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IMPROVED MATRIX AND DETERMINANT METHODS FOR SOLVING NETWORKS

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SUMMARY

It is shown that a linear electrical network, which may include valves and transformers, can be represented topologically by a network composed entirely of elements of a single type. In its most general form an element of this type comprises four nodes and two directed branches. It is also shown that a linear electrical network can be represented algebraically by a matrix, termed an H-matrix. If the elements of the network are shown in the network diagram in the form of elements of the above type, the H-matrix can be written down directly from the network diagram. The value of the usual determinant of the network is equal to the value of any first co-factor of the determinant of the H-matrix. By choosing a suitable co-factor the labour of evaluating the determinant is much reduced. The transfer admittance of the network is treated as a special case of a network element. By the application of what are termed addition operators the transfer admittance can be expressed as a function of any element or elements. The calculation of equivalent networks can be considerably simplified with the aid of H-matrices.

LIST OF SYMBOLS

D = Ordinary determinant of network.
$f_{kj.im}$ = Element corresponding to transfer admittance $Y_{ij.km}$.
g = Mutual admittance of valve.
h = Mutual admittance of element $h_{ij.km}$.
h_{ij} = Matrix or determinant element in row i and column j.
$h_{ij.km}$ = Element of mutual admittance h with current nodes i and k and voltage nodes j and m.
$(h_{ij.km})$ = H-matrix of element $h_{ij.km}$.
H = Determinant of matrix (H).
(H) = H-matrix of network.
H_{ij} = Co-factor of determinant element h_{ij}.
$H_{ij.km}$ = Co-factor of H with respect of determinant elements h_{ij} and h_{km}.
i_k = Current flowing into network through node k.
i_{ik} = Current flowing into element through node k and out through node i.
i'_{ik} = Current flowing into network through node k and out through node i.
$(i + k, j + m)$ = Addition operator.
m = Mutual admittance of transformer.
M = Mutual inductance.
v_j = Potential of node j.
v_{jm} = Voltage between nodes j and m with m positive.
$Y_{ij.km} = i'_{ik}/v_{jm}$ = Transfer admittance of network.

(1) INTRODUCTION

A linear electrical network composed entirely of 2-terminal elements can be represented topologically by a network consisting only of lines and points. In the solution of electrical networks by the nodal method the lines, or branches, are labelled with their admittances. The network may then be solved either by the determinant method or by mathematical trees.[1]

When the electrical network contains valves and transformers, which have more than two terminals, it has not been possible previously to represent the network solely by means of lines and points. In fact valves and transformers have not been given a topological representation, but appear in the network diagram as simplified pictures of the electrical components. The result is that a single uniform rule for setting up the determinant has not been available, while topological methods cannot be applied.

It will be shown that a linear electrical network composed of 2-terminal elements, valves and transformers can be represented topologically by a network composed of elements of a single type. In its most general form such an element comprises four nodes and two directed branches. Thus the network contains only lines and points, albeit the lines representing valves and transformers are necessarily directed lines.

With the aid of a network constructed in this manner the determinant can be set up directly from the network by means of a single rule. Alternatively the network can be solved by means of mathematical trees. In this paper we shall confine our attention to determinants and their matrices.

When all traces of pictorial representation have been eliminated from the network diagram it becomes a mathematical entity which can be operated upon in accordance with appropriate mathematical rules. It is then natural to consider the determinant as an algebraical equivalent of the network, for either can be obtained from the other without considering any electrical significance they may possess. Unfortunately, owing to the absence in the determinant of a row and column corresponding to the so-called ground node, there is no 1 : 1 correspondence between the parts of the network and their algebraic representation in the determinant.

It will be shown that the appropriate algebraic representation of the mathematical network is a special type of matrix, which will be termed an H-matrix, such that there is a 1 : 1 correspondence between the network and the matrix, and such that the sum of the matrices of the elements of the network is equal to the matrix of the network as a whole.

The determinant of an H-matrix will be termed an H-determinant. Such a determinant was employed by Jeans[2] in the solution of passive networks. He showed that the expanded form of the usual determinant of the network was equal to the co-factor of any determinant element in the principal diagonal. We shall show that it is equal to the co-factor of any determinant element. By utilizing the full freedom of choice the determinant can be evaluated more quickly, and with less chance of error, than by the usual method. Thus for a bridge network of six branches, the ordinary method without simplification gives 38 terms of which 22 cancel, whereas the new method gives 18 terms of which only two cancel.

Bode[3] gives formulae which can be used to express the transfer admittance of a network as a function of a single element. However, formulae of this type can apply only in particular

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cases. By the application of what will be termed addition operators to the H-determinant any transfer admittance can be expressed as a function of any element or elements.

It will be shown that a transfer admittance of a network can be included in the H-matrix as a network element. Hence theorems applying to network elements apply also to transfer admittances.

A particular application of H-matrices is to the theory of equivalent networks which in many cases can be found merely by setting up the H-matrices and comparing terms. The presence of valves and transformers makes no difference to the method.

In this paper H-matrices and the concept of the fundamental element will be developed together, and this would appear to be the logical procedure. Nevertheless it should be pointed out that H-matrices can be employed for the purpose of solving networks the elements of which are represented in the conventional manner, while valves and transformers can be represented in terms of the fundamental element merely to assist in the setting up of the classical determinant for the network.

