Welcome to CS103!

- **Three Handouts**
  - Course Information
  - Syllabus
  - Mathematical Prerequisites
  - (Also available online if you'd like!)
- **Today:**
  - Course Overview
  - Introduction to Set Theory
  - The Limits of Computation

Are there “laws of physics”
in computer science?

Key Questions in CS103

- What problems can you solve with a computer?
  - ***Computability Theory***
- Why are some problems harder to solve than others?
  - ***Complexity Theory***
- How can we be certain in our answers to these questions?
  - ***Discrete Mathematics***

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Course Website

http://cs103.stanford.edu

Prerequisite

CS106A

“Prerequisite”

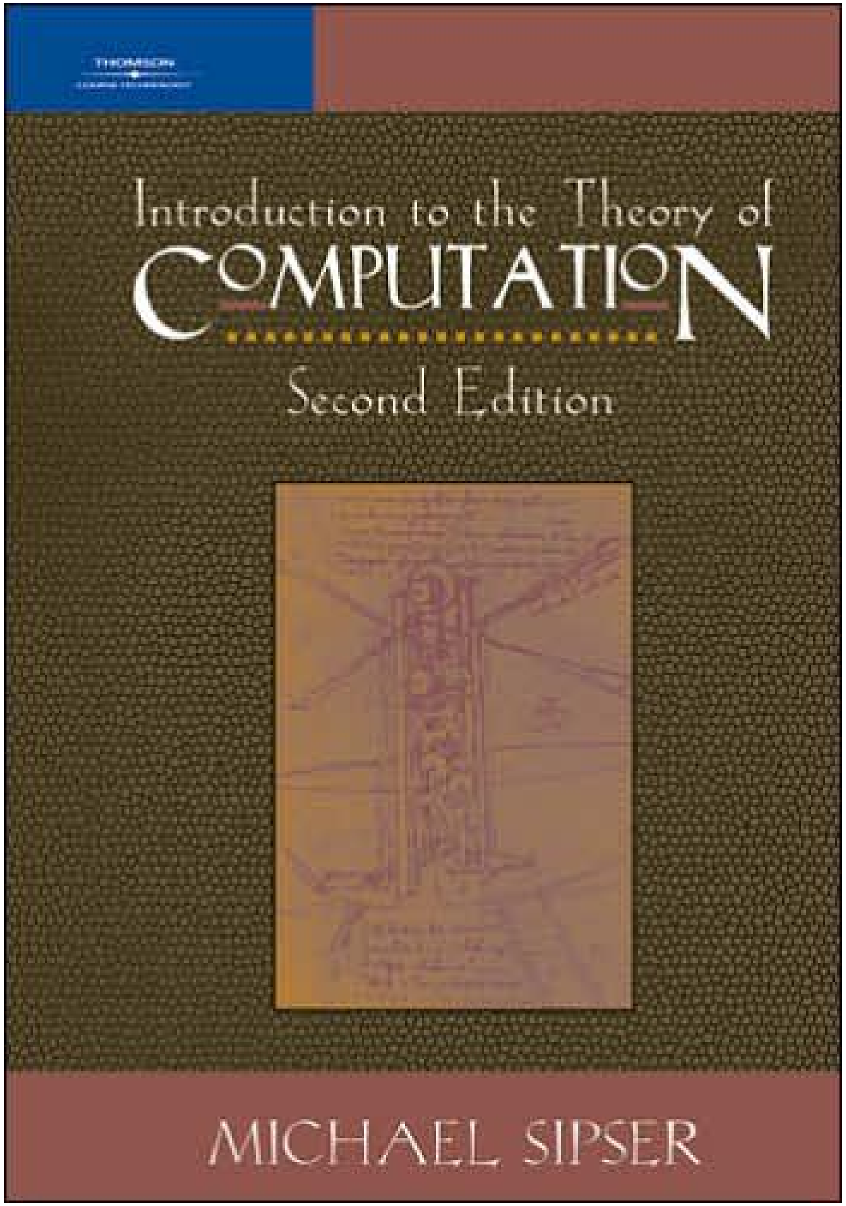
CS106A

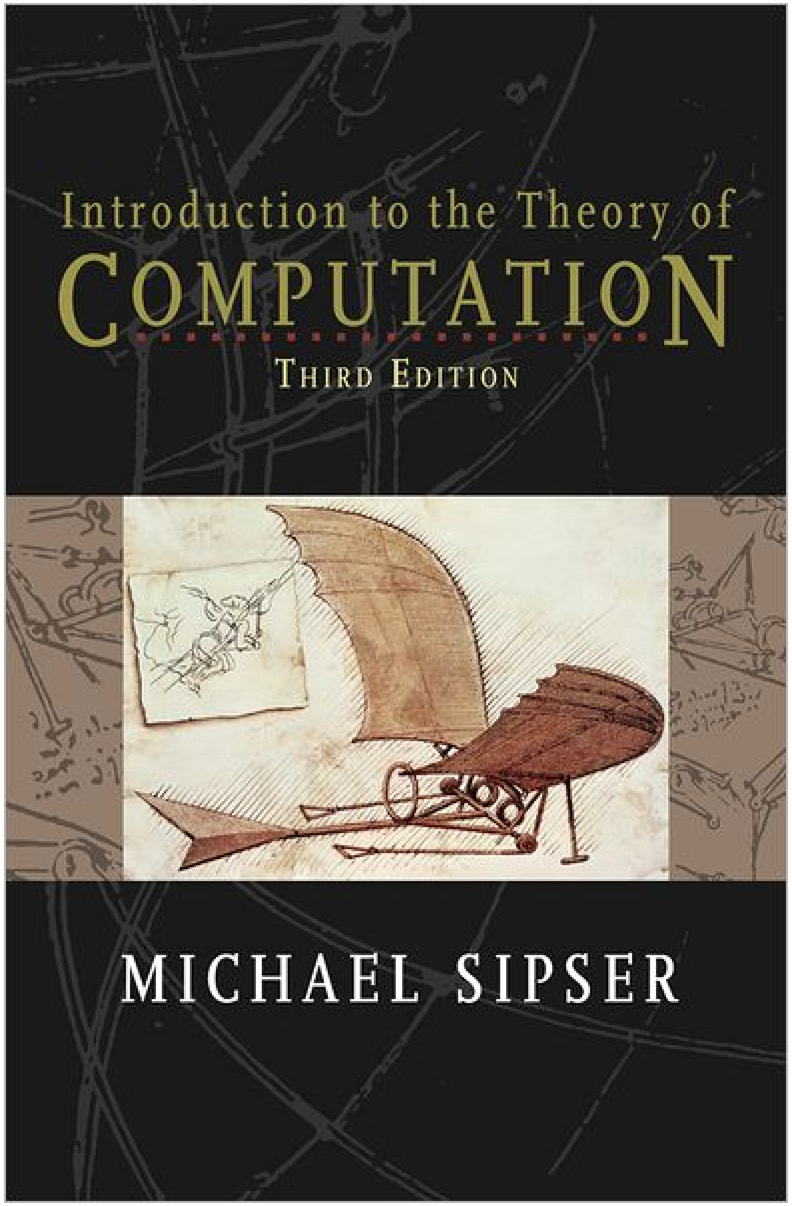
"Prerequisite"

CS106A

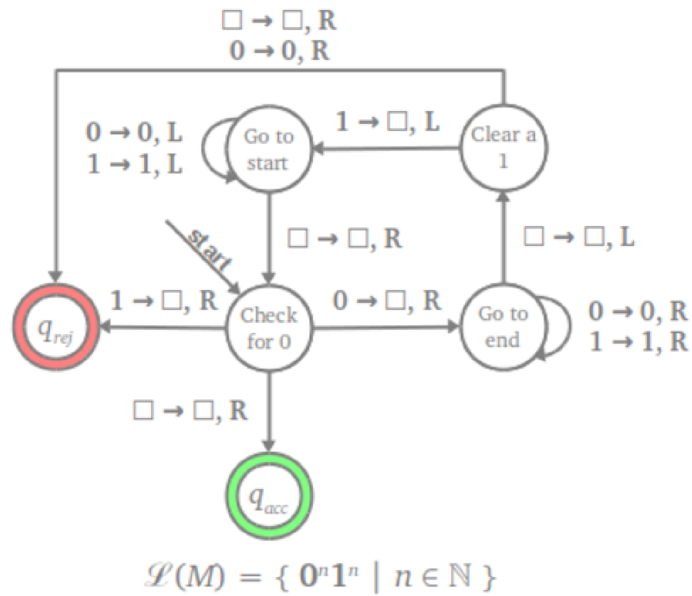
There aren't any math prerequisites for this course – high-school algebra should be enough!

Recommended Reading





Online Course Notes



CS103

Mathematical Foundations of Computing

Announcements

Welcome to CS103!

January 1, 2016

Welcome to CS103, an introduction to discrete mathematics, computability theory, and complexity theory! We have an great quarter ahead of us filled with interesting and exciting results in the power and limits of computation, and I hope that you're able to join us.

Handouts

02: Math Preqreqs
01: Syllabus
00: Course Information

Assignments

Coming soon!

Resources

Course Reader
Lecture Vid...

Lectures

Coming soon!

Grading

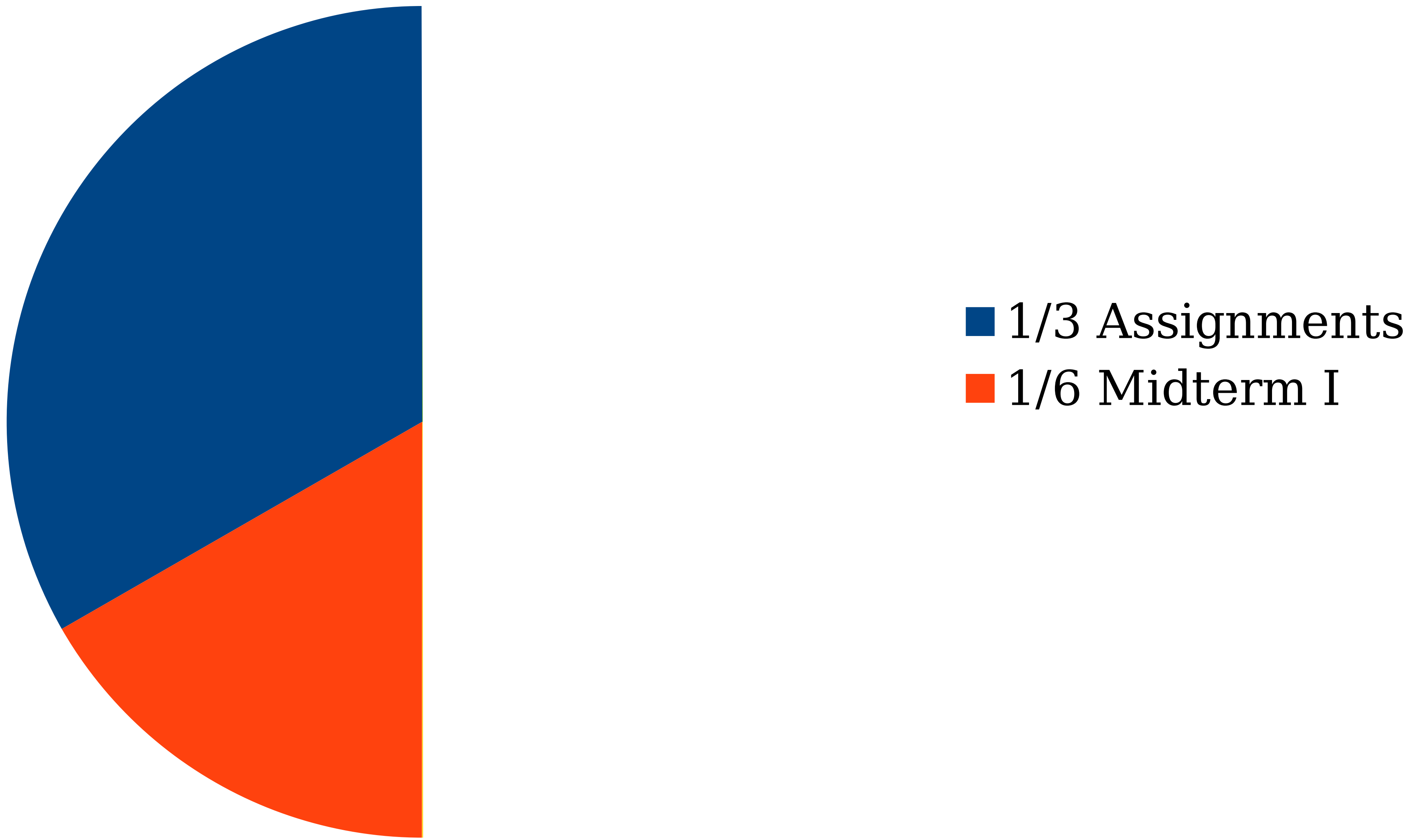
Grading

Grading

Nine Problem Sets

Problem sets may be done individually, in pairs, or in groups of three.

Grading

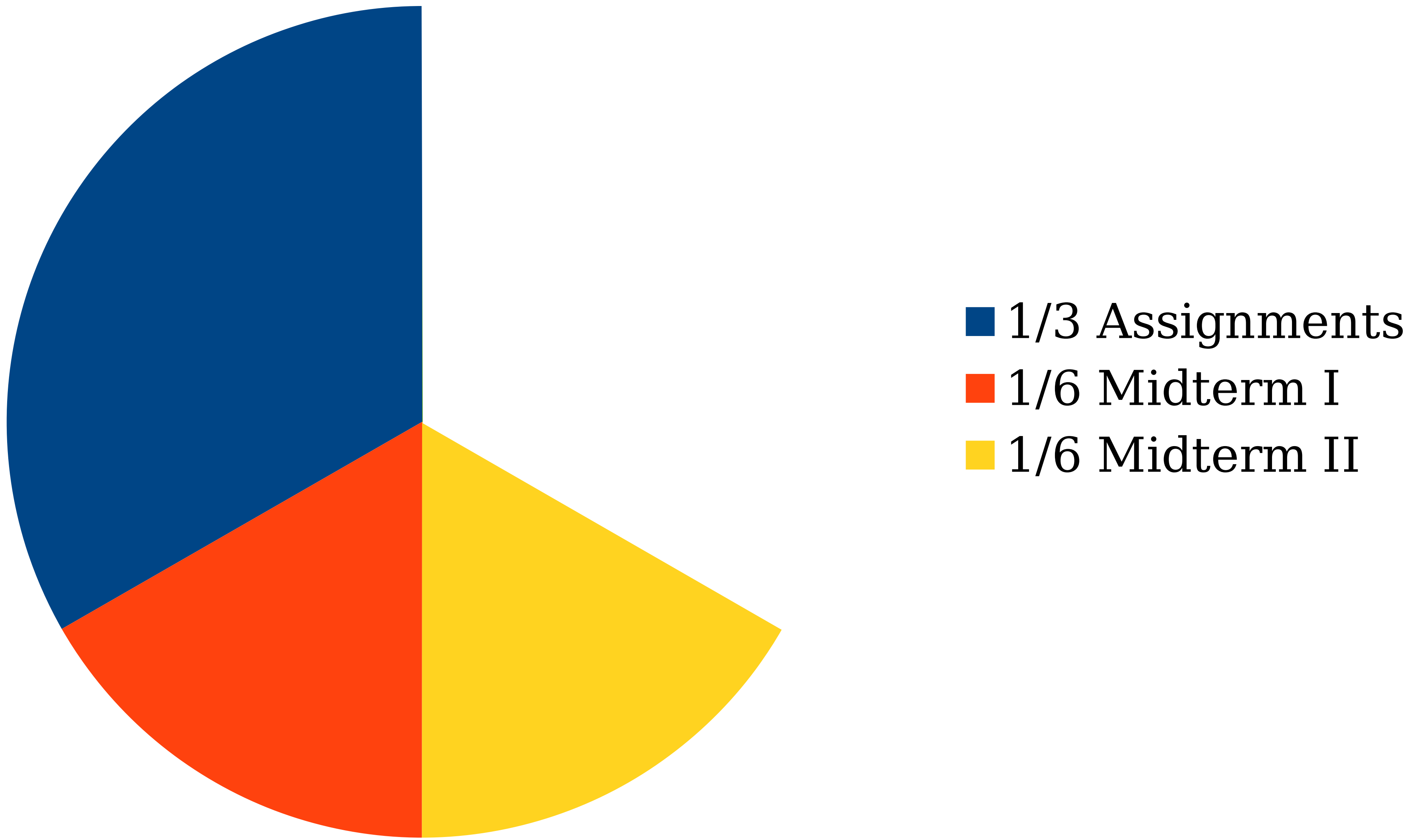


Grading

Midterm I

Monday, February 8
7:00PM – 10:00PM

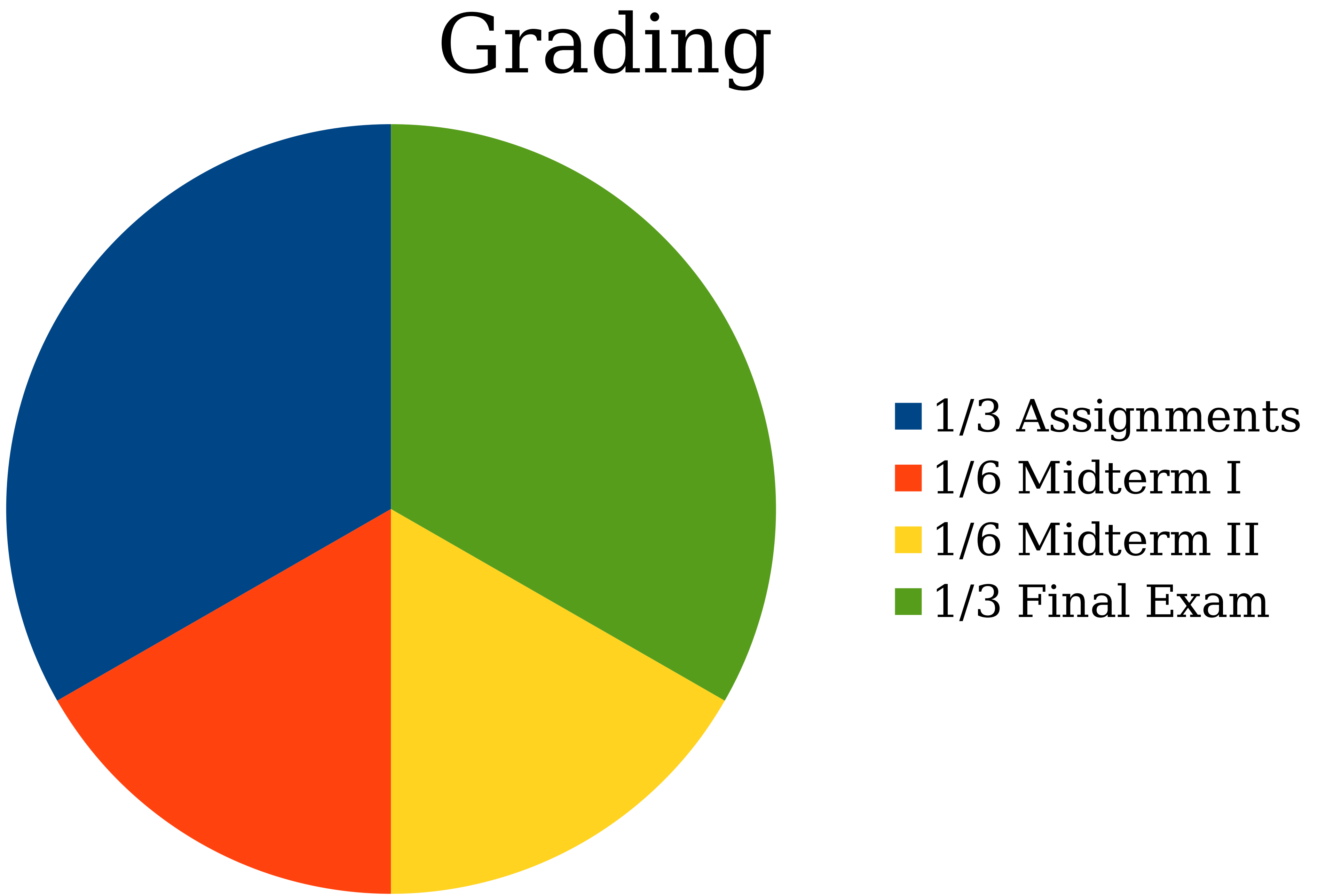
Grading



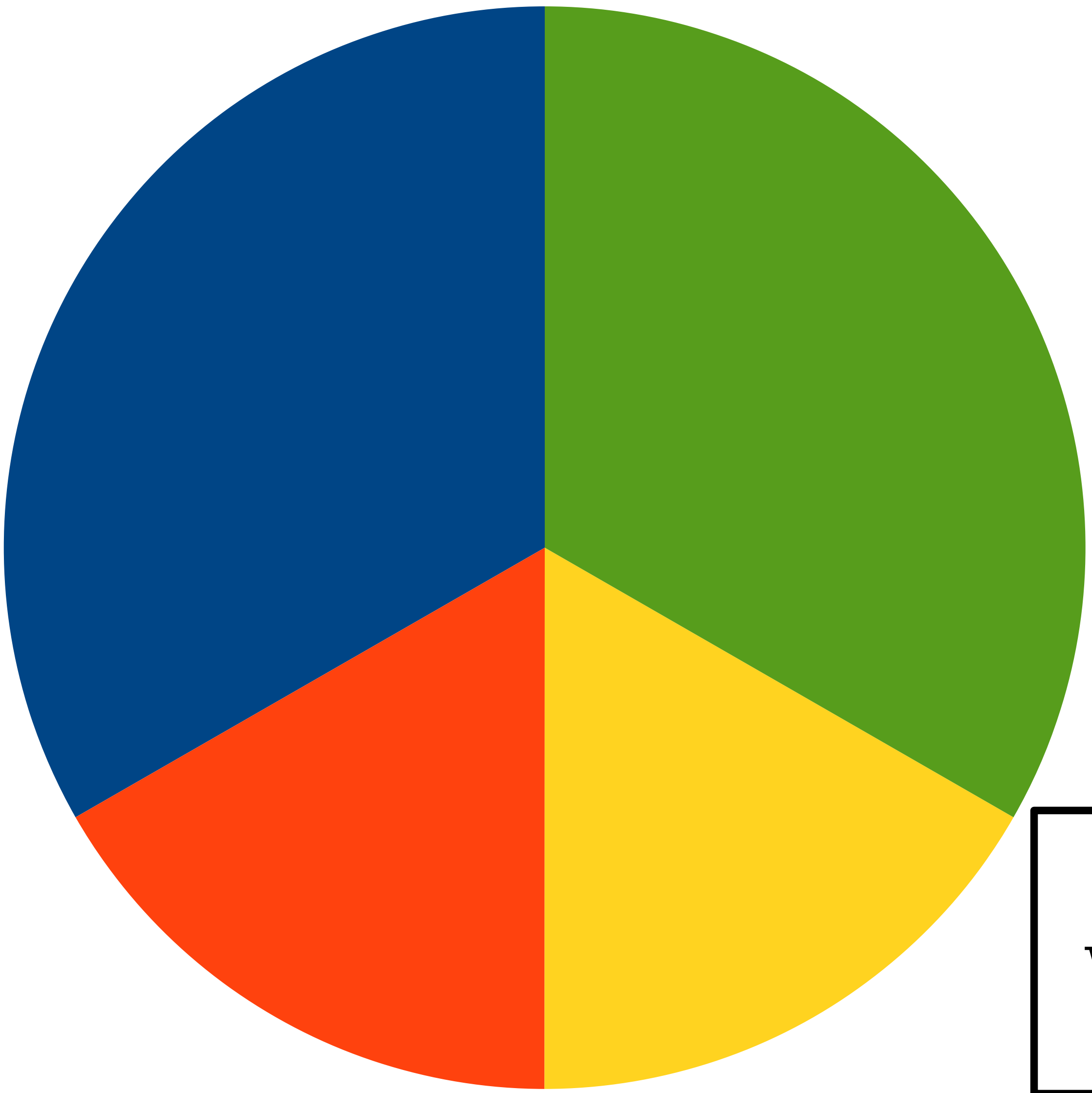
Grading

Midterm II

Monday, February 29
7:00PM – 10:00PM


Grading
1/3 Assignments
1/6 Midterm I
1/6 Midterm II
1/3 Final Exam

Grading



- 1/3 Assignments
- 1/6 Midterm I
- 1/6 Midterm II
- 1/3 Final Exam

Final Exam

Wednesday, March 16
3:30PM – 6:30PM

CS103A

- **CS103A** is an optional, one-unit add-on course to CS103.
- Provides extra review and practice with the material from CS103 and covers general problem-solving techniques useful in discrete math.
- Similar in spirit to Math 51A, Physics 41A, or Chem 31AC.
- We are currently coordinating the room time and location. We'll send out an announcement once everything has been finalized.
- There seems to be some weirdness on Axess where the class claims it's at capacity. It's not. We'll figure out what's going on and get it fixed as soon as we can.

We've got a big journey ahead of us.

Let's get started!

Introduction to Set Theory

“CS103 students”

“All the computers on the Stanford network”

“Cool people”

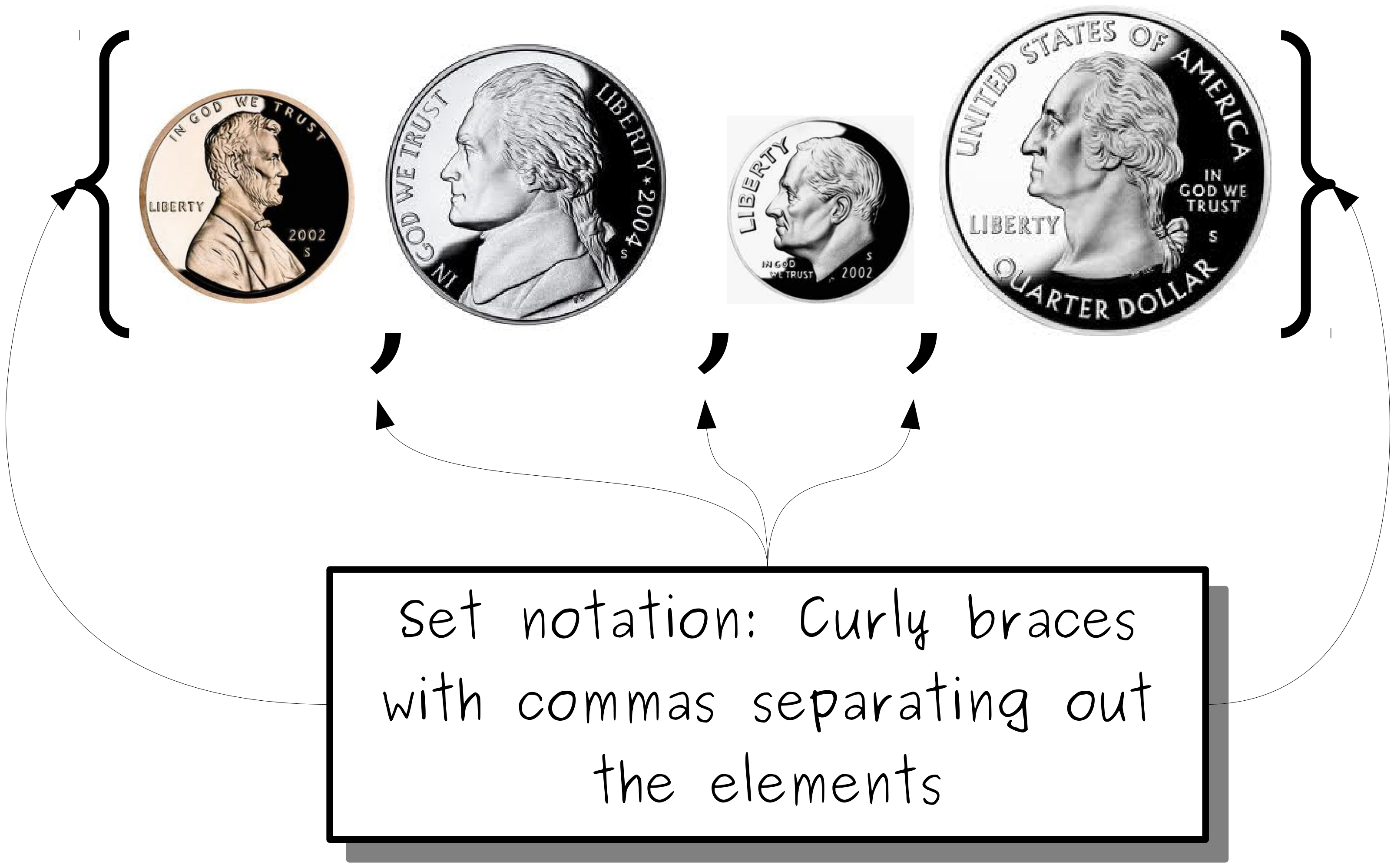
“The chemical elements”

“Cute animals”

“US coins”

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).



