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On Sufficient Solvability Conditions For Neumann Type Problems For Polyharmonic Equation In A Ball

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Abstract. The solvability of one class of Neumann-type problems for non homogeneous polyharmonic equation in the unit ball is investigated. It is established that the set of necessary solvability conditions of the problem obtained earlier is also a set of sufficient conditions. The obtained conditions are in the form of orthogonality of homogeneous harmonic polynomials of certain degrees to linear combinations of boundary functions with coefficients from the Neumann integer triangle.

INTRODUCTION

The classical problems for the polyharmonic equation $\Delta^m u = f(x)$ are the Dirichlet problem (see, for example, [1, 2, 3]) and Neumann problem (see, for example, [4, 5, 6]). The solvability of these problems has been well studied in the theory of boundary value problems. It is established that problems of this type are Fredholm and therefore their solvability for homogeneous boundary conditions is guaranteed by the orthogonality of the right-hand sides to all solutions of the homogeneous adjoint equation. In [7, 8], more general boundary value problems for the polyharmonic equation in a ball with normal derivatives in the boundary conditions are considered.

The solvability conditions in the above papers are in the form of orthogonality of some vector functions depending on the data of the problem or the equality of the ranks of special high-order matrices. In order to establish under what boundary conditions a particular problem of this type is solvable, it is necessary to perform quite complicated calculations.

The class of problems considered in this paper is a natural generalization of the classical formulation of the Neumann problem proposed for the polyharmonic equation by A. V. Bitsadze [4]. In [9], using the Neumann integer triangle [10], some necessary conditions for the solvability of this class of problems are found. In this paper, we prove that the set of necessary solvability conditions of the Neumann-type problems found earlier is also a set of sufficient solvability conditions.

Let $S = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, and $\partial S = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere, where $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. In the unit ball S consider the following class of Neumann-type boundary value problems $\mathcal{N}_k$ [9], depending on a parameter $k \in \mathbb{N}$ (n and m are fixed) for the inhomogeneous polyharmonic equation

$$\Delta^m u = f(x), \quad x \in S; \tag{1}$$

$$\frac{\partial^k u}{\partial \nu^k}|_{\partial S} = \varphi_1(s), \quad \frac{\partial^{k+1} u}{\partial \nu^{k+1}}|_{\partial} S = \varphi_2(s), \ldots, \frac{\partial^{k+m-1} u}{\partial \nu^{k+m-1}}|_{\partial S} = \varphi_m(s), \quad s \in \partial S, \tag{2}$$

where $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative on the unit sphere, functions $f(x)$ and $\varphi_i(s)$ for $i = 1, \ldots, m$ are defined on S and ∂S, respectively.

The class of problems $\mathcal{N}_k$ is a special case of the class of boundary value problems for the polyharmonic equation with high-order normal derivatives proposed in [7]. Problem $\mathcal{N}_0$ is a Dirichlet problem, that is unconditionally solvable, while the problem $\mathcal{N}_1$ coincides with a Neumann problem [4, 5]. The solvability of various Neumann-type problems and their generalizations in the unit ball, excepting for the papers listed above, can be found for the biharmonic equation (in particular, the problems $\mathcal{N}_1$ and $\mathcal{N}_2$) in [11, 12, 13, 14, 15] and for the polyharmonic equation in [16]. In [17, 18], for boundary value problems for the polyharmonic equation with normal derivatives in the boundary conditions, sufficient conditions for the Fredholm property of these problems and a new form of the Fredholm criterion that is equivalent to the complementarity conditions are given.

