Task 1. A subspace W should statisfy 3 definitions

(1) $0 \in W$

(2) $x+y \in W$ whenever $x \in W$ and $y \in W$

(3) $cx \in W$ whenever $c \in F$ and $x \in W$

so we will use the definitions to check whether the following set is a subspace.

W1. (1) $(x,y,z)=(0,0,0)$ $x=0=2y$ (v)

(2) $u, v \in W$, $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$, $u+v=(u_1+v_1, u_2+v_2, u_3+v_3)$

$u_1+v_1 = 2u_2+2v_2 = 2(u_2+v_2)$ (v)

(3) $v \in W$, $v=(x,y,z)$, $cv = c(x,y,z) = (cx,cy,cz)$ $cx = 2\cdot(cy) = c(2y)$ (v)

W1 is a subspace

W2. (1) $(x,y,z)=(0,0,0)$, $y=0$ (v)

(2) $u, v \in W$, $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$ $u+v=(u_1+v_1, u_2+v_2, u_3+v_3)$

$u_2+v_2 = 0+0 = 0 = (u_2+v_2)$ (v)

(3) $v \in W$, $v=(x,y,z)$, $cv = c(x,y,z) = (cx,cy,cz)$ $c\cdot y = cy = 0$ (v)

W_2 is a subspace

W_3 (1) $(x,y,z)=(0,0,0)$, $z=0 \neq 2(x)$

W_3 is not a subspace

W_4 (1) $(x,y,z)=(0,0,0)$ $x=0=y^2$ (v)

(2) $u, v \in W$, $u=(1,1,0)$, $v=(4,2,0)$ $u+v=(5,3,0)$

$u_1+v_1 = 5 \neq (u_2+v_2)^2 = 3^2 = 9$ (x)

W4 is not a subspace

Task 2. Symmetric matrix is a square matrix which $A_{ij}=A_{ji}$

$(A+A^t)_{ij} = A_{ij}+(A^t)_{ij} = A_{ij}+A_{ji} = (A^t)_{ji}+A_{ji} = (A^t+A)_{ji}$

so $A+A^t$ is symmetric for any square matix A

Task 3. if $x=a(y)+b(z)$, x is the linear combination of y and z

(1) $-2x^2+3 = a(x^2+3x)+b(2x^2+4x-1) = (a+2b)x^2+(3a+4b)x+(-b)$

$-b=3 \Rightarrow b=-3$, $a+2b=-2=a-6 \Rightarrow a=4$, $3a+4b=0=3\cdot4+4(-3)$

$a=4, b=-3 \Rightarrow$ True

(2) $x^2+2x-3 = a(-3x^2+2x+1)+b(2x^2-x-1) = (-3a+2b)x^2+(2a-b)x+(a-b)$

$-3a+2b=1$, $2a-b=2$, $a-b=-3 \Rightarrow (2a-b)-(a-b)=a=2-(-3)=5$, $a-b=-3 \Rightarrow b=8$

$\boxed{-3a+2b = -3\cdot5+2\cdot8 = 1}$

$\boxed{a=5, b=8 \Rightarrow \text{True}}$

(3) $3x^2+4x+1 = a(x^2-2x+1)+b(-2x^2-x+1) = (a-2b)x^2+(-2a-b)x+(a+b)$

$a-2b=3$, $-2a-b=4$, $a+b=1 \Rightarrow (a-2b)-(a+b) = -3b = 3-1 = 2 \Rightarrow b=-\frac{2}{3}$, $a+b=1 \Rightarrow a=\frac{5}{3}$

$a-2b = \frac{5}{3} - 2(-\frac{2}{3}) = \frac{5}{3}+\frac{4}{3} = \frac{9}{3} = 3$, $\boxed{-2a-b = -2(\frac{5}{3})-(-\frac{2}{3}) = -\frac{8}{3} \neq 4}$

$\boxed{\text{can't combine to } 3x^2+4x+1 \Rightarrow \text{False}}$

Task 4. if vector V in the Span(S) means V is the linear combination of S

(1) check $(2,-1,1) = a(1,0,2) + b(-1,1,1)$

$a-b=2,\ b=-1,\ 2a+b=1 \Rightarrow (2a+b)+(a-b) = 3a = 3 \Rightarrow a=1,\ b=-1$

$a-b = 1-(-1) = 2,\ 2a+b = 2+(-1) = 1$

$(2,-1,1)$ is in the Span(S)

(2) check $(-1,2,1) = a(1,0,2) + b(-1,1,1)$

$a-b=-1,\ b=2,\ 2a+b=1 \Rightarrow (a-b)+(2a+b) = 3a = 0 \Rightarrow a=0,\ b=2$

$a-b = 0-2 = -2 \neq -1$

can't combine to $(-1,2,1)$, isn't in the Span(S)

(3) check $(-1,1,1,2) = a(1,0,1,-1) + b(0,1,1,1)$

$a=-1,\ b=1,\ a+b=1,\ 2a+b=2 \Rightarrow a+b = (-1)+1 = 0 \neq 1$

can't combine to $(-1,1,1,2)$, isn't in the Span(S)

Task 5. We attempt Martix A is the linear combination of M_1, M_2, M_3,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the equation can be express below:

$$\begin{aligned} a+0+0 &= A_{11} \\ 0+0+c &= A_{12} \\ 0+0+c &= A_{21} \\ 0+b+0 &= A_{22} \end{aligned} \Rightarrow \begin{aligned} a &= A_{11} \\ b &= A_{22} \\ c &= A_{12} = A_{21} \end{aligned}$$

a, b, c can be arbitrary number

$\Rightarrow$ and because A_{12} and A_{21} always be the same

so A can be any symmetric matrix

which means span($\{M_1, M_2, M_3\}$) is the set of all symmetic matrix