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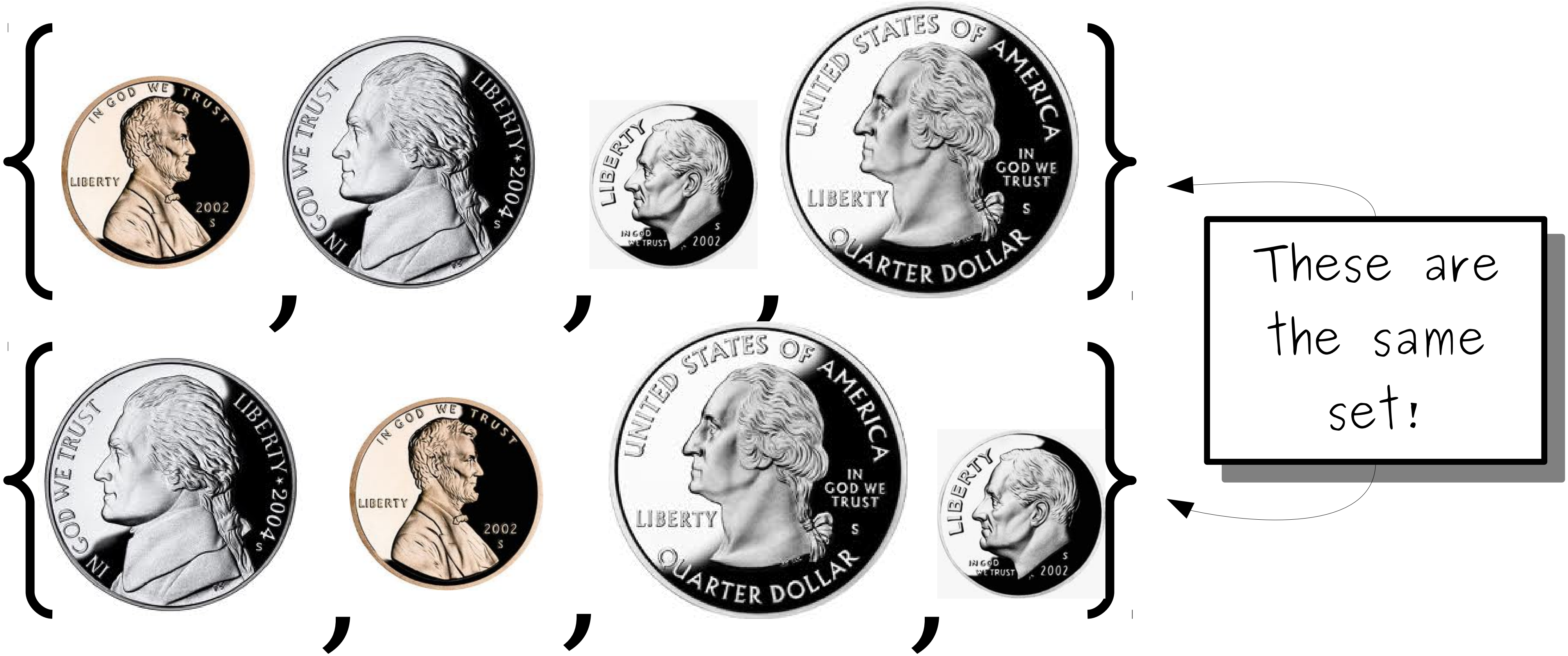
A ***set*** is an *unordered* collection of distinct objects, which may be anything (including other sets).



A ***set*** is an *unordered* collection of distinct objects, which may be anything (including other sets).

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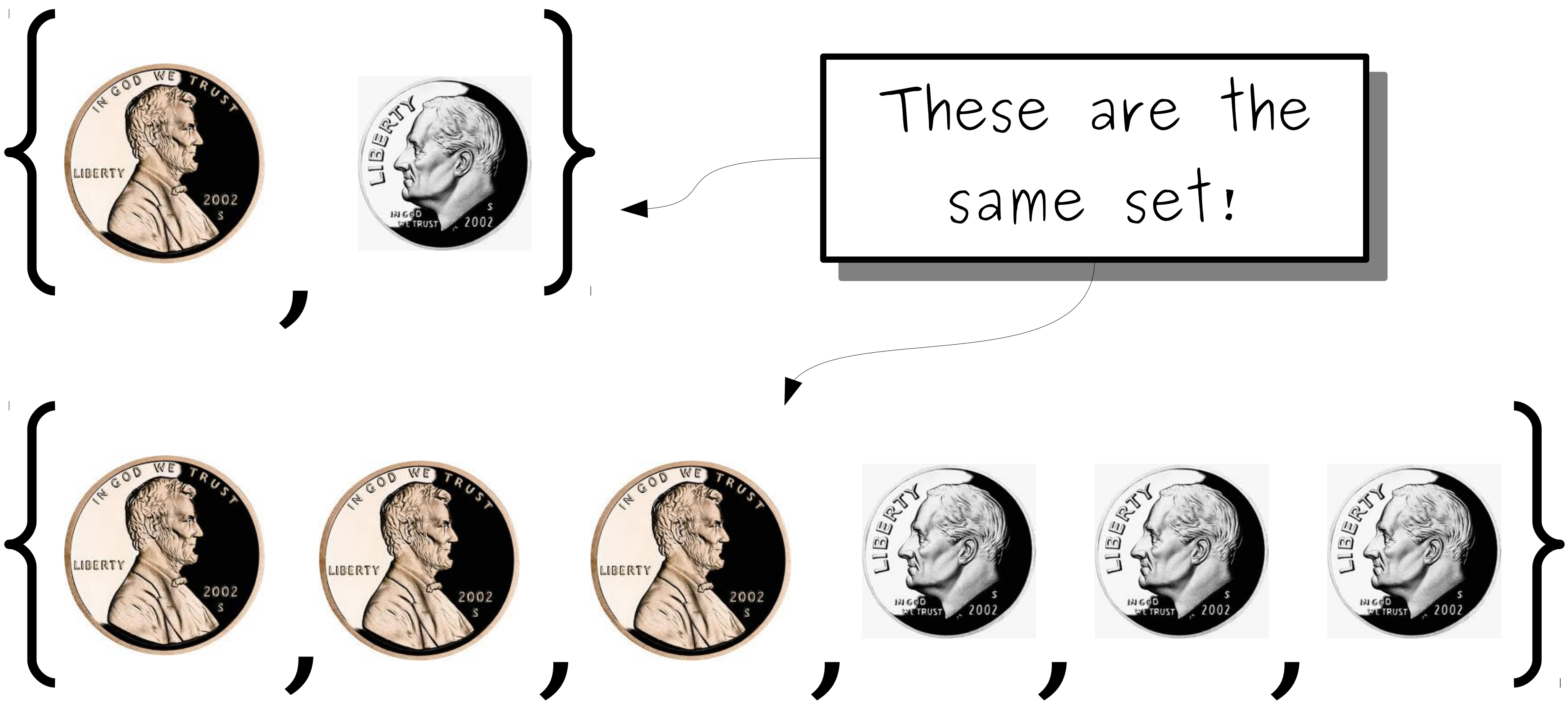
{  ,  }

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).

A ***set*** is an unordered collection of *distinct* objects, which may be anything (including other sets).



A ***set*** is an unordered collection of *distinct* objects, which may be anything (including other sets).

$$\{\ \} = \emptyset$$

The empty set contains no elements.

We use this symbol to denote the empty set.

A ***set*** is an unordered collection of distinct objects, which may be anything (including other sets).

This set contains nothing at all.

This set has one element, which happens to be the empty set.

$$\emptyset \stackrel{?}{=} \{\emptyset\}$$

Are these equal to one another?

This set contains nothing at all.

This set has one element, which happens to be the empty set.

$$\emptyset \neq \{\emptyset\}$$

Are these equal to one another?

This is a number.

This is a set. It contains a number.

$$1 \stackrel{?}{=} \{1\}$$

Are these equal to one another?

This is a number.

This is a set. It contains a number.

$$1 \neq \{1\}$$

Are these equal to one another?

Membership

Membership

{ 
IN GOD WE TRUST
LIBERTY
2004
, 
IN GOD WE TRUST
LIBERTY
2002
, 
UNITED STATES OF AMERICA
IN GOD WE TRUST
LIBERTY
QUARTER DOLLAR
, 
LIBERTY
IN GOD WE TRUST
2002
}

Membership

Is 

in this set?

Membership

$\in$ $\{$


,


,


,


$\}$

Is


in this set?

Membership

$\in \{$ 

, 

, 

, 

$\}$

Is 

in this set?

Membership

$\notin$ $\{$

,

,

,

$\}$

Is

in this set?

Set Membership

- Given a set S and an object x, we write

$$\boldsymbol{x \in S}$$

  if x is contained in S, and

$$\boldsymbol{x \notin S}$$

  otherwise.
- If $x \in S$, we say that x is an ***element*** of S.
- Given any object x and any set S, either $x \in S$ or $x \notin S$.

Infinite Sets

- Some sets contain *infinitely many* elements!
- The set $\mathbb{N}$ = { 0, 1, 2, 3, …} is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set $\mathbb{Z}$ = { …, -2, -1, 0, 1, 2, … } is the set of all the ***integers***.
  - Z is from German "Zahlen."
- The set $\mathbb{R}$ is the set of all ***real numbers***.
  - $e \in \mathbb{R}$, $\pi \in \mathbb{R}$, $4 \in \mathbb{R}$, etc.

Describing Complex Sets

- Here are some English descriptions of infinite sets:

  “The set of all even numbers.”

  “The set of all real numbers less than 137.”

  “The set of all negative integers.”

- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{ \ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

Even Natural Numbers

$$\{ \, n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \, \}$$

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

where

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

where

n is a natural number

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

The set of all n

where

n is a natural number

and n is even

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

where

n is a natural number

and n is even

$$\{\ 0, 2, 4, 6, 8, 10, 12, 14, 16, \dots\ \}$$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\ x \mid \textit{some property } x \textit{ satisfies}\ \}$$

- For example:

$\{\ r \mid r \in \mathbb{R} \text{ and } r < 137\ \}$

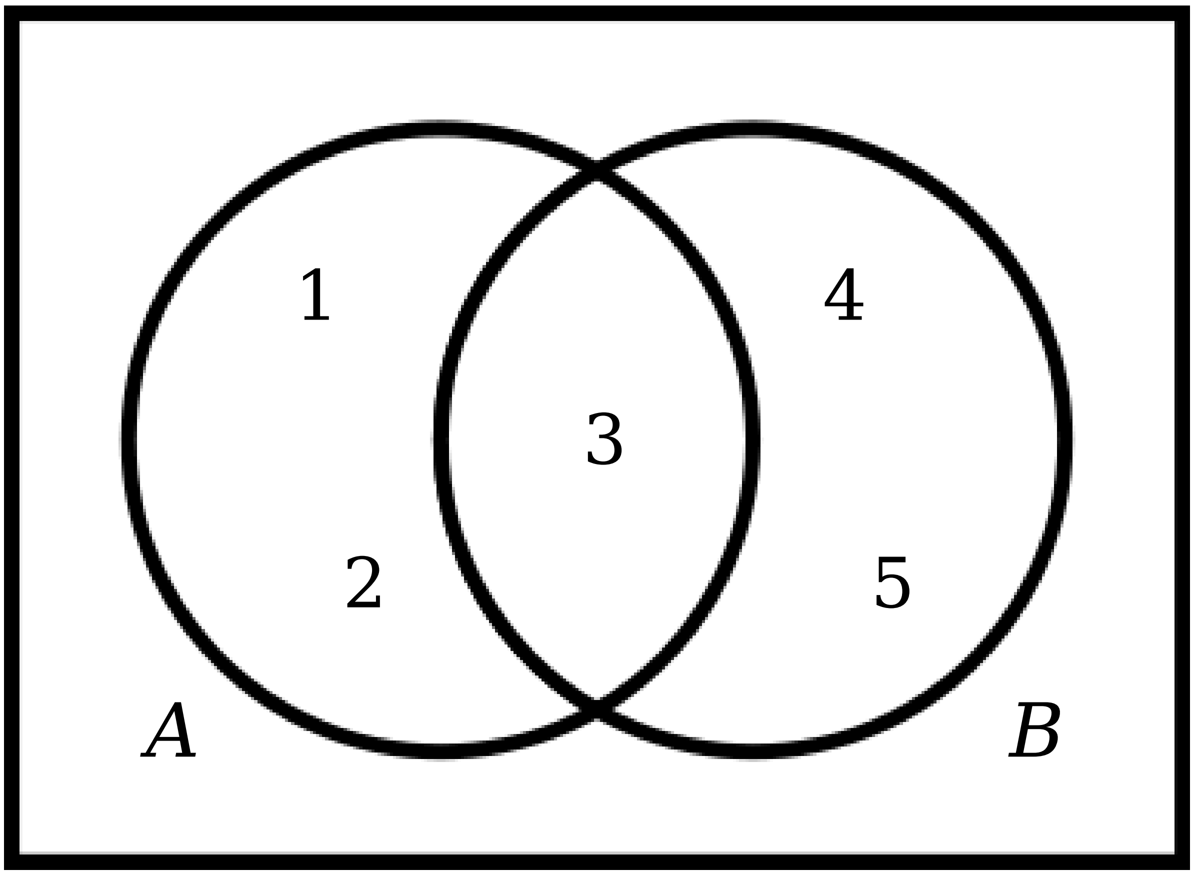
$\{\ n \mid n \text{ is an even natural number}\ \}$

$\{\ S \mid S \text{ is a set of US currency}\ \}$

$\{\ a \mid a \text{ is cute animal}\ \}$

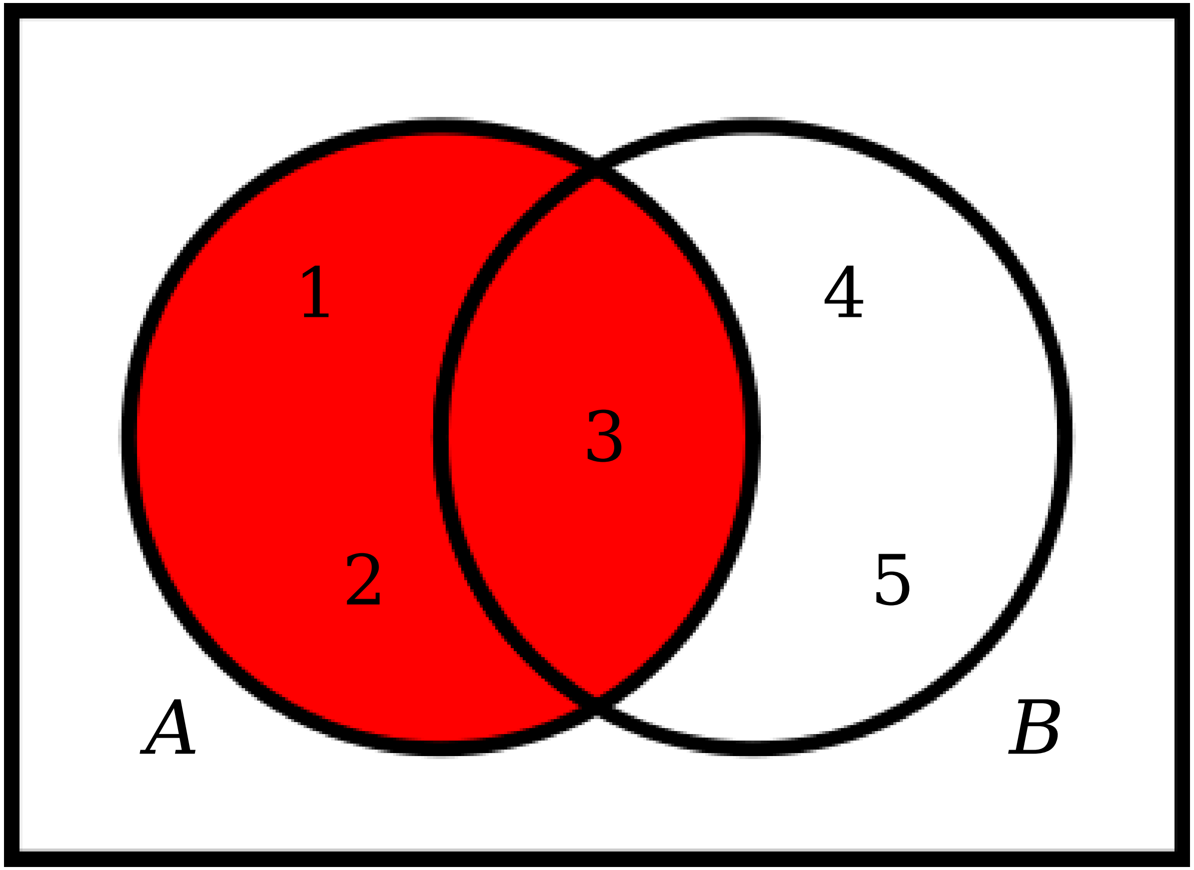
Combining Sets

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



$A = \{\ 1, 2, 3\ \}$
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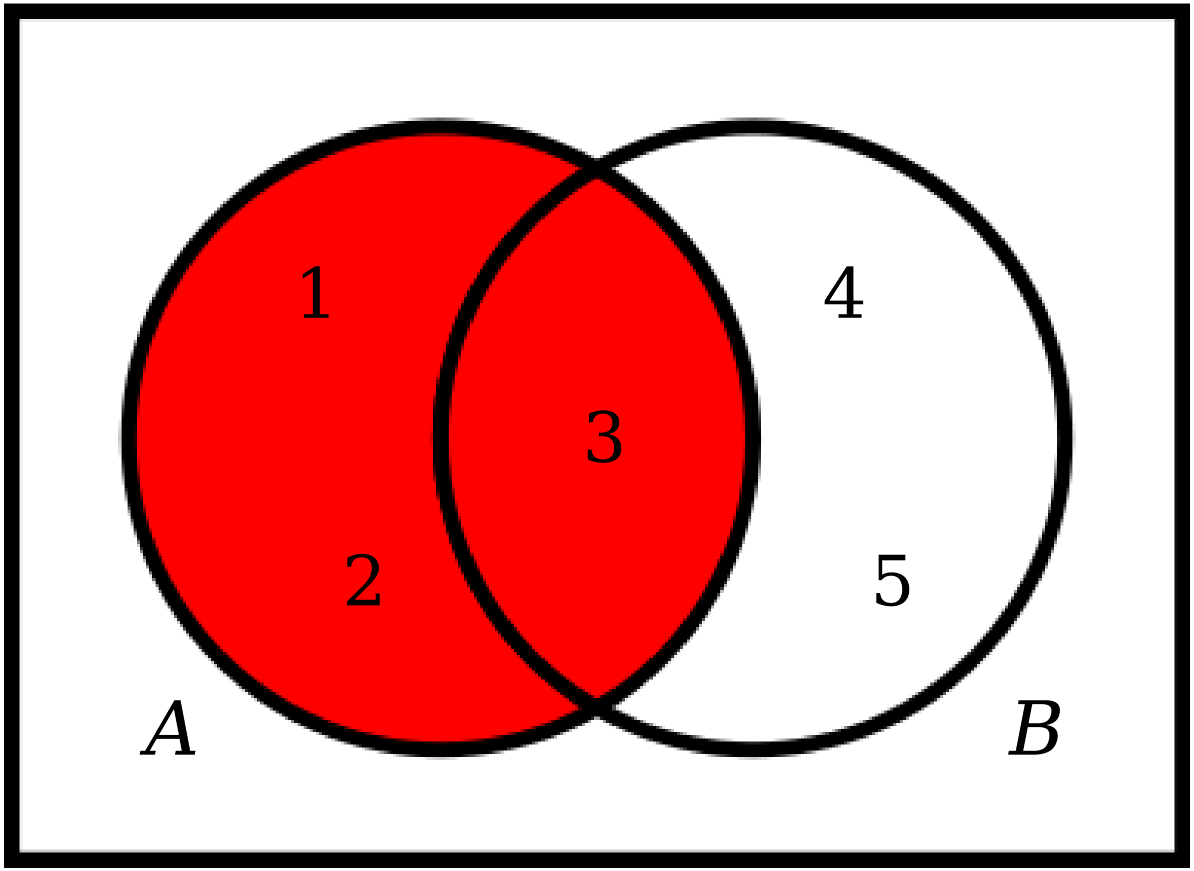
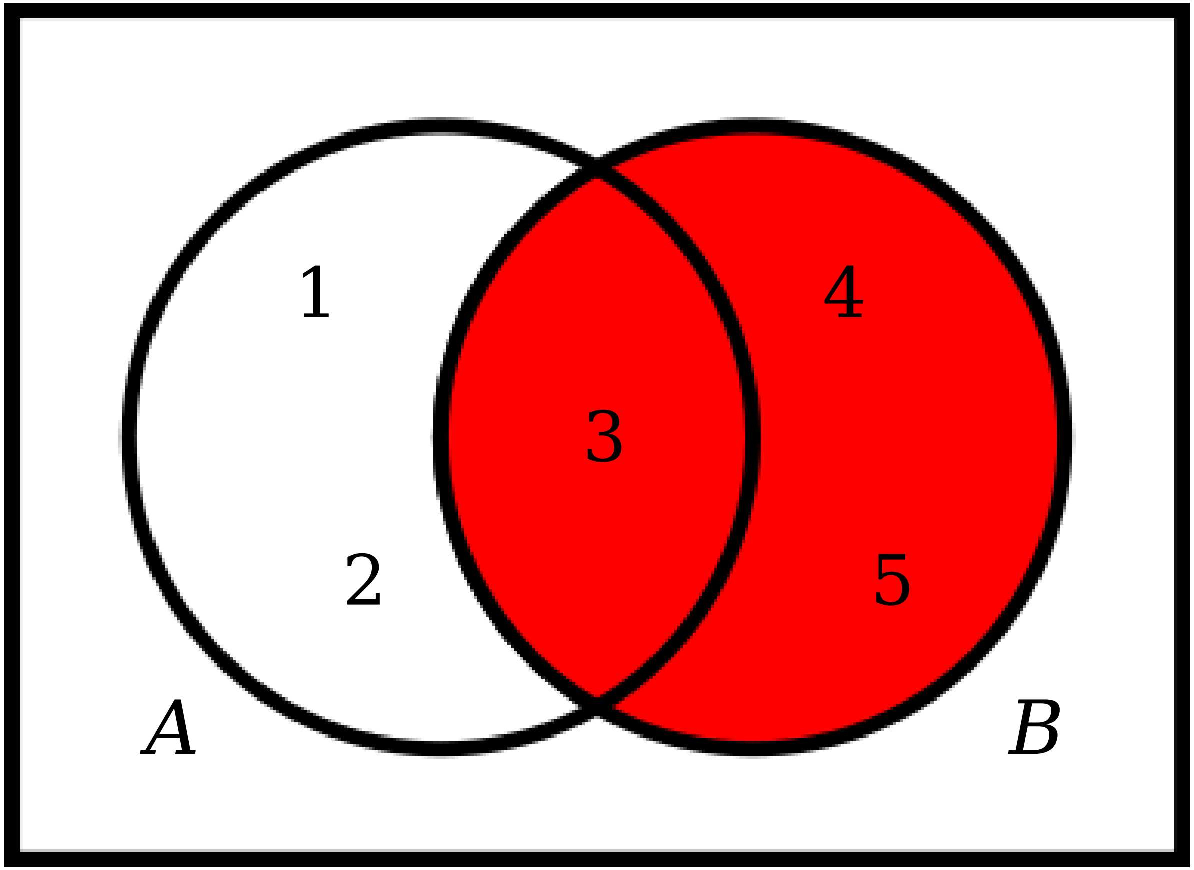
Venn Diagrams



A

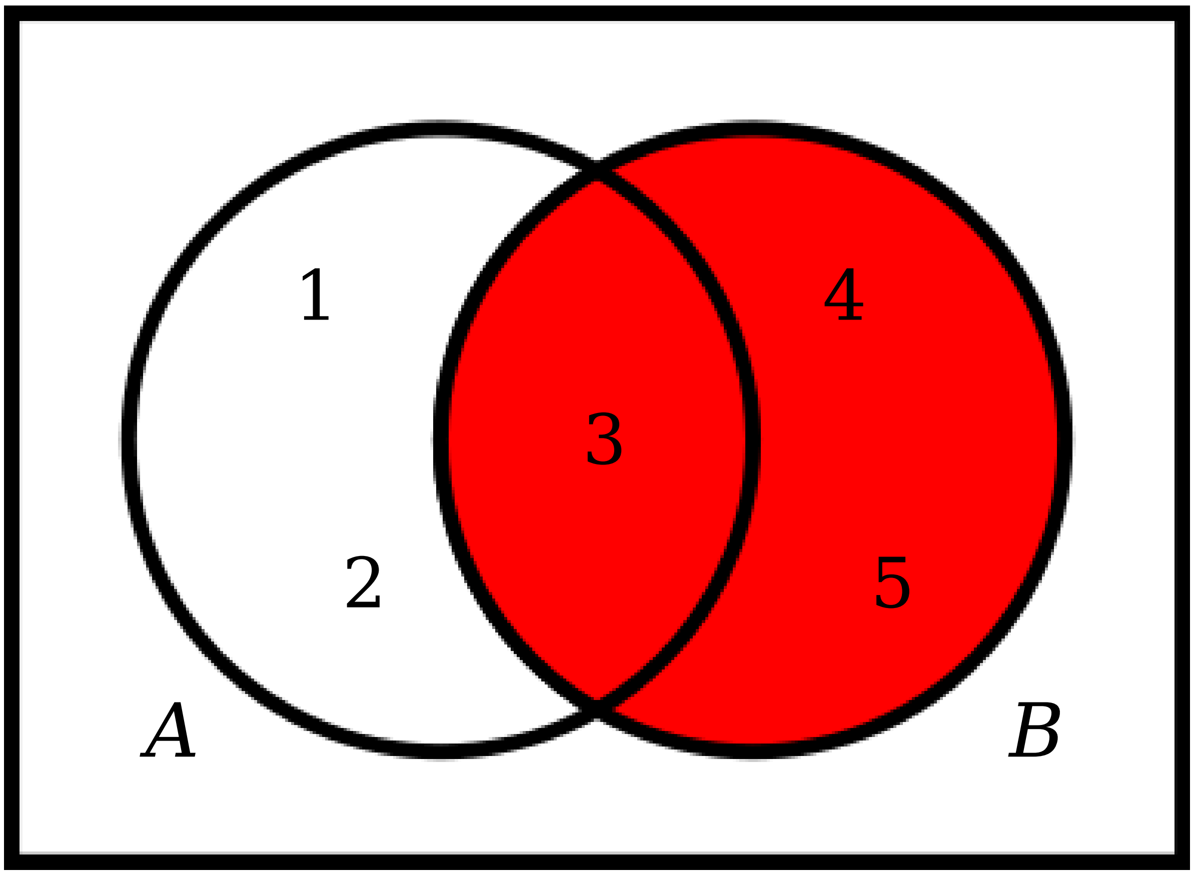
$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



