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Turmetov, B; Karachik, V and Muratbekova, M

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On a Boundary Value Problem for the Biharmonic Equation with Multiple Involutions

Batirkhan Turmetov ^{1,*}, Valery Karachik ^{2,†} and Moldir Muratbekova ^{1,†}¹ Department of Mathematics, Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan 161200, Kazakhstan; moldir_1983@mail.ru² Department of Mathematical Analysis, South Ural State University (NRU), 454080 Chelyabinsk, Russia; karachik@susu.ru

* Correspondence: batirkhan.turmetov@ayu.edu.kz

† These authors contributed equally to this work.

Abstract: A nonlocal analogue of the biharmonic operator with involution-type transformations was considered. For the corresponding biharmonic equation with involution, we investigated the solvability of boundary value problems with a fractional-order boundary operator having a derivative of the Hadamard-type. First, transformations of the involution type were considered. The properties of the matrices of these transformations were investigated. As applications of the considered transformations, the questions about the solvability of a boundary value problem for a nonlocal biharmonic equation were studied. Modified Hadamard derivatives were considered as the boundary operator. The considered problems covered the Dirichlet and Neumann-type boundary conditions. Theorems on the existence and uniqueness of solutions to the studied problems were proven.



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1. Introduction

The concept of a nonlocal operator and the related concepts of a nonlocal differential equation appeared in the theory of differential equations not long ago. For example, in [1], the authors considered equations containing fractional derivatives of the desired function and equations with deviating arguments, in other words, equations that include an unknown function and its derivatives, generally speaking, for different values of arguments. Such equations are called nonlocal differential equations. Among nonlocal differential equations, a special place is occupied by equations in which the deviation of the arguments has an involutive character. Mapping S is usually called an involution if $S^2(x) = S(S(x)) = x$. The theory of equations with involutively transformed arguments and their applications were described in detail in the monographs [2–4]. By now, for differential equations with various types of involution, the well-posedness of boundary and initial-boundary value problems, the qualitative properties of solutions, and spectral issues have been studied quite well [5–16]. In addition, for classical equations, one can study nonlocal boundary value problems of the Bitsadze–Samarskii-type, in which the values of the sought function $u(x)$ at the boundary of the domain are related to the values of $u(Sx)$ [17–19]. Note that the questions about the solvability of the main problems for the nonlocal Poisson equation were studied in [20,21]. Note also that boundary value problems with fractional-order boundary operators for elliptic equations were studied in [22–30].

Applications of the boundary value problems with boundary operators of fractional-order for elliptic equations were given in [31–34].

Let us formulate the problem that is studied in this work. Let Ω be the unit ball in \mathbb{R}^l , $l \geq 2$, and $\partial\Omega$ be the unit sphere. Let also $u(x)$ be a smooth function in the domain

Ω , $\mu \geq 0$, $\alpha \in (m-1, m]$, $m = 1, 2, \dots$, $r = |x| = \sqrt{x_1^2 + \dots + x_n^2}$, $\theta = x/r$, $x \in \Omega$, and $\delta = r \frac{d}{dr}$ be the Dirac operator, where $r \frac{d}{dr} = \sum_{j=1}^l x_j \frac{\partial}{\partial x_j}$. Consider the modified Hadamard integrodifferential operators ([35], p. 116):

$$J_\mu^\alpha[u](x) = \begin{cases} u(x), & \alpha = 0, \\ \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\ln \frac{1}{\tau} \right)^{\alpha-1} \tau^{\mu-1} u(\tau x) d\tau, & \alpha > 0 \end{cases}$$

$$D_\mu^\alpha[u](x) = r^{-\mu} J^{m-\alpha}[\delta^m[\tau^\mu \cdot u]](x), \quad m-1 < \alpha \leq m.$$

Let S_1, \dots, S_n be a set of real symmetric commutative matrices $S_i S_j = S_j S_i$ such that $S_i^2 = I$. Note that, since any transform S_i is isometric, then if $x \in \Omega$ or $x \in \partial\Omega$, then $Sx \in \Omega$ or $Sx \in \partial\Omega$, respectively. For example, matrix S_i can be a matrix of the following linear mapping $S_i x = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_l)$.

Let $a_0, a_1, a_2, a_3, \dots, a_{2^n-1}$ be a set of real numbers. If we write the summation index i in the binary number system $(i_n \dots i_1)_2 \equiv i$, where $i_k = 0, 1$ for $k = 1, \dots, n$, then the coefficients can be written as $a_{(0\dots00)_2}, a_{(0\dots01)_2}, a_{(0\dots10)_2}, a_{(0\dots11)_2}, \dots, a_{(1\dots11)_2}$.

Let us introduce the following nonlocal analogue of the biharmonic operator:

$$L_n u(x) \equiv \sum_{i=0}^{2^n-1} a_i \Delta^2 u(S_n^{i_n} \dots S_1^{i_1} x).$$

Consider the following boundary value problem in the domain Ω .

Problem H. Let $\mu \geq 0$, $0 \leq \alpha \leq 1$. Find the function $u(x) \in C^4(\Omega) \cap C(\bar{\Omega})$, for which functions $D_\mu^\alpha[u](x)$, $D_\mu^{\alpha+1}[u](x)$ are continuous in $\bar{\Omega}$ and satisfy the conditions:

$$L_n u(x) = f(x), \quad x \in \Omega, \quad (1)$$

$$D_\mu^\alpha[u](x) \Big|_{\partial\Omega} = g_0(x), \quad x \in \partial\Omega, \quad (2)$$

$$D_\mu^{\alpha+1}[u](x) \Big|_{\partial\Omega} = g_1(x), \quad x \in \partial\Omega. \quad (3)$$

Remark 1. Note that in the case $a_0 = 1$, $a_k = 0$, $k = 1, 2, \dots, 2^n - 1$, Equation (1) coincides with the classical inhomogeneous biharmonic equation.

Remark 2. Note also that for $x \in \partial\Omega$, the following equalities hold:

$$\delta[u](x)|_{\partial\Omega} = r \frac{du(x)}{dr} \Big|_{\partial\Omega} = \frac{\partial u(x)}{\partial v} \Big|_{\partial\Omega},$$

$$\frac{\partial^2 u(x)}{\partial v^2} \Big|_{\partial\Omega} = r \frac{d}{dr} \left(r \frac{d}{dr} - 1 \right) u(x) \Big|_{\partial\Omega} = r^2 \frac{d^2 u(x)}{dr^2} \Big|_{\partial\Omega},$$

where v is the outer normal to the sphere $\partial\Omega$. Then, for $\alpha = 1$, we obtain:

$$D_\mu^1[u](x) = r^{-\mu} J^0[\delta[\tau^\mu \cdot u]](x) = r^{-\mu} r \frac{d}{dr} [r^\mu u(x)] = r \frac{d}{dr} u(x) + \mu u(x),$$

$$D_\mu^2[u](x) = r^{-\mu} \left(r \frac{d}{dr} \right)^2 [r^\mu u(x)] = r^{1-\mu} \frac{d}{dr} \left(r^{\mu+1} \frac{d}{dr} u(x) + \mu r^\mu u(x) \right)$$

$$= r^2 \frac{d^2 u(x)}{dr^2} + (\mu + 1) r \frac{d}{dr} u(x) + \mu r \frac{d}{dr} u(x) + \mu^2 u(x)$$

$$= r^2 \frac{d^2 u(x)}{dr^2} + (2\mu + 1)r \frac{d}{dr} u(x) + \mu^2 u(x).$$

Thus, if $\mu = 0$, then in the case $\alpha = 0$ in the problem H, we obtain the Dirichlet boundary conditions:

$$u(x) = g_0(x), \quad \frac{\partial u(x)}{\partial \nu} = g_1(x), \quad x \in \partial\Omega,$$

and in the case $\alpha = 1$, boundary conditions of the Neumann-type [36,37]:

$$\frac{\partial u(x)}{\partial \nu} = g_0(x), \quad \frac{\partial^2 u(x)}{\partial \nu^2} + \frac{\partial u(x)}{\partial \nu} = g_1(x), \quad x \in \partial\Omega.$$

Moreover, in the case $\alpha = 1$ and $\mu > 0$, the boundary conditions of the problem H are written as:

$$\frac{\partial u(x)}{\partial \nu} + \mu u(x) = g_0(x), \quad \frac{\partial^2 u(x)}{\partial \nu^2} + (2\mu + 1) \frac{\partial u(x)}{\partial \nu} + \mu^2 u(x) = g_1(x), \quad x \in \partial\Omega,$$

i.e., we obtain a generalization of the third boundary value problem [38].

The article is organized as follows. In Section 1, the main problem is formulated. In Section 2, the properties of matrices of involutive transformations are considered. Section 3 contains well-known statements about the properties of Hadamard integrodifferential operators. In Section 4, the solvability of the Dirichlet problem (case $\alpha = 0$) is studied, and in Section 5, the necessary and sufficient conditions for the solvability of Neumann-type boundary value problems (case $\mu = 0, \alpha = 1$) are found. In Section 6, for the general case, the main boundary value problem is studied.

2. Auxiliary Statements

To study the problem (1)–(3), we need some auxiliary statements. Let us introduce the function:

$$v(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i u(S_n^{i_n} \dots S_1^{i_1} x), \quad (4)$$

where the summation is taken in ascending order with respect to the index i . From this equality, it is easy to conclude that functions of the form $v(S_n^{j_n} \dots S_1^{j_1} x)$, where $j = 0, \dots, 2^n - 1$, can be linearly expressed in terms of functions $u(S_n^{i_n} \dots S_1^{i_1} x)$. If we consider the following vectors of order 2^n :

$$U(x) = \left(u(x), \dots, u(S_n^{i_n} \dots S_1^{i_1} x), \dots, u(S_n^1 \dots S_1^1 x) \right)^T,$$

$$V(x) = \left(v(x), \dots, v(S_n^{j_n} \dots S_1^{j_1} x), \dots, v(S_n^1 \dots S_1^1 x) \right)^T,$$

then this dependence can be expressed in the matrix form:

$$V(x) = A_n U(x), \quad (5)$$

where $A_n = (a_{i,j})_{i,j=0,\dots,2^n-1}$ is the matrix of order $2^n \times 2^n$.

Theorem 1. *The matrix A_n from the equality (5) can be represented in the form:*

$$A_n = (a_{i,j})_{i,j=0,\dots,2^n-1} = (a_{i \oplus j})_{i,j=0,\dots,2^n-1}, \quad (6)$$

where the operation in the subscript of the matrix coefficients is understood in the following sense $i \oplus j \equiv (i)_2 \oplus (j)_2 = ((i_n + j_n \bmod 2) \dots (i_1 + j_1 \bmod 2))_2$, where $(i)_2 = (i_n \dots i_1)_2$ is a

representation of the index in the binary number system. The linear combination of matrices of the form of (6) is a matrix of the form of (6).

