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An inverse problem for a parabolic equation with involution

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An Inverse Problem for a Parabolic Equation with Involution

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Abstract. In this paper, we consider two-dimensional generalization of an analog of parabolic equation. In the paper, by using the Fourier method, we study solvability of inverse problems with the Dirichlet condition.

INTRODUCTION

Concept of a nonlocal operator and the related concept of a nonlocal differential equation appeared in mathematics relatively recently. According to the classification given in the book by Nakhushev [1] such equations include: loaded equations; equations containing fractional derivatives of the desired function; equations with deviating arguments, in other words, equations in which the unknown function and its derivatives enter, generally speaking, for different values of arguments. Among non-local differential equations, a special place is occupied by equations in which deviation of arguments has an involutive character. A mapping I is usually called involution if $I^2 = E$, E is the identity mapping. In this paper, we introduce concept of a nonlocal Laplace operator and investigate spectral questions of some boundary value problems. Questions on solvability of some boundary value problems for a fractional analogue of the nonlocal Laplace equation will be studied in a three-dimensional parallelepiped.

Differential equations with involution have been studied in works of numerous authors [2, 3, 4, 5, 6, 7, 8, 9]. In the work of Linkov [10] boundary value and initial - boundary value problems are investigated in the domain $(t,x): t > 0, -\pi < x < \pi$ for analogues of a parabolic, hyperbolic and elliptic equation with involution

$$u_t(t,x) - u_{xx}(t,x) - \varepsilon u_{xx}(t,-x) = 0,\ \ u_{tt}(t,x) - u_{xx}(t,x) - \varepsilon,\, u_{xx}(t,-x) = 0, u_{tt}(t,x) + u_{xx}(t,x) + \varepsilon u_{xx}(t,-x) = 0.$$

Application of the Fourier method to these problems leads to the one-dimensional spectral problem

$$y''(x) + \varepsilon y''(x) = -\lambda y(x), -\pi < x < \pi$$

with the corresponding boundary value conditions. Eigenfunctions of this problem will be the functions $y_{k,1}(t) = \sin kt, y_{k,1}(t) = \cos(k+0.5)t$ and eigenvalues are $\lambda_k^{(1)} = (1-\varepsilon)k, \lambda_k^{(2)} = (1+\varepsilon)(k+0.5)$.

Moreover, it should be noted here that the eigenfunctions of the equation with involution coincide with the eigenfunctions of the classical equation, i.e. $\varepsilon = 0$, and difference in these problems will be only in eigenvalues. In this paper, we consider two-dimensional generalization of an analog of parabolic equation. In the paper, by using the Fourier method, we study solvability of inverse problems with the Dirichlet condition.

Let us turn to statement of the problem. Let $0 < p, q, T$ be real numbers, $\Pi = \{x = (x_1, x_2) : 0 < x_1 < p, 0 < x_2 < q\}$ be a rectangle, $Q = (0,T) \times \Pi$. For a point $x = (x_1, x_2) \in \Pi$ we consider the mapping

$$S_0x = (x_1, x_2), S_1x = (p - x_1, x_2), S_2x = (x_1, q - x_2), S_3x = (p - x_1, q - x_2).$$

Let a_j be real numbers, $j = \overline{0,3}, \Delta$ be a Laplace operator, acting by the variables x_1 and x_2. For a function $v(x_1, x_2) \in C^2(\Pi)$ we consider the operator $Lv(x) \equiv a_0\Delta v(S_0x) + a_1\Delta v(S_1x) + a_1\Delta v(S_2x) + a_1\Delta v(S_3x)$.

We call the operator L as the nonlocal Laplace operator. If $a_0 = 1, a_j = 0, j = 1,2,3$, then coincides with the usual two-dimensional Laplace operator. In the domain consider the following equation

$$\frac{\partial u(t,x)}{\partial t} = a_0\Delta u(t,S_0x) + a_1\Delta u(t,S_1x) + a_1\Delta u(t,S_2x) + a_1\Delta u(t,S_3x) + f(x), (t,x) \in Q. \tag{1}$$

Here $\Delta u(t,S_jx)$ means $\Delta u(t,S_jx) = \Delta u(t,z)|_{z=S_jx}, j = 0,...,3$. If $a_0 = 1, a_j = 0, j = 1,2,3$, then (1) coincides with the classical parabolic equation. In the domain Q consider the following problem:

Problem ID. Find a solution of the equation (1) in the domain Q, satisfying the condition:

$$u(0,x) = \varphi(x),\, u(T,x) = \psi(x), x \in \bar{\Pi}, \tag{2}$$

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$$u(t,x)=0,\,(t,x)\in\partial\Pi\times[0,T]. \tag{3}$$

By a regular solution to problem ID we mean a pair of functions $(u(x,t),f(x))$ such that $u(t,x)\in C_{t,x}^{1,2}\left(\bar{Q}\right), f(x)\in C(\bar{\Pi})$, satisfying equation (1) and conditions (2),(3) in the classical sense in the domain Q.

