Topics in Liouville quantum gravity and random conformal geometry

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Contents

1 Preliminaries

This section gives an overview of the basic theory on Gaussian free field (GFF) and Schramm-Loewner evolution (SLE). The reference materials are the following lecture notes: [2], [3], [18].

1.1 The continuum Gaussian free field

Let's begin by going back to the first class in quantum field theory, where we tried to establish the real scalar field and it's quantization. Namely, we consider the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi) = -\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2$. For its path integral quantization, we consider the following formal 'functional measure' with $\mathcal{D}\phi$ being a 'uniform measure' over all fields

$$\mathcal{D}\phi(x)e^{i\int d^4x\mathcal{L}(\phi(x),\partial_\mu\phi(x))}$$

For mathematical reasons, and it's connection with statistical mechanics, one is more interested in the case in which spacetime is Euclidean, discrete (When we say discrete, it simply means that the structure is a sub-lattice of $\mathbb{Z}^d$ or a general graph $G = (\mathcal{V}, E)$.) and the mass is 0. We denote the inverse temperature by β, then after recalculation the measure becomes

$$\prod_{x\in\Lambda} d\phi_x e^{-\frac{\beta}{2}\sum_{x\sim y}(\phi_x-\phi_y)^2} \tag{1.1}$$

The above measure on the product space $\mathbb{R}^\Lambda$ is called a discrete Gaussian free field (with zero boundary condition). In fact, this model can be understood by considering a system of vibrating string (or in physics terms, quantum harmonic oscillators). On each vertex x we assign a vibration such that it's fluctuation is described by Gaussian distribution, and it's motion is only affected by the neighboring points, i.e., by $\{y, y \sim x\}$.

Another way to understand this model is via scaling limit of random walk. Consider an integer valued random function $f : \Lambda \to \mathbb{Z}^\Lambda$, such that $f(x) - f(y) \in \{-1, 0, 1\}$ iff $x \sim y$. This amounts to a 'discrete time' simple random walk with integer-valued time $t \in \mathbb{N}$ replaced by the lattice time $t \in \Lambda$. By scaling the values of f, i.e., take $\lim_{\varepsilon\to 0} \varepsilon\mathbb{Z}^\Lambda = \mathbb{R}^\Lambda$, an appropriate choice of f should lead to convergence to the discrete Gaussian free field in distribution. From this perspective, we conclude that discrete Gaussian free field is the lattice-time-analog of Brownian motion. In contrast, if we independently coping Brownian motion on the lattice: $(B_x(t))_{x\in\Lambda}$, we obtain a cylindrical Brownian motion, which is actually the spatially discrete simplification of the spacetime white noise $(W(x,t))_{\mathbb{R}^2\times\mathbb{R}_+}$.

Similar to the construction of spacetime white noise, Gaussian free fields also suffer a similar problem if we take the continuum limit of space Λ: the fields becomes random generalized function! Now we would like to rigorously study the mathematical structure of continuous Gaussian free field in 2d, which is of most interest in the literature of random conformal geometry. Assume $D \subset \mathbb{C}$ is a bounded planar domain and $G(x, y)$ is the Green function defined by:

$$G(x,y) = \pi \int_0^\infty p_t(x,y)\pi_t(x,y)dt = -\log|y-x| - \tilde{G}_x(y) \tag{1.2}$$

where $p_t(x, y) = \frac{1}{2\pi t}\exp\Big(\frac{-|x-y|^2}{2t}\Big)$ is the two dimensional heat kernel, $\pi_t(x, y)$ is the probability that Brownian Bridge starting from x ending at y remains in D in time $[0, t]$. In the second equality, $\tilde{G}_x(y)$ is the harmonic extension to D given the function $-\log|\cdot - x|$ on the boundary. Define a family of compact supported signed measure $\mathcal{M}_0$ on D, such that $\forall \rho \in \mathcal{M}_0$,

$$\iint_{D^2} G(x,y)\rho(dx)\rho(dy) < \infty$$

Starting from this aspect, we make the following definition.

Definition 1.1. *The continuum Gaussian free field (GFF) on D with Dirichlet boundary condition is the zero-mean Gaussian process h indexed by $\mathcal{M}_0$, which has the covariance function:*

$$\operatorname{cov}((h,\rho_1),(h,\rho_2)) = \int_D G(x,y)\rho_1(dx)\rho_2(dy) \tag{1.3}$$

for any $\rho_1,\rho_2 \in \mathcal{M}_0$. Here (h,ρ) is just the random variable indexed by ρ in the process. It can also be informally understood as the test of h against ρ: $\int_D h(z)\rho(dz)$.

In particular, we're interested in the measure $\rho(x)dx \in \mathcal{M}_0$ such that the density satisfies $2\pi\rho = -\Delta f$ for some $f \in H_0^1(D)$. Since $\rho \leftrightarrow f$ and Sobolev Space $H_0^{-1}(D)$ is the dual of $H_0^1(D)$, we regard GFF as a centered Gaussian process $(h,f)_\nabla$ indexed by $H_0^1(D)$, and

$$\operatorname{cov}((h,\rho_1),(h,\rho_2)) = \operatorname{cov}((h,f_1)_\nabla,(h,f_2)_\nabla) = (f_1,f_2)_\nabla := \frac{1}{2\pi}\int_D (\nabla f_1,\nabla f_2)dx$$

These equalities follows from the fact that $G(x,y)$ defined above, is exactly the Green's function for the operator $-(2\pi)^{-1}\Delta$. Thus if we denote $(\cdot,\cdot)_\nabla := \frac{1}{2\pi}(\nabla\cdot,\nabla\cdot)$, it is a well-defined inner product on $H_0^1(D)$ and makes it into a Hilbert space. We then naturally regard h as an isometric embedding of $H_0^1(D)$ into some Gaussian space G, i.e.,

$$h: \quad H_0^1(D) \hookrightarrow G \leq L^2(\Omega,\mathcal{F},\mathbb{P}).$$

From another aspect, instead of interpreting h as a stochastic process, we would like to try understanding GFF as a random variable on $H_0^1(D)$. Take an arbitrary orthonormal basis $\{f_n\}$ chosen under $(\cdot,\cdot)_\nabla$, we see that $(h,f_n)_\nabla$ are i.i.d standard Gaussian. However, the sum $\sum_n (h,f_n)_\nabla f_n$ doesn't converge in $H_0^1(D)$ a.s.. On the upside, $\sum_n (h,f_n)_\nabla (f_n,f)_\nabla$ converges a.s. for all test functions $f \in H_0^1(D)$. Since $H_0^1(D) \supset C_c^\infty(D)$ (compact supported C^∞-function space, which is a topological vector space familiar from real analysis), $h := \sum_n (h,f_n)_\nabla f_n$ converges a.s. in the weak-$\star$ topology of the distribution space $\mathcal{D}_0'(D)$. This shows that continuum zero-boundary GFF is a random generalized function (distribution) on the function space $C_c^\infty(D)$ (*weak sense*). Finally, if we really want to interpretate h as a random element in some Sobolev space equipped with complete norm (*strong sense*), we claim that, for f_n being an orthonormal basis of $H_0^1(D)$, the sum $\sum_n (h,f_n)_\nabla f_n$ converges a.s. in the strong topology of $H_0^s(D)$ for all $s<0$ (see the following Remark for details).

Remark 1.1. *We state some clarifications and supplements to the above.*

(i) A reminder. Take the set of all eigenfunctions of the Laplacian operator on D with 0 boundary condition, normalized them in the $L^2(D)$ norm, we get an L^2-orthonormal basis $\{e_j\}_{j\geq 0}$.

Suppose these eigenvalues are $\{\lambda_j\}_{j\geq 0}$, we can define the inner product $(\cdot,\cdot)_s$ with arbitrary real index $s \in \mathbb{R}$ by $(f,g)_s = \sum_{j\geq 0} \lambda_j^s (f,e_j)_{L^2(D)}(g,e_j)_{L^2(D)}$. The Sobolev space $H_0^s(D)$ is defined by taking the completion of $C_c^\infty(D)$ with respect to the inner product $(\cdot,\cdot)_s$.

(ii) Easy conclusions. $H_0^s(D)$ are Hilbert spaces, with set relation $H_0^s(D) \subset H_0^r(D)$, $s > r$ and $H_0^0(D) = L^2(D)$. The orthonormal basis of $H_0^s(D)$ under the inner product $(\cdot,\cdot)_s$ is given by $\{\lambda_j^{-s/2} e_j\}_{j\geq 0}$. The larger the index s grows, the more regular the function will be.

Since the continuous GFF is a highly 'singular random function', we naturally ask the question: Is it possible to do mollifications just like in the theory of distributions? In particular, we are interested in $h_\varepsilon(z)$, which is the average of h on the set $D \cap \partial B_\varepsilon(z)$. Identically speaking, given the uniform distribution ρ_ε^z on $D \cap \partial B_\varepsilon(z)$, $h_\varepsilon(z) := (h, \rho_\varepsilon^z)$ and the covariance function is given by:

$$G_{\varepsilon_1,\varepsilon_2}(z_1,z_2) := \mathrm{cov}(h_{\varepsilon_1}(z_1), h_{\varepsilon_2}(z_2)) = \iint_{D^2} G(x,y)\rho_{\varepsilon_1}^{z_1}(x)\rho_{\varepsilon_2}^{z_2}(y)dxdy$$

If $\varepsilon < \mathrm{dist}(z,\partial D)$, $h_\varepsilon(z)$ is just the circle average around an interior point z.

Another natural question arises: does $h_\varepsilon(z)$ really belong to the family $\{(h,f)_\nabla\}_{f\in H_0^1}$? To see this, we introduce a function $\xi_\varepsilon^z(y)$ for $y \in D$ by:

$$\xi_\varepsilon^z(y) := -\log\max(\varepsilon, |z-y|) - \tilde{G}_{z,\varepsilon}(y)$$

where $\tilde{G}_{z,\varepsilon}(y)$ is the harmonic extension to D of the restriction of $-\log\max(\varepsilon,|z-y|)$ on ∂D. As an easy consequence, $\xi_\varepsilon^z(y) \to 0$ as $y \to \partial D$. We claim that as a distribution identity, $-\Delta\xi_\varepsilon^z = 2\pi\rho_\varepsilon^z$. Indeed, for the proof, $\forall\varphi \in C_c^\infty(D)$, by Gauss-Green formula:

$$\begin{aligned}
\langle -\Delta\xi_\varepsilon^z, \varphi\rangle &= \langle \xi_\varepsilon^z, -\Delta\varphi\rangle \\
&= \int_D \log\max(\varepsilon,|z-y|)\Delta\varphi(y)dy \\
&= \int_{B_\varepsilon(z)\cap D} \log\varepsilon\Delta\varphi(y)dy + \int_{D/B_\varepsilon(z)} \log|z-y|\Delta\varphi(y)dy \\
&= \int_{\partial(B_\varepsilon(z)\cap D)} \log\varepsilon\frac{\partial\varphi}{\partial\mathbf{n}}ds - \varphi\frac{\partial\log\varepsilon}{\partial\mathbf{n}}ds + \int_{\partial(D/B_\varepsilon(z))} \log|z-y|\frac{\partial\varphi}{\partial\mathbf{n}}ds - \phi\frac{\partial\log|z-y|}{\partial\mathbf{n}}ds \\
&= -\int_{\partial(D/B_\varepsilon(z))} \phi\frac{\partial\log|z-y|}{\partial\mathbf{n}}ds \\
&= \int_{\partial B_\varepsilon(z)\cap D} \varphi ds = \langle 2\pi\rho_\varepsilon^z, \varphi\rangle
\end{aligned}$$

Hence we conclude that $h_\varepsilon(z) = (h, \xi_\varepsilon^z)_\nabla$. The function $G_{\varepsilon_1,\varepsilon_2}(z_1, z_2)$ is equal to $(\xi_{\varepsilon_1}^{z_1}, \xi_{\varepsilon_2}^{z_2})_\nabla$, which is the circle average of $\xi_{\varepsilon_1}^{z_1}$ on $\partial B_{\varepsilon_2}(z_2)$. Due to the mean value property of harmonic function, if $B_{\varepsilon_1}(z_1)$ and $B_{\varepsilon_2}(z_2)$ are disjoint, $G_{\varepsilon_1,\varepsilon_2}(z_1, z_2) = G(z_1, z_2)$.

Proposition 1.1. *The process $h_\varepsilon(z)$ has a modification which is almost surely locally η-Hölder continuous in the pair $(z, \varepsilon) \in \mathbb{C} \times (0, \infty)$ for every $\eta < \frac{1}{2}$.*

Proof. We first show that ξ_ε^z is a Lipschitz function when $z \in D$ and $\varepsilon > \varepsilon_0$, here $\varepsilon_0 > 0$ is arbitrarily fixed. Also, we conclude that the Lipschitz coefficient is universal for ε and z. For relatively small $\varepsilon_0 < \max\{|z-y|, y \in \partial D\}$, we have:

$$\begin{aligned}
|\xi_\varepsilon^z(y_1) - \xi_\varepsilon^z(y_2)| &\leq |\log \frac{\max(\varepsilon, |z-y_2|)}{\max(\varepsilon, |z-y_1|)}| + |\tilde{G}_{z,\varepsilon}(y_2) - \tilde{G}_{z,\varepsilon}(y_1)| \\
&\leq |\log \frac{\max(\varepsilon, |z-y_1| + |y_2-y_1|)}{\varepsilon_0}| + |\nabla \tilde{G}_{z,\varepsilon} \cdot (\mathbf{y}_2 - \mathbf{y}_1)| \\
&\leq \log \frac{\max\{|z-y|, y \in \partial D\} + |y_2-y_1|}{\varepsilon_0} + \sqrt{\sup_{\omega\in D} \frac{\partial \tilde{G}_{z,\varepsilon}}{\partial x}^2 + \sup_{\omega \in D} \frac{\partial \tilde{G}_{z,\varepsilon}}{\partial y}^2}\, |y_2 - y_1|
\end{aligned}$$

The first term has estimation:

$$\log \frac{\max\{|z-y|, y \in \partial D\} + |y_2-y_1|}{\varepsilon_0} \leq \log \frac{\max\{|z-y|, y \in \partial D\}}{\varepsilon_0} + \frac{\varepsilon_0 |y_2 - y_1|}{\max\{|z-y|, y \in \partial D\}}$$

Since $\partial^\alpha \tilde{G}_{z,\varepsilon}$ is harmonic, using harmonic measure for D, we have

$$|\frac{\partial \tilde{G}_{z,\varepsilon}}{\partial x}(a)| \leq |\int_{\partial D} \frac{\mathrm{Re}(z-\omega)}{|z-\omega|^2} h_D(a, d\omega)| \leq \frac{1}{\min\{|z-y|, y \in \partial D\}}$$

We conclude that $\partial^{1,0}\tilde{G}_{z,\varepsilon}$ and $\partial^{0,1}\tilde{G}_{z,\varepsilon}$ uniformly bounded on D. Combining all calculations, we deduce

$$|\xi_\varepsilon^z(y_1) - \xi_\varepsilon^z(y_2)| \leq K(\varepsilon_0, D)|y_1 - y_2|$$

Next, we study the Lipschitz property of $G_{\varepsilon_1,\varepsilon_2}(z_1, z_2)$, by explicit calculation:

$$|G_{\varepsilon_1',\varepsilon_2'}(z_1', z_2') - G_{\varepsilon_1,\varepsilon_2}(z_1, z_2)| \leq |\int \xi_{\varepsilon_1}^{z_1}(y)(\rho_{\varepsilon_2}^{z_2}(y) - \rho_{\varepsilon_2'}^{z_2'}(y))dy| + |\int \rho_{\varepsilon_2'}^{z_2'}(y)(\xi_{\varepsilon_1}^{z_1}(y) - \xi_{\varepsilon_1'}^{z_1'}(y))dy|$$

For the first term, use angular parameters

$$|\int \xi_{\varepsilon_1}^{z_1}(y)(\rho_{\varepsilon_2}^{z_2}(y) - \rho_{\varepsilon_2'}^{z_2'}(y))dy| = \frac{1}{4\pi^2 \varepsilon_2 \varepsilon_2'}|\int_0^{2\pi} \xi_{\varepsilon_1}^{z_1}(z_2 + \varepsilon_2 e^{i\theta})\varepsilon_2 d\theta - \xi_{\varepsilon_1}^{z_1}(z_2' + \varepsilon_2' e^{i\theta})\varepsilon_2' d\theta|$$

$$
\begin{aligned}
&\leq \frac{1}{4\pi^2\varepsilon_0}\int_0^{2\pi} K(\varepsilon, D)(|z_2 - z_2'| + |\varepsilon_2 - \varepsilon_2'|)d\theta \\
&\leq K'(\varepsilon_0, D)(|z_2 - z_2'| + |\varepsilon_2 - \varepsilon_2'|)
\end{aligned}
$$

For the second term, by symmetry of Green function, we have

$$\int \rho_{\varepsilon_2'}^{z_2'}(y)(\xi_{\varepsilon_1}^{z_1}(y) - \xi_{\varepsilon_1'}^{z_1'}(y))dy = \int \xi_{\varepsilon_2'}^{z_2'}(y)(\rho_{\varepsilon_1}^{z_1}(y) - \rho_{\varepsilon_1'}^{z_1'}(y))dy$$

To sum up, $G_{\varepsilon_1,\varepsilon_2}(z_1, z_2)$ is Lipschitz in $(\varepsilon_1, \varepsilon_2, z_1, z_2)$ with uniform coefficient.

By calculating the variance, we have:

$$var(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) = G_{\varepsilon_1,\varepsilon_1}(z_1, z_1) - 2G_{\varepsilon_1,\varepsilon_2}(z_1, z_2) + G_{\varepsilon_2,\varepsilon_2}(z_2, z_2)$$

Thus for all z_1, $z_2 \in D$ and ε_1, $\varepsilon_2 \in [0, \infty)$, there exists a constant K such that:

$$var(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) \leq K(|z_1 - z_2| + |\varepsilon_1 - \varepsilon_2|)$$

By Jensen's inequality, it implies that for all $\alpha > 0$ there exist $K = K(\alpha) > 0$ such that

$$\mathbb{E}(|h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)|^\alpha) \leq K(|z_1 - z_2| + |\varepsilon_1 - \varepsilon_2|)^{\alpha/2}$$

Apply Kolmogorov-Čentov theorem for $n = 3$, α and $\beta = \alpha/2 - 3$, we deduce the $\eta < 1/2$-local Hölder continuity for our Gaussian process $h_\varepsilon(z)$ □

Proposition 1.2. *Write $\mathcal{V}_t = h_{e^{-t}}(z)$, and $t_0^z = \inf\{t : B_{e^{-t}}(z) \subset D\}$. If $z \in D$ is fixed, then the law of*

$$V_t := \mathcal{V}_{t_0^z + t} - \mathcal{V}_{t_0^z}$$

is a standard Brownian Motion in t.

Proof. According to the conformal invariance of Green function, for any bounded two dimensional domain D, Green function takes the form

$$G(x, y) = -\log|x - y| + \log C(x; D) + o(1)$$

where $C(x; D)$ denotes conformal radius viewed from x. For any $\varepsilon_1 \geq \varepsilon_2 > 0$ such that $B_{\varepsilon_1}(z) \subset D$, we have

$$G_{\varepsilon_1,\varepsilon_2}(z, z) = -\log\varepsilon_1 + \log C(z; D)$$

Thus we compute that $cov(V_s, V_t) = s \wedge t$. By previous proposition, V_t is continuous Gaussian, hence a standard Brownian Motion. □

The above circle average mollification mostly apply to interior points. What happens if the center of the circles happen to be on the boundary ∂D? Since we are considering very nice open sets D, the boundary ∂D can be parametrized by piecewise C^1 functions and hence can be further straightened. It is still possible to study a semicircle average process of the GFF.

Proposition 1.3. *Let $z \in \partial D$, suppose there exists a neighborhood of z such that its intersection with ∂D contains a straight line segment $[z-a, z+b]$ with $a, b \in (0,1)$. Then if $t_0 = -\log(a \wedge b)$, then the re-centered semicircle averaged process*

$$\mathcal{W}_t = h^{sc}_{e^{-(t_0+t)}}(z) - h^{sc}_{e^{-t_0}}(z), \quad h^{sc}_{e^{-s}}(z) = \langle h, \xi^{z,sc}_{e^{-s}} \rangle_\nabla, \quad t \geq 0, \ s \geq t_0.$$

has a continuous modification and $\mathcal{W}_t \overset{d}{=} B_{2t}$. Here $\xi^{z,sc}_{e^{-s}}$ is chosen such that $-\frac{1}{2\pi}\Delta\xi^{z,sc}_{e^{-s}}$ is the uniform distribution on the semicircle $\partial B_{e^{-s}}(z) \cap D$. This result is to say, the semicircle average is a Brownian motion rescaled by 2.

The circle average and semicircle average properties actually tells us that zero-boundary GFF is a 'mollifiable' random distribution on all $D \cup \partial D$. In the construction of Liouville quantum gravity area measure and boundary measure (Subsection 2.2), the random density function can only be defined for smooth mollified fields h_ε, then the ill-defined nature for the Liouville measure will be resolved if taking $\varepsilon \to 0$ yields a well-defined limit. A similar method is valid for general log-correlated random distributions and general mollifiers (not just the circle average), the limiting random measure is then called a Gaussian Multiplicative Chaos (GMC).

Recall that the Sobolev spaces $H^s_0(D)$ can be extended to $H^s(D)$ which consists of functions with arbitrary boundary conditions, simply by taking completions of Sobolev norm $(\|\cdot\|^2_{L^2} + \|\nabla \cdot\|^2_{L^2})^{1/2}$ in the space $C^\infty(D)$. Following this idea, we construct the GFF with free boundary condition in a simply connected domain D. Let f_n be an orthonormal basis of $H^1(D)$ under $(\cdot,\cdot)_\nabla$, we immediately see that f_n is only determined up to constant (Compare the difference! In this case $(f,f)_\nabla$ is only a square-seminorm, and the reason for not using the Sobolev norm is that, as before, we care about the projection of GFF onto the eigenfunctions of the Laplacian). If we now write

$$h = \sum_{n\geq 1} \alpha_n f_n, \quad \alpha_n \overset{d}{=} N(0,1), \quad i.i.d.$$

then up to arbitrary constant, h converges a.s. in the Hilbert space topology of $H^s(D)$ for all $s < 0$, and in the weak$-*$ topology of $\mathcal{D}'(D)$, which is the space of continuous linear functional on the space $C^\infty(D)$. Now we define a equivalence relation by $f \sim g$ iff $g = f + c$, the space $H^s(D)/_\sim$ could also be regarded as a quotient Hilbert space. Let π be the quotient map, then a free-boundary Gaussian free field is defined by $\pi \circ h$, and if no confusions occur, it's still denoted by h.

If we do harmonic extensions for the random boundary value of h, we will obtain a random harmonic function in D. Moreover, this is actually an orthorgonal decomposition, in the sense that

$$H^1(D) = H_0^1(D) \oplus H_0^{1,\perp}(D), \quad H_0^{1,\perp}(D) = H_{har}^1(D).$$

Thus a free-boundary Gaussian free field can be decomposed into $h = h_0 + \phi + c$, where h_0 is a sample of zero-boundary GFF, ϕ is an independently sampled harmonic function and c is an arbitrary constant which does not affect the distribution of h. Since the zero-boundary GFF has well-established (semi) circle average process, and any harmonic function has the same average value on (semi) circles with common center, the above orthogonal decomposition with $D = \mathbb{C}$ ($\mathbb{H}$) can be sampled using the following proposition.

Proposition 1.4. *The free-boundary GFF on $\mathbb{C}$ ($\mathbb{H}$) can be sampled according to the following two steps:*

(i) On all circles (semicircles) centered at the origin $\partial B_{e^{-s}}(0)$ ($\partial B_{e^{-s}}(0) \cap \mathbb{H}$), we sample a Brownian motion B_s (B_{2s}) with time index $s \in \mathbb{R}$, which serves as the circle (semicircle) average process for h.

(ii) Independently sample a random field such that its average on each circle (semicircle) are identical. Notice that two such random fields are regarded the same if they differ by an arbitrary constant. For this part of the decomposition, we call it the radial noise of h.

Following such sampling method, the arbitrary constant is inherited by the radial noise and the choice for a specific constant is called an *embedding* of the free-boundary GFF. In the generic case, such sampling can be generalized to an arbitrary random $H^1(\mathbb{C})$ ($H^1(\mathbb{H})$) function.

Proposition 1.5. *Any random element h in $H^1(\mathbb{C})$ ($H^1(\mathbb{H})$) can be sampled according to the following two steps:*

(i) On all circles (semicircles) centered at the origin $\partial B_{e^{-s}}(0)$ ($\partial B_{e^{-s}}(0) \cap \mathbb{H}$), we sample a continuous stochastic process A_s with time index $s \in \mathbb{R}$, which is often taken to be a Ito process and serves as the circle (semicircle) average process for h.

(ii) Independently sample a random field such that its average on each circle (semicircle) are identical. Two such random fields are regarded the same if they differ by an arbitrary constant. For this part of the decomposition, we call it the radial noise of h.

1.2 The Schramm-Loewner evolution: SLE_κ

To begin with, we introduce two important lattice statistical mechanics model in 2d: the Ferromagnetic spin-Ising, and FK-Ising model. After first investigated by Lenz and Ising roughly 100 years ago, the Ising model had become an important toy model in the study of phase transition of matters, renormalization groups, conformal field theory (CFT), stochastic dynamics and even in modern computer sciences. The definition of spin-Ising model is formally given by:

Definition 1.2. *The ferromagnetic Ising model on a finite subgraph $G = (\mathcal{V}, E) \subset \mathbb{Z}^2$ with zero external field, inverse temperature $\beta > 0$, is given by the Gibbs measure on $\{\pm 1\}^{\mathcal{V}}$:*

$$\mu_\beta(\sigma) \propto e^{\beta \sum_{x\sim y} \sigma_x \sigma_y}, \quad \forall \sigma \in \{\pm 1\}^{\mathcal{V}}. \tag{1.4}$$

where $x \sim y$ are neighboring vertices and the proportion on the RHS is given by the inverse of partition function.

The Ising model is shown to possess a first order phase transition in 2d. By a first order phase transition, we mean that certain thermodynamic quantities (e.g., mean-magnetization, free energy.), which are functions of the inverse temperature, obtain a discontinuity on one particular β_c. The β_c is called the critical point of the Ising model, and the 2d case: $\beta_c := \frac{1}{2}\log\big(1+\sqrt{2}\big)$ was first computed by Onsager using exact solutions to partition function. For the high temperature model $\beta < \beta_c$, it was shown that the correlation function $\mu_\beta(\sigma_x\sigma_y)$ decays exponentially fast in the graph distance of x and y. However, the critical case is much more interesting, in the sense that the physics of the model is invariant under rescaling of graph and discrete conformal transformations. This is the first clue of application of critical Ising model in conformal field theory.

Another interesting approach was introduced by Fortuin and Kasteleyn. It shows that spin-Ising model is naturally coupled to a bond percolation on the graph. The state space for this percolation is given by configurations $\omega \in \{0,1\}^E$. All edges are independent, and they can be open (denoted by 1) with probability p, closed (denoted by 0) with probability $1-p$. Furthermore, given a parameter q, we define the FK-measure by:

$$\nu_{p,q}(\omega) \propto p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)} \tag{1.5}$$

where $o(\omega)$ and $c(\omega)$ denote the number of open and closed edges in ω respectively, and $k(\omega)$ counts the number of open clusters. From this, we see that FK-percolation is an example of the random cluster model.

Definition 1.3. *The case $q = 2$ and $p = 1 - e^{-2\beta}$ in (1.5) is called the FK-Ising model.*

The reason why FK-Ising model is of interest is that it naturally correspond to the spin-Ising model. Indeed, we could first sample the FK-Ising with outcome ω and then assign each cluster a spin ± 1 in an i.i.d. fasion. The result is a spin configuration σ such that $\mu_\beta(\sigma) = \nu_{1-e^{-2\beta},2}(\omega)$.

Definition 1.2 can be extended to the case with boundary conditions. To be more specific, choose spin configuration τ on the outer boundary of the graph $\{x \in \mathbb{Z}^d; x \sim G\}$, we define a measure

$$\mu_h^\tau(\sigma) \propto e^{\beta \sum_{x\sim y} \sigma_x \sigma_y} e^{\beta \sum_{x\sim z} \sigma_x \tau_z}, \quad x, y \in \mathcal{V},\ z \notin \mathcal{V},\ \sigma \in \{\pm 1\}^{\mathcal{V}}. \tag{1.6}$$

Suppose we have a bounded simply connected domain $D \subset \mathbb{R}^2$, we approximate it using the measure $D \cap \delta\mathbb{Z}^2$, which is a subgraph of grids. Fix $a, b \in \partial D$, one can always find the closes vertices $a^\delta, b^\delta \in D \cap \delta\mathbb{Z}^2$. For the counterclockwise oriented arcs $(a^\delta b^\delta)$ and $(b^\delta a^\delta)$, we assign boundary conditions $+1$ and -1 respectively, this is called the Dobrushin boundary condition. The upside is that on each sampling, it generates a spin interface, which is a simple curve running from a^δ to b^δ such that its LHS is filled with $+1$ spins and its RHS is filled with -1 spins. For the FK-Ising model, we similarly define the Dobrushin boundary condition by freeing the arc $(a^\delta b^\delta)$ (all edges in between are 0) and wiring the arc $(b^\delta a^\delta)$ (all edges in between are 1). One can also shown that in such boundary condition, we will always obtain cluster interface upon each sampling of the percolation (see Figure 1).

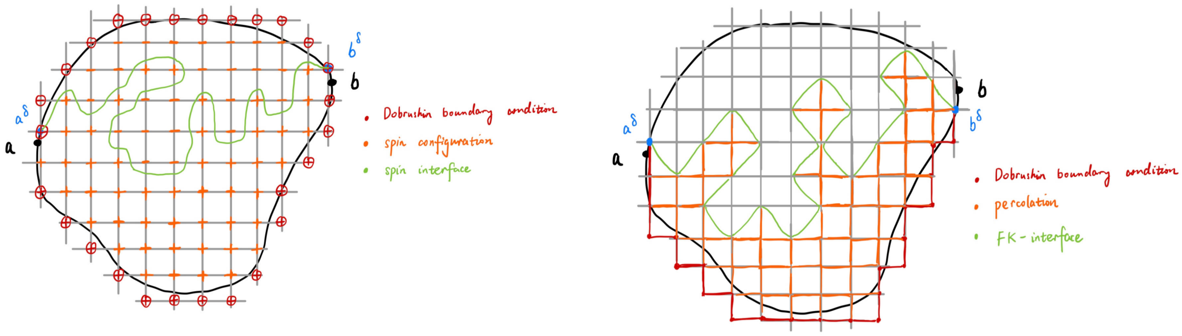
There is a remarkable result showing that if we take the continuum limit $\delta \to 0$, the limiting critical model is a CFT and the interface converges to a random fractal curve with nice property. The topology over all continuous simple curves γ_n and γ in a planar domain is constructed via the metric (It is indeed a Polish space):

$$d(\gamma_n, \gamma) = \inf_{\varphi_1, \varphi_2} \|\gamma_n \circ \varphi_1 - \gamma \circ \varphi_2\|_\infty$$

here the infimum is taken over all all orientation-preserving reparametrizations φ_1, φ_2 and the L^∞ norm is the ordinary one in $\mathbb{R}^2$.

Theorem 1.1. *Let D be a bounded simply connected domain with two distinguished marked points $a, b \in \partial D$. Consider the critical spin-Ising model with Dobrushin boundary condition on the lattice approximation $(D \cap \delta\mathbb{Z}^2, a^\delta, b^\delta)$. The law of the spin interface γ^δ converges weakly as $\delta \to 0$ to the chordal SLE_3 curve runnning from a to b in D.*

Theorem 1.2. *Let D be a bounded simply connected domain with two distinguished marked points $a, b \in \partial D$. Consider the critical FK-Ising model with Dobrushin boundary condition on the lattice approximation $(D \cap \delta\mathbb{Z}^2, a^\delta, b^\delta)$. The law of the percolation interface γ^δ converges weakly as $\delta \to 0$ to the chordal $SLE_{16/3}$ curve runnning from a to b in D.*



(a) spin-Ising interface

(b) FK-Ising interface

Figure 1: Sampling the interfaces

For further details on the critical Ising, FK-Ising models, see [5] and [8].

Question: what exactly is the limiting SLE curve? The Schramm-Loewner evolution (SLE) was first introduced by Oded Schramm in 1999. It was interesting not only because of its natural random conformal invariance, but also due to the conjecture that SLE_κ actually characterises universality classes, in the sense that the continuum limits of different classes of lattice models are conjecture to fall into different universal random objects related to SLE_κ (These different classes are 'labeled by $\kappa > 0$', see Subsection 4.3 for an approach using KPZ relation). For example, we heuristically predict through Figure 1 that SLE_3 is a simple curve while $SLE_{16/3}$ is self-intersection. In the following context of this subsection, we would like to introduce the rigorous mathematical construction of this model (the chordal case), which mainly uses Brownian motion and complex analysis.

Definition 1.4. *A bounded subset $A \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $A = \mathbb{H} \cap \bar{A}$ and $\mathbb{H}/A$ is a non-empty simply connected domain.*

By Riemann mapping theorem, we know that there exists conformal isomorphisms from $\mathbb{H}/A$ to $\mathbb{H}$, but this general existence doesn't provide any useful information about the geometry of A. However, one could construct one particular interesting conformal mapping $g_A(z)$ by restricting that $|g_A(z) - z| \to 0$ as $z \to \infty$. Under this condition, one can show that $g_A(z) - z$ is bounded uniformly in $z \in \mathbb{H}$ and it also admits the series expansion such that

$$g_A(z) = z + \frac{c_A}{z} + O(|z|^{-2}), \quad z \to \infty,\ c_A \geq 0.$$

Moreover, by Schwarz reflection, $g_A(z)$ can be analytically and symmetrically extended to the lower half plane. Hence to sum up, we have the definition below.

Definition 1.5. *Each compact* $\mathbb{H}$*-hull* A *is associated with a unique* mapping out function g_A *with a non-negative* half-plane capacity c_A*, such that:*

(i) It can be continuously extended to a boundary homeomorphism $\partial(\mathbb{H}/A) \to \mathbb{R}$*; (ii) It satisfies the* hydrodynamic normalization $|g_A(z) - z| \to 0$ *at* $z = \infty$*.*

The half-plane capacity c_A, being non-negative, is in fact a nice measurement of the size of A. Now, using a bit of measure theory terms, we denote it to be a non-negative function on hulls: $c_A := \mathrm{hcap}(A)$. In the following discussions, we would like to introduce some of its crucial properties.

(i) Scaling property. For any $r \geq 0$ and a compact $\mathbb{H}$-hull A, we have the identity $\mathrm{hcap}(rA) = r^2 \,\mathrm{hcap}(A)$. This is because one can show that $rg_A(z/r) = g_{rA}(z)$.

(ii) Horizontal translation invariance. Let $x \in \mathbb{R}$, we have $\mathrm{hcap}(A + x) = \mathrm{hcap}(A)$. This follows easily from $g_{A+x}(z) = g_A(z - x) + x$.

