CSC165H1, Problem Set 2

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Question 1

a) This statement is False. The negation of this statement in predicate notation is:

$$\forall x \in \mathbb{Z}, 12 \mid x(x+1)(x+2)(x+3)$$

Proof:
Let $x \in \mathbb{Z}$
Want to show: $\forall x \in \mathbb{Z}, \exists k \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 12k$
Part 1: Show that $\forall x \in \mathbb{Z}, 3 \mid x(x+1)(x+2)(x+3)$, i.e., $\forall x \in \mathbb{Z}, \exists l \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 3l$.
By the Quotient-Remainder Theorem, there are 3 cases.

- Case 1: $3 \mid x : \exists m \in \mathbb{Z}, x = 3m$
  Since $x = 3m$, $x(x+1)(x+2)(x+3) = 3m(3m+1)(3m+2)(3m+3)$
  Let $l = m(3m+1)(3m+2)(3m+3) \in \mathbb{Z}$
  $\implies x(x+1)(x+2)(x+3) = 3l$
  $\implies 3 \mid x(x+1)(x+2)(x+3)$
- Case 2: $3 \nmid x : \exists m \in \mathbb{Z}, x = 3m+1$
  Since $x = 3m+1$,

$$\begin{aligned} x(x+1)(x+2)(x+3) &= (3m+1)(3m+2)(3m+3)(3m+4) \\ &= 3(3m+1)(3m+2)(m+1)(3m+4) \end{aligned}$$

  Let $l = (3m+1)(3m+2)(m+1)(3m+4) \in \mathbb{Z}$,
  $\implies x(x+1)(x+2)(x+3) = 3l$
  $\implies 3 \mid x(x+1)(x+2)(x+3)$
- Case 3: $3 \nmid x : \exists m \in \mathbb{Z}, x = 3m+2$
  Since $x = 3m+2$,

$$\begin{aligned} x(x+1)(x+2)(x+3) &= (3m+2)(3m+3)(3m+4)(3m+5) \\ &= 3(3m+2)(m+1)(3m+4)(3m+5) \end{aligned}$$

  Let $l = (3m+2)(m+1)(3m+4)(3m+5) \in \mathbb{Z}$,
  $\implies x(x+1)(x+2)(x+3) = 3l$
  $\implies 3 \mid x(x+1)(x+2)(x+3)$

So $3 \mid x(x+1)(x+2)(x+3)$ in all 3 cases.
i.e., $\forall x \in \mathbb{Z}, \exists l \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 3l$

Part 2: Show that $\forall x \in \mathbb{Z}, 4 \mid x(x+1)(x+2)(x+3)$, i.e., $\forall x \in \mathbb{Z}, \exists j \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 4j$.
By the Quotient-Remainder Theorem, there are 4 cases.

- Case 1: $4 \mid x : \exists n \in \mathbb{Z}, x = 4n$
  Since $x = 4n$, $x(x+1)(x+2)(x+3) = 4n(4n+1)(4n+2)(4n+3)$
  Let $q = n(4n+1)(4n+2)(4n+3) \in \mathbb{Z}$
  $\implies x(x+1)(x+2)(x+3) = 4q$
  $\implies 4 \mid x(x+1)(x+2)(x+3)$
- Case 2: $4 \nmid x : \exists n \in \mathbb{Z}, x = 4n+1$
  Since $x = 4n+1$,
  $$\begin{aligned} x(x+1)(x+2)(x+3) &= (4n+1)(4n+2)(4n+3)(4n+4) \\ &= 4(4n+1)(4n+2)(4n+3)(n+1) \end{aligned}$$
  Let $q = (4n+1)(4n+2)(4n+3)(n+1) \in \mathbb{Z}$,
  $\implies x(x+1)(x+2)(x+3) = 4q$
  $\implies 4 \mid x(x+1)(x+2)(x+3)$
- Case 3: $4 \nmid x : \exists n \in \mathbb{Z}, x = 4n+2$
  Since $x = 4n+2$,
  $$\begin{aligned} x(x+1)(x+2)(x+3) &= (4n+2)(4n+3)(4n+4)(4n+5) \\ &= 4(4n+2)(4n+3)(n+1)(4n+5) \end{aligned}$$
  Let $q = (4n+2)(4n+3)(n+1)(4n+5) \in \mathbb{Z}$,
  $\implies x(x+1)(x+2)(x+3) = 4q$
  $\implies 4 \mid x(x+1)(x+2)(x+3)$
- Case 4: $4 \nmid x : \exists n \in \mathbb{Z}, x = 4n+3$
  Since $x = 4n+3$,
  $$\begin{aligned} x(x+1)(x+2)(x+3) &= (4n+3)(4n+4)(4n+5)(4n+6) \\ &= 4(4n+3)(n+1)(4n+5)(4n+6) \end{aligned}$$
  Let $q = (4n+3)(n+1)(4n+5)(4n+6) \in \mathbb{Z}$,
  $\implies x(x+1)(x+2)(x+3) = 4q$
  $\implies 4 \mid x(x+1)(x+2)(x+3)$

