Task 1. A set is linear independent (l.i.) if and only if the only representations of 0 as linear combinations of its vectors are trivial representations

(1) $a\begin{bmatrix}1&2\\0&0\end{bmatrix}+b\begin{bmatrix}0&-1\\3&1\end{bmatrix}+c\begin{bmatrix}1&-1\\2&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow \begin{array}{l} a+c=0 \\ 2a-b-c=0 \\ 3b+2c=0 \\ b-c=0 \end{array} \Rightarrow \begin{array}{l} 3b+2c-3(b-c)=5c=0,\ c=0 \\ b-c=0,\ b=0 \\ a+c=0,\ a=0 \end{array}$

$a=b=c=0 \Rightarrow$ l.i.

(2) $a\begin{bmatrix}1&0\\-1&0\end{bmatrix}+b\begin{bmatrix}0&2\\0&4\end{bmatrix}+c\begin{bmatrix}-2&3\\2&6\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow \begin{array}{l} a-2c=0 \\ 2b+3c=0 \\ -a+2c=0 \\ 4b+6c=0 \end{array} \Rightarrow \begin{array}{l} 2(2b+3c)-(4b+6c)=0,\ 2b=3c \\ (a-2c)+(-a+2c)=0,\ a=2c \end{array}$

$a=2c, 2b=3c$, a,b,c can be any number $\Rightarrow$ l.d.

(3) $a(1,0,-2,1)+b(0,-1,1,1)+c(-1,2,1,0)+d(2,1,-4,4)=(0,0,0,0)$

$\Rightarrow \begin{array}{l} a-c+2d=0 \\ -b+2c+d=0 \\ -2a+b+c-4d=0 \\ a+b+4d=0 \end{array} \Rightarrow \begin{array}{l} (a-c+2d)-2(-b+2c+d)=a+2b-5c=0 \\ (-2a+b+c-4d)+(a+b+4d)=-a+2b+c=0 \\ (a+2b-5c)+(-a+2b+c)=4b-4c=0,\ b=c \\ (a+2b-5c)-(-a+2b+c)=2a-6c=0,\ a=3c \end{array} \Rightarrow \begin{array}{l} 3c-c+2d=0 \\ -c+2c+d=0 \\ -6c+c+c-4d=0 \\ 3c+c+4d=0 \end{array} \Rightarrow \begin{array}{l} 2c+2d=0 \\ c+d=0 \\ -4c-4d=0 \\ 4c+4d=0 \end{array} \Rightarrow c=-d$

$a=3c, b=c, c=-d$, a,b,c,d can be any number $\Rightarrow$ l.d.

(4) $a(1,0,-2,1)+b(0,-1,1,1)+c(-1,2,1,0)+d(2,1,2,-2)=(0,0,0,0)$

$\Rightarrow \begin{array}{l} a-c+2d=0 \\ -b+2c+d=0 \\ -2a+b+c+2d=0 \\ a+b-2d=0 \end{array} \Rightarrow \begin{array}{l} (a-c+2d)-(-2a+b+c+2d)=3a-b-2c=0 \\ 2(-b+2c+d)-(-2a+b+c+2d)=2a-3b+5c=0 \\ (a+b-2d)+(-2a+b+c+2d)=-a+2b+c=0 \end{array}$

$\Rightarrow \begin{array}{l} (3a-b-2c)+3(-a+2b+c)=5b+c=0 \\ (2a-3b+5c)+2(-a+2b+c)=b+7c=0 \end{array} \Rightarrow \begin{array}{l} b=c=0 \\ 3a-0-2\cdot 0=0 \\ a=0 \end{array} \Rightarrow \begin{array}{l} 0-0+2d=0 \\ d=0 \end{array}$

$a=b=c=d=0 \Rightarrow$ l.i.