Proof. Let $n = 1$, then we have:

$$A_1 = \begin{pmatrix} a_{0\oplus 0} & a_{0\oplus 1} \\ a_{1\oplus 0} & a_{1\oplus 1} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix},$$

and if $n = 2$, then we obtain:

$$A_2 = \begin{pmatrix} a_{(00)_2 \oplus (00)_2} & a_{(00)_2 \oplus (01)_2} & a_{(00)_2 \oplus (10)_2} & a_{(00)_2 \oplus (11)_2} \\ a_{(01)_2 \oplus (00)_2} & a_{(01)_2 \oplus (01)_2} & a_{(01)_2 \oplus (10)_2} & a_{(01)_2 \oplus (11)_2} \\ a_{(10)_2 \oplus (00)_2} & a_{(10)_2 \oplus (01)_2} & a_{(10)_2 \oplus (10)_2} & a_{(10)_2 \oplus (11)_2} \\ a_{(11)_2 \oplus (00)_2} & a_{(11)_2 \oplus (01)_2} & a_{(11)_2 \oplus (10)_2} & a_{(11)_2 \oplus (11)_2} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}.$$

Consider the function $v(S_n^{i_n} \dots S_1^{i_1} x)$ whose coefficients at $u(S_n^{j_n} \dots S_1^{j_1})$ make up the $i \equiv (i_n \dots i_1)_2$ th row of the matrix A_n :

$$\begin{aligned} v(S_n^{i_n} \dots S_1^{i_1} x) &= \sum_{j \equiv (j_n \dots j_1)_2=0}^{2^n-1=(1\dots 1)_2} a_{(j_n \dots j_1)_2} u(S_n^{j_n} \dots S_1^{j_1} S_n^{i_n} \dots S_1^{i_1} x) \\ &= \sum_{j \equiv (j_n \dots j_1)_2=0}^{(1\dots 1)_2} a_{(j_n \dots j_1)_2} u(S_n^{j_n+i_n \bmod 2} \dots S_1^{j_1+i_1 \bmod 2} x). \end{aligned} \quad (7)$$

Here, the following properties $S_j^2 x = x$ and $S_j S_i x = S_i S_j x$ of matrices S_1, \dots, S_n are taken into account. Let us replace the index $i \oplus j = l$. Then, $l \oplus i = i \oplus j \oplus i = j$, and the correspondence $j \sim l$ is one-to-one. Replacement $j \rightarrow l$ of the index changes only the order of summation in the sum (7). For example, if $i = 1$, then the sequence $j : 0, 1, 2, 3, 4, 5, \dots$ goes to $l = 1 \oplus j : 1, 0, 3, 2, 5, 4, \dots$ After replacing the index, we obtain:

$$v(S_n^{i_n} \dots S_1^{i_1} x) = \sum_{l=0}^{(1\dots 1)_2} a_{(i_n+l_n \bmod 2 \dots i_1+l_1 \bmod 2)_2} u(S_n^{l_n} \dots S_1^{l_1} x),$$

whence $a_{i,l} = a_{(i_n+l_n \bmod 2 \dots i_1+l_1 \bmod 2)_2} = a_{i \oplus l}$, which proves (6).

It is clear that if α, β are constants, then:

$$\alpha(a_{i \oplus j})_{i,j=0,\dots,2^n-1} + \beta(b_{i \oplus j})_{i,j=0,\dots,2^n-1} = (\alpha a_{i \oplus j} + \beta b_{i \oplus j})_{i,j=0,\dots,2^n-1}.$$

The theorem is proven. \square

Corollary 1. The matrix A_n is uniquely determined by its first row $(a_0, a_1, \dots, a_{2^n-1})$.

Indeed, the i th row of the matrix A_n can be written through its first row in the form $(a_{i \oplus 0}, a_{i \oplus 1}, \dots, a_{i \oplus (2^n-1)})$.

This property of the matrix A_n we denote by the equality $A_n \equiv A_n(a_0, \dots, a_{2^n-1})$.

Corollary 2. The matrix A_n has the symmetry property:

$$(a_{i,j})_{i,j=0,\dots,2^n-1} = (a_{j,i})_{i,j=0,\dots,2^n-1} \quad (8)$$

and can be written as:

$$A_n = \begin{pmatrix} A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) & A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) \\ A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) & A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) \end{pmatrix} \quad (9)$$

or more generally in the form of a block matrix A_{n-m} consisting of matrices A_m :

$$A_n = A_{n-m} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(k_n\dots k_{m+1})_2}, \dots, A_m^{(1\dots 1)_2} \right), \quad (10)$$

where $A_m^{(k_n\dots k_{m+1})_2}(a_{(k_n\dots k_{m+1}0\dots 0)_2}, \dots, a_{(k_n\dots k_{m+1}1\dots 1)_2})$ is a matrix of the form of (6) of order 2^m .

Proof. Indeed, since the binary operation $i \oplus j$ is commutative:

$$i \oplus j = (i_n + j_n \bmod 2 \dots i_1 + j_1 \bmod 2)_2 = (j_n + i_n \bmod 2 \dots j_1 + i_1 \bmod 2)_2 = j \oplus i,$$

then the property (8) holds:

$$(a_{i,j})_{i,j=0,\dots,2^n-1} = (a_{i \oplus j})_{i,j=0,\dots,2^n-1} = (a_{j \oplus i})_{i,j=0,\dots,2^n-1} = (a_{j,i})_{i,j=0,\dots,2^n-1}.$$

Further, it is easy to see the validity of the equalities:

$$\left(a_{(0i_{n-1}\dots i_1)_2 \oplus (0j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} = \left(a_{(1i_{n-1}\dots i_1)_2 \oplus (1j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} \quad (11)$$

and:

$$\left(a_{(0i_{n-1}\dots i_1)_2 \oplus (1j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} = \left(a_{(1i_{n-1}\dots i_1)_2 \oplus (0j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1}, \quad (12)$$

from which the property (9) follows. Indeed, if we divide the matrix A_n into four equal-in-size square blocks and consider the lower right block, then its indices are located in the range $(10\dots 0)_2 \leq i, j \leq (11\dots 1)_2$, which means that this block, by virtue of (11), has the form:

$$\begin{aligned} & \left(a_{(1i_{n-1}\dots i_1)_2 \oplus (1j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} \\ &= \left(a_{(0i_{n-1}\dots i_1)_2 \oplus (0j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} = A_{n-1}(a_0, \dots, a_{2^{n-1}-1}), \end{aligned}$$

i.e., the diagonal blocks of the matrix A_n are of the form $A_{n-1}(a_0, \dots, a_{2^{n-1}-1})$.

Similarly, the top right block of A_n has the indices in the range $(00\dots 0)_2 \leq i \leq (01\dots 1)_2, (10\dots 0)_2 \leq j \leq (11\dots 1)_2$, which means this block has the form:

$$\left(a_{(0i_{n-1}\dots i_1)_2 \oplus (1j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} = A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}).$$

By the equality (12), the lower left block of A_n has the form:

$$\begin{aligned} & \left(a_{(1i_{n-1}\dots i_1)_2 \oplus (0j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} \\ &= \left(a_{(0i_{n-1}\dots i_1)_2 \oplus (1j_{n-1}\dots j_1)_2} \right)_{i,j=0,\dots,2^{n-1}-1} = A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}). \end{aligned}$$

Equality (9) is proven. Now, consider a block matrix of the form:

$$A_{n-m} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(k_n\dots k_{m+1})_2}, \dots, A_m^{(1\dots 1)_2} \right) = \left(A_m^{(i_n\dots i_{m+1})_2 \oplus (j_n\dots j_{m+1})_2} \right)_{i,j=0,\dots,2^{n-m}-1}.$$

The elements of its block matrix with the number $(k_n\dots k_{m+1})_2$ can be written as:

$$A_m^{(k_n\dots k_{m+1})_2}(a_{(k_n\dots k_{m+1}0\dots 0)_2}, \dots, a_{(k_n\dots k_{m+1}1\dots 1)_2}) = \left(a_{(k_n\dots k_{m+1}(i_n\dots i_1)_2 \oplus (j_n\dots j_1)_2)} \right)_{i,j=0,\dots,2^m-1}.$$

Consider the element $a_{i,j}$ of the block matrix:

$$A_{n-m} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(k_n\dots k_{m+1})_2}, \dots, A_m^{(1\dots 1)_2} \right).$$

It is located in the block with coordinates $(i_n \dots i_{m+1})_2, (j_n \dots j_{m+1})_2$, and this means it is in the block $A_m^{(i_n\dots i_{m+1})_2 \oplus (j_n\dots j_{m+1})_2}$ and, therefore, has the form:

$$a_{i,j} = a_{((i_n\dots i_{m+1})_2 \oplus (j_n\dots j_{m+1})_2) (i_m\dots i_1)_2 \oplus (j_m\dots j_1)_2} = a_{i \oplus j}.$$

This coincides with Formula (6). The corollary is proven. \square

Theorem 2. *Multiplication of matrices of the form of (6) is commutative. The product of matrices of the form of (6) is again a matrix of the form of (6).*

Proof. Let us prove this property by induction on n . For $n = 1$, it is obviously true that:

$$\begin{aligned} A_1 B_1 &= \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 \\ b_1 & b_0 \end{pmatrix} = \begin{pmatrix} a_0 b_0 + a_1 b_1 & a_0 b_1 + a_1 b_0 \\ a_1 b_0 + a_0 b_1 & a_1 b_1 + a_0 b_0 \end{pmatrix} \\ &= \begin{pmatrix} b_0 a_0 + b_1 a_1 & b_0 a_1 + b_1 a_0 \\ b_1 a_0 + b_0 a_1 & b_1 a_1 + b_0 a_0 \end{pmatrix} = B_1 A_1. \end{aligned}$$

Assuming that the multiplication of matrices A_{n-1} and B_{n-1} of order $n - 1$ is commutative, we prove that the multiplication of matrices A_n and B_n of order n is also commutative. Denote $A_{n-1} = A_{n-1}(a_0, \dots, a_{2^{n-1}-1})$, $\hat{A}_{n-1} = A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1})$. By the property (9):

$$\begin{aligned} A_n B_n &= \begin{pmatrix} A_{n-1} & \hat{A}_{n-1} \\ \hat{A}_{n-1} & A_{n-1} \end{pmatrix} \begin{pmatrix} B_{n-1} & \hat{B}_{n-1} \\ \hat{B}_{n-1} & B_{n-1} \end{pmatrix} = \begin{pmatrix} A_{n-1} B_{n-1} + \hat{A}_{n-1} \hat{B}_{n-1} & A_{n-1} \hat{B}_{n-1} + \hat{A}_{n-1} B_{n-1} \\ \hat{A}_{n-1} B_{n-1} + A_{n-1} \hat{B}_{n-1} & \hat{A}_{n-1} \hat{B}_{n-1} + A_{n-1} B_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} B_{n-1} A_{n-1} + \hat{B}_{n-1} \hat{A}_{n-1} & B_{n-1} \hat{A}_{n-1} + \hat{B}_{n-1} A_{n-1} \\ \hat{B}_{n-1} A_{n-1} + B_{n-1} \hat{A}_{n-1} & \hat{B}_{n-1} \hat{A}_{n-1} + B_{n-1} A_{n-1} \end{pmatrix} = B_n A_n. \end{aligned}$$

