

Task 1. We can use property: T is linear if and only if $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$ to prove that the given transformation is linear or not. Use $N(T) = \{0\}$ to check T is one-to-one or not, use if $T: X \rightarrow Y$ & $\text{rank}(T) = \dim(Y)$ to check T is onto or not.

(1) let $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3)$

Prove: $T(cA+B) = T((ca_1+b_1, ca_2+b_2, ca_3+b_3)) = ((ca_1+b_1) - (ca_2+b_2), 2(ca_3+b_3))$
 $= (ca_1 - ca_2, 2ca_3 + (b_1 - b_2), 2b_3) = c(a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3) = cT(A) + T(B)$

Get $N(T)$ bases & nullity: $T(a_1, a_2, a_3) = (0, 0) \Rightarrow \begin{matrix} a_1 - a_2 = 0 \\ a_3 = 0 \end{matrix} \Rightarrow \begin{matrix} a_1 = a_2 \\ a_3 = 0 \end{matrix}$
 $\Rightarrow N(T) = \text{span}(\{(1, 1, 0)\}), \text{nullity}(T) = 1$

Get $R(T)$ bases & rank: let β in $= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
 $R(T) = \text{span}(T(\beta)) = \text{span}(\{(1, 0), (-1, 0), (0, 2)\}) = \text{span}(\{(1, 0), (0, 1)\})$
the set $\{(1, 0), (0, 1)\}$ is l.i., so it's the $R(T)$ basis, $\text{rank}(T) = 2$

Verify the dimension theorem: $\text{nullity}(T) + \text{rank}(T) = 1 + 2 = 3 = \dim(R^3)$

$N(T)$ isn't $\{0\}$, isn't one-to-one, $\text{rank}(T) = 2 = \dim(R^2)$, is onto

(2) let $A = (a_1, a_2, a_3, a_4, a_5, a_6), B = (b_1, b_2, b_3, b_4, b_5, b_6)$

Prove: $T(cA+B) = T(ca_1+b_1, ca_2+b_2, ca_3+b_3, ca_4+b_4, ca_5+b_5, ca_6+b_6)$
 $= (2(ca_1+b_1) - (ca_2+b_2), (ca_3+b_3) + (ca_4+b_4), 0, 0)$
 $= c(2a_1 - a_2, a_3 + a_4, 0, 0) + (2b_1 - b_2, b_3 + b_4, 0, 0) = cT(A) + T(B)$

Get $N(T)$ bases & nullity: $T(a_1, a_2, a_3, a_4, a_5, a_6) = (0, 0, 0, 0) \Rightarrow \begin{matrix} 2a_1 - a_2 = 0 \\ a_3 + a_4 = 0 \end{matrix} \Rightarrow \begin{matrix} a_1 = \frac{1}{2}a_2 \\ a_3 = -a_4 \end{matrix}$
 $N(T) = \text{Span}(\{(1, 2, -2, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0)\})$ a_5, a_6 can be any number
 $\text{nullity}(T) = 4$

Get $R(T)$ bases & rank: let $\beta = \{(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0)\}$
 $R(T) = \text{span}(T(\beta)) = \text{span}(\{(2, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\})$, $\text{rank}(T) = 3$

Verify the dimension theorem: $\text{nullity}(T) + \text{rank}(T) = 4 + 3 = 7 = \dim(R^6) \Rightarrow$ not pass

$N(T)$ isn't $\{0\}$, isn't one-to-one, $\text{rank}(T) = 3 \neq \dim(R^4)$, isn't onto

(3) let $A = [a_1, a_2], B = [b_1, b_2]$

Prove: $T(cA+B) = T(ca_1+b_1, ca_2+b_2) = [ca_1+b_1+ca_2+b_2, 2(ca_1+b_1) - (ca_2+b_2)]$
 $= c[a_1+a_2, 2a_1-a_2] + [b_1+b_2, 2b_1-b_2] = cT(A) + T(B)$

Get $N(T)$ bases & nullity: $T([a_1, a_2]) = [0, 0] \Rightarrow \begin{matrix} a_1 + a_2 = 0 \\ 2a_1 - a_2 = 0 \end{matrix} \Rightarrow \begin{matrix} a_1 = a_2 = 0 \end{matrix}$
 $N(T) = \{[0, 0]\} = \{0\}$, $\text{nullity}(T) = 0$

Get $R(T)$ bases & rank: let $\beta = \{[1, 0], [0, 1]\}$
 $R(T) = \text{span}(T(\beta)) = \text{span}(\{[1, 2], [1, -1]\})$, $\text{rank}(T) = 2$

Verify the dimension theorem: $\text{nullity}(T) + \text{rank}(T) = 0 + 2 = 2 = \dim(M_{1 \times 2}(R))$

$N(T) = \{0\}$, is one-to-one, $\text{rank}(T) = 2 \neq 3 = \dim(M_{1 \times 3}(R))$, isn't onto

Task 2. We can use property: T is linear if and only if $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$ to check that the given transformation is linear or not. let $A = (a_1, a_2), B = (b_1, b_2)$

$$(1) T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (1, ca_2+b_2) \neq cT(A) + T(B) = c(1, a_2) + (1, b_2) = (c+1, ca_2+b_2)$$

$$(2) T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (ca_1+b_1, (ca_1+b_1)^2) = (ca_1+b_1, ca_1^2+b_1^2+2ca_1b_1) \\ \neq cT(A) + T(B) = c(a_1, a_1^2) + (b_1, b_1^2) = (ca_1+b_1, ca_1^2+b_1^2)$$

$$(3) T(cA+B) = T(ca_1+b_1, ca_2+b_2) = (|ca_1+b_1|, ca_2+b_2) \\ \neq cT(A) + T(B) = c(|a_1|, a_2) + (|b_1|, b_2) = (|ca_1|+|b_1|, ca_2+b_2)$$

Task 3. let $T(v) = Av$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(1,0) = (1,4) \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$\Rightarrow 1 \cdot a + 0 \cdot b = 1, 1 \cdot c + 0 \cdot d = 4 \Rightarrow a=1, c=4, T(1,1) = (2,5) \Rightarrow \begin{bmatrix} 1 & b \\ 4 & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow 1+b=2, 4+d=5 \Rightarrow b=1, d=1 \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$T(2,3) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 8+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} = (5,11)$$

$$\text{let } Av=0 = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x+y=0 \\ 4x+y=0 \end{cases} \Rightarrow \begin{cases} x=-y \\ x=-\frac{1}{4}y \end{cases} \Rightarrow x=y=0$$

the only one solution of v of $Av=0$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow N(T) = \{0\} \Rightarrow T$ is one-to-one

Task 4. Let $T(v) = Av$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $T(1,2,1) = (1,1) \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow a+2b+c=1, d+2e+f=1, T(3,6,3) = (2,1) \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow 3a+6b+3c=2, 3d+6e+3f=1 \Rightarrow 3(d+2e+f)=3 \neq 3d+6e+3f=1$$

so there is no linear transformation T can satisfy both conditions.

Task 5. $T(x,y) = (0,0)$ and $N(T) = \text{span}\{(0,1)\}$, which means any vector of the form $(0,y)$ should be mapped to $(0,0)$.

$$\text{let } T(v) = A \cdot v \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b=0, d=0$$

a, c can be any number

$$A \text{ can be } \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \text{ then } T(x,y) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = (x, 2x)$$