(iii) Markov property. For any two compact $\mathbb{H}-$hulls A and $\tilde{A}$ such that $A \subset \tilde{A}$, we have $g_{\tilde{A}} = g_{g_A(\tilde{A}/A)} \circ g_A$. The reason why this is called a Markov property will be clear after we introduce the SLE_κ. At this stage, we can heuristically understand it as follows: (i) In the process of mapping out $\tilde{A}$, we can use g_A to stop the whole process to the instance that A is exactly absorbed into $\mathbb{R}$; (ii) The remaining portion $g_A(\tilde{A}/A)$ will be another hull, which is due to the boundary homeomorphism; (iii) We then map this remaining part out by $g_{\tilde{A}}$, and the sum of the capacities in the two steps will be equal to the capacity of the full hull $\tilde{A}$. (iv) This cutoff is independent of the choice of $A \subset \tilde{A}$.

(iv) Charaterization via planar Brownian motion: $\mathrm{hcap}(A) = \lim_{s\to\infty} s\mathbb{E}_{is}(\mathrm{Im}\, B_\tau)$. Here the planar Brownian motion is assumed to start from is and τ is the first time that it hits $A \cup \mathbb{R}$. This follows from the definition $\mathrm{hcap}(A) = \lim_{s\to\infty} is(g_A(is) - is)$ and the property that any harmonic function u on a simply connected domain D can be calculated via $u(z) = \mathbb{E}_z(u(B_\tau))$ where τ is the exit time $\inf\{t; B_t \notin A\}$.

(v) Sub-additivity properties. Let A, B and C be compact $\mathbb{H}-$hulls, we then have the inequalities:

$$\mathrm{hcap}(A\cup B) \leq \mathrm{hcap}(A)+\mathrm{hcap}(B), \quad \mathrm{hcap}(A\cup B\cup C)-\mathrm{hcap}(A\cup B) \leq \mathrm{hcap}(A\cup C)-\mathrm{hcap}(A).$$

(vi) Estimations. Given A, we can choose $r > 0$, $x \in \mathbb{R}$ such that $A \subset r\bar{D} + x$, then there exists a universal constant C and we obtain the upper bound

$$\left| g_A(z) - z - \frac{c_A}{z - x} \right| \leq \frac{Crc_A}{|z - x|^2}$$

which holds for all such A, r, x and $|z-x| \geq 2r$.

We now introduce the chordal Loewner theory, which establishes a correspondence between growing evolutions of compact $\mathbb{H}$-hulls $(A_t)_{t\geq 0}$ and continuous driving functions $w_t \in \mathbb{R}$. To be more precise, we first present some essential concepts.

Definition 1.6. *Let $(A_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$-hulls, then it's said to be increasing if A_s is strictly contained in A_t for any $s<t$. For any $s<t$, define $A_{s,t} := g_{A_s}(A_t/A_s)$, then $(A_t)_{t\geq 0}$ is said to satisfy local growth property iff the radius of $A_{s,s+h}$ converges to 0 uniformly in s as $h \to 0$.*

Let $A_{t+} := \cap_{s>t} A_s$, then from local growth property, one can easily derive that $A_t = A_{t+}$ and $t \mapsto \mathrm{hcap}(A_t)$ is a non-negative, continuous and strictly increasing function. In this case, define the unique real number $w_t := \cap_{s>t} \bar{A}_{t,s} \in \mathbb{R}$. Then $(w_t)_{t\geq 0}$ called the Loewner transform of $(A_t)_{t\geq 0}$, which is a continuous function starting from 0.

Since under local growth conditions, $\mathrm{hcap}(A_t)$ is strictly increasing, we introduce a natural reparametrization for the growing compact $\mathbb{H}$-hulls.

Proposition 1.6. *Suppose $T, \tilde{T} > 0$, let $\tau : [0, \tilde{T}) \to [0, T)$ be a homeomorphism (due to connectivity of intervals, it can only be strictly increasing or decreasing). Let $(A_t)_{t\in[0,T)}$ be a family of locally growing compact $\mathbb{H}$-hulls with Loewner transform $(w_t)_{t\in[0,T)}$. Set $(\tilde{A}_t)_{t\in[0,\tilde{T})} := (A_{\tau(t)})_{t\in[0,\tilde{T})}$ and $(\tilde{w}_t)_{t\in[0,\tilde{T})} = (w_{\tau(t)})_{t\in[0,\tilde{T})}$. Then $(\tilde{A}_t)_{t\in[0,\tilde{T})}$ is a family of locally growing compact $\mathbb{H}$-hulls with Loewner transform $(\tilde{w}_t)_{t\in[0,\tilde{T})}$.*

If we set $\tau^{-1} : t \mapsto \mathrm{hcap}(A_t)/2$, then the new family $(\tilde{A}_t)_{t\in[0,\tilde{T})}$ with $(\tilde{w}_t)_{t\in[0,\tilde{T})}$ is said to be parametrized by half-plane capacity*. It's straightforward to see that* $\mathrm{hcap}(\tilde{A}_t) = 2t$.

One remarkable outcome of reparametrization by half-plane capacity is that, we could study the mapping out functions via ODE on the complex plane. This is called the Loewner's theory or Loewner's approach.

Proposition 1.7. *Let $(A_t)_t \geq 0$ be a family of locally growing compact $\mathbb{H}$-hulls, which is also parametrized by half-plane capacity. Suppose $(w_t)_t \geq 0$ is its corresponding Loewner transform. Define $g_t(z) = g_{A_t}(z)$ and threshold $\zeta(z) := \inf\{t \geq 0; z \in A_t\}$, then for any $z \in \mathbb{H}$, the functions $g_t(z)$ is C^1 in the interval $[0, \zeta(z))$ and it satisfies the Loewner differential equation:*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}$$

*In addition, if $\zeta(z) < \infty$, then we have $g_t(z) \to w_t$ as $t \to \zeta(z)$. Moreover, the thresholds $\zeta(z)$ can be symmetrically extended to $\mathbb{C}$, by setting $\zeta^*(z) := \inf\{t \geq 0; z \in A_t \text{ or } z \in A_t^*\}$ (the operator * denotes complex conjugate). Thus for $x \in \mathbb{R}$, $x \in \bar{A}_t$ if and only if $\zeta(x) \leq t$.*

Denote $\mathcal{A}$ to be the set of all compact $\mathbb{H}$-hulls. Let d be a metric such that for any $f_n, f \in C(\mathbb{H}, \mathbb{H})$, $d(f_n, f) \to 0$ if and only if f_n converges to f uniformly on any compact subset of $\mathbb{H}$ as $n \to \infty$. Then we can make $\mathcal{A}$ into a Polish space by defining the Caratheodory metric:

$$d_{CAR}(A, B) = d(g_A^{-1}, g_B^{-1}), \quad \forall A, B \in \mathcal{A}.$$

One can easily see that if $(A_t)_{t\geq 0}$ is a family of locally growing compact $\mathbb{H}$-hulls parametrized by half-plane capacity, then it is in fact a continuous mapping from $[0, \infty)$ to $\mathcal{A}$. Set $C([0, \infty), \mathcal{A})$ to be all such continuous functions and define d_C to be the metric characterizing uniform convergence, then the set of all $(A_t)_{t\geq 0}$ that satisfies our condition, denoted by L, is a closed subset of $(C([0, \infty), \mathcal{A}), d_C)$. Using the topologies given above, we immediately find that

$$\mathcal{L}: \ C([0, \infty), \mathbb{R}) \ni (w_t)_{t\geq 0} \longmapsto (A_t)_{t\geq 0} \in L \subset C([0, \infty), \mathcal{A}) \tag{1.7}$$

is a continuous mapping. We then call $\mathcal{L}$ to be the Loewner map. Combined with Brownian motion, we now state the formal definition of Schramm-Loewner evolution.

Definition 1.7. *Fix $\kappa > 0$, let $W_t := \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion starting from 0. The SLE_κ is a random family of locally growing compact $\mathbb{H}$-hulls parametrized by half-plane capacity, which is given by the mapping:*

$$(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow W_t \xrightarrow{\mathcal{L}} L \subset C([0, \infty), \mathcal{A})$$

Identically speaking, the Loewner transform of SLE_κ is uniquely (in law) given by the stochastic differential equations (SDEs):

$$dg_t(z) = \frac{2}{g_t(z) - W_t} dt, \quad \forall z \in \mathbb{H}, \ W_t \stackrel{d}{=} \sqrt{\kappa} B_t, \ t \leq \zeta(z). \tag{1.8}$$

where $\zeta(z) := \inf\{t \geq 0; z \in A_t\}$ are stopping times with respect to the natural filtration of B_t.

Using the properties the mapping out function, we show that SLE_κ is uniquely characterized by its scaling invariance (CFT nature) and domain Markov property (similar to the one for continuum GFFs, see Subsection 1.1).

Proposition 1.8. *A random variable $(A_t)_{t\geq 0}$ in L is an SLE if and only if $(A_t)_{t\geq 0} \stackrel{d}{=} (\lambda A_{\lambda^2 t})_{t\geq 0}$ and $(A_t)_{t\geq 0} \stackrel{d}{=} (g_{A_s}(A_{s+t}/A_s) - w_s)_{t\geq 0}$ for any fixed $\lambda > 0$ and $s \geq 0$. The former one is called the scaling invariance and the lateral one is called the domain Markov property.*

Proof. Since $(A_t)_{t\geq 0}$ is uniquely characterized by its mapping out functions $(w_t)_{t\geq 0}$, we see that $(A_t)_{t\geq 0}$ satisfies scaling invariance and domain Markov property if and only if $w_t \stackrel{d}{=} \lambda w_{\lambda^{-2}t}$ and $w_t \stackrel{d}{=} w_{s+t} - w_s$. This shows that w_t is a continuous Levy process that is invariant under Brownian rescaling. By Levy-Khinchin Theorem, w_t is a Brownian motion with some diffusion $\kappa \geq 0$. $\square$

As κ grows larger and larger, the 'shape' of the hulls A_t will be more and more complicated. However, there's a theorem by Rohde and Schramm saying that the hulls A_t can be generated by a driving random fractal curve, which is called the trace of SLE.

Theorem 1.3. *Let $(A_t)_{t\geq 0}$ be an SLE_κ with mapping out functions g_t and Loewner transforms $(W_t)_{t\geq 0}$. Then almost surely, the map g_t^{-1} generates a curve $\gamma_t := g_t^{-1}(W_t)$, called the trace of SLE_κ. Moreover, $(A_t)_{t\geq 0}$ is almost surely generated by γ_t in the sense that $\mathbb{H}/A_t$ is the unbounded connected component of $\mathbb{H}/\gamma_{[0,t]}$.*

Intuitively speaking, γ_t is the curve that traces the motion of the 'tip of iceberg'. Last but not least, we show in the following figure and the following theorem that the fractal geometry of the curves γ_t can be classified into three phases.

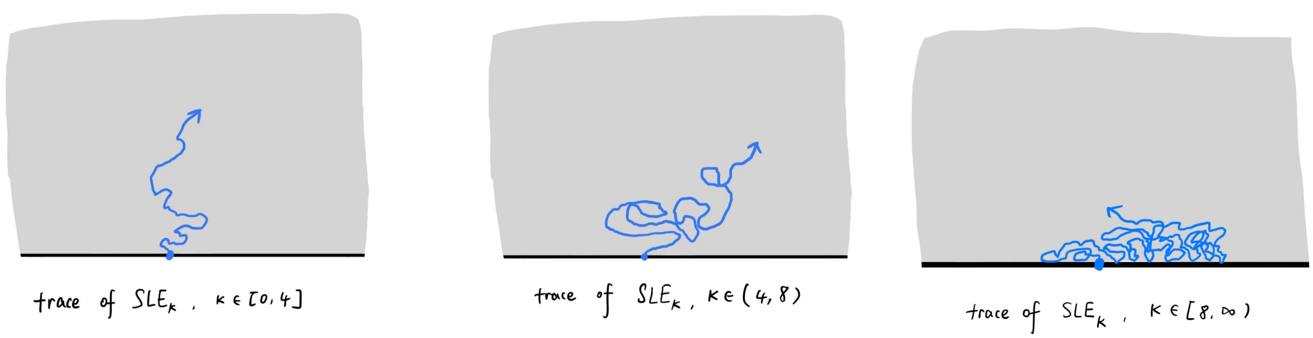


Figure 2: Traces of SLE_κ

Theorem 1.4. *Suppose γ_t is the trace of an SLE_κ with $\kappa \geq 0$, then almost surely, its geometry can be classified into three very different phases:*

(i) For $\kappa \in [0,4]$, γ_t is almost surely a simple curve.

(ii) For $\kappa \in (4,8)$, γ_t is almost surely self-intersecting in $\bar{\mathbb{H}}$ but not space-filling (i.e., there exists $t \neq s$ such that $\gamma_t = \gamma_s$ and $\forall z \in \bar{\mathbb{H}}$, γ_t avoids z a.s.).

(iii) For $\kappa \in [8,\infty)$, γ_t is almost surely a space-filling curve (i.e., we a.s. have $\gamma_{[0,\infty)} = \bar{\mathbb{H}}$).

2 Liouville quantum gravity

This section is organized as follows. In the first subsection, we present basic physics backgrounds on Liouville quantum gravity (LQG) and explain how it is related to the study of random measures. For the second and third subsections, we present two approaches to construct the LQG area measure μ_h. One is via circle averages (a special case of mollification) of the GFF and the other is via orthonormal expansion and rooted random measure Θ.

References for the physics: [7], [16], [19], and [20]. References for the mathematical construction of Liouville quantum gravity: [10].

2.1 2d quantum gravity and bosonic string theory

2.1.1 From Polyakov action to Liouville CFT

Why 2-dimensions? The motivation behind the 2d Euclidean quantum theory is twofold. First, it is still unknown to us how to define a path integral under relativity, not even at a heuristic level, and this is why we introduced Euclidean metrics (one can recover the Lorentzian theory via constructed QFT). Second, the geometry (spacetime) of 2d theory is essentially a Riemann surface, e.g., the sphere $\hat{\mathbb{C}}$, so it's natural to imagine a theory with 'nice conformal symmetry' (the physics invariant under local conformal transformations is called a 2d conformal field theory, aka CFT), leading to various connections with critical statistical mechanics models.

In most theoretical physics textbooks, the explanation on 2d CFT are often intertwined with string theory motivations. So we naturally ask, what is, and why string theory? This is a theory trying to convince people that at super tiny scales (way beyond something measurable under contemporary experimental equipments), elementary particles are in fact vibrating strings, and different vibrations generate different physical properties (mass, momentum, energy, etc.). Mathematically speaking, a motion of a vibrating string is not a trajectory (i.e., worldline, or smooth curves), but an embedded 2d submanifold $\Sigma \hookrightarrow M$ with the topology diffeomorphic to $\mathbb{R} \times [0, 1]$ or $\mathbb{R} \times \mathbb{S}^1$ (called open, closed string worldsheets respectively). Indeed, we can parametrize the worldsheets by introducing the coordinates (τ, σ), where τ denotes the time and σ denotes the spatial coordinate of the string. Then a motion of the string in spacetime is given by a specific smooth embedding map $X(\tau, \sigma)$.

Remark 2.1. *How is string theory related to gravity? From the motivational level, the major difficulty in the study of gravitational field in experiments, somehow matches with the 'craziness* *of string theory': The the 'regular' gravitational effect (i.e., except for black hole structures) in particle physics is super weak compare to other types of interactions depicted in the Standard Model, just like the 'imagined' string structure of point particles being undetectable. Somehow, if we couple matter field with gravitational field (similar to what we did in QED, QCD), and replace the classical particle approach by the string interpretations in the quantization, it solves the problem of gravity's non-renormalizability (a well-constructed QFT has to be renormalizable). Based on such usefulness, string theory is inevitably one of the most important candidate theory for quantum gravity.*

The task of determining the specific type of embedding X reduces to the study of Euler-Langrange equation for strings. This means that we have to first come up with reasonable actions on the set of all embeddings X. Recall that the action is determined by the type of symmetry of the theory, i.e., given a Lie group G, actions should be invariant under certain kind of irreducible representation of G. For our bosonic strings, the action should be proportional to the surface area $A(\Sigma)$ (just like action of point particles is proportional to the length of worldlines), meanwhile invariant under reparametrization of Σ (i.e., invariant under subgroups of diffeomorphisms). For any embedding, Σ is endowed with a unique induced metric given by the gravitational field, lets say $\gamma_{ab} = X^*\eta_{\mu\nu}$, and the area form is invariant under change of coordinates. Based on these heuristics, we have

$$S_{NG} \propto \int d^2x\sqrt{-\det\gamma} \tag{2.1}$$

It is called the Nambu-Goto action for the bosonic strings, and the corresponding Euler-Lagrange equation is given by

$$\partial_a(\sqrt{-\det\gamma}\gamma^{ab}\partial_b X) = 0 \tag{2.2}$$

Following the spirit of the least action principle, we assert that the pursuit for equation of motion in the string theory case, can be mathematically understood as a minimal surface problem.

There's another type of action on the embedding, called Polyakov action (see [16], or [20]), which dynamically matches with Nambu-Goto action but is easier to quantize and possesses new types of symmetry. In Polyakov's approach, the action is given by

$$S_P(X,h) \propto \int d^2x\sqrt{-g}g^{ab}\partial_a X\partial_b X \tag{2.3}$$

where g_{ab} denotes the 2d Riemannian metric, i.e., gravitational field. If we do calculus of variation, the Euler-Lagrange equation will be $\partial_a(\sqrt{-g}g^{ab}\partial_b X) = 0$, plus an additional equation on g. Although Polyakov action is not as 'physical' as Nambu-Goto action, it reproduces the same dynamics, hence still worth studying. If we want to quantize the *moving string in gravity* using

Polyakov action, the path integral measure will be a 'uniform distribution' over all worldsheets embeddings X and all classical gravitational fields g:

$$\mathcal{D}(g)\mathcal{D}(X)e^{-S_P(X,g)} \tag{2.4}$$

Remember in general relativity, an ensemble of point particles contributes to the energy-momentum tensor $T_{\mu\nu}$, and the gravitational field $g_{\mu\nu}$ is characterised by the Ricci tensor $R_{\mu\nu}$, which is determined by $T_{\mu\nu}$ through Einstein's equation. It sounds similar to Newtonian theory in which matter generates gravity. Now in the bosonic string theory, the embedding of worldsheet is equivalent to adding the bosons to the universe, hence the gravitational field will be changed by such an effect. We may imagine that, X represents the matter field in spacetime, the gravity near our matter is characterised by g_{ab}, and the two fields interact with each other through $S_P(X,g)$.

For the classical Polyakov theory, there are **two natural symmetries**: (i) First, the action (2.3) is invariant under reparametrizations, which is also true for Nambu-Goto; (ii) A Riemannian manifold is conformally deformed, or Weyl transformed, if we rescale the metric by $g_{ij}(x) \mapsto e^{\sigma(x)}g_{ij}(x)$. It's calculated that the Polyakov action is an invariant under such deformations on g_{ab}, but the Nambu-Goto action is not. However, for the quantum Polyakov theory, there's a problem, the path integral measure (2.4) is not shown to be Weyl invariant (This is called the *Weyl anomaly*, or conformal anomaly). Thus, there should be some quantum Weyl fluctuations for the gravity.

A Detour : In fact, probabilists are already familiar with such symmetries. In the most general setting, a conformal transformation of manifolds, is a deformation of the type $g_{ij}(x) \mapsto \Lambda(x)g_{ij}(x)$, then Weyl's transform is just a special case. Geometrically, the conformal transformation is a local change of scale (If we define Λ to be a global constant, i.e., a global transform, then the rescaling should be done simultaneously on all spacetime points, which breaks the causal structure of general relativity), and it can be similarly generalized to the discrete case (graphs). A discrete 2d critical Ising model is shown to have such scale invariance, in the sense that under discrete rescaling, the correlation function is unchanged. Furthermore, this scaling invariance of critical lattice model implies that, if we take the continuum limit, the 'shape' of the model will 'conformally converge' to some universality classes (see Subsection 4.3). For instance, in the beginning of Subsection 1.2, we showed that if we suitably couple 2d Ising model with FK percolation (FK-Ising model) on a simply connected domain, the critical percolation interface under Dobrushin boundary condition is discrete conformally invariant, and its scaling limit converges in law to the universality class: chordal Schramm-Loewner evolution $SLE_{16/3}$.

There is still one big difficulty in Polyakov's approach. We don't know how to define a uniform distribution over all (X, g), and this is the problem of *summing over random surfaces* that Polyakov [16], as well as Duplantier & Sheffield mentioned in their papers [10]. Now let's try to study the measures using physical ideas. Although mathematically, $\mathcal{D}(g), \mathcal{D}(X)$ are not understood, they are both physically required to satisfy

$$\int \mathcal{D}(g) e^{-\|g\|_{L^2}^2} = 1, \quad \int \mathcal{D}(X) e^{-\|X\|_{L^2}^2} = 1$$

For the explanation on the L^2 norm, see [7]. If we perform Weyl transform $g_{ab}(x) \mapsto e^{\sigma(x)} g_{ab}(x)$, the S_P term is certainly invariant, and the above conditions forces the measures to changed by

$$\mathcal{D}(X) \to e^{\frac{d}{48\pi} S_L(\sigma, g)} \mathcal{D}(X), \quad \mathcal{D}(g) \to e^{-\frac{26}{48\pi} S_L(\sigma, g)} \mathcal{D}(g) \tag{2.5}$$

where $S_L(\sigma, g)$ is the *Liouville action*:

$$S_L(\sigma, g) = \frac{1}{4\pi} \int d^2 x \sqrt{-g} \big(g^{ab} \partial_a \sigma \partial_b \sigma + Q R_g \sigma + 4\pi \mu e^{\sigma} \big) \tag{2.6}$$

where R_g denotes the Ricci curvature of g_{ab}. The Liouville action can be regarded as a measurement on the effect of gravity's quantum Weyl fluctuation, which is due to the second type of symmetry discussed above. To depict $\mathcal{D}(g)$, we pick a gauge fixing $g_{ab} = e^{\gamma\sigma} \hat{g}_{ab}$ (here $\hat{g}$ is a fixed smooth reference metric), then hopefully we can turn it into a suitable measure over $\sigma(x)$. In the case of $\hat{g}_{ab}$ being Euclidean, it is called a conformal gauge.

Remark 2.2. *The path integral quantization with Liouville action* (2.6) *is called a Liouville conformal field theory (LCFT). We see that the cosmological constant μ makes the path integral renormalizable.*

Is it really the case that $\mathcal{D}(g)\mathcal{D}(X) = \mathcal{D}(\sigma)\mathcal{D}(X) e^{-\frac{26-d}{48\pi} S_L(\sigma, \hat{g})}$, where $c_L = 26 - d$ is the central charge of LCFT? Not exactly correct! The choice of referrence metric $\hat{g}$ is arbitrary, so there should be ghost fields to explain such freedom. To be more precise, we add ghost-antighost coupling (b, c) to our measure:

$$\mathcal{D}(b)\mathcal{D}(c) e^{-S_{gh}(b,c,h)} \tag{2.7}$$

where the ghost action S_{gh} is invariant under Weyl transform. Finally, we claim that a physically satisfied path integral measure for interacting quantum bosonic strings under gravity is given by

$$\mathcal{D}(b)\mathcal{D}(c)\mathcal{D}(X)\mathcal{D}(\sigma) e^{-\frac{c_L}{48\pi} S_L(\sigma, \hat{g})} e^{-[S_P(X,\hat{g}) + S_{gh}(b,c,\hat{g})]} \tag{2.8}$$

If $d = 26$, we no longer need to study LCFT, i.e., Weyl anomaly disappears. This is called the *critical dimension* of string theory (Actually, critical central charge would be a better terminology). We point out that, the 'notion' does have something to do with statistical mechanics, since when Weyl anomaly is gone, the entire quantum matter field X will be scaling invariant, similar to the critical Ising models.

2.1.2 Quantum gravity formalism

So far we have seen how quantum gravity & LCFT naturally arises from the quantization of Polyakov's bosonic string theory. Nevertheless, we would like to present another approach by directly studying the quantum bosonic field propogating through quantum gravity. For simplicity, when we consider quantum gravity (random geometry), we fix the topology of the Riemann surface (given a 2d simply connected topological manifold) and randomly perturb metric tensor g_{ab}. Thus, we take (X, g) to be a coupled quantum field theory on curved spacetime.

For the classical dynamics of (X, g), we apply the usual actions:

$$S_{EH}(g) + S_g(X) \propto \int \sqrt{-g} R_g + \int \frac{1}{2}\sqrt{-g} g^{ab} \partial_a X \partial_b X + \frac{1}{2} m^2 X^2$$

The first term is the Einstein-Hilbert action and the second term is the action of real bosonic field. Interestingly, the format of $S_g(X)$ matches with the Polyakov action (2.3), but here X is just the field on manifold, not smooth embedding of worldsheets.

In 2d, $S_{EH}(g)$ is a topological invariant, hence calculus of variation does not provide non-trivial dynamics, i.e., $g = 0$. Even though it's classically trivial in 2d, the path integral quantization could be difficult, one still needs to modify the conditions. Jachiw and Teitelboim had suggested that one could add a cosmological constant to Ricci curvature $R_g \mapsto R_g - 2\Lambda$, and Polykov also suggested a replacement $R_g \mapsto R_g \Delta^{-1} R_g + 2\Lambda$. The former introduces a new theory called JT gravity while the latter is easier to quantize via light cone gauge. Following our previous discussions, we apply a third approach by setting the Liouville equation $R_g = c$, which means the curvature is constant. As soon as we do Weyl gauge fixing $g = e^{\sigma}\hat{g}$, it allows us to rewrite:

$$S_{EH}(g) \propto \int \sqrt{-g} R_g \propto \int \sqrt{-\hat{g}} e^{\sigma} \tag{2.9}$$

Now, the path intergal measure

$$\mathcal{D}(g)\mathcal{D}(X) e^{-[S_{EH}(g) + S_g(X)]} \tag{2.10}$$

can be reformulated. First of all, we introduce σ:

$$\mathcal{D}(g) = \mathcal{D}_g(\sigma) J(g) = \mathcal{D}_g(\sigma)\mathcal{D}_g(b, c) e^{-S_g^{ghost}(b,c)} \tag{2.11}$$

where $J(\sigma)$ is called the Faddeev-Popov determinant (mathematically, it's just the Jacobian term after we do change of variables). As one can see, the introduction of ghost-antighost fields (b, c) is just an attempt to understand Faddeev-Popov determinant as path integrals. Next, the gauge fixings:

$$\mathcal{D}_g(\sigma) = \mathcal{D}_{\hat{g}}(\sigma) e^{\frac{1}{48\pi}\int \sqrt{-\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + R_{\hat{g}}\sigma)} \tag{2.12}$$

and

$$D_g(b,c)e^{-S_g^{ghost}(b,c)} = \mathcal{D}_{\hat{g}}(b,c)e^{-S_{\hat{g}}^{ghost}(b,c)}e^{-\frac{26}{48\pi}\int\sqrt{-\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\sigma\partial_b\sigma+R_{\hat{g}}\sigma)} \tag{2.13}$$

The reason why we just wrote '$\mathcal{D}(\sigma)$' in the previous context is simply because we assume the $\hat{g}$ is already chosen, but this is in fact a 2-step job. For the matter fields, we get

$$e^{-S_g(X)} = e^{\frac{c_m}{48\pi}\int\sqrt{-\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\sigma\partial_b\sigma+R_{\hat{g}}\sigma)}e^{-S_{\hat{g}}(X)} \tag{2.14}$$

where c_m denotes the central charge for matter field X. Combine (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14), the path integral measure finally becomes

$$\mathcal{D}_{\hat{g}}(b,c)\mathcal{D}_{\hat{g}}(\sigma)\mathcal{D}(X)e^{-[S_{\hat{g}}^{ghost}(b,c)+S_{\hat{g}}(X)]}e^{-\frac{25-c_m}{48\pi}\int\sqrt{-\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\sigma\partial_b\sigma+R_{\hat{g}}\sigma+\mu e^{\sigma})} \tag{2.15}$$

As we can see, $d = 1 + c_m$. If we define Liouville central charge $c_L := (25 - c_m) + 1$, ghost central charge $c_{gh} := -26$, then there's a conservation $c_L + c_{gh} + c_m = 0$. Moreover, the μ in the RHS of (2.15) is chosen to recover the Liouville action defined in (2.6).

Let's go back to the Liouville action, if we want to consider LCFT with the setting

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

the law of σ, i.e., probability measure $\mathcal{D}_{\hat{g}}(\sigma)e^{-S_L(\sigma,\hat{g})}$ would be conformally invariant. Furthermore, by setting $\gamma \in [0,2), \sigma = \gamma\phi$ and $\mu = 0$ (This is called the critical Liouville quantum gravity), the gauge phase ϕ becomes the Gaussian free field on the manifold (to see this, compare the critical density e^{-S} with the density of DGFF (1.1)). We can then choose the conformal gauge $g_{ab} = e^{\gamma\phi}\eta_{ab}$, with η being Euclidean. The reasonability of this choice of gauge is that, for certain family of simply connected Riemann surfaces, we are eligible to apply the Riemann uniformization theorem, and the study of metric tensor g is equivalent to the study of isothermal coordinates: $e^{\lambda(x,y)}dxdy$. Then the behavior of the random metric $g_{ab} = e^{\gamma\phi}\eta_{ab}$ could be characterized by a random measure $e^{\gamma\phi(z)}dz$ (For nice Riemann surfaces, the metric is 'compatible' with the topology, and metric geometry boils down to measure theory) on a 2d isothermal domain $z \in D$. By rewriting ϕ to be the 2d continuum GFF h, we derive the following 'random measure' in question

$$\mu_h(z) = e^{\gamma h(z)}dz \tag{2.16}$$

There are still a few things to be clarified. Firstly, Duplantier and Sheffield [10] had chosen the GFF because of its connection with discrete quantum gravity model, and it may not 100% accurate in the physics context. Secondly, why we study the g_{ab} in the measure theoretic sense, instead regarding it as a 'metric'? This is due to the difficulty brought by the singularity of

GFF. In the next subsection, we will see that $e^{\gamma h}dz$ has to be defined as an a.s. weak limit of a family of probability measures. If we interpretate $e^{\gamma h}dzd\bar{z}$ to be a random metric, then we would have to construct it through 'some a.s. convergence' for metric spaces (maybe Gromov-Hausdorff convergence). This makes the metric space approach a very hard open problem (at least not fully understand to this day, see expository paper [6]).

2.2 Constructing Liouville quantum gravity as a random measure

Following the above discussion, the mathematical structure of Liouville quantum gravity (LQG) is a random measure on domain D with density $e^{\gamma h(z)}dz$. However, this expression is technically invalid since GFF is not a random field. We have to construct LQG measure using the following theorem. Mathematically speaking, we are doing an ε-mollification to to random distribution h while maintaining the expectation of the Radon-Nikodym density. Physically speaking, we are doing an $\varepsilon-$renormalization to GFF to quantize its Liouville CFT.

Theorem 2.1 (Construction of LQG). *Fix $\gamma \in [0, 2)$ and let h be the Dirichlet boundary GFF on a simply connected bounded planar domain D. Then it is a.s. the case that, as $\varepsilon \to 0$ along some subsequence of the dyadics $\varepsilon_k = 2^{-k}$, the random measures $\mu_\varepsilon := \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)}dz$ converge weakly[1] inside D to Liouville quantum gravity (LQG), which is a random measure informally written as $\mu_h := e^{\gamma h(z)}dz$.*

Before starting the proof, we check some calculations and notations. First of all, by Proposition 1.2, we have

$$h_\varepsilon(z) \sim N(0, \log C(z; D) - \log \varepsilon)$$

when $B_\varepsilon(z) \subset D$. Then by characteristic function, we have

$$\mathbb{E}e^{\gamma h_\varepsilon(z)} = \left(\frac{C(z; D)}{\varepsilon}\right)^{\frac{\gamma^2}{2}}$$

and the covariance

$$\mathbb{E}\left(e^{\gamma h_\varepsilon(y)}e^{\gamma h_\varepsilon(z)}\right) = \exp\left\{\left(\frac{\gamma^2}{2}(G_\varepsilon(y, y) + G_\varepsilon(z, z) + G_\varepsilon(y, z))\right)\right\}$$

Whenever $|y - z| \leq 2\varepsilon$, the above is just

$$\mathbb{E}\left(e^{\gamma h_\varepsilon(y)}e^{\gamma h_\varepsilon(z)}\right) = \left(\frac{C(y; D)C(z; D)}{\varepsilon^2}\right)^{\frac{\gamma^2}{2}} e^{\gamma^2 G(y,z)}$$

[1] Almost surely, the random variables μ_{ε_k} converge to μ_h in the Levy-Prokhorov metric.

To get rid of the εs in the terms, we write $\bar{h}_\varepsilon := \gamma h_\varepsilon + \frac{\gamma^2}{2}\log\varepsilon$, then

$$\mathbb{E}\left(e^{\bar{h}_\varepsilon(z)}\right) = C(z;D)^{\frac{\gamma^2}{2}}$$

$$\mathbb{E}\left(e^{\bar{h}_\varepsilon(y)}e^{\bar{h}_\varepsilon(z)}\right) = (C(y;D)C(z;D))^{\frac{\gamma^2}{2}}e^{\gamma^2 G(y,z)}$$

Proof of Theorem 2.1. **Step 1**. First, fix $\gamma \in [0,2)$, for each square area S compactly supported in D, we show that $\mu_{2^{-k}}(S)$ converge to a finite limit as $k \to \infty$. (Observe that due to the continuity of circle average process, $\mu_\varepsilon(S)$ is indeed a random variable.) Without loss of generality, we may assume $S = [0,1]^2$.

For any point $y = (y_1, y_2) \in (0,1)^2$, we have a closest point on the mesh $2^{-k}\mathbb{Z}^2 \cap [0,1]^2$ which is to the left and downside of y, i.e.,

$$\left(\frac{\lfloor 2^k y_1\rfloor}{2^k}, \frac{\lfloor 2^k y_2\rfloor}{2^k}\right)$$

We would like to gather points in S with the form (The translation of the above point to all of the small cubes):

$$\left(\frac{n + 2^k y_1 - \lfloor 2^k y_1\rfloor}{2^k}, \frac{m + 2^k y_2 - \lfloor 2^k y_2\rfloor}{2^k}\right) \qquad n, m = 0, 1, ..., 2^k - 1$$

and denote this set of points by S_k^y (see Figure 3 for an illustration). Define random variables A_k^y and B_k^y by

$$A_k^y := 2^{-2k}\sum_{z\in S_k^y}\exp\{\bar{h}_{2^{-k-1}}(z)\}, \qquad B_k^y := 2^{-2k}\sum_{z\in S_k^y}\exp\{\bar{h}_{2^{-k-2}}(z)\}$$

We claim that

$$\mu_{2^{-k-1}}(S) = \int_S e^{\bar{h}_{2^{-k-1}}(y)}dy = \int_S A_k^y dy, \qquad \mu_{2^{-k-2}}(S) = \int_S B_k^y dy$$

To understand the equality, we fix $\ell \in \{1, ..., 2^{2k}\}$, then there exists a fixed cube Λ_ℓ on the mesh $2^{-k}\mathbb{Z}^2 \cap S$. Now lets move z inside this cube and intergate, meanwhile we take $y \in S$ such that the distance between y and z is always a constant dyadic vector a_j, we obtain

$$\int_S 1_{\Lambda_\ell} e^{\bar{h}_{2^{-k-1}}(z)}dz = \int_S 1_{a_j+\Lambda_\ell} e^{\bar{h}_{2^{-k-1}}(z)}dy$$

For the integration domain, we sum over all cubes on the mesh

$$\int_S e^{\bar{h}_{2^{-k-1}}(z)}dz = \sum_\ell \int_S 1_{\Lambda_\ell} e^{\bar{h}_{2^{-k-1}}(z)}dz = \sum_\ell \int_S 1_{a_j+\Lambda_\ell} e^{\bar{h}_{2^{-k-1}}(y-a_j)}dy = \int_S e^{\bar{h}_{2^{-k-1}}(y-a_j)}dy$$

Since the choice of j is independent of ℓ, we take 'expectation' for the RHS

$$\int_S e^{\bar{h}_{2^{-k-1}}(z)}dz = \frac{1}{2^{2k}}\sum_j \int_S e^{\bar{h}_{2^{-k-1}}(y-a_j)}dy = 2^{-2k}\int_S \sum_{z\in S_k^y} e^{\bar{h}_{2^{-k-1}}(z)}dy$$

This concludes the proof of the first identity, and the other one for B_k^y can be similarly calculated.

