CSC165H1, Problem Set 3

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Question 1

a) i. Proof by induction on n:
Define $P(n) : \sum_{k=1}^{n} k^3 = (\frac{n(n+1)}{2})^2$
Want to show: $\forall n \in \mathbb{N}, P(n)$ is true.
Base case: Prove that $P(0)$ is true.
On the left hand side, $\sum_{k=1}^{0} k^3 = 0$
On the right hand side, $(\frac{0(0+1)}{2})^2 = 0$
So $\sum_{k=1}^{0} k^3 = (\frac{0(0+1)}{2})^2 = 0$, i.e., $P(0)$ is true.
Induction Hypothesis: Let $n = j \in \mathbb{N}$, and assume that $P(j)$ is true.
i.e., we assume that $\sum_{k=1}^{j} k^3 = (\frac{j(j+1)}{2})^2$
Now let $n = j + 1$. We want to show that $P(j + 1)$ is true.
i.e., want to show that $\sum_{k=1}^{j+1} k^3 = (\frac{(j+1)(j+2)}{2})^2$

$$
\begin{aligned}
\sum_{k=1}^{j+1} k^3 &= \sum_{k=1}^{j} k^3 + (j+1)^3 \\
&= (\frac{j(j+1)}{2})^2 + (j+1)^3 && \text{(by the induction hypothesis)} \\
&= \frac{j^2(j+1)^2}{4} + (j+1)^3 && \text{(by arithmetic)} \\
&= \frac{(j+1)^2}{4}(j^2 + 4(j+1)) && \text{(factor out } \tfrac{(j+1)^2}{4}\text{)} \\
&= \frac{(j+1)^2}{4}(j+2)^2 && \text{(since } j^2 + 4(j+1) = (j+2)^2\text{)} \\
&= (\frac{(j+1)(j+2)}{2})^2
\end{aligned}
$$

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ii. Proof by induction on n:
Define $P(n) : \sum_{k=1}^{n} k! \cdot k = (n+1)! - 1$
Want to show: $\forall n \in \mathbb{N}, P(n)$ is true.
Base case: Prove that $P(0)$ is true.

On the left hand side, $\sum_{k=1}^{0} k! \cdot k = 0$, since $0 < 1$
On the right hand side, $(0+1)! - 1 = 1! - 1 = 1 - 1 = 0$
So $\sum_{k=1}^{0} k! \cdot k = (0+1)! - 1 = 0$, i.e., $P(0)$ is true.
Induction Hypothesis: Let $n = j \in \mathbb{N}$, and assume that $P(j)$ is true
i.e., we assume that $\sum_{k=1}^{j} k! \cdot k = (j+1)! - 1$
Now let $n = j+1$. We want to show that $P(j+1)$ is true.
i.e., want to show that $\sum_{k=1}^{j+1} k! \cdot k = (j+1+1)! - 1 = (j+2)! - 1$

$$\begin{aligned}
\sum_{k=1}^{j+1} k! \cdot k &= \sum_{k=1}^{j} k! \cdot k + (j+1)! \cdot (j+1) \\
&= (j+1)! - 1 + (j+1)! \cdot (j+1) && \text{(by the induction hypothesis)} \\
&= (j+1)!(1+j+1) - 1 && \text{(factor out } (j+1)!) \\
&= (j+1)!(j+2) - 1 \\
&= (j+2)! - 1
\end{aligned}$$

$\blacksquare$

b) Proof by induction on n:
Define $P(n)$: n is expressible as a sum of 3's and/or 5's, i.e., $\exists a, b \in \mathbb{N}, n = 3a + 5b$
Want to show: $\forall n \in \mathbb{N}, n \geq 10 \implies P(n)$
Base case: Prove that $P(10), P(11), P(12)$ are true
$P(10)$: Let $a = 0, b = 2, 10 = 5 + 5 = 3 \times 0 + 5 \times 2$, so $P(10)$ is true
$P(11)$: Let $a = 2, b = 1, 11 = 3 + 3 + 5 = 3 \times 2 + 5 \times 1$, so $P(11)$ is true
$P(12)$: Let $a = 4, b = 0, 12 = 3 + 3 + 3 + 3 = 3 \times 4 + 5 \times 0$, so $P(12)$ is true.
Induction Hypothesis: Let $n = k \in \mathbb{N}$. Assume $k \geq 10$ and that $P(k)$ is true. i.e., Assume that k is expressible as a sum of 3's and/or 5's, i.e., $\exists a_0, b_0 \in \mathbb{N}, k = 3a_0 + 5b_0$.
Let $n = k + 3$, we want to show that P(k+3) is true
i.e., want to show that $\exists a, b \in \mathbb{N}, k + 3 = 3a + 5b$
Let $a = a_0 + 1, b = b_0$,

