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# Four Boundary Value Problems for a Nonlocal Biharmonic Equation in the Unit Ball

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**Abstract:** Solvability issues of four boundary value problems for a nonlocal biharmonic equation in the unit ball are investigated. Dirichlet, Neumann, Navier and Riquier–Neumann boundary value problems are studied. For the problems under consideration, existence and uniqueness theorems are proved. Necessary and sufficient conditions for the solvability of all problems are obtained and an integral representations of solutions are given in terms of the corresponding Green's functions.

**Keywords:** nonlocal equation; biharmonic equation; Dirichlet problem; Neumann problem; Navier problem; Riquier–Neumann problem; existence and uniqueness; Green's function

**MSC:** 31B30; 35J40; 35J08

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## 1. Introduction

In [1], the authors considered equations containing fractional derivatives of the desired function and equations with deviating arguments, in other words, equations that include an unknown function and its derivatives, generally speaking, for different values of arguments. Such equations are called nonlocal differential equations.

Among the nonlocal differential equations, a special place is occupied by equations in which deviation of the arguments has an involutive character. A mapping $S$ is called an involution, if $S^2(x) = S(S(x)) = x$. It is known that differential equations containing an involutive deviation of an unknown function or its derivative are some model equations with an alternating deviation of the argument. In the general case, such equations can be attributed to the class of functional differential equations.

Researchers have been studying differential equations with involution for a long time. In 1816, Babbage considered algebraic and differential equations with involution [2]. Monographs by Przeworska-Rolewicz [3] and Wiener [4] have been devoted to the solvability theory of various differential equations with involution. In [5,6], spectral problems for a first-order differential equation with involution were studied. In [7–11], spectral problems for differential operators of the first and second orders with involution were investigated. Some results of studies of spectral problems with involution were used in [12–14] to solve inverse problems.

The series of papers by Cabada and Tojo (see [15,16] and a list of citations therein) was devoted to the creation of the theory of the Green's function of one-dimensional differential equations with involution. The method of representing solutions to boundary value problems based on the construction of the Green's function of the problem is one of the efficient methods for the investigation of solutions to boundary value problems for elliptic equations. This method is also used in the present work. The Green's functions of the Dirichlet, Neumann, Robin and other biharmonic problems in a two-dimensional disc were constructed by means of the Green's harmonic functions of the Dirichlet problem in [17].

Some aspects of the solvability theory of partial differential equations with involution were described in [18–20]. In [21], boundary value problems with involution in the boundary conditions for second and fourth order elliptic equations were studied.

The purpose of this paper is to continue the investigations started earlier on the study of boundary value problems for nonlocal equations. Here, we study four boundary value problems for the biharmonic equation.

The paper is organized as follows. In Section 2, we present the statements of the problems under consideration. Section 3 contains some auxiliary statements obtained by the authors, which are needed in the presentation. In Theorem 2 of Section 4, the uniqueness of solutions to all considered problems is proved. In Section 5, based on Theorem 3, which gives a representation of the solution to the Dirichlet problem for a biharmonic equation in terms of the Green's function $G_4(x,\xi)$, an existence theorem for a solution to the Dirichlet boundary value problem for a nonlocal biharmonic equation is presented. In Section 6, using Lemma 7, necessary and sufficient conditions for the solvability of the Neumann boundary value problem are obtained. An integral representation of the problem's solution is presented. In Section 7, the Navier boundary value problem is studied. In Theorem 6, using the Green's function of the corresponding Navier boundary value problem for a biharmonic equation, an integral representation of a solution to the considered problem is obtained. Finally, in Theorem 7 of Section 8, conditions for the existence of a solution to the Riquier–Neumann boundary value problem are found and an integral representation of the problem's solution is obtained.

## 2. The Problems' Statements

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, $n \geq 2$, $\partial\Omega$ be the unit sphere, and $S$ be a real orthogonal matrix $S \cdot S^T = I$. Suppose also that there exists a natural number $l \in \mathbb{N}$ such that $S^l = I$. Let us call $l$ the order of $S$.

Note that if $x \in \Omega$, or $x \in \partial\Omega$, then for any $k$ the inclusions $S^k x \in \Omega$ or $S^k x \in \partial\Omega$ hold true. This is so because the transformation of the space $\mathbb{R}^n$ with the matrix $S$ saves the norm $|x|^2 = (x,x) = (S^T Sx, x) = (Sx, Sx) = |Sx|^2$. Let us give one simple example of such a matrix $S$.