Step 2. Next, we would like to estimate $\mathbb{E}|\mu_{2^{-k-1}}(S) - \mu_{2^{-k-2}}(S)|$. Indeed, it suffices to estimate $\mathbb{E}|A_k^y - B_k^y|$ and come up with universal bounds for y. Pick any point $z \in S_k^y$, the balls $B_{2^{-k-1}}(z)$ do not overlap. By domain Markov property of GFF, conditioned on $\{h_{2^{-k-1}}(z), z \in S_k^y\}$ (Equivalently, given the values of GFF on $\cup_{z\in S_k^y}\partial B_{2^{-k-1}}(z)$), we have $h_{2^{-k-2}}(z) \overset{d}{=} h_{2^{-k-1}}(z) + W^z_{\log 2}$, where $(W^z)_{z\in S_k^y}$ is another independent 2^{2k}-dimensional Brownian motion. Again, see Figure 3 for an illustration on $k = 2$.

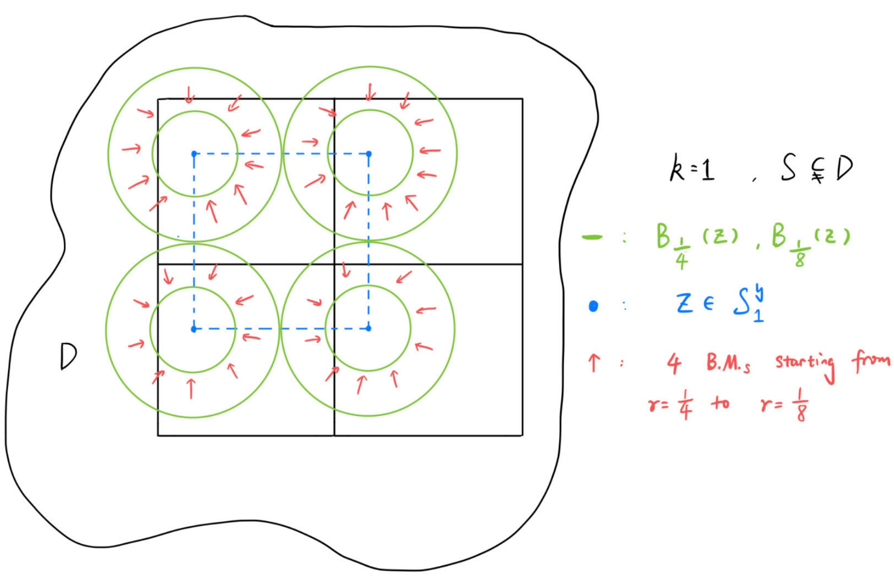


Figure 3

By Jensen's inequality, in order to show $\mathbb{E}|A_k^y - B_k^y|$ to decay very fast, we could instead try to bound $\mathbb{E}(|A_k^y - B_k^y|^2)$. The conditional expectation of $|A_k^y - B_k^y|^2$ given the values of $h_{2^{-k-1}}(z)$ for $z \in S_k^y$ is calculated in the following. The first equality is due to the domain Markov property, which provides the conditional independence.

$$\mathbb{E}\left(|A_k^y - B_k^y|^2|h_{2^{-k-1}}(z)\right) = 2^{-4k}\sum_{z\in S_k^y}\mathbb{E}\left(|e^{\bar{h}_{2^{-k-1}}(z)} - e^{\bar{h}_{2^{-k-2}}(z)}|^2|h_{2^{-k-1}}(z)\right)$$

$$
\begin{aligned}
&= 2^{-4k} \sum_{z \in S_k^y} e^{2\bar{h}_{2^{-k-1}}(z)} \mathbb{E}\left(|1 - e^{\bar{h}_{2^{-k-2}}(z) - \bar{h}_{2^{-k-1}}(z)}|^2 | h_{2^{-k-1}}(z) \right) \\
&= 2^{-4k} \sum_{z \in S_k^y} e^{2\bar{h}_{2^{-k-1}}(z)} \mathbb{E}\left(|1 - 2^{-\frac{\gamma^2}{2}} e^{\gamma(h_{2^{-k-2}}(z) - h_{2^{-k-1}}(z))}|^2 | h_{2^{-k-1}}(z) \right) \\
&= 2^{-4k} \sum_{z \in S_k^y} e^{2\bar{h}_{2^{-k-1}}(z)} \mathbb{E}\left\{ \left(1 - 2^{-\frac{\gamma^2}{2}} e^{\gamma W^z_{\log 2}}\right)^2 \right\} \\
&= 2^{-4k}(2^{\gamma^2} - 1) \sum_{z \in S_k^y} e^{2\bar{h}_{2^{-k-1}}(z)}
\end{aligned}
$$

Note that we have the equality

$$
\mathbb{E}\left[(\varepsilon^2 e^{\bar{h}_\varepsilon(z)})^2 \right] = \varepsilon^{4+\gamma^2} \mathbb{E} e^{2\gamma \bar{h}_\varepsilon(z)} \asymp \varepsilon^{4-\gamma^2}
$$

which holds for any choice of $\varepsilon > 0$. If we substitute $\varepsilon = 2^{-k-1}$, take expectation for $h_{2^{-k-1}}(z)$ and sum over all $2^{2k} = (2\varepsilon)^{-2}$ terms

$$
\mathbb{E}\left(|A_k^y - B_k^y|^2 \right) \asymp \varepsilon^{2-\gamma^2} = 2^{-(k+1)(2-\gamma^2)}
$$

Hence we showed that $\mathbb{E}\left(|A_k^y - B_k^y|^2 \right)$ tends to zero exponentially fast in k, for the case $\gamma^2 \in [0, 2)$. But what about $\gamma^2 \in [2, 4)$?

Step 3. When $\gamma^2 \in [2, 4)$, we separate S_k^y into two parts. Fix some $\gamma < \alpha < 2\gamma$, let $\tilde{S}_k^y$ denote the random set of points $z \in S_k^y$ with the property that $h_\varepsilon(z) > -\alpha \log \frac{\varepsilon}{C(z;D)}$, where $\varepsilon = 2^{-k-1}$ as before (In other words, $\tilde{S}_k^y \subset S_k^y$ is a random subset showing the case that $h_\varepsilon(z)$ is very large). We would like to claim that previous estimation of $\mathbb{E}\left(|A_k^y - B_k^y|^2 \right)$ is too rough and it's better to bound it in two groups: $z \in \tilde{S}_k^y$ and $z \in S_k^y / \tilde{S}_k^y$.

Let $\tilde{A}_k^y$ denote the mean average of $1_{\tilde{S}_k^y} \exp\{\bar{h}_{2^{-k-1}}(z)\}$ over S_k^y, and $\tilde{B}_k^y$ denote the mean average of $1_{\tilde{S}_k^y} \exp\{\bar{h}_{2^{-k-2}}(z)\}$ over S_k^y. Equivalently, we obtain

$$
A_k^y = \tilde{A}_k^y + 2^{-2k} \sum_{z \in S_k^y / \tilde{S}_k^y} \exp\{\bar{h}_{2^{-k-1}}(z)\}, \qquad B_k^y = \tilde{B}_k^y + 2^{-2k} \sum_{z \in S_k^y / \tilde{S}_k^y} \exp\{\bar{h}_{2^{-k-2}}(z)\}
$$

We then prove that $\mathbb{E}\tilde{A}_k^y$ converges to 0 exponentially fast in k. For $\varepsilon > 0$, we denote $\sigma^2 = -\log \frac{\varepsilon}{C(z;D)}$, which is the variance of $h_\varepsilon(z)$ when $B_\varepsilon(z) \subset D$, and calculate

$$
\mathbb{E} e^{\bar{h}_\varepsilon(z)} = C(z;D)^{\gamma^2/2} = \frac{1}{\sqrt{2\pi}\gamma\sigma} \int_{\mathbb{R}} e^\eta e^{-\frac{[\eta - (\gamma^2/2)\log\varepsilon]^2}{2\gamma^2\sigma^2}} d\eta = \frac{\varepsilon^{\gamma^2/2}}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta
$$

We fix an arbitrary $z \in S_k^y$, calculations show that

$$\begin{aligned}
\mathbb{E}1_{\tilde{S}_k^y} \exp\{\bar{h}_\varepsilon(z)\} &= \mathbb{E}1_{\{h_\varepsilon(z) > \alpha\sigma^2\}} e^{\bar{h}_\varepsilon(z)} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} 1_{\{\eta > \alpha\sigma^2\}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta + \frac{\gamma^2}{2}\log\varepsilon} d\eta \\
&= \frac{\int_{\mathbb{R}} 1_{\{\eta > \alpha\sigma^2\}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta}{\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta} \times \mathbb{E}e^{\bar{h}_\varepsilon(z)}
\end{aligned}$$

The last equality is trivially understood as

$$\mathbb{E}1_{\{h_\varepsilon(z) > \alpha\sigma^2\}} e^{\bar{h}_\varepsilon(z)} = \frac{\mathbb{E}1_{\{h_\varepsilon(z) > \alpha\sigma^2\}} e^{\bar{h}_\varepsilon(z)}}{\mathbb{E}e^{\bar{h}_\varepsilon(z)}} \mathbb{E}e^{\bar{h}_\varepsilon(z)}$$

The ratio of integrals refers to the probability of a random variable $X \sim N(\gamma\sigma^2, \sigma^2)$ takes value larger than $\alpha\sigma^2$. Since $P(X > \alpha\sigma^2) = O(e^{-\frac{(\alpha-\gamma)^2\sigma^2}{2}})$, we conclude that all $\mathbb{E}\tilde{A}_k^y$, $\mathbb{E}\tilde{B}_k^y$ and $\mathbb{E}|\tilde{A}_k^y - \tilde{B}_k^y|$ converge to 0 exponentially fast in k.

Next, we try to bound $\mathbb{E}|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)|^2$ by studying points $z \in S_k^y / \tilde{S}_k^y$. Using the relation $\varepsilon^4 \mathbb{E}[1_{\tilde{S}_k^y / S_k^y} (e^{\bar{h}_\varepsilon(z)})^2] = \varepsilon^{4+\gamma^2} \mathbb{E}[1_{\tilde{S}_k^y / S_k^y} e^{2\gamma h_\varepsilon(z)}]$, a similar relation stands

$$\mathbb{E}[1_{\tilde{S}_k^y / S_k^y} e^{2\gamma h_\varepsilon(z)}] = \frac{\int_{\mathbb{R}} 1_{\{\eta < \alpha\sigma^2\}} e^{-\frac{\eta^2}{2\sigma^2}} e^{2\gamma\eta} d\eta}{\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{2\gamma\eta} d\eta} \times \mathbb{E}e^{2\gamma h_\varepsilon(z)}$$

By $\alpha \leq 2\gamma$, another tail estimate implies

$$\varepsilon^4 \mathbb{E}[1_{\tilde{S}_k^y / S_k^y} (e^{\bar{h}_\varepsilon(z)})^2] = O(\varepsilon^{4-\gamma^2} \varepsilon^{(2\gamma-\alpha)^2/2})$$

To proceed, we observe that (use once again the conditioning and the domain Markov property)

$$\begin{aligned}
\mathbb{E}|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)|^2 &\lesssim 2^{-4k} \sum_{z \in S_k^y} \mathbb{E}[1_{\tilde{S}_k^y / S_k^y} (e^{\bar{h}_\varepsilon(z)})^2] \\
&= O(\varepsilon^{4-\gamma^2} \varepsilon^{(2\gamma-\alpha)^2/2})
\end{aligned}$$

Thus by Jensen's inequality, we have

$$\mathbb{E}\left(|A_k^y - B_k^y|\right) \leq \mathbb{E}\tilde{A}_k^y + \mathbb{E}\tilde{B}_k^y + E|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)| = O(\varepsilon^{4-\gamma^2} \varepsilon^{(2\gamma-\alpha)^2/2} \varepsilon^{\frac{(\alpha-\gamma)^2}{2}})$$

Hence in the case of $\gamma^2 \in [2, 4)$, we still have $\mathbb{E}\left(|A_k^y - B_k^y|\right)$ converges to 0 exponentially in k.

Step 4. So far we have shown that $\mu_\varepsilon(S)$ converges to some limit in L^1 along the dyadic sequence $\varepsilon_k = 2^{-k}$. Since D can be built up by disjoint squares, from L^1 convergence, there exists a subsequence of $\mu_\varepsilon(D)$ which almost surely converges to random variable $\mu_h(D)$. Moreover, this implies that μ_ε is a.s. a tight sequence of finite measures, hence a.s., there exists a subsequence weakly converges to μ_h. If we can show that the a.s. weak limit is unique, then the unique μ_h is the desired LQG measure.

Let μ_h and μ'_h be two possible a.s. weak limits. In addition, define a $\pi-$system $\mathcal{A} := \{[x_1, y_1) \times [x_2.y_2) \subset D; x_1, x_2, y_1, y_2 \in \mathbb{Q}\}$. For any $A = [x_1, y_1) \times [x_2.y_2) \in \mathcal{A}$, one can always pick $\mathbb{Q} \ni x'_i < x_i$ and $\mathbb{Q} \ni y'_i > y_i$, $1 \le i \le 2$ such that

$$\mu'_h(A) \le \mu'_h((x'_1, y'_1) \times (x'_2, y'_2))$$

For the RHS, we almost surely apply the Portmanteau theorem

$$RHS \le \liminf_{\varepsilon \to 0} \mu_\varepsilon((x'_1, y'_1) \times (x'_2, y'_2)) \le \mu_h((x'_1, y'_1) \times (x'_2, y'_2))$$

Then we may take supremum over x'_i and infimum over y'_i in $\mathbb{Q}$, which yields $\mu'_h(A) \le \mu_h(A)$. Reciprocally, we also have $\mu_h(A) \le \mu'_h(A)$, which implies almost surely $\mu_h(A) = \mu'_h(A)$. By monotone class theorem, we know that μ_h must almost surely equal to μ'_h and the proof will be complete. $\square$

2.3 Rooted random measure

As discussed in Subsection 1.1, we regard h as the random variable taking values on topological space $(H^1_0(D), weak - \star)$. To make sense of the 'Radon-Nikodym density' $e^{\gamma h(z)}$, we could use not only the mollifications $e^{\gamma h_\varepsilon(z)}$, but also the orthonormal basis of the Sobolev space $\mathbb{H}^1_0(D)$. By a bit of elliptic PDE theory, we know that the orthonormal basis of the $(\cdot, \cdot)_\nabla$ semi-inner product are all C^∞-functions, so their corresponding densities are well-defined.

To achieve this, we would like to introduce a very useful tool. For $\varepsilon > 0$, define the ε-rooted random measure as a probability measure on $D \times H^1_0(D)$, with the form

$$\Theta_\varepsilon := Z_\varepsilon^{-1} e^{\gamma h_\varepsilon(z)} dz dh$$

where Z_ε^{-1} is the partition function, dz denotes the Lebesgue measure and dh is the law of the GFF. We see that the marginal distribution of z is calculated via

$$f(z)dz = Z_\varepsilon^{-1} \mathbb{E}_h e^{\gamma h_\varepsilon(z)} dz \propto C(z; D)^{\gamma^2/2} dz$$

which is deterministic measure. The marginal distribution of h is similarly calculated and has density $Z_\varepsilon^{-1}(\int_D e^{\gamma h_\varepsilon(z)} dz) dh \propto \mu_\varepsilon(D) dh$. Now we can see that, this is not the distribution of

Gaussian free field, in stead, it is absolutely continuous w.r.t. h with density being the 'ε-quantum area of D'.

The aim of introducing $\Theta_\varepsilon(dz, dh)$ is that we want to decompose the sampling of LQG measure μ_h into two steps: (i) First, sample z using a near Lebesgue measure; (ii) Next, sample h according to a near GFF distribution. Since we have shown that the marginal law of h is re-weighted by something still depends on z, in general, the second step could fail. However, there's a light at the end of the tunnel, we may overcome the difficulty using Cameron-Martin formula.

Fact 2.1 (Cameron-Martin)**.** *Suppose $X = (X_1, ..., X_d)$ is a $d-$dimensional Gaussian vector in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with mean $m \in \mathbb{R}^d$ and covariance matrix C. Let $a \in \mathbb{R}^d$ be a fixed reference vector, we define a new probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{e^{\langle a,X\rangle}}{\mathbb{E}e^{\langle a,X\rangle}}$$

Then the random vector X in $(\Omega, \mathcal{F}, \mathbb{Q})$ is identically distributed to random vector $X + a \cdot C$ in $(\Omega, \mathcal{F}, \mathbb{P})$.

Let's apply this result to our case. When sampling points on $D \times H_0^1(D)$ according to measure Θ_ε, it can be decomposed into a two step job. First, sample z by marginal distribution $f(z)dz$, we simultaneously get a point z in the domain and a uniform distribution ρ_ε^z. Next, the sampling of h is based on the conditional distribution $\Theta_\varepsilon(dh|z)$. This sampling, according to Cameron-Martin formula, is equivalent to sampling $h + \gamma\xi_\varepsilon^z$ according to the law of GFF. In other words, by domain Markov property, there exists an independent GFF $\bar{h}$ such that conditioned on $z \in D$, the distribution of h_ε under $\Theta_\varepsilon(z, h)$ is calculated by

$$\begin{aligned}
h_\varepsilon &\overset{d}{=} \langle \bar{h}, \xi_\varepsilon^z\rangle_\nabla + \gamma\langle \xi_\varepsilon^z, \xi_\varepsilon^z\rangle_\nabla \\
&= \langle \bar{h}, \xi_\varepsilon^z\rangle_\nabla + \gamma\langle \xi_\varepsilon^z, \rho_\varepsilon^z\rangle \\
&= \langle \bar{h}, \xi_\varepsilon^z\rangle_\nabla - \int_D \tilde{G}_{z,\varepsilon}(y)\rho_\varepsilon^z(y)dy - \log\varepsilon \\
&= \langle \bar{h}, \xi_\varepsilon^z\rangle_\nabla - \log\varepsilon
\end{aligned}$$

Similar to the arguments in Subsection 2.2, the measures Θ_ε weakly converges to a limit Θ, called the rooted random measure, along a subsequence of the dyadics $\varepsilon_k = 2^{-k}$. Under this limit, $\langle \bar{h}, \xi_\varepsilon^z\rangle_\nabla$ becomes a Brownian Motion with $\varepsilon = e^{-t}$. Thus conditioned on z, the law of $h_{e^{-t}}(z)$ under measure $\Theta(dh|z)$ equals to $B_t + \gamma t$. As a simple application, we establish the following result.

Proposition 2.1. *With Θ probability one, z is a γ-thick point of h, that is,*

$$\liminf_{\varepsilon \to 0} h_\varepsilon(z)/\log \varepsilon^{-1} \geq \gamma.$$

Proof. We see that

$$\liminf_{\varepsilon \to 0} h_\varepsilon(z)/\log \varepsilon^{-1} = \liminf_{t \to \infty} h_{e^{-t}}(z)/t$$

We would like to use dominated convergence theorem, that is

$$\mathbb{E}(h_{e^{-t}}(z)/t) = \mathbb{E}(B_t/t + \gamma) \leq \mathbb{E}(|B_t|/t + \gamma) = \gamma + O(\frac{1}{\sqrt{t}})$$

Thus $\liminf_{\varepsilon \to 0} h_\varepsilon(z)/\log \varepsilon^{-1} = \limsup_{\varepsilon \to 0} h_\varepsilon(z)/\log \varepsilon^{-1} = \gamma$ holds almost surely. □

For any $\tilde{D}$ compactly embedded in D, we can define $\Theta_\varepsilon^{\tilde{D}}$ and $\Theta^{\tilde{D}}$ to be the conditional distribution such that

$$\Theta_\varepsilon^{\tilde{D}}(\cdot) = \frac{\Theta_\varepsilon(\cdot \cap \tilde{D} \times H_0^1(D))}{\Theta_\varepsilon(\tilde{D} \times H_0^1(D))}, \qquad \Theta^{\tilde{D}}(\cdot) = \frac{\Theta(\cdot \cap \tilde{D} \times H_0^1(D))}{\Theta(\tilde{D} \times H_0^1(D))}$$

Finally, we are in the position to establish the second construction of LQG measure μ_h using orthonormal expansion.

Theorem 2.2. *Let $\{f_1, \ldots, f_n, \ldots\}$ be an orthonormal basis of Sobolev space $H_0^1(D)$. Write h as the zero boundary GFF on D and let h^n be the projection of h onto the closed subspace spanned by $\{f_1, \ldots, f_n\}$. Then Liouville quantum gravity measure μ_h is the almost surely the weak limit of μ^n as $n \to \infty$, where*

$$\mu^n = \exp\left\{\gamma h^n(z) - \frac{\gamma^2}{2} \operatorname{Var}(h^n(z)) + \frac{\gamma^2}{2} \log C(z; D)\right\} dz \tag{2.17}$$

Alternatively, for each measurable subset $A \subset D$, we have

$$\mathbb{E}(\mu_h(A)|h^n) = \mu^n(A), \qquad \mathbb{E}(\mu_h(A)) = \int_A C(z; D)^{\frac{\gamma^2}{2}} dz.$$

Proof. **Step 1**. First, we show that the sequence μ^n is indeed a.s. weakly convergent. This is because the density functions are actually uniformly integrable martingales. In fact we have

$$\mathbb{E}\left(\exp\left\{\gamma h^n(z) - \frac{\gamma^2}{2} \operatorname{Var}(h^n(z)) + \frac{\gamma^2}{2} \log C(z; D)\right\}\right) = C(z; D)^{\gamma^2/2}$$

By Fubini's theorem, for any Borel set A, $\mu^n(A)$ is L^1 bounded, hence it's also a uniformly integrable martingale. There exists a corresponding limit random variable such that $\tilde{\mu}_h(A) =$

$\lim_{n\to\infty} \mu^n(A)$ a.s.. Since we are considering random measures on nice 2d domains D, it's not hard to conclude that almost surely, $\tilde{\mu}_h$ is the weak limit. But do we have $\tilde{\mu}_h = \mu_h$?

Step 2. We need to examine that $\mu_h(A) = \tilde{\mu}_h(A)$ for every Borel subset. It suffices to show the identity holds in the case of dyadic squares S compactly embedded in D. Since both μ_h and $\tilde{\mu}_h$ are constructed through h, it's equivalent to show that $\mathbb{E}(\mu_h(S)|h^n) = \mathbb{E}(\tilde{\mu}_h(S)|h^n)$ for all $n \geq 1$. Again by martingale convergence theorem, it remains to show that

$$\mathbb{E}(\mu_h(S)|h^n) = \mu^n(S)$$

Let $h^n_\varepsilon(z)$ denotes the mean value of random function h^n on $\partial B_\varepsilon(z)$ provided that $B_\varepsilon(z) \subset D$. In fact, $h^n_\varepsilon(z)$ is the conditional expectation of $h_\varepsilon(z)$ on h^n. We see that $h^n_\varepsilon(z)$ is a uniformly integrable Gaussian martingale with limit $h_\varepsilon(z)$. Thus by 'discrete' Ito's formula, $\exp\Big\{\gamma h^n_\varepsilon(z) - \frac{\gamma^2}{2}\operatorname{Var}(h^n_\varepsilon(z))\Big\}$ is a convergent exponential martingale. Hence we calculate the conditional expectation

$$\begin{aligned}
\mathbb{E}(\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}|h^n) &= \varepsilon^{\gamma^2/2}e^{\frac{\gamma^2}{2}\operatorname{Var}(h_\varepsilon(z))}\mathbb{E}(e^{\gamma h_\varepsilon(z)-\frac{\gamma^2}{2}\operatorname{Var}(h_\varepsilon(z))}|h^n)\\
&= \varepsilon^{\gamma^2/2}e^{\frac{\gamma^2}{2}\operatorname{Var}(h_\varepsilon(z))}\exp\Big\{\gamma h^n_\varepsilon(z) - \frac{\gamma^2}{2}\operatorname{Var}(h^n_\varepsilon(z))\Big\}\\
&= \exp\Big\{\gamma h^n_\varepsilon(z) - \frac{\gamma^2}{2}\operatorname{Var}(h^n_\varepsilon(z)) + \frac{\gamma^2}{2}\log C(z;D)\Big\}
\end{aligned}$$

The above calculation is based on fixed ε. We see that the RHS of the equations converges since $\langle h^n, \delta_z\rangle = \lim_{\varepsilon\to 0}\langle h^n, \rho^z_\varepsilon\rangle = h^n(z)$ is well defined. We arrive at

$$\lim_{\varepsilon\to 0}\mathbb{E}(\mu_\varepsilon(S)|h^n) = \mu^n(S)$$

Thus we only have to prove the following equality with $\gamma \in [0,2)$.

$$\mathbb{E}(\mu(S)|h^n) = \lim_{\varepsilon\to 0}\mathbb{E}(\mu_\varepsilon(S)|h^n)$$

In other words, the limit in ε is exchangeable with conditional expectation for every n.

Step 3. To do this, we would like to prove the case which $n = 0$, i.e., show that

$$\mathbb{E}(\lim_{\varepsilon\to 0}\mu_\varepsilon(S)) = \lim_{\varepsilon\to 0}\mathbb{E}\mu_\varepsilon(S)$$

Denote $M_\varepsilon = \mu_\varepsilon(S)$, it is enough to prove that M_ε is uniformly integrable in $\varepsilon > 0$. Set $M = \mathbb{E}M_\varepsilon$, we normalize the distribution of the positive random variables M_ε into probability measures $\eta_\varepsilon = M^{-1}M_\varepsilon dM_\varepsilon$ on $\mathbb{R}_+$, such that $1 = \eta_\varepsilon(\mathbb{R}_+)$. Thus it suffices to show that for each $\delta > 0$, we have a constant C independent of ε such that $\eta_\varepsilon([C,\infty)]) < \delta$ for every ε. This would be sufficient to imply that η_ε is tight. On the other hand, given M_ε means that we have already integrated over $z \in S$, this is equivalent to saying that M_ε is a only function of the GFF. Moreover, the distribution of M_ε is already conditioned on $z \in S$, hence determined by Θ_ε^S. We only have to show that for each δ there exist C independent of ε such that

$$\Theta_\varepsilon^S(M_\varepsilon(h) > C) < \delta$$

for each ε. Set $\varepsilon_0 = \sup\{\varepsilon', B_{\varepsilon'}(S) \subset D\}$, we consider the case that $\varepsilon < \varepsilon_0$. We may try to sample (z, h) according to Θ_ε^S using the two-step approach discussed before this theorem. First, sample z according to its marginal distribution, which is supported on S. Then the conditional distribution of h given z is just $\tilde{h} + \gamma\xi_\varepsilon^z$ with $\tilde{h}$ being yet another independent GFF. This second step of sampling provides us a process $\mathcal{B}_t = \tilde{h}_{e^{-t}\varepsilon_0}(z) - \tilde{h}_{\varepsilon_0}(z)$ for all $t \in [0, -\log(\varepsilon/\varepsilon_0)]$.

By Proposition 1.2, the process $\mathcal{B}_t$ is a standard Brownian Motion. The conditional expectation (under the measure Θ_ε^S) of $\tilde{h}$ given the sampling is defined as a random generalized function with support $\bar{B}_{\varepsilon_0}(z)$:

$$\tilde{h}^{\cdot}(\omega) := \mathbb{E}(\tilde{h}(\omega)|z, \mathcal{B}_t) = \begin{cases} \mathcal{B}_{u(\omega)}, & |z-\omega| < \varepsilon_0 \\ 0, & |z-\omega| \geq \varepsilon_0 \end{cases}$$

where the function u is

$$u(\omega) = -\log \frac{|z-\omega| \vee \varepsilon}{\varepsilon_0}$$

Due to the irregular nature of Brownian paths, the smoothness property of $\tilde{h}^{\cdot}(\omega)$ is similar to the one for GFF. When z is fixed, for each ω the mean average of $\tilde{h}^{\cdot}(\cdot)$ on $\partial B_\varepsilon(\omega)$ is denoted as $\tilde{h}_\varepsilon^{\cdot}(\omega)$.

In fact, for $|z-\omega| \leq \varepsilon_0$, we can see that $\tilde{h}_\varepsilon^{\cdot}(\omega)$ is a weighted, non-uniform average of $\mathcal{B}_t$ over $t \in [u_1(\omega), u_2(\omega)]$, where $u_1(\omega) = -\log(\varepsilon_1(\omega)/\varepsilon_0)$, $u_2(\omega) = -\log(\varepsilon_2(\omega)/\varepsilon_0)$, and

$$\varepsilon_1(\omega) = \varepsilon_0 \wedge (|\omega - z| + \varepsilon), \qquad \varepsilon_2(\omega) = (|\omega - z| - \varepsilon) \vee \varepsilon$$

There are at most three different circumstances. In the following figure, we see that as t runs from $u_1(\omega)$ to $u_2(\omega)$, the averaging of $\mathcal{B}_t$ over t is equivalent to the averaging of $\mathcal{B}_{u(\cdot)}$ over the red boundaries. On each point in the red part, we count the contribution from some $\mathcal{B}_t$ which corresponds to some circle center at z passing through the point. It's easy to see that the variance of $\tilde{h}_\varepsilon^{\cdot}(\omega)$ is bounded by $[u_1(\omega), u_2(\omega)]$. For $|z-\omega| \leq \varepsilon_0$, we have $u_1(\omega) \leq u(\omega) \leq u_2(\omega)$ is this area. We want to seek for the following estimation:

$$|u_i(\omega) - u(\omega)| \leq \log 2, \qquad i = 1, 2$$

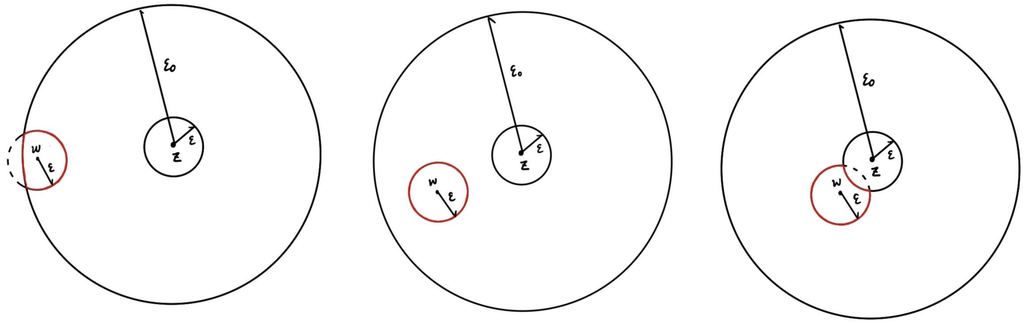


Figure 4

Assume $u(\omega) = -\log(\varepsilon^*/\varepsilon_0)$, the above is valid if we have

$$\varepsilon_1/\varepsilon^* \leq 2, \qquad \varepsilon^*/\varepsilon_2 \leq 2$$

By assuming a further condition $2\varepsilon \leq \varepsilon_0$, which means that ε is very small, the above condition for ε^* indeed holds and we have

$$\mathrm{Var}(\tilde{h}^{\cdot}_{\varepsilon}(\omega)) - u(\omega) = O(\log 2)$$

Meanwhile, it's trivial that $\mathbb{E}(\tilde{h}_{\varepsilon}(\omega)|\tilde{h}^{\cdot}_{\varepsilon}(\omega)) = \tilde{h}^{\cdot}_{\varepsilon}(\omega)$. Now, let's recall that given conditional expectation and conditional variance $\mathbb{E}(Y|X)$ and $\mathrm{Var}(Y|X) = \mathbb{E}((Y - \mathbb{E}(Y|X))^2|X)$, one has the decomposition $\mathrm{Var}(Y) = \mathbb{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathbb{E}(Y|X))$, thus

$$\mathrm{Var}(\tilde{h}_{\varepsilon}(\omega)) = \mathbb{E}(\mathrm{Var}(\tilde{h}_{\varepsilon}(\omega)|\tilde{h}^{\cdot}_{\varepsilon}(\omega))) + \mathrm{Var}(\tilde{h}^{\cdot}_{\varepsilon}(\omega))$$

we have almost surely the case

$$|\,\mathrm{Var}(\tilde{h}_{\varepsilon}(\omega)|\tilde{h}^{\cdot}_{\varepsilon}(\omega)) - \mathrm{Var}(\tilde{h}_{\varepsilon}(\omega)) + u(\omega)| = O(\log 2)$$

which means

$$\begin{aligned}\mathbb{E}(\varepsilon^{\gamma^2/2}e^{\gamma h_{\varepsilon}(\omega)}|z, \mathcal{B}_t) &= \mathbb{E}(\varepsilon^{\gamma^2/2}e^{\gamma \tilde{h}_{\varepsilon}(\omega)+\gamma^2\xi^{z}_{\varepsilon}(\omega)}|z, \mathcal{B}_t) \\ &\asymp \exp\left(\gamma\tilde{h}^{\cdot}_{\varepsilon}(\omega) + \gamma^2 u(\omega)/2\right)\end{aligned}$$

It can be shown that for fixed positive numbers a and b, there is a positive probability for event $A := \{\gamma\mathcal{B}_t < a + bt \text{ for all } t.\}$. Fix b and increase a, we see that the probability of A will converge to 1. Under the assumption $0 \leq \gamma < 2$, we can fix $0 < b < 2 - \gamma^2/2$. By definitions given before, the weighted non-uniform average is bounded by $\gamma\tilde{h}^{\cdot}_{\varepsilon}(\omega) < a + bu_2(\omega) \leq a + bu(\omega) + b\log 2$. Thus we have the bound:

$$\mathbb{E}(\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(\omega)}|z,\mathcal{B}_t,A) = O\left(e^a \exp\left(bu(\omega) + \frac{\gamma^2}{2}u(\omega)\right)\right) = O(e^a|z-\omega|^{-b-\gamma^2/2})$$

for the case that $|z-\omega| < \varepsilon_0$.