$$\begin{aligned}
k + 3 &= 3a_0 + 5b_0 + 3 && \text{(since } k = 3a_0 + 5b_0 \text{ by the induction hypothesis)} \\
&= 3(a_0 + 1) + 5b_0 \\
&= 3a + 5b && \text{(since } a = a_0 + 1, b = b_0)
\end{aligned}$$

$\blacksquare$

c) i. Want to show: $P(2)$ is true
i.e., want to show: $\forall x_1, x_2 \in \mathbb{R}^{\geq 0}, x_1 x_2 \leq (\frac{x_1 + x_2}{2})^2$
Let $x_1, x_2 \in \mathbb{R}^{\geq 0}$.

Want to show $x_1x_2 \leq (\frac{x_1+x_2}{2})^2$:

$$\begin{aligned}
(\frac{x_1+x_2}{2})^2 &= \frac{(x_1+x_2)^2}{4} \\
&\geq \frac{(x_1+x_2)^2-(x_1-x_2)^2}{4} \\
&\text{(Since } (x_1-x_2)^2 \geq 0,\ (x_1+x_2)^2 \geq (x_1+x_2)^2-(x_1-x_2)^2\text{)} \\
&= \frac{4x_1x_2}{4} && \text{(by arithmetic)} \\
&= x_1x_2
\end{aligned}$$

So, $x_1x_2 \leq (\frac{x_1+x_2}{2})^2$ ■

ii. ***PART 1***. Prove for each $n \geq 2$, if $P(2)$ and $P(n)$ are true, then $P(2n)$ is also true.
Let $n \in \mathbb{N}$ and $n \geq 2$.
Assume $P(2)$ and $P(n)$ are true.
Want to show $P(2n)$.
i.e. Want to show: $x_1 \cdots x_{2n} \leq (\frac{x_1+\cdots+x_{2n}}{2n})^{2n}$ where $x_1, \ldots, x_{2n} \in \mathbb{R}^{\geq 0}$

$$\begin{aligned}
x_1 \cdots x_{2n} &= x_1 \cdots x_n x_{n+1} \cdots x_{2n} \\
&\leq (\frac{x_1+\cdots+x_n}{n})^n(\frac{x_{n+1}+\cdots+x_{2n}}{n})^n \\
&\qquad \text{(Since } P(n) \text{ is true by the assumption.)} \\
&\leq (\frac{(\frac{x_1+\cdots+x_n}{n})^n+(\frac{x_{n+1}+\cdots+x_{2n}}{n})^n}{2})^2 \\
&\qquad \text{(Since } P(2) \text{ is true by the assumption.)} \\
&= (\frac{\frac{(x_1+\cdots+x_n)^n}{n^n}+\frac{(x_{n+1}+\cdots+x_{2n})^n}{n^n}}{2})^2 \\
&= (\frac{(x_1+\cdots+x_n)^n+(x_{n+1}+\cdots+x_{2n})^n}{2n^n})^2 \\
&\leq (\frac{(x_1+\cdots+x_n+x_{n+1}+\cdots+x_{2n})^n}{2n^n})^2 \\
&\qquad \text{(Since } x_1, \cdots, x_{2n} \in \mathbb{R}^{\geq 0} \text{ and } n \in \mathbb{N}\text{)} \\
&= \frac{(x_1+\cdots+x_n+x_{n+1}+\cdots+x_{2n})^{2n}}{2n^{2n}} && \text{(By arithmetic)} \\
&= (\frac{x_1+\cdots+x_{2n}}{2n})^{2n}
\end{aligned}$$

Therefore, $P(2n)$ is true. ■

PART 2. Prove that $P(2^m)$ is true for all $m \in \mathbb{N}$, where $m \geq 1$.
We'll prove this using an induction on m.
Base Case: Let $m = 1$.
Since $P(2^1) = P(2)$, we know that $P(2)$ is true from (i).
Inductive Step: Let $m = k \in \mathbb{N}$. Assume $k \geq 1$ and $P(2^k)$ is true.
Let $m = k+1$. Want to show that $P(2^{k+1})$ is true.