$A = \{\ 1, 2, 3\ \}$
$B = \{\ 3, 4, 5\ \}$

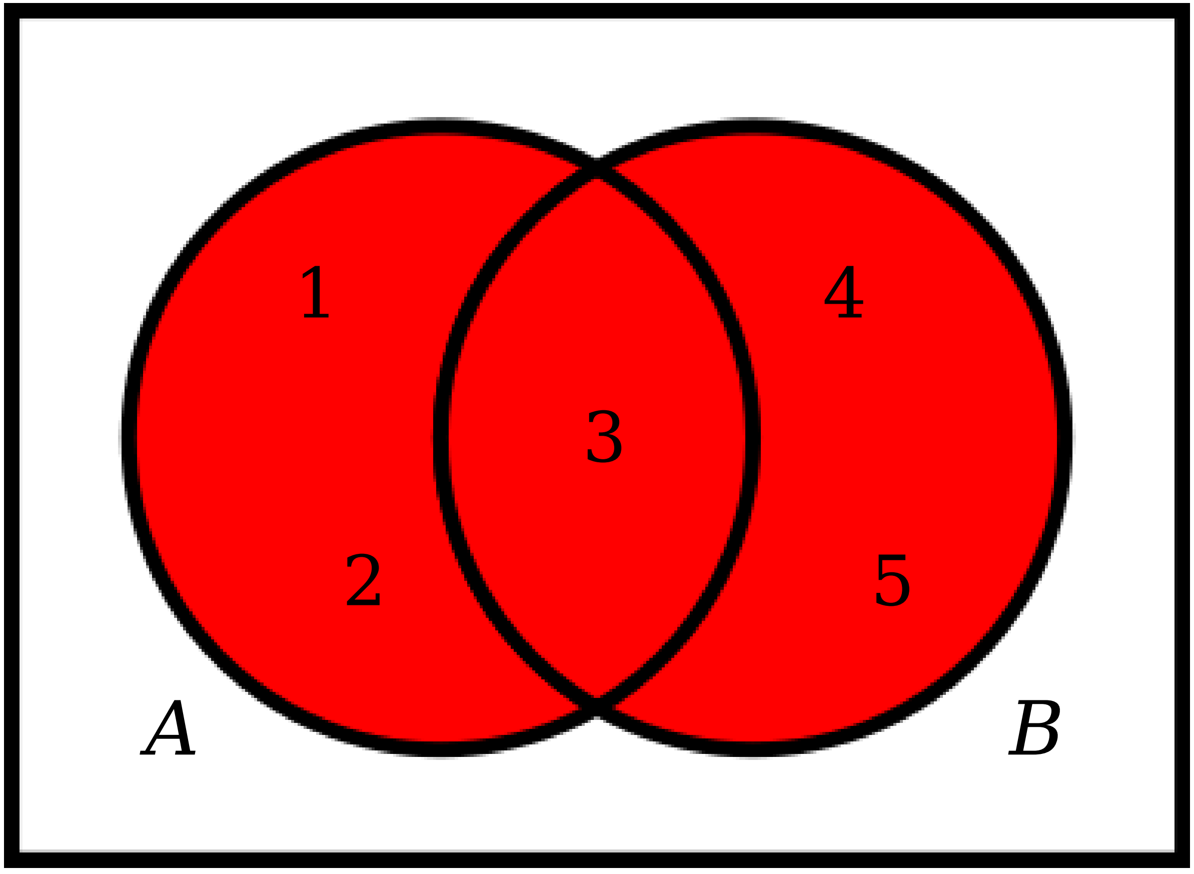
Venn Diagrams



B

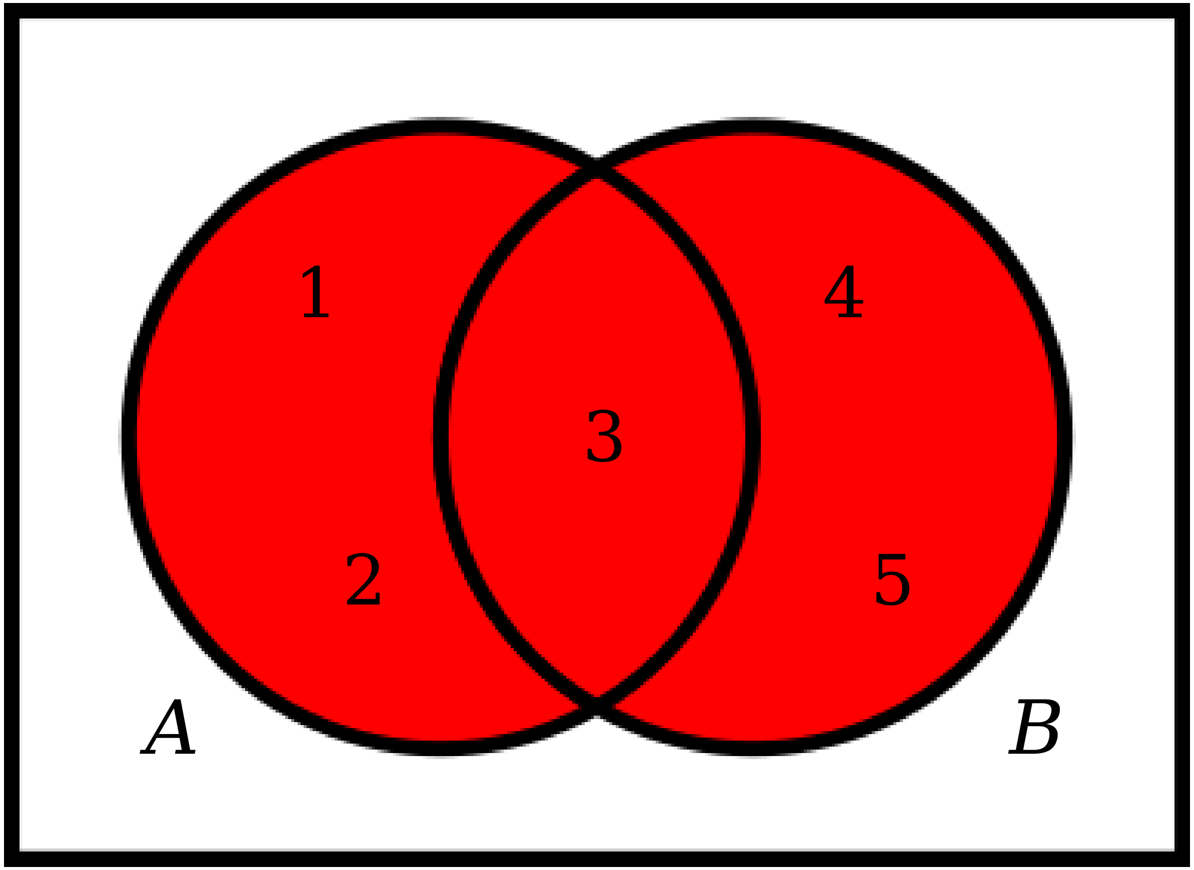
$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



$A = \{\ 1, 2, 3\ \}$
$B = \{\ 3, 4, 5\ \}$

Venn Diagrams



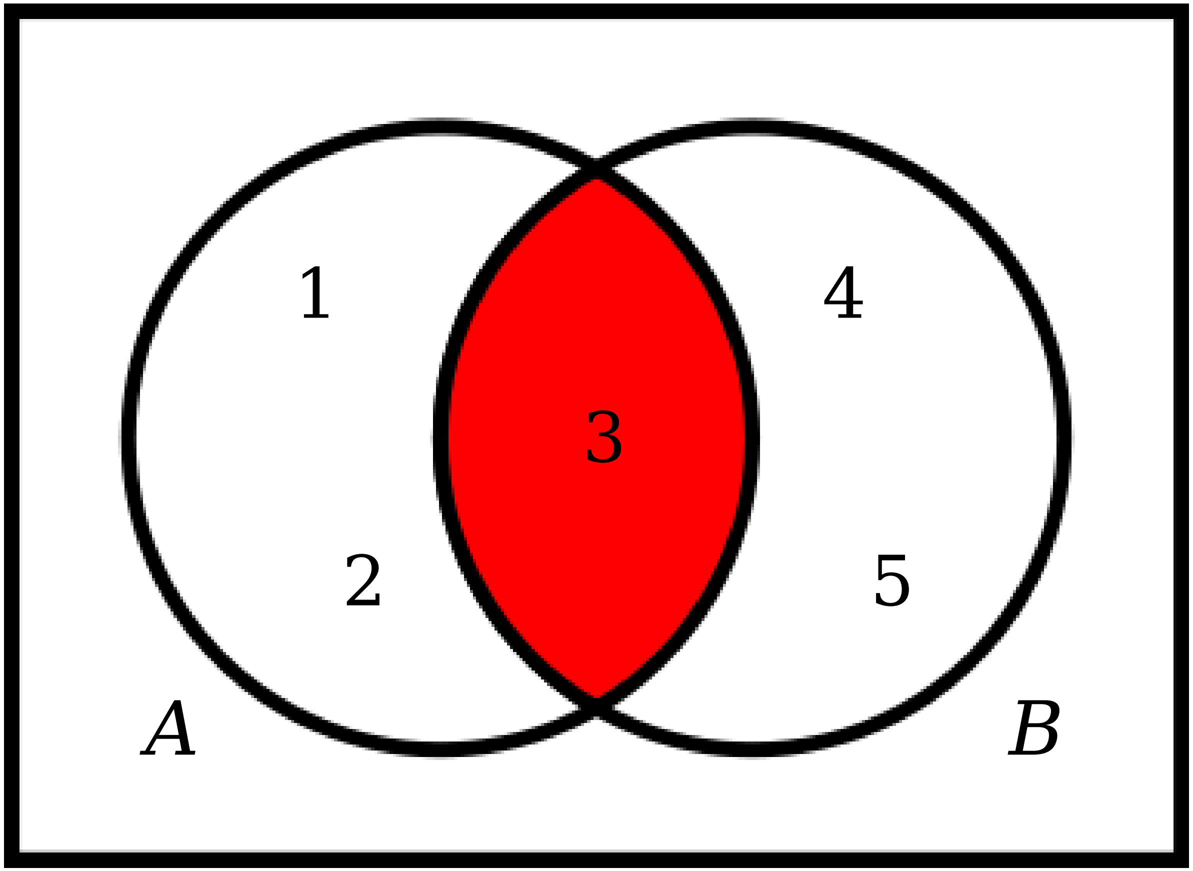
Union

$A \cup B$

{ 1, 2, 3, 4, 5 }

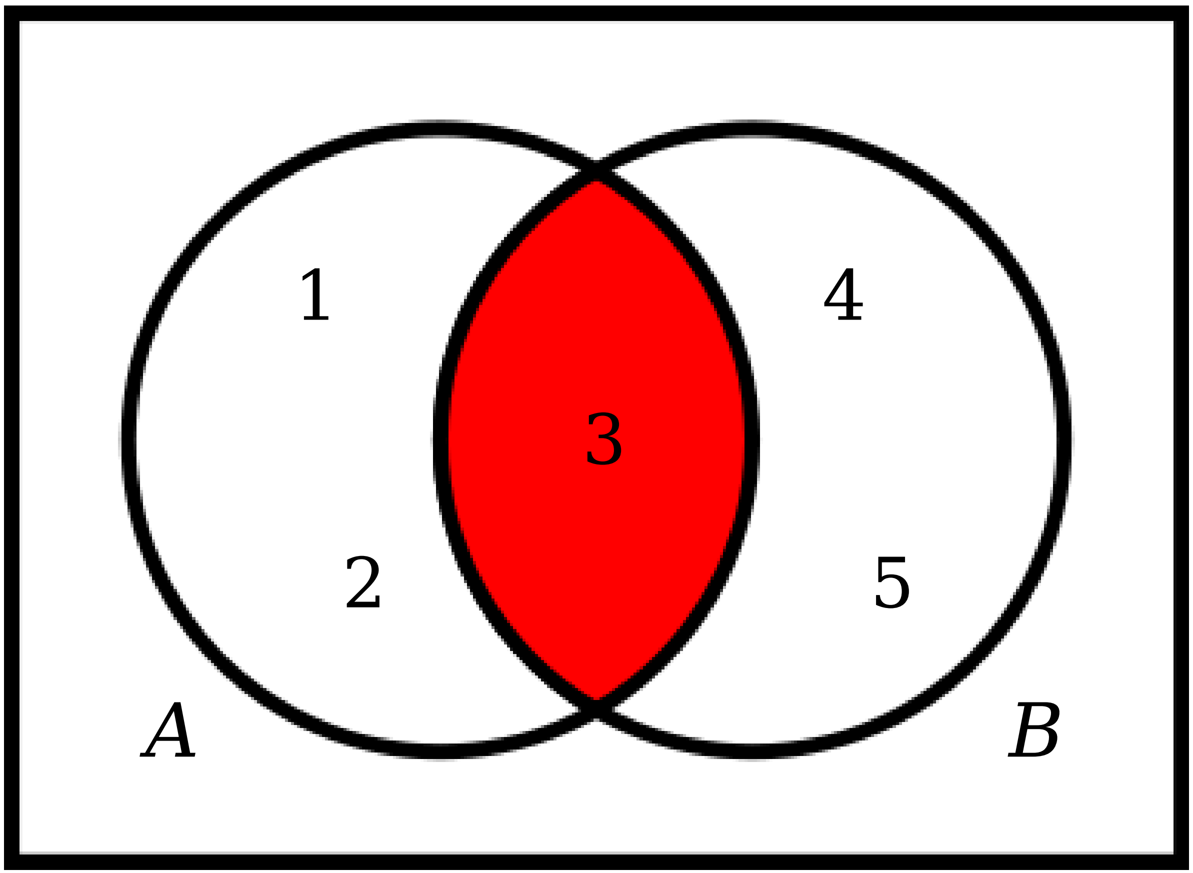
$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



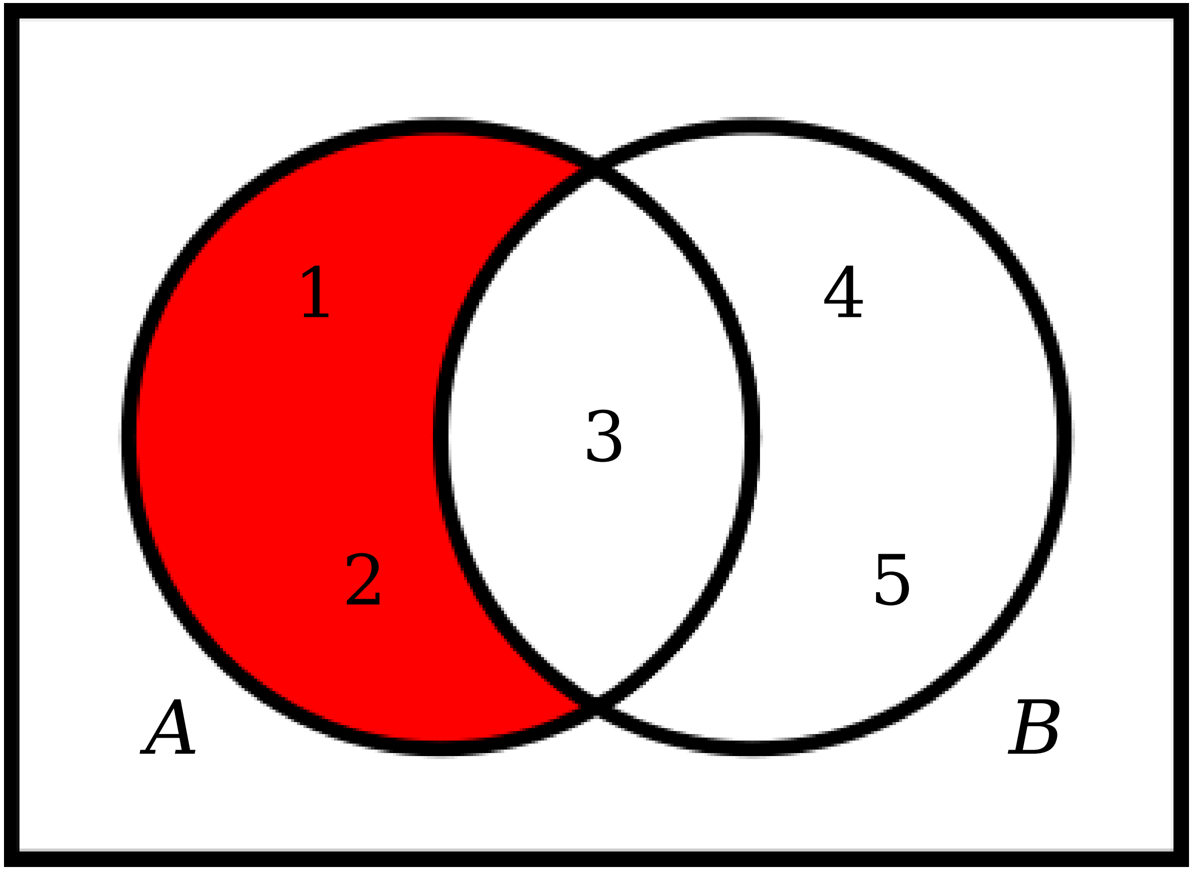
Intersection

$A \cap B$

$\{\ 3\ \}$

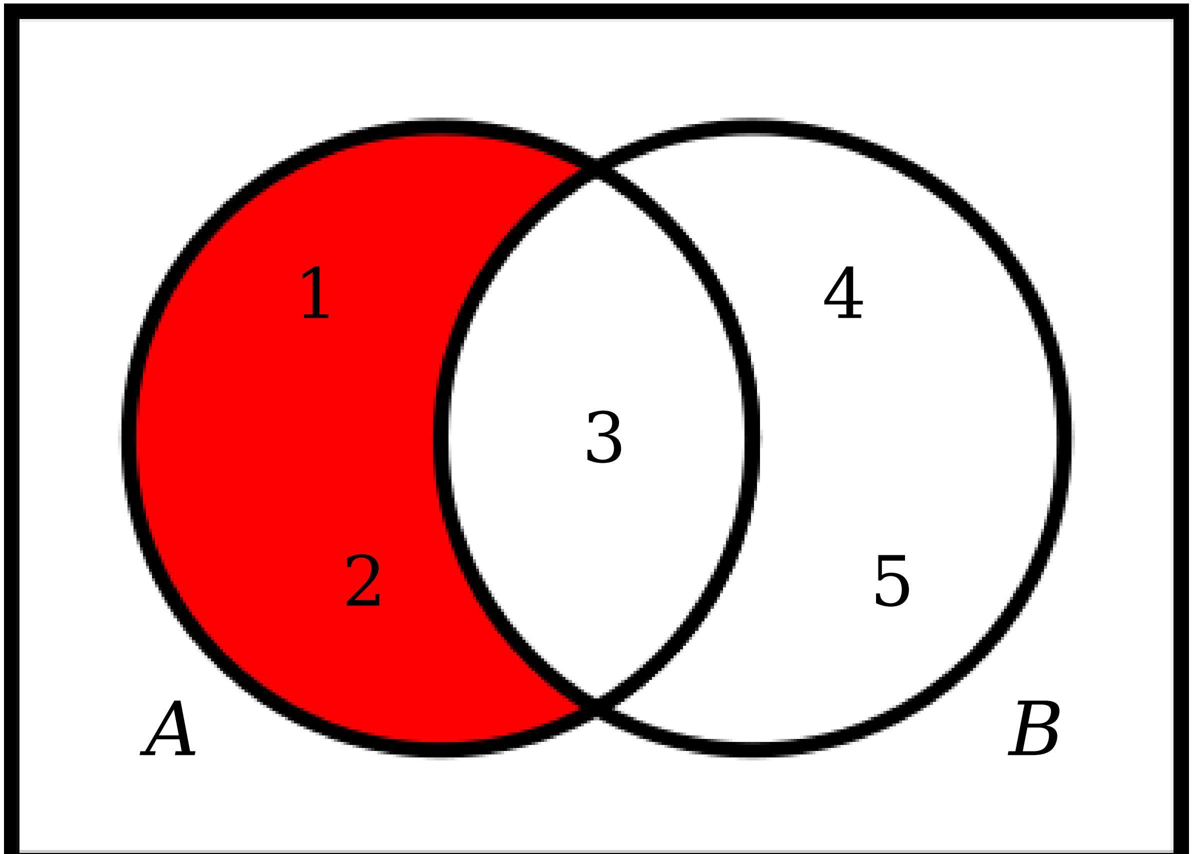
$A = \{\ 1, 2, 3\ \}$
$B = \{\ 3, 4, 5\ \}$

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

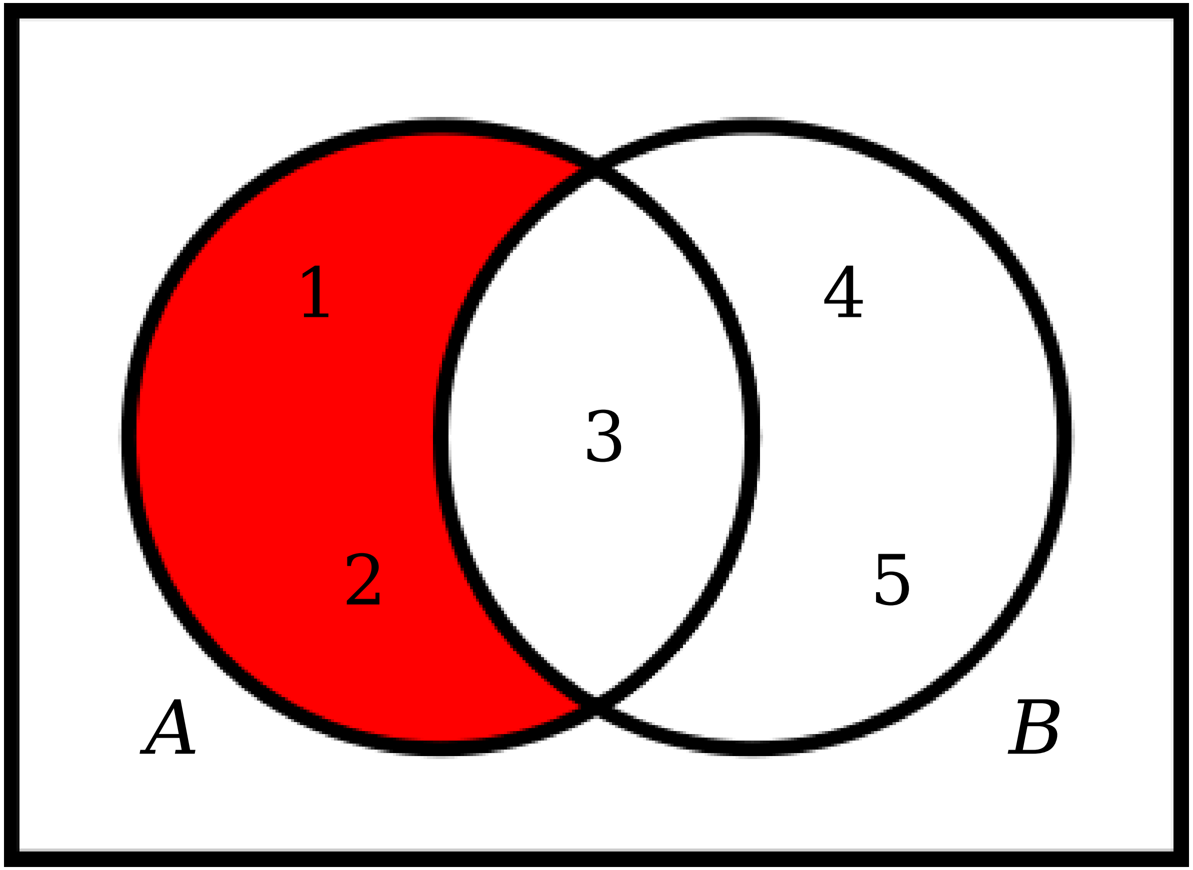
$A - B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

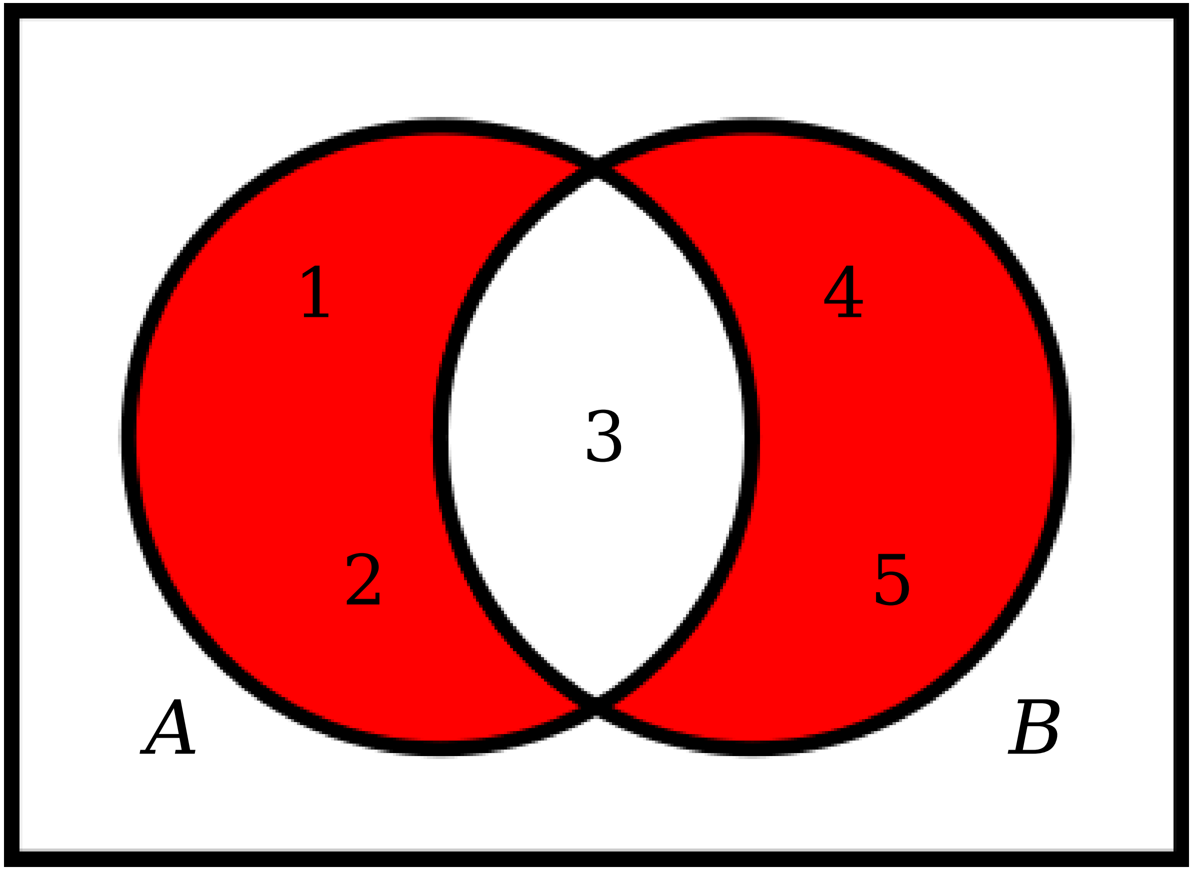
$A \setminus B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$

