Task 1. We can use property: T is linear if and only if $T(cx+y)=cT(x)+T(y)$ for all $x,y \in V$ and $c \in F$ to prove that the given transformation is linear or not. Use $N(T)=\{0\}$ to check T is one-to-one or not, use if $T: X \to Y$ & $rank(T)=dim(Y)$ to check T is onto or not.

(1) let $A=(a_1,a_2,a_3), B=(b_1,b_2,b_3)$

Prove: $T(cA+B)=T((ca_1+b_1, ca_2+b_2, ca_3+b_3))=((ca_1+b_1)-(ca_2+b_2), 2(ca_3+b_3))$
$=(ca_1-ca_2, 2ca_3)+(b_1-b_2, 2b_3) = c(a_1-a_2, 2a_3)+(b_1-b_2, 2b_3)=cT(A)+T(B)$

Get N(T) bases & nullity : $T(a_1,a_2,a_3)=(0,0) \Rightarrow \begin{matrix} a_1-a_2=0 \\ a_3=0 \end{matrix} \Rightarrow \begin{matrix} a_1=a_2 \\ a_3=0 \end{matrix}$
$\Rightarrow N(T)=span(\{(1,1,0)\})$, $nullity(T)=1$

Get R(T) bases & rank : let β in $=\{(1,0,0),(0,1,0),(0,0,1)\}$
$R(T)=span(T(\beta))=span(\{(1,0),(-1,0),(0,2)\})=span(\{(1,0),(0,1)\})$
the set $\{(1,0),(0,1)\}$ is l.i., so it's the R(T) basis, $rank(T)=2$

Verify the dimension theorem: $nullity(T)+rank(T)=1+2=3=dim(R^3)$

N(T) isn't $\{0\}$, isn't one-to-one, $rank(T)=2=dim(R^2)$, is onto

(2) let $A=(a_1,a_2,a_3,a_4,a_5,a_6), B=(b_1,b_2,b_3,b_4,b_5,b_6)$

Prove: $T(cA+B)=T(ca_1+b_1, ca_2+b_2, ca_3+b_3, ca_4+b_4, ca_5+b_5, ca_6+b_6)$
$=(2(ca_1+b_1)-(ca_2+b_2), (ca_3+b_3)+(ca_2+b_2), 0, 0)$
$=c(2a_1-a_2, a_3+a_2, 0, 0)+(2b_1-b_2, b_3+b_2, 0, 0)=cT(A)+T(B)$

Get N(T) bases & nullity : $T(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0) \Rightarrow \begin{matrix} 2a_1-a_2=0 \\ a_3+a_2=0 \end{matrix} \Rightarrow \begin{matrix} a_1=\frac{1}{2}a_2 \\ a_3=-a_2 \end{matrix}$
$N(T)=Span(\{(1,2,-2,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\})$ a_4,a_5,a_6 can be any number
$nullity(T)=4$

Get R(T) bases & rank: let $\beta=\{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0)\}$
$R(T)=span(T(\beta))=Span(\{(2,0,0,0),(-1,1,0,0),(0,1,0,0)\})$, $rank(T)=3$

Verify the dimension theorem: $nullity(T)+rank(T)=4+3\neq 7=dim(R^6) \Rightarrow$ not pass

N(T) isn't $\{0\}$, isn't one-to-one, $rank(T)=3\neq dim(R^4)$, isn't onto

(3) let $A=[a_1,a_2], B=[b_1,b_2]$

Prove: $T(c\cdot A+B)=T([ca_1+b_1, ca_2+b_2])=[ca_1+b_1+ca_2+b_2, 0, 2(ca_1+b_1)-(ca_2+b_2)]$
$=c[a_1+a_2, 0, 2a_1-a_2]+[b_1+b_2, 0, 2b_1-b_2]=c\cdot T(A)+T(B)$

Get N(T) bases & nullity: $T([a_1,a_2])=[0,0,0] \Rightarrow \begin{matrix} a_1+a_2=0 \\ 2a_1-a_2=0 \end{matrix} \Rightarrow a_1=a_2=0$
$N(T)=\{[0,0]\}=\{0\}$, $nullity(T)=0$

Get R(T) bases & rank: let $\beta=\{[1,0],[0,1]\}$
$R(T)=Span(T(\beta))=span(\{[1,0,2],[1,0,-1]\})$, $rank(T)=2$

Verify the dimension theorem: $nullity(T)+rank(T)=0+2=2=dim(M_{1\times2}(R))$

$N(T)=\{0\}$, is one-to-one, $rank(T)=2\neq 3=dim(M_{1\times3}(R))$, isn't onto

Task 2. We can use property: T is linear if and only if $T(cx+y)=cT(x)+T(y)$ for all $x,y\in V$ and $c\in F$ to check that the given transformation is linear or not. let $A=(a_1,a_2)$, $B=(b_1,b_2)$

(1) $T(cA+B)=T(ca_1+b_1, ca_2+b_2)=(1, ca_2+b_2) \neq cT(A)+T(B)=c(1,a_2)+(1,b_2)=(c+1, ca_2+b_2)$

(2) $T(cA+B)=T(ca_1+b_1, ca_2+b_2)=(ca_1+b_1, (ca_1+b_1)^2)=(ca_1+b_1, ca_1^2+b_1^2+2ca_1b_1)$

$\neq cT(A)+T(B)=c(a_1,a_1^2)+(b_1,b_1^2)=(ca_1+b_1, ca_1^2+b_1^2)$

(3) $T(cA+B)=T(ca_1+b_1, ca_2+b_2)=(|ca_1+b_1|, ca_2+b_2)$

$\neq cT(A)+T(B)=c(|a_1|,a_2)+(|b_1|,b_2)=(|ca_1|+|b_1|, ca_2+b_2)$

Task 3. let $T(v)=Av$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(1,0)=(1,4) \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\Rightarrow 1\cdot a+0\cdot b=1,\ 1\cdot c+0\cdot d=4 \Rightarrow a=1, c=4$, $T(1,1)=(2,5) \Rightarrow \begin{bmatrix} 1 & b \\ 4 & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\Rightarrow 1+b=2,\ 4+d=5 \Rightarrow b=1, d=1 \Rightarrow A=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$T(2,3)=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 2+3 \\ 8+3 \end{bmatrix}=\begin{bmatrix} 5 \\ 11 \end{bmatrix}=(5,11)$

let $Av=0=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+y=0 \\ 4x+y=0 \end{matrix} \Rightarrow \begin{matrix} x=-y \\ x=-\frac{1}{4}y \end{matrix} \Rightarrow x=y=0$

the only one solution of v of $Av=0$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow N(T)=\{0\} \Rightarrow T$ is one-to-one

Task 4. we call a fuction $T: R^3\to R^2$ a linear combination from R^3 to R^2 if

(a) $T(x+y)=T(x)+T(y)$ (b) $T(cx)=cT(x)$

let $x=(1,2,1)$, $T(3x)=T(3\times1, 3\times2, 3\times1)=T(3,6,3)=(2,1)\neq 3T(x)=3(1,1)=(3,3)$

Task 5. $T(x,y)=(0,0)$ and $N(T)=\mathrm{span}\{(0,1)\}$, which means any vector of the form $(0,y)$ should be mapped to $(0,0)$.

Let $T(v)=A\cdot v$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b=0, d=0$

a, c can be any number

A can be $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, then $T(x,y)=\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ 2x \end{bmatrix}=(x,2x)$