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Solvability of nonlocal Dirichlet problem for generalized Helmholtz equation in a unit ball

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ABSTRACT
In this paper, we study the solvability of a new class of nonlocal boundary value problems for the generalized Helmholtz equation in the unit ball. These problems are a generalization of the classical Dirichlet boundary value problem to the Helmholtz equation. For the considered problems, existence and uniqueness theorems are proved. Integral representation of solutions is established. Corresponding spectral questions are also investigated, namely, eigenfunctions and eigenvalues of the problem are found.

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1. Introduction

The concept of a nonlocal operator and the related concepts of a nonlocal differential equation and a nonlocal boundary value problem appeared in mathematics relatively recently. For example, in [1] loaded equations, equations containing fractional derivatives of the desired function, equations with deviating arguments, in other words, equations in which the unknown function and its derivatives enter, generally speaking, for different values of arguments are called nonlocal differential equations. Nonlocal boundary value problems for elliptic equations, in which boundary conditions are specified in the form of a connection between values of an unknown function and its derivatives at different points of the domain boundary, are called Bitsadze–Samarskii type problems [2]. Numerous applications of nonlocal boundary value problems for elliptic equations to problems in physics, technology, and other branches of science are described in detail in [3,4]. Solvability of nonlocal boundary value problems for elliptic equations is discussed in [5–8]. Boundary value problems for second- and fourth-order elliptic equations with involution, as special cases of nonlocal problems, are considered in [9–14]. Some nonlocal boundary value problems for the biharmonic equation are discussed in [15].

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In this paper, we investigate the solvability conditions for one class of boundary value problems for a generalized Helmholtz equation in the unit ball with nonlocality both in the equation and in the boundary conditions.

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$, be the unit ball, $\partial\Omega$ be the unit sphere and S be an orthogonal matrix $S \cdot S^T = I$. Moreover, suppose that there exists $m \in \mathbb{N}$ such that $S^m = I$. Note that if $x \in \Omega$, or $x \in \partial\Omega$, then for any $k \in \mathbb{N}$ the following inclusions $S^k x \in \Omega$, or $S^k x \in \partial\Omega$ hold. This is so since the transformation of the space $\mathbb{R}^n$ with the matrix S preserves the norm $|x|^2 = (x, x) = (S^T Sx, x) = (Sx, Sx) = |Sx|^2$.

Let $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_{m-1}$ be some real numbers, $f(x)$ and $g(x)$ be functions defined in Ω and on $\partial\Omega$, respectively. Consider the following nonlocal boundary value problem for the generalized Helmholtz equation.

Problem: *Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying the following conditions*

$$-\sum_{k=0}^{m-1} a_k \Delta u(S^k x) = \lambda u(x) + f(x), \quad x \in \Omega, \tag{1}$$

$$\sum_{k=0}^{m-1} b_k u(S^k x)|_{\partial\Omega} = g(x), \quad x \in \partial\Omega. \tag{2}$$

If $a_0 = b_0 = 1$, $a_k = b_k = 0$, $k = 1, 2, \ldots, m-1$, then we have the classical Dirichlet boundary value problem for the Helmholtz equation. If $\lambda = 0$ and $b_0 = 1$, $b_k = 0$, $k = 1, 2, \ldots, m-1$, then this problem is investigated in [16], and if $\lambda = 0$ and $a_0 = 1$, $a_k = 0$, $k = 1, 2, \ldots, m-1$, it is considered in [17].

2. Auxiliary statements

To study the above posed problem, we need some auxiliary results. The following statement is proved in Lemma 3.1 from [16].

Lemma 2.1: *The operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ commute $\Delta I_S u(x) = I_S \Delta u(x)$ on the functions $u(x) \in C^2(\Omega)$. The operator $\Lambda = \sum_{i=1}^n x_i u_{x_i}(x)$ and the operator I_S also commute $\Lambda I_S u(x) = I_S \Lambda u(x)$ on the functions $u(x) \in C^1(\bar{\Omega})$.*

Lemma 2.2: *Let $\mu_0, \mu_1, \ldots, \mu_{m-1}$ be different mth roots of unity, then*

$$M^{-1} \equiv \begin{pmatrix} \mu_0^0 & \mu_0^1 & \cdots & \mu_0^{m-1} \\ \mu_1^0 & \mu_1^1 & \cdots & \mu_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m-1}^0 & \mu_{m-1}^1 & \cdots & \mu_{m-1}^{m-1} \end{pmatrix}^{-1} = \frac{1}{m} \overline{M}^T.$$

Proof: Let us find element e_{ij} from ith row and jth column in the product of the matrix M and the matrix from the right-hand side of the equality above. We obtain

$$e_{i,j} = \frac{1}{m} \sum_{k=0}^{m-1} \mu_i^k \bar{\mu}_j^k = \frac{1}{m} \sum_{k=0}^{m-1} (\mu_i \bar{\mu}_j)^k = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} = \delta_{i,j},$$

since $\mu_i\bar{\mu}_j$ is also mth root of unity, not equal to 1, if $i \neq j$ and $\mu_i\bar{\mu}_i = |\mu_i|^2 = 1$. Therefore, the product of these matrices is the identity matrix. This proves the lemma. ■

A similar result is established during the proof of Lemma 2.3 from [16]. Consider the following matrix

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{m-1} \\ a_{m-1} & a_0 & \dots & a_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}, \tag{3}$$

which is formed based on coefficients $a_0, a_1, \dots, a_{m-1}$ from (1).