Since we have assumed that S is contained in $B_{\varepsilon_0}(S)$, one can just integrate over S and get

$$\mathbb{E}(\mu_\varepsilon(S)|z,\mathcal{B}_t,A) \leq \int_{B_{\varepsilon_0}(z)} C_0 e^a |z-\omega|^{-b-\gamma^2/2} = C_1(a) \tag{2.18}$$

The integral on the right converges due to the value of b. According to $\lim_{a\to\infty}\Theta^S_\varepsilon(A) = 1$, for each $\delta > 0$, there exist an $a > 0$ such that $\Theta^S_\varepsilon(A) > 1 - \delta/2$. We take such a, set $C = \frac{C_1(a)}{\delta/2}$ and immediately find that, if $\Theta^S_\varepsilon(\mu_\varepsilon(S) > C) \geq \delta$, there's a relation

$$\mathbb{E}(\mu_\varepsilon(S)|z,\mathcal{B}_t,\mathcal{A}) \geq \mathbb{E}(C1_{\{\mu_\varepsilon(S)>C\}}|z,\mathcal{B}_t,\mathcal{A})$$

Since $\Theta^S_\varepsilon(\mu_\varepsilon(S) > C) \geq \delta$ implies $\Theta^S_\varepsilon(\mu_\varepsilon(S) > C, A) + \Theta^S_\varepsilon(\mu_\varepsilon(S) > C, A^c) \geq \delta$, we have $\Theta^S_\varepsilon(\mu_\varepsilon(S) > C, \mathcal{A}) \geq \delta/2$. Continue to calculate:

$$\mathbb{E}(C1_{\{\mu_\varepsilon(S)>C\}}|z,\mathcal{B}_t,A) = \frac{C_1(a)}{\delta/2}\frac{\Theta^S_\varepsilon(\mu_\varepsilon(S) > C, A)}{\Theta^S_\varepsilon(\mathcal{A})}1_A \geq C_1(a)\frac{1_A}{\Theta^S_\varepsilon(A)} \geq C_1(a)1_A$$

which contradicts to (2.18). Thus we must have $\Theta^S_\varepsilon(\mu_\varepsilon(S) > C) < \delta$ and the assertion $\mathbb{E}(\lim_{\varepsilon\to0}\mu_\varepsilon(S)) = \lim_{\varepsilon\to0}\mathbb{E}\mu_\varepsilon(S)$ is proved.

Step 4. Since $\mu_\varepsilon(S)$ and $\mu_h(S)$ are non-negative random variables, due to the L^1 convergence shown in step 3, we apply dominated convergence theorem to conditional expectations:

$$\mathbb{E}(\mu_h(S)|h^n) = \mathbb{E}(\lim_{\varepsilon\to0}\mu_\varepsilon(S)|h^n) = \lim_{\varepsilon\to0}\mathbb{E}(\mu_\varepsilon(S)|h^n)$$

This concludes the proof of the entire theorem. □

3 Supplements to Liouville quantum gravity

The references for this section will be [1], [3] [4], [15] and [20].

3.1 CFT as fixed points of renormalization group

First and foremost, we mention that the description on CFT is not very precise in Subsection 2.1. A field theory with action $S(\phi)$ invariant under local conformal transformations is only a conformal classical field theory. Let's consider a simple ϕ^4-theory with $S(\phi) = \int d^4\phi \left((\partial\phi)^2 + \frac{\lambda}{4!}\phi^4\right)$. By computation, one can show that it is invariant under the scaling $x \mapsto \lambda x$ and $\phi(x) \mapsto \lambda^\delta \phi(\lambda x)$ with $\delta = 1$. If we rephrase it, the ϕ^4-theory is invariant under the rescaling (a special form of conformal transformation) with *scaling exponent* $\delta = 1$. However, this property no longer holds if we study its Euclidean path integral quantization $\int \mathcal{D}(\phi)e^{-S(\phi)}$. The first idea to maintain conformal symmetry under quantum mechanics, is to study massless, local and well renormalized matter fields, and this is one of the primary reasons why we introduced $\sigma = \gamma\phi$ to be the continuum GFF.

How does the conformal symmtry break under quantization? To answer it, we try to construct an interacting quantum field theory $\int \mathcal{D}(\phi)e^{-S_{int}(\phi)}$ via Renormalization Group (RG) flow. First, we lift the field operators to momentum space $\phi(x) \mapsto \hat{\phi}(k)$, i.e., apply Fourier transform. The interpretation on $\hat{\phi}(k)$ is roughly 'a field quanta', aka, particle found at energy-momentum $k = (E, \mathbf{p})$. If we couple $\hat{\phi}(k)$ to other field quantas, we need to introduce a coupling constant g_k. Under the momentum cutoff $|k| \leq \Lambda$, the renormalized path integral is

$$\prod_{|k|\leq\Lambda} \int \mathcal{D}(\hat{\phi}_k)e^{-S_\Lambda(\hat{\phi}_k)}, \quad \Lambda \geq 0. \tag{3.1}$$

where S_Λ denotes the cutoff actions. Meanwhile, we remark that in S_Λ, the coupling constants are also renormalized: $g \mapsto g_\Lambda$. To compute the function of g_Λ on Λ, we do the following trick: suppose $\Lambda' < \Lambda$, we integrate (3.1) against all field quantas $\hat{\phi}_k$ with $\Lambda' \leq |k| \leq \Lambda$, and redefine:

$$e^{-S_{\Lambda'}(\hat{\phi})} := \prod_{\Lambda'\leq|k|\leq\Lambda} \int \mathcal{D}(\hat{\phi}_k)e^{-S_\Lambda(\hat{\phi})} \tag{3.2}$$

The above definition implicitly tells us how the coupling constant is changed if we vary the threshold Λ. If we do (3.2) in an infinitesimal manner, it will be an integration over a thin shell in the momentum space. In general, we derive an evolution of coupling constants:

$$g := \{g_\Lambda\}_{\Lambda\geq 0}$$

This is called a *Renormalization Group Flow.* In fact, the name is quite misleading because it does not form an actual group in the algebraic sense. The parameter Λ is also understood as scales, since choosing $0 \leq \Lambda < \infty$ is equivalent to magnifying the spacetime such that fields with energy larger than Λ disappears from our vision. Define a beta function $\beta(g) := \Lambda\partial_\Lambda g$, then the interacting QFT is invariant under change of scale if and only if β is trivial. If the evolution of coupling constants g^* satisfies $\beta(g^*) = 0$, it is called a *Wilson-Fisher fixed point* and a field theory with RG flow g^* is called a CFT.

Another important application of RG flow is the coarse-graining of Ising model. Consider an Ising-type spin system on $d-$dimensional lattice $\Omega \subset \mathbb{Z}^d$ with spins $\sigma_x \in \mathbb{Z}/2\mathbb{Z} = S^0$, $x \in \Omega$. The partition function is given by

$$\sum_{\sigma \in \{\pm 1\}^\Omega} e^{(\sigma, J\sigma)} = \sum_{\sigma \in \{\pm 1\}^\Omega} e^{-H(\sigma^{(0)}, J^{(0)})}$$

where $J = (J_{x,y})$ is the matrix of coupling constants, H denotes the Hamiltonian. We have also denoted the original spins and couplings by σ^0 and J^0 respectively. Suppose Ω is regular such that it can be decomposed into blocks $\{B_i^{(1)}\}$ of length a. Then for each block $B_i^{(1)}$, we associate it with a new spin variable $\sigma_i^{(1)}$, which is the averaging of the original spins in $B_i^{(1)}$, i.e.,

$$\sigma_i^{(1)} = Z^{(1)} \sum_{x \in B_i^{(1)}} \sigma_x^{(0)}$$

with $Z^{(1)}$ chosen such that $|\sigma_i^{(1)}| \leq 1$. Inductively, we can decompose Ω into blocks of length na with $n \geq 1$, and the new spins are given by

$$\sigma_i^{(n)} = Z^{(n)} \sum_{x \in B_i^{(n-1)}} \sigma_x^{(n-1)}$$

However, we are not constructing the sequence $\{\sigma^{(n)}\}_{n \geq 1}$ for free. One has to maintain the partition function such that

$$\sum_{\sigma^{(n-1)}} e^{-H(\sigma^{(n-1)}, J^{(n-1)})} = \sum_{\sigma^{(n)}} e^{-H(\sigma^{(n)}, J^{(n)})}$$

This is achieved by suitably changing the coupling constants. Finally, we derive a sequence $J := \{J^{(n)}\}_{n \geq 1}$, and this is the RG flow for lattice Ising-type models. Let us denote the flow map by $\mathcal{R}$, then $J^{(n+1)} = \mathcal{R}(J^{(n)})$. If there exists a specific coupling matrix J_c such that $J_c = \mathcal{R}(J_c)$, it will be the Wilson-Fisher fixed point and the system is said to be at criticality. Following our arguments, we have shown that the critical spin systems are discrete CFTs.

3.2 AdS_2/CFT_1, Bulk & Boundary Liouville quantum gravity

A Lorentzian AdS spacetime in $d+1$ dimension (abbreviated by AdS_{d+1}) is given by the 2-component hyperboloid $X_0^2 + X_{d+1}^2 - \sum_{i=1}^d X_i^2 = 1$, which can be understood as a hypersurface in the pseudo-Riemannian manifold $\mathbb{R}^{1,d}$ with induced metric $ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2$. One can see that this metric is indeed invariant under the compact Lie group $SO(2,d)$. Moreover, for the connected component living in the upper half space, we can apply Poincare's coordinate to reparametrize the metric by $ds^2 = \frac{1}{z^2}(dz^2 + dx^2)$. Here x is $d-$dimensional, $z > 0$ is 1-dimensional, and the limit $z \to 0$ is called the asymptotic boundary.

So far one may be wondering why Liouville action could be both a quantum gravity (QG) and a conformal field theory (CFT). In 1997, J. Maldacena initiated a new area of theoretical physics called the AdS/CFT correspondence (also known as the gauge-gravity duality). According to AdS/CFT, QG and CFT are conjectured to have a natural duality, in the sense that a QG on AdS_{d+1} spacetime (Bulk physics) corresponds to a $d-$dimensional CFT_d on the asymptotic boundary of AdS_{d+1} (Boundary physics). For simplicity, in the following context, we apply the physical notation AdS_{d+1}/CFT_d to explain this principle.

Suppose the conformal gauge $\hat{g}_{ab}$ is chosen to be the Euclidean analog of AdS_2 spacetime, the quantization of σ on this hyperboloid is then a $LCFT_2$ called the Bulk Liouville theory. We then ask, what is the corresponding Boundary Liouville theory under AdS_2/CFT_1? One possible solution is to calculate the family of n-point correlation functions $\langle \phi(x_1, z_1) \cdots \phi(x_n, z_n) \rangle$ on AdS_2, and then 'suitably' take $z_1, ..., z_n \to 0$ to see the limiting correlations on the boundary. However, this may not be possible in the general duality setting. We hope it will work in the Liouville theory because the conformal symmetries of $LCFT_2$ may largely restrict the behaviour of the correlations, so a 'highly symmetrical' limit is expected. Indeed, physicists had come up with conjectures saying that, the effective Boundary CFT_1 that we're looking for, is conformally isomorphic to the restriction of an $LCFT_2$ on upper half plane $\mathbb{H}$ to the boundary $\partial\mathbb{H} = \mathbb{R}$.

If the Liouville field σ is simplified to be the 2d continuum free-boundary GFF, then this boundary restriction is actually possible under mathematical sense. Under the random measure interpretation, one expect that the construction of boundary LQG measure is similar to μ_h and the a.s. weak limit has one less Hausdorff dimension.

Theorem 3.1. *For $\gamma \in [0, 2)$, let h be the free-boundary GFF on $\mathbb{H}$ but with additive constant fixed. With slightly abused notations, for any $x \in \mathbb{R}$, let $h_\varepsilon(x)$ denote the semi-circle average of h with center x and radius $\varepsilon > 0$. Then it is a.s. the case that, as $\varepsilon \to 0$ along some subsequence of the dyadic numbers $\varepsilon_k = 2^{-k}$, the random measures $\nu_\varepsilon := \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} h_\varepsilon(x)} dx$ converge weakly to a limiting random measure on $\mathbb{R}$. Such limit is called the Boundary LQG measure, which is informally denoted by $\nu_h := e^{\frac{\gamma}{2} h(x)} dx$.*

Notice that the the factor in the renormalization is $\gamma/2$ rather than γ. This is because: (i) The semi-circle average of h is a Brownian motion rescaled by 2, as explained in Proposition 1.3; (ii) The additional $1/2$ will guarantee that the change of coordinates formula for ν_h shares the same form as μ_h. Remarkably, the second reason shows that the definition of quantum surface equivalence relation for boundary measures ν_h could be identical to Definition 6.1.

3.3 Gaussian multiplicative chaos

Recall the remark drawn in the end of Subsection 2.1, we argued that the construction of LQG using 2d continuum GFF is 'merely' the canonical conjecture by mathematical physicists. A natural generalization is to replace GFF by log-correlated Gaussian fields, i.e., a Gaussian generalized function $(h(z))_{z\in D}$ with the correlation structure:

$$\mathbb{E}(h(x)h(y)) \propto -\log|x-y|.$$

This leads to a more general random measure

$$e^{\gamma h(z)}dz = \lim_{\varepsilon\to 0} e^{\gamma h_\varepsilon(z)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(z)^2]}dz$$

called the Gaussian multiplicative chaos (GMC). We also mentioned before that, the reason for mathematical physicists to impose Liouville field σ to be the GFF is due to the interpretation on discrete quantum gravity. Indeed, the discrete approximation of 2d quantum gravity is a little different from the conformal gauge fixing (LCFT) approach. Instead of quantizing the Liouville action + ghost action + matter field action, we turn to the study on large-N-limit of random planar maps and random matrices.

As a simple example, we consider the 'pure gravity case'. In other words, we want the matter fields to introduce zero degree of freedom, which means the worldsheets X becomes points embedded in the background spacetime. This could be realized as a random triangulation[2] on a Riemann surface, and the *path integral measure* for gravity could be discretized as follows:

$$\int \mathcal{D}g \mapsto \sum_{\substack{\textit{random}\\ \textit{triangulations}}}$$

For the RHS, if the topology is fixed to be S^2, one can show the random triangulation converges to a specific quantum gravity structure when the number of triangulation tends to infinity (large-N-limit), see Subsection 4.3 for an explanation on Le Gall's theorem.

[2]Random triangulation refers to a random packing of equilateral triangles on Riemann surfaces, and when the surface is S^2, it becomes an example of the random planar map. By equilateral, we meant that the lengths of edges of triangles are automatically equal under certain geometry of the surface. So in some sense, each fixed triangulation corresponds to a Riemannian metric and discrete quantum gravity is a uniform distribution over such triangulations.

However, in general, the topology is also random. This means that not only we have to consider random triangulation in the graph theory sense, but also the randomness in the genus h of the surfaces should be counted. Following Polyakov's string theory approach to quantum gravity, the whole picture is actually $\mathcal{D}(g)\mathcal{D}(X)e^{-S_P(X,g)}$, and it is discretized to be the following

$$\sum_{h=0}^{\infty} \kappa^h \int \mathcal{D}(A) e^{-S_{in}}$$

where κ is the coupling constant between strings and spacetime. The $\mathcal{D}(A)$ is a measure over surface areas generated by all possible discretized gravity, hence it counts the randomness of triangulations and matter field X. The S_{in} denotes interactions, which should be, in principal, renormalized by cosmological constant. Similar as before, we would like to fix the topology of spacetime on a given genus h and apply Feynman's perturbative expansion to the latter integral. If X is only a 0-dimensional scalar φ, perturbative expansion leads to the φ^n field theories of the form (notice the Gaussian density at the from)

$$\int e^{-\varphi^2/2} \varphi^n d\varphi, \quad n \geq 1.$$

However, to recover the 2d Riemann surface structure $\mathcal{D}(g)$, physicists had suggested to replace φ by matrix M, thus the integral becomes

$$\int e^{-\operatorname{tr} M^2/2} M^{i_1}_{j_1} \cdots M^{i_n}_{j_n} dM$$

For symmetry requirements, the best candidate for M should be $N \times N$ Hermitian matrices. If we normalize $\int \mathcal{D}(M) = \int e^{-\operatorname{tr} M^2/2} dM \to 1$, the corresponding probability distribution on $M \in \mathbb{R}^{N\times N}$ is called a Gaussian Unitary Ensemble (GUE). Due to unitary symmetry, the GUE distribution can be further factorized into two parts: (i) A Haar measure on the compact Lie group $U(N)$, also called the circular unitary ensemble (CUE); (ii) A permutation invariant probability distribution over the eigenvalues[3].

Over the years, mathematicians had conjectured several variants of central limit theorems (large-N-limits) for such random planar maps and random matrices. As we shall see in the following, beside constructing 2d quantum gravity, there are other useful applications of these 'central limit theorems'. Namely, one could also use random matrices to explain prime number distributions, non-trivial zeros of Riemann zeta function, and many important conjectures in analytic number theory.

[3] It's similar to the conformal gauge fixing. One can imagine that eigenvalues play the same role to the Liouville field. By calculation, the Faddeev-Popov ghost field contribution to the distribution of eigenvalues is a Vandermonde determinant. See (3.3).

3.3.1 Random matrices, log-correlated fields and Riemann zeta function

In this subsection, we will particularly focus on the eigenvalue statistics of the circular unitary ensemble (CUE), which is defined as the unique normalized Haar measure on the compact Lie group $U(N)$. Suppose A is a random matrix drawn from CUE distribution, define its characteristic polynomial to be

$$P_N(A, \theta) := \det(I - Ae^{-i\theta})$$

For each fixed phase θ, it becomes a function on A. In general, for all such functions $f(A)$, one may rewrite it using the random eigenphases of A, i.e., $f(A) := f(\theta_1, ..., \theta_N)$. The reason why this is an equality is because of the diagonalizablility of A and unitary symmetry of CUE (For any fixed unitary matrix U, the measures dA and $dUAU^*$ are identical). The benefit of the eigenphase representation is that, the expectation of $f(A)$ can be calculated via an explicit formula by Weyl:

$$\int_{U_N} f(A)dA = \frac{1}{N!(2\pi)^N} \int_{[0,2\pi]^N} f(\theta_1, ..., \theta_N)|\Delta(e^{i\theta_1}, ..., e^{i\theta_N})|^2 d\theta_1 \cdots d\theta_N \tag{3.3}$$

where the Faddeev-Popov determinant $\Delta(z_1, ..., z_N)$ is, amazingly, the Vandermonde determinant $\prod_{1\leq i<j\leq N}(z_j - z_i)$. Using Weyl's formula, the first integrable quantities of interest are the 2βth moments of the characteristic polynomials with fixed θ:

$$M_N(\beta) := \int_{U_N} |P_N(A, \theta)|^{2\beta} dA$$

Indeed, it was shown that the large-N-limits of these moments are of order $M_N(\beta) \sim c(\beta)N^{\beta^2}$ with $c(\beta)$ expressed by Barnes function. Notice that one has the relation $M_N(\beta) = \mathbb{E}[\exp(2\beta \log |P_N(A, \theta)|)]$, thus this quantitative results on $M_N(\beta)$ leads to the following central limit theorem.

Theorem 3.2. *Suppose A is drawn from the CUE distribution on the unitary group $U(N)$, then for any fixed $\theta \in [0, 2\pi)$, we have the following weak limit as $N \to \infty$:*

$$\lim_{N\to\infty} \frac{\log P_N(A, \theta)}{\sqrt{\frac{1}{2}\log N}} = N_{\mathbb{C}}(0, 1) \tag{3.4}$$

where $N_{\mathbb{C}}(0, 1)$ denotes the complex standard Gaussian distribution.

There's actually a similar central limit theorem in the context of analytic number theory. Recall that the Riemann zeta function is defined via the series expansion $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ in the regime $\mathrm{Re}(s) > 1$. By analytic continuation, one can extend $\zeta(s)$ to the whole complex plane except for a simple pole at $s = 1$. Using such continuation, one can easily deduce that all negative even integers

$\{2k; k \in -\mathbb{N}\}$ are the *trivial zeros* of ζ. For the *non-trivial zeros* ρ_n, we know that they should lie in the critical strip $\{s; 0 < \mathrm{Re}(s) < 1\}$. The Riemann hypothesis states that the non-trivial zeros ρ_n in fact lie on the critical line $\{s; \mathrm{Re}(s) = \frac{1}{2}\}$. Although we don't know much about how to prove the hypothesis, we do have a central limit theorem for the distribution of $\zeta(s)$ on the critical line.

Theorem 3.3. *For any given $T > 0$, suppose t is uniformly distributed on the interval $[T, 2T]$, then we have the following weak limit as $T \to \infty$:*

$$\lim_{T\to\infty} \frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2}\log\log\frac{t}{2\pi}}} = N_{\mathbb{C}}(0,1) \tag{3.5}$$

where $N_{\mathbb{C}}(0,1)$ denotes the complex standard Gaussian distribution.

Loosely speaking, by comparing Theorem 3.2 and Theorem 3.3, one can find a lot of similarities between the critical line distribution of ζ and the characteristic polynomial of A. Indeed, let's rescale the eigenphases by

$$\phi_j := \frac{\theta_j N}{2\pi}$$

then the two-point correlation function for A, which is well-defined on the real line $x \in \mathbb{R}$, is formally given by

$$R_2(A, x) := \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=-\infty}^{+\infty} \delta\left(x + kN - (\phi_i - \phi_j)\right)$$

Similarly, assume the Riemann hypothesis to hold, for the non-trivial zeros $\rho_n = \frac{1}{2} + it_n$, one can do a similar rescaling

$$\omega_n := \frac{t_n}{2\pi}\log\frac{t_n}{2\pi}$$

and define a similar two-point correlation function

$$R_2(\{\omega_n\}, x) := \frac{1}{T}\sum_{\omega_i,\omega_j\in[0,T]}\sum_{k=-\infty}^{+\infty} \delta\left(x + kT - (\omega_i - \omega_j)\right)$$

Based on these ingenius observations, Dyson, Montgomery established the following conjecture.

Theorem 3.4 (Pair correlation conjecture)**.** *Let A, ζ, $\{\phi_n\}$ and $\{\omega_n\}$ be defined as above. For any compactly supported smooth test function $f(x) \in C_c^\infty(\mathbb{R})$, we have the 'large-$N$&$T$-limits':*

$$\lim_{N\to\infty}\int_{U(N)}\int_{-\infty}^{+\infty} f(x)R_2(A,x)dxdA = \lim_{T\to\infty}\int_0^T\int_{-\infty}^{+\infty} f(x)R_2(\{\omega_n\},x)dxd\{\omega_n\}$$

$$= \int_{-\infty}^{+\infty} f(x) \left(\delta(x) + 1 - \left(\frac{\sin \pi x}{\pi x} \right)^2 \right)$$

Here dA denotes the CUE distribution and $d\{\omega_n\}$ denotes the counting measure.

If the above conjecture holds, we could find a handful of very useful illustrations of the non-trivial zeros using random matrix distributions. Thus the investigation upon characteristic polynomials of certain random matrix ensembles is under growing research interests over the last few decades. Apart from the aspects of central limit theorem, what is the asymptotic behavior of the spatial correlations of $\log P_N(A,\theta)$? To answer the question, we state another interesting limit theorem proved by Hughes, Keating and O'Connell.

Theorem 3.5. *Given any N, we regard $\{\log P_N(A,\theta)\}$ as a random field parametrized by $\theta \in S^1$. For each fixed $\varepsilon > 0$, the large-N-limit of the random fields*

$$\log P_N(A,\theta) = (\operatorname{Re}\log P_N(A,\theta), \operatorname{Im}\log P_N(A,\theta))$$

converges weakly in the product Sobolev space $H_0^{-\varepsilon}(S^1) \times H_0^{-\varepsilon}(S^1)$ to a complex-valued Gaussian generalized function

$$Z := (\operatorname{Re} Z, \operatorname{Im} Z) = \sum_{k=1}^{\infty} A_k e^{-ik\theta}$$

where $\{A_k\}_{k\geq 1}$ is an sequence of independent complex Gaussian random variables with distribution $A_k \stackrel{d}{=} N_{\mathbb{C}}(0, \frac{1}{2k})$.

By simple calculations, we obtain

$$\mathbb{E}(\operatorname{Re} Z(\theta) \operatorname{Re} Z(\varphi)) = \mathbb{E}(\operatorname{Im} Z(\theta) \operatorname{Im} Z(\varphi)) = -\frac{1}{2} \log |e^{-i\theta} - e^{i\varphi}|$$

Generally speaking, a *log-correlated Gaussian field* (LCGF) refers to a centered Gaussian distributed random field $X(\ell)$ with ℓ living in a metric space (E,d) such that the correlation function is logarithmic, i.e., $\mathbb{E}(X(\ell)X(\ell')) = -\log d(\ell,\ell') + O(1)$. As we're about to see in the next subsection, the Gaussian multiplicative chaos (GMC) is a special example of LCGF. Apart from the generalization of LCFT in the physics contexts, the above arguments on random matrix theory, number theory and LCGF is our second motivation for the introduction of GMC.

3.3.2 Construction of GMC

This subsection is devoted to a rigorous mathematical construction of the Gaussian multiplicative chaos (GMC), and without causing any confusion, we adopt the same terminologies in Subsection 1.1. Suppose $D \subset \mathbb{R}^2$ is a simply connected bounded domain, we define a generic kernel function $G(x, y)$ for all $x, y \in \bar{D}$ with logarithmic singularity:

$$G(x, y) := -\log(|x-y|) + g(x, y)$$

where $g(x, y) \in C^0(\bar{D} \times \bar{D})$. Similar to the definition of the continuum Gaussian free field, let $\mathcal{M}_0$ denote the set of all compactly supported signed measures ρ in D such that

$$\iint_{D^2} G(x, y)\rho(dx)\rho(dy) < \infty$$

The log-correlated Gaussian field h with Dirichlet boundary condition and covariance G, is defined to be the centered Gaussian process $\{(h, \rho)\}$ indexed by $\rho \in \mathcal{M}_0$ such that

$$\mathrm{cov}((h, \rho_1), (h, \rho_2)) = \iint_{D^2} G(x, y)\rho_1(dx)\rho_2(dy)$$

From a random generalized function perspective, we can use a similar Sobolev space approach to show that h is an a.s. well-defined random element in $H_0^s(D)$ under the usual Hilbert space topology, for all $s < 0$. However, if one demands the $H_0^1(D)$ regularity of h, then it is only possible to construct under the weak-* topology, which means that for any $f \in H_0^1(D)$, the informal summation $\sum_{n\geq 1}(h, f)_\nabla$ may not be an a.s. convergent random series.

Let $\sigma(dz)$ be a Radon measure supported by $\bar{D}$ such that

$$\iint_{\bar{D}^2} \frac{\sigma(dx)\sigma(y)}{|x-y|^\delta} < \infty$$

for some $\delta \in (0, 2)$. Now we would like to take σ as the new reference area measure in D and define a new ε-mollification to the LCGF h. Construct θ to be a regular probability measure supported by the unit disk $\bar{\mathbb{D}}$ such that $\int \log(|x-y|)\theta(dy)$ is uniformly bounded for all $x \in 5\mathbb{D}$. Given any $\varepsilon > 0$ and $x \in \mathbb{D}$, we define the translated-dilated measure $\theta_{\varepsilon,x}$ such that $\theta_{\varepsilon,x}(A) = \theta(\frac{A}{\varepsilon} - x)$ for any Borel measurable set $A \in \mathbb{R}^2$. One could then define the ε-mollification of h via

$$h_\varepsilon(x) := (h, \theta_{\varepsilon,x}), \quad (\varepsilon, x) \in \mathbb{R}_+ \times D.$$

By Kolmogorov-Centsov theorem again, we claim that it is a stochastic process indexed by $\mathbb{R}_+ \times D$, with an η-Holder continuous modification for every $\eta < 1/2$.

Theorem 3.6 (Construction of GMC)**.** *Fix $\gamma \in [0, 2)$, let h be the Dirichlet boundary LCGF with correlation kernel $G(x, y)$ defined on a simply connected bounded planar domain D. Then it is a.s. the case that, as $\varepsilon \to 0$ along some subsequence of the dyadics $\varepsilon_k = 2^{-k}$, the random measures*

$$\mu_\varepsilon := e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}((h_\varepsilon(z))^2)}\sigma(dz)$$

converge weakly inside D to the Gaussian multiplicative chaos (GMC), which is a random measure informally written as $\mu_h := e^{\gamma h(z)}dz$.

We omit the proof of Theorem 3.6 due to its similarity to Theorem 2.1 and Theorem 3.1. In the following context, we would like to present a recent alternative construction by Bourgade and Falconet []. Recall that the large-N-limit of $\log P_N(A,\theta)$ for CUE was proven to have a LCGF structure on the unit cirlce S^1. Since one may informally write $\gamma h = \log e^{\gamma h}$, it is natural to expect that the $P_N(A,\theta)$ for some unitary ensemble A could have certain relation to the GMC measure. Let $\{X_n\}_{n=1}^{N^2}$ be an orthonormal basis for the unitary Lie algebra $\mathfrak{u}(N)$, then given a N^2-dimensional Brownian motion $B_t = \{B_t^n\}_{n=1}^{N^2}$, one can construct the unitary Brownian motion $\tilde{U}_t$ to be the solution to the SDE:

$$d\tilde{U}_t = \tilde{U}_t dB_t - \frac{1}{2}\tilde{U}_t dt$$

In the following construction, we would like to use the time-changed process $U_t := \tilde{U}_{2t}$, i.e., the solution to the SDE

$$dU_t = \sqrt{2}U_t dB_t - U_t dt \tag{3.6}$$

It was shown that under this time rescaling, the dynamics of eigenvalues $\{z_n\}_{n=1}^N$ of U_t are given by the β-Dyson Brownian motion with $\beta = 2$. Similar to the CUE case, define the characteristic polynomial of U_t by $P_N(U_t,\theta) = \det\big(I - U_t e^{-i\theta}\big)$ with $\theta \in [0, 2\pi)$.

Theorem 3.7. *Let $h(t,\theta)$ be a LCGF on the infinite cylinder $\mathcal{C} := \mathbb{R} \times S^1$ with covariance kernel*

$$\mathbb{E}(h(t_1,\theta_1)h(t_2,\theta_2)) := \mathbb{E}(h(z)h(\omega)) = 2\pi(-\Delta_{\mathcal{C}})^{-1}(z,\omega), \quad \Delta_{\mathcal{C}} := \partial_t^2 + \partial_\theta^2.$$

If we run the dynamics U_t up to equilibrium, for any $\gamma \in [0,2)$, we have the following a.s. weak convergence in the large-N-limit:

$$\lim_{N\to\infty} \frac{|P_N(U_t,\theta)|^{\sqrt{2}\gamma}}{\mathbb{E}(|P_N(U_t,\theta)|^{\sqrt{2}\gamma})} dtd\theta = e^{\gamma h(z)}dz = \mu_h(dz), \quad z = (t,\theta) \in \mathcal{C} \tag{3.7}$$

In the RHS, the random measure is the GMC corresponding to h.

In the original presentation of Theorem 3.7 in [4], the definition for h is a $\frac{1}{2}$-correlated field, this causes the resulting parametrization to be $\gamma \in [0, 2\sqrt{2})$. For the consistency with prior context, we prefer the presentation using the LQG parameter regime $\gamma \in [0,2)$. The proof of this result uses a multi-time generalization of the Fisher-Hartwig asymptotics of the Toeplitz determinants. For more details, see Theorem 1.2 in [4].

4 The Knizhnik-Polyakov-Zamolodchikov formula

This section is mainly devoted to an interpretation of the theory of the KPZ formula, which was originated by Knizhnik, Polyakov and Zamolodchikov in 1988. For the first subsection, we explain the derivation of the KPZ formula under the contexts of CFT coupled to quantum gravity. Our methods mostly follows the so-called David-Distler-Kawai approach, and it is different from Knizhnik-Polyakov-Zamolodchikov's original derivation, which uses the light cone gauge quantization of Polyakov quantum gravity action:

$$S_g = \int d^2x\sqrt{-g}(R\Delta^{-1}R + 2\Lambda)$$

and the $\mathfrak{sl}(2,\mathbb{R})$ current algebra symmetry. For the second part, we would like to establish the KPZ formula for the LQG measure μ_h, by applying the mathematical theory developed in Section 2. Finally, we show that the 'KPZ ideology' actually applies in the study of $2d$ critical lattice models, and the KPZ formulas with different scaling exponents actually characterises different universality classes of quantum gravity. The references for this section will be [10], [11], [12] and [20].

4.1 Conformal dimensions of 2d Liouville CFT

Recall that a Wilson-Fisher fixed point of the RG flow is a QFT which is invariant under rescaling. Similarly, one could ask: what will happen if we consider a scale-invariant string theory? First and foremost, a quantum string is a quantization over all possible classical embeddings X of a given Riemann surface (worldsheet) Σ. Due to the topology of Σ, the quantum embedding should be local operators, i.e., $X(z)$ should be a non-constant operator-valued function of $z \in \Sigma$. However, if we consider a 2d CFT on Σ, there's a powerful property, called the *state-operator map*, which shows that the set of all quantum states and the set of all local operators admits a natural isomorphism. Thus, instead of studying the Fock space of quantum states, it suffices to look at the algebraic structure of the family of local, or vertex operators.

For simplicity, we first recall how to calculate the wave function in quantum mechanics, using the path integral approach. Suppose our particle system starts with the wave function $\psi_i(x_i, t_i)$, to calculate the terminal wave function $\psi_f(x_f, t_f)$, we have to take into account all classical paths propagating between spacetime coordinates (x_i, t_i) and (x_f, t_f). The path integral representation writes

$$\psi_f(x_f, t_f) = \int \mathcal{D}(\mathbf{x})G(x_f, t_f; x_i, t_i)\psi_i(x_i, t_i)1_{\{\mathbf{x}|\mathbf{x}(t_i)=x_i, \mathbf{x}(t_f)=x_f\}}$$

where $G(x_f, t_f; x_i, t_i)$ is the Feynman propagator, similar to the Green's function for Gaussian free field (1.2). What is a wave function for a quantum field theory? Following probabilistic interpretation, it should be a functional over all classical fields $\psi(\phi)$ such that it defines a probability measure $\int \psi(\phi)\mathcal{D}(\phi) = 1$. Similarly as before, we do a path integral over all dynamics of classical fields $\phi(x,t)$ such that $\phi(x,t_i) = \phi_i(x)$ and $\phi(x,t_f) = \phi_f(x)$:

$$\psi_f(\phi_f, t_f) = \int \mathcal{D}(\phi) e^{iS[\phi]} \psi_i(\phi_i, t_i) 1_{\{\phi(x,t)|\phi(x,t_i)=\phi_i(x), \phi(x,t_f)=\phi_f(x)\}}$$

where we have written the Feynman propagator explicitly using the action functional $S[\phi]$. So far our calculations work in 'all QFTs'. Now let's consider the case that $(x,t) \in S^1 \times [t_i, t_f] = \Sigma$, i.e., a closed string theory with fixed worldsheet Σ and a fixed metric. Furthermore, assume ϕ is a CFT (for example, the Liouville field σ (2.6)), then we may conformally transform the cylinder $S^1 \times [t_i, t_f]$ into an Annulus, see Figure 5. The integral above becomes

$$\psi_f(\phi_f, r_f) = \int \mathcal{D}(\phi) e^{iS[\phi]} \psi_i(\phi_i, r_i) 1_{\{\text{all fields on the annulus}\}}$$

Since one is interested in the quantum state ψ_f, we may shrink $r_i \to 0$ so the annulus essentially becomes $\mathbb{D}$, see Figure 5 again. The effect of the initial quantum state is now only a re-weighting of path integral at $r = 0$, i.e., it is physically equivalent to a local operator $\mathcal{O}$ evaluated at $z = 0$:

$$\psi_f(\phi_f, r_f) = \int \mathcal{D}(\phi) e^{iS[\phi]} \mathcal{O}(z = 0) 1_{\{\text{all fields on the disk}\}} \tag{4.1}$$

This one-to-one correspondence between quantum state ψ and local operator $\mathcal{O}$ is called the *state-operator map.*

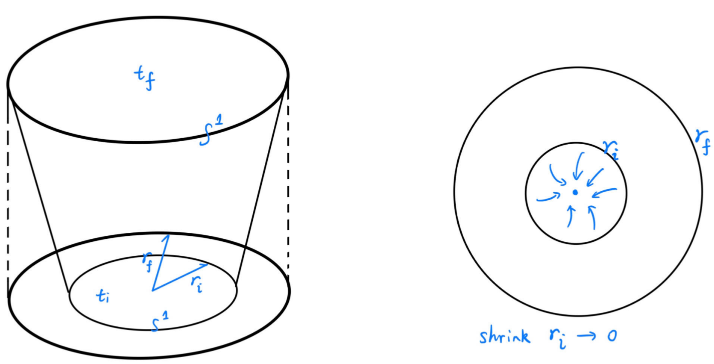


Figure 5: Conformal transformation for path integral.