$$P(2^{k+1}) = P(2^k 2^1)$$

By the assumption, $P(2^k)$ is true. Also, $2^k \in \mathbb{N}$ and $2^k \geq 2$ since $k \geq 1$.
From PART 1, we know that $\forall n \geq 2$, if $P(2)$ and $P(n)$ are true, then $P(2n)$ is true.
Then, because $P(2^k)$ and $P(2)$ are true, $P(2^k 2^1) = P(2^{k+1})$ is also true. $\blacksquare$

iii. Let $k \in \mathbb{N}$ and $k \geq 2$.
Assume $P(k)$ is true.
Want to show $P(k-1)$ is true.
i.e. Want to show: $\forall x_1, \ldots, x_{k-1} \in \mathbb{R}^{\geq 0}, x_1 \cdots x_{k-1} \leq (\frac{x_1+\cdots+x_{k-1}}{k-1})^{k-1}$
Let $x_1, \ldots, x_{k-1} \in \mathbb{R}^{\geq 0}$

$$x_1 \cdots x_k \leq (\frac{x_1 + \cdots + x_k}{k})^k \qquad \text{(by the assumption)}$$

Then, by multiplying $\frac{1}{k}$ to the power of both sides,

$$\begin{aligned}
(x_1 \cdots x_k)^{\frac{1}{k}} &\leq (\frac{x_1 + \cdots + x_k}{k}) \\
&= \frac{1}{k}(\frac{x_1 + \cdots + x_{k-1}}{k-1}(k-1)) + \frac{x_k}{k} && \text{(taking } \tfrac{x_k}{k} \text{ out)} \\
&= \frac{1}{k}((k-1)\frac{x_1 + \cdots + x_{k-1}}{k-1} + x_k) && \text{(taking } \tfrac{1}{k} \text{ out)} \\
&= \frac{1}{k}((k-1)x_k + x_k) && \text{(Since } x_k = \tfrac{(x_1+\cdots+x_{k-1})}{k-1}\text{)} \\
&= \frac{1}{k}((k-1+1)x_k) && \text{(by arithmetic)} \\
&= \frac{1}{k}(k \cdot x_k) && \text{(by arithmetic)} \\
&= x_k && \text{(by arithmetic)}
\end{aligned}$$

Then, we have:

$$\begin{aligned}
(x_1 \cdots x_k)^{\frac{1}{k}} &\leq x_k \\
(x_1 \cdots x_{k-1})^{\frac{1}{k}}(x_k)^{\frac{1}{k}} &\leq x_k \\
(x_1 \cdots x_{k-1})^{\frac{1}{k}} &\leq x_k^{\frac{k-1}{k}}
\end{aligned}$$

Then, by multiplying k to the power of both sides, we get:

$$\begin{aligned}
(x_1 \cdots x_{k-1}) &\leq (x_k)^{k-1} \\
&= (\frac{x_1 + \cdots + x_{k-1}}{k-1})^{k-1} && \text{(Since } x_k = \tfrac{(x_1+\cdots+x_{k-1})}{k-1}\text{)}
\end{aligned}$$

Therefore, $P(k-1)$ is true. $\blacksquare$

iv. Since we know that $P(n) \implies P(n-1)$ for all $n \in \mathbb{N}$ where $n \geq 2$, when $P(2)$ is true, $P(1)$ is true. This case covers when $n = 1$.

Now we want to know why $P(n)$ is true for all $n \in \mathbb{N}$ where $n > 1$.
We know that $P(2^m)$ is true for all $m \in \mathbb{N}$ where $m \geq 1$.
In other words, $P(n)$ is true for every power m of 2, where $n = 2^m$.
Because $P(n) \implies P(n-1)$, $P(n)$ is also true for the intervals that are between each power of 2 (i.e. the intervals that have not been covered by $P(2^m)$), recursively.
Then, $P(n)$ covers both the power of 2 and the intervals between them as well. i.e. P(n) is true for all $n \in \mathbb{N}$ where $n \geq 1$.