Lemma 2.3: *Let μ be a mth root of unity and*

$$\sigma_\mu(A) = \sum_{k=0}^{m-1} a_k \bar{\mu}^k, \tag{4}$$

then for the matrix A from (3), we have the equality

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{m-1} \\ a_{m-1} & a_0 & \dots & a_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} \mu^0 \\ \mu^1 \\ \vdots \\ \mu^{m-1} \end{pmatrix} = \sigma_{\bar{\mu}}(A) \begin{pmatrix} \mu^0 \\ \mu^1 \\ \vdots \\ \mu^{m-1} \end{pmatrix},$$

i.e. the vector $\mathbf{m}_\mu = (\mu^0, \mu^1, \dots, \mu^{m-1})^T$ is an eigenvector of the matrix A, and the number $\sigma_{\bar{\mu}}(A)$ is an eigenvalue corresponding to it.

If $\mu_0, \mu_1, \dots, \mu_{m-1}$ are different mth roots of unity, then eigenvectors $\mathbf{m}_{\mu_0}, \mathbf{m}_{\mu_1}, \dots, \mathbf{m}_{\mu_{m-1}}$ are linear independent and

$$\det A = \prod_{k=0}^{m-1} (\mathbf{a} \cdot \mathbf{m}_{\mu_k}),$$

where $\mathbf{a} = (a_0, a_1, \dots, a_{m-1})^T$.

Proof: It is not hard to see that the element of the ith row in the product of the matrix A and the vector $\mathbf{m}_\mu$ is equal to

$$A_i^{row} \cdot \mathbf{m}_\mu = \sum_{k=0}^{i-1} a_{m+k-i}\mu^k + \sum_{k=i}^{m-1} a_{k-i}\mu^k = \mu^i \sum_{k=0}^{i-1} a_{m+k-i}\mu^{k-i} + \mu^i \sum_{k=i}^{m-1} a_{k-i}\mu^{k-i}$$
$$= \mu^i \left(\sum_{k'=m-i}^{m-1} a_{k'}\mu^{k'} + \sum_{k'=0}^{m-1-i} a_{k'}\mu^{k'} \right) = \mu^i \sum_{k'=0}^{m-1} a_{k'}\mu^{k'} = \sigma_{\bar{\mu}}(A)\mu^i.$$

Here, in the first sum from the second line, the index is replaced by $m + k - i = k'$, and it is taken into account that $\mu^{k'-m} = \mu^{k'}$. In the second sum, the index is replaced by $k - i = k'$. Consequently, $A\mathbf{m}_\mu = \sigma_{\bar{\mu}}(A)\mathbf{m}_\mu$. The first lemma's statement is proved.

Due to Lemma 2.2, the matrix $M = (\mathbf{m}_{\mu_0}, \dots, \mathbf{m}_{\mu_{m-1}})^T$ is invertible, and hence nonsingular, and therefore its row rank is equal to m, that is, rows are linearly independent. Further, it is known that $\det A = \sigma_{\bar{\mu}_0}(A) \cdots \sigma_{\bar{\mu}_{m-1}}(A)$ and since

$$\sigma_{\bar{\mu}}(A) = \sum_{k=0}^{m-1} a_k \mu^k = \mathbf{a} \cdot \mathbf{m}_\mu, \tag{5}$$

we get

$$\det A = \sigma_{\bar{\mu}_0}(A) \cdots \sigma_{\bar{\mu}_{m-1}}(A) = \prod_{k=0}^{m-1} (\mathbf{a} \cdot \mathbf{m}_{\mu_k}).$$

The lemma is proved. ∎

Lemma 2.4: *Let matrix A be constructed based on the numbers $a_0, a_1, \dots, a_{m-1}$, and matrix B be constructed based on the numbers $b_0, b_1, \dots, b_{m-1}$ analogically to (3). Then the matrices A and B commute $AB = BA$ and the matrix AB has the structure of matrices the A and B. If $\mu_0, \mu_1, \dots, \mu_{m-1}$ are different mth roots of unity and $\sigma_{\mu_k}(A) \neq 0$, $k = 0, 1, \dots, m-1$, then*

$$A^{-1} = \frac{1}{m} \left(\sum_{k=0}^{m-1} \frac{1}{\mu_k^{j-i} \sigma_{\bar{\mu}_k}(A)} \right)_{i,j=\overline{0,m-1}}. \tag{6}$$

Proof: Let us write the matrices A and B in the form

$$A = (a_{j-i})_{i,j=\overline{0,m-1}}, \quad B = (b_{j-i})_{i,j=\overline{0,m-1}},$$

where the subindex $j{-}i$ is taken by $\mod m$. Therefore,

$$\begin{aligned}(AB)_{i,j} &= \sum_{k=0}^{m-1} a_{k-i} b_{j-k} = \sum_{k'=-i}^{m-1-i} a_{k'} b_{j-i-k'} = \sum_{k'=-i}^{-1} a_{k'} b_{j-i-k'} + \sum_{k'=0}^{m-1-i} a_{k'} b_{j-i-k'} \\ &= \sum_{k'=m-i}^{m-1} a_{k'} b_{j-i-k'} + \sum_{k'=0}^{m-1-i} a_{k'} b_{j-i-k'} = \sum_{k=0}^{m-1} a_k b_{j-i-k}.\end{aligned}$$

If we denote $c_i = \sum_{k=0}^{m-1} a_k b_{i-k}$, then we get $(AB)_{i,j} = c_{j-i}$ and hence the matrix AB has the structure of the matrices A and B. Since for $i = 0, 1, \dots, m-1$

$$\begin{aligned}(AB)_{0,i} = c_i &= \sum_{k=0}^{i} a_k b_{i-k} + \sum_{k=i+1}^{m-1} a_k b_{i+m-k} = \sum_{k=0}^{i} b_k a_{i-k} \\ &+ \sum_{k=i+1}^{m-1} b_k a_{i+m-k} = \sum_{k=0}^{m-1} b_k a_{i-k} = (BA)_{0,i}\end{aligned}$$

and because elements of the matrices AB and BA are completely determined by their first row, we get $AB = BA$.