Remark 4.1. *Be careful that* (4.1) *only works for CFT. Otherwise, one cannot map a cylinder to a disk without changing the physics.*

We have some more symmetry if we consider a CFT on a **classical** closed string. Indeed, the worldsheet Σ is diffeomorphic invariant, thus one could replace local operator by it's averaging on Σ without changing the physics:

$$\mathcal{O} \longmapsto V \propto \int_{\Sigma} \mathcal{O}(z)dz$$

The operator V is called the *vertex operator* associated to Σ. Hence in this case, the state-operator map is actually between the Fock space and vertex operator algebra.

Definition 4.1 (Conformal dimension)**.** *Suppose* $\mathbf{V}$ *is the vertex operator algebra for a 2d CFT, let* $z \mapsto \tilde{z}$ *be a conformal isomorphism. For a given* $V \in \mathbf{V}$*, if there exists* $\Delta > 0$ *such that the scaling relation*

$$\tilde{V}(\tilde{z}) = \left|\det \frac{\partial z}{\partial \tilde{z}}\right|^{\Delta} V(z)$$

holds, then the positive exponent Δ *is called the conformal dimension (or scaling exponent if the conformal transformation is the rescaling* $z \mapsto \tilde{z} = \lambda z$*) for the vertex operator* V*.*

Let's turn to the study of Lioville CFT on a 2d **quantum** bosonic string. In this case, if we want to conformally deform the vertex operator of σ, the conformal dimension not only depends on LCFT itself, it should also take into account the contribution from the random metric. After we do the Weyl gauge fixing $g = \hat{g}e^{\gamma\sigma}$, by state-operator map, there exists a local operator A such that the physics of LCFT on quantum gravity is depicted by the following vertex operator

$$V_L := \int_{\Sigma} \sqrt{\hat{g}} A(z)dz$$

For the quantum conformal dimension of V_L, we would like to follow the David-Distler-Kawai approach by comparing the classical exponent with its quantum counterpart, which is also the essential idea of KPZ in [12]. First, we vary the metric by coupling a classical field: $\hat{g} \mapsto \hat{g}e^{\gamma\varphi}$, where φ is a smooth function on Σ. There exists a classical conformal dimension Δ_c such that

$$V_L \longmapsto \int_{\Sigma} \sqrt{\hat{g}} e^{\gamma(1-\Delta_c)\varphi} A(z)dz \tag{4.2}$$

Similarly, if we couple the metric to the quantum Liouville field σ, there should also be a quantum conformal dimension Δ_q such that

$$V_L \longmapsto \int_{\Sigma} \sqrt{\hat{g}} e^{\gamma(1-\Delta_q)\sigma} A(z)dz \tag{4.3}$$

If we can derive the KPZ relation between Δ_c and Δ_q, then to calculate Δ_q, we only need to find a reference classical field with known conformal dimension. Physicists first calculated that for the vertex operator of Liouville field

$$v_L := \int_\Sigma e^{\alpha\sigma(z)} dz, \quad \forall \alpha > 0.$$

the conformal dimension is given by $\Delta_\alpha = \frac{\alpha Q}{2} - \frac{\alpha^2}{2}$. Now let's take $\alpha = \gamma(1-\Delta_q)$, following (4.3), the total conformal dimension for $\int_\Sigma e^{\alpha\sigma(z)} A(z) dz$ should be 1, because Liouville field σ and A are now coupled in a conformally invariant manner. But the conformal dimension for $\int_\Sigma e^{\alpha\sigma(z)} A(z) dz$ can also be calculated by first taking into account the contribution Δ_α from the Liouville field alone, then conditioned on the sample of the quantum Liouville field, i.e., given a classical field, the contribution from gravity will be Δ_c. To sum up

$$\Delta_c + \Delta_\alpha = 1$$

Rewriting the above relation yields the KPZ formula:

$$\Delta_c = \Delta_q + \frac{\gamma^2}{4}\Delta_q(\Delta_q - 1) \tag{4.4}$$

where we have used the critical charge $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. This finishes the argument by DDK.

4.2 KPZ formula for Liouville quantum gravity

Under the random measure interpretation of the LQG, we no longer need to calculate complicated and ill-defined intergals of local operators, such as (4.2) and (4.3). Instead, we can just study the relation between the Euclidean area on Σ and the quantum area on Σ measured by μ_h. To simplify even further, our conformal transformations are only taking to be the rescalings, i.e., we are actually proving the KPZ formula only for the scaling exponents.

4.2.1 Statements of KPZ relation

For LQG measure μ_h on a planar domain D, define the isothermal quantum ball $B^\delta(z)$ centered at $z \in D$, such that $\mu(B^\delta(z)) = \delta$. If there exists multiple such balls (i.e., multiple choices on the Euclidean size), let the radius of $B^\delta(z)$ to be $r^\delta := \sup\{r \geq 0, \mu(B_r(z)) = \delta\}$. We see that the measure of the random ball is fixed at δ on LQG but the radius is random. For any subset $X \subset D$, we denote the ε-neighborhood of X as the following set

$$B_\varepsilon(X) := \{z \in D, B_\varepsilon(z) \cap X \neq \varnothing\}$$

Similarly, we define the isothermal quantum δ-neighborhood of X by

$$B^\delta(X) := \{z \in D, B^\delta(z) \cap X \neq \varnothing\}$$

which is obviously a random neighborhood. We see that if μ_0 is the Lebesgue measure, the quantum ball $B^\delta(z)$ determined by μ_0 is just the Euclidean ball $B^\delta(z) = B_{\sqrt{\delta/\pi}}(z)$. Thus we expect that the radii of Euclidean and quantum neighborhoods are the related roughly by $\delta = O(\varepsilon^2)$.

Definition 4.2. *Let $\gamma \in [0, 2)$ and μ_0 denotes the Lebesgue measure. Let $(\mathcal{B}(D), \mathcal{G})$ be the smallest measurable space (Typically, we could take $\mathcal{G}$ to be the power set) such that $\mu_0(\cdot)$ is a measurable function. For a random subset $X \subset D$, we mean a random variable with values in $\mathcal{B}(D)$. In addition, the randomness of X is provided by a distribution independent from LQG. We say X has Euclidean scaling exponent x if*

$$\lim_{\varepsilon \to 0} \frac{\log \mathbb{E}\mu_0(B_\varepsilon(X))}{\log \varepsilon^2} = x$$

We say X has quantum scaling exponent Δ if

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}\mu_h(B^\delta(X))}{\log \delta} = \Delta$$

Theorem 4.1. *Fix $\gamma \in [0, 2)$ and a compact subset $\tilde{D}$ of D. Following the setting of Definition 4.2, if $X \cap \tilde{D}$ has Euclidean scaling exponent $x \geq 0$, then it has a quantum scaling exponent Δ. The two exponents are related by KPZ formula*

$$x = \frac{\gamma^2}{4}\Delta^2 + (1 - \frac{\gamma^2}{4})\Delta$$

In measure theoretic terminologies, the formula above presents a correspondence between the quantum Hausdorff dimension and the Euclidean Hausdorff dimension. It turns out Theorem 1 admits a generalization.

Theorem 4.2. *Let χ be any random measurable subset of the set of all balls of the form $B_\varepsilon(z)$ for $\varepsilon > 0$ and $z \in \tilde{D}$ (which means that χ is a random set in $\mathcal{B}(\mathbb{R}_+ \times \tilde{D})$, we may also check that for fixed $\varepsilon > 0$, the ε-section, χ_ε is a random Borel subset of D). We assume that χ and μ_h are independent. Fix parameter $\gamma \in [0, 2)$, if*

$$\lim_{\varepsilon \to 0} \frac{\log \mathbb{E}\mu_0\{z : B_\varepsilon(z) \in \chi\}}{\log \varepsilon^2} = x$$

we have

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}\mu_h\{z : B^\delta(z) \in \chi\}}{\log \delta} = \Delta$$

and the KPZ relation

$$x = \frac{\gamma^2}{4}\Delta^2 + (1 - \frac{\gamma^2}{4})\Delta \tag{4.5}$$

We see that if $\chi = \{B_\varepsilon(z), B_\varepsilon(z) \cap \tilde{D} \neq \varnothing\}$, Theorem 2 reduces to Theorem 1.

The proof of Theorem 4.2 will be given in the next subsection.

4.2.2 Proofs of KPZ

Proof of Theorem 4.2. For fixed $z \in D$, choose a radius ε_0 such that $B_\varepsilon(z) \subset D$. First, for $\varepsilon \leq \varepsilon_0$, we estimate the conditional expectation of $\mu_h(B_\varepsilon(z))$ conditioned on $h_\varepsilon(z) - h_{\varepsilon_0}(z)$. Recalling Proposition 5, we set f_1 as the first function in an orthonormal basis of $H_0^1(D)$

$$f_1 = \frac{\xi_\varepsilon^z - \xi_{\varepsilon_0}^z}{\|\xi_\varepsilon^z - \xi_{\varepsilon_0}^z\|_\nabla}$$

We have the relation $\|\xi_\varepsilon^z\|_\nabla = \xi_\varepsilon^z(z)$. Since $B_\varepsilon(z) \subset D$, $\xi_\varepsilon^z(y) = G(z,y) + \log|z-y| - \log\max(\varepsilon, |z-y|)$. Thus we have $\xi_\varepsilon^z(z) = -\log\frac{\varepsilon}{C(z;D)}$. In addition, $(\xi_\varepsilon^z, \xi_{\varepsilon_0}^z)_\nabla$ is the circle average of $\xi_{\varepsilon_0}^z(y)$ on $\partial B_\varepsilon(z)$, thus equals to $\xi_{\varepsilon_0}^z(z)$. By calculation

$$\|\xi_\varepsilon^z - \xi_{\varepsilon_0}^z\|_\nabla = \xi_\varepsilon^z(z)^2 - \xi_{\varepsilon_0}^z(z)^2 = -\log\frac{\varepsilon}{C(z;D)} + \log\frac{\varepsilon_0}{C(z;D)} = -\log\frac{\varepsilon}{\varepsilon_0}$$

The projection of GFF on the 1d-Gaussian subspace spanned by f_1 is given by

$$h^1(y) = (h, f_1)_\nabla f_1(y) = (h_\varepsilon(z) - h_{\varepsilon_0}(z))\frac{(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)(y)}{-\log\varepsilon/\varepsilon_0}, \qquad \mathrm{Var}h^1(y) = \frac{(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)^2(y)}{-\log\varepsilon/\varepsilon_0}$$

For the variance, we used the Brownian Motion property discussed in Proposition 2. By Equation (2.17) in Proposition 5, we have

$$\mathbb{E}(\mu_h(B_\varepsilon(z))|h^1) = \mathbb{E}[\mu_h(B_\varepsilon(z))|h_\varepsilon(z) - h_{\varepsilon_0}(z)] = \mu^1(B_\varepsilon(z))$$

where μ^1 is the measure

$$\mu^1(dy) = \exp\left(\gamma h^1(y) - \frac{\gamma^2}{2}\mathrm{Var}h^1(y) + \frac{\gamma^2}{2}\log C(z;D)\right)dy$$

Note that $(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)(y) = -\log\varepsilon/\varepsilon_0$ when $y \in B_\varepsilon(z)$, thus

$$h^1(y) = h_\varepsilon(z) - h_{\varepsilon_0}(z), \quad \mathrm{Var}h^1(y) = -\log\varepsilon/\varepsilon_0, \quad y \in B_\varepsilon(z)$$

Therefore, for $y \in B_\varepsilon(z)$, we have

$$\mu^1(dy) = \mu^0(dy) \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\gamma^2/2} \exp[\gamma(h_\varepsilon(z) - h_{\varepsilon_0}(z))]$$

where

$$\mu^0(dy) := [C(y; D)]^{\gamma^2/2} dy$$

In addition, define the average of conformal radius

$$[C_\varepsilon(z, D)]^{\gamma^2/2} := \frac{\mu^0(B_\varepsilon(z))}{\mu_0(B_\varepsilon(z))} = \frac{1}{\pi\varepsilon^2} \int_{B_\varepsilon(z)} [C(y; D)]^{\gamma^2/2} dy$$

We see that $\lim_{\varepsilon \to 0} C_\varepsilon(z, D) = C(z; D)$. Combining these results altogether

$$\mu^1(B_\varepsilon(z)) = [C_\varepsilon(z; D)]^{\gamma^2/2} \mu_0(B_\varepsilon(z)) \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\gamma^2/2} \exp[\gamma(h_\varepsilon(z) - h_{\varepsilon_0}(z))] v \tag{4.6}$$

$$= \pi \varepsilon^{\gamma Q} \left(\frac{C_\varepsilon(z, D)}{\varepsilon_0}\right)^{\gamma^2/2} \exp[\gamma(h_\varepsilon(z) - h_{\varepsilon_0}(z))] \tag{4.7}$$

where exponent $Q = 2/\gamma + \gamma/2$. So far we have analyzed the quantum measure of $B_\varepsilon(z)$ conditioning on $h_\varepsilon(z) - h_{\varepsilon_0}(z)$.

As an alternative, we might want to study the quantum measure under the conditioning on h_ε. Define $\tilde{f}_1 = \xi_\varepsilon^z / \|\xi_\varepsilon^z\|_\nabla$. By similar calculations as above, we deduce

$$\tilde{h}^1(y) = h_\varepsilon(z) \frac{\xi_\varepsilon^z(y)}{\xi_\varepsilon^z(z)}, \quad \operatorname{Var} \tilde{h}^1(y) = \operatorname{Var} h_\varepsilon(z) \left(\frac{\xi_\varepsilon^z(y)}{\xi_\varepsilon^z(z)}\right)^2 = \frac{\xi_\varepsilon^z(y)^2}{\xi_\varepsilon^z(z)}.$$

For $y \in B_\varepsilon(z)$, we have $\xi_\varepsilon^z(z) - \xi_\varepsilon^z(y) = \log C(z; D) + \tilde{G}_z(y)$. Thus by simple estimation

$$|\xi_\varepsilon^z(z) - \xi_\varepsilon^z(y)| \le C \sup_{y \in B_\varepsilon(z)} |\tilde{G}_z(y) - \tilde{G}_z(z)| \sim O(\varepsilon |\tilde{G}_z'(z)|)$$

This leads to

$$\frac{\xi_\varepsilon^z(y)}{\xi_\varepsilon^z(z)} \sim 1 + O\left(\varepsilon \frac{|\tilde{G}_z'(z)|}{\xi_\varepsilon^z(z)}\right) \sim 1 + O(\frac{\varepsilon}{\log \varepsilon})$$

Thus once again according to Equation (2.17), we have

$$\tilde{\mu}^1(B_\varepsilon(z)) = \mathbb{E}[\mu(B_\varepsilon(z)) | \tilde{h}^1] = \mathbb{E}[\mu(B_\varepsilon(z)) | h_\varepsilon(z)]$$

$$
\begin{aligned}
&= \int_{B_\varepsilon(z)} \exp\left(\gamma h_\varepsilon(z)\left[1+O(\frac{\varepsilon}{\log\varepsilon})\right] + \frac{\gamma^2}{2}\left[1+O(\frac{\varepsilon}{\log\varepsilon})\right]^2 \log\left(\frac{\varepsilon}{C(z;D)}\right) + \frac{\gamma^2}{2}\log C(y;D)\right) dy \\
&\approx \int_{B_\varepsilon(z)} \exp\left(\gamma h_\varepsilon(z) + \frac{\gamma^2}{2}\log\left(\frac{\varepsilon}{C(z;D)}\right) + \frac{\gamma^2}{2}\log C(y;D)\right) dy \\
&= \int_{B_\varepsilon(z)} e^{\gamma h_\varepsilon(z)}\varepsilon^{\gamma^2/2}\mu_0(dy) \approx \pi e^{\gamma h_\varepsilon(z)}\varepsilon^{2+\gamma^2/2}
\end{aligned}
$$

where the approximation is taken by the limit $\varepsilon \to 0$. In conclusion, the conditioned measure is given by

$$
\tilde{\mu}^1(B_\varepsilon(z)) \approx \mu_\odot(B_\varepsilon(z)),\ \ \mu_\odot(B_\varepsilon(z)) := \pi\varepsilon^{\gamma Q}e^{\gamma h_\varepsilon(z)},\ Q = \frac{2}{\gamma}+\frac{\gamma}{2}. \tag{4.8}
$$

Now we compare (4.6) and (4.8), we derive

$$
\mu^1(B_\varepsilon(z)) = \pi\varepsilon_0^2 C_\varepsilon(z;D)^{\gamma^2/2}\frac{\mu_\odot(B_\varepsilon(z))}{\mu_\odot(B_{\varepsilon_0}(z))} \tag{4.9}
$$

Recall Brownian motion modification of the circle average, take $t = -\log(\varepsilon/\varepsilon_0)$ and $V_t = h_\varepsilon(z) - h_{\varepsilon_0}(z)$, we then have

$$
\mu^1(B_\varepsilon(z)) = \pi\varepsilon_0^2 C_\varepsilon(z;D)^{\gamma^2/2}e^{\gamma V_t - \gamma Q t},\ \ \mu_\odot(B_\varepsilon(z)) = \mu_\odot(B_{\varepsilon_0}(z))e^{\gamma V_t - \gamma Q t}. \tag{4.10}
$$

We observe that the exponents are simply Brownian motion with drifts, which are essentially independent of $z \in D$. So far we have done all the preparations for the proof, and we claim that Theorem 4.2 follows easily from Theorem 4.3 below. □

Theorem 4.3. *Let $\tilde{B}^\delta(z)$ denotes the Euclidean ball in D centered at z, with $\delta = e^{\gamma V_t - \gamma Q t}$. For $A = -(\log\delta)/\gamma$, define a stopping time $T_A := \inf\{t : -V_t + Qt = A\}$ (under the randomness of LQG). The radius of the ball is given by e^{-T_A}. Then under the same setting, Theorem 4.2 is equivalent to to the following assertion: If we have Euclidean scaling relation*

$$
\lim_{\varepsilon\to 0}\frac{\log \mathbb{E}\mu_0\{z : B_\varepsilon(z)\in\chi\}}{\log\varepsilon^2} = x
$$

then there's a corresponding quantum scaling relation

$$
\lim_{\delta\to 0}\frac{\log \mathbb{E}\mu_h\{z : \tilde{B}^\delta(z)\in\chi\}}{\log\delta} = \Delta
$$

and the exponents are related by KPZ formula

$$
x = \frac{\gamma^2}{4}\Delta^2 + \left(1-\frac{\gamma^2}{4}\right)\Delta.
$$

Proof of Theorem 4.3. In the following, we only present the exponential martingale strategies for proof, while the another large deviation methods were also presented by [**?**].

Recall the rooted random measure discussed in Section 2.3, the $\Theta(z, dh)$ conditional distribution of $V_t = h_{\varepsilon_0 e^{-t}}(z) - h_{\varepsilon_0}(z)$ given a sample point $z \in D$, is given by $B_t + \gamma t$, where B_t is a standard Brownian motion. Now we can regard $T_A := \inf\{t : -V_t + Qt = A\} \stackrel{d}{=} \inf\{t : B_t(z) + Qt - \gamma t = A\}$ as a random variable under $\Theta(z, h)$, where $B_t(z)$ is a family of (not necessarily independent) standard Brownian motions.

Remark 4.2. *The above is just a transformation of the probability spaces, the distribution is unchanged. Since we have required that the random set X is sampled independently from the LQG measure, then suppose $\mathbb{Q}$ is the randomness for X, we can do informal composition of distributions $\Theta(X(dz), dh) = e^{\gamma h(z)} dh \mathbb{Q}(X \cap dz)$. Conditioned on the samples of z, T_A becomes random variables such that the family $\{T_A(z, \cdot), z \in X\}$ are i.i.d. stopping times, with equal distribution to $\inf\{t : W_t + Qt - \gamma t = A\}$, where W_t is yet another independent standard Brownian motion.*

Here comes the tricky part. Since the random set becomes χ, the distribution informally becomes $\Theta(\chi(\varepsilon, dz), dh)$, with $\varepsilon \geq 0$. Due to $\lim_{t\to\infty} W_t + Qt - \gamma t = -\infty$, a.s., for $A > 0$ large enough, we expect for a sampling on h under $\Theta(\chi(e^{-T_A(z)}, dz), dh)$, e^{-T_A} to be a.s. super small, and uniformly small (due to independence in z). According to the Euclidean scaling relation, we expect $\log \mu_0\{z : B_{e^{-T_A(z)}}(z) \in \chi\}$ to be very close to $x \log e^{-2T_A}$ (temporarily imagine T_A to be small constant). Thus

$$\mathbb{E}\mu_0\{z : B_{e^{-T_A(z)}}(z) \in \chi\} \approx \mathbb{E}e^{-2xT_A}$$

where the LHS is the $\Theta(\chi(e^{-T_A(z)}, dz), dh)$ expectation of the event that the ball centered at z with radius e^{-T_A} is in χ. Denote this expectation by q_A, then for any $0 < x_2 < x < x_1$, we have

$$\mathbb{E}e^{-2x_1 T_A} \leq q_A \leq \mathbb{E}e^{-2x_2 T_A} \tag{4.11}$$

To compute q_A, let $a = Q - \gamma$, then for $\beta \in \mathbb{R}$, consider the exponential martingales $\exp\{\beta B_t(z) - \beta^2 t/2\}$, with z being sampled by χ. Since $B_t(z) + at \leq A$ for all $t \in [0, T_A(z)]$ and $T_A(z)$ is a finite stopping time, one can show that $\beta B_t(z) - \beta^2 t/2$ is a.s. bounded above, and taking expectation with both z and h yields

$$\mathbb{E}[\exp\{\beta B_{T_A} - \beta^2 T_A/2\}] = 1$$

By definition of the stopping times $T_A(z)$, $B_{T_A} = A - aT_A$, we have

$$\mathbb{E}\exp\left[-(\beta a + \beta^2/2)T_A\right] = \exp(-\beta A)$$

If we take $2x = \beta a + \beta^2/2$, the above formulas lead to

$$q_A = \mathbb{E}\exp(-2xT_A) = \exp(-\beta A) = \delta^{\beta/\gamma}$$

Set $\Delta = \beta/\gamma$, we get

$$x = \frac{\beta a}{2} + \frac{\beta^2}{4} = \Delta - \frac{\gamma^2}{4}\Delta + \frac{\gamma^2\Delta^2}{4}$$

which is exactly the KPZ formula.

Note that Equation (4.11) guarantees the existence and uniqueness on the setting of x and Δ. To complete the proof, we keep in mind that $\varepsilon = e^{-T_A}$ and $\delta = e^{\gamma V_{T_A} - \gamma Q T_A}$ and the fact that $\varepsilon \to 0$ amounts to $A \to \infty$, which also amounts to $\delta \to 0$. From Equation (4.10), we get

$$\mu(B^\delta(z)) = \delta \propto \mu^1(B_\varepsilon(z)) = \mu^1(\tilde{B}^\delta(z))$$

keep calculating

$$\begin{aligned}
\mathbb{E}\mu\{z : B^\delta(z) \in \chi\} &= \mathbb{E}\left(\frac{\mu^1\{B_\varepsilon(z) : B_\varepsilon(z) \in \chi\}}{\delta\pi\varepsilon_0^2 C_\varepsilon(z;D)^{\gamma^2/2}}\right) \\
&= \mathbb{E}\mu^1\{z : \tilde{B}^\delta(z) \in \chi\} \\
&= \mathbb{E}\mu_0\{z : B_{e^{-T_A}}(z) \in \chi\} \\
&= q_A \\
&= \delta^\Delta
\end{aligned}$$

The first identity simply follows from the fact that $\mu_h(B^\delta(z))$ are constants, which leads to $\delta\mu_h\{z : B^\delta(z) \in \chi\} = \mu_h\{\cup_z B^\delta(z) : B^\delta(z) \in \chi\}$. Thus, to complete the proof of Theorem 4.3, so as to conclude Theorem 4.2, we only have to take log on both sides. $\square$

4.3 Discrete quantum gravity and universality classes

We will now show that KPZ formula (4.5) not only provided a quantitative comparison between quantum gravity and flat spacetime, it is also a powerful tool in the study of 2d critical statistical mechanics models and could be an important key to solve the whole 2d quantum gravity theory.

For the first reason stated above, the 2d critical lattice models are in general the most difficult to study. Take Ising model on $\mathbb{Z}^2$ for example, many well-known results, such as exponential decay of correlation length, remains unknown at criticality. In these cases, physicists developed a different and rather wild approach. In stead of studying these models on fixed lattices or graphs, we define them on its random analogues, i.e., consider them in the context of discrete quantum gravity (random planar lattices). If it is any easier to do it under quantum gravity, then by KPZ relations, we can get back to the actual Euclidean lattice very easily.

Secondly, no matter in the bosonic string theory approach or the quantization of Polyakov action, the goal of 2d quantum gravity is to construct a uniform distribution over all Riemann surfaces metrics on $\mathbb{S}^2$. Now think about this question as a probabilist, we can try arguments similar to the random walk approximation for Brownian motion, or the definition for discrete GFF (see Subsection 1.1), by constructing a sequence of discrete random planar maps and then show that they converge to a limiting random metric in the Gromov-Hausdorff sense[4]. Similar to the central limit theorem, we expect that the limiting random Riemann surface should not depend too much on the probability distribution of the discrete maps. In fact, this is not exactly true. The parameter γ in the KPZ relation (4.5) actually indicates *universality classes* of quantum gravity, in the sense that, for very different distributions of the sequence of discrete maps, one could get different limiting quantum gravities with different KPZ relations.

Before we start investigating the discrete models, let's introduce some definitions for discrete random objects. One could imagine that, the easiest way to construct a sequence of discrete quantum gravity is to randomly discretize the Riemann sphere $\mathbb{S}^2$ into graphs and then suitably rescale them.

Definition 4.3. *A planar map is defined as a proper graph embedding*[5] *of a finite connected planar graph $G = (\mathcal{V}, E)$ into $\mathbb{S}^2$, modulo orientation preserving homeomorphisms of $\mathbb{S}^2$. A rooted planar map refers to a planar map with one specified oriented edge.*

The planar maps can be further classified according to number of faces and adjacent edges. After we specified such characters to form a finite class of maps, the random planar map is simply a uniform distribution within this class.

Definition 4.4. *For $p \geq 3$, let $\mathbb{M}_n^p$ denote the class of planar maps, such that*

(i) All maps are rooted and have n faces in total.

[4]For any two compact metric spaces X, Y, we construct a reference compact metric space M and two isometric embeddings: $f(X) \hookrightarrow M$, $g(Y) \hookrightarrow M$. Since $f(X)$ and $g(Y)$ are compact subsets living in a common metric space, we compute the Hausdorff distance $d_{HAU}(f(X), g(Y))$ by (5.20). The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of all Hausdorff distances which is taken over all such embedding f, g and all M.

[5]The topological graph embedding of G into a surface M (2d topological manifold) is a mapping of vertices and edges such that: (i) The vertices are mapped into the surface by injection; (ii) Each edge e is mapped to a simple curves (homeomorphic images of $[0, 1]$) on the surface and the end points are exactly the images of end vertices of e; (iii) Any two distinct curves do not intersect in the domain $(0, 1)$.

(ii) Each face has degree p, in the sense that the boundary of each face is composed of p edges.

Such a class of planar maps is called a p-angulations with n faces. The classes $\mathbb{M}_n^p$ are finite sets. Indeed, it was shown by Tutte, 't Hooft, Schaeffer that the cardinality of $\mathbb{M}_n^4$ is given by

$$\#\mathbb{M}_n^4 = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

A random planar map $M_n^p = (\mathcal{V}(M_n^p), d_{gr})$ is a uniformly distribution on the class $\mathbb{M}_n^p$, where $\mathcal{V}(M_n^p)$ denotes the set of vertices for each uniform sampling, and d_{gr} denotes its graph distance[6].

It turns out that if we define a sequence of random planar maps 'very carefully' using the above structure, there exists a limiting random metric (a universality class) on $\mathbb{S}^2$ called the Brownian map. The following remarkable theorem was proved by Le Gall [13].

Theorem 4.4. *For any $p \geq 3$, we set the parameters by*

$$c_p = \begin{cases} 6^{1/4}, & p = 3. \\ \left(\frac{9}{p(p-2)}\right)^{1/4}, & p \geq 4. \end{cases}$$

Then the rescaled sequence of random planar maps:

$$(\mathcal{V}(M_n^p), c_p n^{-1/4} d_{gr})$$

converge in distribution to (m_∞, D^*) in the Gromov-Hausdorff sense[7]*. The limiting space (m_∞, D^*) is called a Brownian map, also known as the pure gravity, and it is a.s. homeomorphic to the Riemann sphere $\mathbb{S}^2$.*

It seems that we have found one universality class of quantum gravity, i.e., the Brownian map, however, we're still not close to the 'full solution'. The original question of 2d quantum gravity is to construct a uniform random metric, but we don't know how to canonically, isometrically embed the space (m_∞, D^*) into $\mathbb{S}^2$. Another interesting question is to find the γ for the Brownian map. To achieve this, we need to consider the local geometries and one natural candidate is to study the shape of the frontiers of a random walk.

[6] For any fixed planar map (i.e., a connected graph constructed on $\mathbb{S}^2$), the graph distance of two vertices $d_{gr}(a, b)$ is given by the minimum length of the curves connecting them, i.e., the length of the graph geodesic computed in the Riemannian metric of the sphere.

[7] The space of all compact metric spaces equipped with Gromov-Hausdorff distance is in fact a Polish space. One could define the Prokhorov distance on the space of all probability measures on the Polish space. Thus the convergence in our context can be explained as convergence of their corresponding probability measures.

Consider a simple random walk (starting from the interior) on a connected graph $G = (\mathcal{V}, E)$ with boundary (maybe random). One is particularly interested in three classes of random subsets of $\mathcal{V}$: (i) The *range* R of the random walk is given by the total trajectory of the walk before its first hitting on a boundary point; (ii) The set if *cut-points* C is given by the vertices $a \in \mathcal{V}$ such that removing a disconnects R into two disjoint connected components; (iii) The set of *frontier-points* F is given by all the vertices on R that are connected to the boundary via paths living in the complement G/R. Now consider the fixed lattices $\Lambda_n := [-n, n]^2$ and a simple random walk X_t starting from 0. If $K_n \subset \Lambda_n$ is a sequence of random subsets, then if the limit exists, we define the discrete Euclidean scaling exponent by

$$x(K) = \lim_{n\to\infty} \frac{\log \mathbb{E}(|K_n|/n^2)}{\log 1/n^2} \tag{4.12}$$

We remark that the proof for the convergence on the RHS is highly non-trivial. Suppose we also run a simple random walk Y_t on random planar maps M_n^p with boundary ∂M_n^p (instead of embedding the p-angulations in a sphere, we embed them in the unit disk $\bar{\mathbb{D}}$). Then for any associated random subsets K_n, the discrete quantum scaling exponent (if it exists) is defined by

$$\Delta(K) = \lim_{n\to\infty} \frac{\mathbb{E}(\log|K_n|/|M_n^p|)}{\log 1/|M_n^p|} = \lim_{n\to\infty} \frac{\mathbb{E}\log|K_n|/n}{\log 1/n} \tag{4.13}$$

It is shown that the two scaling exponent are related to each other via the KPZ formula

$$x = \frac{2}{3}\Delta^2 + \frac{1}{3}\Delta \tag{4.14}$$

In other words, the pure gravity falls into the $\gamma = \sqrt{8/3}$ universality class. Sometimes, we don't need to work under the quantum-to-Euclidean strategy, the opposite direction could be simpler. For example, if we consider the range R of a simple random walk, it's easier to count the vertices in Euclidean geometry and then lift it to quantum gravity by KPZ.

Next, we introduce the coupling of quantum gravity and Ising model, which falls into another universality class. The simplest way to consider the interaction between gravity and fermionic matter is to construct an Ising model on the graph. However, for mathematical use, instead of assigning each vertex with a spin, we attach the spins to the faces of the graph, see Figure 6 for an illustration. Because of nearest neighbor interaction, for a given inverse temperature $\beta > 0$, the Gibbs measure is expressed by

$$\mathbb{P}(M_n^p, \sigma) \propto \exp\left(\beta \sum_{f\sim f'} \sigma_f \sigma_{f'}\right)$$

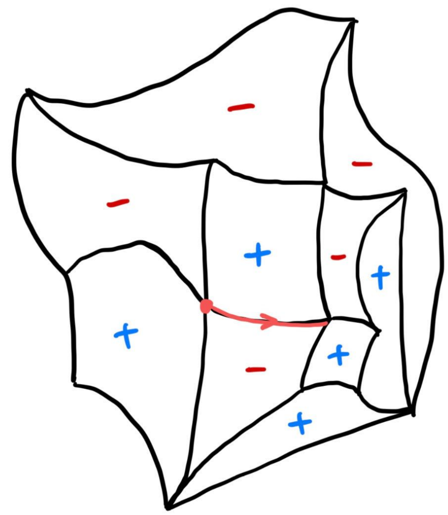


Figure 6: Ising model coupled to a quadrangulation M_n^4.

where the sum is taken over all pairs of adjacent faces. The above construction is fine with fixed p-angulations, but it doesn't show any kind of interaction if M_n^p is a sample of the random planar map. To explain interactions, we introduce another coupling constant $\alpha \in \mathbb{R}$, and for fixed $p \geq 3$, redefine the *quantum grand canonical ensemble* (the number of particles, i.e., faces is flexible, and the graph is random) partition function via

$$Z(\alpha, \beta) = \sum_{n\geq 1} \sum_{\mathbf{m}\in\mathbb{M}_n^p, \sigma\in\{-1,1\}^n} e^{-\alpha|\mathbf{m}|} e^{\beta \sum_{f\sim f'} \sigma_f \sigma_{f'}} = \sum_{n\geq 1} e^{-n\alpha} \sum_{\mathbf{m}_n\in\mathbb{M}_n^p} \sum_{\sigma\in\{-1,1\}^n} e^{\beta \sum_{f\sim f'} \sigma_f \sigma_{f'}}$$

Similar to lattice Ising model, the above ensemble also admits a first order phase transition. The critical point is in fact a 2-vector (α_c, β_c), and it was shown that the analyticity of $Z(\alpha, \beta)$ is broken near (α_c, β_c). Furthermore, if one calculates the marginal distribution of $\mathbf{m}$, one could find that $\mathbf{m}_n$ no longer follows a uniform distribution within each class $\mathbb{M}_n^p$, because of the presence of Ising model. In fact, if we sample an independent random walk on $\mathbf{m}_n$ at criticality and rescale it by $n \to \infty$, the limiting quantum gravity follows a different KPZ relation

$$x = \frac{3}{4}\Delta_{ising}^2 + \frac{1}{4}\Delta_{ising} \tag{4.15}$$

This shows that the Ising unversality class is characterized by $\gamma = \sqrt{3}$. Motivated by Ising model coupled to discrete quantum gravity, it was conjectured that for any critical model in the γ-universality class, the interfaces (e.g., critical percolation interface) are encoded by the corresponding SLE_κ with $\kappa = \gamma^2$. For example, Smirnov had shown the interfaces of 2d critical Ising model and FK-Ising model on the square lattice converges to SLE_3 and $SLE_{16/3}$ respectively, under the scaling limit $mesh \to 0$.