The induction step is proven, and therefore, matrices of the form $A_n(a_0, \dots, a_{2^n-1})$ are commutative. It is not hard to see that:

$$AB = (a_{i \oplus j})_{i,j=0,\dots,2^n-1} (b_{i \oplus j})_{i,j=0,\dots,2^n-1} = \left(\sum_{k=0}^{2^n-1} a_{i \oplus k} b_{k \oplus j} \right)_{i,j=0,\dots,2^n-1}.$$

In the sum from the formula above, let us change the index $k \rightarrow l$, as in Theorem 1, according to the equality $i \oplus k = l$. Then, $l \oplus i = i \oplus k \oplus i = i \oplus i \oplus k = k$, and this means that the correspondence $k \sim l$ is one-to-one. Replacement of the index $k \rightarrow l$ changes only the order of summation in the sum. By virtue of the associativity of the operation \oplus , we have:

$$AB = \left(\sum_{l=0}^{2^n-1} a_l b_{(l \oplus i) \oplus j} \right)_{i,j=0,\dots,2^n-1} = \left(\sum_{l=0}^{2^n-1} a_l b_{l \oplus (i \oplus j)} \right)_{i,j=0,\dots,2^n-1}.$$

The first row of the matrix AB is:

$$(AB)_{i=0} = \left(\sum_{k=0}^{2^n-1} a_k b_{k \oplus j} \right)_{j=0,\dots,2^n-1},$$

and hence, the matrix C of the form of (6), constructed by the first row of AB , is written in the form coinciding with AB :

$$C \equiv \left(\sum_{k=0}^{2^n-1} a_k b_{k \oplus (i+j)} \right)_{j=0, \dots, 2^n-1} = AB.$$

The theorem is proven. \square

Theorem 3. *The eigenvectors of the matrix $A_n(a_0, \dots, a_{2^n-1})$ can be chosen in the form:*

$$\mathbf{a}_n^k = (\mathbf{a}_{n-1}^k, \pm \mathbf{a}_{n-1}^k)^T, k = 0, \dots, 2^{n-1} - 1,$$

where \mathbf{a}_{n-1}^k is the eigenvector of the matrix $A_{n-1}(a_0, \dots, a_{2^{n-1}-1})$, $k = 0, \dots, 2^{n-1} - 1$; for $n = 1$, we have $\mathbf{a}_1^0 = (1, 1)^T$, $\mathbf{a}_1^1 = (1, -1)^T$. The eigenvectors of the matrix A_n are orthogonal. The eigenvalues of the matrix A_n are of the form:

$$\mu_n^{k,\pm} = \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k, k = 0, \dots, 2^{n-1} - 1,$$

where μ_{n-1}^k and $\hat{\mu}_{n-1}^k$ are eigenvalues of the matrices:

$$A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) \quad \text{and} \quad \hat{A}_{n-1} = A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1})$$

corresponding to the eigenvector \mathbf{a}_{n-1}^k , respectively; $\mu_1^0 = a_0 + a_1$, $\mu_1^1 = a_0 - a_1$.

Proof. Let us carry out the proof by induction on n . Suppose that the eigenvectors of the matrix $A_n(a_0, \dots, a_{2^n-1})$ are independent on numbers a_0, \dots, a_{2^n-1} . For $n = 1$, it is obvious that the eigenvectors of the matrix $A_1(a_0, a_1)$ can be chosen in the form $\mathbf{a}_1^+ = (1, 1)^T$, $\mathbf{a}_1^- = (1, -1)^T$, and the eigenvalues corresponding to them have the form $\mu_1^+ = a_0 + a_1$, $\mu_1^- = a_0 - a_1$. For the matrix:

$$A_2 = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_1(a_0, a_1) & A_1(a_2, a_3) \\ A_1(a_2, a_3) & A_1(a_0, a_1) \end{pmatrix}$$

the eigenvectors have the form:

$$\mathbf{a}_2^{(+,+)} = (\mathbf{a}_1^+, \mathbf{a}_1^+)^T, \mathbf{a}_2^{(-,+)} = (\mathbf{a}_1^-, \mathbf{a}_1^-)^T, \mathbf{a}_2^{(+,-)} = (\mathbf{a}_1^+, -\mathbf{a}_1^+)^T, \mathbf{a}_2^{(-,-)} = (\mathbf{a}_1^-, -\mathbf{a}_1^-)^T,$$

or briefly, $\mathbf{a}_2^{(\pm_1, \pm_2)} = (\mathbf{a}_1^{\pm_1}, \pm_2 \mathbf{a}_1^{\pm_1})^T$. The + and - signs in the expressions \pm_1 and \pm_2 are values taken independently of each other. Indeed, the equalities:

$$\begin{aligned} A_2 \mathbf{a}_2^{(\pm_1, \pm_2)} &= \begin{pmatrix} A_1(a_0, a_1) & A_1(a_2, a_3) \\ A_1(a_2, a_3) & A_1(a_0, a_1) \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^{\pm_1} \\ \pm_2 \mathbf{a}_1^{\pm_1} \end{pmatrix} = \begin{pmatrix} A_1(a_0, a_1) \mathbf{a}_1^{\pm_1} \pm_2 A_1(a_2, a_3) \mathbf{a}_1^{\pm_1} \\ A_1(a_2, a_3) \mathbf{a}_1^{\pm_1} \pm_2 A_1(a_0, a_1) \mathbf{a}_1^{\pm_1} \end{pmatrix} \\ &= \begin{pmatrix} (a_0 \pm_1 a_1) \mathbf{a}_1^{\pm_1} \pm_2 (a_2 \pm_1 a_3) \mathbf{a}_1^{\pm_1} \\ (a_2 \pm_1 a_3) \mathbf{a}_1^{\pm_1} \pm_2 (a_0 \pm_1 a_1) \mathbf{a}_1^{\pm_1} \end{pmatrix} = (a_0 \pm_1 a_1 \pm_2 (a_2 \pm_1 a_3)) \begin{pmatrix} \mathbf{a}_1^{\pm_1} \\ \pm_2 \mathbf{a}_1^{\pm_1} \end{pmatrix} \\ &= (a_0 \pm_1 a_1 \pm_2 (a_2 \pm_1 a_3)) \mathbf{a}_2^{(\pm_1, \pm_2)} \end{aligned}$$

are true, and hence, $(\mathbf{a}_1^{\pm_1}, \pm_2 \mathbf{a}_1^{\pm_1})^T$ are the eigenvectors for four different combinations of signs \pm_1 and \pm_2 . It is seen that the eigenvectors $\mathbf{a}_2^{(\pm_1, \pm_2)} = (1, \pm_1 1, \pm_2 1, \pm_2 \pm_1 1)^T$ of the matrix $A_2(a_0, a_1, a_2, a_3)$ do not depend on numbers $\{a_k\}$.

Further, assuming that the eigenvectors $\mathbf{a}_{n-1}^0, \dots, \mathbf{a}_{n-1}^{2^{n-1}-1}$ of the matrix $A_{n-1}(a_0, \dots, a_{2^{n-1}-1})$ do not depend on its coefficients, we prove that this property is also true for the matrix $A_n(a_0, \dots, a_{2^n-1})$.

Let $\mu_{n-1}^0, \dots, \mu_{n-1}^{2^{n-1}-1}$ be the eigenvalues corresponding to the above eigenvectors of the matrix $A_{n-1}(a_0, \dots, a_{2^{n-1}-1})$, independent of its coefficients, then vectors of the form $\mathbf{a}_n^k = (\mathbf{a}_{n-1}^k, \pm \mathbf{a}_{n-1}^k)^T$, where $k = 0, \dots, 2^{n-1}-1$, are the eigenvectors of the matrix $A_n(a_0, \dots, a_{2^n-1})$. Indeed, we have:

$$\begin{aligned} A_n \mathbf{a}_n^k &= A_n(\mathbf{a}_{n-1}^k \pm \mathbf{a}_{n-1}^k) = \begin{pmatrix} A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) & A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) \\ A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) & A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) \end{pmatrix} \begin{pmatrix} \mathbf{a}_{n-1}^k \\ \pm \mathbf{a}_{n-1}^k \end{pmatrix} \\ &= \begin{pmatrix} A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) \mathbf{a}_{n-1}^k \pm A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) \mathbf{a}_{n-1}^k \\ A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1}) \mathbf{a}_{n-1}^k \pm A_{n-1}(a_0, \dots, a_{2^{n-1}-1}) \mathbf{a}_{n-1}^k \end{pmatrix} \\ &= \begin{pmatrix} \mu_{n-1}^k \mathbf{a}_{n-1}^k \pm \hat{\mu}_{n-1}^k \mathbf{a}_{n-1}^k \\ \hat{\mu}_{n-1}^k \mathbf{a}_{n-1}^k \pm \mu_{n-1}^k \mathbf{a}_{n-1}^k \end{pmatrix} = (\mu_{n-1}^k \pm \hat{\mu}_{n-1}^k) \begin{pmatrix} \mathbf{a}_{n-1}^k \\ \pm \mathbf{a}_{n-1}^k \end{pmatrix} = (\mu_{n-1}^k \pm \hat{\mu}_{n-1}^k) \mathbf{a}_n^k, \end{aligned}$$

where $\hat{\mu}_{n-1}^k$ is the eigenvalue of the matrix $A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1})$ corresponding to the eigenvector \mathbf{a}_{n-1}^k . Obviously, there are 2^n vectors of the form $\mathbf{a}_n^k = (\mathbf{a}_{n-1}^k, \pm \mathbf{a}_{n-1}^k)^T$. Therefore, the eigenvalues of the matrix $A_n(a_0, \dots, a_{2^n-1})$ are $\mu_n^{k,\pm} = \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k$.