Let us multiply the matrix A by the matrix given by equality (6). It is easy to see that, according to the formula (5), similarly to the last equality from Lemma 2.2, the following equalities hold

$$(AA^{-1})_{i,j} = \frac{1}{m}\sum_{l=0}^{m-1} a_{l-i} \sum_{k=0}^{m-1} \frac{1}{\mu_k^{j-l}\sigma_{\bar{\mu}_k}(A)}$$
$$= \frac{1}{m}\sum_{k=0}^{m-1} \frac{\mu_k^{i-j}}{\sigma_{\bar{\mu}_k}(A)} \sum_{l=0}^{m-1} \mu_k^{l-i} a_{l-i} = \frac{1}{m}\sum_{k=0}^{m-1} \mu_k^{i-j} = \delta_{i,j},$$

where $i,j = 0,\dots,m-1$. This yields the lemma's statement $AA^{-1} = I$. ■

3. Solvability conditions of the problem

Theorem 3.1: *Let the function $u(x)$ be a solution to the nonlocal Dirichlet problem* (1) *and* (2) *and let μ be some mth root of unity, and the coefficients $a_0, a_1, \dots, a_{m-1}$ of Equation* (1) *be such that $\sigma_\mu(A) \neq 0$, and the coefficients $b_0, b_1, \dots, b_{m-1}$ of condition* (2) *be such that $\sigma_\mu(B) \neq 0$. Then the function*

$$v_\mu(x) = \sum_{k=0}^{m-1} \mu^k u(S^k x), \tag{7}$$

having the smoothness $v_\mu(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, is a solution to the following Dirichlet problem for the Helmholtz equation

$$-\Delta v_\mu(x) = \lambda_\mu v_\mu(x) + f_\mu(x), \quad x \in \Omega, \tag{8}$$
$$v_\mu(x)|_{\partial\Omega} = g_\mu(x), \quad x \in \partial\Omega, \tag{9}$$

where

$$\lambda_\mu = \frac{\lambda}{\sigma_\mu(A)}, \quad f_\mu(x) = \frac{1}{\sigma_\mu(A)} \sum_{k=0}^{m-1} \mu^k f(S^k x), \quad g_\mu(s) = \frac{1}{\sigma_\mu(B)} \sum_{k=0}^{m-1} \mu^k g(S^k s). \tag{10}$$

Proof: Let a function $u(x)$ be a solution to the problem (1) and (2). Obviously in this case $v_\mu(x) \in C^2(\Omega) \cap C(\bar{\Omega})$. Consider the operator $I_S u(x) = u(Sx)$. By Lemma 2.1, operators I_S and Δ commute. Therefore, it is easy to see that in Ω we have the equalities

$$-\Delta\left(a_0 u(S^0 x) + a_1 u(S^1 x) + \dots + a_{m-1} u(S^{m-1} x)\right) = \lambda u(x) + f(x),$$
$$-\Delta\left(a_{m-1} u(S^0 x) + a_0 u(S^1 x) + \dots + a_{m-2} u(S^{m-1} x)\right) = \lambda u(Sx) + f(Sx),$$
$$\dots$$
$$-\Delta\left(a_1 u(S^0 x) + a_2 u(S^1 x) + \dots + a_0 u(S^{m-1} x)\right) = \lambda u(S^{m-1} x) + f(S^{m-1} x),$$

or in the matrix form

$$-A\Delta U(x) = \lambda U(x) + F(x), \quad x \in \Omega, \tag{11}$$

where

$$U(x) = \begin{pmatrix} u(S^0x) \\ u(S^1x) \\ \vdots \\ u(S^{m-1}x) \end{pmatrix}, \quad F(x) = \begin{pmatrix} f(S^0x) \\ f(S^1x) \\ \vdots \\ f(S^{m-1}x) \end{pmatrix}.$$

Let $\mathbf{m}_\mu = (\mu^0, \mu^1, \dots, \mu^{m-1})^T$. If multiply the vector equality (11) scalarly by the vector $\mathbf{m}_\mu$, then we obtain

$$-(\mathbf{m}_\mu, A\Delta U(x)) = \lambda(\mathbf{m}_\mu, U(x)) + (\mathbf{m}_\mu, F(x))$$

and hence

$$-(A^T\mathbf{m}_\mu, \Delta U(x)) = \lambda(\mathbf{m}_\mu, U(x)) + (\mathbf{m}_\mu, F(x)).$$