Question 2

a) i. Want to find x in $(EA)_{16} = (x)_8$:

$$\begin{aligned}
(EA)_{16} &= A \times 16^0 + E \times 16^1 \\
&= 10 \times 16^0 + 14 \times 16^1 && \text{(since } A = 10, E = 14) \\
&= 10 + 14 \times 16 && \text{(by arithmetic)} \\
&= (234)_{10} && \text{(by arithmetic)}
\end{aligned}$$

$(234)_{10} = 3 \times 8^2 + 5 \times 8^1 + 2 \times 8^0$
So $(234)_{10} = (352)_8$
i.e., $(EA)_{16} = (352)_8$
Therefore, $x = 352$.

ii. Want to find x in $(755)_8 = (x)_2$:

$$\begin{aligned}
(755)_8 &= 7 \times 8^2 + 5 \times 8^1 + 5 \times 8^0 \\
&= 7 \times 64 + 40 + 5 && \text{(by arithmetic)} \\
&= (493)_{10}
\end{aligned}$$

$$\begin{aligned}
(493)_{10} &= 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
&= (111101101)_2
\end{aligned}$$

Therefore, $x = 111101101$.

iii. Want to find x in $(9009)_{10} = (x)_{16}$:

$$\begin{aligned}
(9009)_{10} &= 2 \times 16^3 + 3 \times 16^2 + 3 \times 16^1 + 1 \times 16^0 \\
&= (2331)_{16}
\end{aligned}$$

Therefore, $x = 2331$.

b) Want to show: Every fraction p/q (where $p, q \in \mathbb{N}$, $q \neq 0$, and $gcd(p,q) = 1$) has a base-b representation without repeating digits if and only if there exists an $m \in \mathbb{N}$ such that $q \mid b^m$.
Since it's bi-conditional, we will prove both sides.
PART 1. Let $p, q \in \mathbb{N}$. Assume that $q \neq 0$ and $gcd(p,q) = 1$
Let $b \in \mathbb{N}$ and suppose $b \geq 2$
Assume that $\frac{p}{q}$ has a base-b representation without repeating digits, i.e.,

$$\frac{p}{q} = \sum_{i=0}^{k-1} a_i b^i + \sum_{i=1}^{m_1} \frac{y_i}{b^i} \qquad \text{(where } a_i, y_i \in \{0, \ldots, b-1\}\text{, from base b)}$$

We want to show that $\exists m \in \mathbb{N}, q | b^m$
Let $m \geq m_1$

$$\begin{aligned} \frac{p}{q}(b^m) &= b^m (\sum_{i=0}^{k-1} a_i b^i + \sum_{i=1}^{m_1} \frac{y_i}{b^i}) \\ &= b^m (\sum_{i=0}^{k-1} a_i b^i) + \sum_{i=1}^{m_1} \frac{y_i}{b^i}(b^m) \\ &= \sum_{i=0}^{k-1} a_i b^{i+m} + \sum_{i=1}^{m_1} y_i b^{m-i} \end{aligned}$$

Since a_i, y_i, i are all integers, b and m are natural numbers.
Also, because $\forall i \in [0, k-1], i + m \geq 0$, b^{i+m} is an integer.
Since we let $m \geq m_1$,
$\forall i \in [i, m_1], m - i \geq 0$. So, b^{m-i} is an integer as well.
Therefore, we know that $\sum_{i=0}^{k-1} a_i b^{i+m} + \sum_{i=1}^{m_1} y_i b^{m-i}$ is an integer. This means that $\frac{p \cdot b^m}{q}$ is also an integer. Therefore, we have:

$$\begin{aligned} b^m p &= q(\sum_{i=0}^{k-1} a_i b^{i+m} + \sum_{i=1}^{m_1} y_i b^{m-i}) \\ \implies q | b^m p & \qquad \text{(Since } \textstyle\sum_{i=0}^{k-1} a_i b^{i+m} + \sum_{i=1}^{m_1} y_i b^{m-i} \in \mathbb{Z}) \end{aligned}$$

Since $gcd(p,q) = 1, q \nmid p$,

$$\implies q | b^m$$

Therefore, q must divide b^m. ■

PART 2. Let $p, q \in \mathbb{N}$.
Assume $\exists m \in \mathbb{N}, \exists n \in \mathbb{Z}, b^m = qn$.
We want to show: $p \neq 0 \wedge gcd(p,q) = 1 \implies \exists k, m_1 \in \mathbb{Z}^+, \exists\, a_{k-1}, \ldots, a_0, y_1, \ldots, y_{m_1} \in \{0, 1, \ldots, b-1\}, \frac{p}{q} = \sum_{i=0}^{k-1} a_i b^i + \sum_{i=1}^{m_1} \frac{y_i}{b^i}$

Assume $p \neq 0 \wedge gcd(p,q) = 1$.
Let us start with the assumption.

$$\begin{aligned}
b^m &= qn \\
pb^m &= pqn && \text{(multiply both sides by p)} \\
\frac{pb^m}{q} &= pn && \text{(divide both sides by q)}
\end{aligned}$$

Because $b^m \in \mathbb{N}$, $q \in \mathbb{N}$, and $b^m = qn$, we know that $n \in \mathbb{N}$.

