CSC236, A1

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[redacted]

Q1

$P(n)$: Starting from a single group of n coins, the person always wins a total of $\frac{n(n-1)}{2}$ coins.
Base Case: Let $k = 1$.
As soon as the person starts the game, the game is over because there is exactly a group of one coin. Since one coin cannot be divided into smaller groups, the person wins 0 dollars. Also, $\frac{1(1-1)}{2} = 0$.
Therefore, $P(1)$ holds.
Inductive Case: Let $k \in \mathbb{N}, k \geq 1$.
Assume that for all $j \in \mathbb{N}$, $1 \leq j \leq k, P(j)$ holds. [IH]
Want to prove: $P(k+1)$ is true.
i.e. Starting from a group of $k+1$ coins, a person wins exactly $\frac{k(k+1)}{2}$ coins.
Suppose a group of $k+1$ coins are divided into $(k+1-x)$ coins and x coins where $x \in \mathbb{N}$, $1 \leq x < k+1$.
Then, by IH, we get $\frac{x(x-1)}{2}$ dollars for the group of x coins and $\frac{(k-x+1)(k-x)}{2}$ dollars for the group of $(k+1-x)$ coins, since $1 \leq x, k+1-x \leq k$.
Note that the person gets $x(k+1-x)$ dollars as the person divides the group of $(k+1)$ coins at first.
Then, we have $\frac{x(x-1)}{2} + \frac{(k-x+1)(k-x)}{2} + x(k+1-x)$ dollars in total.

$$\begin{aligned}
\frac{x(x-1)}{2} + \frac{(k-x+1)(k-x)}{2} + x(k+1-x) &= \frac{x(x-1)}{2} + \frac{(k-x+1)(k-x)}{2} + \frac{2x(k+1-x)}{2} \\
&= \frac{x^2 - x + k^2 - kx + k - xk + x^2 - x + 2xk + 2x - 2x^2}{2} \\
&= \frac{k^2 + k}{2} \\
&= \frac{k(k+1)}{2}
\end{aligned}$$

■

Q2

Let $B_n = \{2^i : i \in \mathbb{N} \text{ and } 0 \leq i < n\}$, where $n \in \mathbb{N}$.
Let $s \in \mathbb{N}, 0 \leq s < 2^n$.
$P(n)$: B_n contains a subset A such that the sum of all elements in A is equal to S. i.e. $\sum_{x \in A} x = s$
Want to prove: For all $n \in \mathbb{N}, P(n)$ holds.
Assume for a contradiction that there exists a natural number k such that $P(k)$ doesn't hold.
Let S be a set of all natural numbers j such that $j \in S$ iff $P(j)$ does not hold.
By definition of S, $S \subseteq \mathbb{N}$.
By definition of S, S is not empty.
So, by PWO, S has a mimimum element b.
Claim. $b \neq 0$
Proof. We know that $B_0 = \emptyset$. By the definition, we have: $0 \leq s < 2^0$. That is, $0 \leq s < 1$. Since $s \in \mathbb{N}$, $s = 0$. Also, $\sum_{x \in \emptyset} x = 0$. Then, $\sum_{x \in \emptyset} x = s$. So, $P(0)$ holds and by def. of S, $0 \notin S$. Since we assumed that $b \in S$, $b \neq 0$.
Since $b \in \mathbb{N}$ and $b \neq 0$, $b > 0$ and $b - 1 \geq 0$ and so $b - 1 \in \mathbb{N}$.
Since b is the minimum element of S, $b - 1 \notin S$. So, by def. of S, $P(b-1)$ holds.
Then, we have B_{b-1} that contains a subset A' such that the sum, s', of all elements in A' is equal to S where $s' \in \mathbb{N}$ and $0 \leq s' < 2^{b-1}$.
Consider B_b. The sum, s'', of all elements of a subset A'' of B_b will be $s' + 2^{b-1}$.
We have:

$$
\begin{aligned}
&0 \leq s' < 2^{b-1} && \text{(by the def. of } B_{b-1}\text{)} \\
&0 \leq s' + 2^{b-1} < 2^{b-1} + 2^{b-1} \\
&0 \leq s'' < 2^b
\end{aligned}
$$

Thus, B_b contains a subset A'' such that the sum of all elements in A'' is equal to s'' where $0 \leq s'' < 2^b$.
So $P(b)$ holds.
On the other hand, $b \in S$ and by def. of S, $P(b)$ does not hold.
This is a contradiction, so the original assumption must be false. ■