Let us write the matrix A in the form $A = (a_{j-i})_{i,j=\overline{0,m-1}}$, where, similarly to Lemma 2.4, the indices $j-i$ are taken by $\mod m$. Then the ith element of the vector $A^T\mathbf{m}_\mu$, due to equality (4) $\sigma_\mu(A) = \sum_{k=0}^{m-1} a_k\bar{\mu}^k$, has the form

$$(A^T\mathbf{m}_\mu)_i = \sum_{k=0}^{m-1} a_{i-k}\mu^k = \mu^i \sum_{k=0}^{m-1} a_{i-k}\mu^{k-i} = \mu^i \sum_{k=0}^{m-1} a_k\mu^{-k} = \mu^i\sigma_\mu(A) = \sigma_\mu(A)(\mathbf{m}_\mu)_i$$

and then

$$-\sigma_\mu(A)\Delta(\mathbf{m}_\mu, U(x)) = \lambda(\mathbf{m}_\mu, U(x)) + (\mathbf{m}_\mu, F(x)). \tag{12}$$

Therefore, if we use the notations of (7) and (10)

$$v_\mu(x) = (\mathbf{m}_\mu, U(x)), \quad f_\mu(x) = \frac{1}{\sigma_\mu(A)}(\mathbf{m}_\mu, F(x)),$$

we have

$$-\sigma_\mu(A)\Delta v_\mu(x) = \lambda_\mu v_\mu(x) + f_\mu(x).$$

Thus, if $\sigma_\mu(A) \neq 0$ and $\lambda_\mu = \frac{\lambda}{\sigma_\mu(A)}$, then for the functions $v_\mu(x)$ we obtain Equation (8).

Let us investigate the boundary condition (9). Obviously, similarly to (11), from the boundary condition of the original problem (2) we can obtain the equality

$$BU(x)|_{\partial\Omega} = G(x), \quad x \in \partial\Omega,$$

where $G(x) = (g(S^0x), g(S^1x), \dots, g(S^{m-1}x))^T$. Hence, multiplying the obtained vector equality scalarly by the vector $\mathbf{m}_\mu$ and acting similarly to Equation (12), we obtain

$$(\mathbf{m}_\mu, BU(x))|_{\partial\Omega} = (B^T\mathbf{m}_\mu, U(x))|_{\partial\Omega} = \sigma_\mu(B)v_\mu(x)|_{\partial\Omega} = (\mathbf{m}_\mu, G(x)).$$

Since $\sigma_\mu(B) \neq 0$, then we have

$$v_\mu(x)\big|_{\partial\Omega} = g_\mu(x), \quad x \in \partial\Omega,$$

i.e. the boundary condition (9) is satisfied. The theorem is proved. ■

Remark 3.1: If a solution to Equation (1) exists, then the function $v_\mu(x)$ from (7) satisfying Equation (8) must also satisfy the equality

$$v_\mu(Sx) = \sum_{k=0}^{m-1} \mu^k u(S^{k+1}x) = \frac{1}{\mu}\sum_{k=0}^{m-1} \mu^{k+1} u(S^{k+1}x) = \bar{\mu} v_\mu(x).$$

Consider the operator $\Lambda u = \sum_{k=1}^{n} \xi_k u_{\xi_k}$, which on $\partial\Omega$ has the property

$$\frac{\partial u}{\partial \nu} = \Lambda_\xi u, \tag{13}$$

where ν is the unit outward normal to $\partial\Omega$.

Lemma 3.2: *Let the function $v_\mu(x)$ be a solution of the problem* (8) *and* (9)*, where the functions $f_\mu(x)$ and $g_\mu(x)$ satisfy the conditions*

$$f_\mu(Sx) = \bar{\mu} f_\mu(x), \;\; x \in \Omega, \quad g_\mu(Sx) = \bar{\mu} g_\mu(x), \;\; x \in \partial\Omega, \tag{14}$$

and λ_μ is not an eigenvalue of the problem, i.e. the problem (8) *and* (9) *has a unique solution. Then for the function $v_\mu(x)$ we have the equality*

$$v_\mu(Sx) = \bar{\mu} v_\mu(x). \tag{15}$$

Proof: First, note that since $S^T = S^{-1} = S^{m-1}$ and $\mu^{m-1} = 1/\mu = \bar{\mu}$, then the equalities (14) yield

$$f_\mu(S^T x) = \bar{\mu}^{m-1} f_\mu(x) = \mu f_\mu(x), \;\; x \in \Omega, \quad g_\mu(S^T x) = \mu g_\mu(x), \quad x \in \partial\Omega \tag{16}$$

and vice versa.

Let $G(x,\xi)$ be the Green's function of the Dirichlet problem for the Poisson equation in the domain Ω

$$\Delta u(x) = f_1(x), \;\; x \in \Omega, \quad u(x)|_{\partial\Omega} = g_1(x), \quad x \in \partial\Omega.$$

It is well known that the solution to this problem can be written in the form [18]

$$u(x) = -\frac{1}{\omega_n}\int_\Omega G(x,\xi) f_1(\xi)\,\mathrm{d}\xi - \frac{1}{\omega_n}\int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial \nu_\xi} g_1(\xi)\,\mathrm{d}s_\xi,$$

where ω_n is an area of the unit sphere in $\mathbb{R}^n$. Therefore, from Equation (8), taking into account the equality (13), we obtain

$$v_\mu(x) = \frac{1}{\omega_n}\int_\Omega G(x,\xi)(\lambda_\mu v_\mu(\xi) + f_\mu(\xi))\,\mathrm{d}\xi - \frac{1}{\omega_n}\int_{\partial\Omega} g_\mu(\xi)\Lambda_\xi G(x,\xi)\,\mathrm{d}s_\xi. \tag{17}$$

Since the number λ_μ is not an eigenvalue of the Dirichlet problem (8) and (9), the solution of the resulting integral equation is unique. Let us prove that the function $\mu v_\mu(Sx)$ also satisfies the integral Equation (17), and hence $\mu v_\mu(Sx) = v_\mu(x)$.

