Task 1. We can use property: T is linear if and only if $T(cx+y)=cT(x)+T(y)$ for all $x,y\in V$ and $c\in F$ to prove that the given transformation is linear or not. Use $N(T)=\{0\}$ to check T is one-to-one or not, use if $T: X \to Y$ & $\text{rank}(T)=\dim(Y)$ to check T is onto or not.

(1) let $A=(a_1,a_2,a_3), B=(b_1,b_2,b_3)$

Prove: $T(cA+B)=T((ca_1+b_1, ca_2+b_2, ca_3+b_3)) = ((ca_1+b_1)-(ca_2+b_2), 2(ca_3+b_3))$

$=(ca_1-ca_2, 2ca_3)+(b_1-b_2, 2b_3) = c(a_1-a_2, 2a_3)+(b_1-b_2, 2b_3) = cT(A)+T(B)$

Get N(T) bases & nullity: $T(a_1,a_2,a_3)=(0,0) \Rightarrow \begin{matrix} a_1-a_2=0 \\ a_3=0 \end{matrix} \Rightarrow \begin{matrix} a_1=a_2 \\ a_3=0 \end{matrix}$

$\Rightarrow N(T)=\text{span}(\{(1,1,0)\})$, $\text{nullity}(T)=1$

Get R(T) bases & rank: let β in $=\{(1,0,0),(0,1,0),(0,0,1)\}$

$R(T)=\text{span}(T(\beta))=\text{span}(\{(1,0),(-1,0),(0,2)\})=\text{span}(\{(1,0),(0,1)\})$

the set $\{(1,0),(0,1)\}$ is l.i., so it's the R(T) basis, $\text{rank}(T)=2$

Verify the dimension theorem: $\text{nullity}(T)+\text{rank}(T)=1+2=3=\dim(R^3)$

N(T) isn't $\{0\}$, isn't one-to-one, $\text{rank}(T)=2=\dim(R^2)$, is onto

(2) let $A=(a_1,a_2,a_3,a_4,a_5,a_6), B=(b_1,b_2,b_3,b_4,b_5,b_6)$

Prove: $T(cA+B)=T(ca_1+b_1, ca_2+b_2, ca_3+b_3, ca_4+b_4, ca_5+b_5, ca_6+b_6)$

$=(2(ca_1+b_1)-(ca_2+b_2), (ca_3+b_3)+(ca_2+b_2), 0, 0)$

$=c(2a_1-a_2, a_3+a_2, 0, 0)+(2b_1-b_2, b_3+b_2, 0, 0)=cT(A)+T(B)$

Get N(T) bases & nullity: $T(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0) \Rightarrow \begin{matrix} 2a_1-a_2=0 \\ a_3+a_2=0 \end{matrix} \Rightarrow \begin{matrix} a_1=\frac{1}{2}a_2 \\ a_3=-a_2 \end{matrix}$

$N(T)=\text{Span}(\{(1,2,-2,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\})$ a_4,a_5,a_6 can be any number

$\text{nullity}(T)=4$

Get R(T) bases & rank: let $\beta=\{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0)\}$

$R(T)=\text{span}(T(\beta))=\text{Span}(\{(2,0,0,0),(-1,1,0,0),(0,1,0,0)\})$, $\text{rank}(T)=3$

Verify the dimension theorem: $\text{nullity}(T)+\text{rank}(T)=4+3\neq 7=\dim(R^6) \Rightarrow$ not pass

N(T) isn't $\{0\}$, isn't one-to-one, $\text{rank}(T)=3\neq\dim(R^4)$, isn't onto

(3) let $A=[a_1,a_2], B=[b_1,b_2]$

Prove: $T(c\cdot A+B)=T([ca_1+b_1, ca_2+b_2])=[ca_1+b_1+ca_2+b_2, 0, 2(ca_1+b_1)-(ca_2+b_2)]$

$=c[a_1+a_2, 0, 2a_1-a_2]+[b_1+b_2, 0, 2b_1-b_2]=c\cdot T(A)+T(B)$

Get N(T) bases & nullity: $T([a_1,a_2])=[0,0,0] \Rightarrow \begin{matrix} a_1+a_2=0 \\ 2a_1-a_2=0 \end{matrix} \Rightarrow a_1=a_2=0$

$N(T)=\{[0,0]\}=\{0\}$, $\text{nullity}(T)=0$

Get R(T) bases & rank: let $\beta=\{[1,0],[0,1]\}$

$R(T)=\text{Span}(T(\beta))=\text{span}(\{[1,0,2],[1,0,-1]\})$, $\text{rank}(T)=2$

Verify the dimension theorem: $\text{nullity}(T)+\text{rank}(T)=0+2=2=\dim(M_{1\times2}(R))$

$N(T)=\{0\}$, is one-to-one, $\text{rank}(T)=2\neq 3=\dim(M_{1\times3}(R))$, isn't onto

Task 2. We can use property: T is linear if and only if $T(cx+y) = cT(x)+T(y)$ for all $x, y \in V$ and $c \in F$ to check that the given transformation is linear or not. let $A=(a_1,a_2)$, $B=(b_1,b_2)$

(1) $T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (1, ca_2+b_2) \neq cT(A)+T(B) = c(1,a_2)+(1,b_2) = (c+1, ca_2+b_2)$

(2) $T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (ca_1+b_1, (ca_1+b_1)^2) = (ca_1+b_1, ca_1^2+b_1^2+2ca_1b_1)$

$\neq cT(A)+T(B) = c(a_1, a_1^2) + (b_1, b_1^2) = (ca_1+b_1, ca_1^2+b_1^2)$

(3) $T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (|ca_1+b_1|, ca_2+b_2)$

$\neq cT(A)+T(B) = c(|a_1|, a_2) + (|b_1|, b_2) = (|c a_1| + |b_1|, ca_2+b_2)$

Task 3. let $T(v)=Av$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(1,0)=(1,4) \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\Rightarrow 1\cdot a+0\cdot b=1,\ 1\cdot c+0\cdot d=4 \Rightarrow a=1, c=4$, $T(1,1)=(2,5) \Rightarrow \begin{bmatrix} 1 & b \\ 4 & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\Rightarrow 1+b=2, 4+d=5 \Rightarrow b=1, d=1 \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$T(2,3) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 8+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} = (5,11)$

let $Av=0 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+y=0 \\ 4x+y=0 \end{matrix} \Rightarrow \begin{matrix} x=-y \\ x=-\frac{1}{4}y \end{matrix} \Rightarrow x=y=0$

the only one solution of v of $Av=0$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow N(T)=\{0\} \Rightarrow T$ is one-to-one

Task 4. Let $T(v)=Av$ and $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $T(1,2,1)=(1,1) \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow a+2b+c=1, d+2e+f=1$, $T(3,6,3)=(2,1) \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow 3a+6b+3c=2, 3d+6e+3f=1 \Rightarrow 3(d+2e+f)=3 \neq 3d+6e+3f=1$

so there is no linear transformation T can satisfy both conditions.

Task 5. $T(x,y)=(0,0)$ and $N(T)=\text{span}\{(0,1)\}$, which means any vector of the form $(0,y)$ should be mapped to $(0,0)$.

Let $T(v)=A\cdot v$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b=0, d=0$

a, c can be any number

A can be $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, then $T(x,y) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = (x, 2x)$