Note that direct and inverse problems for a parabolic type equation of integer and fractional orders in the one-dimensional case were studied in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], and in the two-dimensional case in [21, 22].

ON EIGENFUNCTIONS AND EIGENVALUES OF THE DIRICHLET BOUNDARY VALUE PROBLEM FOR THE NONLOCAL LAPLACE EQUATION

We consider in Π the following spectral problem: Find a function $v(x)\neq 0$ and a number μ satisfying the conditions

$$-Lv(x)=\lambda v(x), x\in\Pi, \tag{4}$$

$$v(x_1,0)=v(x_1,q)=0, 0\leq x_1\leq p, v(0,x_2,)=v(p,x_2)=0, 0\leq x_2\leq q. \tag{5}$$

First, we present a well-known statement concerning the following spectral problem

$$-\Delta w(x_1,x_2)=\mu w(x_1,x_2),(x_1,x_2)\in\Pi, \tag{6}$$

$$w(x_1,0)=w(x_1,q)=0, 0\leq x_1\leq p, w(0,x_2,)=w(p,x_2)=0, 0\leq x_2\leq q. \tag{7}$$

Let

$$X_k(x_1)=\sqrt{\frac{2}{p}}\sin\frac{k\pi}{p}x_1, Y_m(x_2)\sqrt{\frac{2}{q}}\sin\frac{m\pi}{q}x_2, k,m=1,2,...,$$

$$\nu_k=\left(\frac{k\pi}{p}\right)^2, \sigma_m\left(\frac{m\pi}{q}\right)^2, k,m=1,2,....$$

The following statement is well known (see e.g. [23]).

Lemma 1. *Eigenfunctions and eigenvalues of problem (6),(7) are:*

$$w_{k,m}(x_1,x_2)=X_k(x_1)Y_m(x_2)\equiv\sqrt{\frac{2}{p}}\sin\frac{k\pi}{p}x_1\cdot\sqrt{\frac{2}{q}}\sin\frac{m\pi}{q}x_2, k,m=1,2,..., \tag{8}$$

$$\mu_{k,m}=\nu_k+\sigma_m\equiv\left(\frac{k\pi}{p}\right)^2+\left(\frac{m\pi}{q}\right)^2, k,m=1,2,.... \tag{9}$$

System of the functions $\left\{w_{k,m}(x_1,x_2)\right\}_{k,m=1}^{\infty}$ *forms a complete orthonormal system in the space* $L_2(\Pi)$.

Let $w(x)$ be an eigenfunction of the problem (6),(7). Make the following combinations of these functions:

$$\begin{cases} v_1(x)=\frac{1}{2}\left[\frac{w(x)+w(S_1x)}{2}+\frac{w(S_2x)+w(S_3x)}{2}\right], v_2(x)=\frac{1}{2}\left[\frac{w(x)+w(S_1x)}{2}-\frac{w(S_2x)+w(S_3x)}{2}\right],\\ v_3(x)=\frac{1}{2}\left[\frac{w(x)-w(S_1x)}{2}+\frac{w(S_2x)-w(S_3x)}{2}\right], v_4(x)=\frac{1}{2}\left[\frac{w(x)-w(S_1x)}{2}-\frac{w(S_2x)-w(S_3x)}{2}\right]. \end{cases} \tag{10}$$

Note that the conditions $w(x)|_{\partial\Pi}=0\Rightarrow w(I_jx)\big|_{\partial\Pi}=0, j=1,2,3$ implies $v(I_jx)\big|_{\partial\Pi}=0, j=1,2,3,4$, where $\partial\Pi$ is a boundary of the domain Π.

Consider the following numbers:

$$\varepsilon_1=a_0+a_1+a_2+a_3, \varepsilon_2=a_0+a_1-a_2-a_3, \varepsilon_3=a_0-a_1+a_2-a_3, \varepsilon_4=a_0-a_1-a_2+a_3. \tag{11}$$

It is easy to show that if $w_{k,m}(x) = X_k(x_1) \cdot Y_m(x_2)$ are eigenfunctions of the problem (6),(7), then from (10) we obtain the following system

1) $v_{k,m,1}(x) = X_{2k-1}(x_1) \cdot Y_{2m-1}(x_2)$; 2) $v_{k,m,2}(x) = X_{2k-1}(x_1) \cdot Y_{2m}(x_2), k,m = 1,2,...$;

3) $v_{k,m,3}(x) = X_{2k}(x_1) \cdot Y_{2m-1}(x_2)$; 4) $v_{k,m,4}(x) = X_{2k}(x_1) \cdot Y_{2m}(x_2), k,m = 1,2,....$

The following statement is true.