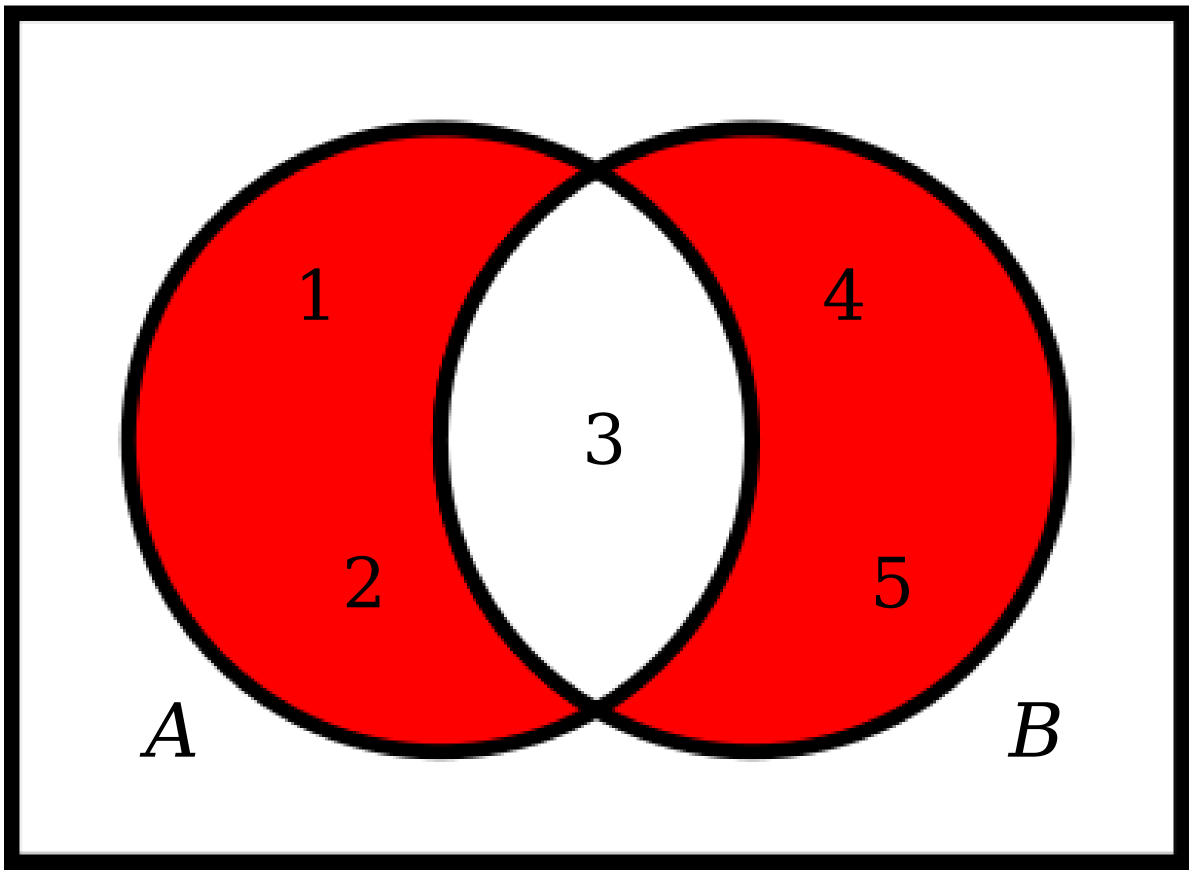
$B = \{ 3, 4, 5 \}$

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Symmetric Difference

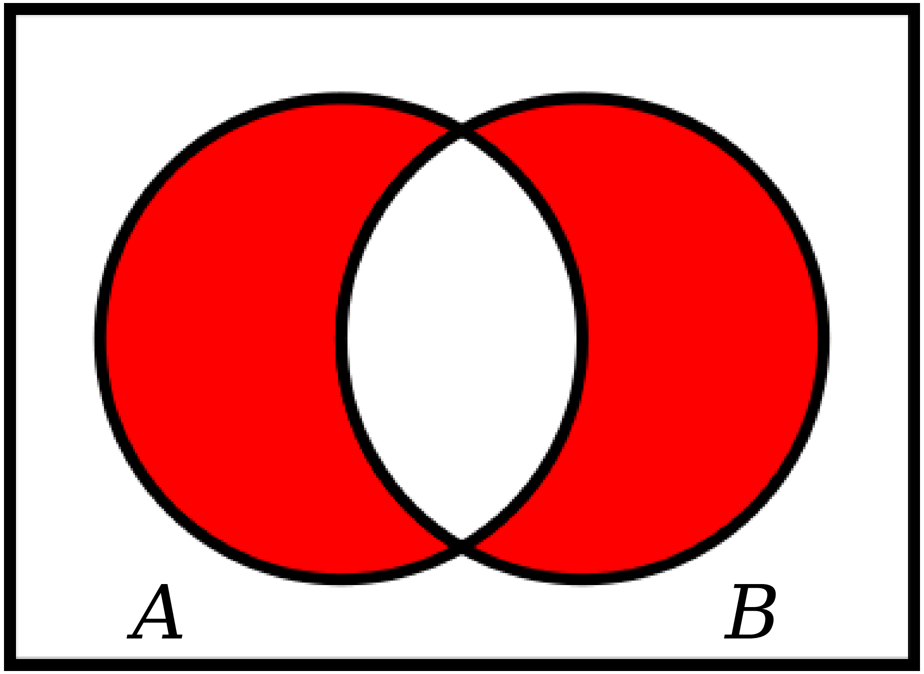
$A \,\Delta\, B$

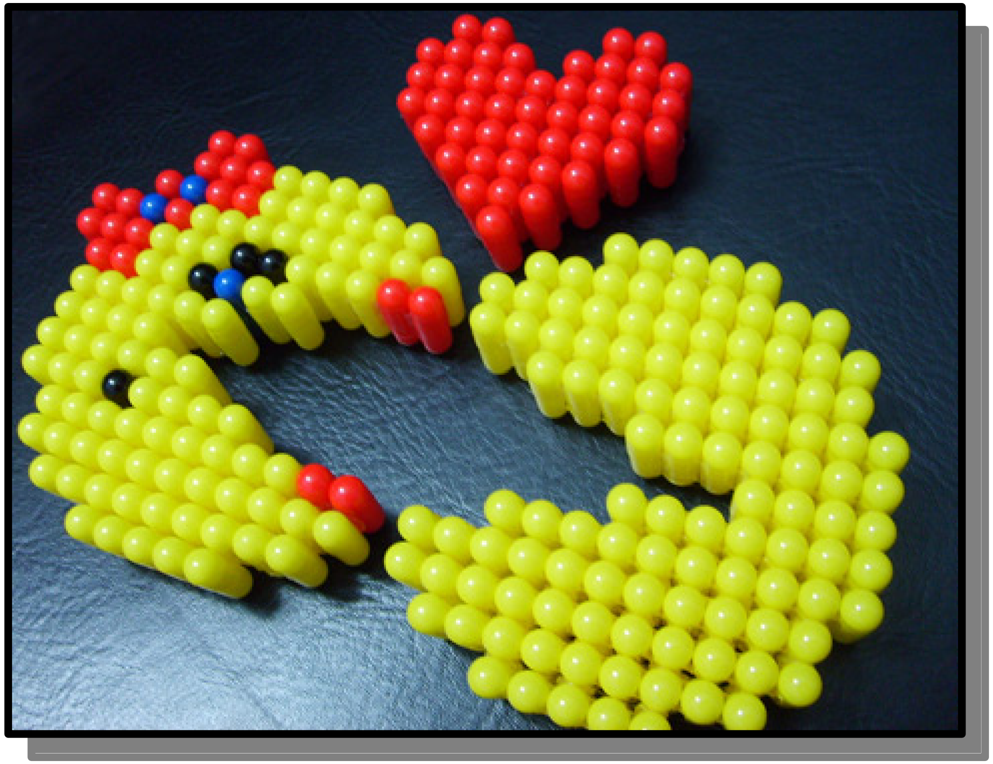
{ 1, 2, 4, 5 }

A = { 1, 2, 3 }

B = { 3, 4, 5 }

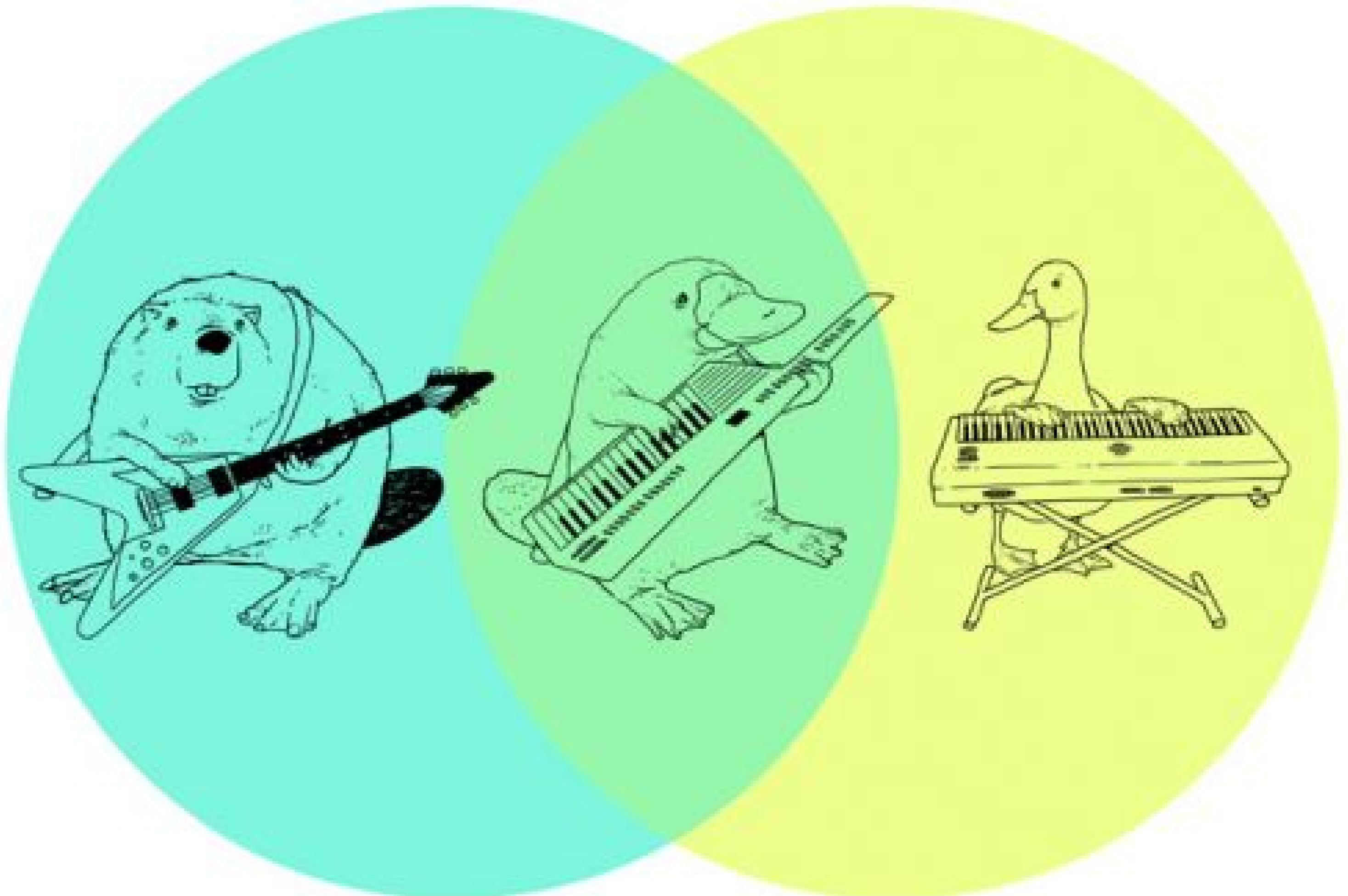
Venn Diagrams



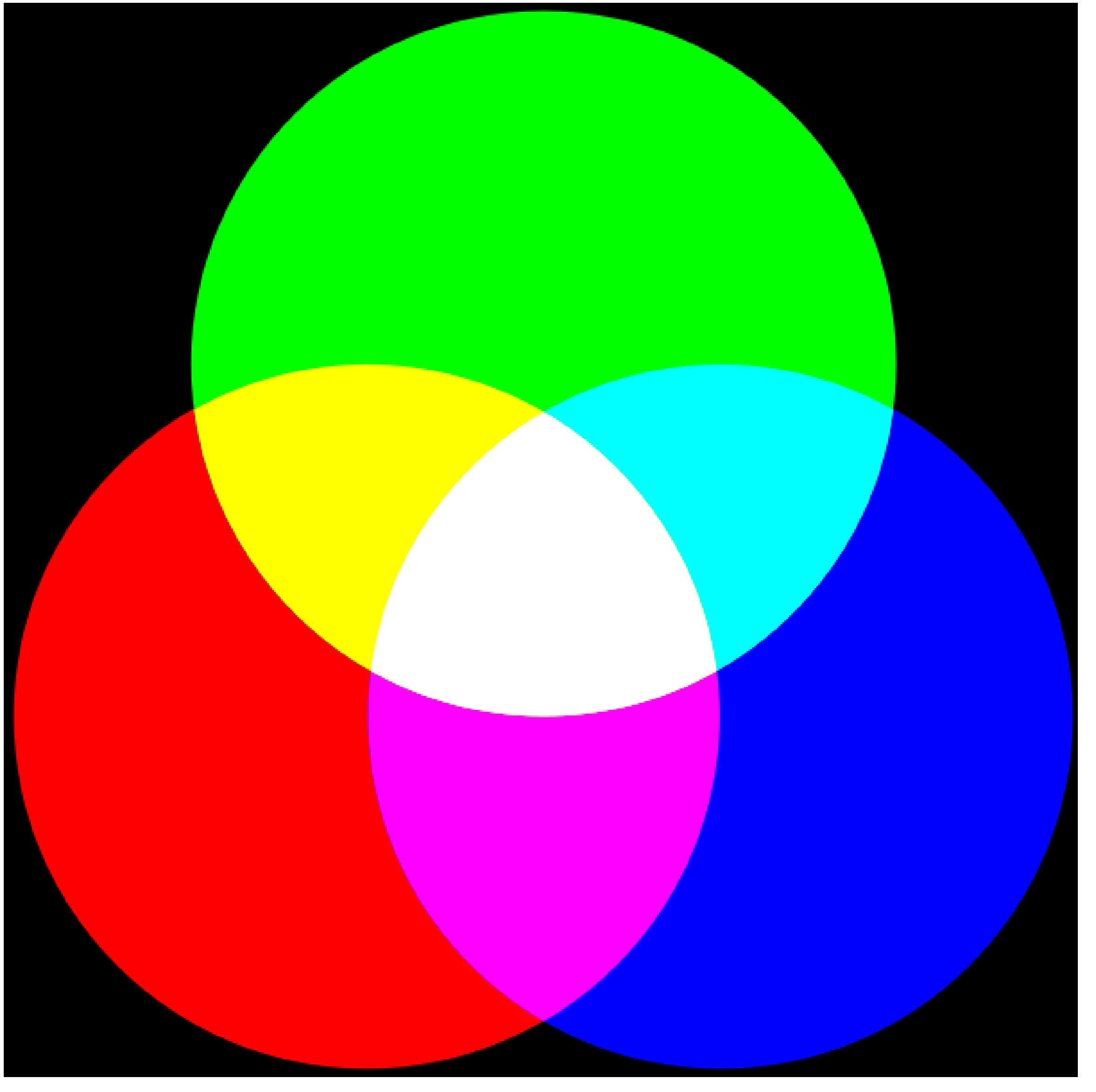


$A \Delta B$

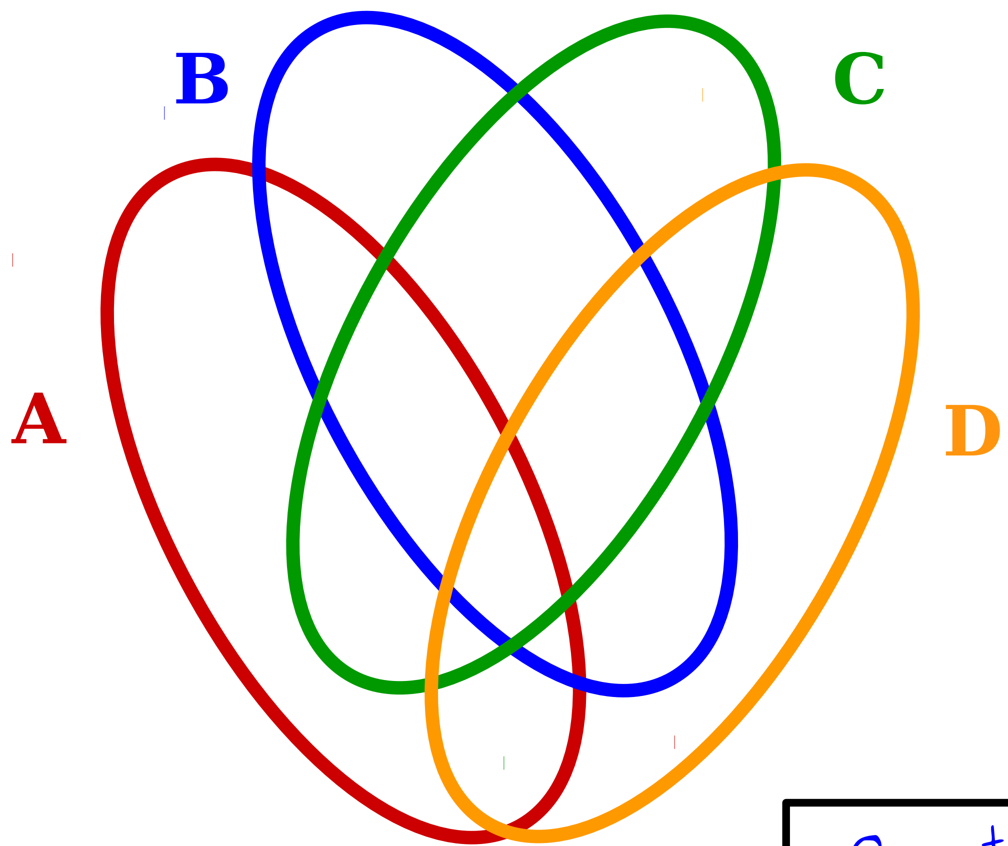
Venn Diagrams



Venn Diagrams for Three Sets

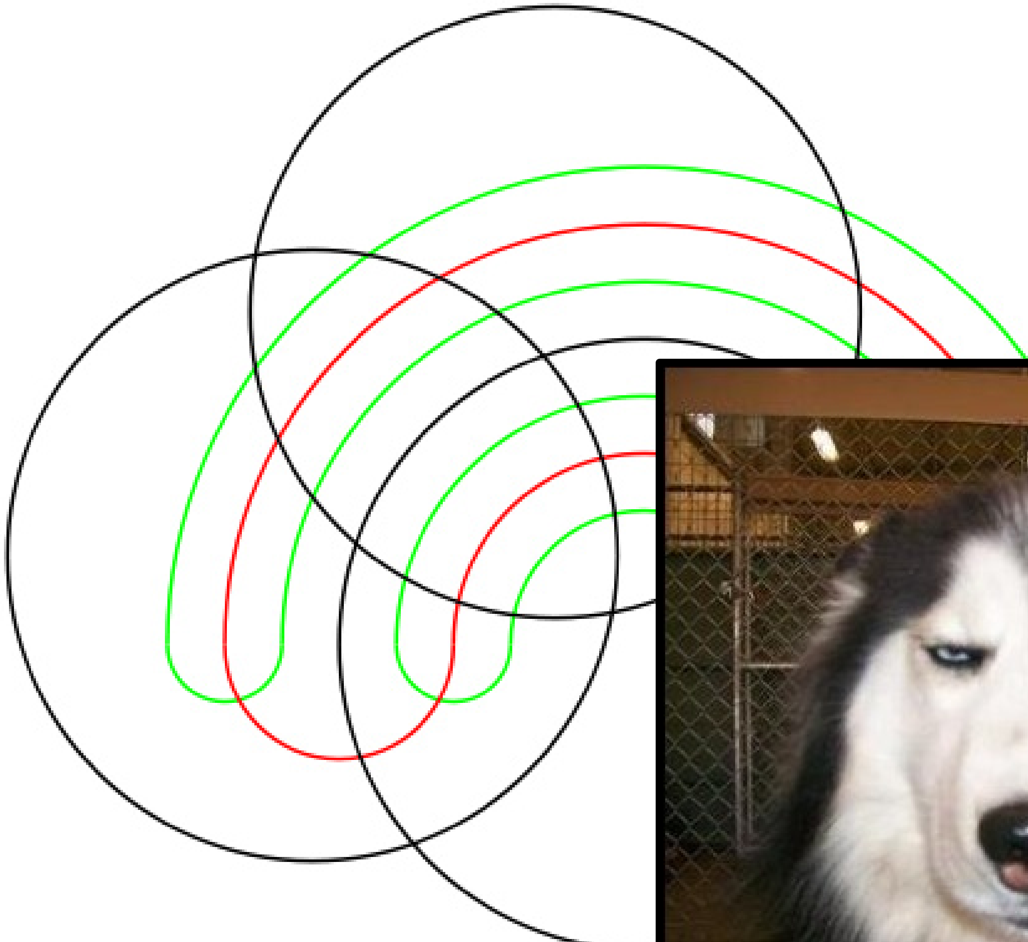


Venn Diagrams for Four Sets



Question to ponder:
why don't we just
draw four circles?