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DIRICHLET PROBLEM $\mathscr{N}_0$

If we use the notations $\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, $t^{[m]} = t(t-1)\dots(t-m+1)$ is mth factorial degree of t ($t^{[0]} = 1$), and the equality $\Lambda^{[i]}u = \frac{\partial^i u}{\partial \nu^i}$ on ∂S, then this problem can be written in the form

$$\Delta^m u = 0, \quad x \in S; \quad u|_{\partial S} = \varphi_1(x), \dots, \Lambda^{[m-1]}u|_{\partial S} = \varphi_m(x), \quad x \in \partial S.$$

It is known that if a function $u(x)$ is m-harmonic in the starlike region $D \subset \mathbb{R}^n$, then it can be expanded by the Almansi formula

$$u(x) = u^{(0)}(x) + |x|^2 u^{(1)}(x) + \dots + |x|^{2m-2} u^{(m-1)}(x), \quad x \in D,$$

where $u^{(i)}(x)$ are "harmonic components" of m-harmonic function $u(x)$.

The following statement about the smoothness of the harmonic components of the solution to the Dirichlet problem is true. Let ε be a small positive number.

Lemma 1. *Let $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$ for $i = 1, \dots, m$, then all harmonic components of the solution of the Dirichlet problem (1), (2) provided that $f = 0$ are such that $u^{(i-1)} \in C^{m-1}(\bar{S})$, $i = 1, \dots, m$.*

In the proof of lemma, we used results of [19].

Consider the following integer triangle [10]

$$\mathbb{P} = \begin{matrix} & & & & 1 & & & & \\ & & & -1 & & 1 & & & \\ & & 3 & & -3 & & 1 & & \\ & -15 & & 15 & & -6 & & 1 & \\ 105 & & -105 & & 45 & & -10 & & 1 \\ & & & & \dots & & & & \\ & \cdots & & p_i^{(k)} = p_{i-1}^{(k-1)} + (i-2k+2)p_i^{(k-1)} & & \cdots & & & \end{matrix},$$

where $p_i^{(k)}$ for $1 \le i \le k$, $k \in \mathbb{N}$ is an element of k-th row of the triangle $\mathbb{P}$, standing in the ith position and $p_1^{(1)} = 1$ is initial condition, whereas $p_i^{(k)} = 0$ for $i > k$ and $0 \ge i$ are boundary conditions for the written recursive equation. The numbers $p_i^{(k)}$ satisfy the equality [12]

$$p_i^{(k)} = (-1)^{k-i} \binom{2k-i-1}{i-1} \frac{(2k-2i+1)!!}{2k-2i+1}.$$

Triangle $\mathbb{P}$ is called the *Neumann triangle* since it plays an important role in the study of the Neumann-type problems $\mathscr{N}_k$.

PROBLEM $\mathscr{N}_k$

Let us now consider the general problem $\mathscr{N}_k$ for a homogeneous polyharmonic equation (1) if $k \le m$. It can be rewritten as

$$\begin{gathered} \Delta^m u = 0, \quad x \in S; \\ \Lambda^{[k]}u|_{\partial S} = \varphi_1(x), \dots, \Lambda^{[m+k-1]}u|_{\partial S} = \varphi_m(x), \quad x \in \partial S. \end{gathered} \tag{3}$$

Consider m-harmonic function $v = \Lambda^{[k]}u$ in S. For the function v, we obtain the following boundary value problem

$$\begin{gathered} \Delta^m v = 0, \quad x \in S, \\ v|_{\partial S} = \varphi_1(x), (\Lambda - k)v|_{\partial S} = \varphi_2(x), \dots, (\Lambda - k)^{[m-1]}v|_{\partial S} = \varphi_m(x), \quad x \in \partial S. \end{gathered} \tag{4}$$

It is not hard to prove the following assertion.

Lemma 2. *Problem (4) is equivalent to the following Dirichlet problem*

$$\begin{gathered}\Delta^m v = 0, \quad x \in S,\\ v|_{\partial S} = \psi_1(x), \Lambda^{[1]} v|_{\partial S} = \psi_2(x), \ldots, \Lambda^{[m-1]} v|_{\partial S} = \psi_m(x), x \in \partial S,\end{gathered} \tag{5}$$

where the functions $\psi_i(x)$, $i = 1, \ldots, m$ are defined from the recurrent equalities

$$\psi_i(x) = \varphi_i(x) - \sum_{j=1}^{i-1} \binom{i-1}{j-1} (-k)^{[i-j]} \psi_j(x).$$

If $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, then $\psi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1, \ldots, m$, which means $v^{(i)} \in C^{m-1}(\bar{S})$.