Theorem 1. *Let $a_j \in R$ be such that $\varepsilon_j \neq 0, j = \overline{1,4}$ and let $w_{k,m}(x) = X_k(x_1) \cdot Y_m(x_2)$ be eigenfunctions of the problem (2.3)-(2.4), and $\mu_{k,m}$ be corresponding eigenvalues. Then system of the functions $v_{k,m,j}(x_1,x_2), k,m = 1,2,..., j = \overline{1,4}$ is eigenfunctions, and $\lambda_{k,m,1} = \varepsilon_1 \cdot \mu_{2k-1,2m-1}, \lambda_{k,m,2} = \varepsilon_2 \cdot \mu_{2k-1,2m}, \lambda_{k,m,3} = \varepsilon_3 \cdot \mu_{2k,2m-1}, \lambda_{k,m,4} = \varepsilon_4 \cdot \mu_{2k,2m}, k,m = 1,2,...,$ are corresponding eigenvalues of the problem (4),(5).*

Proof. The proof of the theorem is carried out by direct application of the operator L to the functions $v_{k,m,j}(x_1,x_2), j = \overline{1,4}$. Moreover, we note that

$$\sin\frac{k\pi}{p}(p-x_1) = (-1)^{k+1}\sin\frac{k\pi}{p}x_1, \sin\frac{m\pi}{q}(q-x_2) = (-1)^{m+1}\sin\frac{m\pi}{q}x_2.$$

Let $j = 1$. Then

$$\begin{aligned}
v_{k,m,1}(p-x_1,x_2) &= \sin\tfrac{(2k-1)\pi}{p}(p-x_1)\cdot\sin\tfrac{(2m-1)\pi}{q}x_2 = X_{2k-1}(x_1)\cdot Y_{2m-1}(x_2)\\
v_{k,m,1}(x_1,q-x_2) &= \sin\tfrac{(2k-1)\pi}{p}x_1\cdot\sin\tfrac{(2m-1)\pi}{q}(q-x_2) = X_{2k-1}(x_1)\cdot Y_{2m-1}(x_2)\\
v_{k,m,1}(p-x_1,q-x_2) &= \sin\tfrac{(2k-1)\pi}{p}(p-x_1)\sin\tfrac{(2m-1)\pi}{q}(q-x_2) = X_{2k-1}(x_1)\cdot Y_{2m-1}(x_2)
\end{aligned}$$

From these equalities, as well as from the equality $(\sin\lambda t)'' = -\lambda^2\sin\lambda t$, applying the operator $-L$ to the function $u_{k,m,1}(x)$, we have

$$Lv_{k,m,1}(x)$$

$$= a_0X''_{2k-1}(x_1)\cdot Y_{2m-1}(x_2) + a_0X_{2k-1}(x_1)\cdot Y''_{2m-1}(x_2) + a_1X''_{2k-1}(p-x_1)\cdot Y_{2m-1}(x_2) + a_1X_{2k-1}(p-x_1)\cdot Y''_{2m-1}(x_2)$$

$$+a_2X''_{2k-1}(x_1)\cdot Y_{2m-1}(q-x_2) + a_2X_{2k-1}(x_1)\cdot Y''_{2m-1}(q-x_2) + a_3X''_{2k-1}(p-x_1)\cdot Y_{2m-1}(q-x_2) + a_3X_{2k-1}(p-x_1)$$

$$\times Y''_{2m-1}(q-x_2) = X_{2k-1}(x_1)(x_1)\cdot Y_{2m-1}(x_2)\left(-a_0\nu_k - a_0\sigma_m - a_1\nu_k + a_1\sigma_m - a_2\nu_k + a_2\sigma_m + a_3\nu_k + a_3\sigma_m\right)$$

$$= X_{2k-1}(x_1)(x_1)\cdot Y_{2m-1}(x_2)\left[-a_0(\nu_k+\sigma_m) - a_1(\nu_k+\sigma_m) - a_2(\nu_k+\sigma_m) - a_3(\nu_k+\sigma_m)\right]$$

$$= -(\nu_k+\sigma_m)(a_0+a_1+a_2+a_3)X_{2k-1}(x_1)(x_1)\cdot Y_{2m-1}(x_2) = -\mu_{k,m}\cdot\varepsilon_1X_{2k-1}(x_1)(x_1)\cdot Y_{2m-1}(x_2) = -\lambda_{k,m,1}v_{k,m,1}(x).$$

Thus, for the function $v_{k,m,1}(x)$ equality $Lv_{k,m,1}(x) = -\lambda_{k,m,1}v_{k,m,1}(x)$ holds, i.e. $v_{k,m,1}(x)$ is the eigenfunction of the operator $-L$, and $\lambda_{k,m,1}$ are the corresponding eigenvalues. For the other functions $v_{k,m,j}(x_1,x_2), j = \overline{2,4}$, the proof is carried out in a similar way. Theorem is proved.