Orthogonality: It is obvious that the eigenvectors $\mathbf{a}_1^+ = (1, 1)^T, \mathbf{a}_1^- = (1, -1)^T$ of the matrix $A_1(a_0, a_1)$ are orthogonal. If the eigenvectors $\mathbf{a}_{n-1}^k, k = 0, \dots, 2^{n-1}-1$ of the matrix $A_{n-1}(a_{2^{n-1}}, \dots, a_{2^n-1})$ are chosen as orthogonal, then the eigenvectors $\mathbf{a}_n^k = (\mathbf{a}_{n-1}^k, \pm \mathbf{a}_{n-1}^k)^T$ of the matrix $A_n(a_0, \dots, a_{2^n-1})$ are also orthogonal:

$$\begin{aligned} \mathbf{a}_n^{k_1} \mathbf{a}_n^{k_2} &= (\mathbf{a}_{n-1}^{k_1}, \pm \mathbf{a}_{n-1}^{k_1})^T (\mathbf{a}_{n-1}^{k_2}, \pm \mathbf{a}_{n-1}^{k_2})^T = \mathbf{a}_{n-1}^{k_1} \mathbf{a}_{n-1}^{k_2} + \mathbf{a}_{n-1}^{k_1} \mathbf{a}_{n-1}^{k_2} = 0, k_1 \neq k_2 \\ \text{and } (\mathbf{a}_{n-1}^k, \mathbf{a}_{n-1}^k)^T (\mathbf{a}_{n-1}^k, -\mathbf{a}_{n-1}^k)^T &= 0. \text{ The theorem is proven. } \square \end{aligned}$$

Corollary 3. Let $k = (k_n, \dots, k_1)_2, k_i = 0, 1$, then the eigenvector of the matrix A_n numbered by k can be written in the form:

$$\begin{aligned} \mathbf{a}_n^k &= (1, (-1)^{k_1}, (-1)^{k_2}, (-1)^{k_2+k_1}, (-1)^{k_3}, (-1)^{k_3+k_1}, (-1)^{k_3+k_2}, \\ &\quad (-1)^{k_3+k_2+k_1}, (-1)^{k_4}, \dots, (-1)^{k_n+\dots+k_1})^T = \left((-1)^{k \otimes m} \right)_{m=0, \dots, 2^n-1}, \end{aligned} \quad (13)$$

where $k \otimes i \equiv (k_n \dots k_1)_2 \otimes (i_n \dots i_1)_2 = k_n i_n + \dots + k_1 i_1$ is a “scalar” product of the indexes $(k)_2$ and $(i)_2$.

The eigenvalue corresponding to the eigenvector $\mathbf{a}_n^{(k_n \dots k_1)_2}$ can be written in a similar form:

$$\mu_n^k \equiv \mu_n^{(k_n \dots k_1)_2} = \sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i = \sum_{i=0}^{2^n-1} (-1)^{k_n i_n + \dots + k_1 i_1} a_{(i_n \dots i_1)_2}. \quad (14)$$

Proof. Let us prove (13). For $n = 1$, we have $\mathbf{a}_1^+ = \mathbf{a}_1^{(0)_2} = ((-1)^{0 \otimes 0}, (-1)^{0 \otimes 1})^T$, $\mathbf{a}_1^- = \mathbf{a}_1^{(1)_2} = ((-1)^{1 \otimes 0}, (-1)^{1 \otimes 1})^T$, and (13) is true. If Formula (13) is true for the vector $\mathbf{a}_{n-1}^{(k_{n-1} \dots k_1)_2}$, then by Theorem 3, we have:

$$\begin{aligned} \left(\mathbf{a}_{n-1}^{(k_{n-1} \dots k_1)_2}, \pm \mathbf{a}_{n-1}^{(k_{n-1} \dots k_1)_2} \right)^T &= \left(\mathbf{a}_{n-1}^{(k_{n-1} \dots k_1)_2}, (-1)^{k_n} \mathbf{a}_{n-1}^{(k_{n-1} \dots k_1)_2} \right)^T \\ &= \left(\left((-1)^{(k_n k_{n-1} \dots k_1)_2 \otimes (0 m_{n-1} \dots m_1)_2} \right)_{m=0, \dots, 2^{n-1}-1}, \right. \\ &\quad \left. \left((-1)^{(k_n k_{n-1} \dots k_1)_2 \otimes (1 m_{n-1} \dots m_1)_2} \right)_{m=0, \dots, 2^{n-1}-1} \right)^T \\ &= \left(\left((-1)^{k \otimes m} \right)_{m=0, \dots, 2^{n-1}-1}, \left((-1)^{k \otimes m} \right)_{m=2^{n-1}, \dots, 2^n-1} \right)^T \\ &= \left(\left((-1)^{k \otimes m} \right)_{m=0, \dots, 2^n-1} \right)^T = \mathbf{a}_n^{(k_n k_{n-1} \dots k_1)_2} \end{aligned}$$

and hence, the formula (13) is also true for the vector $\mathbf{a}_n^{(k_n k_{n-1} \dots k_1)_2} = \mathbf{a}_n^k$.

Let us prove (14). For $n = 1$, we have:

$$\mu_1^{k_1} = a_0 + (-1)^{k_1} a_1 = (-1)^0 a_{(0)_2} + (-1)^{k_1 \cdot 1} a_{(1)_2},$$

where $k_1 = 0, 1$. Assume that the formula (14) is valid for $n = n - 1$, and prove its validity for n . By Theorem 3, changing the notation $\pm = (-1)^{k_n}$, we write:

$$\begin{aligned} \mu_n^{k, \pm} &= \mu_{n-1}^k \pm \hat{\mu}_{n-1}^k = \sum_{i=0}^{2^{n-1}-1} (-1)^{k_{n-1} i_{n-1} + \dots + k_1 i_1} a_{(i_{n-1} \dots i_1)_2} \\ &+ \sum_{i=0}^{2^{n-1}-1} (-1)^{k_n \cdot 1 + k_{n-1} i_{n-1} + \dots + k_1 i_1} a_{(1 i_{n-1} \dots i_1)_2} = \sum_{i=0}^{2^{n-1}-1} (-1)^{k_n \cdot i_n + k_{n-1} i_{n-1} + \dots + k_1 i_1} a_{(i_n i_{n-1} \dots i_1)_2} \\ &+ \sum_{i=2^{n-1}}^{2^n-1} (-1)^{k_n \cdot i_n + k_{n-1} i_{n-1} + \dots + k_1 i_1} a_{(i_n i_{n-1} \dots i_1)_2} = \sum_{i=0}^{2^n-1} (-1)^{k_n \cdot i_n + \dots + k_1 i_1} a_{(i_n i_{n-1} \dots i_1)_2}, \end{aligned}$$

which proves (14). The corollary is proven. \square

Let us denote the operation of taking the adjoint matrix to the matrix $A_n(a_0, \dots, a_{2^n-1})$, for the convenience of presentation, in the form $\text{adj}(A_n(a_0, \dots, a_{2^n-1})) = \overline{A_n}(a_0, \dots, a_{2^n-1})$. Then, by the definition of the adjoint matrix:

$$A_n(a_0, \dots, a_{2^n-1}) \overline{A_n}(a_0, \dots, a_{2^n-1}) = (\det A_n) I_{2^n}. \quad (15)$$

In the case $n = 0$, we assume that $\overline{a_k} = 1$. Obviously, $\overline{\overline{A_n}} = A_n$, and if the matrices A_n and B_n of the form of (6) are nonsingular, then, by virtue of Theorem 2, on the commutativity of A_n and B_n , we have:

$$\begin{aligned} \overline{A_n B_n} &= \overline{B_n A_n} = \det(B_n A_n) (B_n A_n)^{-1} = \det(B_n A_n) A_n^{-1} B_n^{-1} \\ &= \det(A_n) A_n^{-1} \det(B_n) B_n^{-1} = \overline{A_n B_n} \end{aligned}$$

Define the operation of multiplying a matrix C_{k-1} by a block matrix $\begin{pmatrix} A_{k-1}^0 & A_{k-1}^1 \\ A_{k-1}^1 & A_{k-1}^0 \end{pmatrix}$ in the form:

$$C_{k-1} \cdot \begin{pmatrix} A_{k-1}^0 & A_{k-1}^1 \\ A_{k-1}^1 & A_{k-1}^0 \end{pmatrix} \equiv \begin{pmatrix} C_{k-1} & 0 \\ 0 & C_{k-1} \end{pmatrix} \begin{pmatrix} A_{k-1}^0 & A_{k-1}^1 \\ A_{k-1}^1 & A_{k-1}^0 \end{pmatrix} = \begin{pmatrix} C_{k-1} A_{k-1}^0 & C_{k-1} A_{k-1}^1 \\ C_{k-1} A_{k-1}^1 & C_{k-1} A_{k-1}^0 \end{pmatrix}.$$

Note that if $C_{k-1} = C_{k-1}(c_0, \dots, c_{2^{k-1}-1})$, then $C_k = C_k(c_0, \dots, c_{2^{k-1}-1}, 0, \dots, 0)$:

$$C_{k-1} \cdot \begin{pmatrix} A_{k-1}^0 & A_{k-1}^1 \\ A_{k-1}^1 & A_{k-1}^0 \end{pmatrix} = \begin{pmatrix} C_{k-1} & 0 \\ 0 & C_{k-1} \end{pmatrix} \begin{pmatrix} A_{k-1}^0 & A_{k-1}^1 \\ A_{k-1}^1 & A_{k-1}^0 \end{pmatrix} = C_k A_k = A_k C_k = A_k \cdot C_{k-1}$$

and hence, by Theorem 2, the matrix $C_{k-1} \cdot A_k$ has the form of (6).

Theorem 4. The determinant of the matrix $A_n(a_0, \dots, a_{2^n-1})$ can be written in the form:

$$\det A_n(a_0, \dots, a_{2^n-1}) = \prod_{k=0}^{2^n-1} \left(\sum_{i=0}^{2^n-1} (-1)^{k_n i_n + \dots + k_1 i_1} a_{(i_n \dots i_1)_2} \right). \quad (16)$$

For any $m = 0, \dots, n$, the equality holds:

$$\begin{aligned} \overline{A_n}(a_0, \dots, a_{2^n-1}) &= \overline{(\det A_{n-m})} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(k_n\dots k_{m+1})_2}, \dots, A_m^{(1\dots 1)_2} \right) \\ &\quad \times \overline{A_{n-m}} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(k_n\dots k_{m+1})_2}, \dots, A_m^{(1\dots 1)_2} \right), \end{aligned} \quad (17)$$

where $\det A_{n-m}$ is calculated for the block matrix A_{n-m} , first with numerical coefficients $a^{(k_n\dots k_{m+1})_2}$, then the corresponding matrices $A_m^{(k_n\dots k_{m+1})_2}$ are substituted for them, and the adjoint matrix is taken for the resulting matrix. In addition, the adjoint matrix $\overline{A_{n-m}}$ is also constructed for numerical coefficients $a^{(k_n\dots k_{m+1})_2}$, and then, the corresponding matrices $A_m^{(k_n\dots k_{m+1})_2}$ are substituted for them.

