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THE GRAPHS OF ACTIVE NETWORKS

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SUMMARY

The results obtained in previous papers are employed to develop the properties of the graphs of linear networks which may contain valves and transformers. A reduced graph is obtained which facilitates the setting up of the determinant of the network. This graph may be split into two separate graphs, termed the current and voltage graphs, such that there is a 1 : 1 correspondence between every algebraic operation on the H-matrix and every topological operation on the graphs. Thus every problem which can be solved with the one can be solved with the other. The derivative of a graph with respect to a network element corresponds to the derivative of an H-determinant with respect to a network element. With an extended definition of a tree on a network the set of trees on any network is shown to be equal to the nodal determinant of the network. Practical applications of the methods described are outlined.

LIST OF SYMBOLS

D = Ordinary determinant of network.
$f_{kj.im}$ = Fictitious element equal to transfer admittance $Y_{ij.km}$.
g = Mutual admittance of valve.
h = Mutual admittance of element $h_{ij.km}$.
h''_i = Current-node admittance function for the node i.
h'_j = Voltage-node admittance function for the node j.
h_{ij} = Matrix or determinant element in row i and column j.
= Node-pair admittance.
$h_{ij.km}$ = Element of mutual admittance h with current nodes i and k and voltage nodes j and m.
$(h_{ij.km})$ = H-matrix of element $h_{ij.km}$.
H = Determinant of matrix (H).
(H) = H-matrix of network.
H_{ij} = Cofactor of determinant element h_{ij}.
i_i = Current flowing from an external source into node i.
i_{ik} = Current flowing from node k to node i through the branch ik.
$i_{ij.km}$ = Component of current i_{ik} due to voltage v_{jm}.
$(i+k, j+m)$ = Addition operator.
k''_{ik} = Current-branch admittance function for the branch ik.
k'_{jm} = Voltage-branch admittance function for the branch jm.
$k_{ij.km}$ = Branch-pair admittance.
m = Mutual admittance of transformer.
t = Tree on network.
T = Set of trees on network.
v_j = Potential of node j.
v_{jm} = Voltage between nodes j and m, with m positive.
$Y_{ij.km}$ = Transfer admittance of network.

Correspondence on Monographs is invited for consideration with a view to publication.

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(1) INTRODUCTION

An electrical network is an assembly of suitably connected electrical components such as resistors, inductors, capacitors, valves and transformers. The physical network can be represented by a circuit diagram in which each component is denoted by a symbol which is generally a simplified picture of one form of the component itself. The circuit diagram enables the connections of the network to be seen at a glance, while a closer examination reveals the functions of the various parts.

For mathematical purposes the semi-pictorial representation of the network must be replaced by a mathematical representation. This may consist of a diagram, known in topology as a graph, in which the only significant features are the branches, nodes and meshes. Alternatively the representation may be algebraic, e.g. a set of equations, a matrix or a determinant.

It is well known that a linear network comprising 2-terminal elements only can be represented by a graph. With the aid of mathematical trees it is possible to evaluate network admittances directly from the graph. As shown in Reference 1, the quickest way of obtaining the set of trees is to expand the graph as the sum of a number of simpler graphs from which the set of trees can be written down by inspection.

The fact that the graph of a network can be expanded as the sum of a set of simpler graphs is an illustration of the fact that a graph is a mathematical entity, not only in the static sense that it is a representation, but also in the dynamic sense that, like an algebraic expression, it can be operated upon and expanded as a series of terms.

Since an electrical network can be represented either by its graph or by its determinant it must be possible to construct either one of these from the other. Indeed the quickest way of setting up the determinant of a network comprising 2-terminal elements only is to construct the graph first, and then to apply the well-known rule for forming the elements of the determinant from the graph.

In Reference 2 it was shown that 3- or 4-terminal elements can be represented as graphs in terms of an element $h_{ij.km}$ which comprises four nodes and two directed branches. Such elements include transformers, valves and transistors. Hence it is possible to represent any linear network by a graph.

In Reference 2 the graph of a network was utilized solely for the purpose of setting up matrices and determinants. In the ordinary nodal D-determinant the row and column corresponding to the ground node of the network are omitted. If these are included, an enlarged determinant is obtained in which every node in the graph is represented by a row and a column. This determinant was termed an H-determinant. The corresponding H-matrix has the property that the sum of the H-matrices of the elements of the network is equal to the H-matrix of the network as a whole.

To obtain the determinant of a network no equations are required. The procedure is first to set up the graph of the network including a fictitious element f representing the negative of the network admittance it is desired to evaluated. The H-matrix may then be obtained either by adding the H-matrices of the network elements, or by obtaining the matrix elements one at a time with the aid of a rule given in Reference 2, which is an extension of the well-known rule for networks containing passive elements only. The D-determinant is then obtained by deleting one row and one column which may be those containing the greatest number of network elements. The required network admittance is obtained by equating the D-determinant to zero.

The purpose of the paper is to lay down the basis for a theory of graphs of networks containing 3- or 4-terminal elements, such as valves and transformers. This theory is much more extensive than that relating to networks with 2-terminal elements only, and it appears that there are many practical applications. Within the limits of a single paper it has therefore seemed best to deal with the fundamental aspects only, in order that a clear general picture can be obtained.

The development of the theory is considerably facilitated by the correspondence between the graph of the network and its H-matrix and H-determinant. By simplifying the type of graph employed in Reference 2, and then splitting it into two separate graphs, termed the "current" and "voltage" graphs, it is possible to maintain a 1 : 1 correspondence between every operation on the H-matrix and every operation on the graphs. Thus it is possible to take the derivative of a graph with respect to a network element in the same way as it is possible to take the corresponding derivative of an H-determinant.

In complicated networks the labour of evaluating the D-determinant may be very considerable. It may be reduced, however, by expanding the graph of the network into the sum of simpler graphs and adding the simpler determinants so obtained.

The theorem that the D-determinant of a network containing 2-terminal elements only is equal to the set of trees on the network is extended to include networks with 3- or 4-terminal elements. By including a fictitious element to represent the negative of the required network admittance its value can be obtained merely by equating the set of trees to zero. No 2-trees or linkages, as employed in Reference 1, are required.