Venn Diagrams for Five Sets



Venn Diagrams for Seven Sets

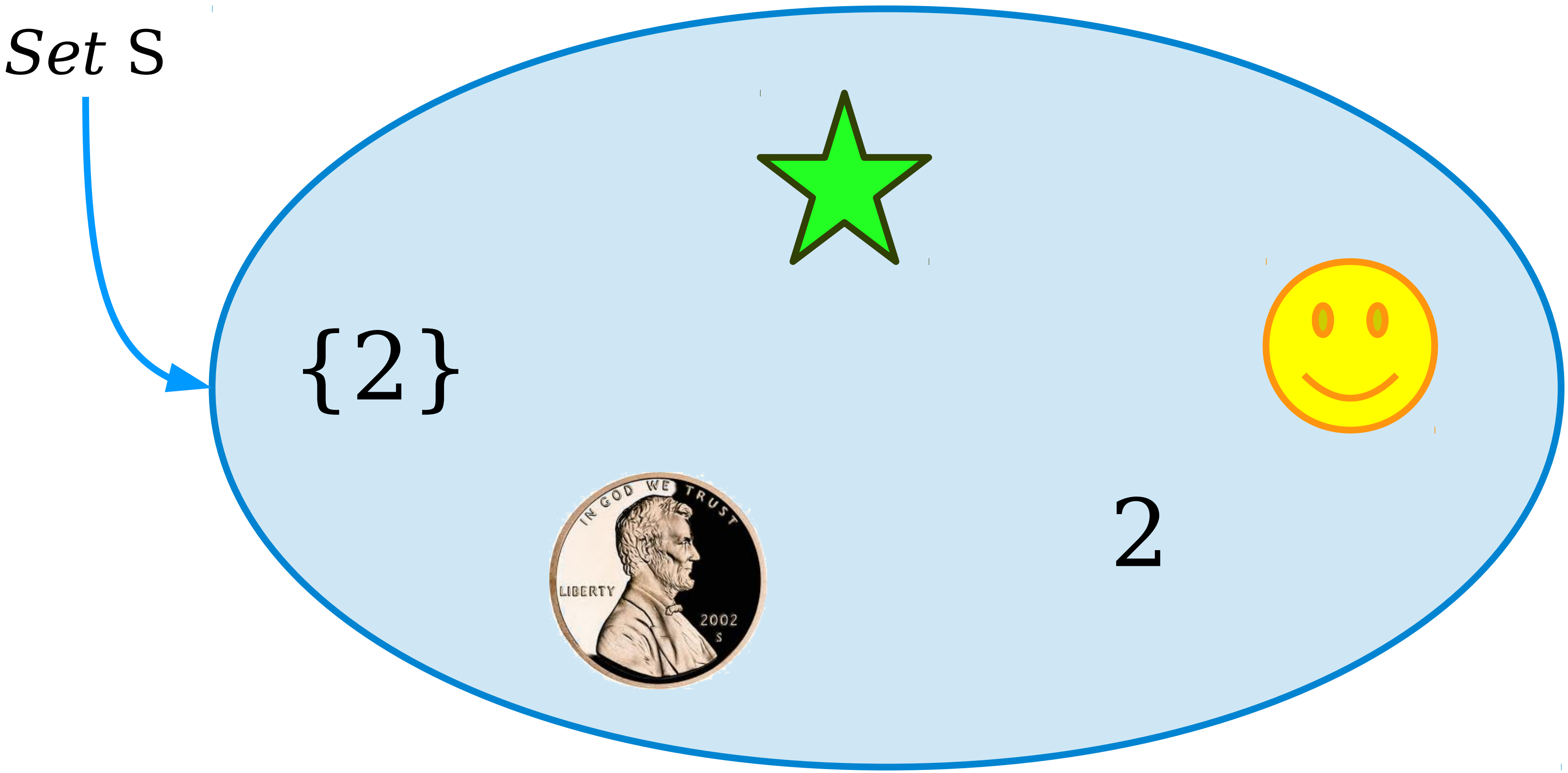
http://moebio.com/research/sevensets/

Subsets and Power Sets

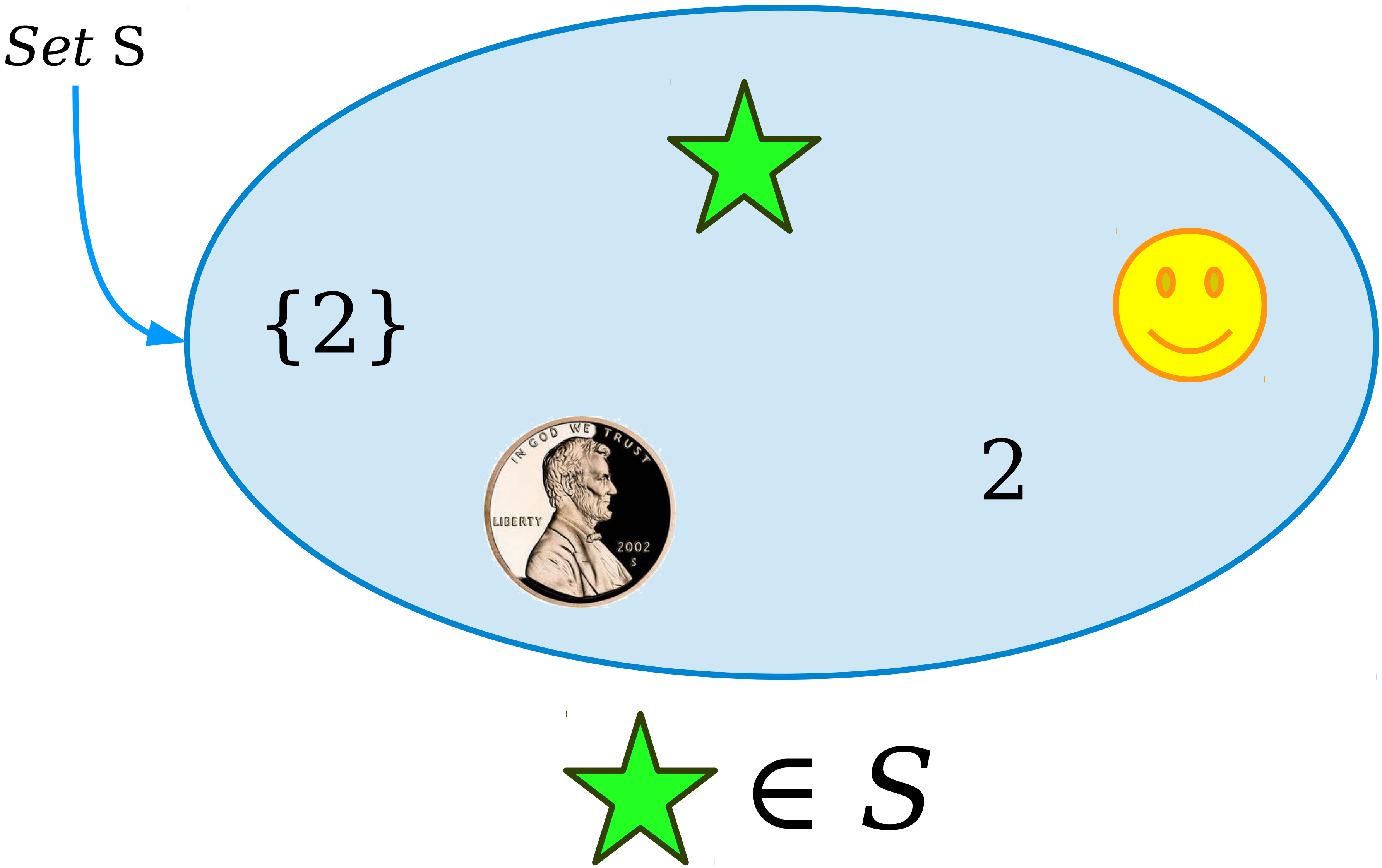
Subsets

- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

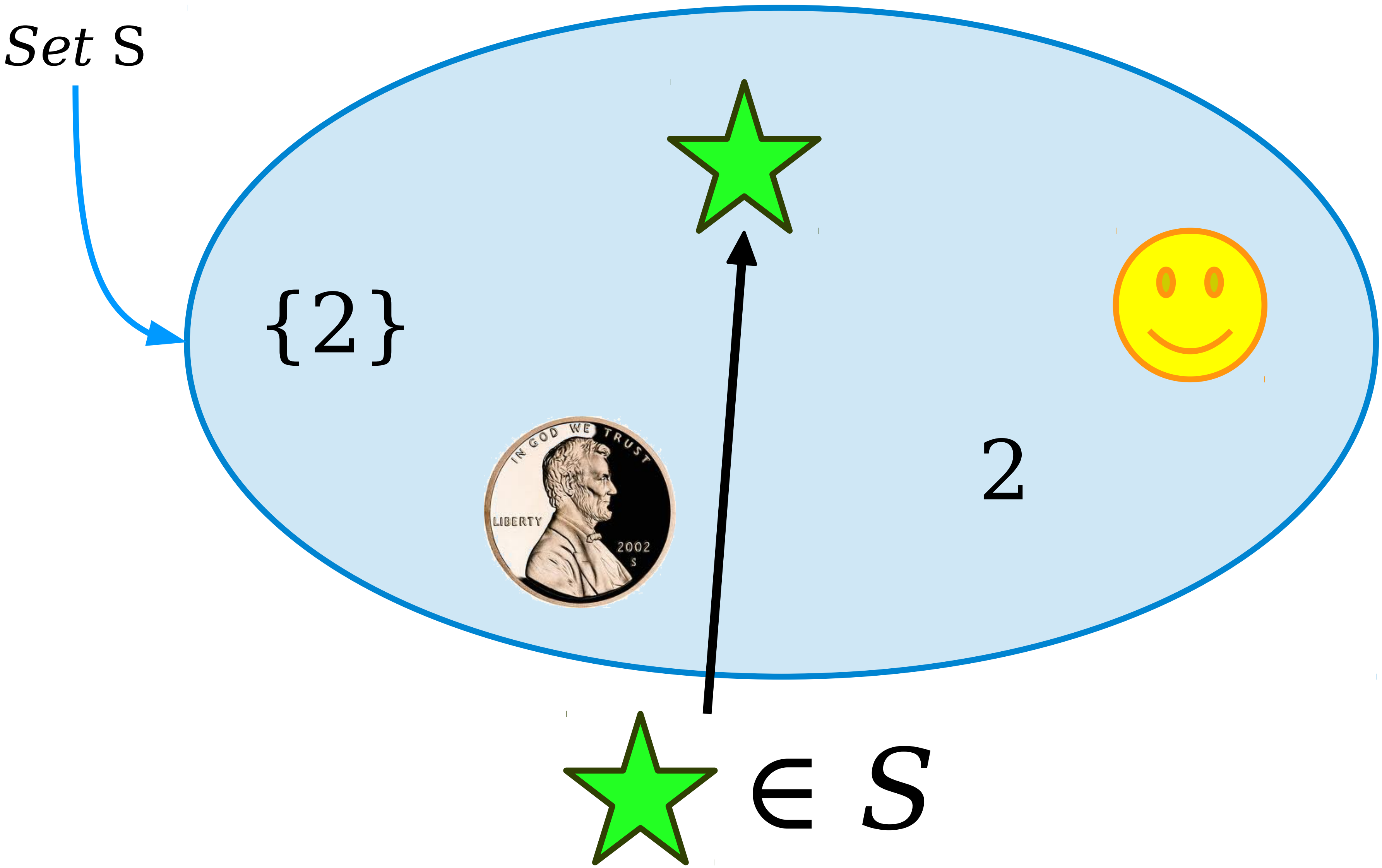
Subsets and Elements



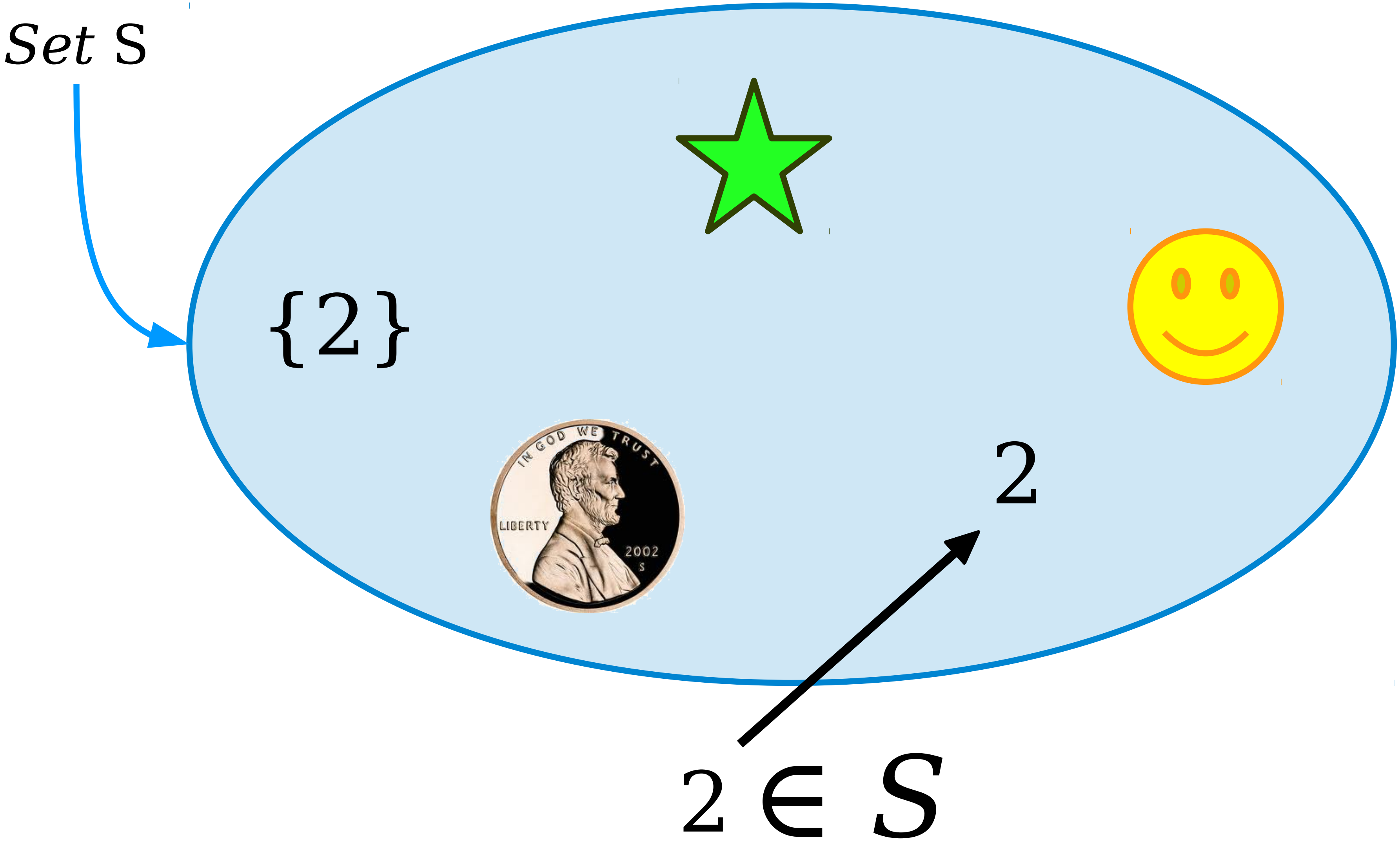
Subsets and Elements



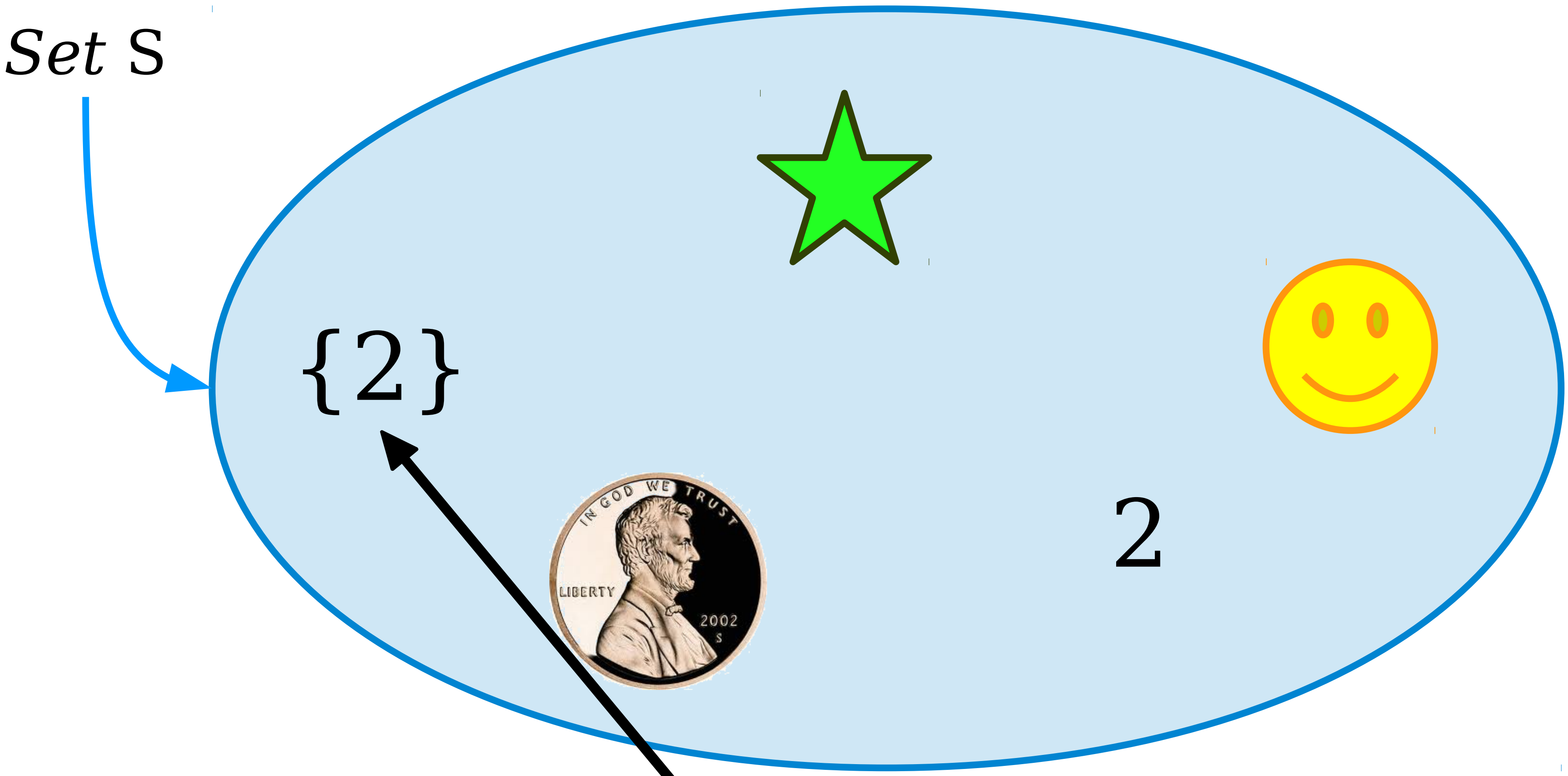
Subsets and Elements



Subsets and Elements

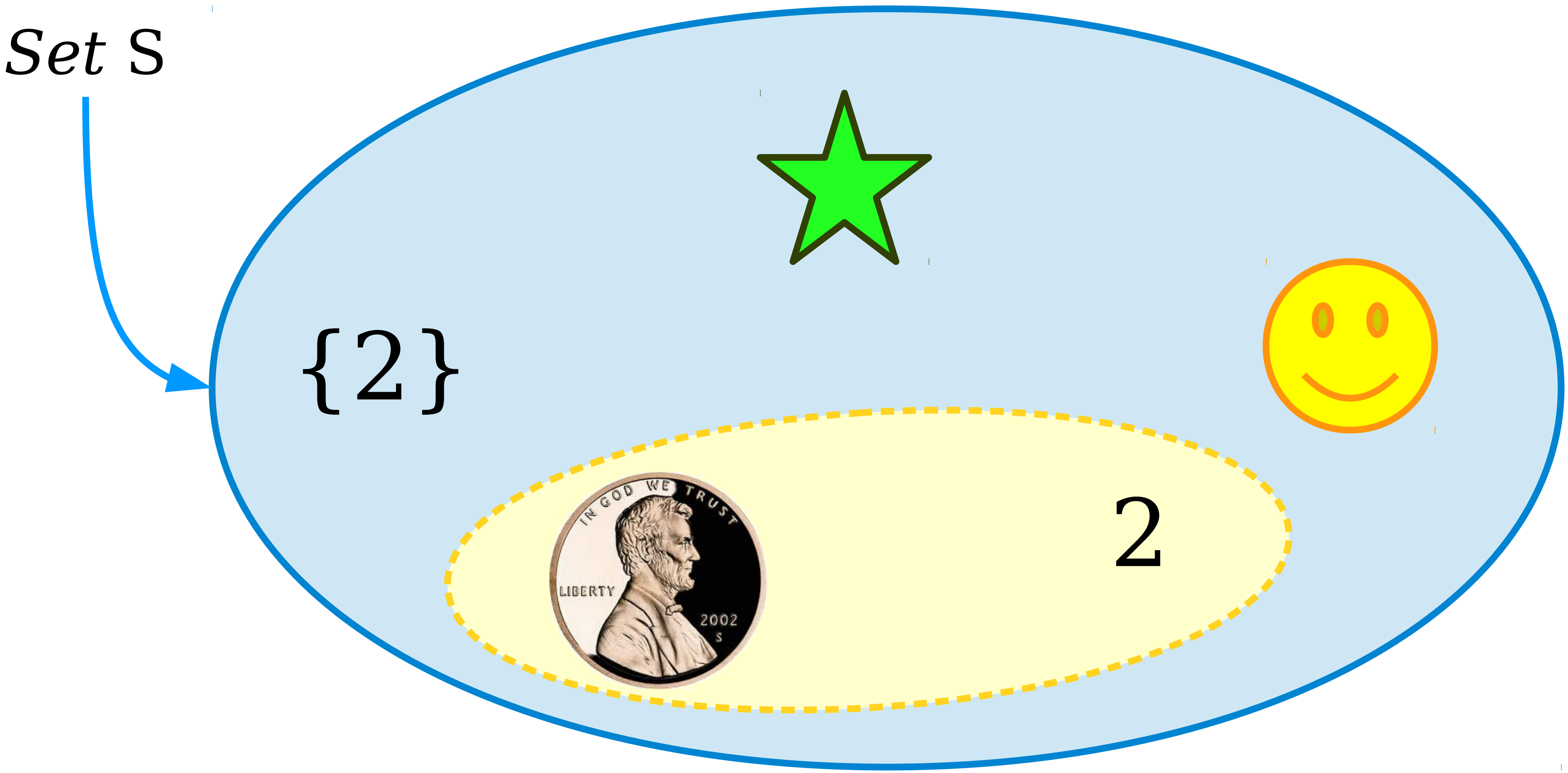


Subsets and Elements

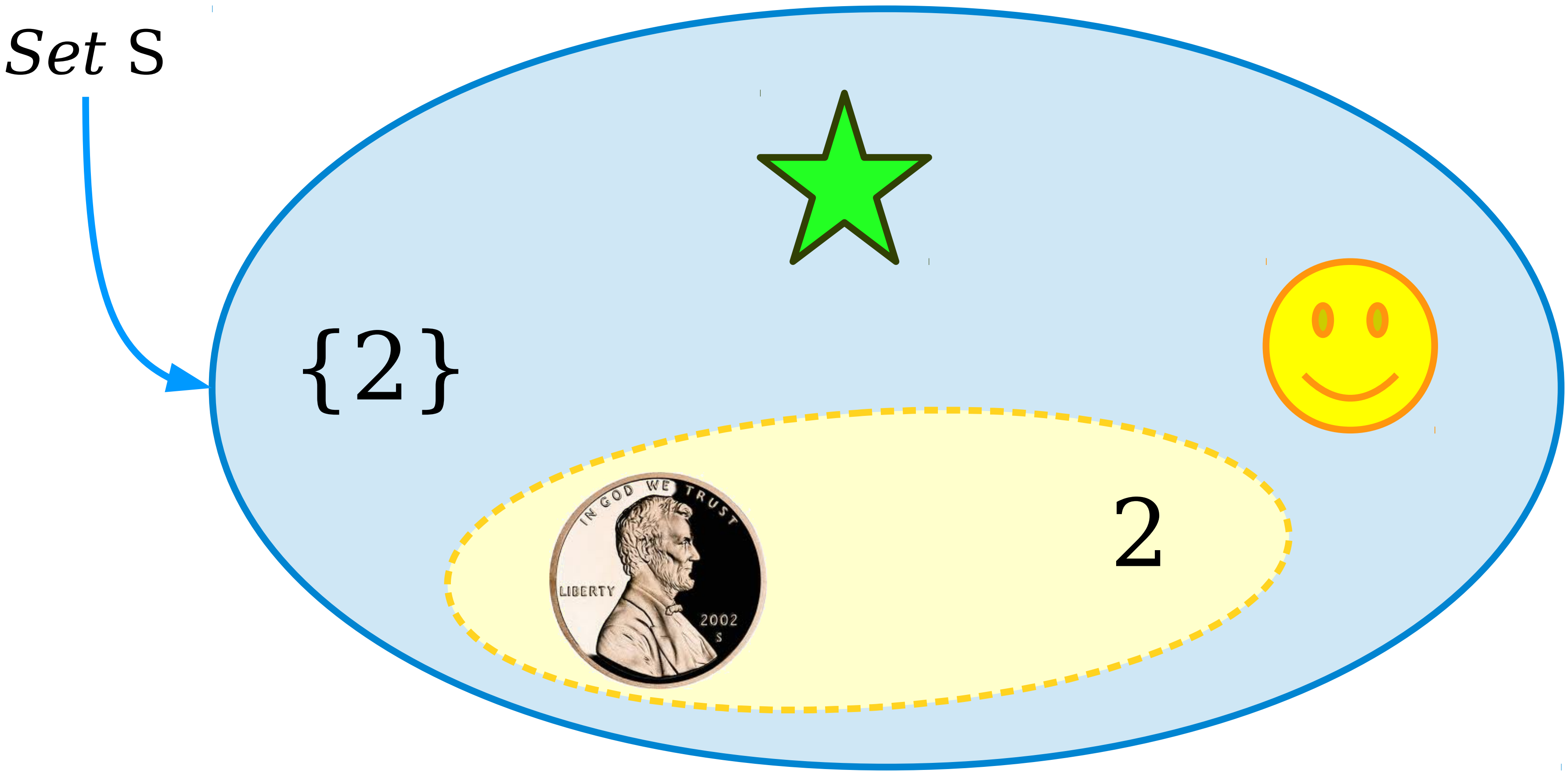


$\{2\} \in S$

Subsets and Elements

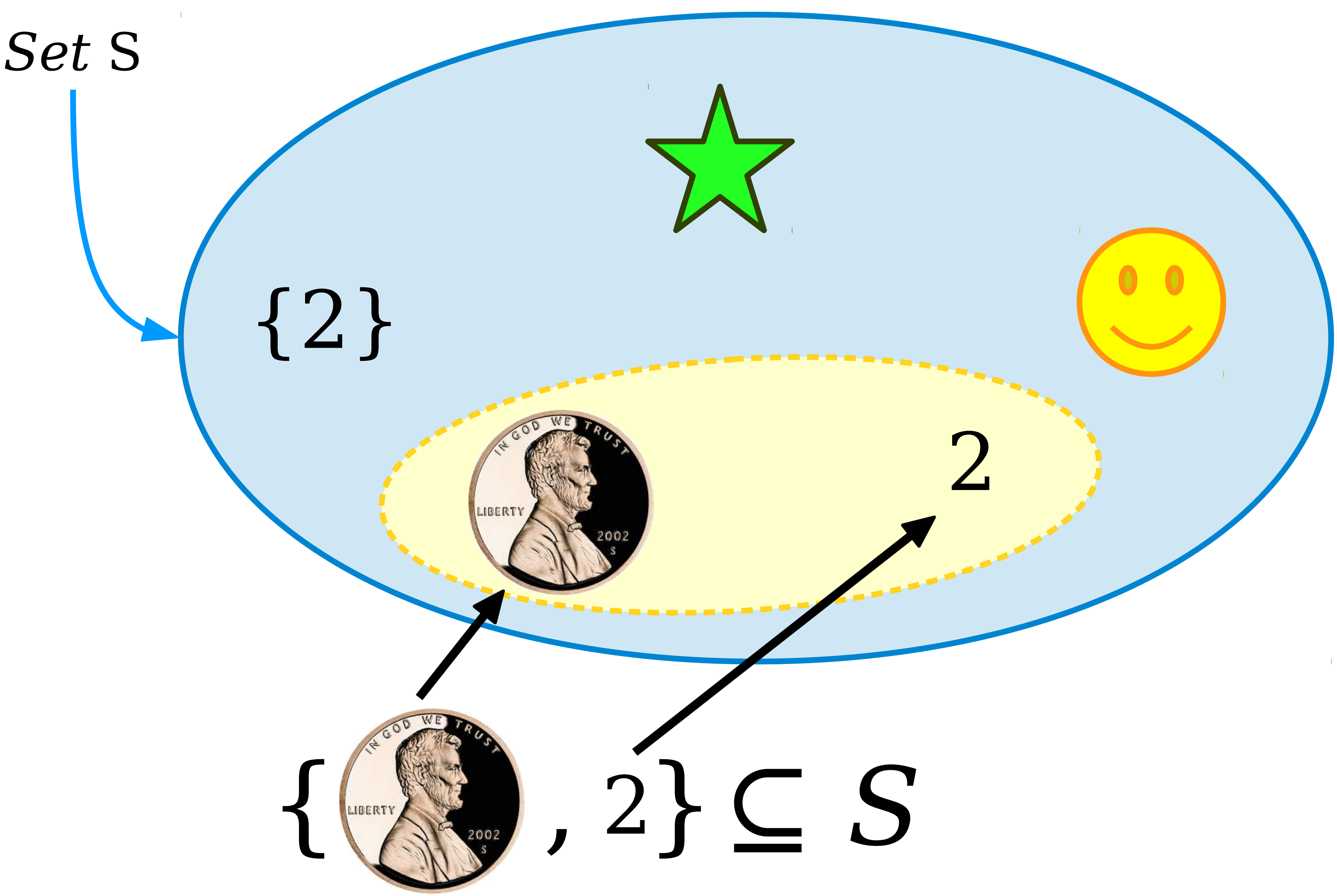


Subsets and Elements

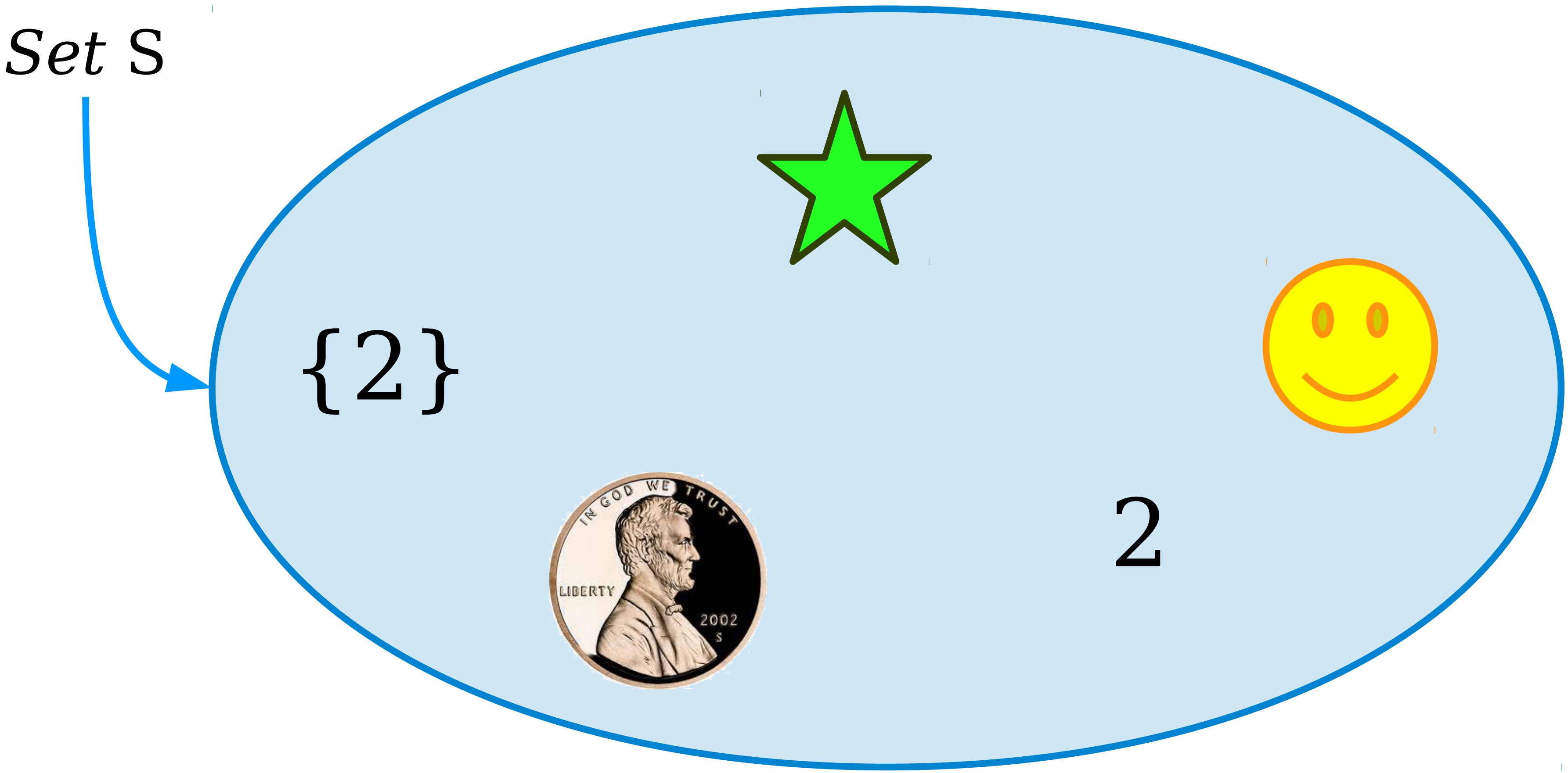


$\{$  $, 2\} \subseteq S$

Subsets and Elements

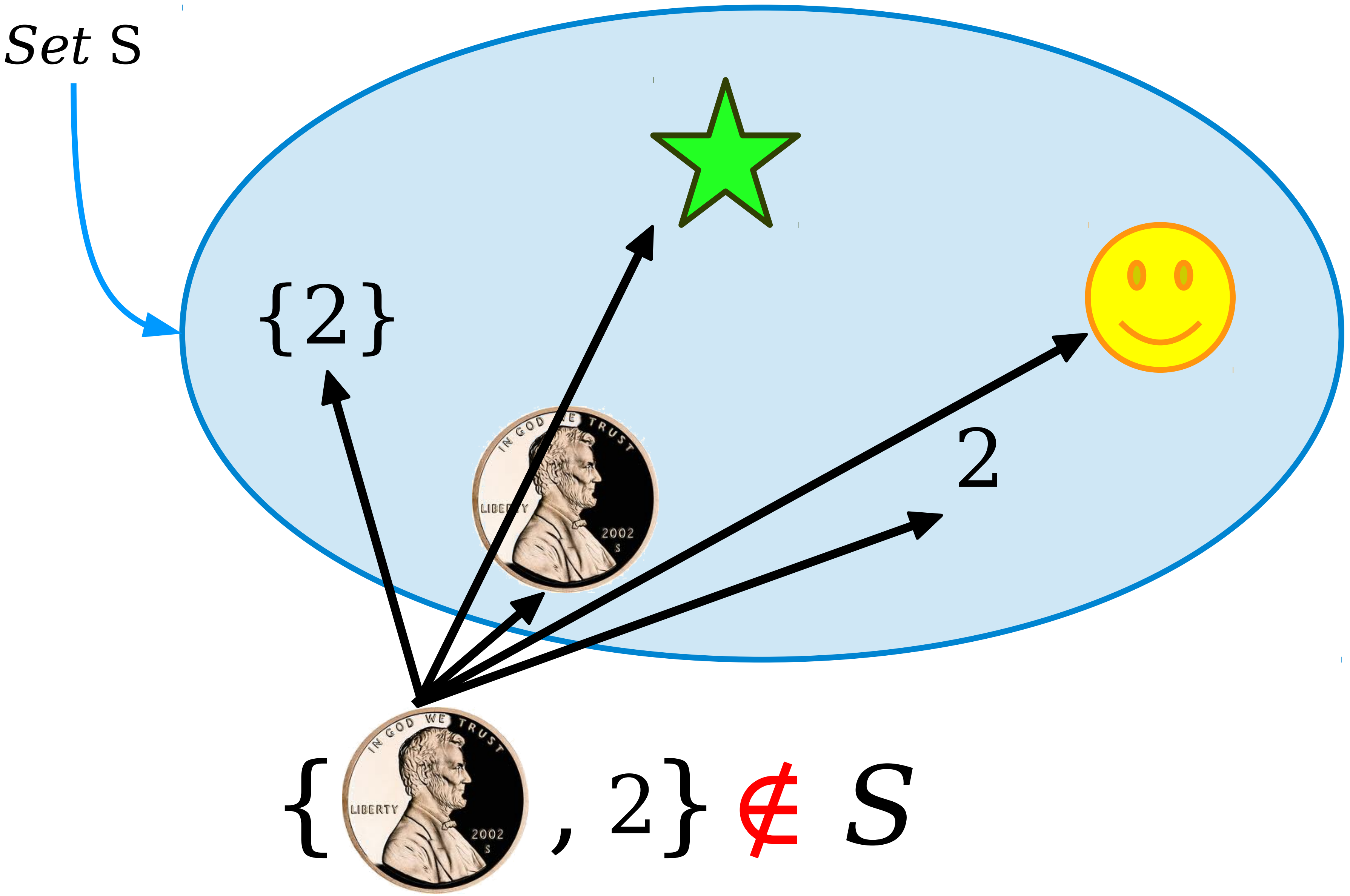


Subsets and Elements

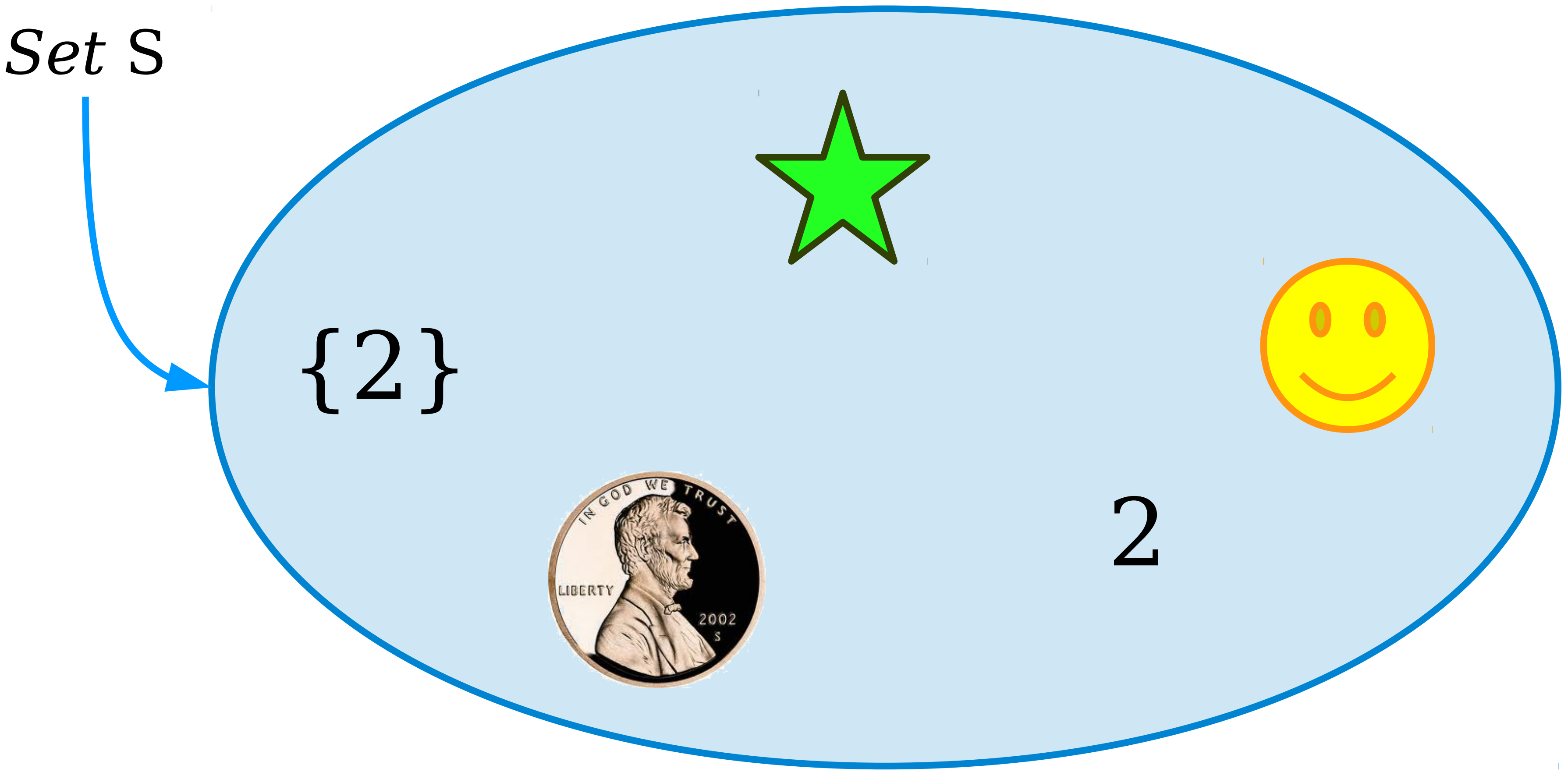


$\{$  $, 2\} \notin S$

Subsets and Elements

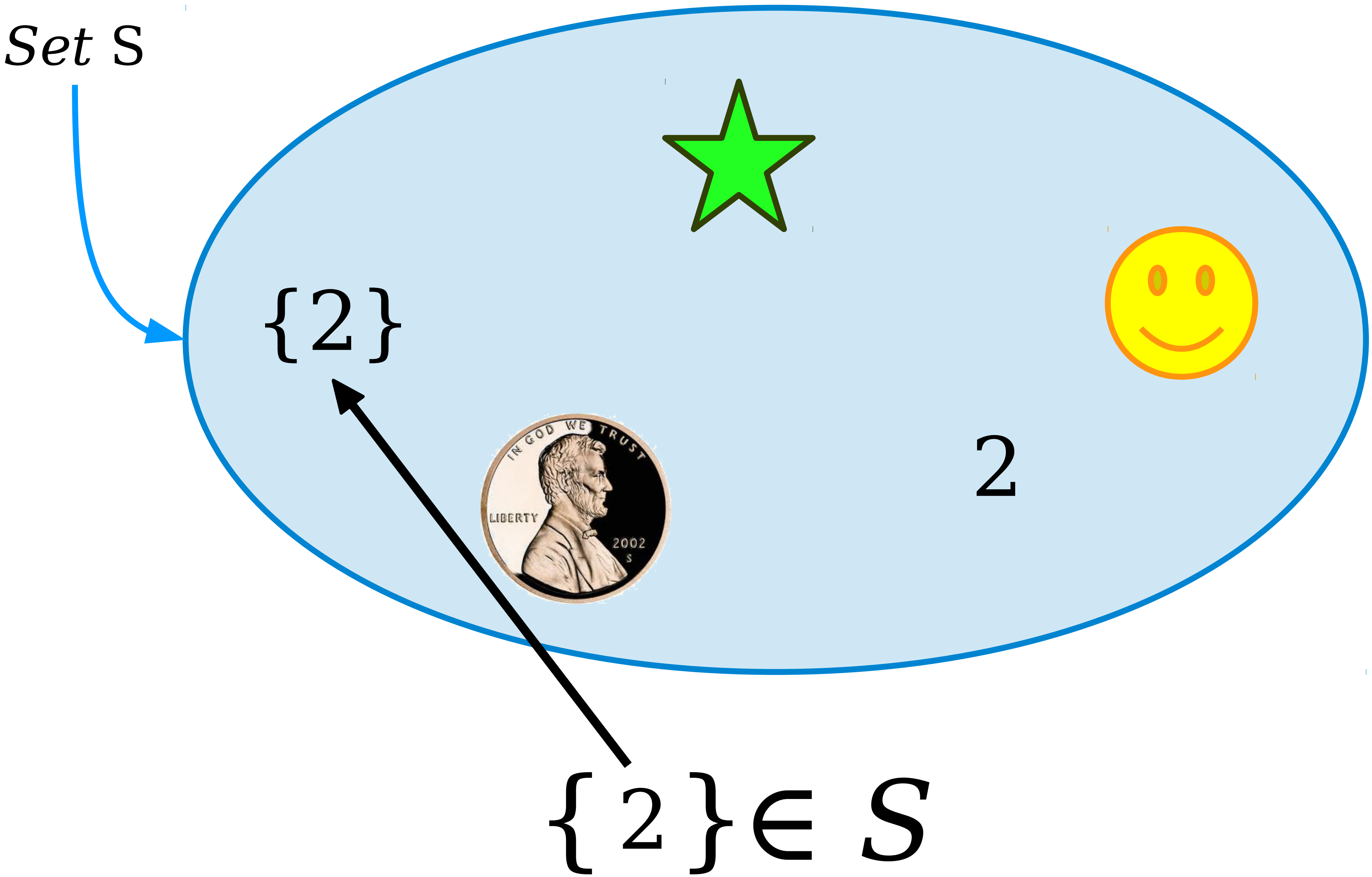


Subsets and Elements

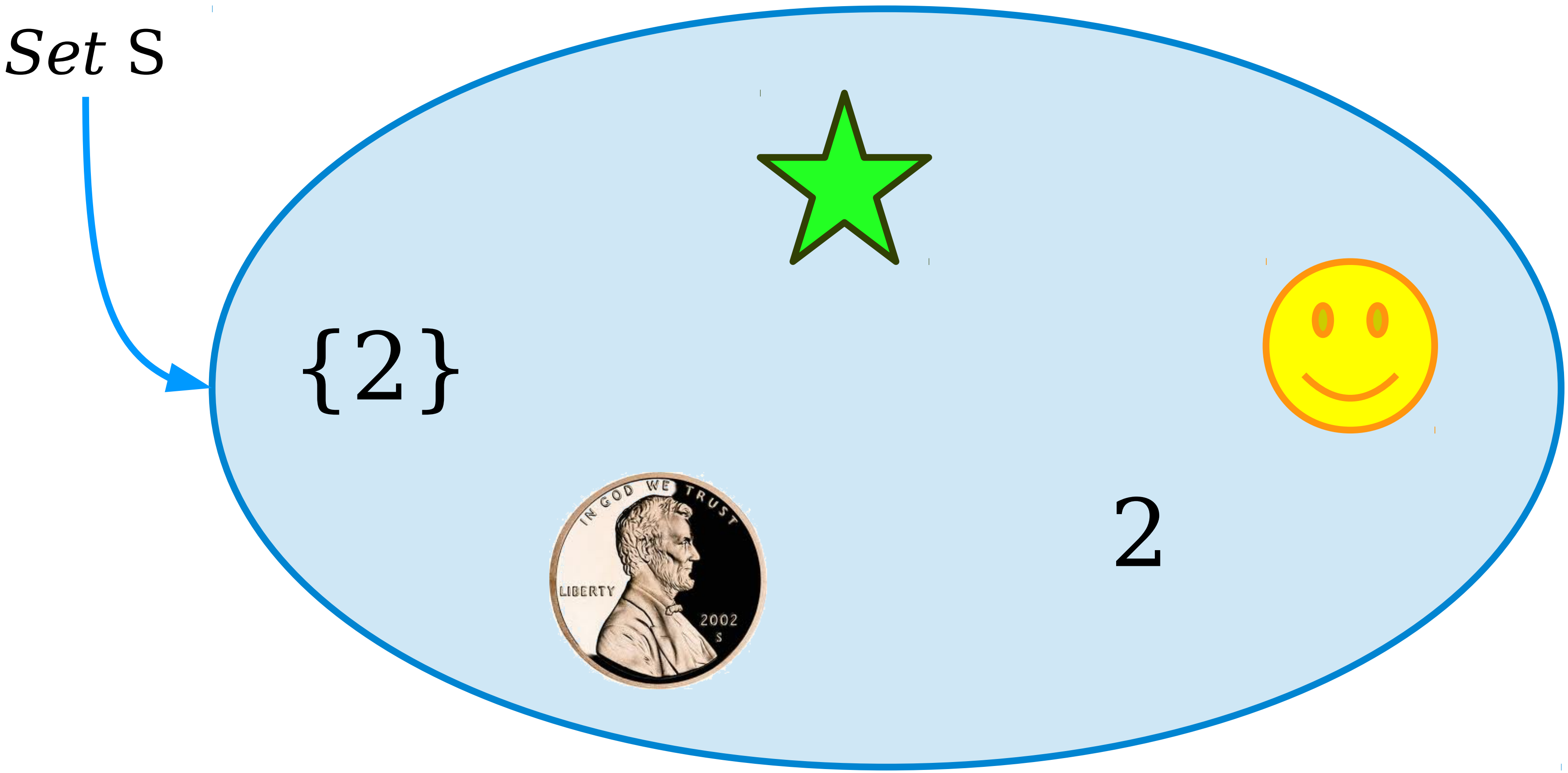


$$\{2\} \in S$$

Subsets and Elements

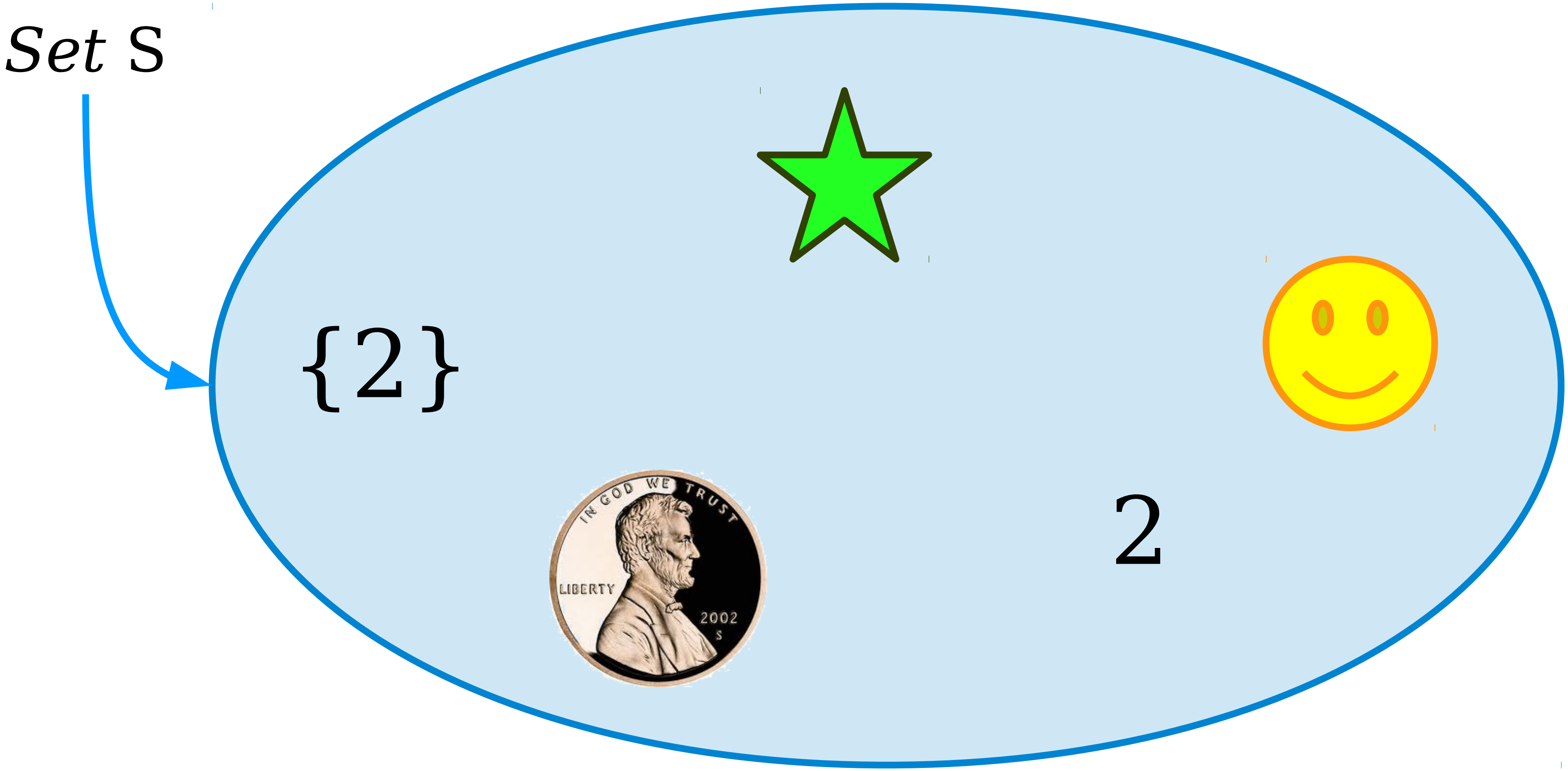


Subsets and Elements



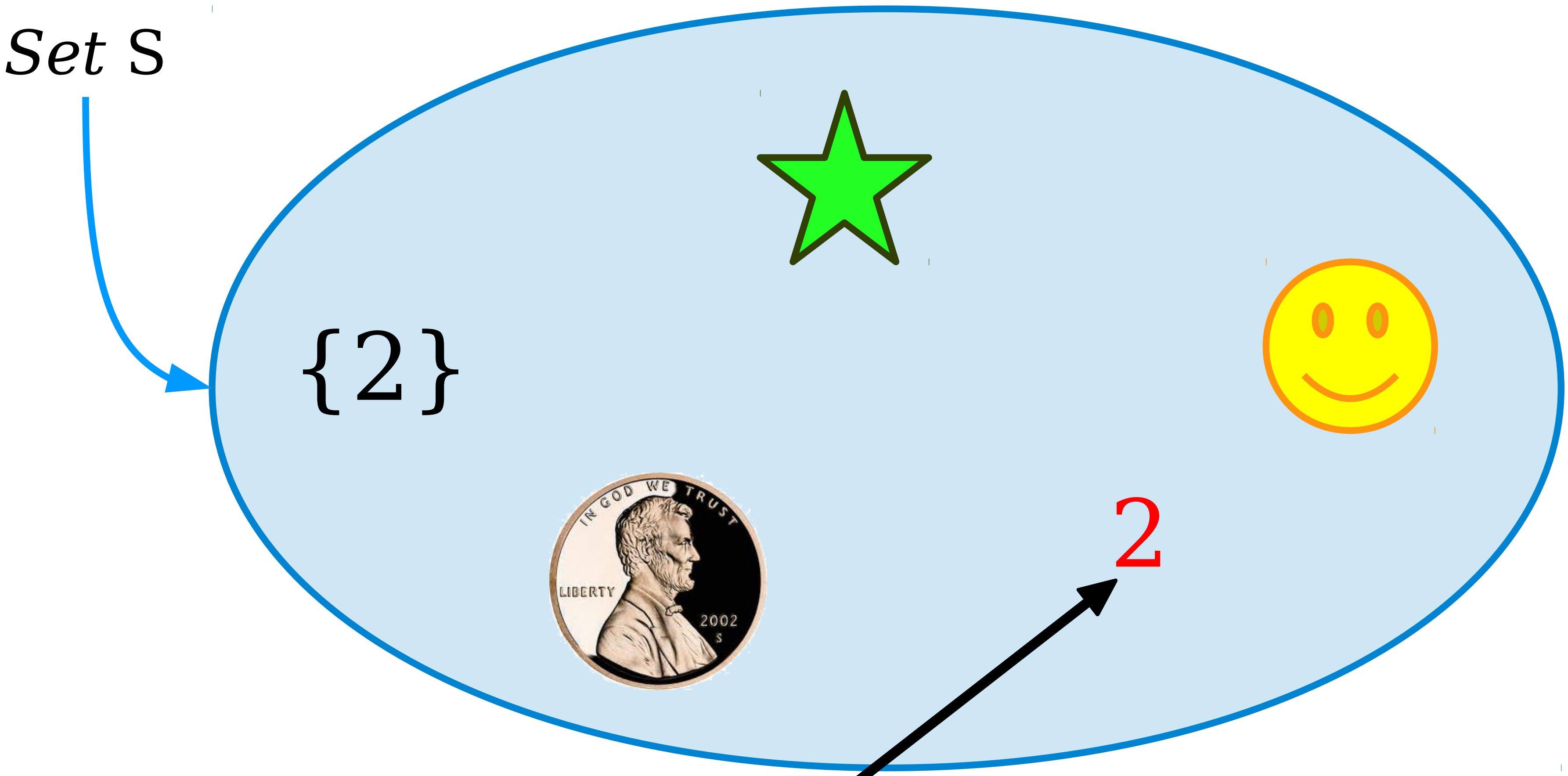