Let us investigate the equation $v = \Lambda^{[k]} u$ for an unknown function $u(x)$ in m-harmonic in S functions because the sufficient solvability conditions of this equation are the sufficient solvability conditions for the problem $\mathscr{N}_k$.

Lemma 3. *Let $v(x)$ be a solution of the problem (4). Equation $v(x) = \Lambda^{[k]} u(x)$ is solvable in m-harmonic in S functions if and only if the m-harmonic in S function $v(x)$ has no terms up to the $(k-1)$th order of smallness inclusively in its expansion in a neighborhood of the original.*

A solution $u(x)$ is unique up to m-harmonic polynomials of degree $k-1$ and satisfies (3). If $v \in C^{m-1}(\bar{S})$ and $u(x)$ exists, then $\Lambda^i u \in C^{m-1}(\bar{S})$ for any $i = 0, \ldots, k$.

INTEGRAL CONDITIONS FOR THE PROBLEM $\mathscr{N}_k$

Let us investigate the necessary solvability conditions of the problem $\mathscr{N}_k$ obtained on the basis of [14,15] in Theorem 1 from [9].

Theorem 1. *Let the functions $\varphi_i(s)$ for $i = 1, 2, \ldots, m$ such that a solution $u(x)$ of the problem $\mathscr{N}_k$ for homogeneous polyharmonic equation (1) exists and it is such that $u \in C^{m+k-1}(\bar{S})$. Then for any $l \in \mathbb{N}_0$ such that $l < k$ the following orthogonality conditions must be met*

$$\begin{gathered}\int_{\partial S} H_l(x)(p_1^{(m-\lambda)} \varphi_{\delta_\lambda+1}(x) + \ldots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(x))\, ds_x = 0,\\ \lambda = \lambda_0, \ldots, \min\{k-l, m\} - 1,\\ \int_{\partial S}\end{gathered} \tag{6}$$

where $p_j^{(i)}$ are elements of the triangle $\mathbb{P}$, $\lambda_0 = [(k-l)/2]$, $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l, $\delta_\lambda = 2\lambda - k + l + 1$, $\sigma_\lambda = k - l - \lambda - 1$. The number of conditions (6) for $l < k \leq 2m$ equals to $N_k = [(k+1)/2][(k+2)/2]$.

Consider the case when the difference $(k-l)$ is an even number, where $l \in \mathbb{N}_0$ and $l < k$. Let us find out what do the conditions of the theorem mean for $l < k \leq m$ in terms of the function $v(x) = \Lambda^{[k]} u(x)$. In what follows, we need a generalized Pochhammer symbol $(a,b)_l = a(a+b)\ldots(a+(l-1)b)$.

Lemma 4. *Let $v(x)$ be a solution of the problem (4) and the boundary functions of the problem $\mathscr{N}_k$ satisfy the conditions of Theorem 1 for even $k-l$ and if $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1, \ldots, m$. Then the following $(k-l)/2$ equalities must be satisfied*

$$\begin{gathered}\int_{\partial S} H_l(x)(\Lambda - k, -2)_{m-(k-l)/2} v\, ds_x = 0,\\ \int_{\partial S} H_l(x)(\Lambda - k + 2, -2)_{m-(k-l)/2+1} v\, ds_x = 0,\\ \ldots\\ \int_{\partial S} H_l(x)(\Lambda - l - 2, -2)_{m-1} v\, ds_x = 0,\end{gathered}$$

where $H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l.

The proof of this lemma is based on the results of [20, 21] and Lemmas 1, 2. Basing on Lemmas 1-4 and results of [22], the following assertion can be proved.