(2) THE H-MATRIX AND H-DETERMINANT

For reference purposes it will be necessary to reproduce some of the diagrams and equations of Reference 2. Fig. 1 shows the

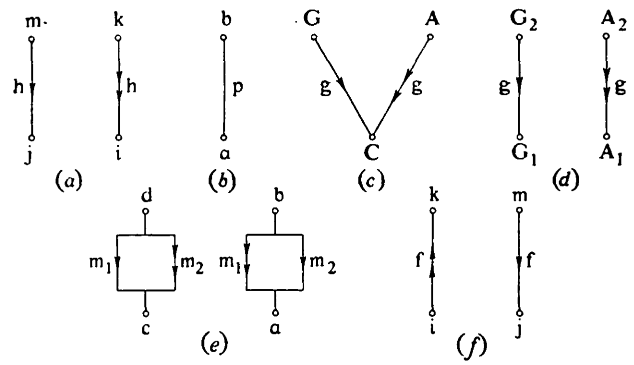

Fig. 1.—Graphs of the network elements.

(a) $h_{ij.km}$
(b) $p_{aa.bb}$
(c) $g_{CC.AG}$
(d) $g_{A1G1.A2G2}$
(e) $(m_{ac.bd}) + (m_{ca.bd})$
(f) $f_{kj.im} = Y_{ij.km}$

graphs of the various network elements. Fig. 1(a) shows the fundamental element $h_{ij.km}$, which is such that a voltage v_{jm} between the nodes j and m, with m positive, produces a current i_{ik} from k to i given by

$$h_{ij.km}v_{jm} = i_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 1(b) shows an ordinary 2-terminal element obtained from $h_{ij.km}$ by identifying i and j and also m and k. Fig. 1(c) shows a valve, and Fig. 1(d) shows a pair of identical valves in push-pull. Fig. 1(e) shows a symmetrical 2-way mutual admittance, which may be that of a transformer. $f_{kj.im}$, shown in Fig. 1(f), is a fictitious element equal to a transfer admittance which it is desired to evaluate.

In the case of valves, passive elements such as the input and output admittances are assumed to be included in the external network. A transformer with primary inductance L_1, secondary inductance L_2 and mutual inductance M is replaced by a mutual admittance m with a shunt admittance p_1 across the primary winding and a shunt admittance q_1 across the secondary winding, where

$$1/p_1 = j\omega L_1(1 - k^2);\ 1/q_1 = j\omega L_2(1 - k^2);\ 1/m = j\omega M(k^2 - 1) \quad . \quad . \quad . \quad . \quad (2)$$

where ω is 2π times the frequency and $k = M/\sqrt{(L_1L_2)}$ is the coupling coefficient. As in the case of valves p_1 and q_1 are assumed to be included in the external network.

Eqn. (3) gives the H-matrix of the element $h_{ij.km}$ from which the matrices of all other elements can be deduced:

$$(h_{ij.km}) = \begin{array}{c|cc} & j & m \\ \hline i & h & -h \\ k & -h & h \end{array} \quad . \quad . \quad . \quad . \quad (3)$$

Fig. 2(a) is the conventional representation of a network fed with a current i and from which it is required to obtain the

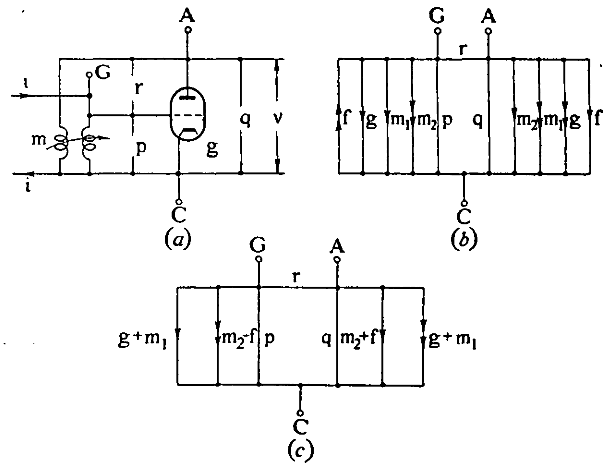

Fig. 2.—Triode network.

output voltage v. The network has been constructed to include a valve of mutual conductance g, a transformer of mutual admittance m and three self-admittances p, q and r. The required transfer admittance is $Y_{CC.GA} = i/v$.

Fig. 2(b) shows the graph of this network in which the values of m_1 and m_2 are both m, and $f_{GC.CA} = Y_{CC.GA}$. The current branch f, with the double arrow, carries the current i which is maintained indefinitely by the output voltage v across the voltage branch f, with the single arrow. The effect of this feedback element $f_{GC.CA}$ is to produce a network in dynamic equilibrium. For such a network the determinant $D = 0$, from which $f_{GC.CA}$, and hence $Y_{CC.GA}$, can be calculated.

The H-matrix of Fig. 2(b) is

$$(H) = \begin{array}{c|ccc} & C & G & A \\ \hline C & p+q+2m+g-f & -p-m-g & -q-m+f \\ G & -p-m+f & p+r & -r+m-f \\ A & -q-m-g & -r+m+g & q+r \end{array} \quad (4)$$

in which each network element appears in the matrix element h_{ij} in accordance with eqn. (3). Each matrix element can be determined separately by the rule that h_{ij} is equal to the sum of all the network elements whose current branches terminate on node i and the voltage branches on node j, the sign being positive if the arrows on both branches point either towards or away from the nodes i and j, and negative otherwise.

The ordinary nodal D-determinant of a network is equal to any first cofactor of the determinant H of the H-matrix, and for Fig. 2 it is therefore given by

$$\partial H/\partial h_{CC} = H_{CC} = D = pq + pr + qr + rg + 2rm - mg - m^2 + f(m + g - r) \quad . \quad (5)$$

which, since the network is in equilibrium, is zero. Hence

$$Y_{CC.GA} = f_{GC.CA} = \frac{pq + pr + qr + rg + 2rm - mg - m^2}{r - m - g} \quad (6)$$

(3) THE GRAPHS OF A NETWORK

In the graph of Fig. 2(b) each network element is shown separately. In this Section it will be shown how such a graph can be reduced to a simpler and more useful form, while other types of graph can be obtained each of which is necessary to the general theory and has its particular applications.

