Task 1.

(1) $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$, $\gamma = \{(1,0),(0,1)\}$

$T(1,0,0) = (1,0) = 1(1,0)+0(0,1)$

$T(0,1,0) = (-1,0) = -1(1,0)+0(0,1) \Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$T(0,0,1) = (0,2) = 0(\quad)+2(\quad)$

(2) $B = \{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$\gamma = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

$T(1,0,0,0,0,0) = (2,0,0,0) = 2(1,0,0,0)+0(0,1,0,0)+0(0,0,1,0)+0(0,0,0,1)$

$T(0,1,0,0,0,0) = (-1,1,0,0) = -1(\quad)+1(\quad)+0(\quad)+0(\quad)$

$T(0,0,1,0,0,0) = (0,1,0,0) = 0(\quad)+1(\quad)+0(\quad)+0(\quad)$

$T(0,0,0,1,0,0) = (0,0,0,0) = 0(\quad)+0(\quad)+0(\quad)+0(\quad)$

$T(0,0,0,0,1,0) = (0,0,0,0) = 0(\quad)+0(\quad)+0(\quad)+0(\quad)$

$T(0,0,0,0,0,1) = (0,0,0,0) = 0(\quad)+0(\quad)+0(\quad)+0(\quad)$

$\Rightarrow [T]_B^\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(3) $B = \{(1,0),(0,1)\}$, $\gamma = \{(1,0,0),(0,1,0),(0,0,1)\}$

$T(1,0) = (2,3,1) = 2(1,0,0)+3(0,1,0)+1(0,0,1) \Rightarrow [T]_B^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

$T(0,1) = (-1,4,0) = -1(\quad)+4(\quad)+0(\quad)$

(4) $e_1 = (1,0,0,\cdots 0)$ $\quad T(e_1) = (0,0,\cdots 1) = e_n$

$e_2 = (0,1,0,\cdots 0)$ $\Rightarrow$ $T(e_2) = (0,0\ldots 1,0) = e_{n-1}$ $\Rightarrow T(e_i) = e_{n-i+1}$

$e_3 = (0,0,1,\cdots 0)$ $\quad T(e_3) = (0,0\ldots 1,0,0) = e_{n-2}$

$e_n = (0,0,0\cdots,1)$ $\quad T(e_n) = (1,0,0\ldots 0) = e_1$

$[T]_B^\gamma = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \Rightarrow$ it's a reverse identity matrix

Task 2

(1) $\beta = \{(1,0),(0,1)\}$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$

$T(1,0) = (1,1,2) \Rightarrow a(1,1,0)+b(0,1,1)+c(2,2,3) = (1,1,2)$

$\begin{array}{l} a+0+2c=1 \\ a+b+2c=1 \\ 0+b+3c=2 \end{array} \Rightarrow \begin{array}{l} a+2c=a+b+2c \\ b=0 \end{array} \Rightarrow \begin{array}{l} 0+3c=2,\ c=\frac{2}{3} \\ a+2\cdot\frac{2}{3}=1,\ a=-\frac{1}{3} \end{array} \Rightarrow a=-\frac{1}{3}, b=0, c=\frac{2}{3}$

$T(0,1) = (-1,0,1) \Rightarrow x(1,1,0)+y(0,1,1)+z(2,2,3) = (-1,0,1)$

$\begin{array}{l} x+0+2z=-1 \\ x+y+2z=0 \\ 0+y+3z=1 \end{array} \Rightarrow \begin{array}{l} y=1 \\ 1+3z=1,\ z=0 \end{array} \Rightarrow \begin{array}{l} x+2\cdot 0=-1 \\ x=-1 \end{array} \Rightarrow x=-1, y=1, z=0$

$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$

(2) $\alpha = \{(1,2),(2,3)\}$, $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$

$T(1,2) = (-1,1,4) = a(1,1,0)+b(0,1,1)+c(2,2,3)$

$\begin{array}{l} a+0+2c=-1 \\ a+b+2c=1 \\ 0+b+3c=4 \end{array} \Rightarrow \begin{array}{l} b=2 \\ 2+3c=4,\ c=\frac{2}{3} \end{array} \Rightarrow \begin{array}{l} a+2\frac{2}{3}=-1 \\ a=-\frac{7}{3} \end{array} \Rightarrow a=-\frac{7}{3}, b=2, c=\frac{2}{3}$

$T(2,3) = (-1,2,7) = x(1,1,0)+y(0,1,1)+z(2,2,3)$

$\begin{array}{l} x+0+2z=-1 \\ x+y+2z=2 \\ 0+y+3z=7 \end{array} \Rightarrow \begin{array}{l} y=3 \\ 3+3z=7,\ z=\frac{4}{3} \end{array} \Rightarrow \begin{array}{l} x+2\cdot\frac{4}{3}=-1 \\ x=-\frac{11}{3} \end{array} \Rightarrow x=-\frac{11}{3}, y=3, z=\frac{4}{3}$

$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$

Task3. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$, and $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki} a_{ik} = \sum_{k=1}^{n} (BA)_{kk} = tr(BA)$$

Task4.

(1) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0[\quad] + 0[\quad] + 1[\quad] + 0[\quad]$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0[\quad] + 1[\quad] + 0[\quad] + 0[\quad]$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0[\quad] + 0[\quad] + 0[\quad] + 1[\quad]$

$T(A) = T\left(\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} = 1[\quad] + (-1)[\quad] + 4[\quad] + 2[\quad]$

$\Rightarrow [T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $[T(A)]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}$

(2) we need to apply T to all basis elements

$f(x) = 1 \Rightarrow T(f(x)) = f(2) = 1$

$f(x) = x \Rightarrow T(f(x)) = f(2) = 2$

$f(x) = x^2 \Rightarrow T(f(x)) = f(2) = 4$

$f(x) = 4x^2 - 2x + 1 \Rightarrow 4 \cdot 4 - 2 \cdot 2 + 1 = 13$

$\Rightarrow [T]_B^\gamma = [1 \ 2 \ 4]$, $[T(f(x))]_B^\gamma = [13]$