1] &\rightarrow [0, 2] \\f(x) &= 2x\end{aligned}$$

$$|[0, 1]| = |[0, 2]|$$

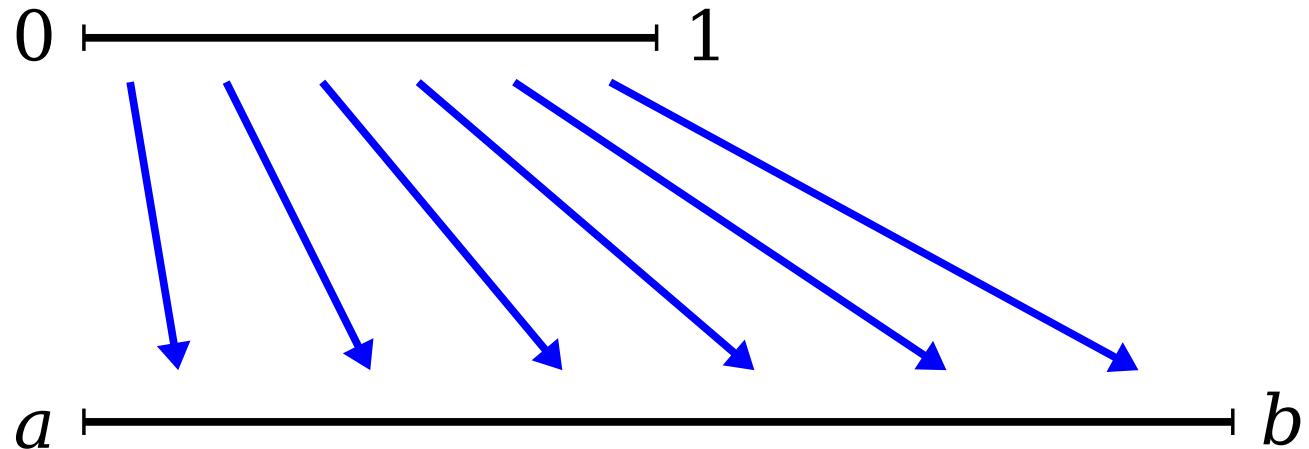
Home on the Range



$$\begin{aligned}f &: [0, 1] \rightarrow [0, k] \\f(x) &= kx\end{aligned}$$

$$|[0, 1]| = |[0, k]|$$

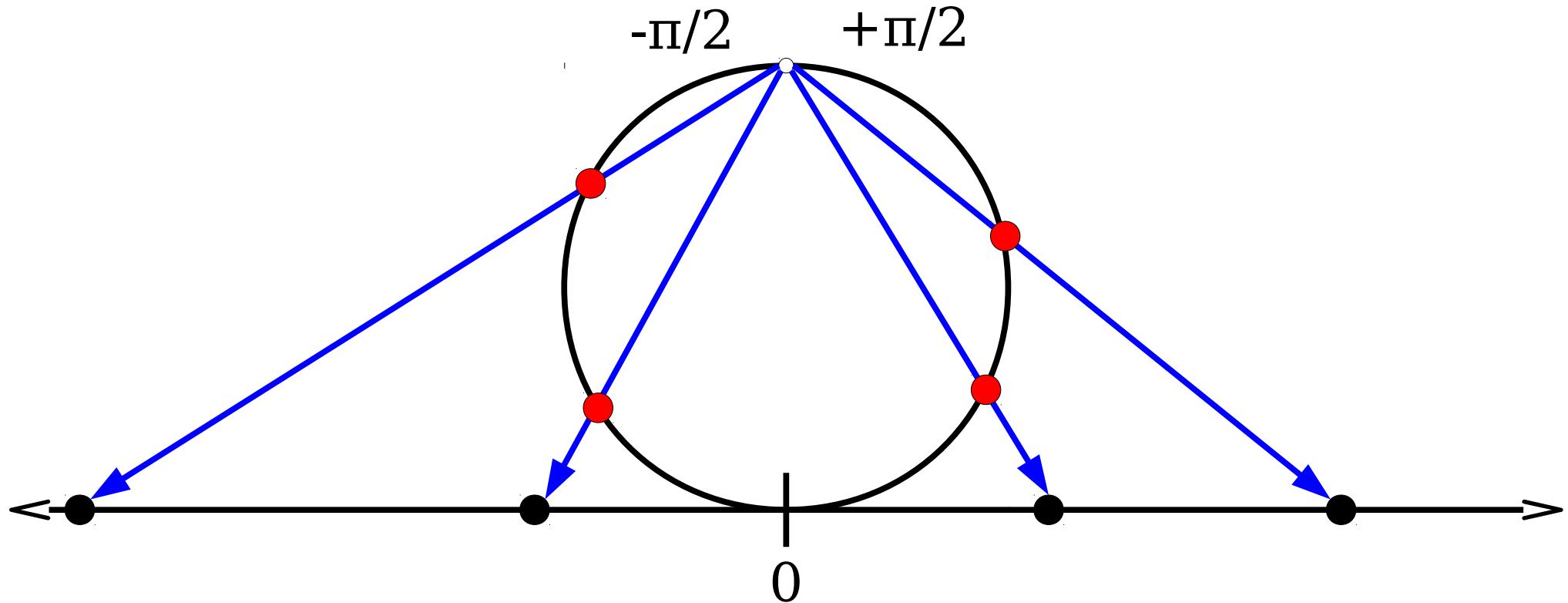
Home on the Range



$$\begin{aligned}f &: [0, 1] \rightarrow [a, b] \\f(x) &= (b - a)x + a\end{aligned}$$

$$|[0, 1]| = |[a, b]|$$

Put a Ring On It

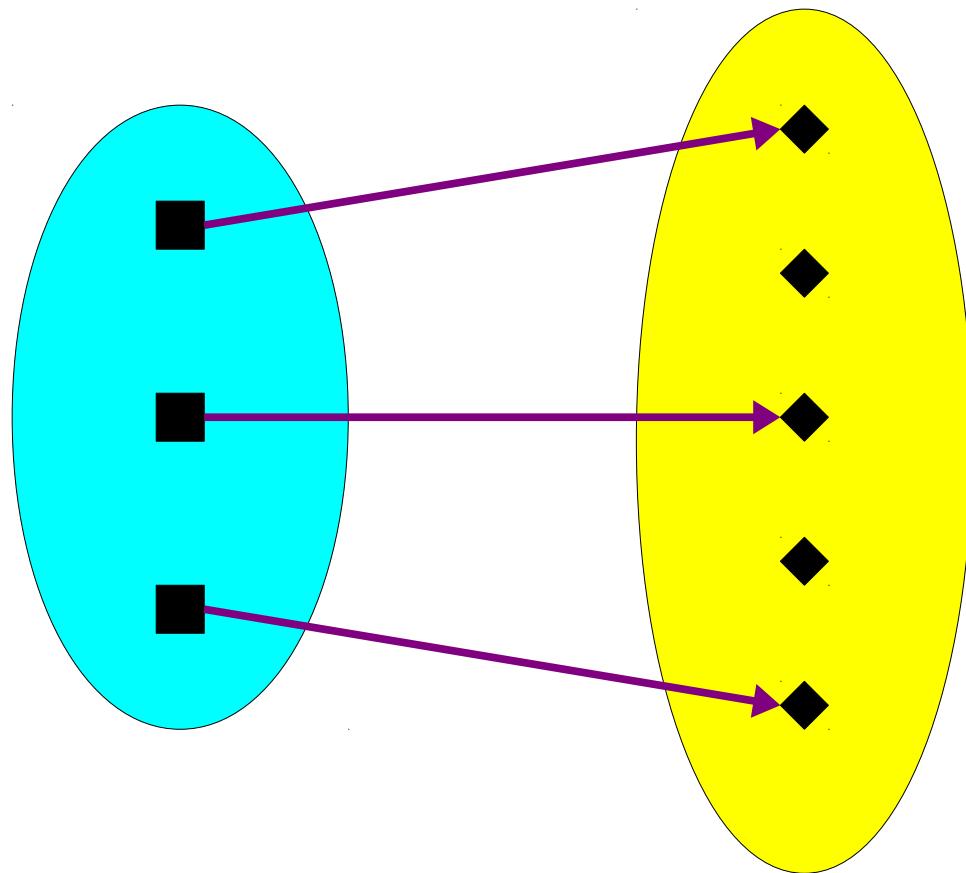


$$\begin{aligned}f : (-\pi/2, \pi/2) &\rightarrow \mathbb{R} \\ f(x) &= \tan x\end{aligned}$$

$$|(-\pi/2, \pi/2)| = |\mathbb{R}|$$

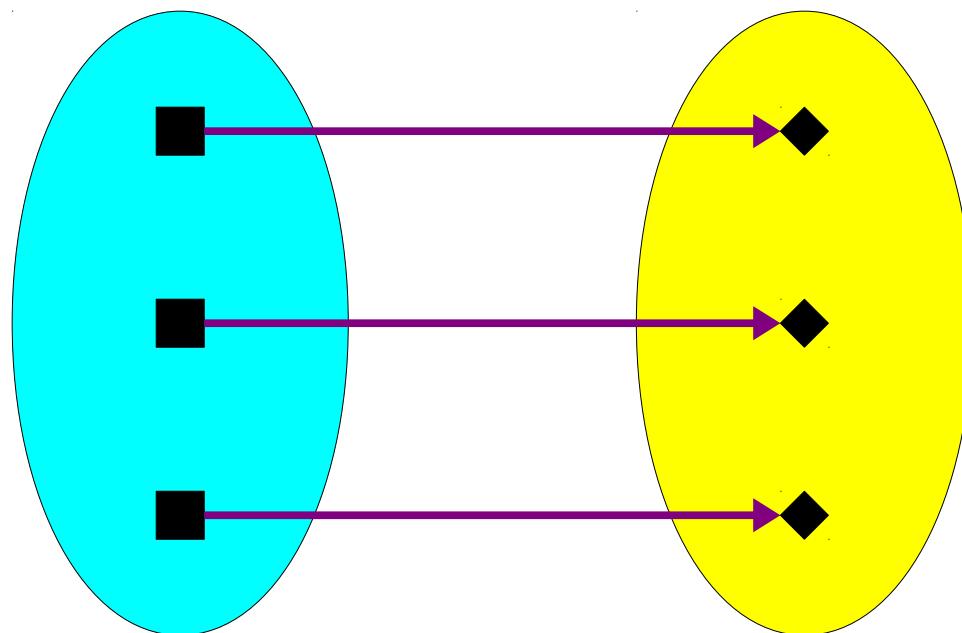
Ranking Cardinalities

- We define $|A| \leq |B|$ as follows:
 $|A| \leq |B|$ if there is an injection $f : A \rightarrow B$



Ranking Cardinalities

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Ranking Cardinalities

- We define $|A| \leq |B|$ as follows:
 $|A| \leq |B|$ if there is an injection $f : A \rightarrow B$
- For any sets A , B , and C :
 - $|A| \leq |A|$.
 - Let $f : A \rightarrow A$ be $f(x) = x$.
 - If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.
 - The composition of two injections is an injection.
 - Either $|A| \leq |B|$ or $|B| \leq |A|$.
 - This one is harder and requires the ***axiom of choice***. We'll just take it on faith. ☺

Theorem (Cantor-Bernstein-Schroeder): If A and B are sets where $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

(This was first proven by Richard Dedekind.)