Since $\bigcup\limits_{j=1}^{4} v_{k,m,j}(x_1,x_2) = \left\{w_{k,m}(x_1,x_2)\right\}_{k,m=1}^{\infty}$, Lemma 1 yields

Corollary 1. *System $v_{k,m,j}(x_1,x_2), j = \overline{1,4}$ forms an orthonormal basis in the space $L_2(\Pi)$*

UNIQUENESS OF SOLUTION TO THE PROBLEM ID

The following theorem holds:

Theorem 2. *Let $\varepsilon_j > 0, j = \overline{1,4}$. If a solution to the ID problem exists, then it is unique.*

Proof. Let us show that the homogeneous problem ($\varphi(x) = \psi(x) = 0$) has only a trivial solution. Suppose the opposite. Let there be two solutions $\{u_1(t,x), f_1(x)\}$ and $\{u_2(t,x), f_2(x)\}$ to the problem ID. Denote $\tilde{u}(t,x) = u_1(t,x) - u_2(t,x)$ and $\tilde{f}(x) = f_1(x) - f_2(x)$. Then functions $\tilde{u}(t,x)$ and $\tilde{f}(x)$ satisfy the equation

$$\frac{\partial\tilde{u}(t,x)}{\partial t} = L_x\tilde{u}(t,x) + \tilde{f}(x) \tag{12}$$

and conditions

$$\tilde{u}(0,x)=0,\ \tilde{u}(T,x)=0, x\in\Pi,\ \tilde{u}(t,x)=0,\ (t,x)\in\partial\Pi\times[0,T]. \tag{13}$$

Now let the functions $\tilde{u}(t,x), \tilde{f}(x)$ be solutions of the problem (12), (13). Consider the functions

$$\tilde{u}_{k,m}^{(j)}(t)=\left(\tilde{u}(t,x), v_{k,m}^{(j)}(x)\right)_{L_2(\Pi)}, j=\overline{1,4}, \tag{14}$$

$$\tilde{f}_{k,m}^{(j)}=\left(\tilde{f}(x), v_{k,m}^{(j)}(x)\right)_{L_2(\Pi)}, j=\overline{1,4}, \tag{15}$$

where $(\xi,\eta)_{L_2(\Pi)}$ means scalar product in the space $L_2(\Pi)$, i.e.

$$(\xi,\eta)_{L_2(\Pi)}=\int_\Pi \xi(x_1,x_2)\eta(x_1,x_2)dx_1dx_2.$$

Differentiating the left and right sides of the equality (14) with respect to t, and taking into account (12), and the conditions (13), to find unknown functions $\tilde{u}_{k,m}^{(j)}(t)$, $j=\overline{1,4}$ and constants $\tilde{f}_{k,m}^{(j)}$, $j=\overline{1,4}$, we get the following problems

$$\frac{du_{k,m}^{(j)}(t)}{dt}+\lambda_{k,m}^{(j)}\cdot u_{k,m}^{(j)}(t)=f_{k,m}^{(j)},\ j=\overline{1,4}, t\in(0,T), \tag{16}$$

$$\tilde{u}_{k,m}^{(j)}(0)=0,\ \tilde{u}_{k,m}^{(j)}(T)=0, i,j=\overline{1,4}. \tag{17}$$

General solution of the equation (16) has the form

$$\tilde{u}_{k,m}^{(j)}(t)=C_{k,m}^{(j)}e^{-\lambda_{k,m}^{(j)}t}+\frac{f_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}}, j=\overline{1,4}.$$

Further, satisfying the condition (17) by this solution, concerning $C_{k,m}^{(j)}$ and $\tilde{f}_{k,m}^{(j)}$, we obtain the system of the form

$$C_{k,m}^{(j)}+\frac{\tilde{f}_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}}=0, C_{k,m}^{(j)}e^{-\lambda_{k,m}^{(j)}T^\alpha}+\frac{\tilde{f}_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}}=0,$$

which has only a trivial solution, i.e. $C_{k,m}^{(j)}=0$, $\tilde{f}_{k,m}^{(j)}=0$. Then $\tilde{u}_{k,m}^{(j)}(t)=0$, and from (14) and (15) it follows that the functions $\tilde{u}(t,x), \tilde{f}(x)$ are orthogonal to all elements of the system $v_{k,m}^{(j)}(x)$, which is complete in $L_2(\Pi)$. Thus, $\tilde{u}(t,x)=0$, $\tilde{f}(x)=0$, i.e. $\tilde{u}(t,x)=0 \Leftrightarrow u_1(t,x)=u_2(t,x)$, $\tilde{f}(x)=0 \Leftrightarrow f_1(x)=f_2(x)$. Uniqueness of solution to the problem ID is proved.