From another perspective, if we sample random planar maps $\mathbf{m}_n$ in the γ-universality class, and then embed them naturally in the sphere $\mathbb{S}^2$, we will get a family of random measures μ_n. It was conjectured by Duplantier and Sheffield that the limit of μ_n will be closely related to the γ-LQG on the sphere[8].

5 A glimpse into imaginary geometry

For the 3 important 2d random geometric objects: SLE, GFF and LQG surfaces, it is shown that their natural probabilistic and geometric structures are strongly related, partly because they possess some natural conformal symmetry, domain Markov properties and they are related to certain critical statistical mechanics models. In this section, we consider in particular the imaginary geometric relation between GFF and SLE, and the materials are mainly from [14].

First, we ask, what is a 2d imaginary geometry, or, where does this terminology come from? Recall the Riemann uniformization theorem allows one to study a simply connected Riemann surface M through its isothermal metric $e^{\lambda}(d^2x + d^2y)$ on a simply connected planar domain $D \subset \mathbb{C}$. We can see that the geometry of the surface is completely encoded by the function λ. Indeed, $-\Delta\lambda$ is the density of the Gaussian curvature of M. Thus it is natural to say that the Riemann surface M has *imaginary geometry* iff the curvature density is imaginary, i.e., $\lambda = ih$, where $h(z)$ is a real-valued H^2 function on D.

Second, we ask, how can we study imaginary geometry? In particular, we want to study the geometry of M by investigating how the curvature of M affect the (local) flow lines. From this perspective, under the 2d settings, we could imagine a smooth function $\lambda = ih$ to represent the shape of a mountain. As an explorer or a professional hiker, one cares about the altitude lines and the gradiant curves. More specifically, to characterise the geometry manifested by the smooth imaginary function ih on $D \subset \mathbb{C}$, we could consider the vector field $e^{i(h/\chi+\theta)}$, where χ is a fixed number and $\theta \in [0, \pi)$ denotes the direction. For each fixed point $z \in \partial D$, we calculate the flow with z being the starting position (imaging tracing the shape of the mountain from z). Indeed, suppose η is the flow line for $\theta = 0$, then it satisfies the ODE

$$\eta'(t) = e^{ih(\eta(t))/\chi}, \quad \eta(0) = z, \quad t > 0. \tag{5.1}$$

[8]The γ-LQG measure on $\mathbb{S}^2$ is informally given by $e^{\gamma h}$, where h is the GFF on the sphere. We consider the Sobolov H^1 norm closure of the space of smooth functions ϕ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} \phi ds = 0$. The GFF on sphere is given by $\sum_{n\geq 1} \alpha_n \phi_n$, where ϕ_n is the orthonormal basis of the above Sobolev space and α_n is a family of i.i.d. standard Gaussian variables. By calculation, one can show that the Green's function is given by $G(x, y) = \log\big(\cot \frac{\theta}{2}\big)$, where θ is the angle between x and y.

This leads naturally to a function-curve coupling (We meant constructing a curve out of a given function). Recall that the Riemann uniformization does not require uniqueness for the parametrization, thus the flow line setting should be, in some sense, invariant under conformal isomorphisms. In 2d, this conformal invariance symmetry is understood as: for any conformal mapping $\psi : \tilde{D} \to D$, the flow lines $\psi^{-1}\eta$ are exactly the ones for $h \circ \psi - \chi \arg \psi'$, i.e.,

$$[\psi^{-1} \circ \eta]' = [\psi^{-1}(\eta)]'\eta' = [\psi'(\eta)]^{-1}\eta' = e^{ih(\eta)/\chi - i \arg \psi'} \tag{5.2}$$

If we denote $\tilde{h} = h \circ \psi - \chi \arg \psi'$, then from the flow lines perspective, we may informally say that imaginary geometries (D, h) and $(\tilde{D}, \tilde{h})$ are conformally equivalent. To classify different imaginary geometries, we follow the spirit in differential geometry: any two isometric Riemannian manifolds are considered **identical** in the *quotient category*. For the purpose of our text, we define our *Imaginary geometry equivalence class* for random surfaces, which incorporate the deterministic setting.

Definition 5.1. *A random Riemann surface modulo imaginary geometry equivalence, with parameter $\chi \in \mathbb{R}$, is a pair (D, h) with D being a simply connected subdomain in $\mathbb{C}$, and h (modulo arbitrary constant) being an random element in $\mathcal{D}_0'(D)$, under the equivalence relation described as follows: Two random surfaces (D', h') and (D, h) are considered equivalent iff there exists a conformal isomorphism such that $\psi : D \to D'$ and $h = h' \circ \psi - \chi \arg |\psi'|$ holds a.s..*

The 'equivalence' in Definition 5.1 actually provided us a useful tool, which is that it suffices to study the case $D = \mathbb{H}$, thanks to the Riemann mapping theorem. To complete the entire picture of the flow line arguement, we ask the reciprocal question: given a (random) curve living in $\mathbb{H}$ starting from 0, how can we determine whether it is the unique flow line of some function on $\mathbb{H}$?

For a simple curve $\eta|_{[0,t]}$ with $\eta(0) = 0$ and $\eta|_{(0,t]} \subset \mathbb{H}$, we consider the corresponding Loewner flow $g|_{[0,t]}$ (t-parametrized unique conformal mappings from $\mathbb{H}/\eta|_{[0,t]}$ to $\mathbb{H}$ satisfying $\lim_{z\to\infty} |g_t(z) - z| = 0$, see Subsection 1.2). Then according to Loewner's theorem, g_t obeys the ODE

$$\partial_t g_t(z) = \frac{2}{g_t - W_t}, \quad g_0 = \mathrm{Id}\,. \tag{5.3}$$

where W_t is a real valued function uniquely determined by g_t and $\eta|_{[0,t]}$. Indeed W_t is exactly the image of the tip of $\eta|_{[0,t]}$ under the mapping g_t, i.e., $W_t = g_t(\eta(t))$. Moreover, we can actually make such geometric picture a bit more symmetric by shifting W_t to 0. Without loss of generality, consider the centered Loewner flow $f_t = g_t - W_t$, then the ODE becomes

$$df_t(z) = \frac{2}{f_t(z)}dt - dW_t, \quad f_0 = \mathrm{Id}\,. \tag{5.4}$$

Without loss of generality, we impose the condition that η starts off with a nearly vertical line, i.e., at $\frac{\pi}{2}$-direction. If $\eta|_{[0,t]}$ is exactly some flow lines corresponding to a nice function (e.g., H^2 regularity), then it should be the integral curve under $e^{i(h/\chi+\pi/2)}$.

More precisely, we propose the statement that $\eta|_{[0,t]}$ is a flow line of function $e^{i(h/\chi+\pi/2)}$ is implied by the relation

$$\lim_{z\to x-} \chi \arg f_t'(z) = -h(x) - \chi\pi/2, \quad \forall x \in \eta((0,t)). \tag{5.5}$$

and

$$\lim_{z\to x+} \chi \arg f_t'(z) = -h(x) + \chi\pi/2, \quad \forall x \in \eta((0,t)). \tag{5.6}$$

where the first limit is given by approaching x from the left, and the second one is by approaching x from the right. To proof this assertion, suppose s_- and s_+ are the two images $f_t(0)$, then $(s_-,0)$ and $(0,s_+)$ present two parametrizations for $\eta((0,t))$. For example, let $\phi(s) = s_+ - s$, then we have $\eta(s) = f_t^{-1}(\phi(s))$, by applying Equation (5.2) to $\psi := f_t^{-1}$

$$(f_t \circ \eta)'(s) = e^{i[h(\eta(s))/\chi - \arg(f_t^{-1})'(s)+\pi/2]} = -1$$

If we choose the argument to be in $(-\pi,+\pi]$, notice that $\arg(f_t^{-1})' = \arg(f_t')^{-1} = -\arg(f_t')$, then we have $h/\chi + \arg(f_t') + \pi/2 = \pi$. In fact, this is not precisely correct, since f_t is not defined for boundary points such as $\eta(s)$, this identity is only attained in the sense of right limit, leading to the completion of the proof for (5.6) (the strategy for (5.5) is similar).

In conclusion, in order to determine that $\eta((0,t))$ is indeed the flow line for some h with certain angle θ, it suffices to check the behavior of its Loewner flow near the curve. This again demonstrates the power of Loewner's methods in complex analysis. In addition, we see that f_t is really not extendable to the boundary, due to the 'height gap' of function h on the two sides of the curve.

For the study of random conformal geometry, we are mainly interested in the case that h is taken to be the Gaussian free field. Following the above explanation on the deterministic case, we embark on our first task in this section: show that SLE curves are indeed the random flow lines of GFF imaginary geometry class! To understand what exactly is a random flow line in such SLE & GFF coupling, we have to do some additional preparations.

5.1 $SLE_\kappa(\underline{\rho})$ and GFF revisited

In this subsection, we would like to introduce a natural variant of SLE_κ, by considering the case in which the driving function has certain interactions with some of its Loewner flow lines. In the context of Subsection 1.2, W_t is a Brownian motion and drives the conformal maps g_t. Suppose now W_t is affected by the motion of some reference points $\{x_1, ..., x_n\} \subset \bar{\mathbb{H}}$, and the entire motion can be solved via the SDEs

$$dW_t = \sum_{i=1}^{n} \operatorname{Re} \frac{-\rho_i}{g_t(x_i) - W_t} dt + \sqrt{\kappa} dB_t \tag{5.7}$$

$$dg_t(x_i) = \frac{2}{g_t(x_i) - W_t} dt, \quad g_0(x_i) = x_i. \tag{5.8}$$

If such stochastic Loewner evolution is defined (up to some stopping time), then the maps g_t is called a forward $SLE_\kappa(\underline{\rho})$ with forcing points $\{x_i\}$ and weights $\{\rho_i\}$. Similarly, if

$$d\tilde{W}_t = \sum_{i=1}^{n} \operatorname{Re} \frac{-\rho_i}{\tilde{g}_t(x_i) - \tilde{W}_t} dt + \sqrt{\kappa} dB_t \tag{5.9}$$

$$d\tilde{g}_t(x_i) = -\frac{2}{\tilde{g}_t(x_i) - \tilde{W}_t} dt, \quad \tilde{g}_0(x_i) = x_i. \tag{5.10}$$

Then $\tilde{g}_t$ is called a reverse $SLE_\kappa(\underline{\rho})$ with forcing points $\{x_i\}$ and weights $\{\rho_i\}$. When first encountering this definition, one may be confused about this setting and the geometric insight seems to be non-obvious. Informally speaking, the centered reverse flow $\tilde{f}_t := \tilde{g}_t - \tilde{W}_t$ could be understood as the reverse mapping of the centered forward flow $f_t := g_t - W_t$, see Figure 7. In the following, based on an alternative explanation, we show that the motion of forcing points in (5.7), (5.8), (5.9), (5.10) could be encoded by Bessel processes.

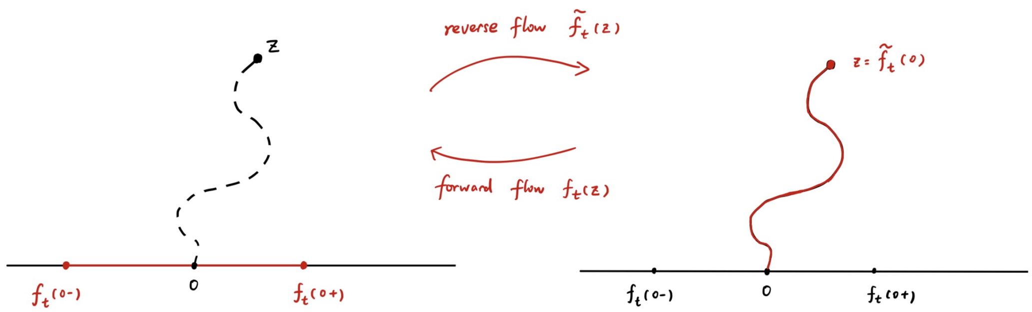


Figure 7: Forward v.s. Reverse

For simplicity, we consider the easiest forward model, which only involves one forcing point $x_0 \geq 0$ and is depicted by the SDE:

$$dW_t = \frac{-\rho}{V_t - W_t} ds + \sqrt{\kappa} dB_t, \quad W_0 = 0, \quad \rho > -2. \tag{5.11}$$

$$dV_t = \frac{2}{V_t - W_t} dt, \quad V_0 = x_0 \geq 0. \tag{5.12}$$

Let $\delta = 1 + \frac{2(\rho+2)}{\kappa} > 1$, we see that $\sqrt{\kappa} X_t := V_t - W_t$ satisfies the SDE:

$$dX_t = dB_t + \frac{\delta - 1}{2X_t} dt \tag{5.13}$$

Identically speaking, X_t is a BES^δ process. Since $\delta > 1$, the corresponding Bessel process should be instantaneously reflecting at 0. This means that as long as $\rho > -2$, the forward $SLE_\kappa(\rho)$ with one forcing point x_0 is a.s. defined, and the strong solutions to (5.11), (5.12) exists with X_t being a semimartingale. Thus the terminology 'forward' means that the Loewner flow line for x_0 is a.s. ahead of the driving function and the relative distance is a Bessel process. For the reverse case, similar arguments shows that the Loewner flow line for x_0 is a.s. behind the driving function, and if we reflect it with respect to 0, the relative distance will be a Bessel process.

However, things gets more complicated if we consider multiple forcing points. For simplicity, we still consider the case in which the forcings are restricted on the real line. Let $q \in \{L, R\}$ be the indicator for left and right, suppose $\underline{x}^L = (x^{k,L} < \cdots < x^{1,L} \leq 0)$ and $\underline{x}^R = (0 \leq x^{1,R} < \cdots < x^{\ell,R})$. For each forcing point $x^{i,q}$, we associate it with a weight $\rho^{i,q} \in \mathbb{R}$ and denote the weight vector by $\underline{\rho}$. Consider the following SDE:

$$W_t = \sum_{q \in \{L,R\}} \sum_i \int_0^t \frac{-\rho_i}{V_s^{i,q} - W_s} ds + \sqrt{\kappa} B_t \tag{5.14}$$

$$V_t^{i,q} = \int_0^t \frac{2}{V_s^{i,q} - W_s} ds + x^{i,q} \tag{5.15}$$

Define a stopping time

$$\tau_c = \inf\{t > 0; \sum_{i, V_t^{i,L} = W_t} \rho^{i,L} \leq -2 \;\; or \;\; \sum_{i, V_t^{i,R} = W_t} \rho^{i,R} \leq -2\} \tag{5.16}$$

We see that, as long as $t < \tau_c$, the 'averaged' position (averaging over the weight $\underline{\rho}$) of forcing points is still 'ahead' of the driving function, and the distance behave as a BES^δ process (an excursion) with $\delta > 1$. But if we exceed the *continuation threshold* given by τ_c, the 'averaged' relative distance is no longer instantaneously reflecting at 0, which means that it is no longer a semimartingale. In fact, the solutions to (5.14) and (5.15) after the continuation threshold do not exist in the strong sense.

Now under these heuristic interpretations, we state the existence theorem for the most general forward and reverse $SLE_\kappa(\underline{\rho})$.

Definition 5.2. *Let B_t denotes the standard Brownian motion, suppose we have continuous processes W_t and $g_t(x_i)$ with $\{x_1, ..., x_n\} \in \bar{\mathbb{H}}$. Then the couple $(W_t, g_t(x_i), i \in [\![1, n]\!])$ is said to describe a forward (resp. reverse) $SLE_\kappa(\underline{\rho})$ if they are adapted to the natural filtration of $(W_t, B_t, g_t(x_i), i \in [\![1, n]\!])$ and up to the continuation threshold given by* (5.16) *(if necessary, take the real part of x_i), we have*

(i) For every stopping time τ such that $W_\tau \neq g_\tau(x_i)$ a.s., the processes $(W_t, g_t(x_i), i \in [\![1, n]\!])$ compose a strong solution to (5.7) *(resp.* (5.9)*) in the random time interval $[\tau, \sigma]$ where σ is the first colliding time (x_i collides into x_j for some $i \neq j$) after τ.*

(ii) The processes $g_t(x_i) - W_t$ are instantaneously reflecting at 0 for all $i \in [\![1, n]\!]$.

(iii) For each t, differential identities (5.8) *(resp.* (5.10)*) hold almost surely.*

Theorem 5.1. *Suppose the processes $(W_t, g_t(x_i), B_t, i \in [\![1, n]\!])$ satisfy Definition 5.2, then the corresponding joint law is uniquely determined. Under this law, the joint process is a continuous Markov process in the state space up to the continuation threshold.*

Next, we revisit the zero-boundary Gaussain free field h on D, and demonstrate some useful 'function properties'. Recall that the continuum GFF is actually the 2-dimensional-time analog of the Brownian motion. Motivated by the Markov property of Brownian motion, we naturally conjecture that given an instance of h on a sub-domain $U \subset D$, the law of h conditioned on the chosen instance should be another independent GFF on D/U. Indeed, this is called the domain (2-dim time domain) Markov property. Moreover, for the analogous strong Markov property for h, see Lemma 5.5 after we introduced the concept of local sets.

Proposition 5.1. *(Domain Markov Property) For a zero-boundary Gaussian free field h on D, suppose $U \subset D$ is an open subset, then there's a decomposition*

$$h = h_U + h_{U^c},$$

such that

(i) h_U is a zero-boundary Gaussian free field on U, which is also a.s. supported on U.

(ii) h_{U^c} is a random distribution which is harmonic in U and independent of h_U.

In other words, given a sampling $\varphi = h(\omega)|_{D/U}$, $\omega \in \Omega$, the conditional law of $h|_U$ is composed of a Gaussian free field in U which is zero on D/U, plus a random harmonic extension of φ (defined on D/U) to U.

The domain Markov property can also be interpreted in the language of Sobolev spaces, and with the help of orthogonal basis, we can explicitly calculate the harmonic part in the decomposition. With the notations defined above, let $H_0^{1,\perp}(U)$ be the set of distributions in $H_0^1(D)$ that is harmonic in U. Suppose $f \in H_0^1(U)$ and $g \in H_0^{1,\perp}(U)$, we see that $(f,g)_\nabla = -(f,\Delta g) = 0$, this explains the reason for the upper index $\perp$. Moreover, one has the orthogonal decomposition $H_0^1(D) = H_0^1(U) \oplus H_0^{1,\perp}(U)$. Thus from Proposition 5.1, we have

$$h = h_U + h_{U^c} = \sum_n \alpha_n^U f_n^U + \sum_n \alpha_n^{U^c} f_n^{U^c}$$

where $\alpha_n^U, \alpha_n^{U^c}$ are i.i.d standard Gaussians, and (f_n^U), $(f_n^{U^c})$ are orthonormal bases for $H_0^1(U)$ and $H_0^{1,\perp}(U)$ respectively.

Similar to the Brownian motion, we would like to introduce 'natural filtration' for the GFF. Let $\mathcal{F}_U^h$ be the sub-σ-algebra generated by the family $(\alpha_n^U f_n^U)$, For a closed or compact set $K \subset D$, we define $\mathcal{F}_{K+}^h$ to be the sub-σ-algebra generated by the family $(\alpha_n^{(D/K)^c} f_n^{(D/K)^c})$. Due to the continuity of trajectories, the natural filtration for the Brownian motion on a complete probability space is proved to be continuous. Similarly, we have the following property for the GFF:

Proposition 5.2. *For any deterministic, closed subset $K \subset D$, we have*

$$\mathcal{F}_{K+}^h = \cap_{U \supset K} \mathcal{F}_U^h. \tag{5.17}$$

where the intersection is taken over all open sets $U \supset K$.

Proof. Without loss of generality, we assume D is a bounded domain, otherwise we can use the conformal invariance of GFF and Riemann mapping theorem to reduce the complexity. Let (ϕ_n) be an orthonormal basis for $H_0^{1,\perp}(D/K)$ and for each n, let $\alpha_n := (h,\phi_n)_\nabla$. We immediately know that $\mathcal{F}_{K+}^h = \sigma(\alpha_n; n \in \mathbb{N})$, and we want to show that $\mathcal{F}_{K+}^h \subset \mathcal{F}_U^h$ for any $U \subset D$ being open neighborhood of $\partial D \cup K$. To prove this, we take a mollifier function ψ and define $\tilde{\phi}_{n,m} = \phi_n * \psi_m$, where ψ_m is a rescaled function with support radius $1/m$. Let $\phi_{n,m}$ be given by $\tilde{\phi}_{n,m}$ minus the harmonic extension of its value on the boundary ∂D, then we see that $\phi_{n,m} = \phi_n$ for all $z \in D$ with $\mathrm{dist}(z, K \cup \partial D) \geq 1/m$. This leads to the result $\alpha_{n,m} := (h,\phi_{n,m})_\nabla$ converges to α_n in L^2 as $m \to \infty$, hence the first step is completed.

Next, we show that $\cap_V \mathcal{F}_V^h$ is equal to $\{\varnothing, \Omega\}$, where $V \subset D$ is any open neighborhood of ∂D. This is because the Green's function for Δ on $D/\bar{V}$ converges to the Green's function on D, and the projection of h on the space $H_0^1(V)$ converges to zero as $\bar{V}$ decreases to the boundary ∂D. This shows that we have $\mathcal{F}_{K+}^h \subset \mathcal{F}_U^h$, where $U \subset D$ is any open neighborhood of K.

To conclude the proposition, it suffices to show $\mathcal{F}_{K+}^h = \cap_{U \supset K} \mathcal{F}_{\bar{U}+}^h$. In fact, for any $\xi \in C_c^\infty(D)$, we only have to show that $\mathbb{E}\big[(h,\xi)_\nabla | \mathcal{F}_{\bar{U}_m+}^h\big]$ converges a.s. to $\mathbb{E}\big[(h,\xi)_\nabla | \mathcal{F}_{K+}^h\big]$ as $m \to \infty$, where U_m is the $1/m$-neighborhood of K. To compute the LHS, we take the orthonormal bases (ϕ_n), $(\tilde{\phi}_n)$ corresponding to the decomposition $H_0^1(D) = H_0^1(D/K) \oplus H_0^1(K)$, and use the formal Parseval identity $(h,\xi)_\nabla = \sum_n (h,\phi_n)_\nabla (\phi_n,\xi)_\nabla + \sum_n (h,\tilde{\phi}_n)_\nabla (\tilde{\phi}_n,\xi)_\nabla$. The proof follows from Martingale convergence theorem, which indicates that the sum over '$\tilde{\phi}_n$-terms' will be vanished under the limit. $\square$

Suppose $K \subsetneqq D$ is a relatively compact subset, and $z_0 \in \partial D/K$, pick a neighborhood $U \ni z_0$ such that it has positive distance from K. It is shown that the projection of h onto $H_0^{1,\perp}(D/K)$ and $H_0^{1,\perp}(D/U)$ are two *almost independent* random distributions, this makes sense because according to domain Markov property, they are 'almost' independent GFFs living in disjoint domains. Here almost independent means that, for almost all instances of the former, the conditional law of the latter is absolutely continuous with respect to its own unconditioned law. Let h_1 be any instance of the projection of h onto $H_0^{1,\perp}(D/K)$, we naturally guess that h_1 should be 'almost constant' near the boundary $\partial D/K$.

Proposition 5.3. *Assume D is a non-trivial simply connected domain and K be a deterministic relatively closed subset. Then a.s. given any instance h_1 defined above, we have*

$$\lim_{D/K \ni z \to z_0} h_1(z) = 0, \quad \forall z_0 \in \partial D/K.$$

Proof. By almost independence, given a deterministic sampling h_1, $h|_U$ can be written as the sum of h_1 and a zero-boundary GFF $\tilde{h}$ restricted to U. Pick any $z_n \in D/K$ approaching z_0, without loss of generality, assume they all lie in U. Construct a sequence of conformal isomorphisms $\varphi_n : U \to \mathbb{D}$, such that $\varphi_n(z_n) = 0$ and $\varphi_n'(z_n) > 0$. Let μ_n be the law of $h \circ \varphi_n^{-1}$, $\tilde{\mu}_n$ be the law of $\tilde{h} \circ \varphi_n^{-1}$, and $\tilde{\mu}_n'$ be the law of $(\tilde{h} + h_1) \circ \varphi_n^{-1}$. Then we claim that almost surely, fix h_1, the laws μ_n and $\tilde{\mu}_n'$ are mutually absolutely continuous, with Radon-Nikodym derivatives 'almost uniform in n' (meaning that for fixed $\varepsilon > 0$, there exists a fixed $\delta > 0$, such that whenever $\mu_n(a) < \delta$, we have $\tilde{\mu}_n'(A) < \varepsilon$).

Next, we define a different kind of mollifier function. For any $\zeta > 0$, let ϕ_ζ be a C_c^∞ function which is non-negative, normalized, radially symmetric and supported in $\mathbb{D}/B(0, 1-\zeta)$. Then $(h \circ \varphi_n^{-1}, \phi_\zeta)_\nabla$ and $(\tilde{h} \circ \varphi_n^{-1}, \phi_\zeta)_\nabla$ converges to 0 in probability as $\zeta \to 0$. By harmonicity, and mean-value property, $(h_1 \circ \varphi_n^{-1}, \phi_\zeta)_\nabla = h_1(z_n)$.

For fixed $\delta > 0$ such that $\mu_n(A) < \delta$ implies $\tilde{\mu}_n'(A) < 1/4$, then for any $a > 0$, we can pick $\zeta_0 > 0$ and n_0, such that for all $\zeta \in (0, \zeta_0)$ and $n \geq n_0$, we have $\mu_n[|(h \circ \varphi_n^{-1}, \phi_\zeta)_\nabla| > a/2] < \delta$, and $\tilde{\mu}_n'[|((\tilde{h} + h_1) \circ \varphi_n^{-1}, \phi_\zeta)_\nabla| > a/2] < 1/4$. By slight modification, we can also have $\tilde{\mu}_n[|(\tilde{h} \circ \varphi_n^{-1}, \phi_\zeta)_\nabla| > a/2] < 1/4$. Since the event $|h_1(z_n)| > a$ is the same as the event $|((\tilde{h} + h_1) \circ \varphi_n^{-1}, \phi_\zeta)_\nabla - (\tilde{h} \circ \varphi_n^{-1}, \phi_\zeta)_\nabla| > a$, and the former event is independent of $\tilde{h}$, we can integrate it out and this yields the inequality

$$\begin{aligned}
1_{\{|h_1(z_n)|>a\}} &= 1_{\{|((\tilde{h}+h_1)\circ\varphi_n^{-1}, \phi_\zeta)_\nabla - (\tilde{h}\circ\varphi_n^{-1}, \phi_\zeta)_\nabla|>a\}} \\
&= \tilde{\mu}_n[|((\tilde{h} + h_1) \circ \varphi_n^{-1}, \phi_\zeta)_\nabla - (\tilde{h} \circ \varphi_n^{-1}, \phi_\zeta)_\nabla| > a] \\
&\leq \frac{1}{2}
\end{aligned}$$

This shows that $|h_1(z_n)| \leq a$ for all $n \geq n_0$, take $n \to \infty$ and $a \to 0$, the proof will be completed.

□

The last property for the GFF shows the consequences of the case when two GFFs are defined on intersecting domains. The expected results will be, roughly speaking, the two distributions are mutually absolutely continuous on the intersections. We now state the proposition without presenting the proof, which is largely based on orthonormal expansion techniques.

Proposition 5.4. *(Absolute Continuity) Let D_1 and D_2 be two intersecting simply connected domains. For $i = 1, 2$, let h_i be GFFs defined on D_i and F_i are arbitrary harmonic functions on D_i. Fix a bounded simply connected open subset $U \subset D_1 \cap D_2$, we have*

(i) If $\operatorname{dist}(U, \partial D_i) > 0$ for all $i = 1, 2$, then the laws of $(h_1 + F_1)|_U$ and $(h_2 + F_2)|_U$ are mutually absolutely continuous.

(ii) Suppose there exists a neighborhood U' of $\bar{U}$ such that $\bar{D}_1 \cap U' = \bar{D}_2 \cap U'$, and $F_1 - F_2$ approaches 0 as one approaches $\partial(D_i \cap U')$. Then the laws of $(h_1 + F_1)|_U$ and $(h_2 + F_2)|_U$ are mutually absolutely continuous.

5.2 Martingale characterization of $SLE_\kappa(\underline{\rho})$ & Local sets of GFF

The definition (Definition 5.2) for $SLE_\kappa(\underline{\rho})$ is in fact not a useful characterization, because in order to check it, one needs to examine a large system of SDEs very carefully. However, there's in fact an alternative martingale charaterization for the criteria listed in Definition 5.2, which reveals some geometry of the harmonic part of the SLE-GFF coupling. In the follow context, we only consider the real valued forcing points (see (5.14) and (5.15)) with $\kappa > 0$. Since in this case, the motion for the forcing is only restricted to the boundary of $\mathbb{H}$, the candidate harmonic random distribution is naturally constructed by such boundary data for the $SLE_\kappa(\underline{\rho}^L;\underline{\rho}^R)$. Let $\mathfrak{h}_t^0$ be the harmonic extension to $\mathbb{H}$ with boundary value:

$$\mathfrak{h}_t^0(x) = \begin{cases} -\lambda\left(1+\sum_{i=0}^{j}\rho^{i,L}\right), & x \in [V_t^{j+1,L}, V_t^{j,L}) \\ \lambda\left(1+\sum_{i=0}^{j}\rho^{i,R}\right), & x \in [V_t^{j,R}, V_t^{j+1,R}) \end{cases} \tag{5.18}$$

Suppose K_t is the stochastic compact h-hall encoded by the centered Loewner flow $f_t := g_t - W_t$, the harmonic random distribution $\mathfrak{h}_t$ defined on the domain $\mathbb{H}/K_t$ is given by the 'pull back' of $\mathfrak{h}_t^0$ in the sense of equivalence of random surfaces (Definition 5.1):

$$\mathfrak{h}_t(z) = \mathfrak{h}_t^0(f_t(z)) - \chi \arg f_t'(z), \quad \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}. \tag{5.19}$$

The next theorem shows that $\mathfrak{h}_t$ is all we need for the study of $SLE_\kappa(\underline{\rho}^L;\underline{\rho}^R)$.

Theorem 5.2. *(Martingale characterization) Suppose we are given a random compact h-hall evolution K_t with end point curve η_t on $\bar{\mathbb{H}}$ starting from 0 ending towards ∞. In addition, suppose also that the driving function W_t for Loewner flow g_t is a continuous stochastic process. Then if there exists particles $x^{i,q}$ on the real line with images $V_t^{i,q} = g_t(x^{i,q})$, weights $\rho^{i,q} \in \mathbb{R}$ and a Brownian motion B_t, the coupled process $(W_t, V_t^{i,q}, B_t)$ satisfies Definition 5.2 (i.e., η_t is a $SLE_\kappa(\underline{\rho}^L;\underline{\rho}^R)$) if and only if $\mathfrak{h}_t(z)$ is a continuous local martingale for each $z \in \mathbb{H}$ until z is absorbed by K_t.*

Proof. We would like to show the reverse statement by checking the conditions in Definition 5.2. In the following sketch of proof, suppose $\mathfrak{h}_t(z)$ is a continuous local martingale for any fixed $z \in \mathbb{H}$ in the random time interval $[0, \inf\{t > 0, \eta_t = z\}]$.

(i) First, the harmonic conjugate $\tilde{\mathfrak{h}}_t(z)$ is in fact another continuous local martingale, this shows that, if we apply Ito's formula to the complex local martingale $\mathfrak{h}_t^*(z) = (-\tilde{\mathfrak{h}}_t(z), \mathfrak{h}_t(z))$, the process W_t must be a semimartingale. Furthermore, if we write decomposition as $W_t =$

$v_t + m_t$, apply again the Ito's formula for $\mathfrak{h}_t^*(z)$ and Levy's characterization, we claim that for any non-colliding ($V_\tau^{i,q} \neq W_\tau, \forall i, q$) stopping time τ, the local martingale part is given by $m_t = \sqrt{\kappa} B_{\tau+t}^\tau$ (here $\tau + t$ must not exceed the nearest colliding time) and W_t is expanded as (5.14). Notice that this only holds locally in some interval $(\tau, \tau + \Delta t)$, and the Brownian motion $B_{\tau+\cdot}^\tau$ is dependent on the choice of stopping time.

(ii) One could imagine that before the continuation threshold, there could be many different colliding times. We have to show that $B_{\tau+\cdot}^\tau$ can be realized by one Brownian motion B_t living on the same probability space with $(W_t, V_t^{i,q})$. This provides a nice coupling $(W_t, V_t^{i,q}, B_t)$, and between each pair of colliding times (τ, σ), we a.s. have

$$W_{\tau+t} - W_\tau = \int_\tau^{\tau+t} \sum_{i,q} \frac{\rho^{i,q}}{W_s - V_s^{i,q}} ds + \sqrt{\kappa}(B_{\tau+t} - B_\tau), \quad t \in [0, \sigma - \tau].$$

To conclude that strong solution exists, we only need to show that B_t is an $\mathcal{F}_t$-Brownian motion, where $\mathcal{F}_t := \sigma(W_s, V_s^{i,q}, B_s, s \leq t)$. This is achieved by showing that $B|_{[\tau,\infty)} - B_\tau$ is a Brownian motion independent of $(W_\tau, V_\tau^{i,q})$.

(iii) Next, we show the instantaneous reflection at 0. Suppose at time t the driving process W_t hits the flow $V_t^{i,q}$ for some forcing point with coordinate (i, q), then since the tip of η_t should be mapped to W_t, it could only be the forcing point $x^{i,q}$ in question, i.e., η_t must hit the real line at $x^{i,q}$. Thus it suffices to show that the 'local time' $\{t; \eta_t \in \mathbb{R}\}$ has zero Lebesgue measure. Recall (see Subsection 1.2) the increase of half-plane capacity $\mathrm{hcap}(\eta|_{[0,t]})$ is continuous and strictly increasing in t, we show that for each fixed $R, \delta > 0$, the capacity $\mathrm{hcap}(\eta|_{[0,t]} \cap \{z = x + iy, x \in (-R, R), y \in (0, \delta)\})$ converges to 0 as $\delta \to 0$. The proof will be complete after we take $R \to \infty$.