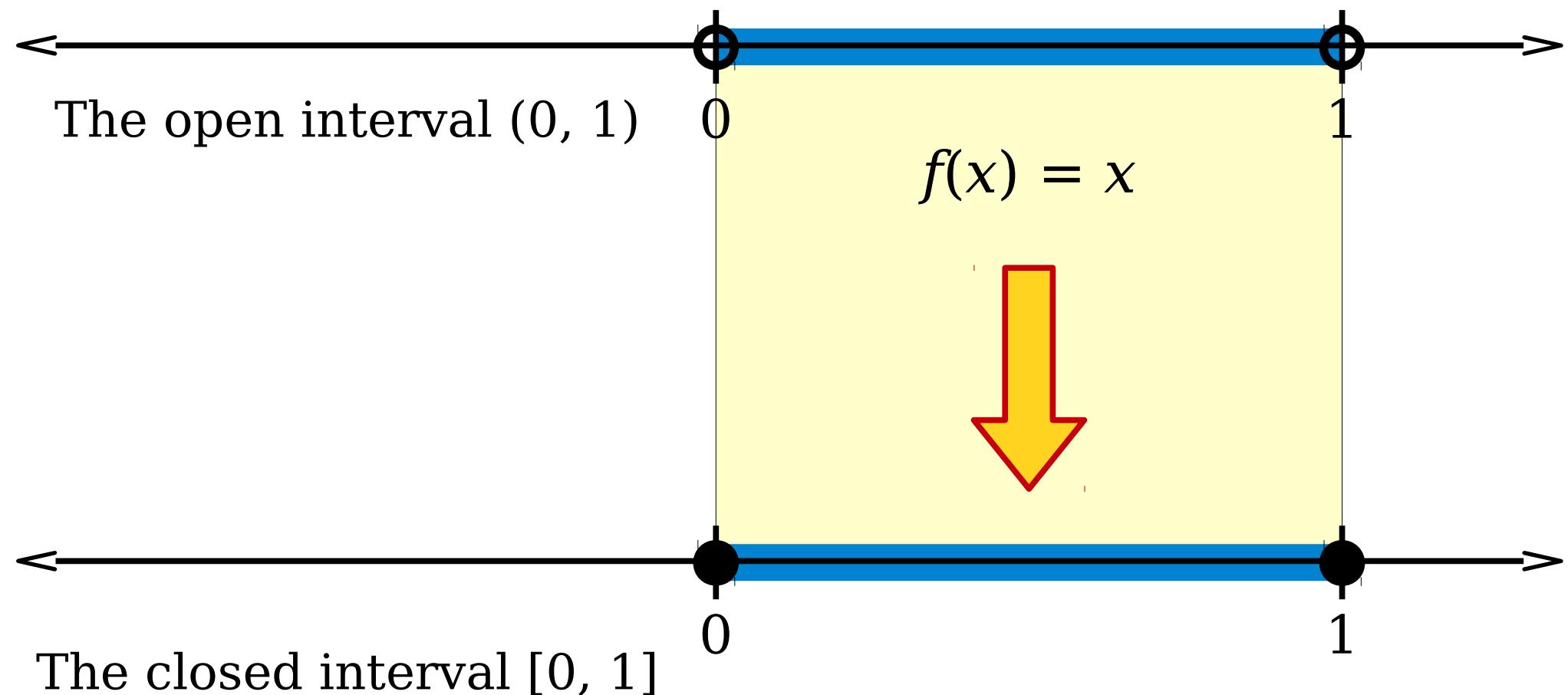
The CBS Theorem

- **Theorem:** If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.
- Isn't this, kinda, you know, obvious?
- Look at the definitions. What does the above theorem actually say?

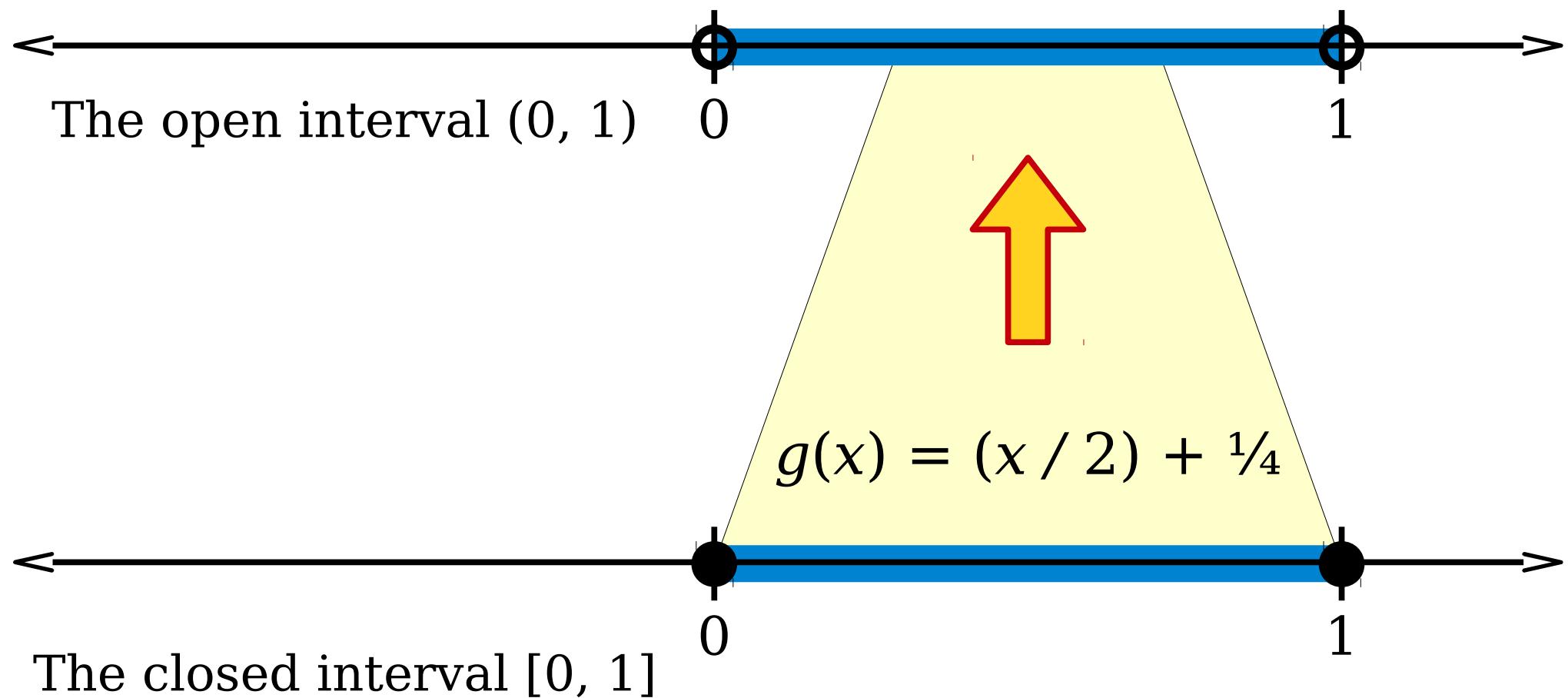
If there is an injection $f : A \rightarrow B$ and an injection $g : B \rightarrow A$, then there must be some bijection $h : A \rightarrow B$.

- This is much less obvious than it looks.

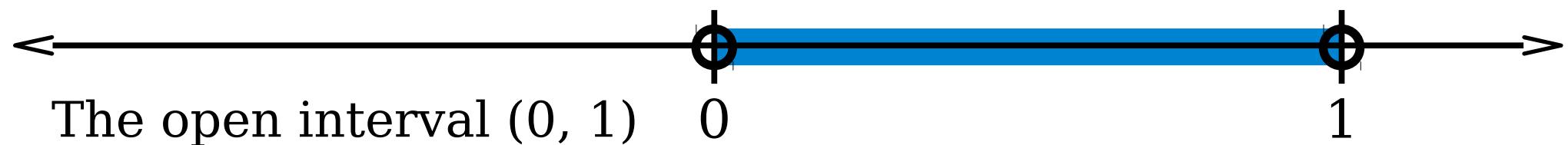
Why CBS is Tricky



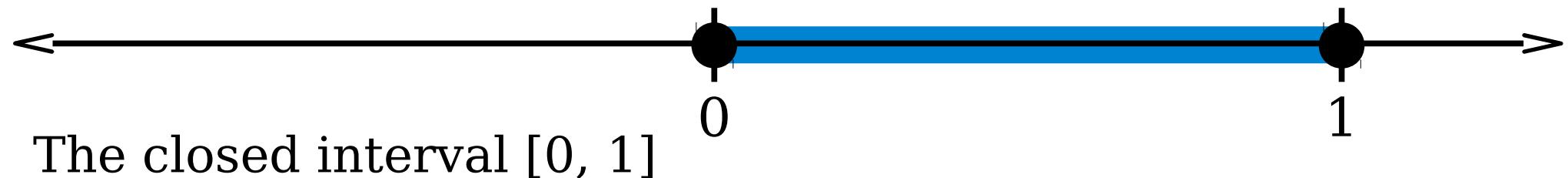
Why CBS is Tricky



Why CBS is Tricky



There has to be a
bijection between these
two sets... so what is it?



Proving CBS

- The proof of the CBS theorem is tricky but really quite beautiful.
- I've included an appendix to this slide deck that outlines the proof.
- Curious? Check it out!