Proof. Since the determinant of a matrix is equal to the product of its eigenvalues, using Formula (14), we obtain:

$$\det A_n(a_0, \dots, a_{2^n-1}) = \prod_{k=0}^{2^n-1} \mu_n^{(k_n\dots k_1)_2} = \prod_{k=0}^{2^n-1} \left(\sum_{i=0}^{2^n-1} (-1)^{k_n i_n + \dots + k_1 i_1} a_{(i_n \dots i_1)_2} \right).$$

Let us prove (17). Note that all operations for constructing matrices in this formula are correct since the determinant of a matrix and the elements of an adjoint matrix are algebraic expressions and all matrices involved in these constructions are commutative.

Denote the matrix on the right side of (17) by M , and multiply it on the right by the matrix A_n represented in the form of (10), then substitute numbers $a^{(k_n\dots k_{m+1})_2}$ instead of matrices $A_m^{(k_n\dots k_{m+1})_2}$; after transformations, return the matrices $A_m^{(k_n\dots k_{m+1})_2}$ back:

$$\begin{aligned} MA_n(a_0, \dots, a_{2^n-1}) &= MA_{n-m} \left(A_m^{(0\dots 0)_2}, \dots, A_m^{(1\dots 1)_2} \right) \\ &= \overline{(\det A_{n-m}^b)} \left(a^{(0\dots 0)_2}, \dots, a^{(k_n\dots k_{m+1})_2}, \dots, a^{(1\dots 1)_2} \right) \\ &\quad \times \overline{A_{n-m}^b} \left(a^{(0\dots 0)_2}, \dots, a^{(k_n\dots k_{m+1})_2}, \dots, a^{(1\dots 1)_2} \right) \\ A_{n-m}^b \left(a^{(0\dots 0)_2}, \dots, a^{(k_n\dots k_{m+1})_2}, \dots, a^{(1\dots 1)_2} \right) &= \left(\overline{\det A_{n-m}} \cdot I_{2^{n-m}}^b \right) \left(\det A_{n-m} \cdot I_{2^{n-m}}^b \right) \\ &= \prod_{k=0}^{2^{n-m}-1} \left(\sum_{i=0}^{2^{n-m}-1} (-1)^{k \otimes i} A_m^{(i_{n-m} \dots i_1)_2} \right) \left(\sum_{i=0}^{2^{n-m}-1} (-1)^{k \otimes i} A_m^{(i_{n-m} \dots i_1)_2} \right) \cdot I_{2^{n-m}}^b \\ &= \prod_{k=0}^{2^{n-m}-1} \det \left(\sum_{i=0}^{2^{n-m}-1} (-1)^{k_{n-m} i_{n-m} + \dots + k_1 i_1} A_m^{(i_{n-m} \dots i_1)_2} \right) I_{2^m} \cdot I_{2^{n-m}}^b \\ &= \prod_{k=0}^{2^{n-m}-1} \det \left(\sum_{i=0}^{2^{n-m}-1} (-1)^{k_{n-m} i_{n-m} + \dots + k_1 i_1} A_m^{(i_{n-m} \dots i_1)_2} \right) I_{2^n} \equiv \lambda I_{2^n}, \end{aligned}$$

where the formula (16) and an equality of the form of (15) are used and $\lambda \in \mathbb{R}$. Here, the matrix I_{n-m}^b has a superscript b indicating that it is a block matrix, each block of size $2^m \times 2^m$. The resulting equality proves (17). The theorem is proven. \square

Corollary 4. *The matrix $\overline{A_n}(a_0, \dots, a_{2^n-1})$ is a matrix of the form of (6). If inverse matrix $A_n^{-1}(a_0, \dots, a_{2^n-1})$ exists, then it also has the form of (6).*

Proof. Obviously, $\overline{A_1}(a_0, a_1) = A_1(a_0, -a_1)$, and therefore, the matrix $\overline{A_1}(a_0, a_1)$ is of the form of (6). Suppose the matrix $\overline{A_{n-1}}(a_0, \dots, a_{2^{n-1}-1})$ for $n > 1$ is of the form of (6). Then, using Formula (17) for $m = n - 1$, we obtain:

$$\begin{aligned}\overline{A_n} &= \overline{(\det A_1)} \left(A_{n-1}^0, A_{n-1}^1 \right) \cdot \overline{A_1} \left(A_{n-1}^0, A_{n-1}^1 \right) \\ &= \overline{\left((A_{n-1}^0)^2 - (A_{n-1}^1)^2 \right)} \cdot A_1 \left(A_{n-1}^0, -A_{n-1}^1 \right).\end{aligned}$$

It is clear that the matrix $\det A_1(A_{n-1}^0, A_{n-1}^1) = (A_{n-1}^0)^2 - (A_{n-1}^1)^2$ has the form of (6), and hence, by the induction hypothesis, the matrix $\overline{\left((A_{n-1}^0)^2 - (A_{n-1}^1)^2 \right)}$ is also a matrix of the form of (6). Since, as said before in Theorem 4, the matrix $C_{n-1} \cdot A_n$ has the form of (6), and the product of the matrices of the form of (6) is again a matrix of the form of (6). The induction step is proven, and therefore, the corollary is true. \square

Corollary 5. *The matrix $\overline{A_n}(a_0, \dots, a_{2^n-1})$ can be calculated using the recurrent formula:*

$$\begin{aligned}\overline{A_n}(a_0, \dots, a_{2^n-1}) &= \overline{(\det A_{n-1})} \left(A_1^{(0\dots 0)_2}, \dots, A_1^{(k_n\dots k_2)_2}, \dots, A_1^{(1\dots 1)_2} \right) \\ &\quad \times \overline{A_{n-1}} \left(A_1^{(0\dots 0)_2}, \dots, A_1^{(k_n\dots k_2)_2}, \dots, A_1^{(1\dots 1)_2} \right) \\ &= \prod_{(k_n\dots k_2)_2=0}^{2^{n-1}-1} \left(\sum_{(i_n\dots i_2)_2=0}^{2^{n-1}-1} (-1)^{k_n i_n + \dots + k_2 i_2} \overline{A_1^{(i_n\dots i_2)_2}} \right) \overline{A_{n-1}}(b_0, \dots, b_{2^{n-1}-1}) \Big|_{b_k=A_1^k},\end{aligned}\quad (18)$$

where $A_1^k = A_1^{(k_n\dots k_2)_2} = A_1(a_{(k_n\dots k_2)_2}, a_{(k_n\dots k_2)_2})$.

This equality follows from (17) when $m = n - 1$, taking into account (16) and the fact that $\overline{A_1 + B_1} = \overline{A_1} + \overline{B_1}$.

3. Properties of Integrodifferential Operators

In this section, we present some well-known statements about fractional integrodifferential operators introduced in Section 1. In [27], the following assertions were proven.

Lemma 1. *Let $\alpha > 0$, $\mu \geq 0$, $0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\bar{\Omega})$, $p \in N_0$. Then:*

- (1) *If $\mu > 0$, then $J_\mu^\alpha[u](x) \in C^{\lambda+p}(\bar{\Omega})$;*
- (2) *If $\mu = 0$, then if the condition $u(0) = 0$ holds, the function $J_0^\alpha[u](x)$ also belongs to the class $C^{\lambda+p}(\bar{\Omega})$, and the equality $J_0^\alpha[u](0) = 0$ is valid.*

Lemma 2. *Let $\mu \geq 0$, $p - 1 < \alpha \leq p$, $p = 1, 2, \dots$, $0 < \lambda < 1$, and $u(x) \in C^{\lambda+q}(\bar{\Omega})$, $q \geq p$. Then, the function $D_\mu^\alpha[u](x)$ belongs to the class $C^{\lambda+q-p}(\bar{\Omega})$, and the equality $D_0^\alpha[u](0) = 0$ holds.*

Lemma 3. *Let $\mu \geq 0$, $p - 1 < \alpha \leq p$, $p = 1, 2, \dots$, $0 < \lambda < 1$ and $u(x) \in C^{\lambda+q}(\bar{\Omega})$, $q \geq l$. Then, for any $x \in \bar{\Omega}$, the following equality holds:*

$$J_\mu^\alpha[D_\mu^\alpha[u]](x) = \begin{cases} u(x), & \mu > 0 \\ u(x) - u(0), & \mu = 0. \end{cases}$$

Lemma 4. Let $\mu \geq 0$, $p - 1 < \alpha \leq p$, $p = 1, 2, \dots, 0 < \lambda < 1$ and $u(x) \in C^{\lambda+q}(\bar{\Omega})$, $q \geq p$. Then, for any $x \in \bar{\Omega}$ for $\mu > 0$, the following equality holds:

$$D_{\mu}^{\alpha} [J_{\mu}^{\alpha}[u]](x) = u(x) \quad (19)$$

For $\mu = 0$, the equality (19) is valid under the additional condition $u(0) = 0$.

Lemma 5. Let $\mu \geq 0$, $p - 1 < \alpha \leq p$, $p = 1, 2, \dots, f(x)$ be a smooth function in the domain $\bar{\Omega}$ and $-\Delta u(x) = f(x)$, $x \in \Omega$. Then, the equality is valid:

$$-\Delta D_{\mu}^{\alpha}[u](x) = F(x), \quad x \in \Omega, \quad (20)$$

where:

$$F(x) = D_{\mu+2}^{\alpha}[f](x). \quad (21)$$

Lemma 6. If $\mu = 0$, $0 < \alpha \leq 1$, then function $F(x)$ from the equality (21) can be represented as:

$$F(x) = \left(r \frac{d}{dr} + 4 \right) f_{1-\alpha}(x), \quad (22)$$

where:

$$f_{1-\alpha}(x) = J_4^{1-\alpha}[f](x). \quad (23)$$

4. The Dirichlet Problem

In this section, we study the problem H for $\alpha = 0$, i.e., the following Dirichlet problem:

$$L_n u(x) = f(x), \quad x \in \Omega, \quad (24)$$

$$u(x) = g_0(x), \quad \frac{\partial u(x)}{\partial \nu} = g_1(x), \quad x \in \partial\Omega. \quad (25)$$

In [20], the following assertion was proven.

Lemma 7. ([20], Lemma 3.1) Let S be an orthogonal matrix, then the operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ commute $\Delta I_S u(x) = I_S \Delta u(x)$ on functions $u \in C^2(\Omega)$. The operator $\delta u(x) = \sum_{i=1}^n x_i u_{x_i}(x)$ and the operator I_S also commute $\delta I_S u(x) = I_S \delta u(x)$ on functions $u \in C^1(\bar{\Omega})$, and the equality $\nabla I_S = I_S S^T \nabla$ is valid.

Corollary 6. If the function $u(x)$ is biharmonic in the domain Ω , then the function $u(S_n^{i_n} \dots S_1^{i_1} x) = I_{S_n^{i_n} \dots S_1^{i_1}} u(x)$, and hence, the function $v(x)$ from (4) is also biharmonic in Ω .