Consider the function

$$F(x) = \mu v_\mu(Sx) - \frac{1}{\omega_n}\int_\Omega G(x,\xi)(\lambda_\mu \mu v_\mu(S\xi) + f_\mu(\xi))\,\mathrm{d}\xi + \frac{1}{\omega_n}\int_{\partial\Omega} g_\mu(\xi)\Lambda_\xi G(x,\xi)\,\mathrm{d}s_\xi$$

and put here $x = S^T x$. Then we have

$$F(S^T x) = \mu v_\mu(x) - \frac{1}{\omega_n}\int_\Omega G(S^T x,\xi)\left(\lambda_\mu \mu v_\mu(S\xi) + f_\mu(\xi)\right)\mathrm{d}\xi + \frac{1}{\omega_n}\int_\Omega \Lambda_\xi G(S^T x,\xi) g_\mu(\xi)\,\mathrm{d}s_\xi.$$

Make the change of variables $y = S\xi$ under the integral sign which gives $\xi = S^T y$. Then, due to commutability of the operators Λ_ξ and I_S (see Lemma 2.1) and the equality $\int_{\partial\Omega}\varphi(Sx)\,\mathrm{d}s_x = \int_{\partial\Omega}\varphi(x)\,\mathrm{d}s_x$ (see [15, Lemma 5.1]), we obtain

$$F(S^T x) = \mu v_\mu(x) - \frac{1}{\omega_n}\int_\Omega G(S^T x, S^T y)(\mu\lambda_\mu v_\mu(y) + f_\mu(S^T y))\left|\frac{\partial\xi}{\partial y}\right|\mathrm{d}y + \frac{1}{\omega_n}\int_{\partial\Omega} g_\mu(S^T\xi)\Lambda_\xi G(S^T x, S^T\xi)\,\mathrm{d}s_\xi.$$

Now, if we take into account the equalities (16) and also that $|\frac{\partial\xi}{\partial y}| = |S^T| = 1$, we obtain

$$F(S^T x) = \mu v_\mu(x) - \mu\frac{1}{\omega_n}\int_\Omega G(S^T x, S^T y)(\lambda_\mu v_\mu(y) + f_\mu(y))\,\mathrm{d}y + \mu\frac{1}{\omega_n}\int_{\partial\Omega} g_\mu(\xi)\Lambda_\xi G(S^T x, S^T\xi)\,\mathrm{d}s_\xi.$$

It is known that for Ω

$$G(x,\xi) = E(x,\xi) - E\left(\frac{x}{|x|}, |x|\xi\right),$$

where

$$E(x,\xi) = \begin{cases} \dfrac{1}{n-2}|x-\xi|^{2-n}, & n > 2 \\ -\ln|x-\xi|, & n = 2 \end{cases}$$

is an elementary solution to the Laplace equation. Then it is easy to see that due to orthogonality of the matrix S we have

$$E(S^T x, S^T y) = \frac{1}{n-2}\left|S^T x - S^T y\right|^{2-n} = \frac{1}{n-2}\left|x-y\right|^{2-n} = E(x,y),$$

$$E\left(\frac{S^T x}{|S^T x|}, |S^T x| S^T y\right) = E\left(\frac{S^T x}{|x|}, |x| S^T y\right)$$

$$= \frac{1}{n-2}\left|S^T\left(\frac{x}{|x|} - |x|y\right)\right|^{2-n} = E\left(\frac{x}{|x|}, |x|y\right),$$

if $n > 2$. This yields $G(S^Tx, S^Ty) = G(x,y)$. Therefore, due to (17), we get

$$F(S^Tx) = \mu\left(v_\mu(x) - \frac{1}{\omega_n}\int_\Omega G(x,y)(\lambda_\mu v_\mu(y) + f_\mu(y))\,\mathrm{d}y\right.$$
$$\left.+\frac{1}{\omega_n}\int_{\partial\Omega} g_\mu(\xi)\Lambda_\xi G(x,\xi)\,\mathrm{d}s_\xi\right) = 0$$

for $x \in \bar{\Omega}$ and hence, due to bijectively of the mapping $S : \bar{\Omega} \to \bar{\Omega}$, we have $F(x) = 0$ for $x \in \bar{\Omega}$. Thus, the function $\mu v_\mu(Sx)$ is also a solution to the integral equation (17), and therefore, as noted above, $\mu v_\mu(Sx) = v_\mu(x)$. This immediately implies (15). ■

The converse statement to Theorem 3.1 is also true.

Theorem 3.3: *Let functions $v_{\mu_0}(x), v_{\mu_1}(x), \dots, v_{\mu_{m-1}}(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, where μ_0, $\mu_1, \dots, \mu_{m-1}$ are different mth roots of unity, be solutions to the Dirichlet problems* (8) *and* (9) *in which*

$$\sigma_{\mu_k}(A) = \sum_{i=0}^{m-1} a_i\bar{\mu}_k^i \neq 0, \quad \sigma_{\mu_k}(B) = \sum_{i=0}^{m-1} b_i\bar{\mu}_k^i \neq 0,$$

functions $f_{\mu_k}(x)$ and $g_{\mu_k}(x)$ be defined through the functions $f(x)$ and $g(x)$ by the equalities similar to (10)

$$f_{\mu_k}(x) = \frac{1}{\sigma_{\mu_k}(A)}\sum_{i=0}^{m-1} \mu_k^i f(S^ix), \quad g_{\mu_k}(x) = \frac{1}{\sigma_{\mu_k}(B)}\sum_{i=0}^{m-1} \mu_k^i g(S^ix).$$