So $4 \mid x(x+1)(x+2)(x+3)$ in all 4 cases.
i.e., $\forall x \in \mathbb{Z}, \exists j \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 4j$.

Since we've shown above that $\forall x \in \mathbb{Z}, 3 \mid x(x+1)(x+2)(x+3)$,
so $\forall x \in \mathbb{Z}, \exists j \in \mathbb{Z}$:

$$\begin{aligned}
& 3 \mid x(x+1)(x+2)(x+3) \\
\implies & 3 \mid 4j \quad (\text{since } \forall x \in \mathbb{Z}, \exists j \in \mathbb{Z}, x(x+1)(x+2)(x+3) = 4j) \\
\implies & 3 \mid j \qquad (\text{since } 3 \nmid 4) \\
\implies & \exists g \in \mathbb{Z}, j = 3g \\
\implies & \exists g \in \mathbb{Z}, 3 \mid 4(3g) \;\; (\text{since } 3 \mid 4j \text{ and } j = 3g) \\
\implies & \exists g \in \mathbb{Z}, 3 \mid 12g
\end{aligned}$$

Therefore, $\forall x \in \mathbb{Z}, \exists g \in \mathbb{Z}$, $x(x+1)(x+2)(x+3) = 12g$,
i.e., $\forall x \in \mathbb{Z}, 12 \mid x(x+1)(x+2)(x+3)$. ■

b) This statement is True. This statement in predicate notation is:

$$\forall x \in \mathbb{R}, x \geq 6 \implies 4x^2 - 3\lfloor x \rfloor^2 \geq 9$$

Proof:
Let $x \in \mathbb{R}$, and assume $x \geq 6$
Want to show: $4x^2 - 3\lfloor x \rfloor^2 \geq 9$
Since $x \in \mathbb{R}$,

$$\begin{aligned}
& 0 \leq x - \lfloor x \rfloor \leq 1 && (\text{by Fact 1}) \\
\implies & x - 1 \leq \lfloor x \rfloor \leq x && (\text{by arithmetic}) \\
\implies & (x-1)^2 \leq \lfloor x \rfloor^2 \leq x^2
\end{aligned}$$
$$(\text{since } x \geq 6 > 0, \text{ and } y = x^2 \text{ is a monotonically increasing function when } x > 0)$$

Therefore,

$$\begin{aligned}
4x^2 - 3\lfloor x \rfloor^2 &\geq 4x^2 - 3x^2 && (\text{since } \lfloor x \rfloor^2 \leq x^2) \\
&= x^2 && (\text{by arithmetic}) \\
&\geq 36 && (\text{since } x \geq 6) \\
&\geq 9 && (\text{since } 36 \geq 9)
\end{aligned}$$

Therefore, $4x^2 - 3\lfloor x \rfloor^2 \geq 9$. ■

c) This statement is False. The negation of this statement in predicate notation is:

$$\begin{aligned}
& \forall g : \mathbb{R} \to \mathbb{R}, \forall h : \mathbb{R} \to \mathbb{R}, (\forall x \in \mathbb{R}, (g(-x) = -g(x)) \;\wedge (h(-x) = -h(x))) \\
\implies & ((\exists x \in \mathbb{R}, (g(x) - h(x)) \neq (g(-x) - h(-x))) \vee (\exists k \in \mathbb{R}, \forall x \in \mathbb{R}, g(x) - h(x) = k))
\end{aligned}$$