(iv) Last, we show that $V_t^{1,L}$ is almost surely the Loewner flow for the forcing point $x^{1,L}$. We already have that on each time interval (s, t) where none of the endpoints are colliding times, $V_t^{1,L}$ almost surely solves (5.15), and $V_t^{1,L}$ is non-increasing for all $t \geq 0$. This gives us the non-decreasing process

$$I_t := \int_0^t \frac{2}{V_s^{1,L} - W_s} ds - V_t^{1,L} \geq 0, \quad a.s..$$

To show that I_t is a.s. 0, we first observe that I_t can only grow 'around' colliding times $\{t; V_t^{1,L} = W_t\}$, otherwise the SDE holds and I_t remains constant. We take any $z_0 \in \mathbb{H}$ and $\varepsilon > 0$, construct a bounded stopping time T such that a.s. in $[0, T]$, $\mathrm{Im}(g_t(z_0))$ is lower bounded, $\mathfrak{h}_t(z_0)$ has bounded change. Take time instances $0 < s_1 < t_1 < \cdots < s_k < t_k < T$ where s_i is the colliding time for $x^{1,L}$ and t_i is the immediate subsequent time that $V_t^{1,L}$ has distance ε from W_t. We claim that the total amount of change for I_t in $\cup_i(s_i, t_i)$ is of same order (in ε) as the net change of $\mathfrak{h}_t(z_0)$. Since $\mathfrak{h}_t(0)$ is a bounded local martingale in $[0, T]$, use a simple Borel-Cantelli argument, one can show that its net fluctuation converges to 0 as $\varepsilon \to 0$. Repeat this methodology after T, we see that $I_t = 0, a.s.$.

So far all three conditions in Definition 5.2 are satisfied, and the proof is complete. □

For the second part of this subsection, we briefly survey the theory of local sets of GFF. Suppose $D \subset \mathbb{C}$ is a bounded domain and h is a GFF on D. Let Γ denotes the set of all non-empty closed subsets of $\bar{D}$ with the requirement that $\forall A \in \Gamma$, we have $\partial D \subset A$. The set Γ can be endowed with a Hausdorff distance:

$$d_{\mathrm{HAU}}(A, B) := \max\left[\sup_{x\in A} \mathrm{dist}(x, B), \sup_{y\in B} \mathrm{dist}(y, A)\right], \quad \forall A, B \in \Gamma. \tag{5.20}$$

One can show that $(\Gamma, d_{\mathrm{HAU}})$ is a compact metric space, let $\mathcal{G}$ denotes the corresponding Borel σ-algebra. We make the following formal definition.

Definition 5.3. *Suppose we're given a 2-component coupled (means that the joint distribution is a suitable 'mixture' of the marginals) random element (A, h), where A takes value in Γ and h is a GFF on D. Then A is called the* local set *of h iff there exists a 2-component coupled random element (A, h_1) such that h_1 is a random distribution harmonic in D/A and*

$$(A, h) \stackrel{d}{=} (A, h_1 + h_2) \tag{5.21}$$

where h_2 is another independent zero-boundary GFF on $\mathrm{int}(D/A)$ *which is sampled after the sampling of the random compact set A.*

Next, we try to make sense of the restriction $h|_A$ of a GFF onto the local set. We will eventually see that, the correct interpretaion to this informal projection is given by the conditional expectation $\mathcal{C}_A = \mathbb{E}(h|A)$. Fix any $\delta > 0$, let A_δ denote the closed subset of D such that it contains all the points whose distance from A is at most δ, then we note that the mappings

$$(A, h) \to (A_\delta, h) \to h_{\mathrm{int}(A_\delta)}$$

are all continuous with respect to the topology $(d_{\mathrm{HAU}}, weak - *)$, hence are measurable. Let $\mathcal{A}_\delta$ denote the sub-σ-algebra generated by the random distribution $h_{\mathrm{int}(A_\delta)}$, and define $\mathcal{A} = \cap_{\delta>0}\mathcal{A}_\delta$.

We see that $\mathcal{A}$ is the smallest σ-algebra such that A and the values of h in an infinitesimal neighborhood of A are all measurable. A reader with keen observation might notice the similarity of $\mathcal{A}$ and $\mathcal{F}_\tau$ where τ is a stopping time. Let $\mathcal{C}_A = \mathbb{E}(h|\mathcal{A})$, and the conditional expectation is still a random distribution, then according to Definition 5.3, we heuristically assert $\mathcal{C}_A \overset{d}{=} h_1$. To be more rigorous, for any test function $\phi \in C_c^\infty(D)$, we have $(\mathcal{C}_A, \phi) = \mathbb{E}[(h,\phi)|\mathcal{A}] = (h_1, \phi)$. To sum up, we have exactly $\mathcal{A} = \sigma(h_1)$. This relation matches with Proposition 5.2, in the case that A is deterministic. We then have the following result which in fact presents characterisations of the local sets from different perspectives.

Proposition 5.5. *Given a coupling of local sets and zero-boundary GFF: (A, h), the following assertions are equivalent:*

(i) For any deterministic open subdomain $U \subset D$, conditioned on the projection of h to $H_0^{1,\perp}(U)$, the event $\{A \cap U = \varnothing\}$ is independent of $h|_U$. Identically speaking, given any instance of h, the conditional probability of $\{A \cap U = \varnothing\}$ is only determined by the value of such instance in D/U.

(ii) For any $U \subset D$ as above, we define another random element in Γ by

$$\tilde{A} = \begin{cases} A, & A \cap U = \varnothing. \\ \varnothing & A \cap U \neq \varnothing. \end{cases}$$

Then conditioned on the projection of h to $H_0^{1,\perp}(U)$, the random element $(1_{\{A\cap U \neq \varnothing\}}, \tilde{A})$ is independent of $h|_U$.

(iii) (Stochastic Domain Markov Property, in comparison with Proposition 5.1) Conditioned on $\mathcal{A}$, there exists a regular condition law of h that is identically distributed to $h_1 + h_2$, where $h_2 \overset{d}{=} h_{\mathrm{int}(D/A)}$ (similar to $h_{\mathrm{int}(A_\delta)}$) is a zero-boundary GFF on D/A. Meanwhile h_1 is harmonic in D/A, and can be understood as a measurable function (distribution valued function, and since all spaces have topologies, measurability is understood in the Borel σ-algebra) of the pair (A, h) such that the total composition is $\mathcal{A}$-measurable.

(iv) A is a local set of h.

The GFF & local set coupling can be heuristically understood as measurable function restricted to a measurable sets. Thus one expect that there could be a series of measure theory flavoured nice properties. First of all, local sets are 'closed under finite unions'.

Proposition 5.6. *Let (A_1, h) and (A_2, h) be two local sets for one GFF h on D. We can construct a random element $A = A_1 \tilde{\cup} A_2$ in Γ. The distribution of A is given as follows: first sample h, then conditioned on this sampling, we sample A_1 and A_2 independently according to their respective conditional laws. Then we conclude that (A, h) is also a local set.*

We see that if the local sets A_1 and A_2 are determined by h, then Proposition 5.6 reduces to the trivial fact that Borel sets are closed under finite union. So instead of this triviality, which would be out of interest, we focus on the near-trivial case. A local set (A, h) is called almost surely determined by h if there exists a modification of A which is $\sigma(h)$-measurable. Note that under this proposition, we have naturally required that A_1 and A_2 are independently conditioned on an instance of h. In this setting, the situations on different connected components 'cutout by local sets' are 'independent'.

Proposition 5.7. *Let D be a bounded simply connected domain and A_1, A_2 be two connected local sets that are conditionally independent given h. Suppose C is a $\sigma(A_1)$-measurable open sets in D/A_1 which can be expressed as union of components that don't intersect with A_2 (conditioned on A_1, randomly pick the components, avoiding A_2). Then we have $\mathcal{C}_{A_1 \tilde{\cup} A_2}|_C = \mathcal{C}_{A_1}|_C$ a.s.. In particular, conditioned on $\mathcal{A}_1$, $h|_C$ is independent of the pair $(h|_{D/C}, A_2)$.*

The next result provides us a nice property for the restriction of GFF h to different local sets A_1, A_2.

Proposition 5.8. *Let D be a bounded simply connected domain and A_1, A_2 be two connected local sets. The random distribution $\mathcal{C}_{A_1 \tilde{\cup} A_2} - \mathcal{C}_{A_2}$ is a.s. harmonic in $D/(A_1 \tilde{\cup} A_2)$ and moreover:*

(i) If $z \in D/(A_1 \tilde{\cup} A_2)$ converges to a point in one connected components of A_2/A_1, then $\mathcal{C}_{A_1 \tilde{\cup} A_2}(z) - \mathcal{C}_{A_2}(z)$ tends to 0 a.s..

(ii) If $z \in D/(A_1 \tilde{\cup} A_2)$ converges to a point in one connected components of $A_2 \cap A_1$ which has positive distance from either A_2/A_1 or A_1/A_2, then $\mathcal{C}_{A_1 \tilde{\cup} A_2}(z) - \mathcal{C}_{A_2}(z)$ tends to 0 a.s..

Last, we show that if a local set A is 'thin enough', then it does not 'affect' the distribution of the GFF. More precisely, h is almost surely determined by its restriction to D/A.

Proposition 5.9. *Let (A, h) be a local set such that for any compact subset $K \subset D$, there exists a sequence of positive numbers $\delta_k \to 0$ as $k \to \infty$, such that for each k, the size of the smallest finite cubic cover of $A \cap K$, i.e., $\#\{\xi_n + \delta_k[0,1]^2; \xi_n \in \delta_k \mathbb{Z}^2, (\xi_n + \delta_k[0,1]^2) \cap A \cap K \neq \varnothing\}$, is of order $o(\delta_k^{-2}(\log \delta_k^{-1})^{-1})$. Then the instance of h is a.s. determinede by its restriction to D/A.*

Proof. Before we get to the proof, let's settle some notations. For a given compact set $K \subset D$, $\delta > 0$, and $\phi \in C_c^\infty(D)$ with $\operatorname{supp}\phi \subset K$, we define $\mathcal{D}_\delta$ to be the set of all half-open cubes on the lattice $\delta\mathbb{Z}^2$ that are contained in K, define A_δ to be the union of the subfamily of $\mathcal{D}_\delta$ consisting of cubes with non-empty intersection with A.

It is straightforward to see that for mollified functions $\phi_\delta = \bar{\phi} * \eta_{\frac{\delta}{4}}$, where $\bar{\phi} = \sum_{\chi_n \in \mathcal{D}_\delta} \fint \phi \chi_n$ denotes the cube-averaged function, we have convergence for almost surely all samples of A:

$$\lim_{\delta\to 0} \phi_\delta = \phi, \quad \lim_{\delta\to 0} \phi_\delta 1_{A_\delta^c} = \lim_{\delta\to 0} \phi_\delta, \quad a.s..$$

in the topology of $C_c^\infty(D)$. Note that the second convergence follows from our assumption on A. By definition of GFF, we have corresponding convergence:

$$\lim_{\delta\to 0}(h, \phi_\delta)_\nabla = (h,\phi)_\nabla, \quad \lim_{\delta\to 0} |(h|_{D/A}, \phi_\delta 1_{A_\delta^c})_\nabla - (h,\phi_\delta)_\nabla| = \lim_{\delta\to 0} |(h, \phi_\delta 1_{A_\delta^c})_\nabla - (h,\phi_\delta)_\nabla|.$$

both in the L^2 sense and in the a.s. sense (possibly along a subsequence).

By an 'almost' circle average argument, we claim that the variance of $(h, \delta^{-2} 1_{\chi_n})_\nabla$ is of order $O(\log\delta^{-1})$. Apply Gaussian tail bound, we see that uniformly in n and constant $c > 0$

$$\mathbb{P}\left((h, \delta^{-2}1_{\chi_n})_\nabla \geq c\log\delta^{-1}\right) \sim O(\sqrt{\delta})$$

Then by Borel-Cantelli lemma, there exists a subsequence $\delta_k \to 0$ such that $(h, \delta_k^{-2} 1_{A_{\delta_k}^c})_\nabla \leq c\log\delta_k^{-1}$ almost surely for all k. This give the following bound

$$|(h, \phi_{\delta_k} 1_{A_{\delta_k}^c})_\nabla - (h, \phi_{\delta_k})_\nabla| \leq \lim_{k\to\infty} \|\phi\|_\infty \times o(\delta_k^{-2}(\log\delta_k^{-1})^{-1}) \times \delta_k^2 \times c\log\delta_k^{-1} = 0, \quad a.s..$$

We then have an almost sure convergence

$$\lim_{k\to\infty}(h|_{D/A}, \phi_{\delta_k} 1_{A_{\delta_k}^c})_\nabla = \lim_{k\to\infty}(h|_{D/A}, \phi_{\delta_k})_\nabla = (h,\phi)_\nabla$$

The result follows since ϕ is arbitrary. □

5.3 Forward $SLE_\kappa(\underline{\rho})$ & GFF coupling

Under the definitions (5.14) and (5.15), there's a theorem by Miller and Sheffield in [14] saying that, the forward $SLE_\kappa(\underline{\rho})$ curves are naturally coupled to the imaginary geometry of the free boundary GFF on $\mathbb{H}$.

Theorem 5.3 (Forward SLE-GFF coupling)**.** *Fix any $\kappa > 0$, $\lambda = \frac{\pi}{\sqrt{\kappa}}$ and $(\rho^{i,L}, \rho^{i,R})$. Let K_t be the compact-$\mathbb{H}$-hull at time t of the forward $SLE_\kappa(\underline{\rho})$ process diven by* (5.14) *and* (5.15)*. In* *addition, let $\mathcal{F}_t$ be the natural filtration generated by $(W_t, B_t, g_t(x_i), i \in [\![1, n]\!])$, $\mathfrak{h}_t(z)$ be the random harmonic function in $\mathbb{H}$ given by* (5.19).

Suppose $\tilde{h}$ is a zero boundary GFF on $\mathbb{H}$, define $h = \tilde{h} + \mathfrak{h}_0$, then there exist a coupling (K, h), i.e., a joint law of (K_t) and h, such that: For any $\mathcal{F}_t$-stopping time τ which is a.s. smaller than the continuation threshold, the K_τ is a local set for h and the conditional law of $h|_{\mathbb{H}/K_\tau}$ given $\mathcal{F}_\tau$ equals to the law of $\mathfrak{h}_\tau + \tilde{h} \circ f_\tau$.

In the statement given above, the random generalized function $\mathfrak{h}_\tau + \tilde{h} \circ f_\tau$ lies in the same imaginary geometry class with h, and the conformal isomorphism between them is given by the centered forward flow f_τ. Equivalently speaking, Theorem 5.3 shows that if an observer is sitting on the tip of the forward $SLE_\kappa(\underline{\rho})$, then during the entire evolution process, the imaginary geometry in his vision will always be the the same (in the sense of Definition 5.1) as h.

Proof of Theorem 5.3. **Step 1**. Given $z \in \mathbb{H}$, let $\tau_z = \inf\{t \geq 1; g_t(z) \in \mathbb{R}\}$, then for any $t < \tau_z$ which is also below the continuation threshold, we apply the Ito's formula and get

$$df_t(z) = \left(\frac{2}{f_t(z)} - \sum_{q \in \{L,R\}} \sum_i \frac{\rho^{i,R}}{W_t - V_t^{i,q}} \right) dt - \sqrt{\kappa} dB_t \tag{5.22}$$

$$d \log f_t(z) = \left(\frac{4-\kappa}{2f_t^2(z)} - \sum_{q \in \{L,R\}} \sum_i \frac{\rho^{i,R}}{f_t(z)(W_t - V_t^{i,q})} \right) dt - \frac{\sqrt{\kappa}}{f_t(z)} dB_t \tag{5.23}$$

For the derivatives $f_t'(z)$ and $\log f_t'(z)$, we also get

$$df_t'(z) = -\frac{2f_t'(z)}{f_t^2)(z)} dt, \quad d \log f_t'(z) = -\frac{2}{f_t^2)(z)} dt \tag{5.24}$$

By martingale characterization, we know that $\mathfrak{h}_t(z)$ is a continuous local time up to the $\mathcal{F}_t$-stopping time τ_z. Let $\mathfrak{h}_t^*(z)$ be the complex local martingale as defined in the proof of Theorem 5.2, then the calculations given above leads us to

$$d\mathfrak{h}_t^*(z) = \frac{2}{f_t(z)} dB_t, \quad d\mathfrak{h}_t(z) = d \operatorname{Im} \mathfrak{h}_t^*(z) = \operatorname{Im} \left(\frac{2}{f_t(z)} \right) dB_t. \tag{5.25}$$

Step 2. Since $\mathfrak{h}_t(z)$ is a continuous local martingale, we would like to compute its quadratic variation. Using Dambis-Dubin-Schwarz theorem, we know that $\mathfrak{h}_t(z)$ is the time-change of a Brownian motion. However, it's better to do this by first constructing the variation process

$d\langle \mathfrak{h}_t(z), \mathfrak{h}_t(y)\rangle$ for any $y, z \in \mathbb{H}$. Let $G(y, z)$ be the Green's function for operator Δ in $\mathbb{H}$, according to (1.2), we know that

$$G(y, z) = \log |y - \bar{z}| - \log |y - z| \tag{5.26}$$

For any y, z in the unbounded component of $\mathbb{H}/K_t$ and $t < \tau_y \wedge \tau_z$, we define $G_t(y, z) := G(f_t(y), f_t(z))$. By direct computation using (5.26), we know that

$$dG_t(y, z) = -\operatorname{Im}\left(\frac{2}{f_t(y)}\right)\operatorname{Im}\left(\frac{2}{f_t(z)}\right)dt$$

Notice that it matches perfectly with the Ito's formula for $\mathfrak{h}_t(z)$, thus we get

$$d\langle \mathfrak{h}_t(z), \mathfrak{h}_t(y)\rangle = -dG_t(y, z)$$

Let ϕ be an arbitrary smooth compactly supported functions in $\mathbb{H}$, for any fixed open domain $U \subset \mathbb{H}$ such that $\operatorname{supp}(\phi) \subset U$, define an $\mathcal{F}_t$-stopping time $\tau_U := \inf\{t \geq 0; K_t \cap U \neq \varnothing\}$, we can construct the propagation functional (due to its similarity to the Feynman propagator in QFT):

$$E_t(\phi) := \iint_{\mathbb{H}^2} G_t(y, z)\phi(y)\phi(z)dydz$$

As we're about to see

$$d\langle(\mathfrak{h}_t, \phi), (\mathfrak{h}_t, \phi)\rangle = -dE_t(\phi), \quad \forall t < \tau_U.$$

Indeed, since $\mathfrak{h}_t(z)$ is a local martingale for all z, by Fubini's theorem, we deduce that $(\mathfrak{h}_t, \phi)$ is also a local martingale. By uniqueness of quadratic variation, it suffices to show that $(\mathfrak{h}_t, \phi)^2 + E_t(\phi)$ is a local martingale. This is obvious because we can apply Fubini's theorem to integrate the local martingale $\mathfrak{h}_t(z)\mathfrak{h}_t(y) + G_t(y, z)$ against ϕ.

Step 3. For any $\mathcal{F}_t$-stopping time $\tau \leq \tau_U$ a.s., we construct a random field $h_{U,\tau}$ on U using the following procedures. First, conditioned on each sample of the stopped curve K_τ, we construct $h^\tau + \mathfrak{h}_\tau$ where h^τ is a new zero boundary GFF on the domain $\mathbb{H}/K_\tau$. Then, we restrict the random field on U and define $h_{U,\tau} := h^\tau + \mathfrak{h}_\tau|_U$. Claim: $h_{U,\tau}$ is identically distributed to $h|_U$.

To prove the claim, notice that both $h|_U$ and $h_{U,\tau}$ are random generalized distribution in the domain U, we only need to show that $(h_{U,\tau}, \phi) \overset{d}{=} (h, \phi)$ for any $\phi \in C_c^\infty(U)$. The most efficient way to compare probability distributions is to compute characteristic functions. Since almost surely, the Green's function for the domain $\mathbb{H}/K_t$ is given by $G_\tau(y, z)$, we have $\operatorname{Var}((h^\tau, \phi)) = E_\tau(\phi)$. Consequently, for any $t \in \mathbb{R}$

$$\mathbb{E}[\exp(it(h_{U,\tau}, \phi))] = \mathbb{E}\left(\mathbb{E}[\exp(it(h^\tau, \phi))|\mathcal{F}_\tau]\exp(it(h_\tau, \phi))\right)$$

Due to conditional independence (definition of h^τ), the RHS equals to

$$\mathbb{E}\left[\exp\left(it(h_\tau,\phi)-\frac{t^2E_\tau(\phi)}{2}\right)\right]=\mathbb{E}\left[\exp\left(it(h_0,\phi)-\frac{t^2E_0(\phi)}{2}\right)\right]$$

where we have used optional stopping theorem. Since $\tilde{h}$ and $\mathfrak{h}_0$ are independently sampled, the above formula is just $\mathbb{E}[\exp(it(h,\phi))]$. So far the proof of claim is complete.

Step 4. The results proved in step 3 shows that their exists a joint law $(h_{U,\tau}, W_{t\wedge\tau_U}, V^{i,q}_{t\wedge\tau_U}, B_{t\wedge\tau_U})$ such that $h_{U,\tau}\stackrel{d}{=}h|_U$. Now we would like to extend it to n stopping times. To be more specific, we construct $\mathcal{F}_t$-stopping times τ_k such that $\tau_k\le\tau_U$ a.s., $k=1,...,n$. Similar as before, we claim that there exists a joint law $((h_{U,\tau_k})_{k=1}^n, W_{t\wedge\tau_U}, V^{i,q}_{t\wedge\tau_U}, B_{t\wedge\tau_U})$ such that $h_{U,\tau_k}\stackrel{d}{=}h|_U$ for all $k=1,...,n$. This means that, the random fields $(h_{U,\tau_k})_{k=1}^n$ can be simultaneously constructed using the same sampling of $(W_{t\wedge\tau_U}, V^{i,q}_{t\wedge\tau_U}, B_{t\wedge\tau_U})$. For simplicity, one may first prove the $n=2$ case. The general $n\in\mathbb{N}$ can be deduced using an induction arguments, see Lemma 3.12 in [14].

Step 5. The next generalization is to consider n arbitrary open domains $U_1,...,U_n\subset\mathbb{H}$, and $\mathcal{F}_t$-stopping times $\tau_1,...,\tau_n$ such that $\tau_k\le\tau_{U_k}$ a.s. for all $k=1,...,n$. After proving this case for arbitrary $n\in\mathbb{N}$, we use the Kolmogorov extension theorem to get the final result:

For any sequence of open sets $U_k\subset\mathbb{H}$ and any sequence of $\mathcal{F}_t$-stopping times τ_k such that $\mathbb{P}(\tau_k\le\tau_{U_k})=1$, one can show that there exists a joint law $(h_{U_k,\tau_k}, W_{t\wedge\tau_{U_k}}, V^{i,q}_{t\wedge\tau_{U_k}}, B_{t\wedge\tau_{U_k}})_{k=1}^\infty$ such that for each $k\in\mathbb{N}$ we have $h_{U_k,\tau_k}\stackrel{d}{=}h|_{U_k}$. See Lemma 3.13, 3.14 in [14].

Step 6. The result shown in step 5 only holds for $\tau_k\le\tau_{U_k}$, so the main task is to extend it to arbitrary stopping times. First of all, one can always construct the sequence $(U_k,\tau_k)_{k=1}^n$ such that

$$\{(U,r\wedge\tau_U);U=\cup_{j=1}^\ell B(p_j,\ p_j),p_j,q_j,r\in\mathbb{Q}_+,\ell\in\mathbb{N}\}\subset(U_k,\tau_k)_{k=1}^n$$

Suppose τ is any $\mathcal{F}_t-$stopping time, given $\mathcal{F}_\tau$, we pick a function $\phi\in C_c^\infty(\mathbb{H})$ such that a.s. $\mathrm{supp}(\phi)\cap K_\tau=\varnothing$. Due to the continuity of the filtration $\mathcal{F}_t$, one can define a sequence of $\mathcal{F}_t$-stopping time $(\sigma_n)_{n\ge1}$ such that

$$\sigma_n:=\inf\{\tau_k;\tau_k\ge\tau,\ \mathrm{supp}(\phi)\subset U_k,k=1,...,n\}$$

where $\inf\varnothing:=\infty$. For any $1\le k\le n$, optional sampling theorem implies that $\{\sigma_n=\tau_k\}$ is an $\mathcal{F}_{\tau_k}$-measurable event. Fix $t\in\mathbb{R}$, on the event $\{\sigma_n\}$, we have

$$\mathbb{E}[\exp(it(h,\phi))|\mathcal{F}_{\sigma_n}]=\sum_{k=1}^n\mathbb{E}[\exp(it(h,\phi))|\mathcal{F}_{\sigma_n}]1\{\sigma_n=\tau_k\}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \mathbb{E}[\exp(it(h,\phi))|\mathcal{F}_{\sigma_k}]1\{\sigma_n = \tau_k\} \\
&= \sum_{k=1}^{n} \exp\left(it(\mathfrak{h}_{\tau_j},\phi) - \frac{\theta^2 E_{\tau_j}(\phi)}{2}\right) 1\{\sigma_n = \tau_k\} \\
&= \exp\left(it(\mathfrak{h}_{\sigma_n},\phi) - \frac{\theta^2 E_{\sigma_n}(\phi)}{2}\right)
\end{aligned}
$$

where in the second to last equality, we have applied step 5. This shows that the conditional law of (h, ϕ) on $\mathcal{F}_{\sigma_n}$ is equal to the law of $(h^{\sigma_n} \circ f_{\sigma_n} + \mathfrak{h}_{\sigma_n}, \phi)$. Since σ_n a.s. decreases to τ as $n \to \infty$, and the filtration $\mathcal{F}_t$ is right continuous, by backward martingale convergence theorem, we have

$$
\lim_{n\to\infty} \mathbb{E}[\exp(it(h,\phi))|\mathcal{F}_{\sigma_n}] = \mathbb{E}[\exp(it(h,\phi))|\mathcal{F}_{\tau}], \quad a.s. \ \& \ L^1.
$$

which leads to

$$
\lim_{n\to\infty} \exp\left(it(\mathfrak{h}_{\sigma_n},\phi) - \frac{\theta^2 E_{\sigma_n}(\phi)}{2}\right) = \exp\left(it(\mathfrak{h}_{\tau},\phi) - \frac{\theta^2 E_{\tau}(\phi)}{2}\right), \quad a.s. \ \& \ L^1.
$$

Thus, we have conclude that the conditional law of (h, ϕ) given $\mathcal{F}_\tau$ is equal to the law of $(h^\tau \circ f_\tau + \mathfrak{h}_\tau, \phi)$.

Step 7. The last task to do is to show that for each τ, the compact set K_τ is local for h. Let $\downarrow$ denotes projection, $\mathscr{L}$ denotes probability distribution. According to Proposition 5.5, we only need to show that the conditional law $\mathscr{L}(h \downarrow H(U) \mid h \downarrow H^\perp(U), \mathcal{F}_\tau)$ is equal to the conditional law $\mathscr{L}(\tilde{h}|_U \mid \tau \le \tau_U)$. This is similarly given by another application of the backward martingale convergence theorem. Now the proof is complete. □

The forward *SLE-GFF* coupling allows us to define the flow line for the Gaussian free field.

Definition 5.4 (Flow line)**.** *Let $\kappa \in (0,4)$ and $\theta \in [0,\pi)$, an $SLE_\kappa(\underline{\rho})$ curve K_t is called a $\theta-$angled flow line of the free-boundary Gaussian free field $h + \theta\chi$ with boundary condition* (5.18), *if $(K_t, h + \theta\chi)$ is coupled together in the sense of Theorem 5.3.*

For the $SLE_\kappa(\underline{\rho})$ curves with $\kappa > 4$, we define another concept called the counter flow line.

Definition 5.5 (Counter flow line)**.** *Let $\kappa' \in (4.\infty)$, the trace K_t of an $SLE_{\kappa'}(\underline{\rho})$ (if it exists) is called a counter flow line of the free-boundary Gaussian free field h with boundary condition* (5.18), *if $(K_t, -h)$ is coupled together in the sense of Theorem 5.3.*

So far we have officially established the relationship between the imaginary geometries of SLE and GFF. Namely, there exists a joint probability distribution for $SLE_\kappa(\underline{\rho})$ and h such that, up to each stopping time before the continuation threshold, the K_t curve are a.s. the 'straight lines' in the geometry of $e^{i(h/\chi+\theta)}$. Finally, such imaginary geometry relation is preserved under conformal isomorphisms, in the sense of Definition 5.1.

6 Quantum surfaces

Having already established Liouville quantum gravity (LQG) as our first and simplest model of random surfaces (recall Subsection 2.2), we now generalise it to arbitrary (D, h), where h is some random distributions modulo arbitrary constant. Topologically speaking, if h is free-boundary-valued, then the random surfaces parametrized by a simply connected domain can only be classified into two cases: either diffeomorphic to the upper half plane (randon surfaces with boundary, and you could imagine, if h vanishes on the boundary, then the surface boundary can be ignored), or diffeomorphic to the whole plane (random surfaces without boundary).

Similar to the classification of imaginary geometries (Definition 5.1), if we want to study the conformal structure of random surfaces, we have to develop another equivalence class. This is actually motivated by the conformal invariance of GFF. Suppose $\psi : D \to \tilde{D}$ is a conformal isomorphism of two planar simply connected domains and $\tilde{h}$ is a GFF on $\tilde{D}$, then $h = \tilde{h} \circ \psi$ is a GFF on D and the two corresponding $\gamma - LQG$ measures μ_h and $\mu_{\tilde{h}}$ are related through

$$\mu_h = e^{-\gamma Q \log|\psi'|}(\mu_{\tilde{h}} \circ \psi), \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{6.1}$$

To proof this equation, we decompose the GFFs by $\tilde{h} = \sum_{n\geq 1} \tilde{\alpha}_n \tilde{f}_n$, where $\tilde{\alpha}_n$ are i.i.d. standard Gaussians and $\tilde{f}_n$ is the orthonomal basis in $\mathbb{H}_0^1(\tilde{D})$. Notice that the pullback functions $f_n := \tilde{f}_n \circ \psi$ are also the orthonormal basis of $H_0^1(D)$. Thus we would like to apply Proposition 2.2 and denote the nth-truncated structures by (h^n, μ_h^n) and $(\tilde{h}^n, \mu_{\tilde{h}}^n)$ accordingly. By change of variable formula, we obtain

$$\begin{aligned}
\mu_h^n(D) &= \int_D \exp\Big\{(\gamma h^n(z) - \frac{\gamma^2}{2}\operatorname{Var}(h^n(z)) + \frac{\gamma^2}{2}\log C(z; D))\Big\} dz \\
&= \int_D \exp\Big\{(\gamma \tilde{h}^n(\psi(z)) - \frac{\gamma^2}{2}\operatorname{Var}[\tilde{h}^n(\psi(z))] + \frac{\gamma^2}{2}\log C(z; D))\Big\} dz \\
&= \int_{\tilde{D}} \exp\Big\{(\gamma \tilde{h}^n(\tilde{z}) - \frac{\gamma^2}{2}\operatorname{Var}(\tilde{h}^n(\tilde{z})) + \frac{\gamma^2}{2}\log C(\psi^{-1}(\tilde{z}); D))\Big\} \frac{d\tilde{z}}{|\psi'(\psi^{-1}(\tilde{z}))|^2}
\end{aligned}$$

By definition of the conformal radius, we have $\log C(\psi^{-1}(\tilde{z}); D)) = \log C(\tilde{z}; \tilde{D}) - \log|\psi'(z)|$. Inserting it back to the calculation, we have

$$\mu_h^n(D) = \int_{\tilde{D}} \mu_{\tilde{h}}^n(d\tilde{z}) e^{-(2+\frac{\gamma^2}{2})\log|\psi'|}$$

Take $n \to \infty$, we will see that $\tilde{h} \circ \psi + Q \log|\psi'|$ and $\tilde{h}$ yields conformally identical random measures on D and $\tilde{D}$ respectively. Informally speaking, we have almost surely that $e^{\gamma(h+Q\log|\psi'|)}dz = e^{\gamma\tilde{h}}d\tilde{z}$. Motivated by this calculation, we now present the definition of the quantum surface equivalence class.

Definition 6.1. *A random Riemann surface modulo quantum surface equivalence, with parameter $\gamma \in [0, 2)$, is a pair (D, h) with D being a simply connected subdomain in $\mathbb{C}$, and h (modulo arbitrary constant) being an random element in $\mathcal{D}_0'(D)$, under the equivalence relation described as follows: Two random surfaces $(\tilde{D}, \tilde{h})$ and (D, h) are considered equivalent iff there exists a conformal isomorphism such that $\psi : D \to \tilde{D}$ and $h = \tilde{h} \circ \psi + Q \log|\psi'|$, where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.*

Compare Definition 6.1 and Definition 5.1, in this section, instead of classifying random surfaces by different imaginary geometries, we prefer to classify them according to different $\gamma-$LQG measures.

Let's start with surfaces parametrized by $\mathbb{H}$. The most natural candidate for our generalized model is that h being the free boundary GFF. For this definition, the biggest drawback is that the surface is not scale invariant, which means it falls out of the scope of CFT. In addition, we would like to run SLE_κ curves independently on this quantum surface, hence it's better to introduce two different reference boundary points to distinguish the starting and finishing position. To resolve the issues, we define a quantum surface $(\mathbb{H}, \tilde{h}, 0, \infty)$ by introducing radial logarithmic drift to the free-boundary GFF $h(z) - \alpha \log|z| := \tilde{h}(z)$ which has singularities at 0 and ∞, and such surfaces (D, h, a, b) with two special points are called doubly marked. The above definition is called a *unscaled quantum wedge.*

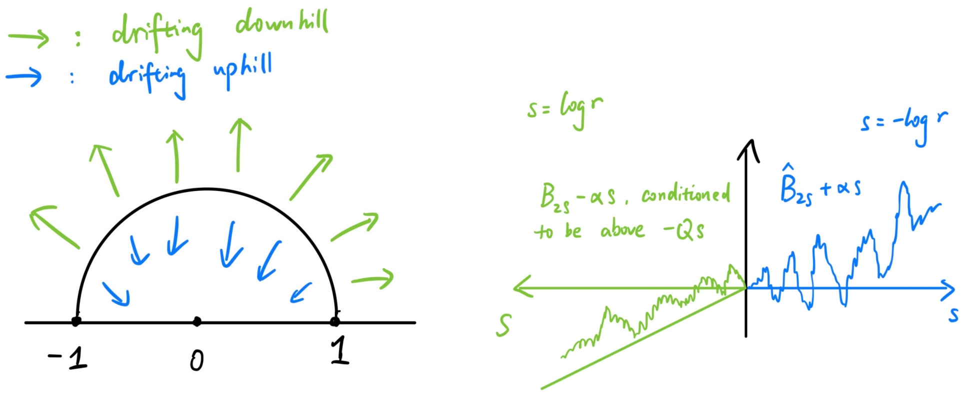
However, even we can run indepedent SLE_κ on unscaled quantum wedge, this quantum surface is still not a CFT and the two marked points are not well-distinct. To improve the situation, we take the circle average embedding (see Subsection 6.2) and restrict h such that it looks slightly differently at 0 and ∞. This leads to the thick quantum wedge, which is of primary significance in the following context.