Then, since $p \in \mathbb{N}$ and $n \in \mathbb{N}$, we know that $pn \in \mathbb{N}$.

Then by the fact that every natural number has a base-b representation, pn has a base b representation: $pn = \sum_{i=0}^{k_0-1} c_i b^i$, where $c_i \in \{0, 1, \ldots, b-1\}$. We let the max of i to be $k_0 - 1$. Here, k_0 is a natural number and is always greater than m because we can always just add 0s to the representation to make it a valid base b representation. Let $m_1 = m$ and $k = k_0 - m$. Finally, we have pn:

$$\begin{aligned}
pn &= \sum_{i=0}^{k_0-1} c_i b^i && \text{(where } c_i \in \{0,1,\ldots,b-1\}\text{)} \\
\frac{pb^m}{q} &= \sum_{i=0}^{k_0-1} c_i b^i && \text{(Since } \tfrac{pb^m}{q} = pn\text{)} \\
\frac{p}{q} &= \frac{1}{b^m}\sum_{i=0}^{k_0-1} c_i b^i && \text{(divide both sides by } b^m \text{ to get } \tfrac{p}{q}\text{)} \\
&= \sum_{i=0}^{k_0-1} c_i b^{i-m} \\
&= \sum_{i=0}^{m-1} c_i b^{i-m} + \sum_{i=m}^{k_0-1} c_i b^{i-m} && \text{(Since } k_0 \text{ is always greater than } m\text{)} \\
&= \sum_{i=0}^{m-1} c_{m-(m-i)} b^{-(m-i)} + \sum_{i=m}^{k_0-1} c_i b^{i-m} && \text{(by manipulating the bounds and terms)} \\
&= \sum_{i=1}^{m} c_{m-i} b^{-i} + \sum_{i=0}^{k_0-1-m} c_{m+i} b^i && \text{(by manipulating the bounds and terms)}
\end{aligned}$$

Then, we have:

$$\frac{p}{q} = \sum_{i=1}^{m} c_{m-i} b^{-i} + \sum_{i=0}^{k_0-m-1} c_{m+i} b^i$$

,where c_{m-i}, $c_{m+i} \in \{0, 1, \ldots, b-1\}$, which is the same domain as where a_i and y_i in the "Want to show" statement belong to.

Then, finally, we have:

$$\frac{p}{q} = \sum_{i=0}^{k-1} a_i b^i + \sum_{i=1}^{m_1} y_i b^{-i} \qquad \text{(Since } m = m_1 \text{ and } k_0 - m = k\text{)}$$

So, we conclude that $\frac{p}{q}$ has no repeating digits. $\blacksquare$

Question 3

a) i. Translation of statement:

$$\forall \epsilon \in \mathbb{R}^+, n \in \mathcal{O}(n^{1+\epsilon})$$

Expanding the definition of big-oh:

$$\forall \epsilon \in \mathbb{R}^+, \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n \leq c \cdot n^{1+\epsilon}$$

Proof: Let $\epsilon \in \mathbb{R}^+$. Let $c = 1$. Let $n_0 = 1$. Let $n \in \mathbb{N}$ and assume $n \geq n_0$.
We will show $n \leq c \cdot n^{1+\epsilon}$ holds:

$$\begin{aligned} \epsilon &> 0 \\ 1 + \epsilon &> 1 \\ n^{1+\epsilon} &> n^1 && \text{(since } n \geq 1\text{)} \\ n &\leq 1 \cdot n^{1+\epsilon} \\ n &\leq c \cdot n^{1+\epsilon} && \text{(from c = 1)} \end{aligned}$$