(2) THE ELEMENTS OF A NETWORK

(2.1) Physical Elements and Mathematical Elements

The physical elements of a network are those essential electrical parts which cannot be broken down into simpler parts. The physical elements with which we shall be concerned are resistors, capacitors, inductors, transformers and valves, all operating in a linear manner.

A physical network can be represented by a mathematical network. This may be broken down into mathematical elements which may, or may not, correspond to the physical elements. It will be shown that a physical network composed of the elements listed above can be represented by a mathematical network composed entirely of elements of a single type. Henceforth the term element will be reserved for a mathematical element.

(2.2) The Fundamental Element

Consider an element denoted by $h_{ij.km}$ with unidirectional transmission such that a voltage v_{jm} between the nodes j and m, with m positive, produces a current i_{ik} given by

$$h_{ij.km}v_{jm} = i_{ik} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which flows into the element through node k and out of the element through node i.

Such an element can be represented as in Fig. 1 in which the double arrow from k to i represents the direction of current flow

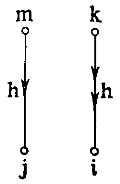


Fig. 1.—$h_{ij.km}$.

and the single arrow from m to j represents the direction from positive to negative voltage. The branch ik will be termed the current branch of the element and the nodes i and k the current nodes of the element, while the branch jm will be termed the voltage branch of the element and the nodes j and m the voltage nodes of the element. The branches are labelled with the symbol h which denotes the mutual admittance of the pair of branches.

The element $h_{ij.km}$ will be termed the fundamental element, and it will be shown that it is the only type of element required in the mathematical network. Since every physical element can be considered as a network in its own right it will first be necessary to show how the physical elements can be represented in terms of a fundamental element.

(2.3) Ordinary Elements

Any 2-terminal network can be represented by a single branch terminating on two nodes, and, mathematically, this represents a single element. This will be termed an ordinary element to distinguish it from those which require two branches and more than two nodes for their representation and which will be termed non-ordinary elements. The branch of an ordinary element will be termed an ordinary branch and the branches of non-ordinary elements will be termed non-ordinary branches.

If, in the element $h_{ij.km}$, the nodes i and j are identified and the nodes k and m are also identified, the two branches appear in parallel and the mutual admittance becomes a self-admittance. The two branches can therefore be replaced by a single branch

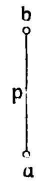


Fig. 2.—Ordinary branch $p_{aa.bb}$.

as in Fig. 2 which represents an ordinary element of admittance p the element being $p_{aa.bb}$. The arrow will be omitted as its direction is immaterial.

(2.4) The Valve

If the nodes i and j of the element $h_{ij.km}$ are identified the element represents a valve such that i and j become the cathode C, m becomes the grid G and k becomes the anode A. Hence a valve, normally represented as in Fig. 3(a), is represented by branches as in Fig. 3(b), where g is the mutual admittance which,

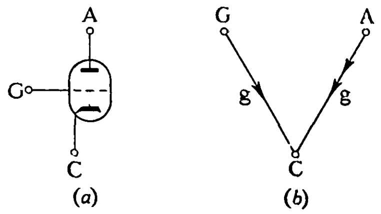


Fig. 3.—Triode valve $g_{CC.AG}$.

at low frequencies, becomes the mutual conductance. The element is written $g_{CC.AG}$. The passive admittances of the valve are not included since they can be more conveniently considered as part of the external network.

(2.5) Valves in Push-Pull

Two valves with an impedance in the cathode lead which is so large that the cathode current can be neglected are normally represented as in Fig. 4(a), but can be represented by branches as in Fig. 4(b), the element being written $g_{A1G1.A2G2}$. Thus a pair of valves so connected is represented by a single mathematical element.

(2.6) The Transformer

The mutual admittance $h_{ij.km}$ is a one-way mutual admittance. If a second element $h_{ji.mk}$ is added, with the current and voltage

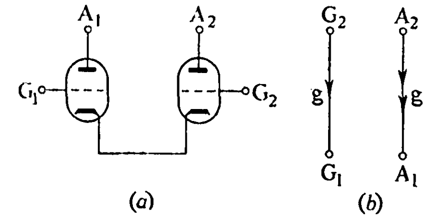


Fig. 4.—Pair of valves in push-pull with a large impedance in the common cathode lead $g_{A1G1.A2G2}$.

branches interchanged, but with the same mutual admittance, then a symmetrical 2-way mutual admittance is obtained. The most common example is the admittance of a transformer.

Fig. 5(*a*) shows the normal representation of a transformer, while Fig. 5(*b*) shows the corresponding branch representation.

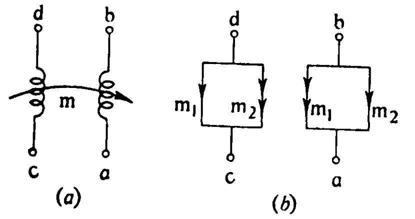


Fig. 5.—Symmetrical two-way mutual admittance $(m_{ac.cd})+(m_{ca.db})$.