$$\{2\} \subseteq S$$

Subsets and Elements



$$\{2\} \subseteq S$$

Subsets and Elements



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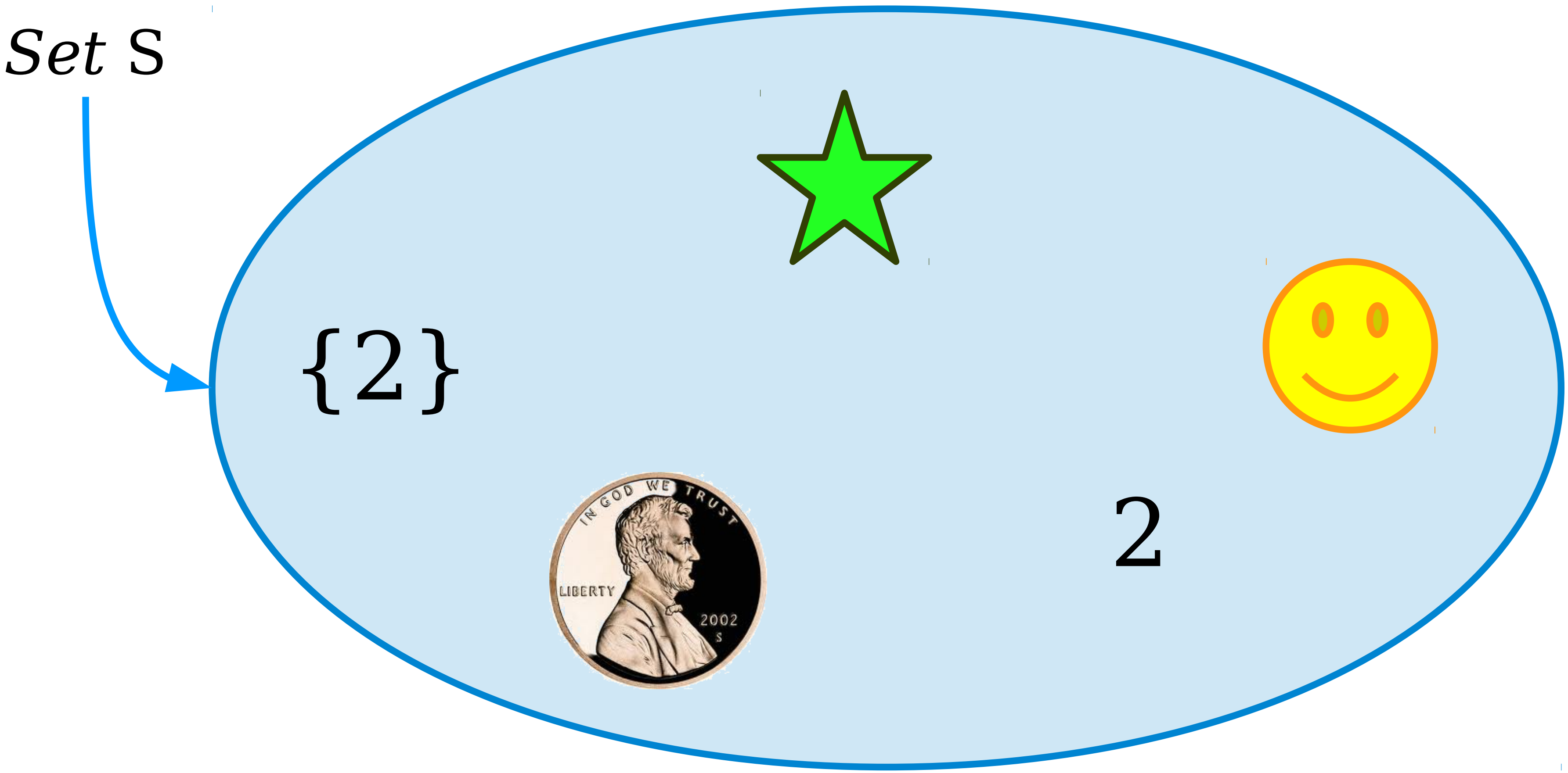
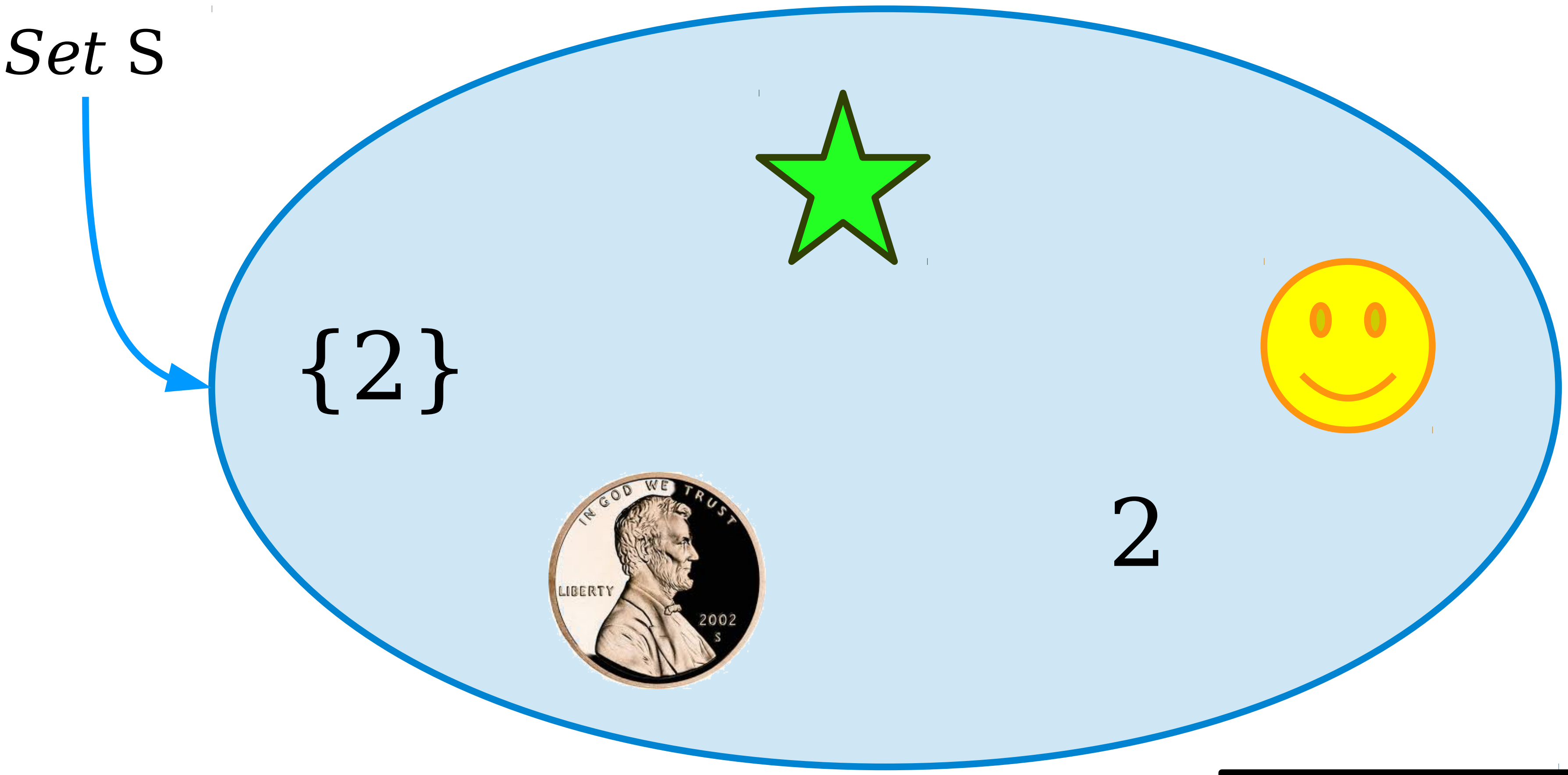
Subsets and Elements



$$2 \nsubseteq S$$

Subsets and Elements



$$2 \not\subseteq S$$

(Since 2 isn't a set.)

Subsets and Elements

- We say that $x \in S$ if, among the elements of S, one of them is *exactly* the object x.
- We say that $T \subseteq S$ if every element of T is also an element of S. (T has to be a set for the statement $T \subseteq S$ to be meaningful.)
- Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
- Don't worry if you're still wrapping your head around this – it's tricky! You'll get to play around with this on Problem Set One.

What About the Empty Set?

- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Are there any sets S where $\emptyset \subseteq S$?
- Equivalently, is there a set S where the following statement is true?

"All elements of Ø are also elements of *S*"

- **Yes!** In fact, this statement is true for every choice of S!

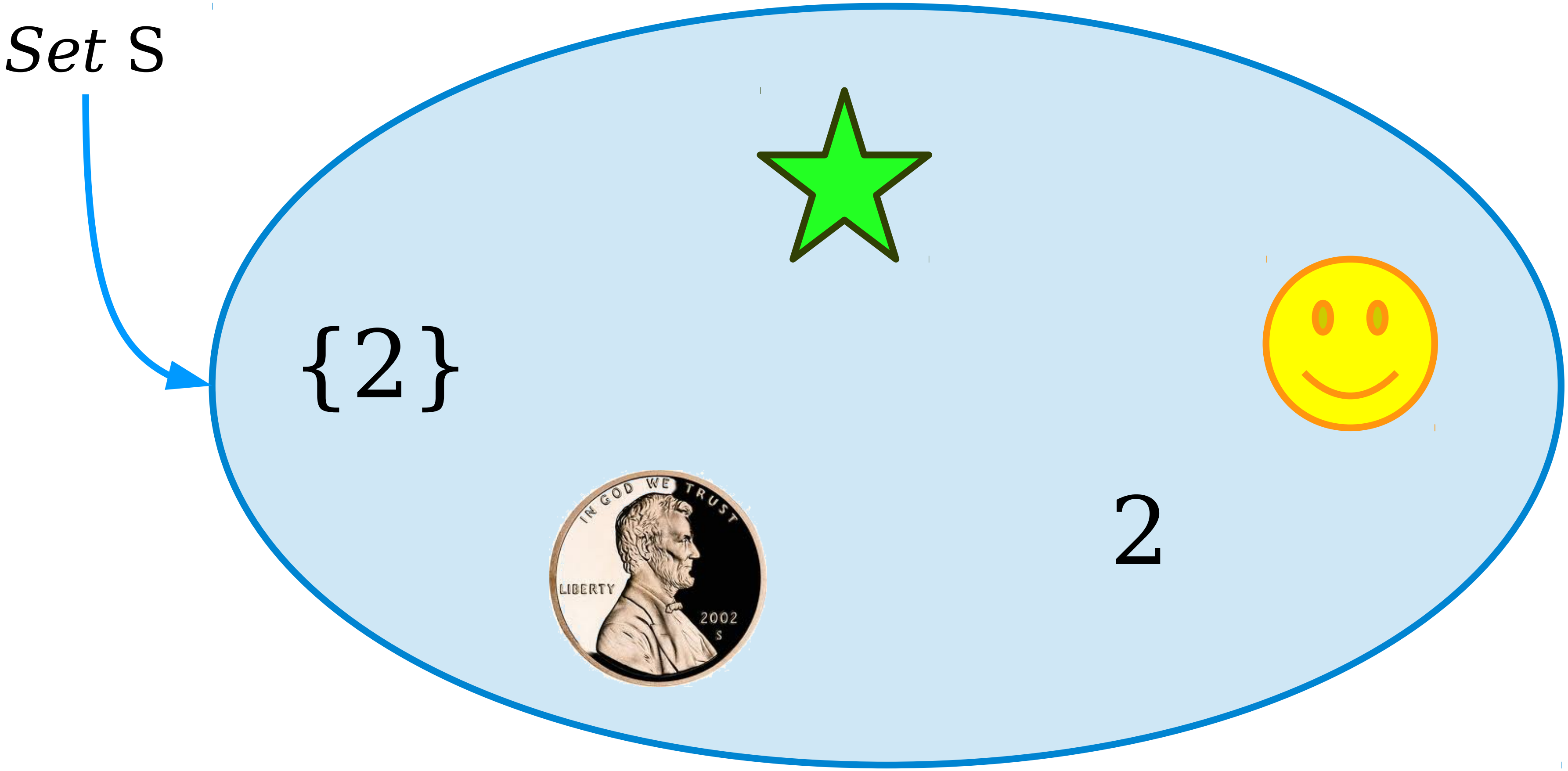
Vacuous Truth

- A statement of the form

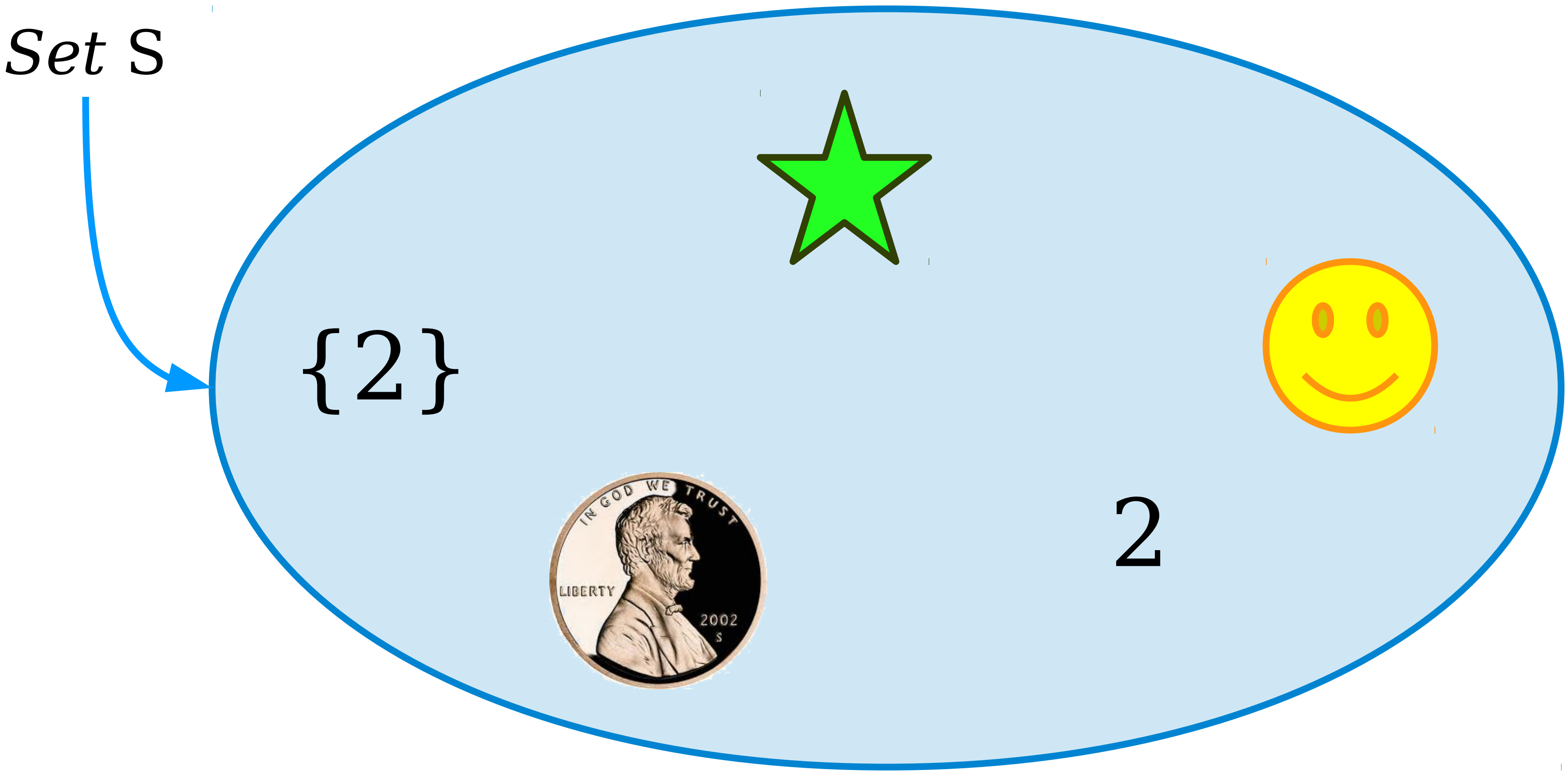
