Task1. If a linear transformation T is invertible, it must be one-to-one (1-1) and onto

(1) Let β be ordered basis of V, $\beta = \{(1,0),(0,1)\}$

$R(T) = Span(T(\beta)) = Span(\{(1,2,1),(-1,4,0)\})$, $rank(T) = 2$

$nullity(T) = dim(R^3) - rank(T) = 3-2=1$, not 1-1 $\Rightarrow$ T isn't invertible

(2) Consider $T(a_1,a_2,a_3) = (3a_1-2a_3, a_2, 3a_1+4a_2) = (0,0,0)$

$$\begin{cases} 3a_1 - 2a_3 = 0 \\ a_2 = 0 \\ 3a_1+4a_2 = 0 \end{cases} \Rightarrow \begin{matrix} 3a_1+4\cdot 0=0 \\ a_1=0 \end{matrix} \Rightarrow \begin{matrix} 3\cdot 0-2a_3=0 \\ a_3=0 \end{matrix} \Rightarrow a_1=a_2=a_3=0$$

$\Rightarrow N(T)=\{0\}$, $nullity(T)=0$, $rank(T) = dim(R^3)-nullity(T)=3$, T is 1-1 and onto $\Rightarrow$ T is invertible

(3) Let β be ordered basis of V, $\beta = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

$R(T) = Span(T(\beta)) = Span\{1, 2x, x^2, x^2\} = Span(\{1, x, x^2\})$, $rank(T)=3$

$nullity(T) = dim(M_{2\times 2}(R)) - rank(T) = 4-3=1$, not 1-1 $\Rightarrow$ T isn't invertible

Task2. V is isomorphic to W if and only if $dim(V) = dim(W)$

(1) $dim(R^3) = 3 \neq dim(P_3(R)) = 4$, isn't isomorphic

(2) $dim(R^4) = 4 = dim(P_3(R))$, is isomorphic

(3) Let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, $tr(A)=0$ means $a+d=0 \Rightarrow d=-a$

so A can be express as $\begin{bmatrix}a&b\\c&-a\end{bmatrix}$, a,b,c can be any number

$dim(V) = 3 = dim(R^3)$, is isomorphic

Task3.

(1) Let's assume A is invertible at first $\Rightarrow AA^{-1} = I \Rightarrow A(AA^{-1}) = A(I)$

$\Rightarrow (AA)A^{-1} = A \Rightarrow 0\cdot A^{-1} = A = 0$, but if $A=0$, it's not invertible, because there is no matrix B satisfied $A\cdot B = 0B = I$, so A isn't invertible

(2) Let's assume A is invertible at first $\Rightarrow AB=0 \Rightarrow A^{-1}(AB) = A^{-1}0$

$\Rightarrow (A^{-1}A)B = 0 \Rightarrow B=0$, B can't be nonzero matrix, so A isn't invertible

Task4. Let's check the 2 directions below are correct, then we can prove the statement is true.

(1) If T is isomorphic, then $T(\beta)$ is a basis of W:

T is onto, which means $T(\beta)$ can span W

T is 1-1, which means $T(\beta)$ is linear independent

$\Rightarrow T(\beta)$ is a basis of W

(2) If $T(\beta)$ is a basis of W, then T is isomorphic:

Since $T(\beta)$ is basis, so it's linear independent and can span W,

linear independent means T is 1-1, can span W means T is onto $\Rightarrow$ T is isomorphic