The branches are labelled m_1 and m_2, to avoid confusion, but of course the mutual admittance $m = 1/j\omega M$, where M is the mutual inductance, is the same in either direction. Thus a transformer is represented by a pair of mathematical elements $m_{ac.bd}$ and $m_{ca.db}$ which, as will be shown later, can be written $(m_{ac.bd}) + (m_{ca.db})$.

In practice a mutual inductance is always associated with leakage inductances in series with the primary and secondary. It would be possible to consider the admittances of these inductances as part of the external network, which would mean that a transformer would be represented by a network with six nodes. It is therefore usual to convert the series admittances into parallel admittances. If before transformation p' is the admittance in series with the primary, q' is the admittance in series with the secondary, and m' is the symmetrical 2-way mutual admittance $1/j\omega M$, then

$$p = \frac{p'm'^2}{m'^2 - p'q'};\ q = \frac{q'm'^2}{m'^2 - p'q'};\ m = \frac{m'p'q'}{p'q' - m'^2} \quad . \quad (2)$$

where, after transformation, p is the admittance in parallel with the primary, q is the admittance in parallel with the secondary, and m is the symmetrical 2-way mutual admittance, which will in general be complex. The admittances p and q are not included in the branch representation since they can more conveniently be considered as part of the external network.

(3) THE COMBINATION OF ELEMENTS TO FORM A NETWORK

(3.1) The Representation of a Network

Fig. 6(*a*) shows a network which has been constructed to include three self-admittances p, q and r, a triode valve of mutual admittance g and a two-way mutual admittance m. The last may represent a mutual admittance with admittances in series with the primary and secondary which has been transformed in accordance with eqn. (2), the parallel admittances being included in p and q.

By replacing the valve and transformer by their branch representations the network of Fig. 6(*b*) is obtained.

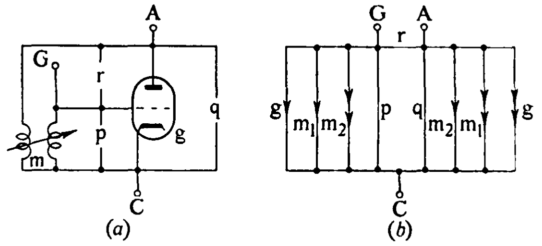


Fig. 6.—Triode valve associated with various admittances.

(3.2) The Network Equations

In order to obtain the network equations it is necessary to rewrite eqn. (1) in terms of the voltages v_j and v_m of the individual voltage nodes and of the currents i_i and i_k flowing into the element through the individual current nodes. We then have two equations

$$\left.\begin{aligned} h_{ij.km}v_j - h_{ij.km}v_m &= i_i \\ -h_{ij.km}v_j + h_{ij.km}v_m &= i_k \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

These are the equations for a network comprising only a single element. If the element $h_{ij.km}$ is connected to other elements to form the network we shall have a set of equations, one for each node of the network. The sum of all the currents, such as i_i, entering the network must, of course, be zero.

Now consider the coefficient of the voltage v_j of the jth node in the equation for the ith node. This will include all elements with a current node i and a voltage node j, i.e. all elements of the types $h_{ij.km}$, $h'_{kj.im}$, $h''_{im.kj}$ and $h'''_{km.ij}$ for all values of the suffices k and m. Since an interchange of either the voltage nodes or the current nodes of any element causes a change of sign, elements of the first and last type will appear with a positive sign and elements of the second and third type will appear with a negative sign. The equation to the ith node will include on its left-hand side the voltages of all the voltage nodes of elements of which i is a current node. Hence the equation for the ith node is

$$\sum_i \sum_{k,m} (h_{ij.km} - h'_{kj.im} - h''_{im.kj} + h'''_{km.ij})v_j = i_i \quad . \quad (4)$$

where i_i is the current flowing from an external source into node i, and the four elements in the brackets include all those which possess current nodes i and k and voltage nodes j and m, the sign being given by the number of inversions of the suffices relative to the element $h_{ij.km}$.

Let h_{ij} be defined by the equation

$$h_{ij} = \sum_{k,m} (h_{ij.km} - h'_{kj.im} - h''_{im.kj} + h'''_{km.ij}) \quad . \quad . \quad (5)$$

where h_{ij} will be termed the mutual admittance of the node pair ij in contradistinction to $h_{ij.km}$, which can be termed the mutual admittance of the branch pair ik and jm.

The set of eqn. (4) can then be written

$$\sum h_{ij}v_j = i_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which represents an equation for each value of the suffix i.

Thus the mutual admittance h_{ij} is equal to the current which flows into the node i for unit change of the voltage v_j. The network is constructed from elements of the type $h_{ij.km}$, so that these will be termed network elements; but in order to form a matrix from the network it is considered to be composed of elements of the type h_{ij}, so that these will be termed matrix elements.

The number of equations is equal to the number of nodes,

V, in the network. The network may consist of a number of separate parts coupled only by mutual admittances. If so, the number of nodes, and hence the number of equations, can be reduced without loss of generality by identifying (i.e. short-circuiting) appropriate nodes in the different parts of the network until a single network is obtained. It will be assumed henceforth that this has been done.