EXISTENCE OF SOLUTION TO THE PROBLEM ID

The following statement is true.

Theorem 3. *Let $\varepsilon_j>0$, $j=\overline{1,4}$, $\varphi(x),\psi(x)\in C^3(\bar{\Pi})$, $\frac{\partial^4\varphi(x)}{\partial x_r^3\partial x_s}$, $\frac{\partial^4\psi(x)}{\partial x_r x_s^3}\in C(\bar{\Pi})$, $r,s=1,2$ and functions $\varphi(x),\psi(x)$, $\frac{\partial\varphi(x)}{\partial x_r\partial x_s}$, $\frac{\partial\psi(x)}{\partial x_r\partial x_s}$, $r,s=1,2$ satisfy the boundary value condition (1.3). Then solution to the problem ID exists, and is represented in the form*

$$u(t,x)=\varphi(x)-\sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\frac{1-e^{-\lambda_{k,m,j}t}}{1-e^{-\lambda_{k,m,j}T}}\left[\psi_{k,m,j}-\varphi_{k,m,j}\right]v_{k,m,j}(x), \tag{18}$$

$$f(x) = -L\varphi(x) - \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}\left(\frac{\varphi_{k,m,j}-\psi_{k,m,j}}{1-e^{-\lambda_{k,m,j}T}}\right)v_{k,m,j}(x). \tag{19}$$

Proof. Since system of eigenfunctions $v_{k,m,j}(x_1,x_2)$, $j=\overline{1,4}$ of the problem (2.1), (2.2) forms orthonormal basis in $L_2(\Pi)$, then solution $(u(x,t),f(x))$ of the problem (1.1) - (1.4) can be represented as a series expansion in this system, that is

$$u(t,x) = \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}u_{k,m,j}(t)v_{k,m,j}(x),\ f(x) = \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}f_{k,m,j}v_{k,m,j}(x), \tag{20}$$

where $u_{k,m,j}(t)-$ unknown functions, $f_{k,m,j}-$ unknown numbers, $k,m\in N$, $j=\overline{1,4}$. Putting (4.3) and (4.4) into the equation (1.1), to find functions $u_{k,m,j}(t)$ and constants $f_{k,m,j}$, we get the following equations

$$\frac{d}{dt}u_{k,m,j}(t) + \lambda_{k,m,j}\cdot u_{k,m,j}(t) = f_{k,m,j},\ j=\overline{1,4}, t\in(0,T),$$

where $\lambda_{k,m,j} = \varepsilon_j\cdot\mu_{k,m}$. Solving these equations, we have

$$u_{k,m,j}(t) = C_{k,m,j}e^{-\lambda_{k,m,j}t} + \frac{f_{k,m,j}}{\lambda_{k,m,j}}, j=\overline{1,4}, \tag{21}$$

where $C_{k,m,j}-$ unknown numbers. To find these constants, we use conditions (2). For this, we assume that the functions $\varphi(x)$ and $\psi(x)$ are expanded in a Fourier series in the system $v_{k,m,j}(x_1,x_2)$, $j=\overline{1,4}$, i.e.

$$\varphi(x) = \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\varphi_{k,m,j}v_{k,m,j}(x), \psi(x) = \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\psi_{k,m,j}v_{k,m,j}(x),$$

where

$$\varphi_{k,m,j} = \left(\varphi(x), v_{k,m,j}(x)\right)_{L_2(\Pi)}, \psi_{k,m,j} = \left(\psi(x), v_{k,m,j}(x)\right)_{L_2(\Pi)}, j=\overline{1,4}. \tag{22}$$

Then conditions (2) concerning $u_{k,m,j}(t)$ have the form $u_{k,m,j}(0)=\varphi_{k,m,j}$, $u_{k,m,j}(T)=\psi_{k,m,j}$, $j=\overline{1,4}$.
Taking into account it, from (21) we obtain that

$$C_{k,m,j} + \frac{f_{k,m,j}}{\lambda_{k,m,j}} = \varphi_{k,m,j}, C_{k,m,j}e^{-\lambda_{k,m,j}T} + \frac{f_{k,m,j}}{\lambda_{k,m,j}} = \psi_{k,m,j}, j=\overline{1,4}. \tag{23}$$

Hence, we find $C_{k,m,j}$ and $f_{k,m,j}$, $j=\overline{1,4}$,

$$C_{k,m,j} = \frac{\varphi_{k,m,j}-\psi_{k,m,j}}{1-e^{-\lambda_{k,m,j}T}}, j=\overline{1,4},\ f_{k,m,j} = \lambda_{k,m,j}\left(\varphi_{k,m,j}-C_{k,m,j}\right), j=\overline{1,4}. \tag{24}$$