An Application

The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

The Cartesian Product

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$$\begin{array}{c} \{0, 1, 2\} \\ A \end{array} \quad \begin{array}{c} \{a, b, c\} \\ B \end{array}$$

The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

$$\begin{matrix} \{0, 1, 2\} \\ A \end{matrix} \times \begin{matrix} \{a, b, c\} \\ B \end{matrix} =$$

The Cartesian Product

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$$\{0, 1, 2\} \times \{a, b, c\} =$$

A B

| a | b | c |
|-----|-----|-----|
| 0 | | |
| 1 | | |
| 2 | | |

The Cartesian Product

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$$\begin{matrix} \{0, 1, 2\} \times \{a, b, c\} \\ A \qquad \qquad \qquad B \end{matrix} =$$

| | <i>a</i> | <i>b</i> | <i>c</i> |
|---|----------------|----------------|----------------|
| 0 | (0, <i>a</i>) | (0, <i>b</i>) | (0, <i>c</i>) |
| 1 | (1, <i>a</i>) | (1, <i>b</i>) | (1, <i>c</i>) |
| 2 | (2, <i>a</i>) | (2, <i>b</i>) | (2, <i>c</i>) |

The Cartesian Product

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$$\left\{ 0, 1, 2 \right\}_A \times \left\{ a, b, c \right\}_B = \left\{ (0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \right\}$$

The Cartesian Product

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- We denote $A^2 = A \times A$

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The Cartesian Product

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$$\left\{ 0, 1, 2 \right\} \times \left\{ 0, 1, 2 \right\} = \left\{ \begin{array}{l} (0, 0), (0, 1), (0, 2), \\ (1, 0), (1, 1), (1, 2), \\ (2, 0), (2, 1), (2, 2) \end{array} \right\}$$

A A

The Cartesian Product

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$$\left\{ 0, 1, 2 \right\}^2 = \left\{ (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2) \right\}$$

A^2

The Cartesian Product

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- We denote $A^2 = A \times A$

If you've taken Math 51, this is where we get the notation \mathbb{R}^2 and \mathbb{R}^3 from!

$$\left\{ 0, 1, 2 \right\}^2 = \left\{ (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2) \right\}$$

A^2

What is $|\mathbb{N}^2|$?

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f: \mathbb{N} \rightarrow \mathbb{N}^2$

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f: \mathbb{N} \rightarrow \mathbb{N}^2$

| \mathbb{N} | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|
| | (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| | (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| | (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| \mathbb{N}^2 | (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| | (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| | (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| | (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f: \mathbb{N} \rightarrow \mathbb{N}^2$

| \mathbb{N} | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|
| | (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| | (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| | (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| \mathbb{N}^2 | (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| | (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| | (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| | (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f: \mathbb{N} \rightarrow \mathbb{N}^2$

$$f(n) = (0, n)$$

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f : \mathbb{N} \rightarrow \mathbb{N}^2$

$$f(n) = (0, n)$$

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$|\mathbb{N}| \leq |\mathbb{N}^2|$$

Find an injection $f : \mathbb{N} \rightarrow \mathbb{N}^2$

$$|\mathbb{N}^2| \leq |\mathbb{N}|$$

$$f(n) = (0, n)$$

| | | | | | | | | |
|--------------|---|---|---|---|---|---|---|-----|
| \mathbb{N} | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|--------------|---|---|---|---|---|---|---|-----|

| | | | | | | | | |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|
| \mathbb{N}^2 | (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| | (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| | (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| | (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| | (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| | (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| | (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$\begin{array}{lll} |\mathbb{N}| \leq |\mathbb{N}^2| & \text{Find an injection } f: \mathbb{N} \rightarrow \mathbb{N}^2 & \mathbf{f(n) = (0, n)} \\ |\mathbb{N}^2| \leq |\mathbb{N}| & \text{Find an injection } f: \mathbb{N}^2 \rightarrow \mathbb{N} & \end{array}$$

\mathbb{N}

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|

 \mathbb{N}^2

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|
| (0, 0) | (0, 1) | (0, 2) | (0, 3) | (0, 4) | (0, 5) | (0, 6) | (0, ...) |
| (1, 0) | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) | (1, ...) |
| (2, 0) | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) | (2, ...) |
| (3, 0) | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) | (3, ...) |
| (4, 0) | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) | (4, ...) |
| (5, 0) | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) | (5, ...) |
| (..., 0) | (..., 1) | (..., 2) | (..., 3) | (..., 4) | (..., 5) | (..., 6) | ... |

$$\begin{aligned}|\mathbb{N}| &\leq |\mathbb{N}^2| \\ |\mathbb{N}^2| &\leq |\mathbb{N}|\end{aligned}$$

$$\begin{aligned}\text{Find an injection } f : \mathbb{N} \rightarrow \mathbb{N}^2 \\ \text{Find an injection } f : \mathbb{N}^2 \rightarrow \mathbb{N}\end{aligned}$$

$$\begin{aligned}f(n) &= (0, n) \\ f(a, b) &= 2^a 3^b\end{aligned}$$

Counterintuitive result: $|\mathbb{N}| = |\mathbb{N}^2|$

Time-Out for Announcements!

Problem Set Three

- Problem Set Two was due at 3:00PM today. Feel free to use late days if you'd like!
- Problem Set Three goes out today.
 - The checkpoint is due on Monday of next week.
 - Remaining problems due on Friday.
- Explore properties of functions, cardinality, and equivalence relations!
- Two questions will require topics from next lecture. They're clearly marked as such.

Undergraduate Research Panel

- The CS Course Advisor is organizing an undergraduate research panel next **Tuesday, January 26 at 5:30PM** in Gates 219.
- Interested in getting involved in research? Curious about CURIS? Stop on by!
- Please RSVP using [this link](#) so they know how much food to get.

Your Questions

“How do you get started on a side project?”

It really depends on the project! The first step is figuring out what you want to make. I find it hard to just say “I want to make X” in isolation; there usually needs to be a good reason to make thing X.

For technical projects, getting the environment set up is often the first step. Keep an eye out for HackOverflow later this year as a great way to learn how to do this, or search online for tips on how to get set up. Alternatively, consider taking CS108!

Also, keep in mind that you’re not expected to be working on side projects all the time! You’ve got enough on your plate as is. Do what you think is going to be the most fun, not what you think other people want you to do.

“What are your long term goals?”

Professionally, there are a few areas I want to continue to focus on (making CS accessible and welcoming to everyone, making CS theory interesting and exciting to a broad audience, etc.), though the specific ways I'm working on those goals change and morph over time.

Back to CS103!

Unequal Cardinalities

- Recall: $|A| = |B|$ if the following statement is true:

There exists a bijection $f : A \rightarrow B$

- What does it mean for $|A| \neq |B|$ to be true?
Every function $f : A \rightarrow B$ is not a bijection.
- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from A to B can be injective and surjective.

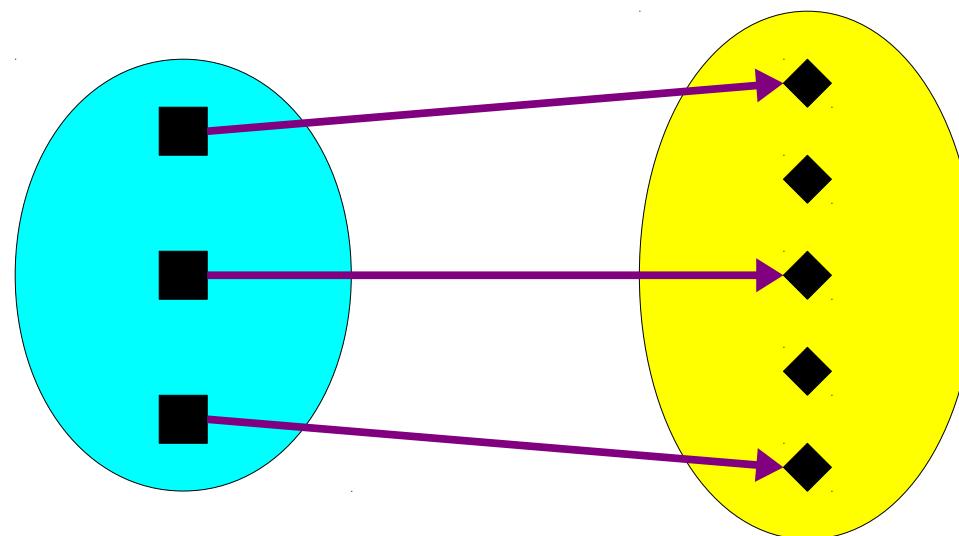
Comparing Cardinalities

- Formally, we define $<$ on cardinalities as

$$|A| < |B| \text{ if } |A| \leq |B| \text{ and } |A| \neq |B|$$

- In other words:

- There is an injection from A to B .
- There is no bijection between A and B .



Comparing Cardinalities

- Formally, we define $<$ on cardinalities as

$$|A| < |B| \text{ if } |A| \leq |B| \text{ and } |A| \neq |B|$$

- In other words:
 - There is an injection from A to B .
 - There is no bijection between A and B .
- **Theorem:** For any sets A and B , exactly one of the following is true:

$$|A| < |B| \quad |A| = |B| \quad |A| > |B|$$

Putting it Together: *Cantor's Theorem*

Cantor's Theorem

- **Cantor's Theorem** is the following:
If S is a set, then $|S| < |\wp(S)|$
- This is how we concluded that there are more problems to solve than programs to solve them.
- We informally sketched a proof of this in the first lecture.
- Let's now formally prove Cantor's Theorem.