(3.1) The Reduced Graph

In Fig. 2(b) it will be seen that both g and m_1 are of the form $h_{CC.AG}$, while $-f$ and m_2 are of the form $h_{CC.GA}$. Such elements are in parallel, and from the point of view of the graph, they constitute a single element. We shall therefore introduce a new symbol k to denote the sum of elements which have both their current and voltage branches and parallel. Thus we have

$$k_{ij.km} = \sum_h h_{ij.km} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which gives $k_{CC.AG} = g + m_1$ and $k_{CC.GA} = -f + m_2$.

Fig. 2(c) is the graph of the network in terms of the k's, which will be termed *branch-pair admittances*. Thus $k_{CC.AG}$ is the admittance of the branch pair comprising the current branch CA and the voltage branch CG.

The graph of Fig. 2(c) can be further simplified by replacing each set of parallel branches by a single branch as in Fig. 3, which will be termed the *reduced graph* of the network.

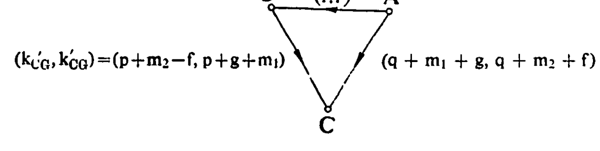

Fig. 3.—The reduced graph.

The first step in the reduction is to cause all arrows on parallel branches to point in the same direction, the signs of the branches being changed where necessary. In the example given no change is required, since the sign of f has already been changed in the process of replacing network elements by branch-pair admittances. Each branch in the reduced graph is then denoted by a pair of terms enclosed in a bracket, the first term being the sum of the current branches and the second the sum of the voltage branches. If the direction of the arrow on the branch is reversed, all the signs in the brackets must be reversed.

In order to maintain the 1 : 1 correspondence between the graph and the H-matrix it is convenient to impose a further condition, namely that the arrows on the branches always point from a higher node to a lower node, a node being higher if it corresponds to a row or column occurring later in the H-matrix.

It will be appreciated that the reduced graph is not only simpler than the previous forms but renders it easier to apply the rule given in the previous Section for obtaining the H-matrix from the graph. As will be shown it is also of greater importance in the theory of graphs.

(3.2) The Current and Voltage Graphs

Each branch on the reduced graph can be considered as being composed of two branches—a composite current branch and a composite voltage branch. Hence the reduced graph can be replaced by two graphs, a *current graph* as in Fig. 4(a) composed of current branches only, and a *voltage graph* composed of voltage branches only.

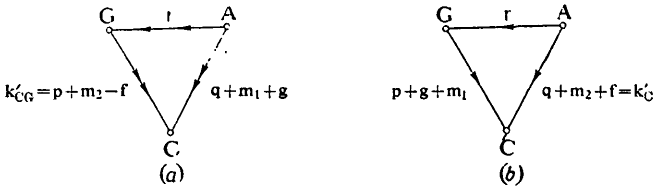

Fig. 4.—Current and voltage graphs.

The total current in each composite current branch is the sum of the currents in the component current branches. Hence if each branch of the current graph is labelled with the total current taken by the branch, the graph completely describes the properties of the network so far as currents are concerned. The only condition which must be imposed on the current graph considered alone is that Kirchhoff's law for currents must be obeyed.

If each node of the voltage graph is labelled with its voltage, this graph completely describes the properties of the network so far as voltages are concerned. The only condition which must be imposed on the voltage graph considered alone is that Kirchhoff's law for voltages must be obeyed.

The current and voltage networks are linked by the admittances of the elements. Thus the element m_2 is associated with the current branch CG and the voltage branch CA. Hence the current flowing from G to C due to this element is m times the voltage of A relative to that of C.

The nodes on the reduced graph correspond both to the rows and columns of the H-matrix. But the nodes on the current graph correspond to the rows, and the nodes on the voltage graph correspond to the columns, of the H-matrix. Hence, as will appear later, the correspondence between the current and voltage graphs and the H-matrix extends even to operations on the determinant of the H-matrix in which the rows are different from the columns. This is a very important advantage, since an essential feature of the theory of graphs is its correspondence with the theory of H-matrices and determinants.

(3.3) The Nodal Graphs

The rule for obtaining the matrix elements from the graph of the network depends on associating each element with a pair of nodes. This association may be made most directly from the nodal graphs of the network.

Fig. 5 shows what will be termed the *reduced nodal graph* in which each node is labelled with the sum of the expressions for the branches terminating on the node, the sign being positive if the arrow on the branch points towards the node and negative if it points away. It will be observed that all information relating

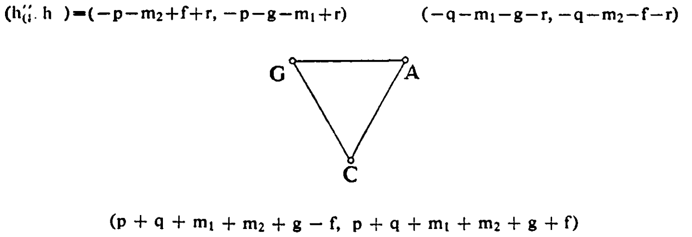

Fig. 5.—Reduced nodal graph.

to the network is retained without the necessity for putting arrows on the branches. In fact there is no need to show the branches at all.

The *current and voltage nodal graphs* are shown in Fig. 6. Since no arrows appear it is necessary to label the current graph i and the voltage graph v.