Putting expressions for $u_{k,m,j}(t)$ and $f_{k,m,j}$ into (20), we obtain

$$u(t,x) = \varphi(x) + \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}(e^{-\lambda_{k,m,j}t}-1)C_{k,m,j}v_{k,m,j}(x), \tag{25}$$

$$f(x) = \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}\left(\varphi_{k,m,j}-C_{k,m,j}\right)v_{k,m,j}(x). \tag{26}$$

Thus, we have found a formal form of the solution of the problem in the form of series (25) and (26), where the coefficients $C_{k,m,j}$ are calculated by formulas (24). By direct calculation it is easy to show that the functions $u(t,x)$, $f(x)$, defined by series (25) and (26), satisfy equation (1) and conditions (2),(3). It remains to prove legality of these actions. For this, we will show that $u(x,t) \in C^{1,2}_{t,x}\left(\bar{Q}\right)$, $f(x) \in C(\bar{\Pi})$. Furthermore, C will mean an arbitrary constant, value of which is not of interest to us. Let $g(x_1,x_2)$ be a function that satisfies conditions of the theorem. Then

$$\int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2 = \int_0^q\int_0^p g(x_1,x_2)\frac{d}{dx_1}\left(-\frac{p}{k\pi}cos\frac{k\pi}{p}x_1\right)\sin\frac{m\pi}{q}x_2dx_1dx_2 = \frac{C}{k^3m}g^{3,1}_{k,m},$$

where

$$g^{3,1}_{k,m} = \int_0^q\int_0^p \frac{\partial^4 g(x_1,x_2)}{\partial x_1^3\partial x_2}cos\frac{k\pi}{p}x_1cos\frac{m\pi}{q}x_2dx_1dx_2. \tag{27}$$

Therefore, we have the equality:

$$\int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2 = \frac{C}{k^3m}g^{3,1}_{k,m}, \int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2 = \frac{C}{km^3}g^{1,3}_{k,m},$$

where

$$g^{1,3}_{k,m} = \int_0^q\int_0^p \frac{\partial^4 g(x_1,x_2)}{\partial x_1\partial x_2^3}cos\frac{k\pi}{p}x_1cos\frac{m\pi}{q}x_2dx_1dx_2. \tag{28}$$

Now we will use the obtained equalities to estimate $C_{k,m,j}$. Suppose that $j=1$. Integrating by parts the expressions for the coefficients $C_{k,m,1}$ from formulas (24), we obtain

$$\begin{aligned}C_{k,m,1} = \frac{\varphi_{k,m,1}-\psi_{k,m,1}}{1-e^{-\lambda_{k,m,1}T}} &= \frac{C}{1-e^{-\lambda_{k,m,1}T}}\int_0^q\int_0^p[\varphi(x_1,x_2)-\psi(x_1,x_2)]sin\frac{(2k-1)\pi}{p}x_1sin\frac{(2m-1)\pi}{q}x_2dx_1dx_2\\ &= \frac{C}{\left(1-e^{-\lambda_{k,m,1}T}\right)(2k-1)^3(2m-1)}\left(\varphi^{3,1}_{k,m,1}-\psi^{3,1}_{k,m,1}\right),\end{aligned} \tag{29}$$

and

$$C_{k,m,1} = \frac{C}{\left(1-e^{-\lambda_{k,m,1}T}\right)(2k-1)(2m-1)^3}\left(\varphi^{1,3}_{k,m,1}-\psi^{1,3}_{k,m,1}\right). \tag{30}$$

where the coefficients $\varphi^{3,1}_{k,m,1}, \psi^{3,1}_{k,m,1}, \varphi^{1,3}_{k,m,1}, \psi^{1,3}_{k,m,1}$ are defined as in formulas (27) and (28). Similarly, we can write the coefficients $C_{k,m,j}, j=2,3,4$. Further, for the function $v_{k,m,1}(x)$ we have

$$\frac{\partial^2 v_{k,m,1}(x)}{\partial x_1^2} = -\left(\frac{(2k-1)\pi}{p}\right)^2 sin\frac{(2k-1)\pi}{p}x_1sin\frac{(2m-1)\pi}{q}x_2,$$

$$\frac{\partial^2 v_{k,m,1}(x)}{\partial x_2^2} = -\left(\frac{(2m-1)\pi}{q}\right)^2 sin\frac{(2k-1)\pi}{p}x_1sin\frac{(2m-1)\pi}{q}x_2$$