The feed current to any one node can always be deduced from the currents feeding the remaining nodes, for the total feed current to the network must be zero. Hence one of the equations must be redundant. Moreover, no generality is lost if the potential of one of the nodes is made equal to zero. Thus the number of independent equations for a connected network and the number of independent nodal voltages are both equal to $V - 1$.

Thus it would be possible to delete the equation for any feed current such as i_p, and also to delete any nodal voltage such as v_q. However, to do so would result in a loss of symmetry, and, as will be shown, considerable advantages are to be gained by retaining the complete set of equations as in eqn. (6).

(4) THE *H*-MATRIX

The coefficients h_{ij} of eqn. (6) can be expressed in the form of a matrix in which the typical matrix element h_{ij} appears at the intersection of row i and column j. This matrix will be termed the H-matrix of the network and will be written (H).

(4.1) The *H*-Matrices of the Network Elements

Since a single network element is a network in its own right there must be an H-matrix for every such element. It follows from eqn. (3) that the H-matrix of the fundamental element $h_{ij.km}$ is given by

$$(h_{ij.km}) = \begin{array}{ccc} & j & m \\ i & h & -h \\ k & -h & h \end{array} \quad . \quad . \quad . \quad . \quad (7)$$

By comparing eqn. (7) and Fig. 1 it will be seen that h is positive in the matrix element h_{pq} if both arrows point towards or away from p and q, and negative otherwise.

The matrix of an ordinary element as in Fig. 2 is given by

$$(p_{aa.bb}) = \begin{array}{ccc} & a & b \\ a & p & -p \\ b & -p & p \end{array} \quad . \quad . \quad . \quad . \quad (8)$$

The matrix of a valve as in Fig. 3 is given by

$$(g_{CC.AG}) = \begin{array}{ccc} & C & G \\ C & g & -g \\ A & -g & g \end{array} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The matrix of a symmetrical two-way mutual admittance is given by

$$(m_{ac.bd}) + (m_{ca.db}) = \begin{array}{ccccc} & a & b & c & d \\ a & 0 & 0 & m & -m \\ b & 0 & 0 & -m & m \\ c & m & -m & 0 & 0 \\ d & -m & m & 0 & 0 \end{array} \quad . \quad (10)$$

The curved brackets will be omitted if it is clear from the context that the array is a matrix. The rows will be labelled with the letters corresponding to the current nodes, and the columns will be labelled with the letters corresponding to the voltage nodes.

(4.2) The *H*-Matrix of a Network

The physical operation of connecting elements to form a network corresponds to the algebraic operation of combining the matrices of the elements of a network to form a single matrix of the coefficients h_{ij} of eqn. (6). Thus all that is necessary to form the matrix of the network is to label the rows and columns with the letters denoting the nodes of the network and to insert the network elements in the appropriate positions. Thus we have

Theorem (1).—The H-matrix of a network is equal to the sum of the matrices of its network elements, i.e.

$$(H) = \sum_{i,j,k,m} (h_{ij.km}) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where, to be formally correct, the matrix of an element belonging to a network is assumed to include all the rows and columns corresponding to the nodes of the network, the matrix elements being zero except for those belonging to the element itself.

Corollary.—The H-matrix of a composite network is equal to the sum of the H-matrices of its parts.

Eqn. (12) gives the H-matrix of the network of Fig. 6.

$$(H) = \begin{array}{cccc} & C & G & A \\ C & p+q+2m+g & -p-m-g & -q-m \\ G & -p-m & p+r & -r+m \\ A & -q-m-g & -r+m+g & q+r \end{array} \quad . \quad (12)$$

With the aid of eqns. (8), (9) and (10) it can be shown that (H) is formed by the addition of the matrices of the network elements. However, it is convenient to employ a rule whereby the H-matrix may be set up, matrix element by matrix element.

Rule for Setting up H-Matrix.—The matrix element h_{ij} is equal to the sum, with appropriate signs, of all the network elements, the current branches of which terminate on node i and the voltage branches of which terminate on node j. For ordinary elements the sign is always positive when they appear in the principal diagonal and negative when they do not. For non-ordinary elements the sign is positive if the arrows on both branches of the element point either towards or away from the nodes i and j, and negative if one arrow points towards i and the other away from j or if one arrow points away from i and the other towards j.

When the matrix has been set up by the above rule it is advisable to check it with the aid of theorem (1). Alternatively theorem (1) can be used to set up the matrix.

(5) THE *H*-DETERMINANT AND ITS DERIVATIVES

The determinant of an H-matrix will be termed an H-determinant and written H. The derivatives of H will be studied first. It will be shown later that, with the aid of such derivatives, any transfer admittance of a network can be expressed as a function of any selected network elements or expanded as a function of all the elements of the network.

If H is considered as a function of its determinant elements such as h_{ij}, then, as is well known, $\partial H/\partial h_{ij}$ equals H_{ij} and $\partial^2 H/\partial h_{ij}\partial h_{km}$ equals $H_{ij.km}$, where H_{ij} is the co-factor of h_{ij}, and $H_{ij.km}$ is a further co-factor with respect to h_{km}.