The Key Step

- We need to show the following:
If S is a set, then $|S| \neq |\wp(S)|$.
- To do this, we need to prove this statement:
Given any set S and any function $f : S \rightarrow \wp(S)$, the function f is not a bijection.

x_0

x_1

x_2

x_3

x_4

x_5

\dots

$$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

| | | | | | | |
|-------|-------|-------|-------|-------|-------|---------|
| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|-------|-------|-------|-------|-------|-------|---------|

$$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

\dots

| | | | | | | |
|--------------|--------------|--------------|--------------|--------------|--------------|-----|
| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
| \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | ... |

$$x_0 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|----------|----------|----------|----------|----------|----------|-----|
| Y | N | Y | N | Y | N | ... |
| Y | N | N | Y | Y | N | ... |

$$x_0 \longleftrightarrow \{ x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_3 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|----------|----------|----------|----------|----------|----------|-----|
| Y | N | Y | N | Y | N | ... |

$x_0 \longleftrightarrow \{$

$x_1 \longleftrightarrow \{$

$x_2 \longleftrightarrow \{ \quad \quad \quad x_4, \quad \quad \quad \dots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$

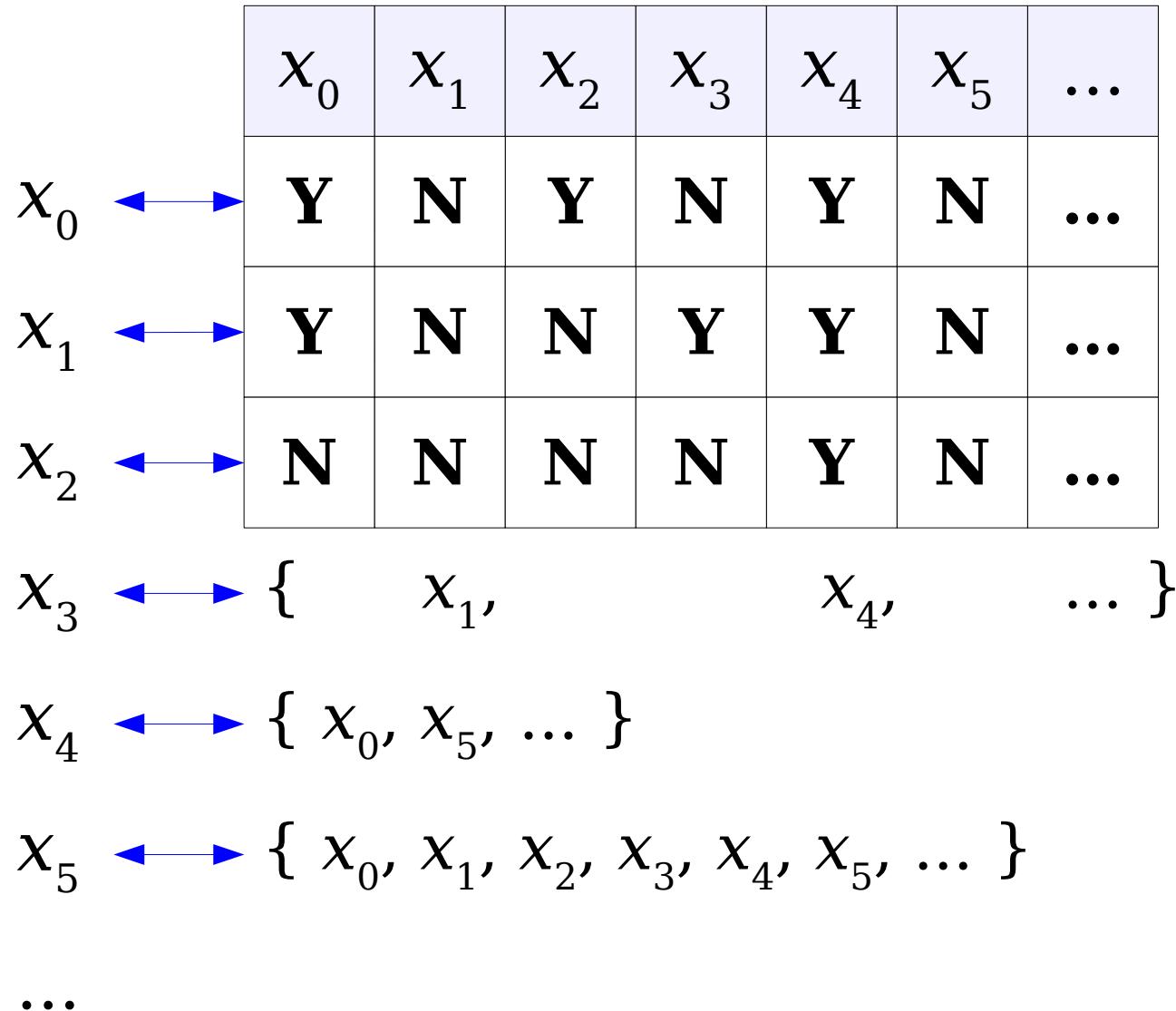
$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

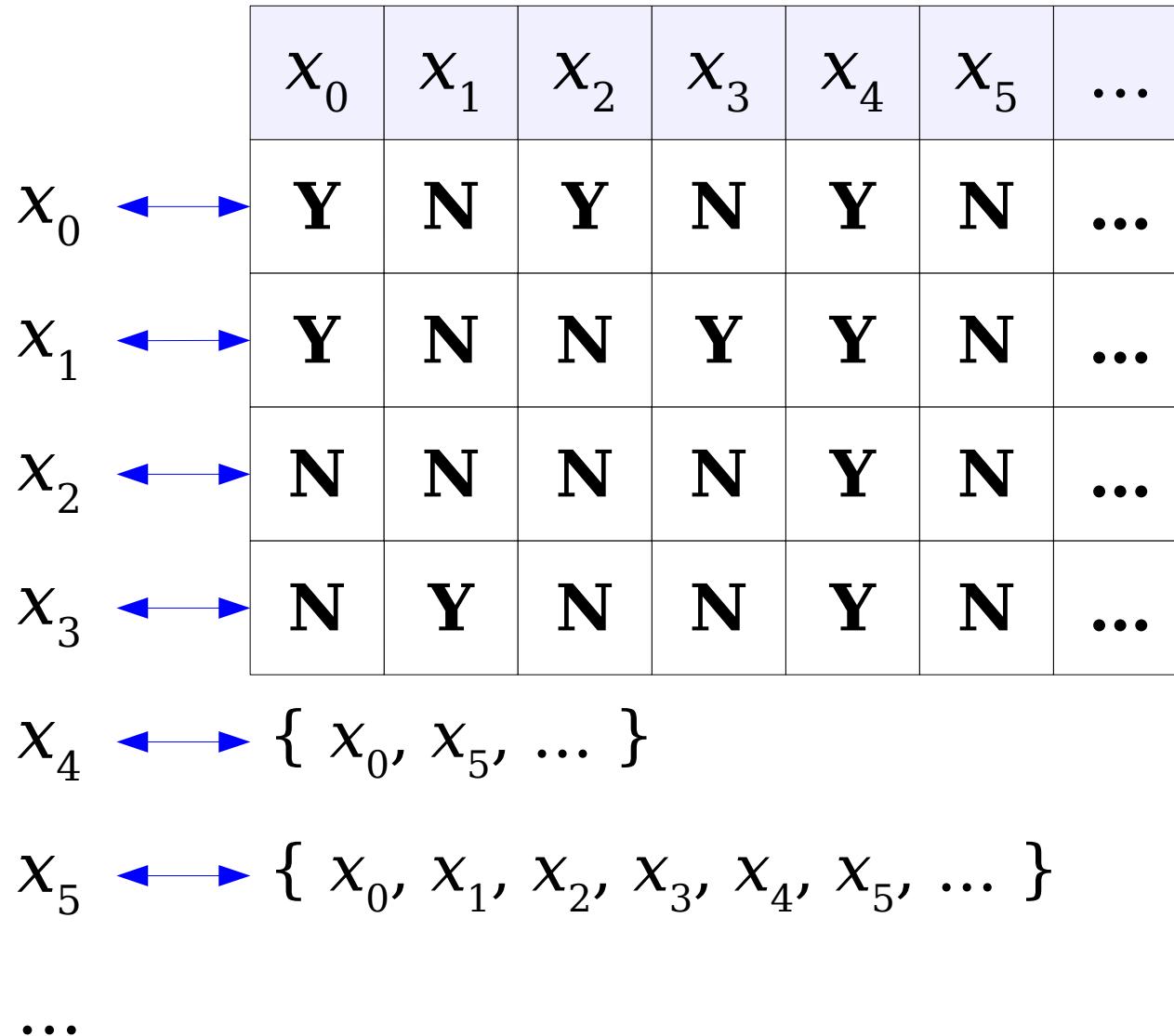
$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

...

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|--------------|---|--------------|--------------|--------------|--------------|--------------|
| \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | ... |
| x_1 | \mathbf{Y} | \mathbf{N} | \mathbf{N} | \mathbf{Y} | \mathbf{Y} | \mathbf{N} |
| x_2 | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{Y} | \mathbf{N} |
| x_3 | $\{ x_1, x_4, \dots \}$ | | | | | |
| x_4 | $\{ x_0, x_5, \dots \}$ | | | | | |
| x_5 | $\{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$ | | | | | |
| ... | | | | | | |

$x_0 \longleftrightarrow \{ x_1, x_4, \dots \}$
 $x_1 \longleftrightarrow \{ x_0, x_5, \dots \}$
 $x_2 \longleftrightarrow \{ x_1, x_4, \dots \}$
 $x_3 \longleftrightarrow \{ x_0, x_5, \dots \}$
 $x_4 \longleftrightarrow \{ x_1, x_2, x_3, x_4, x_5, \dots \}$
 \dots





| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|--------------|---|--------------|--------------|--------------|--------------|--------------|
| \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | \mathbf{Y} | \mathbf{N} | ... |
| x_1 | \mathbf{Y} | \mathbf{N} | \mathbf{N} | \mathbf{Y} | \mathbf{Y} | \mathbf{N} |
| x_2 | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{Y} | \mathbf{N} |
| x_3 | \mathbf{N} | \mathbf{Y} | \mathbf{N} | \mathbf{N} | \mathbf{Y} | \mathbf{N} |
| x_4 | \mathbf{Y} | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{N} | \mathbf{Y} |
| x_5 | $\{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$ | | | | | |
| ... | | | | | | |

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... | | |
|-------|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | ↔ | Y | N | Y | N | Y | N | ... |
| x_1 | ↔ | Y | N | N | Y | Y | N | ... |
| x_2 | ↔ | N | N | N | N | Y | N | ... |
| x_3 | ↔ | N | Y | N | N | Y | N | ... |
| x_4 | ↔ | Y | N | N | N | N | Y | ... |
| x_5 | ↔ | Y | Y | Y | Y | Y | Y | ... |
| ... | | | | | | | | |

| x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... | | |
|-------|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | ↔ | Y | N | Y | N | Y | N | ... |
| x_1 | ↔ | Y | N | N | Y | Y | N | ... |
| x_2 | ↔ | N | N | N | N | Y | N | ... |
| x_3 | ↔ | N | Y | N | N | Y | N | ... |
| x_4 | ↔ | Y | N | N | N | N | Y | ... |
| x_5 | ↔ | Y | Y | Y | Y | Y | Y | ... |
| ... | | ... | ... | ... | ... | ... | ... | |

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

$\left\{ \begin{array}{ccccccc} x_0, & & & & & x_5, & \dots \end{array} \right\}$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

Flip all Y's to N's and vice-versa to get a new set

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

x_1' , x_2' , x_3' , x_4' , ...