Proof. Indeed, the matrix $S_n^{i_n} \dots S_1^{i_1}$ is symmetric and orthogonal as $(S_n^{i_n} \dots S_1^{i_1})^2 = I$, and hence, by virtue of Lemma 7:

$$\Delta^2 u(x) = 0 \Rightarrow \Delta^2 I_{S_n^{i_n} \dots S_1^{i_1}} u(x) = I_{S_n^{i_n} \dots S_1^{i_1}} \Delta^2 u(x) = 0$$

and $\Delta^2 v(x) = 0$. It follows that if the function $u(x)$ is biharmonic in Ω , it satisfies the homogeneous Equation (1) in Ω . The corollary is proven. \square

The converse assertion is also valid.

Lemma 8. Let the function $u \in C^4(\Omega)$ satisfy homogeneous Equation (1) and $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, for $k = 0, \dots, 2^n - 1$, then the function $u(x)$ is biharmonic in Ω .

Proof. Let the function $u \in C^4(\Omega)$ satisfy homogeneous Equation (1). Consider the function $v(x)$ from (4):

$$v(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i u(S_n^{i_n} \dots S_1^{i_1} x).$$

It is evident that $v(x) \in C^4(\Omega)$ and $\Delta^2 v(x) = 0$, $x \in \Omega$, i.e., the function $v(x)$ is biharmonic in Ω . By virtue of Corollary 6, functions $v(S_n^{i_n} \dots S_1^{i_1} x)$ are also biharmonic in Ω . Further, as stated in (5), for vectors:

$$U(x) = (u(x), \dots, u(S_n^{i_n} \dots S_1^{i_1} x), \dots, u(S_n^1 \dots S_1^1 x))^T,$$

$$V(x) = (v(x), \dots, v(S_n^{j_n} \dots S_1^{j_1} x), \dots, v(S_n^1 \dots S_1^1 x))^T$$

the equality $V(x) = A_n U(x)$ holds, where $A_n = A_n(a_0, \dots, a_{2^n-1})$ is a matrix of the form of (6). By the condition of the lemma and by virtue of Theorem 4, the determinant of this system does not vanish. By Corollary 4, the matrix A_n^{-1} also has the form of (6). Let us introduce the notation $B_n(b_0, \dots, b_{2^n-1}) = A_n^{-1}(a_0, \dots, a_{2^n-1})$. If the first row of the matrix B_n is written as $b = (b_0, \dots, b_{2^n-1})^T$, we obtain:

$$u(x) = b \cdot V(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} b_i v(S_n^{i_n} \dots S_1^{i_1} x). \quad (26)$$

This immediately implies that the function $u(x)$ is biharmonic in Ω . The lemma is proven. \square

Theorem 5. If the coefficients of the operator L_n satisfy the condition $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, for $k = 0, \dots, 2^n - 1$ and a solution to the Dirichlet problem (24) and (25) exists, then it is unique.

Proof. Let us prove that the homogeneous problem (24) and (25) has only a zero solution, and hence, the solution to the inhomogeneous problem (24) and (25), if it exists, is unique. Let $u(x)$ be a solution to the homogeneous problem (24) and (25). By Lemma 8, the function $u(x)$ is biharmonic in Ω and satisfies the homogeneous conditions (24). Therefore, the function $u(x)$ is a solution to the following Dirichlet problem:

$$\Delta^2 u(x) = 0, \quad x \in \Omega; \quad u(x)|_{\partial\Omega} = 0, \quad \frac{\partial u(x)}{\partial \nu} \Big|_{\partial\Omega} = 0.$$

By virtue of the uniqueness of the solution to this Dirichlet problem, we have $u(x) \equiv 0$ in Ω . The theorem is proven. \square

Let us consider a theorem on the existence of a solution to the problem (24) and (25).

Theorem 6. If coefficients of the operator L_n satisfy the condition $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, $k = 0, \dots, 2^n - 1$, then for functions $f(x) \in C^\lambda(\bar{\Omega})$, $g_0(x) \in C^{\lambda+2}(\partial\Omega)$ and $g_1(x) \in C^{\lambda+1}(\partial\Omega)$, $0 < \lambda < 1$, the solution to the problem (24) and (25) exists, is unique, and can be represented as:

$$u(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} b_i v(S_n^{i_n} \dots S_1^{i_1} x), \quad (27)$$

where the function $v(x)$ is a solution to the Dirichlet problem:

$$\Delta^2 v(x) = f(x), \quad x \in \Omega; \quad v(x) = \tilde{g}_0(x), \quad \frac{\partial v(x)}{\partial \nu} = \tilde{g}_1(x), \quad x \in \partial\Omega, \quad (28)$$

with functions:

$$\tilde{g}_0(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i g_0(S_n^{i_n} \dots S_1^{i_1} x), \quad \tilde{g}_1(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i g_1(S_n^{i_n} \dots S_1^{i_1} x) \quad (29)$$

on the boundary, while b_i , $i = 0, \dots, 2^n - 1$ are the coefficients of the first row of the matrix $B_n(b_0, \dots, b_{2^n-1})$ inverse to the matrix $A_n = A_n(a_0, \dots, a_{2^n-1})$.

Proof. Let $u(x)$ be a solution to the problem (24) and (25). Consider the function $v(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i u(S_n^{i_n} \dots S_1^{i_1} x)$. Then, for the function $v(x)$, by virtue of Lemma 7, we obtain the following boundary value problem:

$$\begin{aligned} \Delta^2 v(x) &= f(x), \quad x \in \Omega, \\ \frac{\partial^j v(x)}{\partial \nu^j} \Big|_{\partial\Omega} &= \delta^j \sum_{i=0}^{2^n-1} a_k u(S_n^{i_n} \dots S_1^{i_1} x) \Big|_{\partial\Omega} = \sum_{i=0}^{2^n-1} a_k I_{S_n^{i_n} \dots S_1^{i_1}} \delta^j u(x) \Big|_{\partial\Omega} \\ &= \sum_{i=0}^{2^n-1} a_k I_{S_n^{i_n} \dots S_1^{i_1}} g_j(x) = \sum_{i=0}^{2^n-1} a_k g_j(S_n^{i_n} \dots S_1^{i_1} x) = \tilde{g}_j(x), \end{aligned}$$

where $j = 0, 1$. If $g_j(x) \in C^{2+\lambda-j}(\partial\Omega)$, $j = 0, 1$, then it is obvious that:

$$\tilde{g}_j(x) = \sum_{i=0}^{2^n-1} a_k g_j(S_n^{i_n} \dots S_1^{i_1} x) \in C^{2+\lambda-j}(\partial\Omega).$$

It is known (see, for example, [39]) that for given functions $f(x)$ and $\tilde{g}_j(x) = \sum_{i=0}^{2^n-1} a_k g_j(S_n^{i_n} \dots S_1^{i_1} x)$, $j = 0, 1$, a solution to the Dirichlet problem (28) exists and is unique. As in the case of the equality (5) between functions $v(x)$ and $u(x)$, we have the algebraic relation $V(x) = A_n U(x)$. If $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, then, by virtue of the equality (26), the unknown function $u(x)$ is uniquely determined through the function $v(x)$. On the contrary, let the function $v(x)$ be a solution to the problem (28). Let us show that the function $u(x)$ defined by Formula (27) satisfies all conditions of the problem (24) and (25). Indeed, if $f(x) \in C^\lambda(\bar{\Omega})$, $g_j(x) \in C^{2+\lambda-j}(\partial\Omega)$, $j = 0, 1$, then we obtain $v(x) \in C^4(\Omega) \cap C^{2+\lambda}(\bar{\Omega})$. From here, it follows that $u(x) \in C^4(\Omega) \cap C^{2+\lambda}(\bar{\Omega})$. Therefore, according to Lemma 7, in Ω , we have the equalities:

$$\begin{aligned} \Delta^2 u(x) &= \sum_{j=0}^{2^n-1} b_j \Delta^2 v(S_n^{j_n} \dots S_1^{j_1} x) = \sum_{j=0}^{2^n-1} b_j I_{S_n^{j_n} \dots S_1^{j_1}} \Delta^2 v(x) = \sum_{j=0}^{2^n-1} b_j I_{S_n^{j_n} \dots S_1^{j_1}} f(x) \\ &= \sum_{j=0}^{2^n-1} b_j f(S_n^{j_n} \dots S_1^{j_1} x). \end{aligned}$$

Let us denote $w(x) = \sum_{j=0}^{2^n-1} b_j f(S_n^{j_n} \dots S_1^{j_1} x)$ and consider the function $\hat{u}(x) = \sum_{i=0}^{2^n-1} a_i w(S_n^{i_n} \dots S_1^{i_1} x)$. Then, for vectors:

$$W(x) = (w(x), \dots, w(S_n^{i_n} \dots S_1^{i_1} x), \dots, w(S_n^1 \dots S_1^1 x))^T,$$

$$F(x) = (f(x), \dots, f(S_n^{j_n} \dots S_1^{j_1} x), \dots, f(S_n^1 \dots S_1^1 x))^T$$

the equalities $W(x) = B_n F(x)$ and $\hat{U}(x) = A_n W(x)$ hold, and hence, $A_n W(x) = A_n B_n F(x) = F(x)$. Thus:

$$\sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i \Delta^2 u(S_n^{i_n} \dots S_1^{i_1} x) \equiv \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i w(S_n^{i_n} \dots S_1^{i_1} x)$$

$$= (A_n W(x))_{j=0} = (A_n B_n F(x))_{j=0} = F(x)_{j=0} = f(x).$$

Therefore, the function $u(x)$ satisfies Equation (24).