Proof:
Let $g : \mathbb{R} \to \mathbb{R}, h : \mathbb{R} \to \mathbb{R}$

Let $x \in \mathbb{R}$, assume $g(-x) = -g(x)$ and $h(-x) = -h(x)$
Want to show: $(\exists x \in \mathbb{R}, (g(x) - h(x)) \neq (g(-x) - h(-x))) \vee (\exists k \in \mathbb{R}, \forall x \in \mathbb{R}, g(x) - h(x) = k)$
Let $f(x) = g(x) - h(x)$, then:

$$\begin{aligned}
f(-x) &= g(-x) - h(-x) \\
&= -g(x) - (-h(x)) && \text{(since } \forall x \in \mathbb{R},\ g(-x) = -g(x) \text{ and } h(-x) = -h(x)) \\
&= -g(x) + h(x) && \text{(by arithmetic)} \\
&= -(g(x) - h(x)) && \text{(by arithmetic)} \\
&= -f(x) && \text{(since } f(x) = g(x) - h(x))
\end{aligned}$$

So $\forall x \in \mathbb{R}, f(-x) = -f(x)$
i.e., $\forall x \in \mathbb{R}, g(-x) - h(-x) = -(g(x) - h(x))$

Since g, h are any odd functions from $\mathbb{R}$ to $\mathbb{R}$, there can be only two cases in terms of the relationship between g and h:
Case 1: $\forall x \in \mathbb{R}, g(x) = h(x)$, i.e., $\forall x \in \mathbb{R}, f(x) = g(x) - h(x) = 0$
In this case, since $-x \in \mathbb{R}, f(-x) = 0$.
So $\forall x \in \mathbb{R}, f(-x) = -f(x) = f(x) = 0$, i.e., $\exists k = 0, \forall x \in \mathbb{R}, g(x) - h(x) = 0$.

Case 2: $\exists x \in \mathbb{R}, g(x) \neq h(x)$, i.e., $\exists x \in \mathbb{R}, f(x) = g(x) - h(x) \neq 0$
So let $x = x_0 \in \mathbb{R}$, such that $f(x_0) \neq 0$.
Since $\forall x \in \mathbb{R}, f(-x) = -f(x)$, so $f(-x_0) = -f(x_0)$.
Since $f(x_0) \neq 0$, so $f(x_0) \neq -f(x_0)$.
Since $-f(x_0) = f(-x_0)$, so $f(x_0) \neq f(-x_0)$.
i.e., $\exists x = x_0 \in \mathbb{R}, (g(x_0) - h(x_0)) \neq (g(-x_0) - h(-x_0))$. ∎

Question 2

a)

- $$G_0(S,T) \implies G_0(S,T)$$

  $G_0(S,T) \implies G_0(S,T)$ is equivalent to $\neg G_0 \vee G_0$. $\neg G_0 \vee G_0$ always outputs True. Equivalently, the implication statement will also always output True.

- $$G_0(S,T) \implies G_1(S,T)$$

  Assume every x is greater than every y. Then for every x, there must be at least one y that is less than x. Therefore, if the hypothesis is true, the conclusion of the implication statement is also true.

- $$G_0(S,T) \implies G_2(S,T)$$

Assume every x is greater than every y. Then, there must be always at least one x that is greater than every y. Therefore, if the hypothesis is true, the conclusion of the statement is also true.

- $$G_0(S,T) \implies G_3(S,T)$$
  Assume every x is greater than every y. Then, there must be always at least one x that is greater than at least one y. Therefore, if the hypothesis is true, the conclusion of the statement is also true.

- $$G_1(S,T) \implies G_1(S,T)$$
  $G_1(S,T) \implies G_1(S,T)$ is equivalent to $\neg G_1 \vee G_1$. $\neg G_1 \vee G_1$ always outputs True. Equivalently, the implication statement will also always output True.

- $$G_1(S,T) \implies G_3(S,T)$$
  Assume every x is greater than at least one y (let this y be y_0). Then, that means that there exists at least one x that is greater than y_0. Therefore, if the hypothesis is true, the conclusion of the statement is also true.

- $$G_2(S,T) \implies G_2(S,T)$$
  $G_2(S,T) \implies G_2(S,T)$ is equivalent to $\neg G_2 \vee G_2$. $\neg G_2 \vee G_2$ always outputs True. Equivalently, the implication statement will also always output True.

- $$G_2(S,T) \implies G_3(S,T)$$
  Assume at least one x is greater than every y (let this x be x_0). Then, that means that there exists at least one x, x_0, that is greater than at least one y. Therefore, if the hypothesis is true, the conclusion of the statement is also true.