$\blacksquare$

ii. Expanded statement:

$$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \log_2 n \leq c \cdot n$$

Proof: Let $c = 1$. Let $n_0 = 1$. Let $n \in \mathbb{N}$. Assume $n \geq n_0$.
We will show $\log_2 n \leq c \cdot n$ holds.
Fact 1 states that $\forall n \in \mathbb{Z}^+, n \leq 2^n$, since the n we picked is ≥ 1, by subbing in the n we picked into fact 1 directly, we have:

$$\begin{aligned} n &\leq 2^n && \text{(fact 1)} \\ \log_2 n &\leq \log_2 2^n && \text{(taking log base 2 of both sides, by fact 2)} \\ \log_2 n &\leq n && \text{(by log arithmetic)} \\ \log_2 n &\leq 1 \cdot n \\ \log_2 n &\leq c \cdot n && \text{(from } c = 1\text{)} \end{aligned}$$

$\blacksquare$

iii. Expanded statement:

$$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 2^n \leq c \cdot n!$$

Let $c = 1$. Let $n_0 = 4$.
Let $n \in \mathbb{N}$. Assume $n \geq n_0$
Want to show: $2^n \leq c \cdot n!$
i.e., Want to show $2^n \leq n!$
Proof by induction: Define $P(n) : 2^n \leq n!$
Base case: $n = 4$

$$\begin{aligned} 2^4 &\leq 4! \\ 16 &\leq 24 && \text{(base case holds.)} \end{aligned}$$

Inductive Step: Let $k \in \mathbb{N}$. Assume $k \geq 4$ and that $P(k)$ holds. i.e., $2^k \leq k!$
We will show $P(k+1)$:
$P(k+1)$ means: $2^{k+1} \leq (k+1)!$
from the induction hypothesis:

$$\begin{aligned} 2^k &\leq k! \\ 2 \cdot 2^k &\leq 2 \cdot k! \\ 2^{k+1} &\leq (k+1) \cdot k! && \text{(from } k \geq 4 > 2\text{)} \end{aligned}$$

$$(k+1)k! = (k+1)!$$

Therefore:

$$2^{k+1} \leq (k+1)!$$

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b) $f(n) \in \mathcal{O}(\log_2 n)$ means:

$$\exists c_1, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq c_1 \cdot \log_2 n$$

$2^{f(n)} \in \mathcal{O}(n^c)$ means:

$$\exists c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow 2^{f(n)} \leq c_2 \cdot n^c$$

Proof: Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $f(n) \in \mathcal{O}(\log_2 n)$.
Let $c = c_1$. Let $c_2 = 1$. Let $n_1 = n_0$. Let $n \in \mathbb{N}$ and assume $n \geq n_1$. We will show: $2^{f(n)} \leq c_2 \cdot n^c$

Using fact 3 and $f(n) \in \mathcal{O}(\log_2 n)$:

$$2^{f(n)} \leq 2^{c_1 \cdot \log_2 n}$$

$$2^{c_1 \cdot \log_2 n} = (2^{\log_2 n})^{c_1} = n^{c_1}$$

Therefore:

$$2^{f(n)} \leq n^{c_1}$$

Since $c = c_1$:

$$2^{f(n)} \leq n^c$$

Since $c_2 = 1$:

$$2^{f(n)} \leq c_2 \cdot n^c$$

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c) Proof: Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$.
Assume:

$$\exists n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq 1$$

Want to show:

$$\exists c, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow e^{f(n)} - 1 \leq c \cdot f(n)$$

Let $c = e - 1$. Let $n_1 = n_0$ from the eventually dominated assumption. Let $n \in \mathbb{N}$. Assume $n \geq n_1$ (therefore $n \geq n_0$). We will show $e^{f(n)} - 1 \leq c \cdot f(n)$

Using the series given, substitute $f(n)$ for x:

$$\begin{aligned}
e^{f(n)} - 1 &= f(n) + \frac{f(n)^2}{2!} + \frac{f(n)^3}{3!} + \cdots \\
&= f(n)(1 + \frac{f(n)}{2!} + \frac{f(n)^2}{3!} + \cdots) \\
&\leq f(n)(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots) \\
&\qquad\qquad\qquad \text{(since } \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq 1 \text{ by the assumption)} \\
&= f(n)(\sum_{k=0}^{\infty} \frac{1}{k!} - 1) \\
&= f(n)(e - 1) \qquad \text{(from the definition of Euler's number, } e = \textstyle\sum_{k=0}^{\infty} \frac{1}{k!}\text{)} \\
&= f(n)(c)
\end{aligned}$$

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