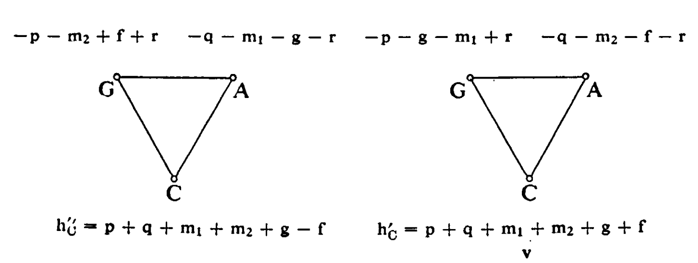

Fig. 6.—Current and voltage nodal graphs.

The matrix element h_{ij} is obtained from the nodal graphs merely by taking the elements common to the current node i and the voltage node j and affixing a positive sign if the sign of the element is the same for both nodes and a negative sign otherwise. This may be checked by obtaining eqn. (4) from Fig. 5 or 6.

The element h_{ij} will also be termed the *node-pair admittance* since, as will be shown in the next Section, it bears the same relation to the nodal graphs as the branch-pair admittance bears to the branch graphs.

(4) THE ALGEBRA OF THE GRAPHS

The algebra of the graphs deals with the relations between the various admittances and admittance functions appearing in the graphs and with the relations between the graphs and the H-matrix. It also deals with the correspondences between operations on the graphs and on the H-matrix, but the treatment of this aspect will be postponed to later Sections.

(4.1) Branch Admittances

In Fig. 4 the component of current flowing from G to C due to the voltage v_{CA} is

$$i_{CC.GA} = (m - f)v_{CA} = k_{CC.GA}v_{CA} \quad . \quad . \quad . \quad (8)$$

while the total current flowing through the branch from G to C is

$$\begin{aligned} i_{CG} &= \sum_{j,m} h_{Cj.Gm}v_{jm} \\ &= (m - f)v_{CA} + pv_{CG} \\ &= k_{CC.GA}v_{CA} + k_{CC.GG}v_{CG} \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

so that $k_{CC.GA} = \partial i_{CG}/\partial v_{CA}$ and, in general,

$$k_{ij.km} = \sum_{h} h_{ij.km} = \partial i_{ik}/\partial v_{jm} \quad . \quad . \quad . \quad (10)$$

in which it is necessary to interpret the partial derivative as implying that all voltages other than v_{jm} are ineffective in producing currents. The interpretation that all voltages other than v_{jm} are zero is not possible, as voltages round a mesh are not independent.

The expression which has been used as a label for a current branch in the current graph will be termed the *current-branch admittance function* and written k''_{ik} for the current branch ik with the arrow pointing towards i. Thus $k''_{CG} = p + m_2 - f$. The expression which has been used as a label for a voltage branch in the voltage graph will be termed the *voltage-branch admittance function* and written k'_{jm} for the voltage branch jm with the arrow pointing towards j. Thus $k'_{CA} = q + m_2 + f$.

The expression in the brackets used in the single reduced graph as a label for a branch can therefore be written (k''_{ik}, k'_{ik}). Thus the branch CG in Fig. 3 would appear as $(k''_{CG}, k'_{CG}) = (p + m_2 - f, p + g + m_1)$.

$k_{ij.km}$ is the sum of the elements common to k''_{ik} and k'_{jm}, the sign being positive for an element which has the same sign in k''_{ik} and k'_{jm}, and negative for an element which has the opposite sign. This relationship can be expressed in the form

$$k_{ij.km} = k''_{ik} \cdot k'_{jm} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which the dot indicates a type of multiplication in which $A.A = A$ and $A.B = 0$, the signs being combined by the ordinary rule for multiplication. Thus in Fig. 4

$$k_{CC.GA} = k''_{CG} \cdot k'_{CA} = (p + m_2 - f) \cdot (q + m_2 + f) = m_2 - f \quad . \quad . \quad . \quad . \quad (12)$$

We also have

$$k''_{ik} = \sum_{j,m} k_{ij.km}; \quad k'_{jm} = \sum_{i,k} k_{ij.km} \quad . \quad . \quad . \quad (13)$$

bearing in mind that a reversal of any pair of letters denoting a branch reverses the sign.

(4.2) Node Admittances

The expression which has been used as a label for a node in the current nodal graph will be termed the *current-node admittance function* and written h''_i for the node i. Thus $h''_G = -p - m_2 + f + r$. The expression which has been used as a label for a node in the voltage nodal graph will be termed the *voltage-node admittance function* and written h'_j for the node j. Thus $h'_A = -q - m_2 - f - r$.

The expression in the brackets used in the single nodal graph as a label for a node can therefore be written (h''_i, h'_i). Thus the node G in Fig. 5 is

$$(h''_G, h'_G) = (-p - m_2 + f + r, -p - g - m_1 + r)$$

From the way in which the nodal admittance functions were derived in Section 3.3, we have

$$h''_i = \sum_k k''_{ik}; \quad h' = \sum_m k'_{jm} \quad . \quad . \quad . \quad . \quad (14)$$

again remembering that a reversal of any pair of letters denoting a branch reverses the sign.

The rule given in Section 3.3 for forming the matrix element, or node-pair admittance, h_{ij}, leads to the equation

$$h_{ij} = h''_i \cdot h'_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which the dot has exactly the same significance as in eqn. (11) for the branch-pair admittance. Thus

$$h_{GA} = (-p - m_2 + f + r) \cdot (-q - m_2 - f - r) = m - f - r \quad . \quad . \quad . \quad . \quad (16)$$

in accordance with eqn. (4).

(4.3) Derivation of the *H*-Matrix

In Reference 2 it was shown that the nodal equation for the *i*th node of a network can be written in the form

$$\sum_j \sum_{k,m} (h_{ij.km} - h'_{kj.im} - h''_{im.kj} + h'''_{km.ij}) v_j = i_i \quad . \quad (17)$$

where i_i is the current flowing from an external source into node i. For a network in equilibrium $i_i = 0$.

It is now possible to write this equation in the simpler form

$$\sum_j \sum_{k,m} k_{ij.km} v_j = i_i \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The quantity h_{ij} is defined by the equation

$$h_{ij} = \sum_{k,m} k_{ij.km} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The complete set of equations, one for each node, can then be written

$$\sum_j h_{ij} v_j = i_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which represents an equation for each value of the suffix i. As before, the right-hand side of all these equations is zero for a network in equilibrium.