Flip all Y's to N's and vice-versa to get a new set

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

Which row in the table is paired with this set?

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

Which row in the table is paired with this set?

| | | | | | | | |
|----------|---|---|---|---|---|---|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|---|---|---|---|---|---|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

Which row in the table is paired with this set?

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

Which row in the table is paired with this set?

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

Which row in the table is paired with this set?

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

Which row in the table is paired with this set?

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

Which row in the table is paired with this set?

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

Which row in the table is paired with this set?

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|----------|----------|----------|----------|----------|----------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

What set is this?

| | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|----------|----------|----------|----------|----------|----------|----------|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | |
|---|---|---|---|---|---|-----|
| N | Y | Y | Y | Y | N | ... |
|---|---|---|---|---|---|-----|

What set is this?

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|----------|----------|----------|----------|----------|----------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots | ... | ... | ... | ... | ... | ... | ... |

| | | | | | | |
|----------|----------|----------|----------|----------|----------|---------|
| N | Y | Y | Y | Y | N | \dots |
|----------|----------|----------|----------|----------|----------|---------|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|----------|----------|----------|----------|----------|----------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots | ... | ... | ... | ... | ... | ... | ... |

$f(x_0)$

| | | | | | | |
|----------|----------|----------|----------|----------|----------|---------|
| N | Y | Y | Y | Y | N | \dots |
|----------|----------|----------|----------|----------|----------|---------|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|----------|----------|----------|----------|----------|----------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots | \dots | \dots | \dots | \dots | \dots | \dots | \dots |

A blue arrow points from the cell containing 'Y' at (x_0, x_0) to the cell containing 'N' at (x_0, x_1) . A blue curved arrow originates from the cell containing 'Y' at (x_0, x_0) and points towards the bottom right corner of the grid.

$x_0 \in f(x_0)?$

Below the main grid, there is a horizontal row of seven cells:

| | | | | | | |
|----------|----------|----------|----------|----------|----------|---------|
| N | Y | Y | Y | Y | N | \dots |
|----------|----------|----------|----------|----------|----------|---------|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots | | | | | | | |
|---|---------|---------|---------|---------|---------|---------|---------|---|---|---|---|---|---|-----|
| x_0 | Y | N | Y | N | Y | N | \dots | | | | | | | |
| x_1 | Y | N | N | Y | Y | N | \dots | | | | | | | |
| x_2 | N | N | N | N | Y | N | \dots | | | | | | | |
| x_3 | N | Y | N | N | Y | N | \dots | | | | | | | |
| x_4 | Y | N | N | N | N | Y | \dots | | | | | | | |
| x_5 | Y | Y | Y | Y | Y | Y | \dots | | | | | | | |
| \dots | \dots | \dots | \dots | \dots | \dots | \dots | \dots | | | | | | | |
|  <table border="1"> <tr> <td>N</td> <td>Y</td> <td>Y</td> <td>Y</td> <td>Y</td> <td>N</td> <td>...</td> </tr> </table> | | | | | | | | N | Y | Y | Y | Y | N | ... |
| N | Y | Y | Y | Y | N | ... | | | | | | | | |

$x_0 \notin f(x_0) ?$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|---------|---------|---------|---------|---------|---------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots |

$f(x_1)$

| | | | | | | |
|---|---|---|---|---|---|---------|
| N | Y | Y | Y | Y | N | \dots |
|---|---|---|---|---|---|---------|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|-------|-------|-------|-------|-------|-------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots | ... | ... | ... | ... | ... | ... | ... |

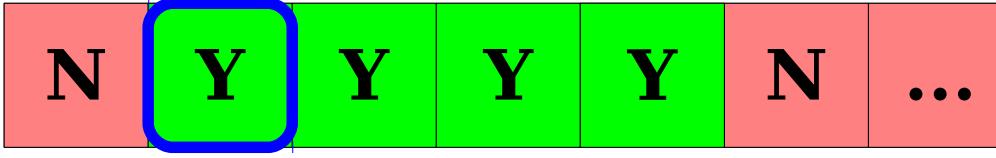
$x_1 \in f(x_1)?$

The matrix is a 7x8 grid. Rows are labeled x_0 through x_5 , and columns are labeled x_0 through x_5 . The values in the grid are as follows:

- Row x_0 : Y, N, Y, N, Y, N, ...
- Row x_1 : Y, N, N, Y, Y, N, ... (The cell (x1, x1) is highlighted with a green box)
- Row x_2 : N, N, N, N, Y, N, ...
- Row x_3 : N, Y, N, N, Y, N, ...
- Row x_4 : Y, N, N, N, N, Y, ...
- Row x_5 : Y, Y, Y, Y, Y, Y, ...
- Row \dots : ..., ..., ..., ..., ..., ..., ...

A blue arrow points from the question $x_1 \in f(x_1)?$ to the cell (x1, x1). The cell (x1, x1) is highlighted with a green box.

The bottom row of the matrix is shown separately as a horizontal sequence: N, Y, Y, Y, Y, N, ...

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|--|---------|---------|---------|---------|---------|---------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots | \dots | \dots | \dots | \dots | \dots | \dots | \dots |
|  | | | | | | | |
| N Y Y Y Y N ... | | | | | | | |

$x_1 \notin f(x_1) ?$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | \dots |
|---------|---------|---------|---------|---------|---------|---------|---------|
| x_0 | Y | N | Y | N | Y | N | \dots |
| x_1 | Y | N | N | Y | Y | N | \dots |
| x_2 | N | N | N | N | Y | N | \dots |
| x_3 | N | Y | N | N | Y | N | \dots |
| x_4 | Y | N | N | N | N | Y | \dots |
| x_5 | Y | Y | Y | Y | Y | Y | \dots |
| \dots |

$f(x_2)$

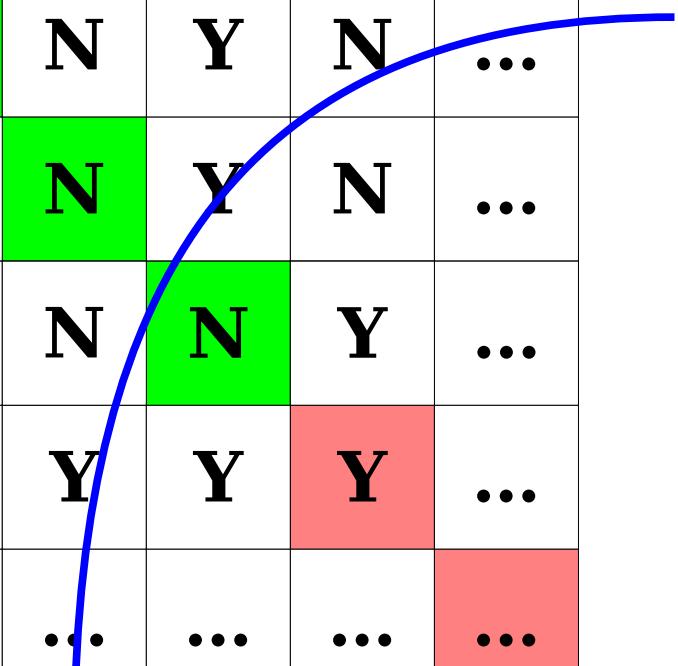
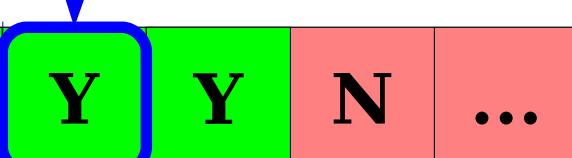
N **Y** **Y** **Y** **Y** **N** \dots

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|-------|-------|-------|-------|-------|-------|-------------------|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | Y | N | ... | $x_2 \in f(x_2)?$ |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

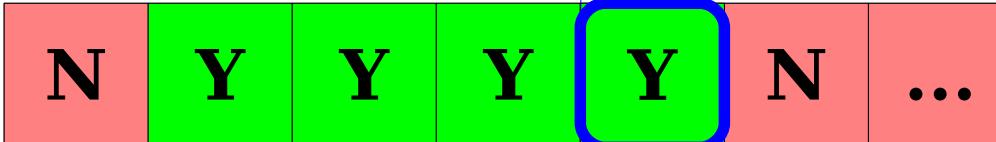
| | | | | | | |
|---|---|---|---|---|---|-----|
| N | Y | Y | Y | Y | N | ... |
|---|---|---|---|---|---|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|--|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |
|  | | | | | | | |
| N Y Y Y Y N ... | | | | | | | |

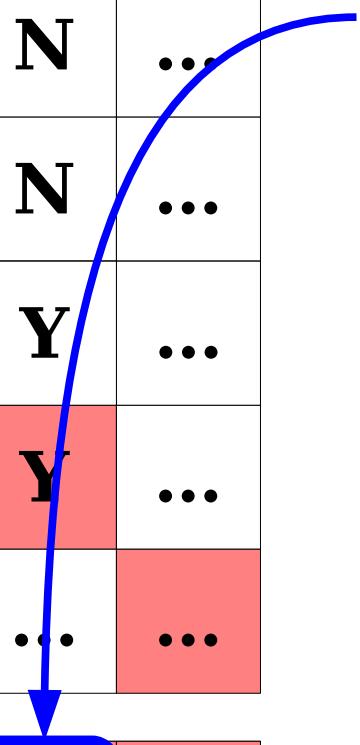
$x_2 \notin f(x_2) ?$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|--|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |
|  | | | | | | | |
|  | | | | | | | |
| N Y Y Y Y N ... | | | | | | | |

$x_3 \notin f(x_3) ?$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|--|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |
|  | | | | | | | |

$x_4 \notin f(x_4) ?$

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... | | | | | | | |
|--|-------|-------|-------|-------|-------|-------|-----|---|---|---|---|---|---|-----|
| x_0 | Y | N | Y | N | Y | N | ... | | | | | | | |
| x_1 | Y | N | N | Y | Y | N | ... | | | | | | | |
| x_2 | N | N | N | N | Y | N | ... | | | | | | | |
| x_3 | N | Y | N | N | Y | N | ... | | | | | | | |
| x_4 | Y | N | N | N | N | Y | ... | | | | | | | |
| x_5 | Y | Y | Y | Y | Y | Y | ... | | | | | | | |
| ... | ... | ... | ... | ... | ... | ... | ... | | | | | | | |
|  | | | | | | | | | | | | | | |
| <table border="1"> <tr> <td>N</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td>N</td><td>...</td></tr> </table> | | | | | | | | N | Y | Y | Y | Y | N | ... |
| N | Y | Y | Y | Y | N | ... | | | | | | | | |