**Example 1.** *In the case of $n = 2$, the matrix $S$ can be chosen in the form $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is clear that $S \cdot S^T = I$. In this case $l = 4$.*

Let $a_1, a_2, \ldots, a_l$ be some real numbers, $f(x)$ and $g(s)$ be functions, given on $\Omega$ and $\partial\Omega$, respectively. Introduce the nonlocal operator

$$Lu(x) \equiv \sum_{k=1}^{l} a_k \Delta^2 u\left(S^{k-1}x\right).$$

If the matrix $S$ is taken from Example 1, then the operator $L$ has the form

$$Lu(x) = a_1\Delta^2 u(x_1,x_2) + a_2\Delta^2 u(-x_2,x_1) + a_4\Delta^2 u(-x_1,-x_2) + a_4\Delta^2 u(x_2,-x_1).$$

In the unit ball $\Omega$, consider the following boundary value problems.

**Dirichlet problem**. *Find a function $u(x) \in C^4(\Omega) \cap C^1(\bar{\Omega})$, satisfying the conditions*

$$Lu(x) = f(x),\ x \in \Omega, \tag{1}$$

$$u|_{\partial\Omega} = g_0(x),\quad \frac{\partial u}{\partial \nu}|_{\partial\Omega} = g_1(x),\ x \in \partial\Omega, \tag{2}$$

where $\nu$ is the outer normal to sphere $\partial\Omega$.

**Neumann problem** [22]. *Find a function $u(x) \in C^4(\Omega) \cap C^2(\bar{\Omega})$, satisfying Equation (1) and the boundary conditions*

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega} = h_1(x), \quad \frac{\partial^2 u}{\partial \nu^2}|_{\partial\Omega} = h_2(x), \; x \in \partial\Omega. \tag{3}$$

**Navier problem** [23]. *Find a function $u(x) \in C^4(\Omega) \cap C^2(\bar{\Omega})$, satisfying Equation (1) and the boundary conditions*

$$u|_{\partial\Omega} = r_0(x), \quad \Delta u|_{\partial\Omega} = r_1(x), \; x \in \partial\Omega. \tag{4}$$

**Riquier–Neumann problem** [24,25]. (in [17] it is also called the Neumann-2 problem). *Find a function $u(x) \in C^4(\Omega) \cap C^3(\bar{\Omega})$, satisfying Equation (1) and the boundary conditions*

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega} = p_0(x), \quad \frac{\partial \Delta u}{\partial \nu}|_{\partial\Omega} = p_1(x), \; x \in \partial\Omega. \tag{5}$$

In the case where $a_1 \neq 0, \; a_k = 0, \; k = 2, 3, \ldots, l$ we obtain the classical Dirichlet, Neumann, Navier and Riquier–Neumann boundary value problems for the biharmonic equation. It should be noted that in the case $n = 2$ in [26], for the classical Laplace equation, nonlocal boundary value problems with some matrix $S$ were studied.

## 3. Auxiliary Statements

In this section, we present some auxiliary statements regarding a special form of matrices. Consider the matrix $A$, consisting of the coefficients $a_1, a_2, \ldots, a_l$ of the operator $L$

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

**Lemma 1** ([27]). *Let $\lambda_1 = e^{i\frac{2\pi}{l}}$ be the primitive lth root of unity. Then*

$$\det A = \prod_{k=1}^{l} \left( a_1 \lambda_1^k + \ldots + a_l \lambda_l^k \right),$$

*where $\lambda_k = e^{i\frac{2\pi k}{l}} = \lambda_1^k$, $k = 1, \ldots, l$.*

Clearly, $\lambda_k^n = (\lambda_1^k)^n = (\lambda_1^n)^k = \lambda_n^k$. Denote

$$\mu_k = a_1 \lambda_0^k + \ldots + a_l \lambda_{l-1}^k = \sum_{q=1}^{l} a_q \lambda_{q-1}^k = \sum_{q=1}^{l} a_q \lambda_k^{q-1}. \tag{6}$$

**Example 2.** *Let $l = 3$. Then, $\lambda_1 = e^{i\frac{2\pi}{3}}$, and hence $\lambda_k = e^{i\frac{2\pi k}{3}}$. In this case we have*

$$\det A = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$$
$$= \left(a_1 + a_2 e^{i\frac{2\pi}{3}} + a_3 e^{i\frac{4\pi}{3}}\right)\left(a_1 + a_2 e^{i2\frac{2\pi}{3}} + a_3 e^{i2\frac{4\pi}{3}}\right)(a_1 + a_2 + a_3)$$
$$= \left(a_1 + a_2 e^{i\frac{2\pi}{3}} + a_3 e^{i\frac{4\pi}{3}}\right)\left(a_1 + a_2 e^{i\frac{4\pi}{3}} + a_3 e^{i\frac{2\pi}{3}}\right)(a_1 + a_2 + a_3)$$
$$= (a_1 + a_2 + a_3)(a_1^2 + a_2^2 + a_3^2 - a_2 a_3 - a_1 a_2 - a_1 a_3) = a_1^3 + a_2^3 + a_3^3 - 3a_1 a_2 a_3.$$

**Lemma 2** ([27])**.** *Let the numbers $\mu_k$ in (6) be nonzero $\mu_k \neq 0$, $k = 1, \ldots, l$. Then, there exists an inverse to the matrix $A$, which is given by the formula*

$$A^{-1} = \frac{1}{l} \mathrm{M}_+ \mathrm{diag}^{-1}(\mu_1, \ldots, \mu_l) \mathrm{M}_-{}^T,$$

*where*

$$\mathrm{M}_+ = \left(\lambda_{i-1}^{j}\right)_{i,j=1,l}, \quad \mathrm{M}_- = \left(\lambda_{i-1}^{-j}\right)_{i,j=1,l}.$$

Based on these two lemmas it is not hard to prove the statement.