Let moreover, the numbers $\lambda_{\mu_k} = \lambda/\sigma_{\mu_k}(A)$ for $k = 0, 1, \dots, m-1$ be not eigenvalues corresponding to the problem (8) *and* (9), *then the function*

$$u(x) = \frac{1}{m}\sum_{k=0}^{m-1} v_{\mu_k}(x) \tag{18}$$

is a solution to the problem (1) *and* (2). *Under the assumptions made, the solution to the problem* (1) *and* (2) *is unique.*

Proof: Let the function $u(x)$ be taken in the form of (18). It is obvious that $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$. Further, it is easy to see that the functions $f_{\mu_k}(x)$ and $g_{\mu_k}(x)$ satisfy the equalities (14). Indeed, if $\mu = \mu_k$, we have

$$f_\mu(Sx) = \frac{1}{\sigma_\mu(A)}\sum_{i=0}^{m-1} \mu^i f(S^{i+1}x) = \frac{1}{\mu\sigma_\mu(A)}\sum_{i=0}^{m-1} \mu^i f(S^ix) = \bar{\mu}f_\mu(x), \quad x \in \Omega, \tag{19}$$

and similarly

$$g_\mu(Sx) = \bar{\mu}g_\mu(x), \quad x \in \partial\Omega.$$

Since $\lambda_{\mu_k} = \lambda/\sigma_{\mu_k}(A)$ for $k = 0, 1, \ldots, m-1$ are not eigenvalues of the corresponding problem (8) and (9), conditions of Lemma 3.2 hold. By this lemma the equality (15) $v_\mu(Sx) = \bar{\mu} v_\mu(x)$ holds. Thus, for $x \in \Omega$ we have

$$u(S^j x) = \frac{1}{m} \sum_{k=0}^{m-1} \bar{\mu}_k^j v_{\mu_k}(x) \tag{20}$$

and, moreover, due to (19) we get

$$\Delta v_\mu(S^j x) = \lambda_\mu v_\mu(S^j x) + f_\mu(S^j x) = \bar{\mu}^j(\lambda_\mu v_\mu(x) + f_\mu(x)).$$

By using the equalities (20) obtained above, due to Equation (8), we find

$$\begin{aligned}
&-\Delta\left(\sum_{j=0}^{m-1} a_j u(S^j x)\right)\\
&= -\sum_{j=0}^{m-1} a_j \frac{1}{m} \sum_{k=0}^{m-1} \bar{\mu}_k^j \Delta v_{\mu_k}(x)\\
&= \frac{\lambda}{m} \sum_{k=0}^{m-1} v_{\mu_k}(x) \frac{1}{\sigma_{\mu_k}(A)} \sum_{j=0}^{m-1} a_j \bar{\mu}_k^j + \frac{1}{m} \sum_{j=0}^{m-1} a_j \sum_{k=0}^{m-1} \bar{\mu}_k^j f_{\mu_k}(x)\\
&= \frac{\lambda}{m} \sum_{k=0}^{m-1} v_{\mu_k}(x) + \frac{1}{m} \sum_{k=0}^{m-1} \sigma_{\mu_k}(A) f_{\mu_k}(x) = \lambda u(x) + \frac{1}{m} \sum_{i=0}^{m-1} f(S^i x) \sum_{k=0}^{m-1} \mu_k^i\\
&= \lambda u(x) + f(x),
\end{aligned}$$

i.e. Equation (1) is satisfied in Ω.

Let us check the boundary condition (2). Using the equalities (20) and (9), we can write

$$\begin{aligned}
\sum_{j=0}^{m-1} b_j u(S^j x)\Bigg|_{\partial\Omega} &= \sum_{j=0}^{m-1} b_j \frac{1}{m} \sum_{k=0}^{m-1} \bar{\mu}_k^j v_{\mu_k}(x)\Bigg|_{\partial\Omega} = \frac{1}{m} \sum_{j=0}^{m-1} b_j \sum_{k=0}^{m-1} \bar{\mu}_k^j g_{\mu_k}(x)\\
&= \frac{1}{m} \sum_{k=0}^{m-1} g_{\mu_k}(x) \sum_{j=0}^{m-1} b_j \bar{\mu}_k^j = \frac{1}{m} \sum_{k=0}^{m-1} \sigma_{\mu_k}(B) g_{\mu_k}(x)\\
&= \frac{1}{m} \sum_{k=0}^{m-1} \sum_{i=0}^{m-1} \mu_k^i g(S^i x) = \sum_{i=0}^{m-1} g(S^i x) \frac{1}{m} \sum_{k=0}^{m-1} \mu_k^i = g(x).
\end{aligned}$$

Thus, the function $u(x)$ defined by the formula (18) is the solution to the problem (1) and (2). Now suppose that the solution to the problem (1) and (2) is not unique. Then $u(x)$ – the difference of two solutions to this problem is a solution to the problem (1) and (2) for $f(x) = 0, g(x) = 0$ and hence $f_{\mu_k}(x) = 0, g_{\mu_k}(x) = 0$. Since λ_{μ_k} are not eigenvalues to the corresponding problem (8) and (9), then $v_{\mu_k}(x) = 0$ and hence according to (18) we get $u(x) = 0$. This proves the uniqueness of the solution to the problem (1) and (2). The theorem is proved. ■