$x_5 \notin f(x_5) ?$

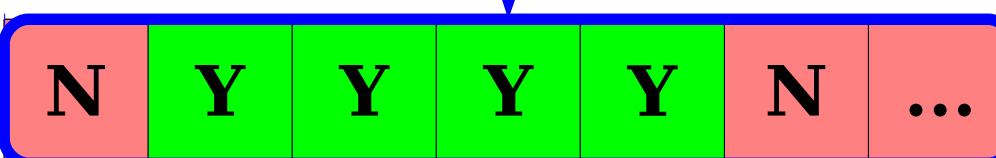
| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

$x \notin f(x) ?$

| | | | | | | | |
|---|---|---|---|---|---|---|-----|
| N | Y | Y | Y | Y | Y | N | ... |
|---|---|---|---|---|---|---|-----|

| | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | ... |
|-------|-------|-------|-------|-------|-------|-------|-----|
| x_0 | Y | N | Y | N | Y | N | ... |
| x_1 | Y | N | N | Y | Y | N | ... |
| x_2 | N | N | N | N | Y | N | ... |
| x_3 | N | Y | N | N | Y | N | ... |
| x_4 | Y | N | N | N | N | Y | ... |
| x_5 | Y | Y | Y | Y | Y | Y | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

$\{ x \in S \mid x \notin f(x) \}$



The Diagonal Set

- Let $f: S \rightarrow \wp(S)$ be an arbitrary function from S to $\wp(S)$.
- Define the set D as follows:

$$D = \{ x \in S \mid x \notin f(x) \}$$

(“The set of all elements x where x is not an element of the set $f(x)$.”)

- This is a formalization of the set we found in the previous picture.
- Using this choice of D , we can formally prove that no function $f: S \rightarrow \wp(S)$ is a bijection.

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Suppose for the sake of contradiction that there is a bijective function $f : S \rightarrow \wp(S)$. Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

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$$y \in D \text{ iff } y \notin f(y). \quad (2)$$

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The Diagonal Argument

- ***This proof is tricky.*** It's one of the hardest proofs we're going to encounter over the course of this quarter.
- To help you wrap your head around how it works, we've asked you a few questions about it on Problem Set Three.
- Don't panic if you don't get it immediately; you'll get a really good understanding of how it works if you play around with it.

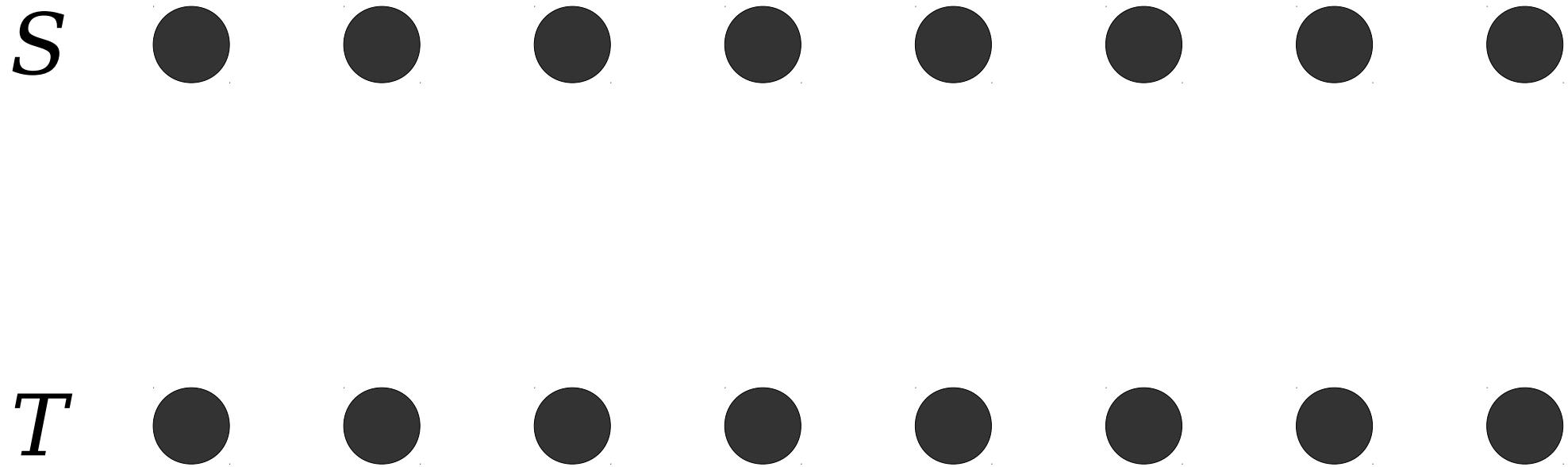
Why All This Matters

- Diagonal arguments come up all the time in computer science and formal logic.
- Some examples:
 - Later in the quarter, we'll use diagonalization to find specific examples of problems that cannot be solved by computers.
 - In complexity theory, diagonal arguments give rise to the **time hierarchy theorem**, which proves that certain problems are fundamentally hard to solve.
 - In formal logic, **Gödel's incompleteness theorem** uses diagonalization to establish limits on what formal logic can tell us.
- Want to learn more? Take Phil 152, CS154, and CS254!