The reference matertial for this section will be [17].

6.1 Quantum wedges and quantum cones

We first study the thick quantum wedge, a random surface diffeomorphic to $\mathbb{H}$ with Gaussian fluctuation and $\alpha < Q$ logarithmic singularities at 0 and ∞.

Definition 6.2. *Fix $0 < \alpha < Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, an $\alpha-$thick quantum wedge is a doubly marked quantum surface $(\mathbb{H}, h, 0, \infty)$ such that the circle average process of h is given by the following figure 8. The*



(a) geometrics (b) sample paths

Figure 8: circle average of α-thick quantum wedge

radial noise of h is chosen such that the average on the unit semicircle $\partial B(0,1) \cap \mathbb{H}$ is zero. Such choice of the arbitrary constant is called a circle average embedding.

Following Definition 6.2, the α-thick quantum wedge is uniquely characterised by the global conformal invariance (a global CFT) of its corresponding quantum measure. To be more specific, if we multiply the quantum measure $e^{h(z)}dz$ by an arbitrary positive number, we can always do a random scaling $z \to z/\varepsilon$ such that the circle average embedding of the α-thick quantum wedge is recovered.

Proposition 6.1. *The following are the 'hidden reasons' for Definition 6.2.*

(i) Let $(\mathbb{H}, h, 0, \infty)$ denote the circle average embedding of an α-thick quantum wedge. For the random surface $(\mathbb{H}, h^f, 0, \infty)$ where h^f is a free-boundary GFF plus a drift $-\alpha \log|\cdot|$, the circle average embedding of $(\mathbb{H}, h^f + C/\gamma, 0, \infty)$ converges weakly in the space of distributions to $(\mathbb{H}, h, 0, \infty)$.

(ii) Fix any $C \in \mathbb{R}$, and let $(\mathbb{H}, \tilde{h}, 0, \infty)$ be the circle average embedding of $(\mathbb{H}, h + C/\gamma, 0, \infty)$, then $(\mathbb{H}, \tilde{h}, 0, \infty) \overset{d}{=} (\mathbb{H}, h, 0, \infty)$. This implies that it is globally conformal invariant.

Proof. Suppose we are given a random surface $(\mathbb{H}, h^f, 0, \infty)$ where h^f is a free-boundary GFF plus a drift $-\alpha \log|\cdot|$, we ask: What will happen if we add an arbitrary constant? How to rescale the surface by $\varepsilon \in [0,1]$ to recover circle average embedding. Since a scaling $z \to z/\varepsilon$ results in $h^f(\varepsilon\cdot)$+Qlog ε, for any sample path, all points are translated to the left by $-\log\varepsilon$, and down by $-Q\log\varepsilon$, see figure 9 for a complete picture.

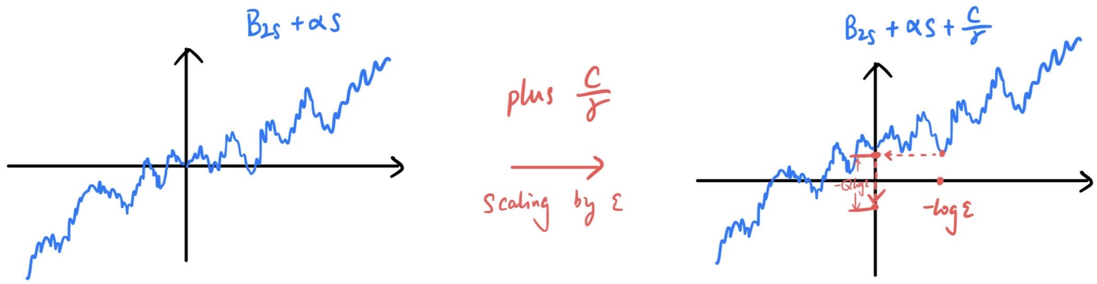
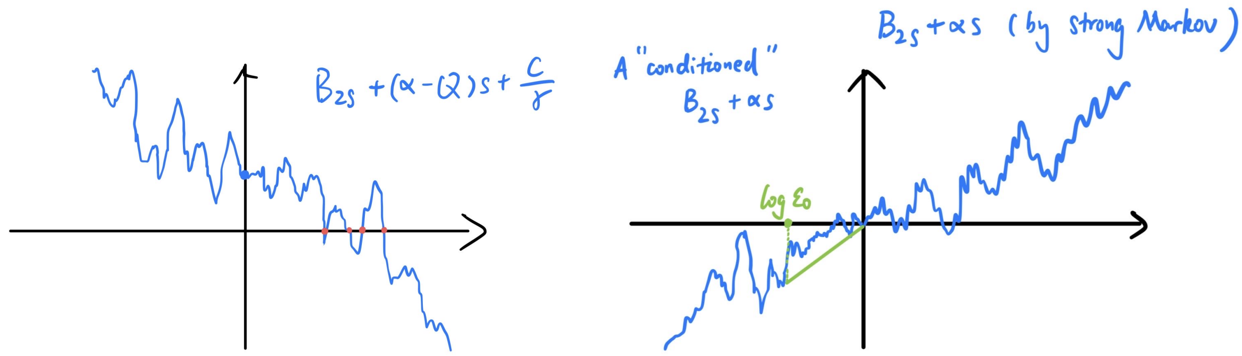


Figure 9: transformation

Now, since we can do translation for all $\varepsilon \in [0,1]$, all points lying on the right half plane represent different amount of scaling, and they have *equal rights* to be moved to the $y-$axis. What are the desired scales (or reference points) to recover circle average embedding? Obviously, such reference points should be moved to 0, which means their heights and C/γ should compensate with their downward translations. Thus, to determine the correct scales, we move all the points $(s, B_{2s}+\alpha s+C/\gamma)$ down by Qs and see how many of them hit the x-axis, and these are the candidate scales to recover circle average embedding. Mathematically, we recover a downward drifted Brownian motion. There is one particularly interesting scale, which is given by the a.s. positive stopping time $\tau = \inf\{t \geq 0; B_{2t}+(\alpha-Q)t+C/\gamma\}$, and it actually correspond to the largest $\varepsilon_0 = e^{-\tau}$. We perform the ε_0-rescaling to every sample path, the result is illustrated in the (b) of Figure 10.

Now if we send $C \to +\infty$, the $\log\varepsilon_0$ will be sent to $-\infty$ a.s.. Thus under the topology of local uniform convergence, the circle average processes of $(\mathbb{H}, h^f + C/\gamma, 0, \infty)$ under *circle average embedding* converges to the $\alpha-$thick quantum wedge a.s.. Lastly, we explain why an unscaled quantum wedge with logarithmic singularities fail to be a global CFT. If we multiply its quantum measure by constant, then no matter doing what kind of rescaling and the circle average embedding, there will always be an additional 'excursion' appearing on the left half plane, which means that it could never be identically distributed to drifted Brownian motion. $\square$



(a) finding the candidate scales (b) rescaled process

Figure 10: Illustration

If we 'glue' the the boundary of $\alpha-$thick quantum wedge, we're expecting to obtain a quantum surfaces with local Gaussian distribution and logarithmic singularities at $0\&\infty$, which is also diffeomorphic to $\mathbb{C}$ and rescaling invariant. Indeed, this is 'almost' the definition of quantum cones.

Definition 6.3. *Fix $0<\alpha<Q=\frac{2}{\gamma}+\frac{\gamma}{2}$, an α-quantum cone is a doubly marked quantum surface $(\mathbb{C},h,0,\infty)$ such that the circle average process of h is given by the figure 8 but with $B_{2s},\hat{B}_{2s}$ replaced by $B_s,\hat{B}_s$. The radial noise of h is chosen such that the average on the unit circle $\partial B(0,1)$ is zero. Such choice of the arbitrary constant is called a circle average embedding.*

Then one can imagine that we have an analog of Proposition 6.1.

Proposition 6.2. *The following are the 'hidden reasons' for Definition 6.3.*

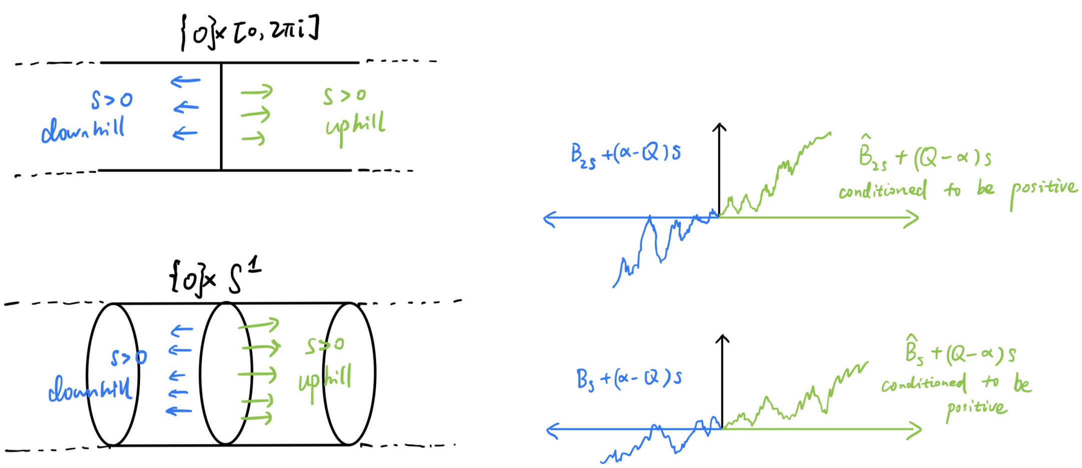
(i) Let $(\mathbb{C},h,0,\infty)$ denote the circle average embedding of an α-thick quantum wedge. For the random surface $(\mathbb{C},h^f,0,\infty)$ where h^f is a whole-plane free-boundary GFF plus a drift $-\alpha\log|\cdot|$, the circle average embedding of $(\mathbb{C},h^f+C/\gamma,0,\infty)$ converges weakly in the space of distributions to $(\mathbb{C},h,0,\infty)$.

(ii) Fix any $C\in\mathbb{R}$, and let $(\mathbb{C},\tilde{h},0,\infty)$ be the circle average embedding of $(\mathbb{C},h+C/\gamma,0,\infty)$, then $(\mathbb{C},\tilde{h},0,\infty)\overset{d}{=}(\mathbb{C},h,0,\infty)$. This implies that it is globally conformal invariant.

If we take the conformal isomorphism $z\to\log z$, then $\mathbb{H}$ and $\mathbb{C}$ are mapped to the strip $\{u+[0,2\pi i];u\in\mathbb{R}\cup\infty\}$ and cylinder $\{u\times S^1;u\in\mathbb{R}\cup\infty\}$ respectively. Following this transformation, we introduce the strip & cylinder parametrization of the quantum wedge & cone.

Remark 6.1. *Before we establish our construction, we remark that the results in Proposition 1.4 and Proposition 1.5 admit natural generalizations. Namely, in order to sample a 'random H^1 function' on the strip, or the cylinder, we can first sample a continuous Ito process in the $x-$direction, then we add the sample of noise on each vertical line or circle to maintain common vertical average. A bit formally speaking, the Sobolev space admits an orthogonal decomposition into the subspace of 'horizontal functions' and the subspace of 'common-mean vertical perturbations'.*

Even if we don't decompose the Sobolev space in the spirit of the above remark, we can still construct strip & cylinder parametrization thorough definition of random surfaces and the circle average embedding, see Figure 11 below.



(a) geometrics

(b) sample paths

Figure 11: strip & cylinder parametrizations

Having established the strip parametrization, we could use it to generalize the definition of quantum wedge to the $\alpha \in [Q, Q+\gamma/2)$ case, and for the $\alpha > Q$ case, the random surface will be called the thin quantum wedge. As we can see from the sample trajectory shown in Figure 11, the evolution towards the right is positive and upward drifted. So it's natural to imagine whether we can find some connections with BES^δ processes.

Indeed, if we denote the process $X_t = B_{2t} + at$ with $a = Q - \alpha > 0$, then following Ito's formula, the reparametrization of $Z_t = \exp(\gamma X_t/2)$ by its quadratic variation process is a BES^δ with $\delta = 2 + \frac{2a}{\gamma}$. This encourages us to sample the 'horizontal process' of a quantum wedge on strip through Bessel processes, instead of Brownian motion with drift.

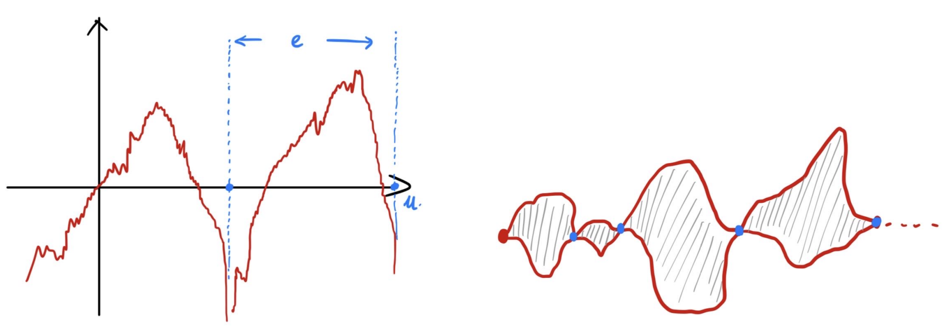
So let's 'reprogram everything backwards', suppose Z is a Bessel process BES^δ with $\delta > 0$, then we reparametrize the process $2\gamma^{-1}\log Z$ such that the quadratic variation is exactly $2t$. There are two consequences in total:

(i) If $\delta = 2 + \frac{2a}{\gamma} \geq 2$, which means $a = Q - \alpha \geq 0$, the process Z only has one 'infinite excursion' (loosely speaking) and a.s. does not hit zero. In this case, if we sample a random surface on the strip with horizontal process given by the reparametrization of $2\gamma^{-1}\log Z$, we get an α-thick quantum wedge.

(ii) If $1 < \delta = 2 + \frac{2a}{\gamma} < 2$, which means $\alpha \in (Q, Q + \gamma/2)$, the process Z is instantaneously reflecting at 0, and the corresponding random surface will be an $\alpha-$thin quantum wedge. If the horizontal process hits 0 at time t, the left side of 0 will be falling down to negative infinity, while the right side of 0 will emerge from negative infinity until it reaches one peak.

For the 'thin case', if we consider the Ito excursion decomposition of Z from 0 to positive infinity, we will get a Possion Point Process (PPP) on the topological space $\mathcal{E} \times (0, \infty)$, where $\mathcal{E}$ is the space of all excursions e and $u \in (0, \infty)$ is the local time at 0 which is constant in each excursion. But for the completeness of our context, we first define the thin wedge using our 'old language'.

Definition 6.4. *Fix any $\alpha \in (Q, Q + \gamma/2)$. Consider the strip parametrization of random surfaces as stated in Remark 6.1. We first construct a BES^{δ} process Z with $\delta = 2(Q - \alpha)/2 + 2$ such that $Z_0 = 1$, then the horizontal process is given by the reparametrization of $2\gamma^{-1}\log Z$ such that its quadratic variation is $2t$. The vertical perturbation noise is sampled from the corresponding part in another independent free-boundary GFF (i.e., vertical part of the GFF) on the strip (with arbitrary constant fixed such that the average on $[0, 2\pi i]$ is 0). Couple these two parts together in the Sobolev space $H^1(\mathbb{R} \times (0, 2\pi i))$, the resulting random surface with doubly marked point $\pm\infty$ is called an $\alpha-$thin quantum wedge.*



(a) horizontal process (b) geometric shape of $\alpha-$quantum wedge

Figure 12: strip parametrization of $\alpha-$quantum wedge

The reason for the name 'thick wedge' is clearly illustrated in Figure 12. Indeed, the Bessel process Z hits 0 infinitely many times, hence each bubble in the the sampling of the thin wedge can not grow very large as in the case of thick wedge (because the two boundaries of the bubble has to reunite with each other after certain amount of time). Using the random distribution constructed on each bubble, we can define an '$\gamma - LQG$-like' random surface area.

Proposition 6.3. *The law of the '$\gamma-LQG$-like' random area of the bubbles in the $\alpha-$thin quantum wedge coincides with the law of the lengths of Bessel excursions of the process Z.*

Notice that in Definition 6.4, the process X_t is one-sided and the embedding is chosen such that the first zero of X_t is fixed at the origin. In fact, for the thin quantum wedge, the embedding (horizontal fixing) is not quite as useful as the PPP interpretation. Now we are finally in the position to state an alternative definition using this 'new language'.

Definition 6.5. *Fix $\alpha \in (Q, Q+\gamma/2)$ and let Z be a BES^δ as in Definition 6.4. Suppose ν_δ^{BES} is the Ito measure for the Bessel excursions for Z, let $\sum_{i\geq 1} \delta_{(e_i,u_i)}$ denote the PPP of the excursion of Z in the state space $\mathcal{E} \times (0,\infty)$ with intensity measure $d\lambda \otimes d\nu_\delta^{BES}$.*

For each $i \geq 1$, we construct a space of surfaces $\mathcal{S}_i$ associated to the ith excursion, which is given by: (i) First, reparametrize all $2\gamma^{-1}\log e_i$ to have quadratic variation $2t$; (ii) Second, use it as the horizontal process and add the vertical noise from another independent free-boundary GFF (with arbitrary constant fixed such that the average on $[0, 2\pi i]$ is 0). To sum up, each $\mathcal{S}_i$ is equipped with a probability distribution and the family $\{\mathcal{S}_i\}_{i\geq 1}$ is independent. The $\alpha-$thin quantum wedge is the PPP

$$\sum_{i\geq 1} \delta_{(u_i,S_i)}, \quad u_i > 0,\ S_i \in \mathcal{S}_i. \tag{6.2}$$

The intensity measure of the PPP, i.e., the intensity (not the distribution) of the quantum wedge, is determined by ν_δ^{BES} and the distribution of GFF. Each space of surfaces $\mathcal{S}_i$ is called a bead of the quantum wedge.

6.2 Quantum disk and quantum sphere

It is seen from Definition 6.5 that each bead of the thin quantum wedge is homeomorphic to a disk, instead of a wedge. There is also a natural question to ask: what are the laws of these beads?

Definition 6.6. *Fix $\alpha \in (Q, Q+\gamma/2)$, the infinte $\alpha-$quantum disk $\mathcal{M}_\alpha^{disk}$ is defined via the following procedures:*

(i) *Take* $\delta = 2+\frac{2(Q-\alpha)}{\gamma}$, *we sample a* BES^{δ} *process* Z. *Then one can obtain horizontal process by suitably reparametrize* $2\gamma^{-1}\log Z$ *such that its quadratic variation is* $2t$. *If* ν_{δ}^{BES} *denotes the Ito measure, we see that it derives another measure* ν_{α}, *which is the probability distribution for the horizontal processes.*

(ii) *We take the measure* $= \nu_{\alpha} \otimes \mathbb{P}$ *where* $\mathbb{P}$ *denotes the law of a free-boundary GFF on the strip. If we reverse the procedure described in Remark 6.1 by sampling the two parts in the decomposition via* ν_{α}^{disk}, *then the pushforward of* ν_{α}^{disk} *will be a probability measure* μ_{α}^{disk} *on the space of distributions.*

(iii) *Use* μ_{α}^{disk} *to construct a random distribution on the strip. The embedding is given by fixing 0 to be the peak of the horizontal process. In conclusion, this quantum surface parametrized by the cylinder with marked points* $\pm\infty$ *is the so-called quantum disk.*

One can comprehend the structure of quantum disk as a finite-volume variant of the quantum wedge. Intuitively speaking, if we take a quantum wedge and horizontally compress the marked points $\pm\infty$ until their distance is finite, then it is 'visible' (because of finiteness) that the two boundaries of the wedge are glued together into new end points and the topology becomes a disk. Similar jobs could be done to the quantum cone, the effect will be gluing the boundaries of a finite cylinder into two end points, i.e., the topology becomes a sphere.

In the following context, we use similar procedure to define the $\alpha-$quantum sphere. But before we embark on this generalization, we remark that the parameters are chosen in the domain $\alpha \in (Q, Q+\gamma/4)$. The reason for the slight change is that, the quadratic variation for the horizontal process of the quantum cone is t, thus the reparametrization of the corresponding α-log-Bessel process is different (because Ito's formula is changed).

Definition 6.7. *Fix* $\alpha \in (Q, Q+\gamma/4)$, *the* $\alpha-$*quantum sphere* $\mathcal{M}_{\alpha}^{sphere}$ *is defined via the following procedures:*

(i) *Take* $\delta = 2+\frac{4(Q-\alpha)}{\gamma}$, *we sample a* BES^{δ} *process* Z. *Then one can obtain horizontal process by suitably reparametrize* $2\gamma^{-1}\log Z$ *such that its quadratic variation is* t. *If* ν_{δ}^{BES} *denotes the Ito measure, we see that it derives another measure* ν_{α}', *which is the probability distribution for the horizontal processes.*

(ii) *We take the measure* $\nu_{\alpha}^{sphere} = \nu_{\alpha}' \otimes \mathbb{P}'$ *where* $\mathbb{P}'$ *denotes the law of a free-boundary GFF on the cylinder. If we reverse the procedure described in Remark 6.1 by sampling the two parts* *in the decomposition via ν_α^{sphere}, then the pushforward of ν_α^{sphere} will be a probability measure μ_α^{sphere} on the space of distributions.*

(iii) Use μ_α^{sphere} to construct a random distribution on the cylinder. The embedding is given by fixing 0 to be the peak of the horizontal process. In conclusion, this quantum surface parametrized by the cylinder with marked points $\pm\infty$ (now become interior points of the topological sphere) is the so-called quantum sphere.

At last, we would like to introduce two useful conditioned distribution for the quantum disks and spheres.

Definition 6.8. *Let ν_h and μ_h denote the $\gamma-LQG$ boundary measure and the $\gamma-LQG$ area measure respectively (see Subsection 2.2 and Subsection 3.2).*

(i) For fixed $\ell > 0$, the ℓ-boundary-length α-quantum disk S is given by the regular conditional distribution of the random measure $\mathcal{M}_\alpha^{disk}$ given that $\nu_h(\partial S) = \ell$.

(ii) For fixed $A > 0$, the A-surface-area α-quantum sphere C is given by the regular conditional distribution of the random measure $\mathcal{M}_\alpha^{sphere}$ given that $\mu_h(C) = A$.

The proof for the existence of regular conditional distribution is given by [9].

6.3 Reverse $SLE_\kappa(\underline{\rho})$ & GFF coupling

Similar to the forward $SLE_\kappa(\underline{\rho})$ & GFF coupling discussed in Subsection 5.3, one can actually couple the geometry of GFF with reverse $SLE_\kappa(\underline{\rho})$ curves. Unlike the forward case, which was coupled in the sense of imaginary geometry, the reverse coupling is constructed in the sense of quantum surfaces and we don't need to introduce the structure of local sets of GFF. In the following, we would like to present the most general theorem for $SLE_\kappa(\underline{\rho})$ with forcing points living in the upper half plane, while due to similarity to the forward coupling, the proof will be omitted.

Let $\kappa > 0$ and $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ be fixed parameters. Recall Subsection 5.3, the Green's function for the upper half plane is given by

$$G(y,z) = \log|y-\bar{z}| - \log|y-z|, \quad y,z \in \mathbb{H}.$$

We also construct a harmonic function via

$$\tilde{\mathfrak{h}}_0(z) := \frac{2}{\sqrt{\kappa}} \log|z| \tag{6.3}$$

In the following contexts, we always assume in priori that there's a well-defined reverse $SLE_\kappa(\underline{\rho})$, given by (5.9) and 5.10 in the sense of Definition 5.2, with forcing points $x_1, ..., x_k \in \mathbb{H}$ and weights $\rho_1, ..., \rho_k \in \mathbb{R}$. Let $\tilde{f}_t := \tilde{g}_t - \tilde{W}_t$ be the centered reverse Loewner flow, then we could define the pullback functions via

$$\tilde{G}_t(y, z) := G(\tilde{f}_t(y), \tilde{f}_t(z)), \quad \tilde{\mathfrak{h}}_t(z) := \tilde{\mathfrak{h}}_0(\tilde{f}_t(z)) + Q \log |\tilde{f}_t'(z)|.$$

Combining them together, we construct the full harmonic part by

$$\hat{\mathfrak{h}}_t(z) := \tilde{\mathfrak{h}}_t(z) + \frac{1}{2\sqrt{\kappa}} \sum_{i=1}^{k} \rho_i \tilde{G}_t(x_i, z). \tag{6.4}$$

Theorem 6.1 (Reverse SLE-GFF coupling)**.** *Suppose we are given a free-boundary GFF h on $\mathbb{H}$ which is independent to our reverse $SLE_\kappa(\underline{\rho})$. Let $\tau > 0$ be an arbitrary finite stopping time with respect to the filtration generated by $(\tilde{W}_t, \tilde{g}_t(x_i), B_t)$, which is also a.s. smaller than the continuation threshold (similar to* (5.16)*). Then the random surfaces encoded by $\hat{\mathfrak{h}}_0 + h$ and $\hat{\mathfrak{h}}_\tau + h \circ \tilde{f}_\tau$ fall into the same quantum surface equivalence class defined by Definition 6.1. This implies that for any time $t \geq 0$, we have*

$$\hat{\mathfrak{h}}_0 + h \stackrel{d}{=} \hat{\mathfrak{h}}_t + h \circ \tilde{f}_t \tag{6.5}$$

where both sides are defined modulo arbitrary distributions. The joint law of $(\tilde{f}, \hat{\mathfrak{h}} + h)$ described above is called the reverse coupling of $SLE_\kappa(\underline{\rho})$ and GFF.

Remember that the forward coupling is aimed to show that SLE curves are indeed the random imaginary geometry flow lines of the GFF, we assert that the reverse coupling is something even more useful. It shows that when we 'zip' two independent Liouville quantum gravity surfaces together along their boundaries, the interface, or i.e., the 'quantum zipper', is exactly an independent SLE curve. To show this marvelous result in the next subsection, we rephrase the above general theorem for the simplest unforced coupling, which serves as the main ingredient.

Corollary 6.1. *Suppose γ is the trace of an SLE_κ with $\kappa > 0$, we construct h to be another independent free-boundary GFF on $\mathbb{H}$. If $f_t(z)$ is the reverse Loewner flow corresponding to γ_t, we write the harmonic functions*

$$\mathfrak{h}_0(z) := \frac{2}{\sqrt{\kappa}} \log |z|, \quad \mathfrak{h}_t(z) := \mathfrak{h}_0(f_t(z)) + Q \log |f_t'(z)|, \quad Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}.$$

Then $\mathfrak{h}_0 + h$ and $\mathfrak{h}_t + h \circ f_t$ fall into the same quantum surface equivalence class. This is to say, for any time $t > 0$, we have

$$\mathfrak{h}_0 + h \stackrel{d}{=} \mathfrak{h}_t + h \circ f_t$$

where both sides are defined modulo arbitrary constants.

6.4 Zipping up quantum surfaces along the SLE curve

In this subsection, we would like to show some beautiful geometric results which further demonstrate the fact that: (i) The κ-universality class for SLE curves corresponds to te $\sqrt{\kappa} = \gamma$-universality class for quantum surfaces; (ii) The SLE_κ curves can be regarded as the natural interfaces for quantum surfaces. Namely, suppose η_t denotes the trace of SLE_κ for $\kappa \in (0, 4)$, we would like to construct a coupling of (h, η). However, this coupling is not understood that η is the flow line for h, instead, η is the quantum boundary of two independent quantum surfaces encoded by h. Thus heuristically speaking, given two quantum surfaces, one can zip them together along the trace SLE curve η_t. The entire zipping process is proved to be stationary and the surfaces h_t will always remain in the quantum surface equivalence class of h_0 for all $t \in \mathbb{R}$. See Figure 13.

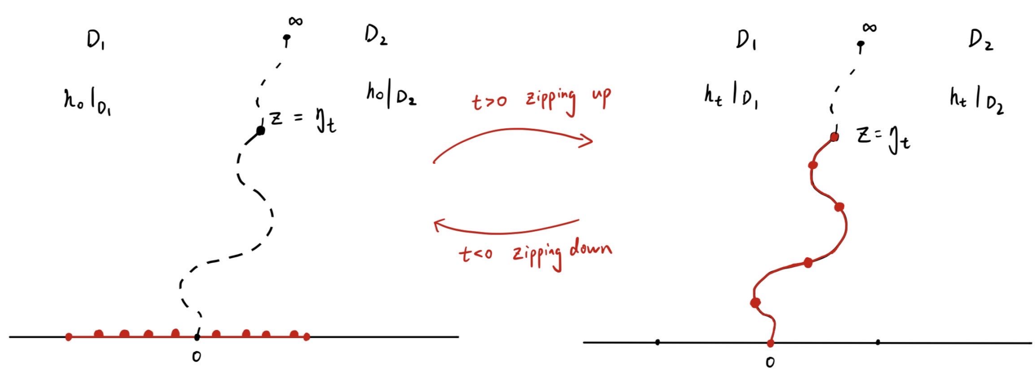


Figure 13: Quantum zipper

From now on, we fix $\kappa \in [0, 4)$ and $\gamma \in [0, 2)$ such that $\kappa = \gamma^2$. First of all, we state that if we zip two independent γ-LQG surfaces (with log-singularity) along the capacity of SLE_κ, we get a nice two-sided stationary process.

Theorem 6.2 (Capacity zipper for γ-LQG)**.** *There exists a two-sided stationary process $(h_t, \eta_t)_{t\in\mathbb{R}}$, in the state space $H^1(\mathbb{H}) \times L$ (recall Subsection 1.2) such that:*

(i) h_0 is identically distributed to $h + \mathfrak{h}_0$, where h is a free-boundary GFF and $\mathfrak{h}_0$ is defined as (6.3)*. Meanwhile, η_0 is a chordal SLE_κ curve running from* 0 *to* ∞*, which is also independent of h_0.*

(ii) Let $\tilde{f}_t$ be the centered reverse Loewner flow associated with η_0. For any $t > 0$ we have the identity $(h_{-t}, \eta_{-t}) = (h \circ \tilde{f}_t + \mathfrak{h}_t, \tilde{f}_t(\eta_0))$. To be more specific, the first identity $h_{-t} = h \circ \tilde{f}_t + \mathfrak{h}_t$

is understood in the sense of Definition 5.1, while the second identity means that η_{-t} and $\tilde{f}_t(\eta_0)$ are identically distributed random variables in state space L.

(iii) *For each $t \in \mathbb{R}$, there exists an isomorphism $\mathcal{Z}_t^{cap}$ defined as $\mathcal{Z}_t^{cap}(h_0, \eta_0) := (h_t, \eta_t)$. The family of all isomorphism $(\mathcal{Z}_t^{cap})_{t\in\mathbb{R}}$ is called the capacity quantum zipper. In addition, it has a natural group structure*

$$\mathcal{Z}_{t+s}^{cap} = \mathcal{Z}_t^{cap}\mathcal{Z}_s^{cap}, \quad \forall s, t \in \mathbb{R}.$$

The reason why this is called a capacity zipper is that, due to the reverse SLE-GFF coupling, $\mathcal{Z}_{-t}^{cap}$ with $t > 0$ corresponds to 'zipping down' the quantum surface by t half-plane capacity units. This follows directly from the second assertion in the above theorem. For the zippings $\mathcal{Z}_t^{cap}$ with $t > 0$, one can also infer from the group structure that it corresponds to 'zipping up' the quantum surface by t half-plane capacity.

Next, we would like to show that a similar zipper group can be performed to the quantum wedges parametrized by $\mathbb{H}$. But before we jump into the formal statements, we remark that for general quantum surfaces, the Bulk-Boundary physics could be similarly argued as in Subsection 3.2. Thus one can still construct a boundary quantum measure μ_h with h being the thick quantum wedge under circle average embedding, using the same renormalization scheme. In the following statement, let $[H^1(\mathbb{H}), 0, \infty]$ denotes the set of all doubly-marked Riemann surfaces on $\mathbb{H}$.

Theorem 6.3 (Length zipper for $(\gamma - \frac{2}{\gamma})$-quantum wedge)**.** *There exists a doubly-marked two-sided stationary process $(\mathbb{H}, h_t, 0, \infty, \eta_t)_{t\in\mathbb{R}}$ in the state space $[H^1(\mathbb{H}), 0, \infty] \times L$ such that*

(i) *$(\mathbb{H}, h_0, 0, \infty)$ is identically distributed to a $(\gamma - \frac{2}{\gamma})$-quantum wedge under the circle average embedding. Meanwhile, η_0 is a chordal SLE_κ curve running from 0 to ∞, which is also independent of h_0.*

(ii) *Let ν_{h_0} denotes the quantum boundary measure associated to h_0. For any fixed $t > 0$, there exists a random $t^* < 0$ such that a.s., the centered reverse Loewner flow maps both $[t^*, 0)$ and $(0, t]$ to the same portion of the SLE curve, denoted by $\eta_0([0, t])$. Due to the mirror symmetry about the imaginary axis, we may define stopping times using the right portion:*

$$\sigma(t) := \inf\{s \geq 0 : \nu_{h_0}(\eta_0([0, s])) = \nu_{h_0}(\eta_0([s^*, 0])) \geq t\}, \quad \forall t > 0.$$

Then we have the identity $(h_{-t}, \eta_{-t}) = (h \circ \tilde{f}_{\sigma(t)} + \mathfrak{h}_{\sigma(t)}, \tilde{f}_{\sigma(t)}(\eta_0))$. As before, the first part of the identity is understood in the sense of Definition 5.1, while the second part is understood as identically distributed random variables in state space L.

(iii) For each $t \in \mathbb{R}$, there exists an isomorphism $\mathcal{Z}_t^{len}$ defined as $\mathcal{Z}_t^{len}(h_0, \eta_0) := (h_t, \eta_t)$. The family of all isomorphism $(\mathcal{Z}_t^{len})_{t \in \mathbb{R}}$ is called the length quantum zipper. In addition, it has a natural group structure

$$\mathcal{Z}_{t+s}^{len} = \mathcal{Z}_t^{len} \mathcal{Z}_s^{len}, \quad \forall s, t \in \mathbb{R}.$$

Similar to the previous interpretations, for any $t > 0$, the length zipper $\mathcal{Z}_t^{len}$ corresponds to 'zipping up' the quantum surfaces by t quantum length units under the measure ν_{h_0}. For for any $t < 0$, the length zipper $\mathcal{Z}_t^{len}$ corresponds to 'zipping down' the quantum surfaces by t quantum length. In addition, if we parametrize our quantum wedge using the $\alpha < Q$, we see that

$$\alpha = -\frac{2}{\sqrt{\kappa}} + \gamma$$

Indeed, the $-\frac{2}{\sqrt{\kappa}}$ arises from the harmonic part of the random generalized function h_0, and the γ term is the contribution from LQG and the log-singularity. In general, these two quantum zipper theorems demonstrates that there is a conformally invariant way to decorate LQG-like quantum surfaces with SLE curves, and the corresponding KPZ universality classes for these two random geometric objects also match perfectly. For a detailed proof of Theorem 6.2 and Theorem 6.3, we refer to [3].

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