  "All objects of type *P* are also of type *Q*"

  is called ***vacuously true*** if there are no objects of type P.
- Vacuously true statements are true *by definition*. This is a convention used throughout mathematics.
- Some examples:
  - All unicorns are pink.
  - All unicorns are blue.
  - Every element of Ø is also an element of S.

Subsets and Elements

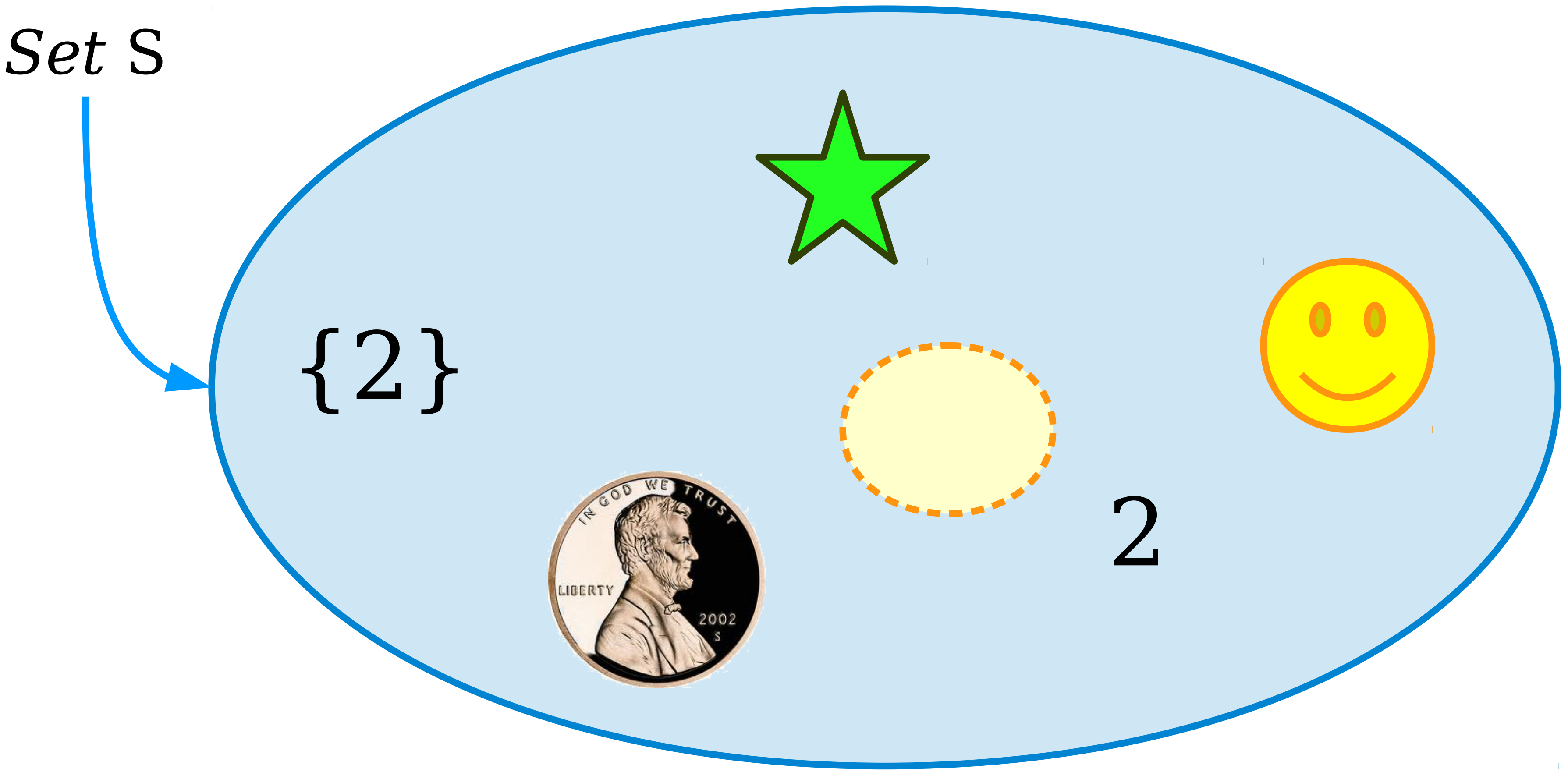


Subsets and Elements



$$\emptyset \subseteq S$$

Subsets and Elements



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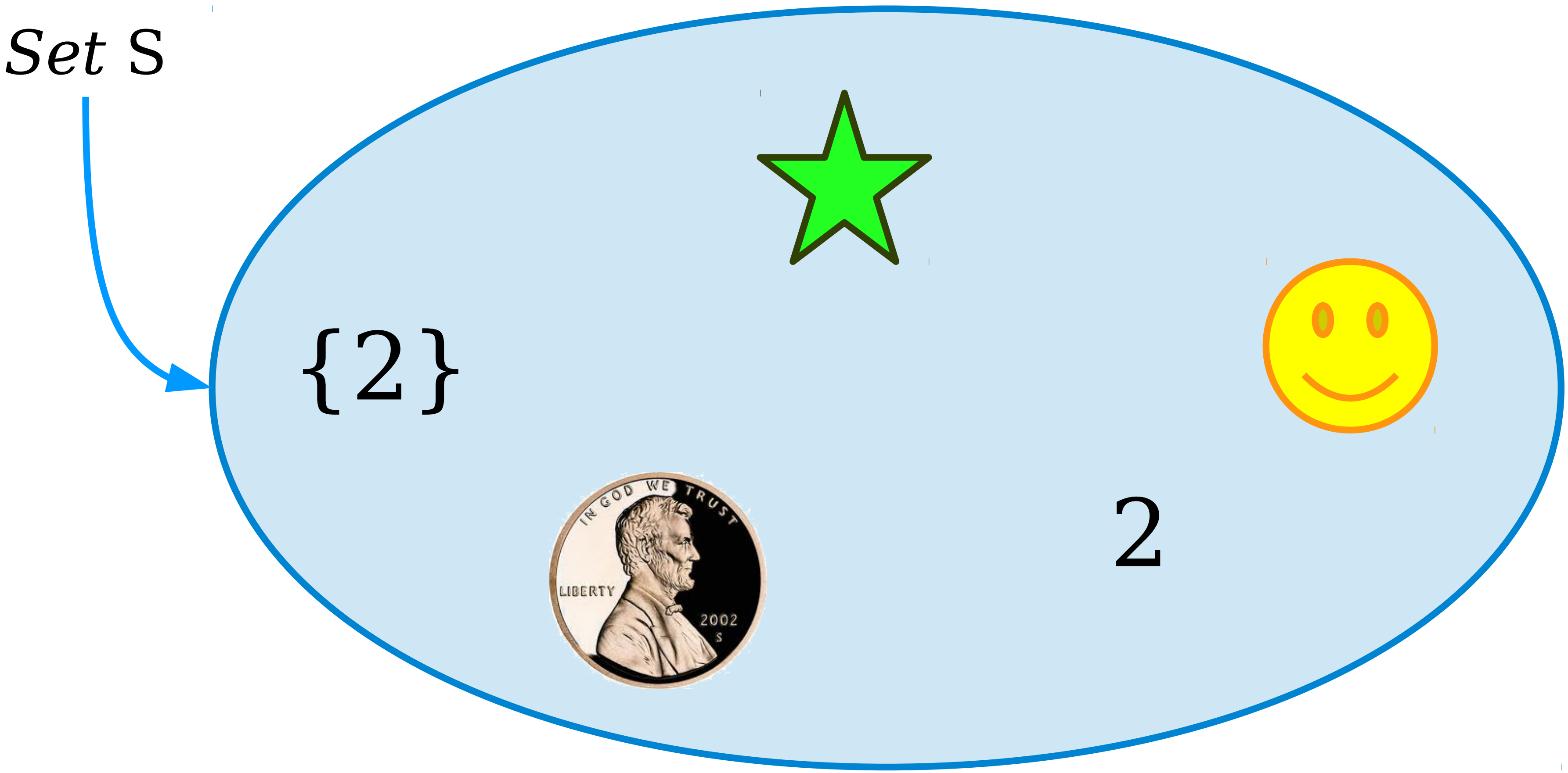
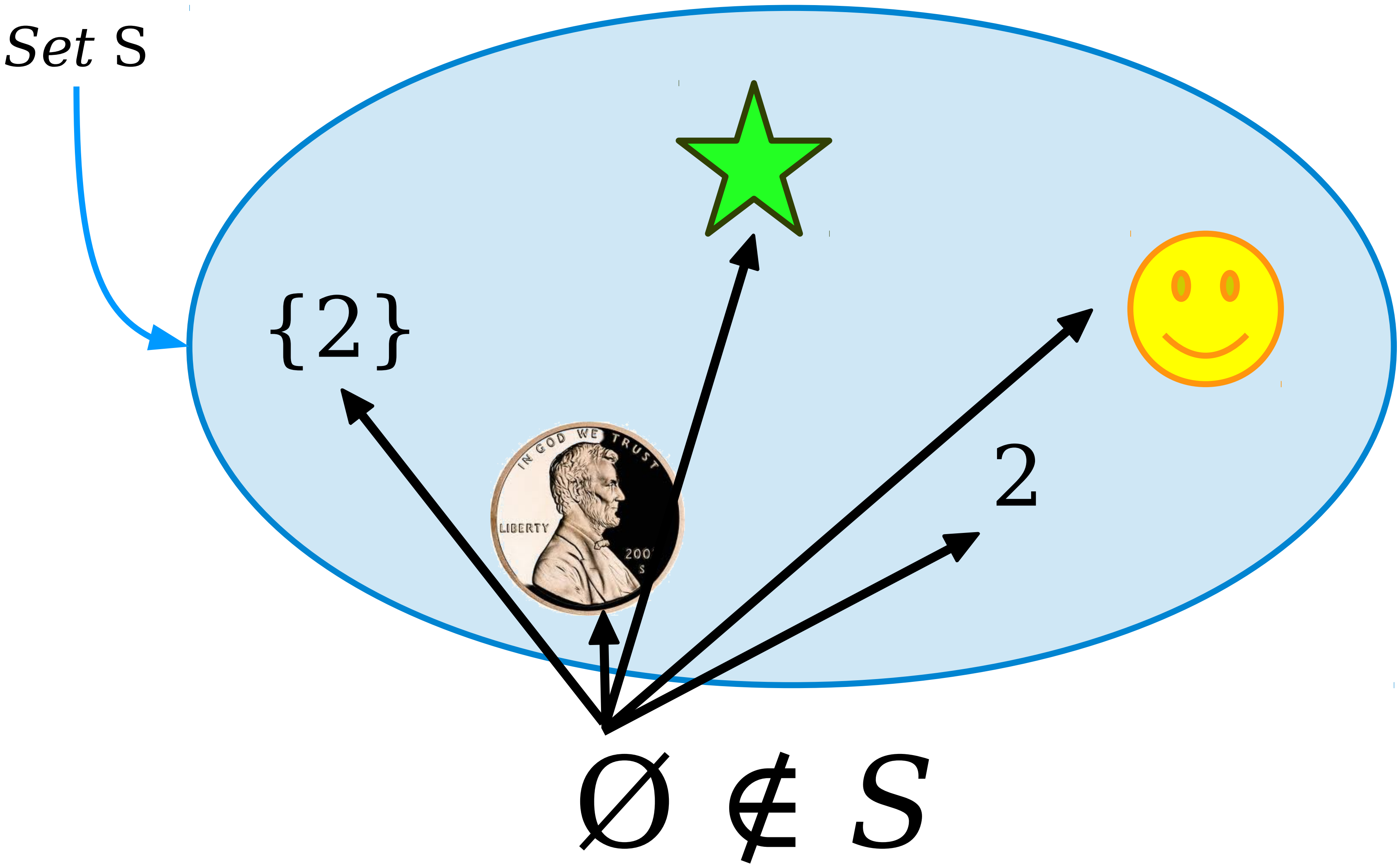
Subsets and Elements



$$\emptyset \notin S$$

Subsets and Elements



$S = \{$ ,  $\}$

$\wp(S) = \{ \emptyset,$ $\{$  $\}$, $\{$  $\}$, $\{$ ,  $\} \}$

$\wp(S)$ is the ***power set*** of S (the set of all subsets of S)

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Cardinality

Cardinality

Cardinality

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by writing $|S|$.
- Examples:
  - $|\{a, b, c, d, e\}| = 5$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$

The Cardinality of $\mathbb{N}$

- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.