Estimate the coefficients $(2k-1)^2C_{k,m,1}, (2m-1)^2C_{k,m,1}$. By using representation (29), we have

$$(2k-1)^2\left|C_{k,m,1}\right| = \left|\frac{C(2k-1)^2}{\left(1-e^{-\lambda_{k,m,1}T}\right)(2k-1)^3(2m-1)}\left(\varphi^{3,1}_{k,m,1}-\psi^{3,1}_{k,m,1}\right)\right| \le \frac{C}{(2k-1)(2m-1)}\left(\left|\varphi^{3,1}_{k,m,1}\right|+\left|\psi^{3,1}_{k,m,1}\right|\right).$$

Similarly, from (30) we obtain

$$(2m-1)^2\left|C_{k,m,1}\right| = \left|\frac{C(2m-1)^2\left(\varphi^{1,3}_{k,m,1}-\psi^{1,3}_{k,m,1}\right)}{\left(1-e^{-\lambda_{k,m,1}T}\right)(2k-1)(2m-1)^3}\right| \le \frac{C}{(2k-1)(2m-1)}\left(\left|\varphi^{1,3}_{k,m,1}\right|+\left|\psi^{1,3}_{k,m,1}\right|\right).$$

Similar estimates are valid for the coefficients $(2k-1)^2C_{k,m,j}, (2m-1)^2C_{k,m,j}$, $j = 2,3,4$. Since the system $w_{k,m}(x_1,x_2) = \cos\frac{(2k-1)\pi}{p}x_1 \cdot \cos\frac{(2m-1)\pi}{q}x_2$ is orthonormal and $\frac{\partial^4\varphi(x_1,x_2)}{\partial x_1^3\partial x_2}, \frac{\partial^4\psi(x_1,x_2)}{\partial x_1^3\partial x_2} \in L_2(\Pi)$, then applying the Cauchy-Schwarz and Bessel inequalities, we have

$$\sum_{k,m=1}^{\infty}(2k-1)^2\left|C_{k,m,1}\right| \le \sqrt{\sum_{k,m=1}^{\infty}\frac{1}{(2k-1)^2(2m-1)^2}}\left(\sqrt{\sum_{k,m=1}^{\infty}\left|\varphi^{3,1}_{k,m,1}\right|^2}+\sqrt{\sum_{k,m=1}^{\infty}\left|\psi^{3,1}_{k,m,1}\right|^2}\right) < \infty,$$

$$\sum_{k,m=1}^{\infty}(2m-1)^2\left|C_{k,m,1}\right| \le \sqrt{\sum_{k,m=1}^{\infty}\frac{1}{(2k-1)^2(2m-1)^2}}\left(\sqrt{\sum_{k,m=1}^{\infty}\left|\varphi^{1,3}_{k,m,1}\right|^2}+\sqrt{\sum_{k,m=1}^{\infty}\left|\psi^{1,3}_{k,m,1}\right|^2}\right) < \infty.$$

Convergence of the series corresponding to the coefficients $(2k-1)^2C_{k,m,j}, (2m-1)^2C_{k,m,j}$, $j = 2,3,4$, is proved similarly. Then the series, obtained from (24) by differentiation twice with respect to x_1 converges absolutely and uniformly in the domain $\bar{Q}$. Consequently, the sum of this series represents a continuous function in the domain $\bar{Q}$, i.e $u_{x_1x_1}(t,x) \in C\left(\bar{Q}\right)$. In a similar way it can be shown that $u_{x_2x_2}(t,x) \in C\left(\bar{Q}\right)$ and $u_t(t,x) \in C\left(\bar{Q}\right)$. Now let us prove that $f(x) \in C\left(\bar{Q}\right)$. For this, due to (26), it is necessary to study convergence of the series

$$\sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}\varphi_{k,m,j}v_{k,m,j}(x), \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}C_{k,m,j}v_{k,m,j}(x).$$

We have already shown convergence of the second series, and convergence of the first series is proved in a similar way. Indeed, for example, in the case $j = 1$ we have

$$\varphi_{k,m,1} = \frac{2}{\sqrt{pq}}\int_0^q\int_0^p \varphi(x_1,x_2) sin\frac{(2k-1)\pi}{p}x_1 sin\frac{(2m-1)\pi}{q}x_2 dx_1dx_2 = \frac{C}{(2k-1)^3(2m-1)}\varphi^{3,1}_{k,m}.$$

Consequently, we have that

$$\varphi_{k,m,1} = \frac{C}{(2k-1)^3(2m-1)}\varphi^{3,1}_{k,m,1}, \varphi_{k,m,1} = \frac{C}{(2k-1)(2m-1)^3}\varphi^{1,3}_{k,m,1}.$$

From this, we obtain absolute and uniform convergence of the series $\sum\limits_{k,m=1}^{\infty}\lambda_{k,m,1}\varphi_{k,m,1}v_{k,m,1}(x)$. In a similar way, we can show convergence of the series $\sum\limits_{k,m=1}^{\infty}\lambda_{k,m,j}\varphi_{k,m,j}v_{k,m,j}(x)$ for $j = 2,3,4$. Then the series (26) converges absolutely and uniformly in $\bar{\Pi}$, i.e $f(x) \in C\left(\bar{\Pi}\right)$. Further, by using equalities (24), we obtain representations of solutions to the problem in the form (18) and (19). Theorem is proved.