4. Eigenfunctions and eigenvalues of the nonlocal problem

By virtue of Theorem 3.3 proved above, a solution to the homogeneous problem (1) and (2) can be represented through the solutions of the homogeneous problem (8) and (9), which have only a discrete spectrum. Therefore, it is natural to investigate the existence of eigenfunctions of the homogeneous problem (1) and (2)

$$-\sum_{k=0}^{m-1} a_k \Delta u(S^k x) = \lambda u(x), \quad x \in \Omega, \tag{21}$$

$$\sum_{k=0}^{m-1} b_k u(S^k x) = 0, \quad x \in \partial\Omega. \tag{22}$$

Theorem 4.1: *Let $\mu_0, \mu_1, \ldots, \mu_{m-1}$ be different mth roots of unity, numbers $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_{m-1}$ be such that $\sigma_{\mu_k}(A) \neq 0$ and $\sigma_{\mu_k}(B) \neq 0$ for $k = 0, 1, \ldots, m-1$, where numbers $\sigma_{\mu_k}(A)$ and $\sigma_{\mu_k}(B)$ are defined in* (4). *Then $\lambda = \lambda^d \sigma_{\mu_k}(A)$, where λ^d is an eigenvalue of the homogeneous problem* (8) *and* (9), *is an eigenvalue of the problem* (21) *and* (22) *and the system of eigenfunctions of the problem* (21) *and* (22) *is complete in $L_2(\Omega)$.*

Proof: Assume that a solution of the problem (21) and (22) exists for some λ. Apply Theorem 3.3 to the problem (21) and (22) for $\mu = \mu_k$, $k = 0, 1, \ldots, m-1$. In this case from the formula (10) it follows that $f_{\mu_k}(x) = g_{\mu_k}(x) = 0$, $\lambda_{\mu_k} = \lambda/\sigma_{\mu_k}(A)$, and the problem (8) and (9) is reduced to the problem

$$-\Delta v(x) = \lambda v(x), \ \ x \in \Omega, \quad v(x) = 0, \ \ x \in \partial\Omega, \tag{23}$$

solution of which – $v_\mu(x)$ exists and is given by the formula (7). If $v_\mu(x) \neq 0$, then λ_{μ_k} must coincide with some eigenvalue of the Dirichlet problem (23), i.e. $\lambda_{\mu_k} = \lambda^{(k)}$.

Let λ^d be an eigenvalue of the problem (23), and $v(x)$ be an eigenfunction of this problem corresponding to it. It is clear that $\lambda^d > 0$. Denote

$$v_k(x) = \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i v(S^i x). \tag{24}$$

It is easy to see that, due to Lemma 2.1 and the equality $S(\partial\Omega) = \partial\Omega$, the functions $v_k(x)$ for $k = 0, 1, \ldots, m-1$ are also eigenfunctions of the problem (23)

$$-\Delta v_k(x) = -\frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i \,(\Delta v)\,(S^i x) = \lambda^d \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i v(S^i x) = \lambda^d v_k(x),$$

$$v_k(x)|_{\partial\Omega} = \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i v(S^i x)|_{\partial\Omega} = 0.$$

The following representation takes place

$$v(x) = \sum_{k=0}^{m-1} v_k(x). \tag{25}$$

Indeed,

$$\sum_{k=0}^{m-1} v_k(x) = \sum_{k=0}^{m-1} \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i v(S^i x) = \frac{1}{m} \sum_{i=0}^{m-1} v(S^i x) \frac{1}{m} \sum_{k=0}^{m-1} \mu_k^i = v(x),$$

since

$$\sum_{k=0}^{m-1} \mu_k^i = \begin{cases} m, & i = 0 \\ 0, & i = 1, 2, \ldots, m-1 \end{cases}.$$

Note that the formula (24) can give null functions (see Example 4.2 below). The functions $v_k(x)$ also have the property

$$v_k(Sx) = \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^i v(S^{i+1} x) = \bar{\mu}_k \frac{1}{m} \sum_{i=0}^{m-1} \mu_k^{i+1} v(S^{i+1} x) = \bar{\mu}_k v_k(x).$$

Therefore, since $v_k(x)$ is the eigenfunction of the problem (23), we have

$$-\Delta \left(\sum_{j=0}^{m-1} a_j v_k(S^j x) \right) = -\sum_{j=0}^{m-1} a_j \bar{\mu}_k^j \Delta v_k(x)$$

$$= \lambda^d v_k(x) \sum_{j=0}^{m-1} a_j \bar{\mu}_k^j = \lambda^d \sigma_{\mu_k}(A) v_k(x),$$

i.e. the function $v_k(x)$ satisfies Equation (21) at $\lambda = \lambda^d \sigma_{\mu_k}(A)$. Moreover,

$$\sum_{i=0}^{m-1} b_i v_k(S^i x)|_{\partial\Omega} = v_k(x)|_{\partial\Omega} \sum_{i=0}^{m-1} b_i \bar{\mu}_k^i = \sigma_{\mu_k}(B) v_k(x)|_{\partial\Omega} = 0$$

and, hence $v_k(x)$ is the eigenfunction of the problem (21) and (22) for $\lambda = \lambda^d \sigma_{\mu_k}(A)$. It is known (see, e.g. [19]), that the problem (23) has a complete in $L_2(\Omega)$ orthogonal system of eigenfunctions $\{v^{(j)}(x),\ j \in \mathbb{N}\}$ corresponding to the eigenvalues $\{\lambda^{(j)}(x),\ j \in \mathbb{N}\}$. By virtue of the formula (25), each such a function is a linear combination of the eigenfunctions $v_{k,j}(x)$ of the problem (21) and (22) for $\lambda = \lambda^{(j)} \sigma_{\mu_k}(A)$

$$v^{(j)}(x) = \sum_{k=0}^{m-1} v_{k,j}(x).$$

Therefore, the system of eigenfunctions $\{v_{k,j}(x)\}$ of the problem (21) and (22) is complete in $L_2(\Omega)$.