Task 2. In a set, if one vector can be other vectors linear combination, we will say the set is linear dependent (l.d.), so we can first define 2 arbitrary vectors, then do a linear combination of these 2 vector, then we can get a l.d. vector set.

ex. $V_1=(1,0,1,)$, $V_2=(3,2,1)$, $V_3=1(1,0,1)+2(3,2,1)=(7,4,3)$

Task 3, (1) False, ex. V is R^2, basis can be $\{(1,0),(0,1)\}$, $\{(1,1),(1,2)\}$ ...

(2) True

(3) False, ex. V is R^2 which $\dim(V)=2$, one subset of V with 2 vectors is $\{(1,1),(0,0)\}$, but this is not a basis since it's not l.i.

Task 4. To determine a set can be a basis of $P_2(R)$ or not, we can check

(1) vector number of set must be $\dim(P_2(R)) = 3$

(2) vectors in set should be linear independent (l.i.)

(1) $a(1-x^2)+b(2+5x+x^2)+c(-4x+3x^2)=0$

$\Rightarrow \begin{cases} a+2b = 0 \\ 5b-4c=0 \\ -a+b+3c=0 \end{cases} \Rightarrow (a+2b)+(-a+b+3c)=3b+3c=0$, $3b+3c=5b-4c=0, b=c=0 \Rightarrow a+2\cdot 0=0$, $a=0 \Rightarrow a=b=c=0 \Rightarrow$ l.i. is basis

(2) $a(2-4x+x^2)+b(3x-x^2)+c(6-x^2)=0$

$\Rightarrow \begin{cases} 2a+6c=0 \\ -4a+3b=0 \\ a-b-c=0 \end{cases} \Rightarrow (2a+6c)+6(a-b-c)=8a-6b=0$, $8a-6b=-4a+3b=0 \Rightarrow a=\frac{3}{4}b \Rightarrow 2(\frac{3}{4}b)+6c=\frac{3}{2}b+6c=0$, $\frac{3}{4}b-b-c=-\frac{1}{4}b-c=0$, $b=-4c$

$a=\frac{3}{4}b, b=-4c$, a,b,c can be any number $\Rightarrow$ l.d. isn't bais

(3) $a(1+2x-x^2)+b(1+2x^2)+c(2+x+x^2)=0$

$\Rightarrow \begin{cases} a+b+2c=0 \\ 2a+c=0 \\ -a+2b+c=0 \end{cases} \Rightarrow 2(a+b+2c)-(2a+c)=2b+c=0$, $(a+b+2c)+(-a+2b+c)=3b+3c=0$, $b=c=0 \Rightarrow a+0+2\cdot 0=0$, $a=0 \Rightarrow a=b=c=0 \Rightarrow$ l.i is basis

Task 5. since $a_1=a_3+a_4$, $(a_1,a_2,a_3,a_4,a_5)=(a_3+a_4,a_2,a_3,a_4,a_5)$

a_2,a_3,a_4,a_5 are free and a_1 is depends on a_3 and a_4, 4 variable generate W

then we can check the dependency

let $a_2=1$, other be 0 $\Rightarrow (a_1,a_2,a_3,a_4,a_5)=(0,1,0,0,0)$

$a_3=1$, other be 0 $\Rightarrow (a_1,a_2,a_3,a_4,a_5)=(1,0,1,0,0)$

$a_4=1$, other be 0 $\Rightarrow (a_1,a_2,a_3,a_4,a_5)=(1,0,0,1,0)$

$a_5=1$, other be 0 $\Rightarrow (a_1,a_2,a_3,a_4,a_5)=(0,0,0,0,1)$

$a_2(0,1,0,0,0)+a_3(1,0,1,0,0)+a_4(1,0,0,1,0)+a_5(0,0,0,0,1)=(0,0,0,0,0)$

$\Rightarrow \begin{cases} a_3+a_4=0 \\ a_2=0 \\ a_3=0 \\ a_4=0 \\ a_5=0 \end{cases} \Rightarrow a_2=a_3=a_4=a_5=0 \Rightarrow$ l.i. is basis $\Rightarrow \dim(W)=4$