From eqn. (20) it follows that

$$h_{ij} = \partial i_i / \partial v_j \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

in which the voltages of all the nodes except the jth are held constant.

By comparing eqns. (11) and (15) it will be seen that the relation between branch-pair admittances and the branch admittance functions is similar to the relation between node-pair admittances and node admittance functions. By comparing eqns. (10) and (21) it will be seen that the relation between branch-pair admittances and the branch currents and voltages is similar to the relation between node-pair admittance and the nodal currents and voltages.

The rule given in Section 2 for deriving the matrix elements from the network elements is equivalent to the first part of eqn. (10) together with eqns. (13), (14) and (15). By employing the branch-pair admittance graph of Fig. 2(*c*), eqn. (10) is no longer required, and by employing the reduced graph of Fig. 3, eqn. (13) need not be used, while by utilizing the reduced nodal graph of Fig. 5 only eqn. (13) is necessary.

In practical work with the graphs and *H*-matrices it has been found most convenient to set up the *H*-matrix from either the reduced branch graph or the current- and voltage-branch graphs.

(5) CORRESPONDING OPERATIONS ON *H*-DETERMINANTS AND GRAPHS

(5.1) The Operator $\partial H/\partial h_{ij.km}$

It was shown in Reference 2 that the derivative of an *H*-determinant with respect to the network element $h_{ij.km}$ is given by

$$\partial H / \partial h_{ij.km} = (i + k, j + m)H \quad . \quad . \quad . \quad (22)$$

where $(i + k, j + m)$ is an addition operator which deletes row k, replaces row i by the sum of the rows i and k, performs the corresponding operations on the columns j and m and affixes the sign $(-1)^{k+m}$. A derivative may be taken with respect to an element which does not appear in H, since it is only necessary to state that such a derivative is given by the corresponding addition operation.

By comparing the *H*-determinant with its graphs it can be seen that the operation of adding rows in the *H*-determinant corresponds to short-circuiting the corresponding nodes in the current graph, while the operation of adding columns corresponds to the operation of short-circuiting the corresponding nodes in the voltage graph. Affixing the sign $(-1)^{k+m}$ to the *H*-determinant corresponds to affixing the sign $(-1)^k$ to the current graph and the sign $(-1)^m$ to the voltage graph.

An operation on a graph corresponds to an operation on the *H*-determinant if the resulting determinant is the *H*-determinant of the resulting graph. It was shown in Reference 2 that the derivative of an *H*-determinant with respect to a network element is also an *H*-determinant. If the sign of the *H*-determinant of a pair of current and voltage graphs is defined as the product oı the signs of the graphs it can easily be proved that the *H*-determinant of the graph formed in the manner described above is the derivative of the *H*-determinant of the original graph.

If H in eqn. (22) represents the current and voltage graphs and $i + k$ denotes the operation of short-circuiting the nodes i and k in the current graph and affixing the sign $(-1)^k$, and $j + m$ denotes the operation of short-circuiting the nodes j and m in the voltage graph and affixing the sign $(-1)^m$, it follows that the resulting graphs are the derivatives of the original graphs with respect to the network element $h_{ij.km}$.

The derivative of the *H*-determinant of the network of Fig. 2 with respect to the element g is given by

$$\partial H/\partial g = (C + A,\ C + G)H = - \begin{array}{ccc} & C+G & A \\ C+A & -r+m-f & r-m+f \\ G & r-m+f & -r+m-f \end{array} \Bigg\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The corresponding current and voltage graphs are derived from Fig. 4 and consist of single branches as shown in Fig. 7,

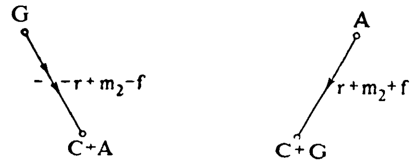

Fig. 7.—$\partial H/\partial g$ = Derivative of the graphs of Fig. 4 with respect to $g_{CC.AG}$.

a negative sign being affixed to the current graph and a positive sign to the voltage graph. The elements g and m_1, which are in parallel in both graphs, are short-circuited. The branch of the element q and the voltage branch of the element p are also short-circuited. Hence these elements must be deleted. This is in accordance with the rule, given in Reference 2, that, in taking the derivative with respect to a given network element, all elements which are in either current or voltage parallel with it are eliminated.

The nodes formed by short-circuiting two nodes are denoted by the sum of the letters denoting these nodes, the order being the same as in the operator.

It is possible to continue to take derivatives of an *H*-determinant with respect to successive network elements until only a single row and a single column remain with a single determinant element of zero. This can be illustrated from eqn. (23) by taking a further derivative with respect to r, the operator being $(G + C + A,\ C + G + A)$. The corresponding operations on the current and voltage graphs leave only a single node in each graph, the determinant being that described above.

In taking the derivative with respect to an ordinary element $h_{aa.bb}$ the nodes to be short-circuited are the same in the current and voltage graphs. The operation can therefore be applied to a single reduced graph such as that of Fig. 3. The mathematical operation of taking a derivative then corresponds to the electrical operation of short-circuiting two nodes on the network. It is also equivalent to making the admittance of the element $h_{aa.bb}$ infinite.

It can easily be shown that the mathematical operation of taking the derivative with respect to any element $h_{ij.km}$ is equivalent to making the mutual admittance of this element infinite. Thus eqn. (23) and Fig. 7 represent the result of making the slope g of the valve infinite. The D-determinant of the H-determinant of eqn. (23) is any first cofactor of H and is therefore, on taking the sign into account, $r - m + f$. This is the coefficient of g in eqn. (5), as it should be. But it is also the D-determinant when g is infinite. It follows that as g is increased indefinitely the transfer admittance approaches the value $Y_{CC.GA} = m - r$.

(5.2) The Operator $h_{ij.km}\partial/\partial h_{ij.km}$

The operator $h_{ij.km}\partial/\partial h_{ij.km}$ adds all the network elements in row k, except $h_{ij.km}$, to row i, and all the network elements in column m, except $h_{ij.km}$, to column j. The number of rows and columns is unchanged and no question of sign arises. The D-determinant of the resulting H-determinant gives the set of terms containing $h_{ij.km}$ in the D-determinant of the original network.