H can also be considered as a function of the network elements such as $h_{ij.km}$, and it should be possible therefore to find the derivatives of H with respect to these elements. However, a certain difficulty must be disposed of first. The sum of all the elements in any row or column of H is necessarily zero. Thus the rows and columns are linearly dependent, and the value of the H-determinant is identically zero. However, we shall not be concerned with the value of an H-determinant, but only with the values of those derivatives which involve at least one differentiation with respect to a determinant element. Under these conditions it may be shown that H may be safely differentiated with respect to network elements, it being understood that these derivatives will be subsequently differentiated with respect to a determinant element before an evaluation is made.

The addition operator $(i + k, j + m)$ will be defined as the operator which, operating on the determinant H, deletes row k, replaces row i by the sum of the rows i and k, performs the corresponding operations on the columns j and m and affixes the sign $(-1)^{k+m}$, i.e. the sign normally affixed to the co-factor H_{km}.

Theorem (2).—The operation of taking the derivative of the determinant H with respect to the network element $h_{ij.km}$ is equivalent to applying to H the addition operator $(i + k, j + m)$

i.e.
$$\partial H/\partial h_{ij.km} = (i + k, j + m)H \quad . \quad . \quad . \quad (13)$$

$(i + k, j + m)H$ is an H-determinant, and successive derivatives of H can be taken with respect to network elements. Thus

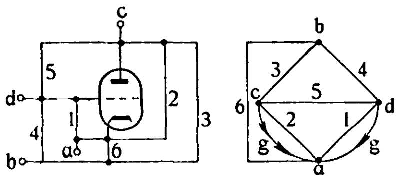
$$\partial^2 H/\partial h_{ij.km}\partial h_{pq.rs} = (p + r, q + s)(i + k, j + m)H \quad . \quad (14)$$

Proof.—Since the network element $h_{ij.km}$ appears in H in the four positions ij, kj, im and km, the sign being positive for the first and last, and negative for the other two, we have

$$\begin{aligned}\partial H/\partial h_{ij.km} &= \partial H/\partial h_{ij} - \partial H/\partial h_{kj} - \partial H/\partial h_{im} + \partial H/\partial h_{km}\\ &= \quad H_{ij} \quad - \quad H_{kj} \quad - \quad H_{im} \quad + \quad H_{km}\\ &= (i + k, j + m)H\end{aligned}$$

The operation $(i + k, j + m)$ converts H into a determinant with one less row and one less column, such that all elements of the forms $h_{iv.kw}$ and $h_{uj.xm}$ are eliminated. But all other elements remain. Hence $(i + k, j + m)H$ is an H-determinant and further derivatives with respect to network elements can be taken.

Elements of the form $h_{iv.kw}$ have their current branches in parallel with the current branch of $h_{ij.km}$. Such elements will be said to be in current parallel. Elements of the form $h_{uj.xm}$ have their voltage branches in parallel with the voltage branch of $h_{ij.km}$. Such elements will be said to be in voltage parallel. Thus the operation $(i + k, j + m)$ eliminates $h_{ij.km}$ and all elements in current or voltage parallel with it.



(*a*) (*b*)

Fig. 7.—Single-valve network.

Fig. 7(*a*) shows a single valve network which is redrawn in Fig. 7(*b*) with the valve represented by its branches. The H-determinant is given by eqn. (15) and the derivative $\partial H/\partial g$ by eqn. (16).

$$H = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1+2+6+g & -6 & -2 & -g-1 \\ b & -6 & 3+4+6 & -3 & -4 \\ c & -2-g & -3 & 2+3+5 & -5+g \\ d & -1 & -4 & -5 & 1+4+5 \end{array} \quad . \quad (15)$$

$$\partial H/\partial g = (a+c, a+d)H = - \begin{array}{c|ccc} & a+d & b & c \\ \hline a+c & 6-5 & -3-6 & 3+5 \\ b & -4-6 & 3+4+6 & -3 \\ d & 4+5 & -4 & -5 \end{array} \quad . \quad (16)$$

The effect of the operation is to produce another H-determinant with a negative sign and one less row and one less column. The element $g_{aa.cd}$ is eliminated together with element 2, which is in current parallel, and element 1, which is in voltage parallel.

A further derivative may be taken with respect to the element 6, giving

$$\begin{aligned}&\partial^2 H/\partial g\partial 6\\ &= (a+b, a+b)(a+c, a+d)H = - \begin{array}{c|cc} & a+d+b & c \\ \hline a+c+b & -5 & 5 \\ d & 5 & -5 \end{array} \quad . \quad (17)\end{aligned}$$

In applying addition operators in succession, an instruction such as "add row b to row a" is to be taken to mean "add the row which includes the letter b to the row which includes the letter a"; similarly with the columns.

A further, and final, derivative may be taken with respect to the network element 5 giving

$$\partial^3 H/\partial g\partial 6\partial 5 = \begin{array}{c|c} & a + d + b + c \\ \hline a + c + b + d & 0 \end{array} \quad . \quad . \quad . \quad (18)$$

A derivative may be taken with respect to an element which does not appear in H since it is only necessary to state that such a derivative is given by the corresponding addition operation. With this understanding H may be expanded in terms of any of the network elements by means of the general equation

$$H = [\Pi(1 + h_{ij.km}\partial/\partial h_{ij.km})]H_0 \quad . \quad . \quad . \quad (19)$$

where the product is taken for all the elements in terms of which the expansion is made, and $H_0 = H$ when all these elements are put equal to zero. The equation follows from the fact that H is a linear function of the network elements. Thus the H of eqn. (15) can be written

$$H = H_0 + g\partial H_0/\partial g + 6\partial H_0/\partial 6 + 6g\partial^2 H_0/\partial g\partial 6 \quad . \quad (20)$$

where $H_0 = H(g = 6 = 0)$.