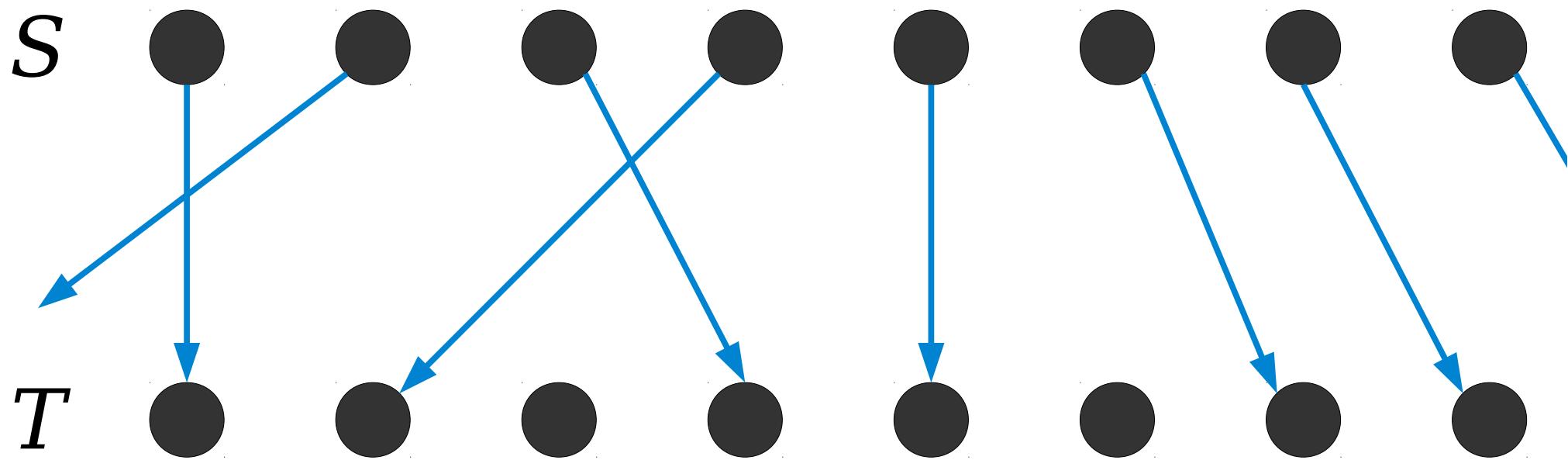
Appendix: *Proving CBS*

Proving CBS, Intuitively

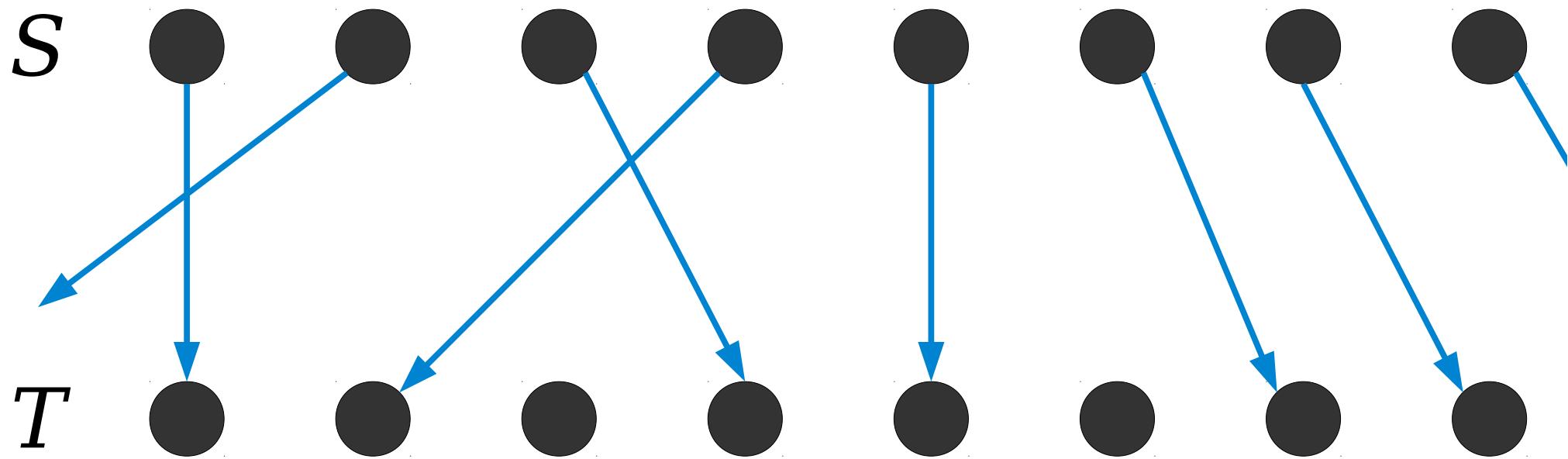
Proving CBS, Intuitively



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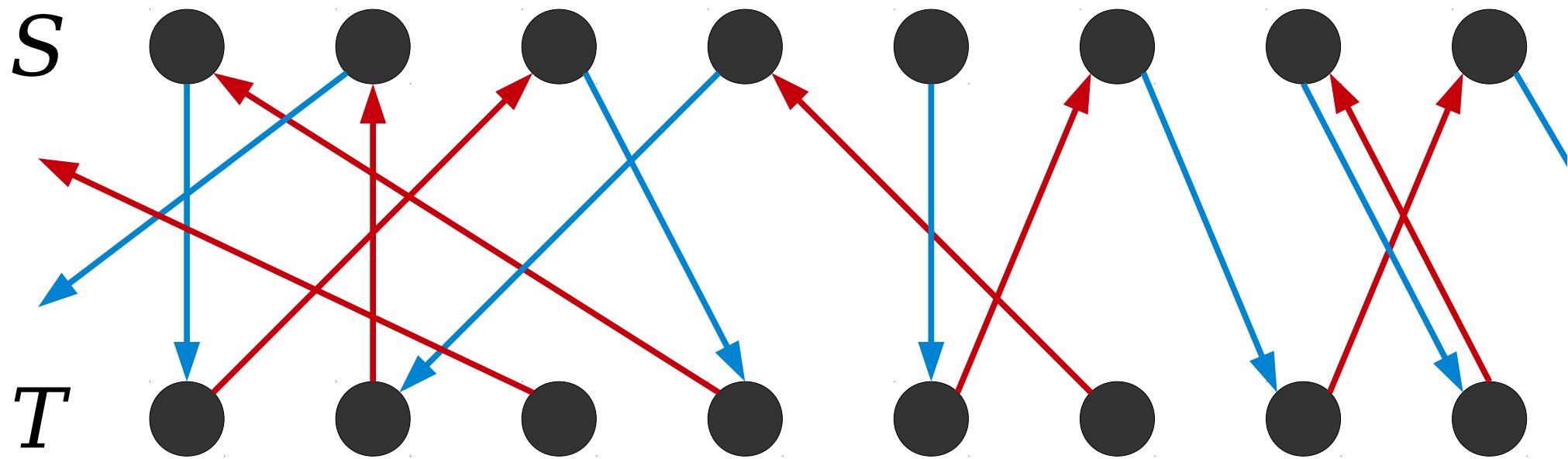


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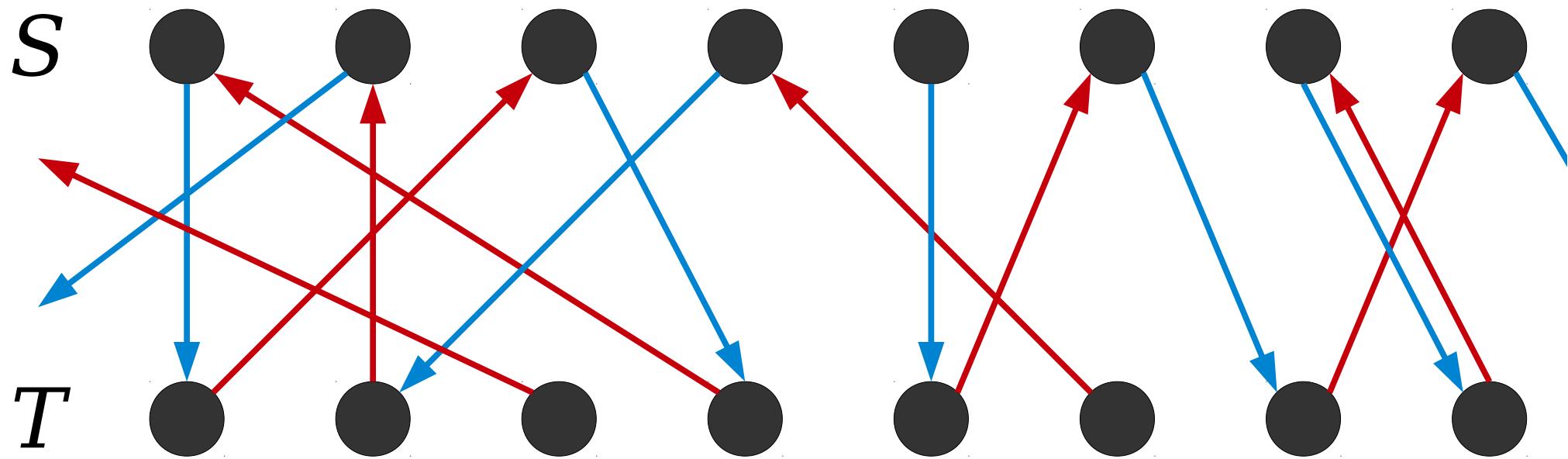
Blue lines represent the injection $f : S \rightarrow T$