From eqn. (4) we have

$$(g_{CC.AG}\partial/\partial g_{CC.AG})H = \begin{array}{c|ccc} & C & G & A \\ \hline C & -r+m-f+g & -g & r-m+f \\ G & r-m+f & 0 & -r+m-f \\ A & -g & g & 0 \end{array} \quad . \quad . \quad . \quad . \quad (24)$$

The D-determinant is $g(r - m + f)$, which is the set of terms containing g in eqn. (5).

The operation $h_{ij.km}\partial/\partial h_{ij.km}$ on the current and voltage graphs is a transfer operation, which leaves $h_{ij.km}$ unchanged, but transfers to node i the ends of all other current branches which are normally connected to node k, and transfers to node j the ends of all voltage branches which are normally connected to node m.

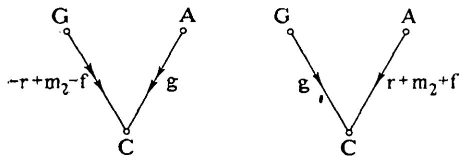

Fig. 8.—Result of operation $g_{CC.AG}\partial/\partial g_{CC.AG}$ on the graphs of Fig. 4.

Fig. 8 shows the effect of the operation $g_{CC.AG}\partial/\partial g_{CC.AG}$ on the graphs of Fig. 4.

(5.3) The Operator $(1 + h_{ij.km}\partial/\partial h_{ij.km})$

H can be expanded in terms of the network element $h_{ij.km}$ by the equation

$$H = H_0 + (h_{ij.km}\partial/\partial h_{ij.km})H \quad . \quad . \quad . \quad (25)$$

where $H_0 = H(h_{ij.km} = 0)$.

But as it is possible to take the derivative with respect to an element which is not actually present in H_0, a more convenient form is

$$H = (1 + h_{ij.km}\partial/\partial h)H_0 \quad . \quad . \quad . \quad . \quad (26)$$

in which the superfluous suffix of h is omitted.

By taking first cofactors we obtain the expansion of D in terms of $h_{ij.km}$ as follows:

$$D = D_0 + (h_{ij.km}\partial/\partial h)D \quad . \quad . \quad . \quad . \quad (27)$$

where $D_0 = D(h_{ij.km} = 0)$.

Corresponding to these expansions of the determinants we have the expansions of the current and voltage graphs. Fig. 9 shows the expansion of the graphs of Fig. 4 in terms of g.

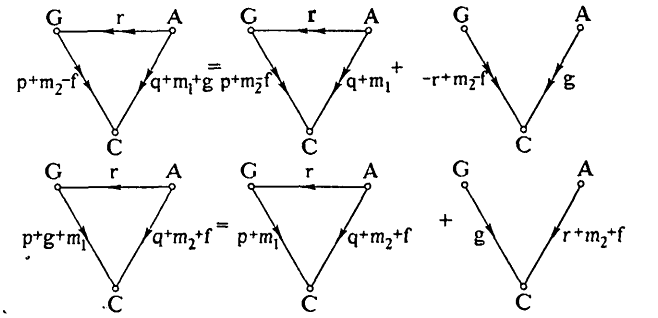

Fig. 9.—$H = H_0 + g_{CC.AG}\partial H/\partial g_{CC.AG}$.

It is possible to continue the expansion in terms of other elements. The resulting expansion of the current and voltage graphs corresponds to the network equations described in Reference 1, but generalized to include all elements of the form $h_{ij.km}$.

(6) EQUIVALENCE OF THE D-DETERMINANT TO THE SET OF TREES ON A NETWORK

It is known that the nodal D-determinant of a network, all the elements of which are of the form $h_{ii.jj}$, is equal to the set of trees on the graph of the network. It will now be shown that, with an appropriate definition of a tree, the theorem can be extended to networks containing elements of the general form $h_{ij.km}$.

(6.1) Trees on the Current and Voltage Graphs

On a graph, all the elements of which are of the form $h_{ii.jj}$, a tree is defined as a set of branches which connect all the nodes but enclose no meshes. It is easily shown that on such a graph any tree can be formed by a succession of transfer operations of the form $h_{ii.jj}\partial/\partial h$. Moreover, on the current and voltage graphs of any network the more general operations $h_{ij.km}\partial/\partial h$, if continued until no meshes or parallel branches remain, give rise to a pair of trees, one on the current graph and the other on the voltage graph.

Thus the graphs of Fig. 8 represent the first step in the formation of a pair of trees. A second and final step consists in the

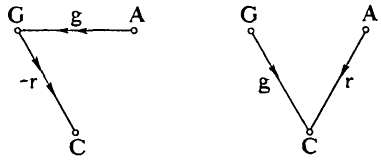

Fig. 10.—The tree rg.

operation $r_{GC.CA}\partial/\partial r$, which gives the pair of trees shown in Fig. 10. The H-determinant of the pair of trees is given by

$$(r_{GC.CA}\partial/\partial r)(g_{CC.AG}\partial/\partial g)H = \begin{array}{c|ccc} & C & G & A \\ \hline C & -r & 0 & r \\ G & r+g & -g & -r \\ A & -g & g & 0 \end{array} \quad . \quad (28)$$

It has already been shown that the operation $h_{ij.km}\partial/\partial h$ gives an H-determinant, the D-determinant of which is the set of terms in the D-determinant of the original network which contains $h_{ij.km}$. Hence a succession of operations of this type continued until no further elements remain must give a single term in D. Thus the D-determinant of eqn. (28) gives the single term rg of eqn. (5).

We define a tree on the graphs of a network as a set of network elements such that the current branches form a tree on the current graph, and the voltage branches form a tree on the voltage graph. Thus the set of elements r and g form a tree on the graphs of Fig. 4. Another tree is formed by the elements f and m_1. Indeed, as will be proved later, every term in D of eqn. (5) corresponds to a tree on the graphs of Fig. 4.

