CSC236H Exercise 2

Youngsin Ryoo

IMPORTANT:

- The question(s) in the "**Exercises**" section are worth 1% of your total mark in the course.
- You may work in groups of no more than **two** students, and you should produce a single solution in a PDF file, submitted to MarkUs. Handwritten submissions are accepted as long as they are neat and legible..
- The problems in the "**Practice Problems**" section are **optional**.
  You DO NOT need to submit your answers to those questions.
- Start working on the exercises *before* the tutorial. During the tutorial you may ask the TAs for hints if you are stuck. You may also ask them to comment on your answers, which you may then change accordingly.
- Note that even though you have the opportunity to get help from your TA to improve your answers, it's **your responsibility** to make sure that your submission is correct and complete.

Exercises

1. Let G be a set defined as follows:

   - if x is a propositional variable, then $x \in G$;
   - if $f_1, f_2 \in G$, then $\neg f_1 \in G$, $(f_1 \vee f_2) \in G$, and $(f_1 \wedge f_2) \in G$;
   - nothing else belongs to G.

   Use structural induction to prove that for every $f \in G$, there exists $f' \in G$ such that f and f' are logically equivalent, and f' does not contain $\wedge$ symbol.

   (Recall that propositional formulas f_1 and f_2 are *logically equivalent* if f_1 and f_2 evaluate to the same value, no matter how their variables are set.)

$P(f)$: There exists f' s.t. f and f' are logically equivalent, and f' does not contain $\wedge$ symbol.

Base Case : let x be a propositional variable.

Let $f = x$

Now let $f' = x$.

Then f and f' are logically equivalent and f' does not contain $\wedge$ symbol since x is a (propositional) variable.

So $P(x)$ holds.

Inductive step : let $f_1, f_2 \in G$.

Assume $P(f_1)$ and $P(f_2)$ hold. [IH]

i.e. There exists f_1' s.t. f_1 and f_1' are logically equivalent and f_1' doesn't contain $\wedge$ symbol!

There exists f_2' s.t. f_2 and f_2' are logically equivalent and f_2' doesn't contain $\wedge$ symbol!

WTP ① $P(\neg f_1)$ ② $P(f_1 \vee f_2)$ ③ $P(f_1 \wedge f_2)$ are true.

Case ① :

By IH, there exists f_1' s.t. f_1 and f_1' are logically equivalent and f_1' doesn't contain $\wedge$ symbol.

For $f = \neg f_1$, let $f' = \neg f_1'$

By IH, since f_1 and f_1' are logically equivallent, $\neg f_1$ and $\neg f_1'$ are logically equivalent.

B/c f_1' doesn't contain $\wedge$ symbol, (by IH),

$\neg f_1'$ doesn't contain $\wedge$ symbol.

Therefore, $P(\neg f_1)$ holds.

Case 2 :

By IH, there exists f_1' and f_2' s.t. f_1 and f_1' and f_2 and f_2' are logically equivalent and f_1' and f_2' don't contain $\wedge$ symbol.

For $f = f_1 \vee f_2$, let $f' = f_1' \vee f_2'$

By IH, f_1 and f_1' are logically equivalent and f_2 and f_2' are logically equivalent.

So, $(f_1 \vee f_2)$ and $(f_1' \vee f_2')$ are logicaly equivalent.

Also, by IH, f_1' and f_2' don't contain $\wedge$ symbol. So, $f' = f_1' \vee f_2'$ doesn't contain $\wedge$ symbol as well.

Therefore, $P(f_1 \vee f_2)$ holds.

Case 3 :

For $f = f_1 \wedge f_2$, let $f' = \neg(\neg f_1' \vee \neg f_2')$

By IH, f_1 and f_1' are logically equivalent.

By IH, f_2 and f_2' are logically equivalent.

So, $f_1 \wedge f_2 = f_1' \wedge f_2'$ (logically equivalent)

$\neg f_1 \vee \neg f_2 = \neg f_1' \vee \neg f_2'$ (applied $\neg$)

$f_1 \wedge f_2 = \neg(\neg f_1' \vee \neg f_2')$ (applied $\neg$)

Then, f and f' are logically equivalent.

By IH, f_1' and f_2' don't contain $\wedge$ symbol,

So $\neg(\neg f_1' \vee \neg f_2')$ doesn't contain $\wedge$ symbol.

Therefore, $P(f_1 \wedge f_2)$ holds.

$\blacksquare$

Practice Problems

- Let F be a set defined as follows:
  - any tree consisting of a single node is an element of F;
  - if $T_1, T_2 \in F$, so is a binary tree consisting of a root with T_1 and T_2 as sub-trees;
  - nothing else belongs to F.

  Use structural induction to prove that every $T \in F$ has exactly one more leaf than interior nodes.