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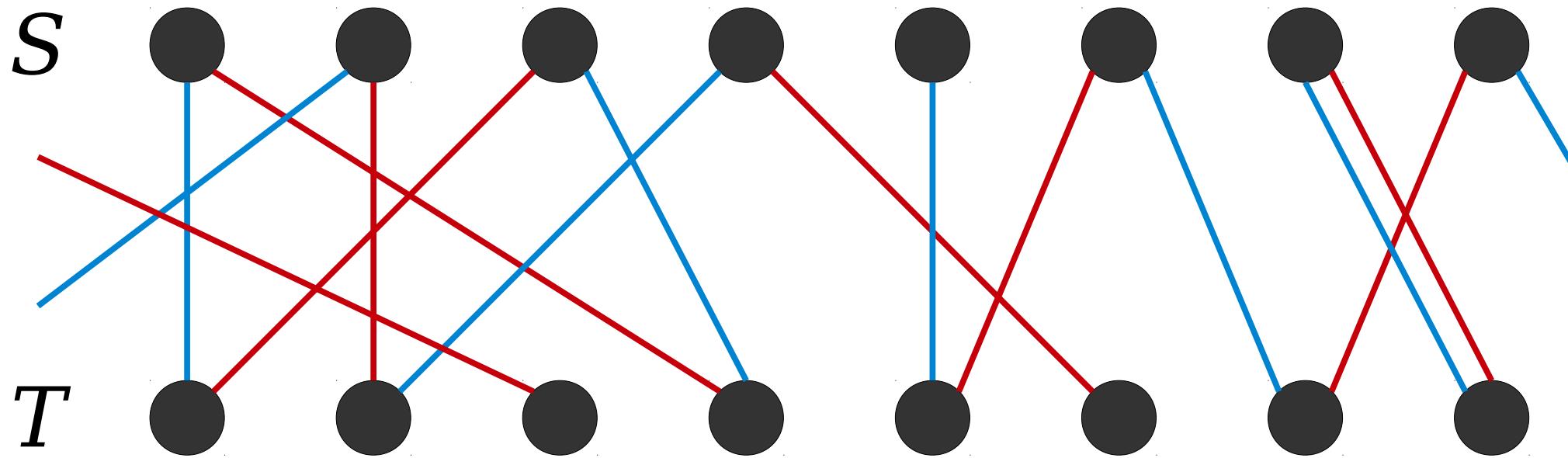
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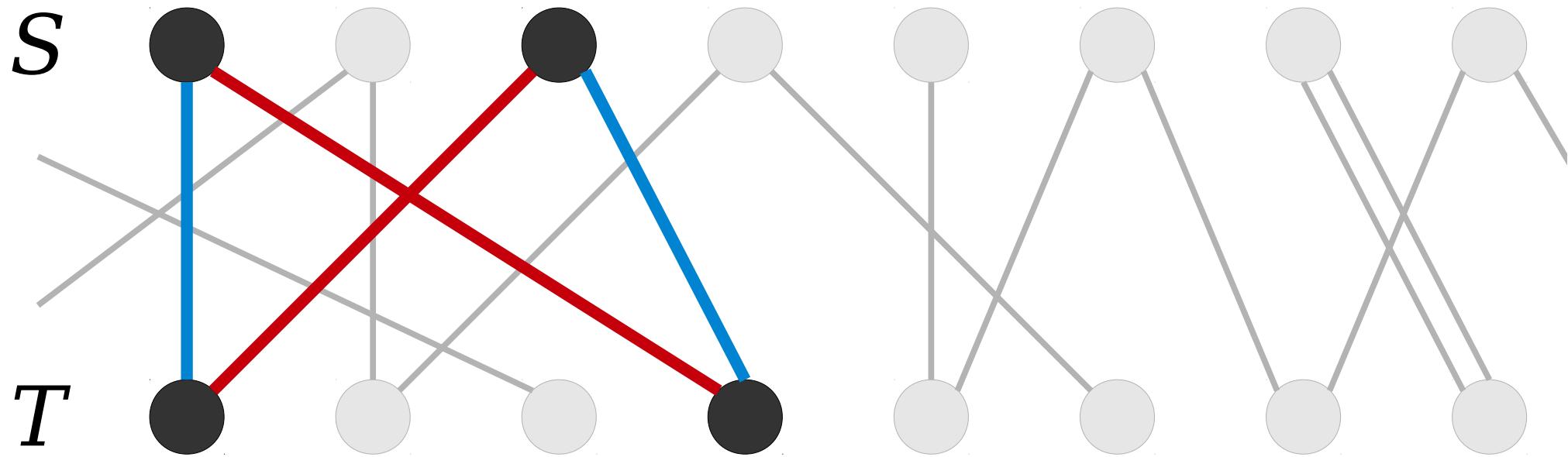
Blue lines represent the injection $f : S \rightarrow T$
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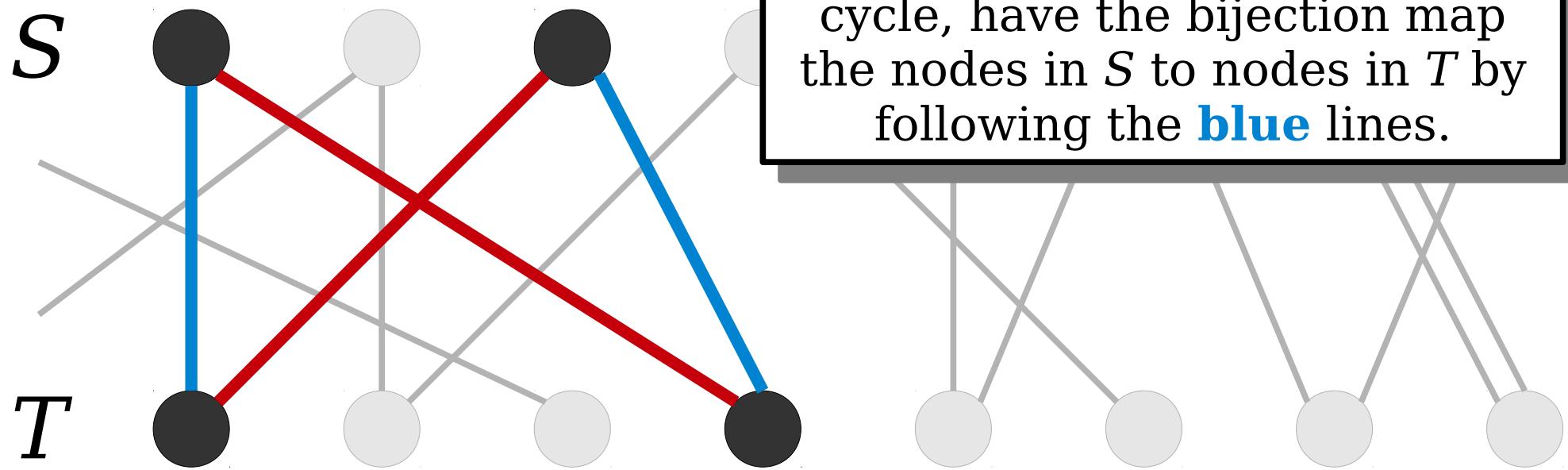
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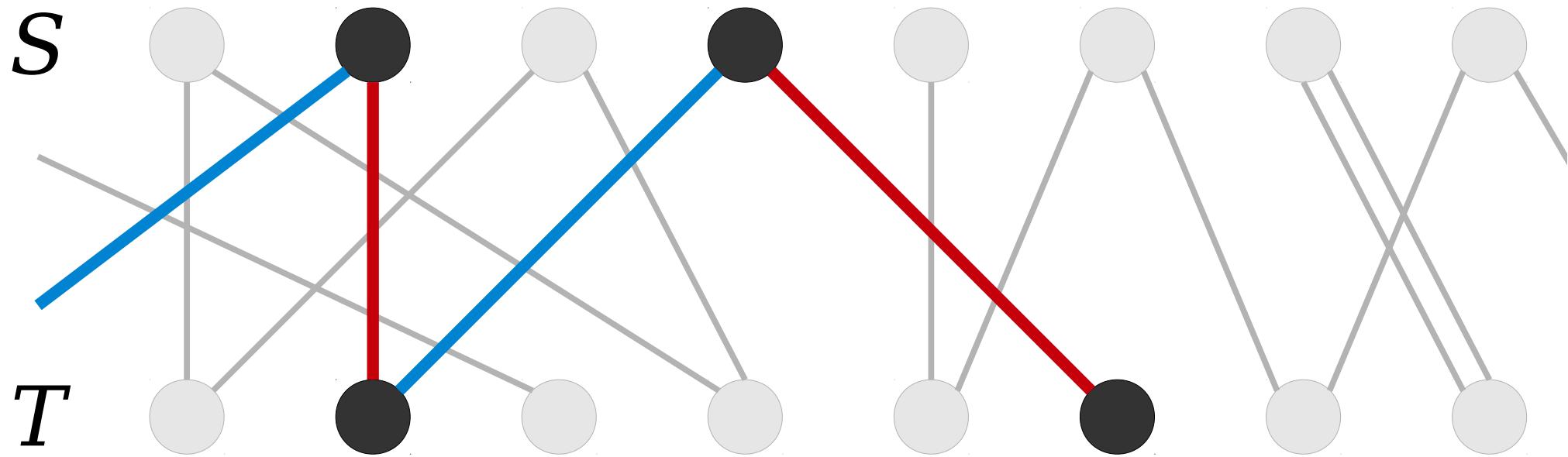
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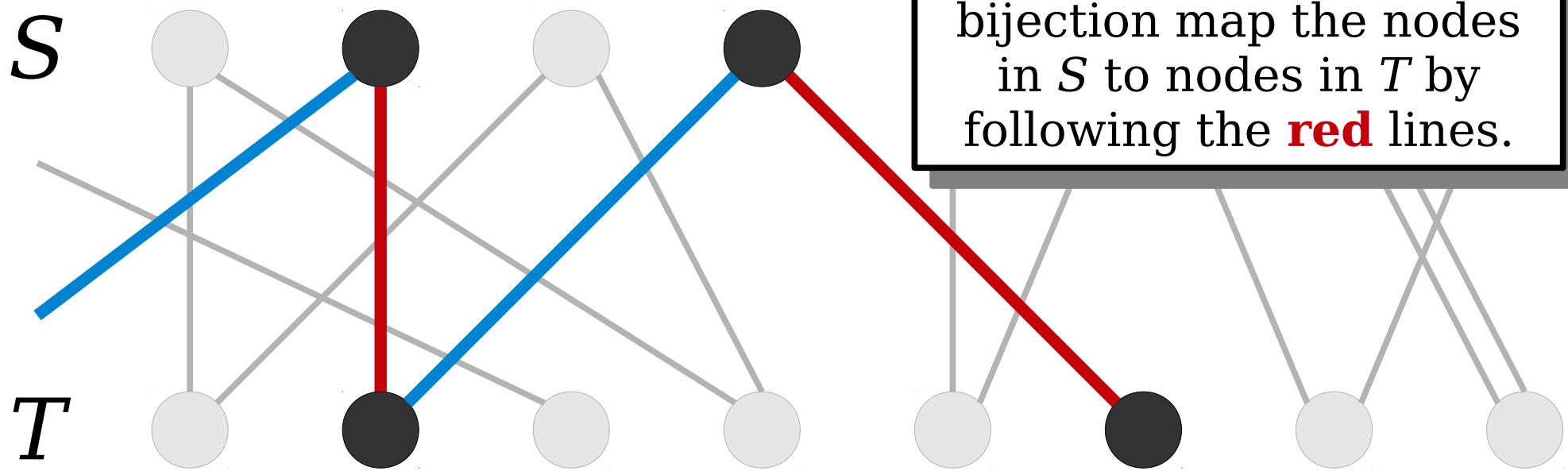
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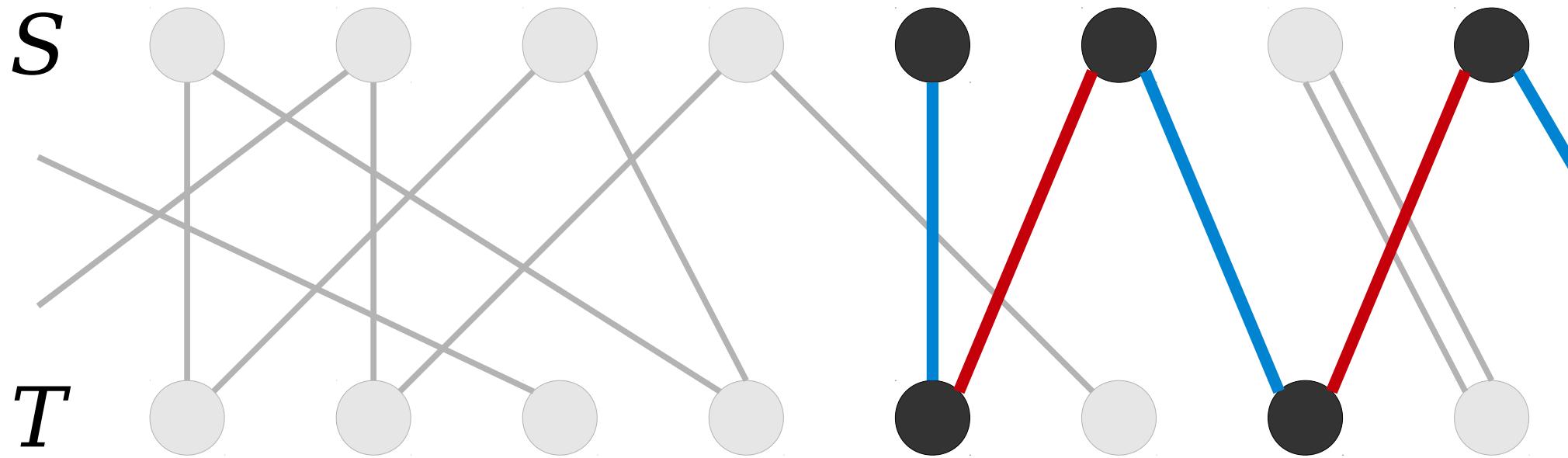
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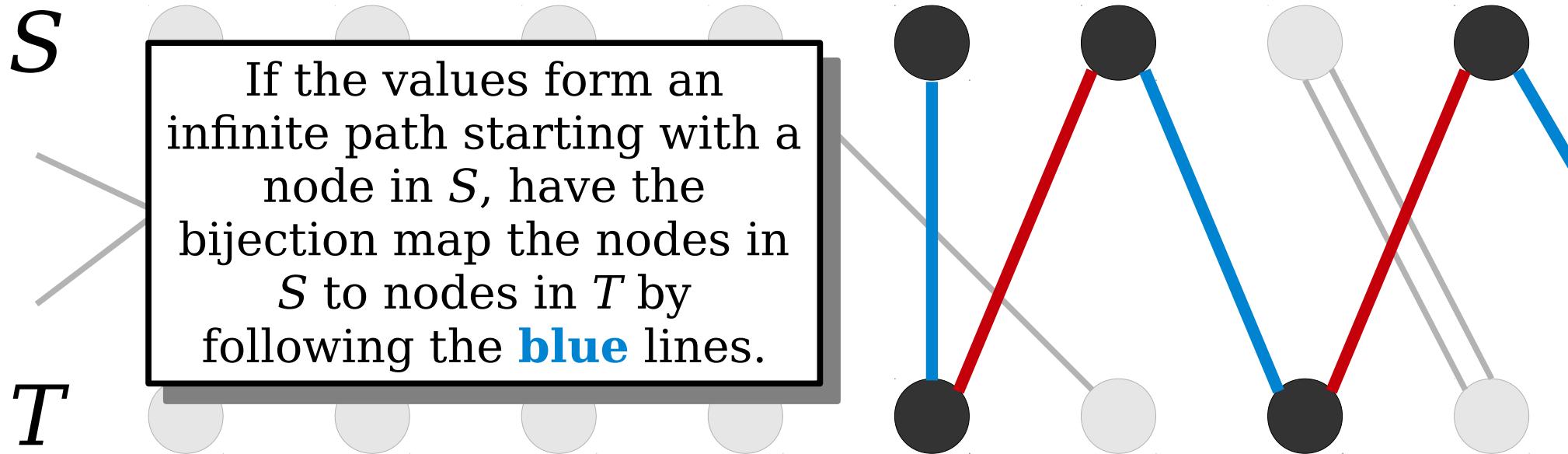
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Why This Matters

- Don't worry too much about the specifics of this proof. Think of it as more of a “math symphony.” ☺
- ***Fun Challenge Problem:*** Find an explicit bijection $f: [0, 1] \rightarrow (0, 1)$.