In the case of a network composed entirely of elements of the form $h_{ii.jj}$ the current and voltage graphs are identical, and the definition of a tree given above reduces to the standard definition. In the more general case the graphs are not identical and neither are the trees on the graphs. Thus if a tree includes the element $h_{ij.km}$ the tree on the current graph will include the branch ik, while the corresponding branch on the tree on the voltage graph will be jm.

For a network composed entirely of elements of the form $h_{ii.jj}$ the value t of a tree on the graph of the network is defined as the product of its elements. This is equal to the D-determinant of the tree, which in turn is equal to one of the terms in the D-determinant of the network. The set T of trees on the network is then defined as the sum of the individual trees, from which follows the theorem that the set of trees on a network is equal to the nodal D-determinant of the network,

i.e. $$T = \Sigma t = D \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It is required to extend this theorem to the general case of a network composed of elements of the type $h_{ij.km}$. We commence by defining the value t of a tree, not directly in terms of the graphs, but as the D-determinant of the graphs of the tree.

At this point it is necessary to draw attention to the fact that the tree rg of Fig. 10 is not identical with the tree rg on the graphs of Fig. 4. Nevertheless it can be shown that the value of the D-determinant of the tree of Fig. 10 is the same as the value of the D-determinant of the tree rg on the graphs of Fig. 4. Hence the values of the trees are the same. The tree rg of Fig. 4 can be isolated by the operation $rg(C + A, C + G)(A + G, G + A)$, the first operation being $(A + G, G + A)$, which can be shown to be equivalent to the operation of eqn. (28). It can be shown that operations always exist whereby a tree can be isolated in its original form, but that, in general, other operations exist whereby a tree can be formed having the same value but with its branches connected between different nodes.

It thus follows that D is the sum of terms each of which is given by a succession of operations of the type $h_{ij.km}\partial/\partial h$, and that each tree is also given by a succession of operations of this type. Hence the value of each tree is equal to one of the terms in D, and D itself is equal to the set of trees T. Hence eqn. (29) is true in the general case.

It remains to define the value of a tree in terms of the graphs of the network, so that it is unnecessary to refer to determinants in order to ascertain the value of the set of trees, and then to give a rule whereby T can be obtained without enumerating the individual trees.

(6.2) The Value of a Tree

For a network composed entirely of elements of the form $h_{ii.jj}$ the determinant of a tree is simply the product of the elements of the tree. In the general case the determinant is equal to the product of the elements, but with a sign which must be determined. If the determinant is expanded by means of addition operators it may be shown that the process is equivalent to the following definition for the value of the equivalent tree.

Multiply the branches in each graph together and, provisionally, give the tree a positive sign if the signs of the products are the same, and a negative sign otherwise. In each graph, associate each branch with a node. If the arrow on the branch points away from the node write down a negative sign. If the total number of negative signs is odd change the sign of the tree; otherwise make no change. Finally, assign a convenient order to the nodes and write down the branches of the tree on each graph in the order of the associated nodes, including, for each graph, a dummy branch to correspond with the node which, it will be found, is not included. If the relative inversions of the branches are odd, change the sign of the tree; otherwise make no change.

Thus for the tree rg of Fig. 4 we write $r(G), g(C), d(A)$ for the current graph and $r(G), g(C), d(A)$ for the voltage graph, where d is the dummy branch. The value of the tree is therefore rg. For the tree m_1m_2 we write $m_1(C)$, $-m_2(G)$, $d(A)$ for the current graph and $m_1(C)$, $-d(G) - m_2(A)$, for the voltage graph. The value of the tree is therefore $-m_1m_2 = -m^2$, corresponding to the term in eqn. (5).

It will be appreciated that, owing to the need for ascertaining the sign, the process of enumerating the individual trees is an even more tedious process than for a passive network.

(6.3) Evaluation of a Set of Trees

In Reference 1 it was shown that, with the aid of network equations, the set of trees on a network could be evaluated more rapidly than by the enumeration of the individual trees. This is also the case for networks containing elements of the more general type $h_{ij.km}$. However, the expansion should always be in terms of ordinary elements.

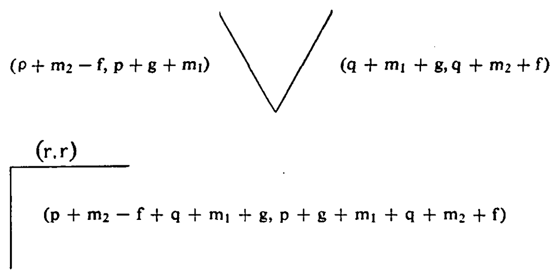

Fig. 11.—Expansion of the graphs of Fig. 3.

Thus Fig. 11 shows the expansion of the graph of Fig. 3 in terms of the ordinary element r, which gives a pair of trees in which, however, the branches are composite. Since the corresponding current and voltage graphs are identical there is no need to show the arrows or to label the nodes, the sign of a term being determined merely by the signs of the branches and the number of relative inversions. Formally we write

$$T = (p + m_2 - f)(q + m_1 + g) \,.\, (p + g + m_1)(q + m_2 + f) + r(p + m_2 - f + q + m_1 + g) \,.\, r(p + g + m_1 + q + m_2 + f) \quad . \quad . \quad . \quad . \quad (30)$$

in which the dot indicates a form of multiplication in which, if A and B are the two factors to be multiplied, $A.B = 0$ and $A.A = \pm 1$, the sign being positive if the relative inversions of the individual factors are even, and negative if they are odd. Thus we have for one of the terms in the first dot product $m_2g \,.\, gm_2 = -mg$, and for the complete expression

$$T = D = rp + 2rm - rf + rq + pq - m^2 - mg + mf + gf + rg \quad . \quad . \quad . \quad . \quad (31)$$

which is the same as in eqn. (5). It will be noted that the suffices of m must be retained in eqn. (30) but are not required in eqn. (31).

(6.4) An Example

The following example is of some intrinsic interest, and shows how the tree method can be used to advantage.