Let us check the boundary conditions (25) of the studied problem. Taking into account that:

$$(\tilde{G}_0(x))_{j=0} = \tilde{g}_0(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i g_0(S_n^{i_n} \dots S_1^{i_1} x) = (A_n G_0(x))_{j=0}$$

for $x \in \partial\Omega$, according to Lemma 7, we obtain:

$$\frac{\partial^k u(x)}{\partial v^k} = \delta^k u(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} b_i \delta^k v(S_n^{i_n} \dots S_1^{i_1} x) = (B_n \delta^k V(x))_{j=0}$$

$$= (B_n \tilde{G}_k(x))_{j=0} = (B_n A_n G_k(x))_{j=0} = g_k(x),$$

where $k = 0, 1$, and hence, the boundary conditions (25) for the function $u(x)$ are satisfied. The theorem is proven. \square

5. The Neumann Problem

In this section, we first consider the following analogue of the Neumann problem:

$$L_n u(x) = f(x), \quad x \in \Omega, \quad (30)$$

$$\frac{\partial u(x)}{\partial v} = g_0(x), \quad \frac{\partial^2 u(x)}{\partial v^2} = g_1(x), \quad x \in \partial\Omega. \quad (31)$$

First, let us consider a property of the solution to the Dirichlet problem for the classical biharmonic equation. Consider the following problem:

$$\Delta^2 w(x) = F(x), \quad x \in \Omega; \quad w(x) = h_0(x), \quad \frac{\partial w(x)}{\partial v} = h_1(x), \quad x \in \partial\Omega, \quad (32)$$

where the function $F(x)$ has the form $F(x) = (\delta + 4)f(x)$, and functions $\tilde{h}_0(x)$ and $\tilde{h}_1(x)$ are defined by the equalities:

$$\tilde{h}_0(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_0(S_n^{i_n} \dots S_1^{i_1} x), \quad \tilde{h}_1(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_1(S_n^{i_n} \dots S_1^{i_1} x).$$

Let us assume that functions $f(x), h_0(x), h_1(x)$ are quite smooth. First, we study the problem for the homogeneous equation, i.e., consider the problem:

$$\Delta^2 w(x) = 0, x \in \Omega; \quad w(x) = \tilde{h}_0(x), \frac{\partial w(x)}{\partial \nu} = \tilde{h}_1(x), x \in \partial\Omega. \quad (33)$$

It is known (see, for example, [36]) that the solution to the problem (33) can be represented as $w(x) = w_0(x) + (1 - |x|^2)w_1(x)$, where the functions $w_0(x), w_1(x)$ are solutions to the following problems:

$$\Delta w_0(x) = 0, x \in \Omega; \quad w(x) = \tilde{h}_0(x), x \in \partial\Omega. \quad (34)$$

$$\Delta w_1(x) = 0, x \in \Omega; \quad w_1(x) = -\frac{1}{2} \left[\tilde{h}_1(x) - \frac{\partial w_0(x)}{\partial \nu} \right], x \in \partial\Omega. \quad (35)$$

If we represent the solutions to the problems (34) and (35) in the form of the Poisson integral, then we obtain:

$$w_0(x) = \int_{\partial\Omega} P(x, y) \tilde{h}_0(y) ds_y = \int_{\partial\Omega} P(x, y) \left[\sum_{i=(i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_0(S_n^{i_n} \dots S_1^{i_1} y) \right] ds_y,$$

$$w_1(x) = -\frac{1}{2} \int_{\partial\Omega} P(x, y) \left[\sum_{i=(i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_1(S_n^{i_n} \dots S_1^{i_1} y) - \frac{\partial w_0(y)}{\partial \nu} \right] ds_y,$$

where $P(x, y) = \frac{1}{\omega_l} \frac{1-|x|^2}{|x-y|^l}$ is the Poisson kernel of the Dirichlet problem. Further, it is obvious that $P(0, y) = \frac{1}{\omega_l}$, and for the harmonic function $w_0(x)$ the equality:

$$\int_{\partial\Omega} \frac{\partial w_0(y)}{\partial \nu} ds_y = 0.$$

holds. Moreover, in [18] (Lemma 5.1), it was proven that:

$$\int_{\partial\Omega} g(S_i y) ds_y = \int_{\partial\Omega} g(y) ds_y.$$

Then, for the function $w(x) = w_0(x) + (1 - |x|^2)w_1(x)$, we obtain:

$$w(0) = w_0(0) + w_1(0) = \frac{1}{\omega_l} \int_{\partial\Omega} \left[\sum_{i=(i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_0(S_n^{i_n} \dots S_1^{i_1} y) \right] ds_y$$

$$- \frac{1}{2\omega_l} \int_{\partial\Omega} \left[\sum_{i=(i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i h_1(S_n^{i_n} \dots S_1^{i_1} y) \right] ds_y = \frac{1}{\omega_l} \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} \left[h_0(y) - \frac{1}{2} h_1(y) \right] ds_y$$

$$= \frac{1}{2\omega_l} \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [2h_0(y) - h_1(y)] ds_y.$$

Thus, if $w(x)$ is a solution to the problem (33), then the equality:

$$w(0) = \frac{1}{2\omega_l} \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [2h_0(y) - h_1(y)] ds_y \quad (36)$$

is valid.

Let us study the problem (32) when the boundary conditions are homogeneous, i.e., consider the problem:

$$\Delta^2 w(x) = (\delta + 4)f(x), \quad x \in \Omega; \quad w(x) = 0, \quad \frac{\partial w(x)}{\partial \nu} = 0, \quad x \in \partial\Omega. \quad (37)$$

In [27], it was proven that the solution to the problem (37) satisfies the equality:

$$w(0) = \frac{1}{4\omega_l} \int_{\Omega} (1 - |x|^2) f(x) dx. \quad (38)$$

Thus, using the equalities (36) and (38) to solve the problem (32), we obtain:

$$w(0) = \frac{1}{4\omega_l} \int_{\Omega} (1 - |x|^2) f(x) dx + \frac{1}{2\omega_l} \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [2h_0(y) - h_1(y)] ds_y.$$

Hence, it follows that for the condition $w(0) = 0$ to be satisfied, it is necessary and sufficient that the equality:

$$\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx + \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [2h_0(y) - h_1(y)] ds_y = 0 \quad (39)$$

be satisfied. Thus, we proved the following assertion.

Lemma 9. *If the function $w(x)$ is a solution to the problem (32), then for the condition $w(0) = 0$ to be satisfied, it is necessary and sufficient to satisfy the equality (39).*

Let us consider an analogue of the Neumann problem (30) and (31). The following assertion is valid.

Theorem 7. *Let the coefficients of the operator L_n satisfy the condition $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, for $k = 0, \dots, 2^n - 1$ and $F(x) \in C^{\lambda+1}(\bar{\Omega})$, $g_0(x) \in C^{\lambda+4}(\partial\Omega)$, $g_1(x) \in C^{\lambda+3}(\partial\Omega)$, $0 < \lambda < 1$. Then, for the solvability of the problem (30) and (31), it is necessary and sufficient that the condition:*

$$\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx = \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [g_1(x) - g_0(x)] dS_x \quad (40)$$

be satisfied. If a solution to the problem exists, it is unique up to a constant term and belongs to the class $C^{\lambda+4}(\bar{\Omega})$.

Proof. Assume that a solution $u(x)$ to the problem (30) and (31) exists, and let $\delta = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$.

Let us apply the operator δ to the function $u(x)$ and denote $v(x) = \delta[u](x)$. Note that for $x \in \partial\Omega$, the equalities hold:

$$\begin{aligned} \left. \frac{\partial u(x)}{\partial \nu} \right|_{\partial\Omega} &= \delta[u](x)|_{\partial\Omega} = v(x)|_{\partial\Omega}; \\ \left. \frac{\partial^2 u(x)}{\partial \nu^2} \right|_{\partial\Omega} &= (\delta - 1)\delta[u](x)|_{\partial\Omega} = \delta[v](x) - v(x)|_{\partial\Omega}. \end{aligned} \quad (41)$$

Then, taking into account the equality $\Delta^2 \delta[u](x) = (\delta + 4)\Delta^2 u(x)$, where $x \in \Omega$ and the commutability of the operators δ and I_{S_i} , we obtain:

$$\begin{aligned} L_n v(x) &= \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i \Delta^2 I_{S_n^{i_n} \dots S_1^{i_1}} \delta u(x) = (\delta + 4) \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i \Delta^2 u(S_n^{i_n} \dots S_1^{i_1} x) \\ &= (\delta + 4)L_n u(x) = (\delta + 4)f(x) \equiv F(x). \end{aligned}$$

From the boundary conditions of the problem (30) and (31), taking into account the equalities (41), we obtain:

$$v(x)|_{\partial\Omega} = g_0(x); \frac{\partial v(x)}{\partial \nu} \Big|_{\partial\Omega} = g_1(x) + g_0(x).$$

Thus, if $u(x)$ is a solution to the problem (30) and (31), then for the function $v(x) = \delta[u](x)$, we obtain the following Dirichlet problem:

$$L_n v(x) = (\delta + 4)f(x) \equiv F(x), \quad x \in \Omega; \quad v(x)|_{\partial\Omega} = g_0(x), \quad \frac{\partial v(x)}{\partial \nu} \Big|_{\partial\Omega} = g_1(x) + g_0(x). \quad (42)$$

Moreover, from the equality $v(x) = \delta[u](x)$, it follows that the solution to this problem must satisfy the condition $v(0) = 0$. Let us find out when this condition is fulfilled. We introduce the notation:

$$w(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i v(S_n^{i_n} \dots S_1^{i_1} x). \quad (43)$$

If $v(x)$ is a solution to the problem (42), then the function $w(x)$ from (43) satisfies the conditions of the problem (32) with the functions:

$$\tilde{h}_0(x) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i g_0(S_n^{i_n} \dots S_1^{i_1} x), \quad \tilde{h}_1(x) = \tilde{h}_0(x) + \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i g_1(S_n^{i_n} \dots S_1^{i_1} x).$$

From (43), it follows that:

$$w(0) = \sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i v(0) = \left(\sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i \right) v(0).$$

Then, under the condition $\sum_{i \equiv (i_n \dots i_1)_2=0}^{(1\dots1)_2} a_i \neq 0$, the equality $v(0) = 0$ is satisfied if and only if the condition $w(0) = 0$ is satisfied. In turn, by the assertion of Lemma 9, for the equality $w(0) = 0$ to hold, it is necessary and sufficient to satisfy the condition (39). As in our case:

$$2h_0(y) - h_1(y) = 2g_0(x) - [g_1(x) + g_0(x)] = g_0(x) - g_1(x),$$

then the condition for the solvability of the problem (30) and (31) can be rewritten as (40). Thus, the necessity of fulfilling the condition (40) for the existence of a solution to the problem (30) and (31) is proven.

Let us show that the fulfillment of the condition (40) is also sufficient for the existence of a solution to the problem (30) and (31). To do this, consider in Ω the following Neumann problem with respect to the function $z(x)$:

$$\Delta^2 z(x) = F(x), \quad x \in \Omega, \quad (44)$$

$$\begin{aligned} z(x)|_{\partial\Omega} &= \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i g_0(S_n^{i_n} \dots S_1^{i_1} x) \equiv \tilde{h}_0(x), \\ \frac{\partial z(x)}{\partial \nu} \Big|_{\partial\Omega} &= \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i g_1(S_n^{i_n} \dots S_1^{i_1} x) \equiv \tilde{h}_1(x). \end{aligned} \quad (45)$$

It is known (see, for example, [36]) that the solvability condition of this problem can be written as:

$$\frac{1}{2} \int_{\Omega} (1 - |x|^2) F(x) dx = \int_{\partial\Omega} [\tilde{h}_1(x) - \tilde{h}_0(x)] dS_x. \quad (46)$$

By virtue of [18] (Lemma 5.1), $\int_{\partial\Omega} g(S_i x) dS_x = \int_{\partial\Omega} g(x) dS_x$, whence it follows that:

$$\sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} a_i \int_{\partial\Omega} g_0(S_n^{i_n} \dots S_1^{i_1} x) dS_x = \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} g(x) dS_x$$

and therefore, the condition (37) can be rewritten in the form of (40). If this condition is satisfied, then a solution to the problem (44) and (45) exists and is unique up to a constant term.

