Task 1. In a set, if one vector can be other vectors linear combination, we will say the set is linear dependent (l.d.) otherwise is linear independent (l.i.)

(1) $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{matrix} 0+b=1 \\ (-a)+(-b)=2 \\ 3a+2b=0 \\ a-b=0 \end{matrix} \Rightarrow \begin{matrix} b=1 \\ -a+(-1)=2 \\ 3a+2\cdot 1=0 \\ a-1=0 \end{matrix} \Rightarrow \begin{matrix} b=1 \\ a=-3 \\ a=-\frac{2}{3} \\ a=1 \end{matrix} (X)$

can't combine to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow$ l.i.

(2) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} + b\begin{bmatrix} -2 & 3 \\ 2 & 6 \end{bmatrix} \Rightarrow \begin{matrix} 0-2b=1 \\ 2a+3b=0 \\ 0+2b=-1 \\ 4a+6b=0 \end{matrix} \Rightarrow \begin{matrix} b=-\frac{1}{2} \\ 2a+3(-\frac{1}{2})=0 \\ b=-\frac{1}{2} \\ 4a+6(-\frac{1}{2})=0 \end{matrix} \Rightarrow \begin{matrix} b=-\frac{1}{2} \\ a=\frac{3}{4} \end{matrix}$

$a=\frac{3}{4}, b=-\frac{1}{2} \Rightarrow$ l.d.

(3) $(1,0,-2,1) = a(0,-1,1,1) + b(-1,2,1,0) + c(2,1,-4,4)$

$\Rightarrow \begin{matrix} 0-b+2c=1 \\ -a+2b+c=0 \\ a+b-4c=-2 \\ a+0+4c=1 \end{matrix} \Rightarrow \begin{matrix} b=2c-1 \\ a=-4c+1 \\ -(-4c+1)+2(2c-1)+c=9c-3=0 \\ (-4c+1)+(2c-1)-4c=-6c=-2 \end{matrix} \Rightarrow \begin{matrix} c=\frac{1}{3} \\ b=-\frac{1}{3} \\ a=-\frac{1}{3} \end{matrix}$

$a=-\frac{1}{3}, b=-\frac{1}{3}, c=\frac{1}{3} \Rightarrow$ l.d.

(4) $(1,0,-2,1) = a(0,-1,1,1) + b(-1,2,1,0) + c(2,1,2,-2)$

$\Rightarrow \begin{matrix} 0-b+2c=1 \\ -a+2b+c=0 \\ a+b+2c=-2 \\ a+0-2c=1 \end{matrix} \Rightarrow \begin{matrix} b=2c-1 \\ a=2c+1 \\ -(2c+1)+2(2c-1)+c=3c-3=0 \\ (2c+1)+(2c-1)+2c=6c=-2 \end{matrix} \Rightarrow \begin{matrix} c=1 \\ c=-\frac{1}{3} \end{matrix} (X)$

can't combine to $(1,0,-2,1) \Rightarrow$ l.i.

Task 2. In a set, if one vector can be other vectors linear combination, we will say the set is linear dependent (l.d.), so we can first define 2 arbitrary vectors, then do a linear combination of these 2 vector, then we can get a l.d. vector set.

ex. $V_1=(1,0,1)$, $V_2=(3,2,1)$, $V_3=1(1,0,1)+2(3,2,1)=(7,4,3)$

Task 3, (1) False, ex. V is $R^2$, basis can be $\{(1,0),(0,1)\}$, $\{(1,1),(1,2)\}$ ...

(2) True

(3) False, ex. V is $R^2$ which $\dim(V)=2$, one subset of V with 2 vectors is $\{(1,1),(0,0)\}$, but this is not a basis since it's not l.i.

Task 4. To determine a set can be a basis of $P_2(R)$ or not, we can check

(1) vector number of set must be $\dim(P_2(R)) = 3$

(2) vectors in set should be linear independent (l.i.)

(1) $1-x^2 = a(2+5x+x^2) + b(-4x+3x^2) \Rightarrow \begin{matrix} 2a+0=1 \\ 5a+0=0 \\ a+3b=-1 \end{matrix}$ (X)

can't combine to $1-x^2 \Rightarrow$ l.i., is basis

(2) $2-4x+x^2 = a(3x-x^2) + b(6-x^2) \Rightarrow \begin{matrix} 0+6b=2 \\ 3a+0=-4 \\ -a-b=1 \end{matrix} \Rightarrow \begin{matrix} b=\frac{1}{3} \\ a=\frac{-4}{3} \end{matrix}$ (O)

$a = -\frac{4}{3}, b = \frac{1}{3} \Rightarrow$ l.d., isn't basis

(3) $1+2x-x^2 = a(1+2x^2) + b(2+x+x^2) \Rightarrow \begin{matrix} a+2b=1 \\ 0+b=2 \\ 2a+b=-1 \end{matrix} \Rightarrow \begin{matrix} b=2 \\ a+2\cdot 2=1 \\ 2a+2=-1 \end{matrix} \Rightarrow \begin{matrix} b=2 \\ a=-3 \\ a=-\frac{3}{2} \end{matrix}$ (X)

can't combine to $1+2x-x^2 \Rightarrow$ l.i. is basis

Task 5. since $a_1 = a_3 + a_4$, $(a_1, a_2, a_3, a_4, a_5) = (a_3+a_4, a_2, a_3, a_4, a_5)$

$a_2, a_3, a_4, a_5$ are free and $a_1$ is depends on $a_3$ and $a_4$, 4 variable generate W

then we can check the dependency

let $a_2 = 1$, other be $0 \Rightarrow (a_1, a_2, a_3, a_4, a_5) = (0,1,0,0,0)$

$a_3 = 1$, other be $0 \Rightarrow (a_1, a_2, a_3, a_4, a_5) = (1,0,1,0,0)$

$a_4 = 1$, other be $0 \Rightarrow (a_1, a_2, a_3, a_4, a_5) = (1,0,0,1,0)$

$a_5 = 1$, other be $0 \Rightarrow (a_1, a_2, a_3, a_4, a_5) = (0,0,0,0,1)$

$(0,1,0,0,0) = a(1,0,1,0,0) + b(1,0,0,1,0) + c(0,0,0,0,1)$

$\Rightarrow \begin{matrix} a+b+0=0 \\ 0+0+0=1 \\ a+0+0=0 \\ 0+b+0=0 \\ 0+0+c=0 \end{matrix}$ (X) $\Rightarrow$ can't combine to $(0,1,0,0,0) \Rightarrow$ l.i. is basis $\Rightarrow \dim(W) = 4$