Wheeler[3] has introduced the concept of an ideal transformer-repeater which has the following properties:

(*a*) It has two pairs of terminals which will be referred to as A and B.
(*b*) If A (or B) is on open-circuit the admittance of B (or A) is zero.
(*c*) If A (or B) is on closed circuit the impedance of B (or A) is zero.
(*d*) The current multiplication from A to B is a and that from B to A is $1/a$, where a may be complex.
(*e*) The voltage multiplication from A to B is b and that from B to A is $1/b$, where b may be complex.

It follows that on open- or closed-circuit no power is taken by the device from a source at either A or B, that the impedance transformation from A to B is b/a and that from B to A is a/b, and that the power gain from A to B is ab and that from B to A is $1/ab$.

It will be shown that this concept can be realized as a limiting case, with appropriate limitations on the upper frequency, by means of the network shown in Fig. 12(*a*), in which p and q

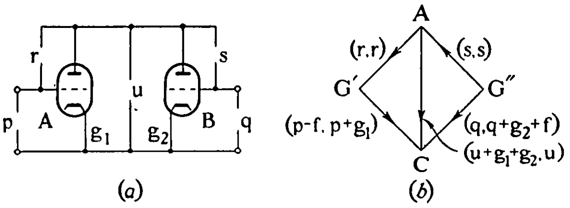

Fig. 12.—Ideal generalized transformer-repeater.

are external admittances, r and s are small admittances, u is a coupling admittance and g_1 and g_2 are large mutual admittances. The valves shown may each be replaced by several valves in tandem whereby the values of g_1 and g_2 may be made complex and as large as required.

The reduced graph is shown in Fig. 12(*b*), in which $f_{G'C.CG''}$ is the transfer admittance from A to B. This can be expanded in terms of r and s and solved as in the previous example. However, the graph can be considerably simplified by taking advantage of the fact that g_1 and g_2 are very large.

It follows that all the terms in the determinant can be neglected with the exception of those containing g_1 or g_2. Hence in the graph the only trees which need to be considered are those containing g_1 or g_2. But the only current branch containing g_1 or g_2 is $u + g_1 + g_2$, in which u can be neglected, which leaves $g_1 + g_2$. It follows that the element u does not appear in the determinant, and hence the voltage branch u can be deleted.

We now expand the network in terms of g_1 and g_2, as explained in Section 5.3. The end A of the current branch r must be transferred to C so that the current branch r becomes in parallel with the current branch $p - f$, but with its sign reversed owing to the opposite direction of the arrows. Similarly the current branch s appears in parallel with the current branch q with the same sign. If the voltage part of the graph is expanded in terms of $p + g_1$, the branch r takes the place of the voltage branch u with the same sign. The voltage branch $q + g_2 + f$ will be unchanged. The voltage branch s can also be left unchanged.

We have now obtained a current tree and a voltage graph. In order to make these identical it will be convenient to expand the latter in terms of the branch $q + g_2 + f$. This is quite a permissible operation although its only effect is to transfer the end G'' of the voltage branch s to C, when the branch becomes in parallel with r but of opposite sign. This gives the single tree of Fig. 13.

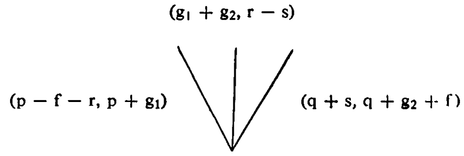

Fig. 13.—Graph of Fig. 12(*b*) when g_1 and g_2 are large.

The value of this tree is

$$T = D = (p - f - r)(g_1 + g_2)(q + s).$$
$$(p + g_1)(r - s)(q + g_2 + f)$$
$$= pg_2 s + rg_1 q + fg_1 s \quad . \quad (32)$$

By putting $D = 0$ and $p = 0$ we obtain the transfer admittance

$$f = Y_{CC.G'G''} = -qr/s \quad . \quad . \quad . \quad . \quad (33)$$

and by symmetry

$$Y_{CC.G''G'} = -ps/r \quad . \quad . \quad . \quad . \quad (34)$$

It is easily seen that by deleting f and replacing p by $-f'$ we obtain the input admittance at A

$$Y_{CC.G'G'} = qrg_1/sg_2 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

which is infinite when $1/q = 0$ and zero when $q = 0$.

From eqn. (33) the current gain is $a = -s/r$, while from eqns. (33) and (35) the voltage gain is $b = -g_1/g_2$. The conditions for the ideal transformer-repeater are therefore satisfied.

(7) PRACTICAL APPLICATIONS

In dealing with the graphs of networks it has been necessary to lay the main stress on the theoretical relations between the various aspects. A brief summary will now be given of the practical applications of the results obtained in this and the two preceding papers.[1,2]

To solve simple networks consisting of 2-terminal elements only, the choice of method is of little importance and the usual D-determinant method is as good as any other. If the network is more complicated the H-determinant method is generally quicker, since the row and column deleted to give the D-determinant may be chosen as those containing the greatest number of network elements. The H-determinant method also has an advantage when the transfer admittance involves four terminals, since only a single cofactor is required.

The advantage of determinant methods as compared with tree methods is that fewer rules are required. If the labour involved in expanding the determinant is considerable the tree method is quicker, since there are no terms to be cancelled in the D-determinant.

For networks containing valves or transformers it has been found of considerable advantage to construct the reduced graph of the network as shown in Fig. 3. Either the D-determinant or the H-determinant can be set up from the graph, but the latter has generally been found most useful. If a transfer admittance is required, the fictitious element f may be included in the graph when it becomes unnecessary to take cofactors. The network admittance across any 2-terminal element p can always be found by replacing p by $p - f'$ in the D-determinant and equating it to zero.

If valves or transformers are present the tree method is inevitably more complicated and is to be recommended only in special cases such as the example given. However, the labour of expanding the determinant of a very complicated network may be considerably reduced by expanding the graph as the sum of simpler graphs and then adding the determinants of these graphs.

Equivalent networks can be found with the aid of *H*-matrices as described in Reference 2. It has also been found that useful equivalent networks can be obtained directly from the graphs.

(8) ACKNOWLEDGMENTS

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