- $$G_3(S,T) \implies G_3(S,T)$$
  $G_3(S,T) \implies G_3(S,T)$ is equivalent to $\neg G_3 \vee G_3$. $\neg G_3 \vee G_3$ always outputs True. Equivalently, the implication statement will also always output True.

b)

- $$G_0([3,5],[0,2])$$
  $$\forall x \in [3,5], \forall y \in [0,2], x > y$$
  This statement is True. The proof is:
  Let $x \in [3,5]$. Let $y \in [0,2]$. Want to show $x > y$.

  $$5 \geq x \geq 3 \text{ and } 2 \geq y \geq 0.$$
  $$\begin{aligned} 5 \geq x &\geq 3 \\ &> 2 \\ &\geq y \end{aligned}$$

Therefore, $x > y$ ■

- $$G_0(\mathbb{R}, \mathbb{R})$$

  $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y$$

  This statement is False. The disproof is:
  Want to show the negation of the original statement $\neg G_0(\mathbb{R}, \mathbb{R})$: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x \leq y$ is true.
  Let $x = 3$. Let $y = 4$.

  $$3 \leq 4.$$

  So, $x \leq y$
  Since the negation of the original statement is true, the original statement is false.
  ■

- $$G_1((0, 1], (0, 1))$$

  $$\forall x \in (0, 1], \exists y \in (0, 1), x > y$$

  This statement is True. The proof is:
  Let $x \in (0, 1]$. Let $y = \frac{x}{2}$.
  Want to show $x > y$.

  $$\begin{aligned} 1 \geq x &> 0 \\ x &> \frac{x}{2} && \text{(Since } 1 \geq x > 0\text{)} \\ &= y \end{aligned}$$

  So, $x > y$. ■

- $$G_2(\mathbb{Z}, (-\infty, 0))$$

  $$\exists x \in \mathbb{Z}, \forall y \in (-\infty, 0), x > y$$

  This statement is True. The proof is:
  Let $x = 1$. Let $y \in (-\infty, 0)$.
  Want to show $x > y$.

  $$\begin{aligned} x &= 1 \\ &> 0 \\ &> y && \text{(Since } 0 > y\text{)} \end{aligned}$$

  So, $x > y$. ■

- $$G_2(\{10\}, \varnothing)$$
  $$\exists x \in \{10\}, \forall y \in \varnothing, x > y$$

  This statement is True. The proof is:
  Assume $\neg G_2(10, \varnothing)$ is True.
  i.e., assume $\neg G_2(\{10\}, \varnothing)$: $\forall x \in \{10\}, \exists y \in \varnothing, x \geq y$ is True
  Want to show $x \leq y$, i.e., $\neg G_2(10, \varnothing)$ is False
  Let x $\in \{10\}$. This means that $x = 10$.
  However, since $y \in \varnothing$, it means that there does not exist y in $\varnothing$, such that $x \geq y$.
  So $\neg G_2(10, \varnothing)$ is False.
  Since $\neg G_2(\{10\}, \varnothing)$ is false, $G_2(\{10\}, \varnothing)$ is true. ■

- $$G_3([0, 1] \cap \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$$
  $$\exists x \in [0, 1] \cap \mathbb{Q}, \exists y \in \mathbb{R} \setminus \mathbb{Q}, x > y$$

  This statement is True. The proof is:
  Let $x = 1$. Let $y = -\pi$.
  Want to show $x > y$.

  $$\begin{aligned} 1 &> -\pi \\ &\approx -3.14 \end{aligned}$$

  So, $x > y$. ■

Question 3

a) Suppose $f(x) = (x + \frac{c}{x})^b$, where $b, c \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Original statement: $\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*$, if $f(x)$ has no constant term, then b is an odd number.

Proof by contrapositive, and thus the statement becomes, $\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*$, if b is an even number, then $f(x)$ has at least one constant term.

Let $b, c \in \mathbb{Z}^+$. Let $x \in \mathbb{R}^*$. Assume b is an even number. We will show that $f(x)$ has at least one constant term.