Note that the functions $v_k(x)$ and $v_s(x)$ in the expansion (25) are orthogonal if $k \neq s$. Indeed, if $k \neq s$ we have

$$\int_\Omega v_k(x) \overline{v_s(x)}\, \mathrm{d}x = \frac{1}{m^2} \int_\Omega \sum_{i,j=0}^{m-1} \mu_k^i \bar{\mu}_s^j v(S^i x) \overline{v(S^j x)}\, \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{m^2}\sum_{i,j=0}^{m-1}\mu_k^i\bar{\mu}_s^j\int_\Omega v(S^ix)\bar{v}(S^jx)\,\mathrm{d}x \\
&= \sum_{l=0}^{m-1}\sum_{i-j=l(\mod m)}^{m-1}\frac{\mu_k^i\bar{\mu}_s^j}{m^2}\int_\Omega v(S^ix)\overline{v(S^jx)}\,\mathrm{d}x \\
&= \sum_{l=0}^{m-1}\frac{\mu_k^l}{m^2}\sum_{j=0}^{m-1}(\mu_k\bar{\mu}_s)^j\int_\Omega v(S^{l+j}x)\overline{v(S^jx)}\,\mathrm{d}x \\
&= \frac{1}{m}\sum_{l=0}^{m-1}\mu_k^l\int_\Omega v(S^ly)\overline{v(y)}\,\mathrm{d}y\frac{1}{m}\sum_{j=0}^{m-1}(\mu_k\bar{\mu}_s)^j = 0,
\end{aligned}
$$

since $\det|S| = 1$ and $\mu_k\bar{\mu}_s \neq 1$ is also mth root of unity. The theorem is proved. ■

Remark 4.1: Since the orthogonal system of eigenfunctions of the problem (23) has the form (see, for example, [20,21])

$$v^{(\lambda_i,k,j)}(x) = g_{n+2k}\left(\lambda_i|x|^2\right)H_k^{(j)}(x) \equiv \frac{1}{|x|^{n/2-1}}J_{k+n/2-1}\left(\sqrt{\lambda_i}|x|\right)H_k^{(j)}\left(\frac{x}{|x|}\right), \quad j\in\mathbb{N}_0,$$

where $J_\nu(x)$ is the Bessel function of the first kind, $\sqrt{\lambda_i}$ is a root of Bessel function $J_\nu(x)$, $\{H_k^{(j)}(x)\}$ is a system of orthogonal on $\partial\Omega$ homogeneous harmonic polynomials of order k (see, for example, [22]), and $|x| = |Sx|$, then the expansion (25) rather refers to the expansion of the homogeneous harmonic polynomial $H_k(x)$.

Example 4.2: Consider the case $S = -I$ and $a_0 \neq \pm a_1$, $b_0 = 1, b_1 = 0$, i.e. consider the problem

$$a_0\Delta u(x) + a_1\Delta u(-x) = \lambda u(x),\ \ x\in\Omega, \quad u(x)|_{\partial\Omega} = 0. \tag{26}$$

In this case $m = 2$, $\mu_0 = 1$, $\mu_1 = -1$, and by the formula (4) we get $\sigma_{\mu_0}(A) = a_0 + a_1$, $\sigma_{\mu_1}(A) = a_0 - a_1$. Let $v(x)$ be a solution to the problem (23)

$$\Delta v(x) + \mu u(x) = 0,\ \ x\in\Omega, \quad v(x)|_{\partial\Omega} = 0. \tag{27}$$

Then $\mu > 0$, and the eigenfunction of this problem

$$v(x) = \frac{1}{|x|^{n/2-1}}J_{k+n/2-1}\left(\sqrt{\mu}|x|\right)H_k\left(\frac{x}{|x|}\right)$$

is either even or odd depending on the parity of $k\in\mathbb{N}_0$ (see [13], Lemma 1]). Therefore, only one of the functions given by the equalities (24)

$$u_0(x) = \tfrac{1}{2}\left[v(x) + v(-x)\right], \quad u_1(x) = \tfrac{1}{2}\left[v(x) - v(-x)\right].$$

is nonzero. Theorem 4.1 implies, but it is easy to verify directly, that these functions are eigenfunctions of the problem (26). If $H_k(x)$ is an even polynomial, then $u_0(x) = v(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_0 = \mu\sigma_{\mu_0}(A) = \mu(a_0 + a_1)$ and $u_1(x) = 0$. If $H_k(x)$ is an odd polynomial, then $u_1(x) = v(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_1 = \mu\sigma_{\mu_1}(A) = \mu(a_0 - a_1)$. Thus, the eigenfunctions of the problem (26) coincide with the eigenfunctions of the problem (27), and therefore the system of eigenfunctions of the nonlocal problem (26) is orthogonal and complete in $L_2(\Omega)$.

Remark 4.2: Using the results of [23] and Remark 4.1, we can be convinced of the statement similar to Example 4.2: The eigenfunctions of the problem (21) and (22) coincide with the eigenfunctions of the problem (23), and therefore the system of eigenfunctions of the nonlocal problem (21) and (22) is a basis in $L_2(\Omega)$.

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