As previously explained, we are not concerned with the values of H and its derivatives with respect to network elements, but only with the values of further derivatives with respect to determinant elements. Since derivatives with respect to network elements are linear functions of derivatives with respect to determinant elements, all operations of differentiation with respect to either network elements or determinant elements are commutative.

(6) THE *D*-DETERMINANT AND ITS DERIVATIVES

(6.1) The *D*-Determinant

It will be recognized that the co-factor H_{uu} is the determinant D of a network with ground node u, and it is well known that the value of D is independent of u. We shall now show that the value of D is equal to that of any first co-factor of H.

Theorem (3).—All the first co-factors of an H-determinant are equal, i.e.

$$D = H_{uv} = \partial H/\partial h_{uv} \quad \text{for any value of } u \text{ and } v \quad . \quad . \quad (21)$$

Proof.—The value of a determinant is unaltered by adding to any row a linear combination of other rows. Hence any row i in H_{uv} may be replaced by the sum of all the rows in H_{uv}. But the sum of all the rows in H_{uv} is equal to minus the deleted row u. Therefore row i may be replaced by minus the deleted row u. Now transfer this row to the position of the original row u and prefix the sign $(-1)^{u+i}$ to the co-factor. It is easily shown that these operations have not altered the value of H_{uv} but have converted it into H_{iv}. It can be proved in a similar way that $H_{iv} = H_{ij}$ where j is any column.

The same result could have been obtained from the set of eqns. (6) by omitting the equation for the node u, making the potential of node v equal to zero and then obtaining D from the remaining set of equations.

The advantage of writing down the complete determinant H and then deleting one row and one column is that the row and column deleted can be so chosen that the resultant determinant D can be expanded with the fewest number of cancellations. In most cases this implies that the row and column deleted are those containing the largest number of entries. This advantage will be demonstrated for the determinant of eqn. (15). The row containing the largest number of entries is row a and the column which then contains the largest number of entries is column d. Hence we unite

$$D = H_{ad} = - \begin{array}{c} \\ b \\ c \\ d \end{array} \begin{array}{ccc} a & b & c \\ -6 & 3+4+6 & -3 \\ -2-g & -3 & 2+3+5 \\ -1 & -4 & -5 \end{array} \quad . \quad (22)$$

which gives

$$D = 6[4(2+3+5) + 35] + (2+g)[5(3+4+6) + 34] + 1[(4+6)(2+3+5) + 3(2+5)] \quad . \quad (23)$$

Only two terms cancel, i.e. 133 and -133, whereas if the best choice of a ground node had been made, i.e. node a, and the usual method had been employed without simplifying the determinant, it would have been necessary to cancel 22 terms.

(6.2) The Derivatives of the *D*-Determinant

Theorem (4).—The coefficient in D of any product of network elements is equal to the derivative with respect to any determinant element of the derivative of H with respect to these network elements. Thus for a single network element

$$\partial D/\partial h_{ij.km} = \partial^2 H/\partial h_{ij.km}\partial h_{uv} = |(i+k, j+m)H|_{uv} \quad . \quad (24)$$

for any value of u and v, and for a number of network elements

$$\left(\Pi\frac{\partial}{\partial h_{ij.km}}\right)D = \left|\left(\Pi\frac{\partial}{\partial h_{ij.km}}\right)H\right|_{uv} = |[\Pi(i+k, j+m)]H|_{uv} \, . \, (25)$$

Proof.—Since $D = \partial H/\partial h_{uv}$ and derivatives with respect to determinant elements and network elements are commutative, the first equations of eqns. (24) and (25) follow. The second equations are merely restatements in terms of addition operators.

Theorem (5).—The coefficient of a single network element in D is equal to the derivative of H with respect to the two determinant elements in which the network element appears with a positive sign, i.e.

$$\partial D/\partial h_{ij.km} = \partial^2 H/\partial h_{ij}\partial h_{km} = H_{ij.km} \quad . \quad . \quad (26)$$

Proof.—Put $u = i$ and $v = j$ in eqn. (24).

Theorem (6).—The coefficient of n network elements in D can be found by performing $n - j$ addition operations corresponding to $n - 1$ of the elements, giving an H-determinant H', and then taking the second co-factor $H'_{ij.km}$ where $h_{ij.km}$ is the final element. Thus for the coefficient of two network elements we have

$$\partial^2 D/\partial h_{ij.km}\partial h_{pq.rs} = |(i+k, j+m)H|_{pq.rs} = |(p+r, q+s)H|_{ij.km} \quad . \quad (27)$$

Proof.—The theorem follows from theorems (4) and (5).

Theorem (7).—The coefficient of n connected ordinary elements in D is $H_{ii.jj\ldots}$, where the suffices include all the $n+1$ nodes on which the branches terminate.