CONCLUSION

We considered two-dimensional generalization of an analog of parabolic equation. Using the Fourier method, here, we studied solvability of inverse problems with the Dirichlet condition.

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REFERENCES

1. A.M.Nahushev, *Equations of Mathematical Biology* (Publishing house for foreign literature, Moscow, 1995) [Russian].
2. A.A.Andreev, "Analogs of classical boundary value problems for a second-order differential equation with deviating argument," Differential Equations **40**, 1192–1194 (2004), [Russian].
3. S. K. N.Al-Salti and M.Kirane, "Initial - boundary value problems for a time-fractional differential equation with involution perturbation," Mathematical Modelling of Natural Phenomena **14**, 1–15 (2019).
4. A. Ashyralyev and A. Sarsenbi, "Well-posedness of a parabolic equation with involution," Numerical Functional Analysis and Optimization **38**, 1295–1304 (2017).
5. A. Ashyralyev and A. Sarsenbi, "Well-posedness of an elliptic equation with involution," Electronic Journal of Differential Equations **2015**, 1–8 (2015).
6. M. Burlutskaya and A. Khromov, "Fourier method in an initial-boundary value problem for a first-order partial differential equation with involution," Computational Mathematics and Mathematical Physics **51**, 2102–2114 (2011).
7. A. Cabada and F. Tojo, "On linear differential equations and systems with reflection," Applied Mathematics and Computation 2017; 305: 84-102. **305**, 84–102 (2017).
8. A. Kopzhassarova and A. Sarsenbi, "Basis properties of eigenfunctions of second-order differential operators with involution," Abstr Appl Anal. **6**, 1–3 (2012).
9. A. S. V.V. Karachik and B. Turmetov, "On solvability of the main boundary value problems for a non-local poisson equation," Turkish journal of mathematics **43**, 1604–1625 (2019).
10. A. Linkov, "Substantiation of a method the fourier for boundary value problems with an involute deviation," Vestnik Samarskogo Universiteta. Estestvenno-Nauchnaya Seriya **2**, 60–66 (1999), [Russian].
11. M. M. I.A. Kaliev and O. Fattahova, "Inverse problem for forward-backward parabolic equation with generalized conjugation conditions," Ufa Math J. , 34–42 (2001).
12. I. Kaliev and M. Sabitova, "Problems of determining the temperature and density of heat sources from the initial and final temperatures," J Appl Ind Math **4**, 332–339 (2010).
13. M. Kirane and N. Al-Salti, "Inverse problems for a nonlocal wave equation with an involution perturbation," Journal of Nonlinear Sciences and Applications **9**, 1243–1251 (2016).
14. M. Kirane and A. Malik, "Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time," Appl Math Comput. **218**, 163–170 (2011).
15. B. S. M. Kirane and B. Torebek, "Determination of an unknown source term temperature distribution for the sub-diffusion equation at the initial and final data," Electronic Journal of Differential Equations **2017**, 1–13 (2017).
16. B. Torebek and R.Tapdigoglu, "Some inverse problems for the nonlocal heat equation with caputo fractional derivative," Mathematical Methods in the Applied Sciences **40**, 6468–6479 (2017).
17. I. Orazov and M. Sadybekov, "One nonlocal problem of determination of the temperature and density of heat sources," Russian Math. **56**, 60–64 (2012).
18. I. Orazov and M. Sadybekov, "On a class of problems of determining the temperature and density of heat sources given initial and final temperature," Sib Math J. **53**, 146–151 (2012).
19. O. I. K.M. Furati and M. Kirane, "An inverse problem for a generalized fractional diffusion," Appl Math Comput. **249**, 24–31 (2014).
20. D. L. H. Nguyen and V.T.Nguyen, "Regularized solution of an inverse source problem for a time fractional diffusion equation," Appl Math Modell. **40**, 8244–8264 (2016).
21. B. S. M. Kirane and M. Al-Gwaiz, "An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions," Mathematical Methods in the Applied Sciences **36**, 1056–1069 (2012).
22. S. Malik and S. Aziz, "An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions," Computers & Mathematics with Applications **73**, 2548 – 2560 (2017).
23. S. G. Mikhlin, *Linear Partial Differential Equations* (Vysshaya Shkola, Moscow, 1977) [Russian].