From the binomial theorem, we have:

$$(x + \frac{c}{x})^b = \sum_{k=0}^{b} \binom{b}{k} \cdot x^k \cdot (\frac{c}{x})^{b-k}$$

b being even means that $\exists m \in \mathbb{N}, 2m = b$. Consider the term where $k = 2m - k$, therefore, $k = m$. We would have:

$$\binom{2m}{k} \cdot x^k \cdot (\frac{c}{x})^{2k-k}$$

$$\binom{2m}{k} \cdot x^k \cdot (\frac{c^k}{x^k})$$
$$\binom{2m}{k} \cdot c^k$$

The term of $\binom{2m}{k} \cdot c^k$ is not based on x as it does not contain it:

$$\frac{(2m)!}{k!(2m-k)!} \cdot c^k$$

(since $2m = b, b \in \mathbb{Z}^+$, and $k = m, m \in \mathbb{N}$, so $2m \in \mathbb{N}, k \in \mathbb{N}$, and $k \leq 2m$.)
Therefore it is a constant term.
Since the contrapositive is true, the original statement is true. ■

b) Let $f(n,k) = \binom{n}{k}$, where $n, k \in \mathbb{N}$, we will show that it is an onto function.

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}^+$ being an onto function means that:

$\forall x \in \mathbb{Z}^+, \exists (n,k) \in \mathbb{N}\times\mathbb{N}, x = f(n,k) = \binom{n}{k}$ (by the definition of onto given in Worksheet 2)

Let $x \in \mathbb{Z}^+$. Let $n = x$ and $k = 1$.

$$\begin{aligned}
x &= n \\
x &= \frac{n}{1!} && (1! = 1) \\
x &= \frac{n}{k!} && (k = 1) \\
x &= \frac{n(n-1)!}{k!(n-1)!} && (\text{multiply } \tfrac{(n-1)!}{(n-1)!}) \\
x &= \frac{n!}{k!(n-1)!} && (\text{since } n(n-1)! = n!) \\
x &= \frac{n!}{k!(n-k)!} && (\text{since k} = 1)
\end{aligned}$$

Since $n = x, x \in \mathbb{Z}^+$ and $k = 1$, so $n \in \mathbb{N}$ and $k \in \mathbb{N}$ and $k \leq n$.
Therefore, $f(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
Therefore, $x = f(n,k)$
Thus, for all x values in $\mathbb{Z}^+$, there exist $(n,k) \in \mathbb{N} \times \mathbb{N}$ such that $x = f(n,k) = \binom{n}{k}$.
Therefore, the function is onto.

c) Let $n, k \in \mathbb{Z}^+$. Assume $k < n$. Want to show: $\binom{n}{k} \div \binom{n-1}{k} > 1$

We know: $n, k \in \mathbb{N}$, and specifically $n > 0, k > 0, n > k$

$$\begin{aligned}
n &> k \\
n - k &> 0 \\
n > n - k &> 0 && (\text{Since } k > 0 \text{ and } n > 0)
\end{aligned}$$

From this, we know that $n > n-k$, and we can have:

$$\begin{aligned}
n &> n-k \\
\frac{n}{n-k} &> \frac{n-k}{n-k} && \text{(Since } n-k>0\text{)} \\
\frac{n}{n-k} &> 1 && \left(\tfrac{n-k}{n-k}=1\right) \\
\frac{n(n-k-1)!}{(n-k)!} &> 1 && \text{(multiplying } \tfrac{(n-k-1)!}{(n-k-1)!} \text{ on the left hand side)} \\
\frac{n!(n-1-k)!}{(n-k)!(n-1)!} &> 1 && \text{(multiplying } \tfrac{(n-1)!}{(n-1)!} \text{ on the left hand side)} \\
\frac{n!}{(n-k)!}\cdot\frac{(n-1-k)!}{(n-1)!} &> 1 \\
\frac{n!}{k!(n-k)!}\cdot\frac{k!(n-1-k)!}{(n-1)!} &> 1 && \text{(multiplying } \tfrac{k!}{k!} \text{ on left hand side)} \\
\frac{n!}{k!(n-k)!}\div\frac{(n-1)!}{k!(n-1-k)!} &> 1 && \text{(switching multiplication into division)} \\
\binom{n}{k}\div\binom{n-1}{k} &> 1
\end{aligned}$$

(since we know that $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \leq n \implies \binom{n}{k} = \frac{n!}{k!(n-k)!}$, and here we have $n, k \in \mathbb{N}$, and $k < n$)

Therefore, it satisfies $\binom{n}{k} \div \binom{n-1}{k} > 1$. ■