Proof.—An addition operation with respect to an ordinary branch is equivalent to closing the branch in the network. Thus the addition operations necessary to find the coefficient of the set of elements is equivalent to reducing the set to a single node, say q. In the corresponding H-determinant all the rows and columns corresponding to the nodes of the elements will have to be deleted, with the exception of row q and column q. To find the coefficient of the set of elements in D the final step is to take any first co-factor for which we choose the co-factor of the determinant element h_{qq}. This proves the theorem.

The theorem may be extended to non-ordinary elements for which both the current branches and also the voltage branches are connected, but considerable care must then be taken over signs. The coefficient of two or more unconnected elements cannot be found by taking a co-factor of H or D.

It follows from theorem (4) and eqn. (19) that D can be expanded in terms of any of its elements with the aid of the equation

$$D = |[\Pi(1 + h_{ij.km}\partial/\partial h_{ij.km}]H_0|_{uv} \quad . \quad . \quad . \quad (28)$$

By way of illustration, for the network of Fig. 7 we have, with the aid of eqn. (20),

$$D = D_0 + g[5(3+4) + 34] + 6[(2+3)(1+4+5) + 5(1+4)] + 6g5 \quad . \quad (29)$$

where

(*a*) $D_0 = D(g = 6 = 0)$ can be obtained from eqn. (15).

(*b*) The coefficient of g can be obtained by taking the co-factor in eqn. (16) with respect to the determinant element h_{aa} with the element 6 put equal to zero.

(*c*) The coefficient of 6 can be obtained by taking the co-factor with respect to the elements h_{aa} and h_{bb} in eqn. (15) with $g = 0$.

(*d*) The coefficient of $6g$ is obtained by taking any first co-factor in eqn. (17).

(7) THE TRANSFER ADMITTANCE OF A NETWORK

The transfer admittance $Y_{ij.km}$ of a network is defined by the equation

$$Y_{ij.km} = i'_{ik}/v_{jm} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where i'_{ik} is a feed current which flows into the network at node k and out at node i and causes a voltage v_{jm} to be developed between nodes j and m with m positive. These relations are shown in Fig. 8(a).

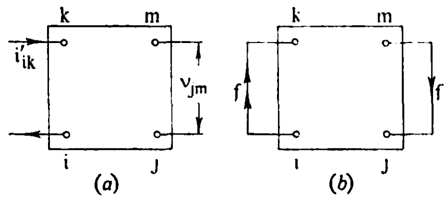


Fig. 8.—$Y_{ij.km} = i'_{k}/v_{jm} = f_{kj.im}$.

Now let the feed current be replaced instantaneously by a network element $f_{kj.im}$ such that the mutual admittance f is equal to the transfer admittance $Y_{ij.km}$, as shown in Fig. 8(b). Such an element may not be realizable physically, but this is immaterial since we are concerned with it only as a mathematical device. The feed current i'_{ik} and all the other voltages and currents in the network will then be maintained indefinitely, the only difference being that i'_{ik} is produced by a network element instead of by an external source.

Such a network is in equilibrium, and eqn. (6) becomes

$$\sum_j h_{ij}v_j = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

This represents a set of homogenous equations of which we know that one is superfluous. Hence the rank of the H-determinant is two less than the number of its rows and columns. The conditions for a network to be in equilibrium must therefore be that

$$H_{uv} = D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

But D is a linear function of $f_{kj.im}$, so that we can write

$$A + Bf_{kj.im} = 0 \quad . \quad . \quad . \quad . \quad . \quad (33)$$

It follows that

$$Y_{ij.km} = -A/B = \frac{-D_0}{\partial D/\partial f_{kj.im}} \quad . \quad . \quad . \quad (34)$$

where $D_0 = D(f_{kj.im} = 0)$.

With the aid of Section 6 both the numerator and the denominator can be expanded as functions of one or more network elements. Suppose it is required to find the transfer admittance $Y_{ac.bd}$ of the network of Fig. 7. The first step is to add an element $f_{bc.ad}$ as in Fig. 9. The H-determinant is

$$H = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1+2+6+g & -6 & -2-f & -1-g+f \\ b & -6 & 3+4+6 & -3+f & -4-f \\ c & -2-g & -3 & 2+3+5 & -5+g \\ d & -1 & -4 & -5 & 1+4+5 \end{array} \quad . \quad (35)$$

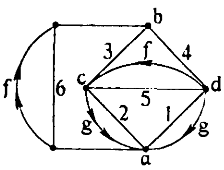


Fig. 9.—Network of Fig. 7 with additional element $f_{bc.ad} = Y_{ac.bd}$.

The determinant D can be expanded in the form

$$D = D_0 + g|(a+c, a+d)H_0|_{ab} + f_{bc.ad}[|H_0|_{bc.ad} + g|(a+c, a+d)H_0|_{bc.ad}]$$
$$= D_0 + g[5(4+6) + 3(4+5)] + f_{bc.ad}(-24 + 13 - 4g) \quad . \quad . \quad . \quad . \quad (36)$$

$$\text{Hence} \quad Y_{ac.bd} = \frac{D_0 + g[5(4+6) + 3(4+5)]}{24 - 13 + 4g} \quad . \quad . \quad (37)$$

Of course, it is not necessary to employ the concept of a network in equilibrium in order to find an expression for the transfer admittance. The advantages are that it leads to eqn. (31), which is the simplest form for the set of equations for a network. Moreover, it proves that a transfer admittance is of the same form as a mutual admittance, so that it can be represented in the network in the same way and theorems which apply to mutual admittances, including all those in the paper, apply also to transfer admittances.

