CSC236H Exercise 1

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IMPORTANT:

- The question(s) in the "**Exercises**" section are worth 1% of your total mark in the course.

- You may work in groups of no more than **two** students, and you should produce a single solution in a **PDF file**, submitted to MarkUs. Handwritten submissions are accepted as long as they are neat and legible.

- The problems in the "**Practice Problems**" section are **optional**.
You DO NOT need to submit your answers to those questions.

- Start working on the exercises *before* the tutorial. During the tutorial you may ask the TAs for hints if you are stuck. You may also ask them to comment on your answers, which you may then change accordingly.

- Note that even though you have the opportunity to get help from your TA to improve your answers, it's **your responsibility** to make sure that your submission is correct and complete.

Exercises

1. Assume $x \in \mathbb{R}$ and $(x + \frac{1}{x}) \in \mathbb{Z}$. Use induction to prove that for all $n \in \mathbb{N}$

$$(x^n + \frac{1}{x^n}) \in \mathbb{Z}.$$

Practice Problems

- Use induction to prove that $3^{2n} - 1$ is divisible by 8, for all $n \in \mathbb{N}$.

1. Assume $x \in \mathbb{R}$ and $(x + \frac{1}{x}) \in \mathbb{Z}$. Use induction to prove that for all $n \in \mathbb{N}$

$$(x^n + \frac{1}{x^n}) \in \mathbb{Z}.$$

Let $x \in \mathbb{R}$.

$P(n) : \left(x^n + \frac{1}{x^n}\right) \in \mathbb{Z}$

We will use complete induction to prove $P(n)$ holds for all $n \geq 0$.

Base Case 1: Let $k = 0$.

$$x^k + \frac{1}{x^k} = x^0 + \frac{1}{x^0} = 1 + 1 = 2$$

Then, $2 \in \mathbb{Z}$.

$P(0)$ holds.

Base Case 2: Let $k = 1$

$$x^k + \frac{1}{x^k} = x + \frac{1}{x} \in \mathbb{Z}$$ by the assumption.

Then, $P(1)$ holds.

Inductive Step: Let $k \in \mathbb{N}$. and $k \geq 1$

Assume for all $j \in \mathbb{N}$, $0 \leq j \leq k$, $P(j)$ holds. [IH]

WTP: $P(k+1)$ holds.

i.e. $\left(x^{k+1} + \frac{1}{x^{k+1}}\right) \in \mathbb{Z}$

Since $0 \leq k \leq k$, $x^k + \frac{1}{x^k} \in \mathbb{Z}$.

Then, $(x+\frac{1}{x})(x^k + \frac{1}{x^k}) \in \mathbb{Z}$ because $x + \frac{1}{x} \in \mathbb{Z}$ by the assumption.

$$(x+\frac{1}{x})(x^k + \frac{1}{x^k}) = x^{k+1} + \frac{x}{x^k} + \frac{x^k}{x} + \frac{1}{x x^k}$$

$$= x^{k+1} + x^{1-k} + x^{k-1} + \frac{1}{x^{k+1}}$$

$$= x^{k+1} + \frac{1}{x^{k+1}} + x^{1-k} + x^{k-1}$$

$$= x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}}$$

Here, by [I.H], $0 \leq k-1 \leq k$, so

$(x^{k-1} + \frac{1}{x^{k-1}}) \in \mathbb{Z}$.

We know that $(x+\frac{1}{x})(x^k + \frac{1}{x^k}) \in \mathbb{Z}$.

Since we also know that $(x^{k-1} + \frac{1}{x^{k-1}}) \in \mathbb{Z}$,

$\left((x+\frac{1}{x})(x^k + \frac{1}{x^k}) - (x^{k-1} + \frac{1}{x^{k-1}})\right) \in \mathbb{Z}$.

That is, $\left(x^{k+1} + \frac{1}{x^{k+1}}\right) \in \mathbb{Z}$.

Therefore, $P(k+1)$ holds. $\blacksquare$