Further, as in the case of the Dirichlet problem, the solution to the Neumann problem (30) and (31) can be found by the formula:

$$u(x) = \sum_{i \equiv (i_n \dots i_1)_2 = 0}^{(1\dots1)_2} b_i v(S_n^{i_n} \dots S_1^{i_1} x). \quad (47)$$

Indeed, if we consider the vector $Z = (z(x), \dots, z(S_n^{j_n} \dots S_1^{j_1} x), \dots, z(S_n^1 \dots S_1^1 x))^T$, then from the vector equality $U = A_n^{-1}Z$, if the conditions of the theorem are satisfied, we can determine the vector $U(x) = (u(x), \dots, u(S_n^{i_n} \dots S_1^{i_1} x), \dots, u(S_n^1 \dots S_1^1 x))^T$. As $A_n U = Z$, then the function $u(x)$ is uniquely determined through the function $z(x)$ from (44) and (45) by Formula (47). In a similar way, as in the case of the Dirichlet problem, it can be shown that the function (47) satisfies all conditions of the problem (30) and (31). The theorem is proven. \square

Further, we define the solvability condition for the problem H in the case $\alpha = 1, \mu = 0$. In this case, the boundary conditions are written as:

$$\frac{\partial u(x)}{\partial \nu} = g_0(x), \quad \frac{\partial^2 u(x)}{\partial \nu^2} + \frac{\partial u(x)}{\partial \nu} = g_1(x), \quad x \in \partial\Omega. \quad (48)$$

The second condition from (48), taking into account the first condition, can be rewritten in the form:

$$\frac{\partial^2 u(x)}{\partial \nu^2} = g_1(x) - g_0(x).$$

Then, the problem under consideration is equivalent to the problem (30) and (31) with the following boundary conditions:

$$\frac{\partial u(x)}{\partial \nu} = g_0(x), \quad \frac{\partial^2 u(x)}{\partial \nu^2} = g_1(x) - g_0(x), \quad x \in \partial\Omega.$$

Therefore, Theorem 7 implies the following assertion.

Corollary 7. Let $\alpha = 1$, $\mu = 0$, the coefficients of the operator L_n , satisfy the condition $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, for $k = 0, \dots, 2^n - 1$ and $0 < \lambda < 1$, $f(x) \in C^{\lambda+1}(\bar{\Omega})$, $g_0(x) \in C^{\lambda+4}(\partial\Omega)$, $g_1(x) \in C^{\lambda+3}(\partial\Omega)$. Then, for the solvability of the problem H , it is necessary and sufficient that the condition:

$$\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx = \left(\sum_{i=0}^{2^n-1} a_i \right) \int_{\partial\Omega} [g_1(x) - 2g_0(x)] dS_x \quad (49)$$

be satisfied. If a solution to the problem exists, it is unique up to a constant term and belongs to the class $C^{\lambda+4}(\bar{\Omega})$.

6. The General Case of the Main Problem

In this section, we consider the general case of the problem H . The following assertion is true.

Theorem 8. Let $0 < \alpha \leq 1$, $\mu \geq 0$, the coefficients of the operator L_n , satisfy the condition $\sum_{i=0}^{2^n-1} (-1)^{k \otimes i} a_i \neq 0$, for $k = 0, \dots, 2^n - 1$ and $f(x) \in C^{\lambda+1}(\bar{\Omega})$, $g_0(x) \in C^{\lambda+4}(\partial\Omega)$, $g_1(x) \in C^{\lambda+3}(\partial\Omega)$, $0 < \lambda < 1$. Then:

(1) If $\mu > 0$, then a solution to the problem H exists, is unique, belongs to the class $C^{\lambda+4}(\bar{\Omega})$, and can be represented in the form:

$$u(x) = J_{\mu}^{\alpha}[v](x), \quad (50)$$

where $v(x)$ is a solution of the following Dirichlet problem:

$$L_n v(x) = D_{\mu+4}^{\alpha}[f](x), \quad x \in \Omega; \quad v(x) = g_0(x), \quad \frac{\partial v(x)}{\partial \nu} = g_1(x), \quad x \in \partial\Omega; \quad (51)$$

(2) If $\mu = 0$, then for solvability of the problem H , it is necessary and sufficient that the condition:

$$\frac{1}{2} \int_{\Omega} (1 - |x|^2) f_{1-\alpha}(x) dx = \left(\sum_{k=1}^l a_k \right) \int_{\partial\Omega} [g_1(x) - 2g_0(x)] dS_x \quad (52)$$

be satisfied, where $f_{1-\alpha}(x)$ is defined by the equality $f_{1-\alpha}(x) = J_4^{1-\alpha}[f](x)$.

If a solution to the problem exists, it is unique up to a constant term, belongs to the class $C^{\lambda+4}(\bar{\Omega})$, and can be represented as:

$$u(x) = C + J_0^{\alpha}[v](x), \quad C = \text{const},$$

where $v(x)$ is a solution to the problem (51) for $\mu = 0$.

Proof. Let the function $u(x)$ be a solution to the problem H . We apply the operator D_{μ}^{α} to this function and denote $v(x) = D_{\mu}^{\alpha}[u](x)$. Let us apply the operator Δ^2 to this function. Then, by virtue of the equality (20), $\Delta^2 v(x) = D_{\mu+4}^{\alpha}[\Delta^2 u](x)$. From here, it follows that for any $k = 0, \dots, 2^n - 1$, $\Delta^2 v(S_n^{i_n} \dots S_1^{i_1} x) = D_{\mu+4}^{\alpha}[\Delta^2 u](S_n^{i_n} \dots S_1^{i_1} x)$. Hence,

$$L_n v(x) = \sum_{i=0}^{2^n-1} a_i \Delta^2 D_{\mu}^{\alpha} u(S_n^{i_n} \dots S_1^{i_1} x) = D_{\mu+4}^{\alpha} \left[\sum_{i=0}^{2^n-1} a_i \Delta^2 u(S_n^{i_n} \dots S_1^{i_1} x) \right] = D_{\mu+4}^{\alpha}[f](x),$$

where $x \in \Omega$.

In [27], it was proven that the function $D_\mu^{\alpha+1}[u](x)$ can be represented as $D_\mu^{\alpha+1}[u](x) = \delta[D_\mu^{\alpha+1}[u]](x)$. Then, from the boundary conditions (2) and (3), it follows that:

$$v(x)|_{\partial\Omega} = D_\mu^\alpha[u](x)|_{\partial\Omega} = g_0(x), \quad \frac{\partial v(x)}{\partial\nu}|_{\partial\Omega} = \delta[D_\mu^\alpha[u]](x)|_{\partial\Omega} = g_1(x).$$

Thus, if $u(x)$ is a solution to the problem H , then for the function $v(x) = D_\mu^\alpha[u](x)$, we obtain the Dirichlet problem (51). If $f(x) \in C^{\lambda+1}(\bar{\Omega})$, then by virtue of Lemma 2, $D_{\mu+4}^\alpha[f](x) \in C^\lambda(\bar{\Omega})$. Moreover, by the condition of the theorem $g_0(x) \in C^{\lambda+4}(\partial\Omega)$, $g_1(x) \in C^{\lambda+3}(\partial\Omega)$. Then, by virtue of the assertion of Theorem 6, a solution to the problem (51) exists, is unique, and belongs to the class $C^{\lambda+4}(\bar{\Omega})$. If we apply the operator J_μ^α to both sides of the equality $v(x) = D_\mu^\alpha[u](x)$, then by virtue of Lemma 3 in the case $\mu > 0$, we obtain $u(x) = J_\mu^\alpha[v](x)$, that is, if a solution to the problem H exists, then the representation (50) is valid for it.

Let, on the contrary, the function $v(x)$ be a solution to the problem (51). Let us show that the function $u(x) = J_\mu^\alpha[v](x)$ satisfies all the conditions of the problem H . Indeed, if we apply the operator L_n to this function, then we obtain:

$$L_n u(x) = L_n[J_\mu^\alpha[v]](x) = J_{\mu+4}^\alpha[L_n v](x) = J_{\mu+4}^\alpha[D_{\mu+4}^\alpha[f]](x) = f(x), \quad x \in \Omega,$$

i.e., the function $u(x) = J_\mu^\alpha[v](x)$ satisfies Equation (1). Let us check the fulfillment of the boundary conditions (2) and (3). By the assertion of Lemma 4, namely by virtue of the equality (19) in the case $\mu > 0$, the equalities:

$$\begin{aligned} D_\mu^\alpha[u](x)|_{\partial\Omega} &= D_\mu^\alpha[J_\mu^\alpha[v]](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = g_0(x), \\ D_\mu^\alpha[u](x)|_{\partial\Omega} &= r \frac{\partial}{\partial r} D_\mu^\alpha[J_\mu^\alpha[v]](x)|_{\partial\Omega} = r \frac{\partial v(x)}{\partial r}|_{\partial\Omega} = \frac{\partial v(x)}{\partial\nu}|_{\partial\Omega} = g_1(x), \end{aligned}$$

are satisfied, i.e., the boundary conditions are also satisfied.

It remains to investigate the case $\mu = 0$. In this case, the function $v(x)$ —a solution to the problem (51)—must satisfy the additional condition $v(0) = 0$.

As the function $F(x) = D_{\mu+4}^\alpha[f](x)$ by virtue of Lemma 6 is represented as $F(x) = (\delta + 4)f_{1-\alpha}(x)$, $f_{1-\alpha}(x) = J_4^{1-\alpha}[f](x)$, then, by Lemma 9, for the equality $v(0) = 0$ to hold, it is necessary and sufficient that condition (39) be satisfied. In our case, $2h_0(y) - h_1(y) = 2g_0(x) - g_1(x)$, and therefore, the condition (39) can be rewritten in the form of (52). Thus, the necessity of the fulfillment of the condition (52) for the existence of a solution to the problem H is proven. The other part of the theorem can be proven in the same way as in the case $\mu > 0$. The theorem is proven. \square

Remark 3. If $\alpha = 1$ and $\mu = 0$, then $f_0(x) = J_4^0[f](x) \equiv f(x)$, and the solvability condition (52) coincides with the condition (49).

7. Conclusions

Summarizing the investigation carried out, we note that with the help of transformations of the involution-type, a nonlocal analogue of the biharmonic operator was introduced. Then, for the corresponding biharmonic equation with multiple involutions, the solvability of the boundary value problems with a fractional-order boundary operator having a derivative of the Hadamard-type was investigated. Theorems on the existence and uniqueness of solutions to the problems under consideration were proven. Exact solvability conditions were found depending on the order of the boundary operators. In this work, we extended the correct statement of the boundary value problems for some class of nonlocal partial differential equations. In the future, we plan to investigate spectral questions for operators with multiple involutions.

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