The Cardinality of $\mathbb{N}$

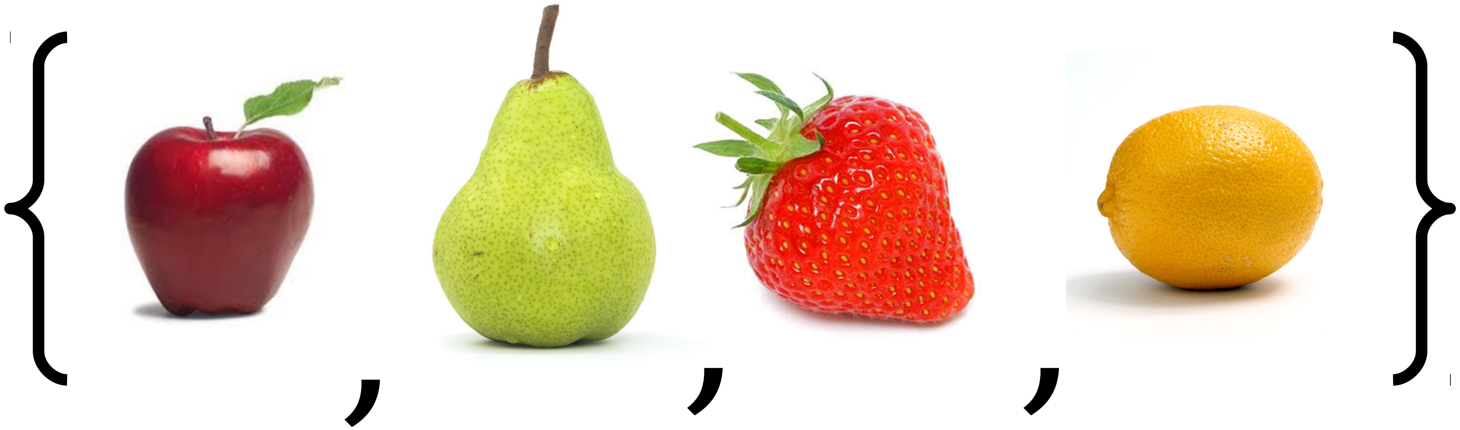
- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define $\aleph_0 = |\mathbb{N}|$.
  - $\aleph_0$ is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”

Consider the set

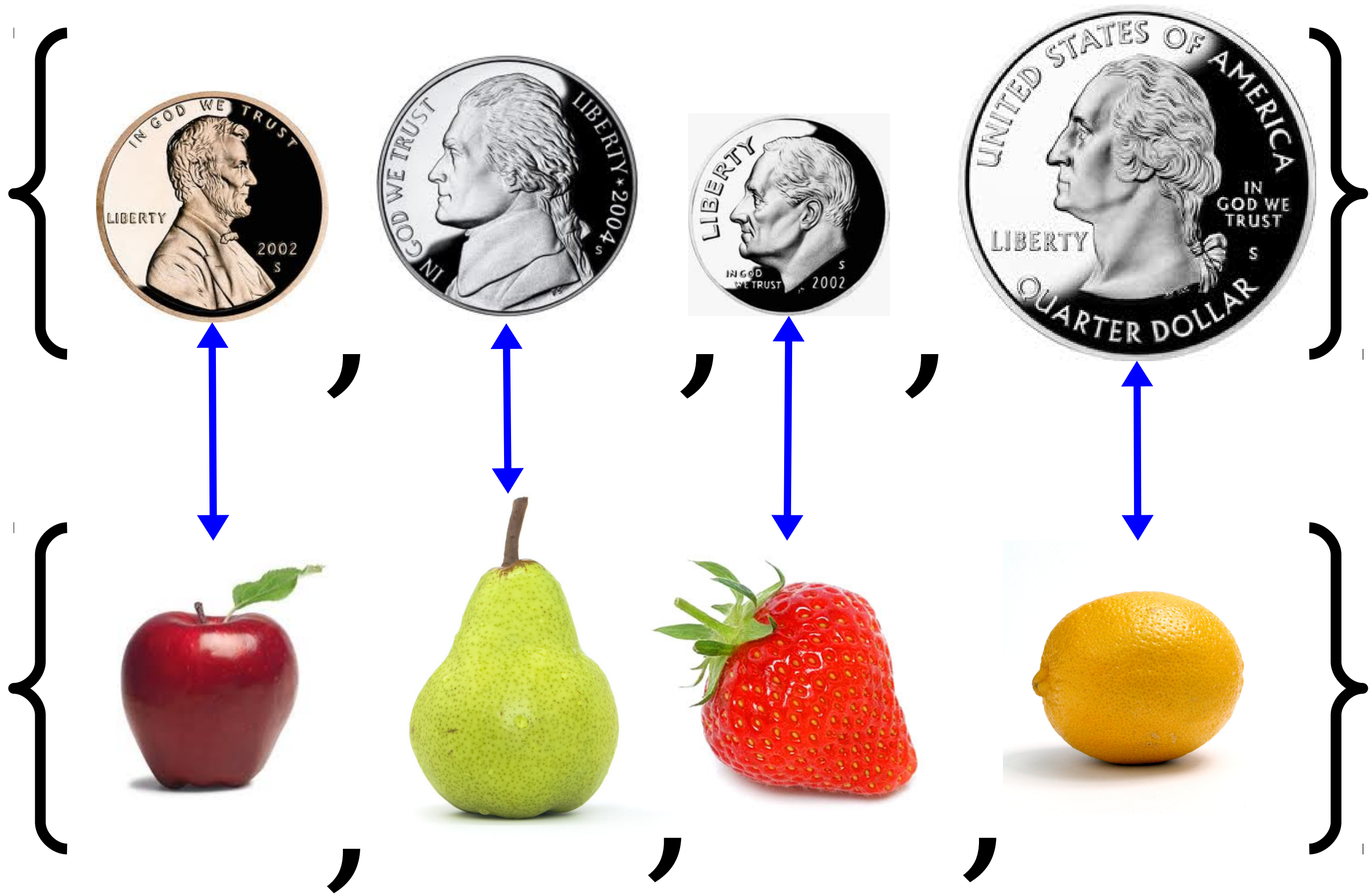
$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even }\}$$

What is $|S|$?

How Big Are These Sets?

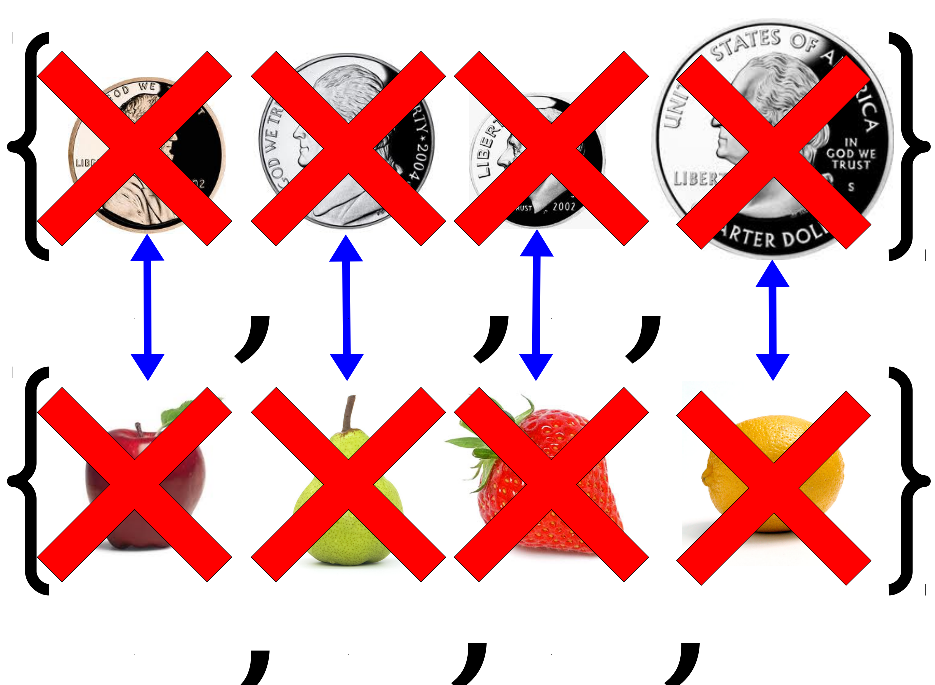


How Big Are These Sets?



Comparing Cardinalities

- *By definition,* two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered.
- The intuition:



Comparing Cardinalities

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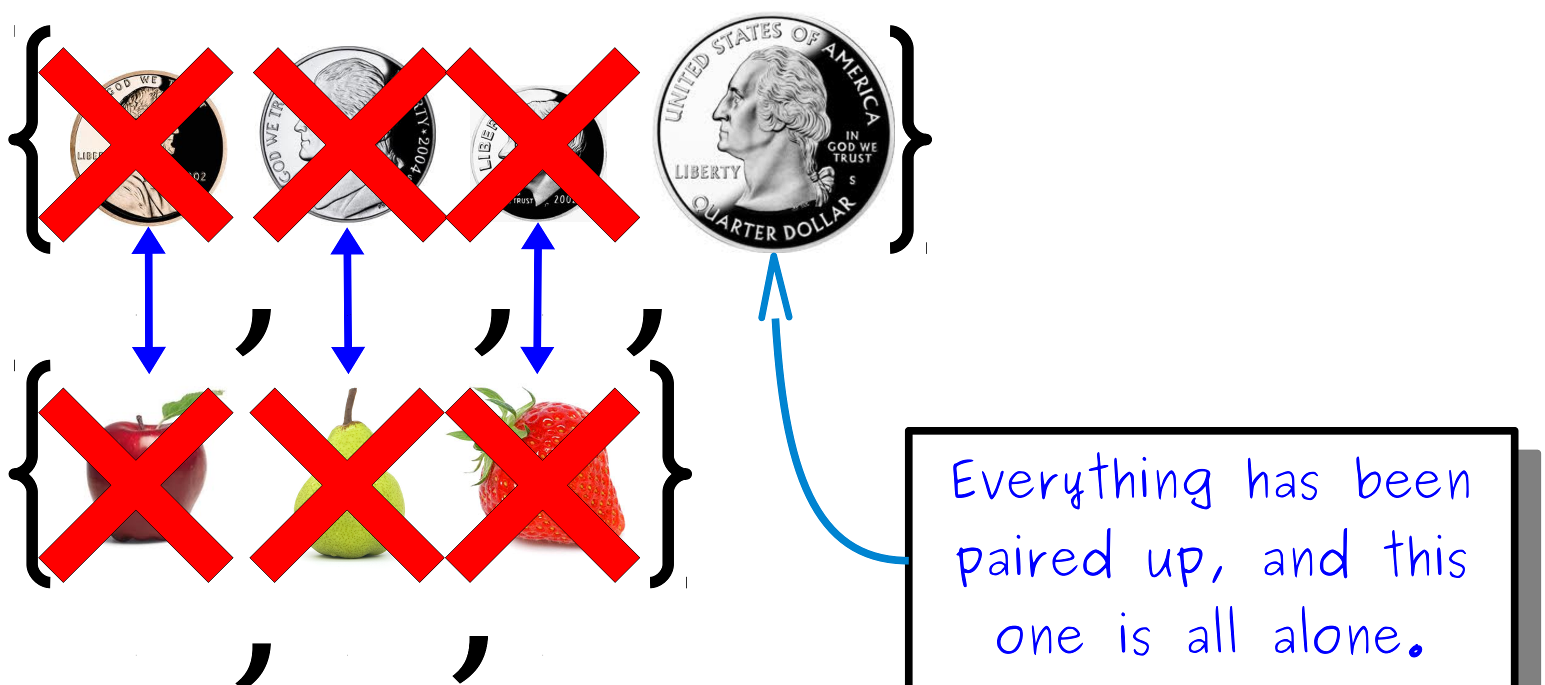

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
S	0		2		4		6		8	...

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even }\}$$

Two sets have the same size if there is a way to pair their elements off without leaving any elements uncovered

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S	0		2		4		6		8	…

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even }\}$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
S	0	2	4	6	8	10	12	14	16	…

$$n \leftrightarrow 2n$$

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even }\}$$

$$|S| = |\mathbb{N}| = \aleph_0$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	4	…

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
$\mathbb{Z}$					0	1	2	3	4	…
	…	-3	-2	-1						

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	1	2	3	4	5	6	7	8	...
	...	-3	-2	-1						

Infinite Cardinalities

$$
\begin{array}{cccccccccccc}
\mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Z} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
 & \ldots & -3 & -2 & -1 & & & & & &
\end{array}
$$

Two sets have the same size if *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	4	…

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
$\mathbb{Z}$										
	…	-3	-2	-1	0	1	2	3	4	…

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0		1		2		3		4	...
	...	-3	-2	-1						

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0		1		2		3		4	...
	...	-3	-2	-1						

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	↕		↕		↕		↕		↕	
$\mathbb{Z}$	0		1		2		3		4	...
	...	-3	-2	-1						

Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
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Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
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Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

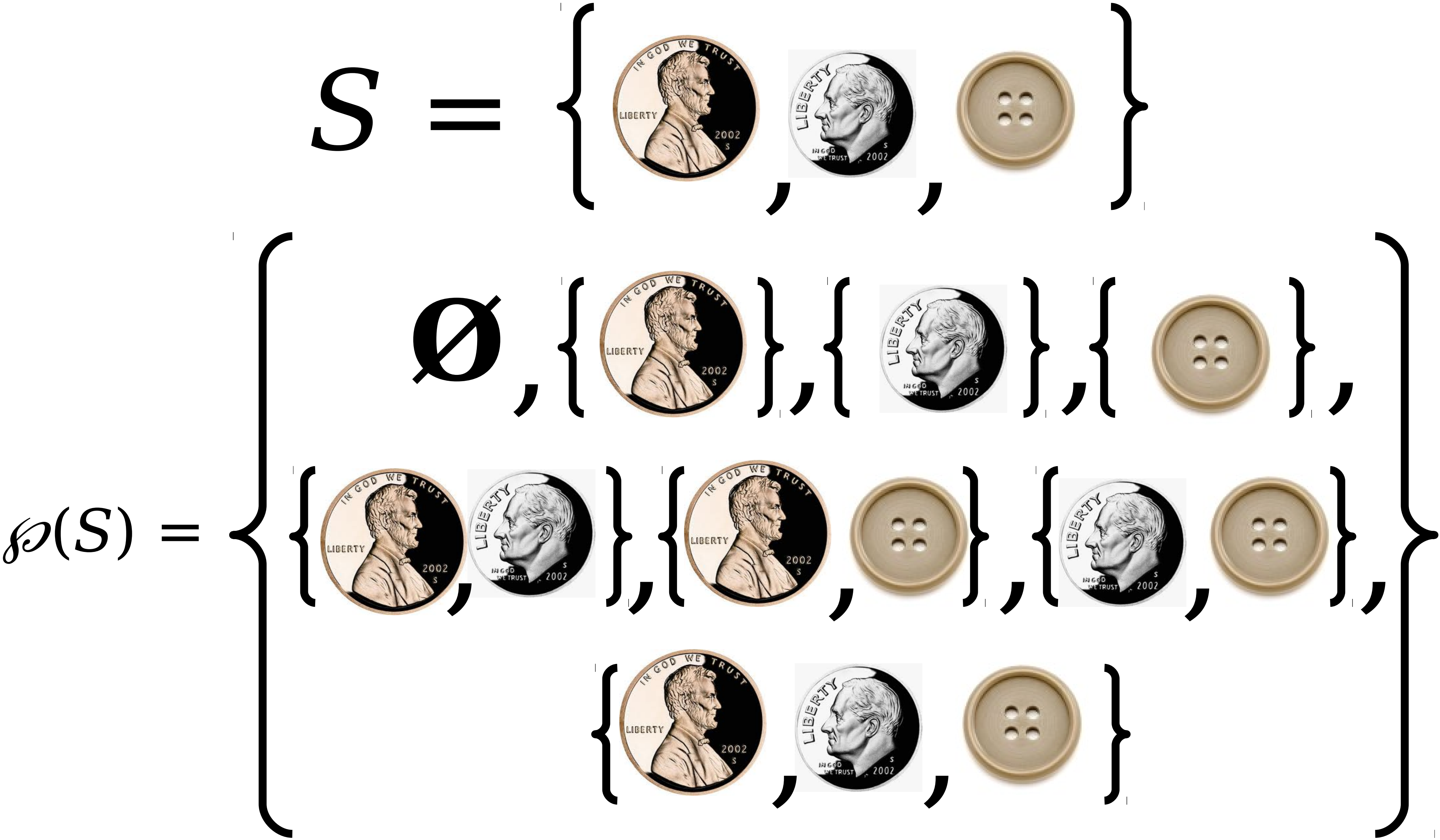
Important Question:

Do all infinite sets have the same cardinality?

$S = \{$  ,  $\}$

$\wp(S) = \{ \emptyset, \{$  $\}, \{$  $\}, \{$  ,  $\} \}$

$$|S| < |\wp(S)|$$



$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$
$$\varnothing,$$
$$\{a\}, \{b\}, \{c\}, \{d\},$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$
$$\{a, b, c, d\}$$
$$\}$$

$$|S| < |\wp(S)|$$

If $|S|$ is infinite, what is the relation between $|S|$ and $|\wp(S)|$?

Does $|S| = |\wp(S)|$?

If $|S| = |\wp(S)|$, we can pair up the elements of S and the elements of $\wp(S)$ without leaving anything out.

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If $|S| = |\wp(S)|$, we can pair up the elements of S and the subsets of S without leaving anything out.

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If $|S| = |\wp(S)|$, we can pair up the elements of S and the subsets of S without leaving anything out.

What would that look like?

x_0

x_1

x_2

x_3

x_4

x_5

$\ldots$

$$x_0 \longleftrightarrow \{\ x_0,\ x_2,\ x_4,\ \ldots\ \}$$

$$x_1 \longleftrightarrow \{\ x_0,\ x_3,\ x_4,\ \ldots\ \}$$

$$x_2 \longleftrightarrow \{\ x_4,\ \ldots\ \}$$

$$x_3 \longleftrightarrow \{\ x_1,\ x_4,\ \ldots\ \}$$

$$x_4 \longleftrightarrow \{\ x_0,\ x_5,\ \ldots\ \}$$

$$x_5 \longleftrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$$

$$\ldots$$

x_0	x_1	x_2	x_3	x_4	x_5	…

$$x_0 \longleftrightarrow \{\ x_0,\ x_2,\ x_4,\ \ldots\ \}$$

$$x_1 \longleftrightarrow \{\ x_0,\ x_3,\ x_4,\ \ldots\ \}$$

$$x_2 \longleftrightarrow \{\ x_4,\ \ldots\ \}$$

$$x_3 \longleftrightarrow \{\ x_1,\ x_4,\ \ldots\ \}$$

$$x_4 \longleftrightarrow \{\ x_0,\ x_5,\ \ldots\ \}$$

$$x_5 \longleftrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**

$$x_1 \longleftrightarrow \{\, x_0, x_3, x_4, \ldots \,\}$$

$$x_2 \longleftrightarrow \{\, x_4, \ldots \,\}$$

$$x_3 \longleftrightarrow \{\, x_1, x_4, \ldots \,\}$$

$$x_4 \longleftrightarrow \{\, x_0, x_5, \ldots \,\}$$

$$x_5 \longleftrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \ldots \,\}$$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \leftrightarrow \{\ x_4, \ldots\ \}$

$x_3 \leftrightarrow \{\ x_1, x_4, \ldots\ \}$

$x_4 \leftrightarrow \{\ x_0, x_5, \ldots\ \}$

$x_5 \leftrightarrow \{\ x_0, x_1, x_2, x_3, x_4, x_5, \ldots\ \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \longleftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \longleftrightarrow \{ \; x_4, \; \dots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \leftrightarrow \{ \quad x_1, \quad x_4, \quad \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
$x_4 \leftrightarrow$	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…							

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\left\{ \boldsymbol{x_0}, \qquad \boldsymbol{x_5}, \ \ldots \right\}$$

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**N**	**N**	**N**	**Y**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

Y	**N**	**N**	**N**	**N**	**Y**	**...**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**
	Y	**N**	**N**	**N**	**N**	**Y**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Flip all Y's to N's and vice-versa to get a new set

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\{\, \boldsymbol{x_1},\ \boldsymbol{x_2},\ \boldsymbol{x_3},\ \boldsymbol{x_4},\ \ldots \,\}$$

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Which row in the table is paired with this set?

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**Y**	**Y**	**N**	...

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**...**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**...**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**...**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**N**	**N**	**N**	**Y**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	...
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	...
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	...
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	...
...	...	...	...	...	...	...	...
	Y	**N**	**Y**	**Y**	**Y**	**N**	...

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**...**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**...**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**...**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

Y	**N**	**Y**	**Y**	**Y**	**N**	**...**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**…**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**…**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**…**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**Y**	**Y**	**Y**	**N**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	...
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	...
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	...
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	...
...	...	...	...	...	...	...	...

Y	**N**	**Y**	**Y**	**Y**	**N**	...

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	**...**
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	**...**
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	**...**
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

Y	**N**	**Y**	**Y**	**Y**	**N**	**...**

The Diagonalization Proof

- No matter how we pair up elements of S and subsets of S, the complemented diagonal won't appear in the table.
  - In row n, the nth element must be wrong.
- No matter how we pair up elements of S and subsets of S, there is *always* at least one subset left over.
- This result is ***Cantor's theorem***: Every set is strictly smaller than its power set:

If S is a set, then $|S| < |\wp(S)|$.

Infinite Cardinalities

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$

$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$

$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$

$$\dots$$

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

What does this have to do with computation?

“The set of all computer programs”

“The set of all problems to solve”

Where We're Going

- A ***string*** is a sequence of characters.
- We're going to prove the following results:
  - There are ***at most*** as many programs as there are strings.
  - There are ***at least*** as many problems as there are sets of strings.
- This leads to some *incredible* results – we'll see why in a minute!

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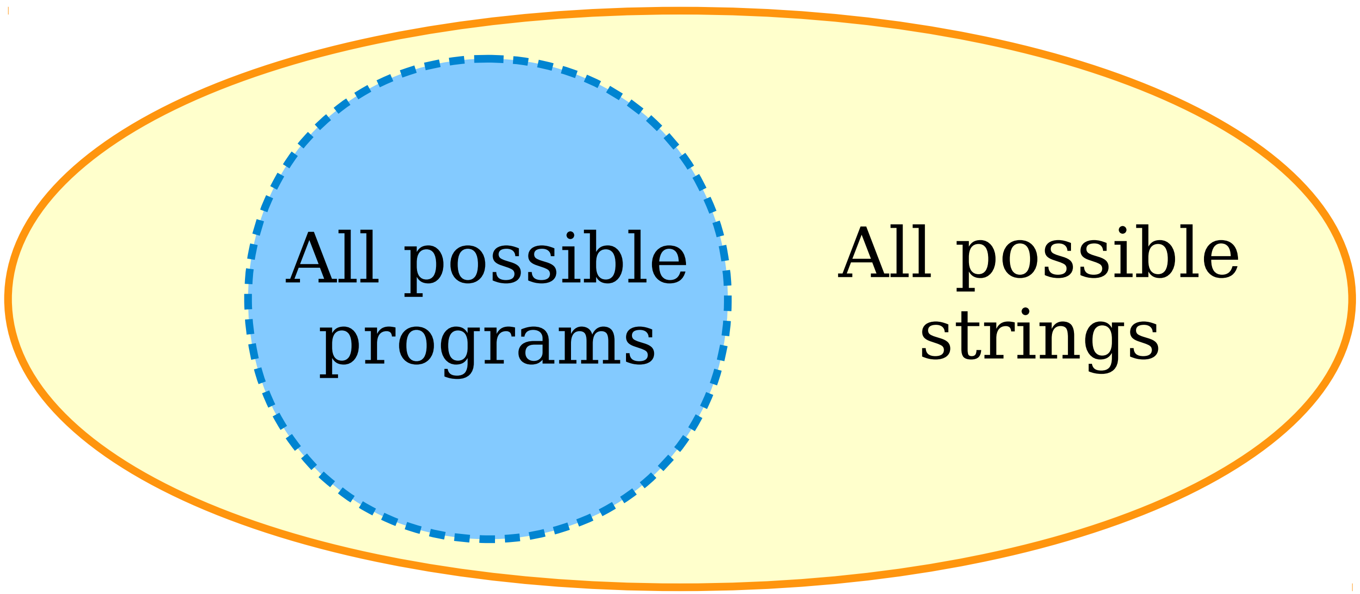
Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



|Programs| ≤ |Strings|

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- This leads to some *incredible* results – we'll see why in a minute!

Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let S be any set of strings. This set S gives rise to a problem to solve:

Given a string w, determine whether $w \in S$.

Strings and Problems

Given a string w, determine whether $w \in S$.

- Suppose that S is the set

$$S = \{ \text{"}a\text{"}, \text{"}b\text{"}, \text{"}c\text{"}, \dots \text{"}z\text{"} \}$$

- From this set S, we get this problem:

Given a string w, determine whether w is a single lower-case English letter.

Strings and Problems

Given a string *w*, determine whether $w \in S$.

- Suppose that S is the set

$$S = \{ \text{"0"}, \text{"1"}, \text{"2"}, \ldots, \text{"9"}, \text{"10"}, \text{"11"}, \ldots \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* represents a natural number.

Strings and Problems

Given a string *w*, determine whether $w \in S$.

- Suppose that S is the set

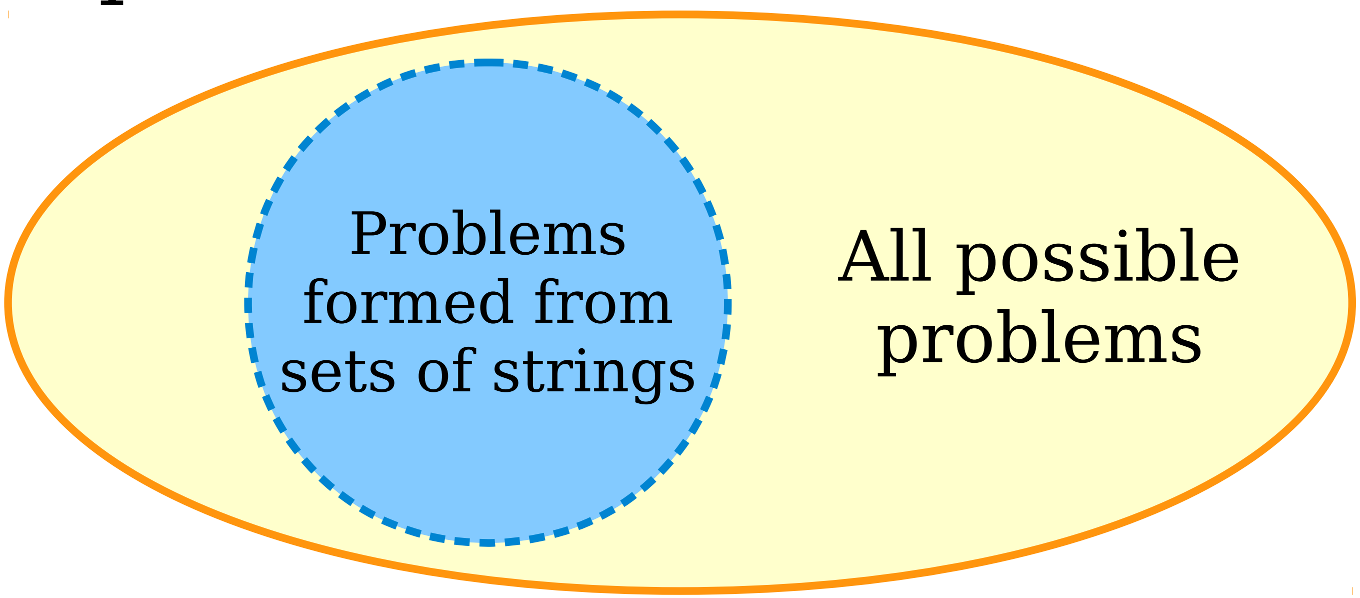
$$S = \{ \ p \mid p \text{ is a legal Java program } \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* is a legal Java program.

Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



|Sets of Strings| ≤ |Problems|

Where We're Going

- A ***string*** is a sequence of characters.
- We're going to prove the following results:
  - There are ***at most*** as many programs as there are strings. ✓
  - There are ***at least*** as many problems as there are sets of strings.
- This leads to some *incredible* results – we'll see why in a minute!

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Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings.

|Programs| ≤ |Strings| < |Sets of Strings| ≤ |Problems|

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|Programs| < |Problems|

There are more problems to solve than there are programs to solve them.

|Programs| < |Problems|

It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible, but we don't know *which* problems are impossible!

We need to develop a more nuanced understanding of computation.

Where We're Going

- ***What makes a problem impossible to solve with computers?***
  - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
  - How do you know when you're looking at an impossible problem?
  - Are these real-world problems, or are they highly contrived?
- ***How do we know that we're right?***
  - How can we back up our pictures with rigorous proofs?
  - How do we build a mathematical framework for studying computation?

Next Time

- **Mathematical Proof**
  - What is a mathematical proof?
  - How can we prove things with certainty?