Theorem 2. *Let the boundary functions of the problem $\mathcal{N}_k$ satisfy the conditions of Theorem 1 for even $k-l$, have the smoothness $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1,\ldots,m$, and the m-harmonic function $v(x)$ that is a solution of the problem (4) has the following Almansi expansion*

$$v(x) = v^{(0)}(x) + |x|^2 v^{(1)}(x) + \ldots + |x|^{2m-2} v^{(m-1)}(x), \quad x \in S.$$

Then for even $k-l$ the following equalities hold

$$v_l^{(0)}(x) = v_l^{(1)}(x) = \ldots = v_l^{((k-l)/2-1)}(x) = 0.$$

The case when the number $k-l$ is odd, for $l \in \mathbb{N}_0$ and $l < k$ is considered similarly.

MAIN RESULTS

Let us prove that the necessary solvability conditions of the problem $\mathcal{N}_k$ obtained in Theorem 1 for $k \leq m$ are also the sufficient conditions if the boundary functions of the problem have the required smoothness. Based on Lemmas 1-4 and Theorem 1, 2 the following assertion can be proved.

Theorem 3. *Let $k \leq m$ and $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1,\ldots,m$ then the necessary and sufficient solvability conditions of the problem $\mathcal{N}_k$ for homogeneous polyharmonic equation (1) are $N_k = [(k+1)/2][(k+2)/2]$ conditions of the form*

$$\int_{\partial S} H_l(x)(p_1^{(m-\lambda)} \varphi_{\delta_\lambda+1}(x) + \ldots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(x))\, ds_x = 0 \qquad (7)$$

for $l = 0,1,\ldots,k-1$, where

$$\lambda = [(k-l)/2],\ldots,k-l-1, \quad \delta_\lambda = 2\lambda - k + l + 1, \quad \sigma_\lambda = k - l - \lambda - 1,$$

$H_l(x)$ is arbitrary homogeneous harmonic polynomial of degree l, and $p_i^{(m)}$ are numbers from the Neumann triangle $\mathbb{P}$. Solution of the problem $\mathcal{N}_k$ exists and is unique up to m-harmonic polynomials of degree $k-1$.

Remark 1. *Conditions (7) for $k = 1$ (in this case we have $N_1 = 1$ condition) coincide with the condition for the problem $\mathcal{N}_1$ obtained in [*10*], and for $k = 2$ (in this case we have $N_2 = 2$ conditions) coincide with the conditions for the problem $\mathcal{N}_2$ obtained in [*6, 11*].*

Similar result for inhomogeneous polyharmonic equation (1) and polynomial right-hand side $f(x)$ has the form.

Theorem 4. *Let $k \leq m$ and $\varphi_i \in C^{2m-i-1+\varepsilon}(\partial S)$, $i = 1,\ldots,m$ then the necessary and sufficient solvability conditions for the problem (1),(2) are $N_k = [(k+1)/2][(k+2)/2]$ conditions of the form*

$$\int_{\partial S} H_l(x)(p_1^{(m-\lambda)} \varphi_{\delta_\lambda+1}(x) + \ldots + p_{m-\lambda}^{(m-\lambda)} \varphi_{m-\sigma_\lambda}(x))\, ds_x = \int_S \frac{(1-|x|^2)^{m-1}}{(2m-2)!!} H_l(x) \hat{f}(x)\, dx$$

for every $l = 0,1,\ldots,k-1$, where

$$\lambda = [(k-l)/2],\ldots,k-l-1, \quad \delta_\lambda = 2\lambda - k + l + 1, \quad \sigma_\lambda = k - l - \lambda - 1,$$

$H_l(x)$ is an arbitrary homogeneous harmonic polynomial of degree l, coefficients $p_j^{(i)}$ are the elements of the Neumann integer triangle $\mathbb{P}$,

$$\hat{f}(x) = (\Lambda,-1)_{i+k-1}/(\Lambda - l,-2)_m f(x).$$

Solution of the problem $\mathcal{N}_k$ exists and is unique up to m-harmonic polynomials of degree $k-1$.

Proof of this theorem is based on Theorem 3.

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