Theorem (8).—No elements which are in current or voltage parallel can appear as a product in the expansion of the D-determinant.

Proof.—The theorem follows from the proof of theorem (2).

Corollary.—No such elements can appear as a product in the numerator or denominator of a transfer admittance, nor can elements, the current branches of which appear across the input of a network or the voltages branches of which appear across the output of a network, appear in the denominator of the transfer admittance.

(8) EQUIVALENT NETWORKS

(8.1) Theorems

Two networks are said to be equivalent with respect to a given set of terminals if the networks are indistinguishable by any measurements made between the terminals. Two network diagrams, which may contain non-physical elements but which behave mathematically as though the networks are equivalent, are said to be mathematically equivalent.

In setting up the H-matrix of a network, nodes which are also terminals may be grouped together, and the matrix elements defined by these nodes may be separated by a dotted line as in eqn. (38). The portion of the matrix separated is a sub-matrix. Co-factors of the H-determinant in which only rows and columns of the sub-matrix are deleted will be referred to for convenience as co-factors of the sub-matrix.

Theorem (9).—Two networks are equivalent with respect to a given set of terminals if, and only if, the corresponding co-factors of their sub-matrices are in constant ratio. This ratio is equal to the ratio of the D-determinants of the two networks when all the terminals on each network are short-circuited.

Proof.—Let any transfer admittance $Y_{ij.km}$ be represented by the addition of the corresponding element $f_{kj.im}$ to the sub-matrix. The D-determinants of both networks will then be zero. If the transfer admittances of the two networks are to be the same, the corresponding terms in their D-determinants can differ only by a multiplying constant. But if suitable elements are present, every co-factor of either of the sub-matrices can be a coefficient in the corresponding D-determinant. Hence the corresponding co-factors in the two H-determinants can differ only by a constant multiplying constant. This constant must be equal to the ratio of those particular co-factors for which all the rows and columns of the sub-matrices are deleted. This is equal to the ratio of the D-determinants of the two networks when all the terminals are identified, i.e. short-circuited.

The minimum network equivalent to a given network will be defined as that equivalent network which possesses no nodes other than the terminals. The sub-matrix of the minimum equivalent network will clearly be identical with the H-matrix itself.

Theorem (10).—The matrix element h_{ij} of the minimum network equivalent to a given network is equal to the co-factor of the sub-matrix of the given network—in which all the rows and columns of the sub-matrix are deleted except row i and column j—divided by the D-determinant of the given network when all its terminals are short-circuited.

Proof.—This theorem is only a particular case of theorem (9).

(8.2) Example

Hsu[4] has shown how the familiar Y-delta transformation for passive networks can be extended to networks containing a valve. As an example of the simplicity of the H-matrix method,

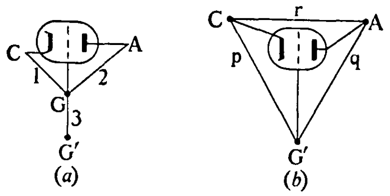


Fig. 10.—Y-delta transformation.

we shall prove his formula for what he terms the common grid transformation. Fig. 10 shows the Y-delta transformation with the valve represented in the orthodox way, and Fig. 11 shows the same transformation with the valve represented by branches.

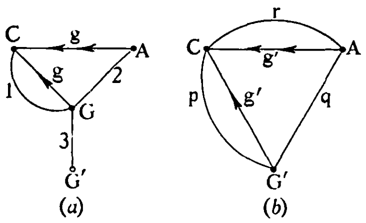


Fig. 11.—Y-delta transformation with valve represented by branches.

The H-matrix of Fig. 11(a) is

$$(H_1) = \begin{array}{c|ccc|c} & C & A & G' & G \\ \hline C & 1+g & 0 & 0 & -1-g \\ A & -g & 2 & 0 & g-2 \\ G' & 0 & 0 & 3 & -3 \\ \hline G & -1 & -2 & -3 & 1+2+3 \end{array} \quad . \quad (38)$$

The H-matrix of Fig. 11(b) is

$$(H_2) = \begin{array}{cccc} & C & A & G' \\ C & p+r+g' & -r & -p-g' \\ A & -r-g' & r+q & -q+g' \\ G' & -p & -q & p+q \end{array} \quad . \quad (39)$$

The common denominator of all the elements in (H_2) is the co-factor obtained by deleting all the rows and columns of the sub-matrix enclosed in the dotted lines. This is $1+2+3$. The co-factor corresponding to $-r$ in the position CA in eqn. (39) is that obtained by deleting all the rows and columns of the sub-matrix except row C and column A. This is $-2(1+g)$. Proceeding in a similar way we obtain

$$p = 13/1+2+3; \quad q = 23/1+2+3;$$
$$r = 2(1+g)/1+2+3; \quad p+g' = 3(1+g)/1+2+3;$$
$$g' = 3g/1+2+3 \quad . \quad . \quad . \quad . \quad . \quad (40)$$

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