Q3

$Z(v)$: number of occurrences of 0 in the binary string v
$O(v)$: number of occurrences of 1 in the binary string v
$P(w)$: For every proper prefix u of the string w, $O(u) \le Z(u)$
Want to prove: For all $w \in S$, $P(w)$ holds.
Base Case: Let $w = 1$.
There is only one character. There is no proper prefix for w, as $\epsilon \notin S$.
Therefore, $P(1)$ holds true vacuously.
Inductive Step: Let $w_1, w_2 \in S$.
Assume $P(w_1)$ and $P(w_2)$ hold. [IH]
i.e. w_1 has a proper prefix u_1 such that $O(u_1) \le Z(u_1)$.
w_2 has a proper prefix u_2 such that $O(u_2) \le Z(u_2)$
Want to prove: $P(0 \cdot w_1 \cdot w_2)$ holds.
Let $w_1 = u_1 v_1$ and $w_2 = u_2 v_2$. Here, u_1 and u_2 are the proper prefixes of w_1 and w_2, respectively.
Let $w = 0 \cdot w_1 \cdot w_2$.
That is, $w = 0 \cdot u_1 \cdot v_1 \cdot u_2 \cdot v_2$.
Let $u' =$ prefix of w.
Case 1. Let $u' = 0$.
Then, $O(u') = 0$ and $Z(u') = 1$.
Therefore, $O(u') \le Z(u')$, so $P(w)$ holds true.
Case 2. Let $u' = 0 \cdot u_1$.
By IH, $O(u_1) \le Z(u_1)$.
Since we have an additional 0 at the front, $O(u_1) < Z(u_1) + 1$.
So, $O(u') \le Z(u')$.
Therefore, $P(w)$ holds true.
Case 3. Let $u' = 0 \cdot u_1 \cdot v_1$.
By the definition of the proper prefix, v_1 can have a minimum length of 1 character.
There are two subcases we can consider for v_1.
Subcase 1. Let $v_1 = 0$.
Then, the number of occurrences of 1 for v_1 is 0 and the number of occurrences of 0 for v_1 is 1. Since there is also 0 at the front, we have:

$$\begin{aligned} O(u_1) &\le Z(u_1) && \text{(By IH)} \\ 0 + O(u_1) + 0 &< 1 + Z(u_1) + 1 \\ O(u') &< Z(u') \end{aligned}$$

Subcase 2. Let $v_1 = 1$.
Then, the number of occurrences of 1 for v_1 is 1 and the number of occurrences of 0 for v_1 is 0. Since there is also 0 at the front, we have:

$$\begin{aligned} O(u_1) &\le Z(u_1) && \text{(By IH)} \\ 0 + O(u_1) + 1 &\le 1 + Z(u_1) + 0 \\ O(u') &\le Z(u') \end{aligned}$$

Therefore, $O(u') \leq Z(u')$ for both subcases. So, $P(w)$ holds true.
Case 4. Let $u' = 0 \cdot u_1 \cdot v_1 \cdot u_2$.
By the definition of the proper prefix, v_1 can have a minimum length of 1 character.
There are two subcases we can consider for v_1.
Subcase 1. Let $v_1 = 0$.
Then, the number of occurrences of 1 for v_1 is 0 and the number of occurrences of 0 for v_1 is 1. Since there is also 0 at the front, we have:

$$\begin{aligned}
O(u_1) + O(u_2) &\leq Z(u_1) + Z(u_2) \qquad \text{(By IH, } O(u_1) \leq Z(u_1) \text{ and } O(u_2) \leq Z(u_2)\text{)} \\
0 + O(u_1) + 0 + O(u_2) &< 1 + Z(u_1) + 1 + Z(u_2) \\
O(u') &< Z(u')
\end{aligned}$$

Subcase 2. Let $v_1 = 1$.
Then, the number of occurrences of 1 for v_1 is 1 and the number of occurrences of 0 for v_1 is 0. Since there is also 0 at the front, we have:

$$\begin{aligned}
O(u_1) + O(u_2) &\leq Z(u_1) + Z(u_2) \qquad \text{(By IH, } O(u_1) \leq Z(u_1) \text{ and } O(u_2) \leq Z(u_2)\text{)} \\
0 + O(u_1) + 1 + O(u_2) &\leq 1 + Z(u_1) + 0 + Z(u_2) \\
O(u') &\leq Z(u')
\end{aligned}$$

Therefore, $O(u') \leq Z(u')$ for both subcases. So, $P(w)$ holds true. ∎