**Theorem 1.** *Let $\mu_k = a_1 \lambda_k^0 + \ldots + a_l \lambda_k^{l-1} \neq 0$, $k = 1, \ldots, l$, where $\{\lambda_k\}$ are the $l$th roots of unity. Then, the solution to the system of algebraic equations $Ab = g$ can be written in the form*

$$b = (b_i)_{i=1,\ldots,l} = \frac{1}{l} \left( \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{i-j} g_j \right)_{i=1,\ldots,l}.$$

**Proof.** Using Lemma 2, we can write

$$b = A^{-1} g = \frac{1}{l} \mathrm{M}_+ \mathrm{diag}^{-1}(\mu_1, \ldots, \mu_l) \mathrm{M}_-{}^T \cdot (g_j)_{j=1,\ldots,l}$$
$$= \frac{1}{l} \mathrm{M}_+ \mathrm{diag}^{-1}(\mu_1, \ldots, \mu_l) \left( \sum_{j=1}^{l} \lambda_{j-1}^{-i} g_j \right)_{i=1,\ldots,l} = \frac{1}{l} \mathrm{M}_+ \left( \sum_{j=1}^{l} \frac{1}{\mu_i} \lambda_{j-1}^{-i} g_j \right)_{i=1,\ldots,l}$$
$$= \left( \frac{1}{l} \sum_{k=1}^{l} \lambda_{i-1}^{k} \sum_{j=1}^{l} \frac{1}{\mu_k} \lambda_{j-1}^{-k} g_j \right)_{i=1,\ldots,l} = \left( \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} (\lambda_{i-1}/\lambda_{j-1})^k g_j \right)_{i=1,\ldots,l}$$
$$= \left( \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_{i-j}^{k} g_j \right)_{i=1,\ldots,l}.$$

The theorem is proved. □

## 4. Uniqueness

In order to study the uniqueness of the solution to Equations (1) and (2), we start with the following statement.

**Lemma 3** ([28])**.** *The operator $I_S u(x) = u(Sx)$ and the Laplace operator $\Delta$ commute: $\Delta I_S u(x) = I_S \Delta u(x)$. Operators $\Lambda u = \sum_{i=1}^{n} x_i u_{x_i}(x)$ and $I_S$ also commute: $\Lambda I_S u(x) = I_S \Lambda u(x)$ and the equality $\nabla I_S = I_S S^T \nabla$ holds true.*

**Corollary 1.** *If a function $u(x)$ is biharmonic in $\Omega$, then the function $u(Sx) = I_S u(x)$ is also biharmonic in $\Omega$.*

**Corollary 2.** *If a function $u(x)$ is biharmonic in $\Omega$, then it satisfies the homogeneous Equation (1) in $\Omega$.*

Indeed, according to Lemma 3, for any $x \in \Omega$, we have

$$Lu(x) = \sum_{k=1}^{l} a_k \Delta^2 u\left(S^{k-1}x\right) = \sum_{k=1}^{l} a_k \Delta^2 I_{S^{k-1}} u(x) = \sum_{k=1}^{l} a_k I_{S^{k-1}} \Delta^2 u(x) = 0.$$

The converse statement is also true. Let

$$c_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1} \mu_k} \tag{7}$$

where $j = 1, 2, \dots, l$. Introduce the operators

$$I_L v(x) = \sum_{k=1}^{l} a_k I_{S^{k-1}} v(x) = \sum_{k=1}^{l} a_k v(S^{k-1}x), \quad J_L v(x) = \sum_{k=1}^{l} c_k v(S^{k-1}x). \tag{8}$$

Then the operator $L$ of Equation (1) can be written as $Lu = I_L \Delta^2 u$.

**Lemma 4.** *Let the function $u \in C^4(\Omega)$ satisfy the homogeneous Equation (1) and $\mu_k \neq 0$, $k = 1, \dots, l$. Then, the function $u(x)$ is biharmonic in $\Omega$.*

**Proof.** Let the function $u \in C^4(\Omega)$ satisfy the homogeneous Equation (1). Denote

$$v(x) = \sum_{k=1}^{l} a_k u(S^{k-1}x) = I_L u(x). \tag{9}$$

It is obvious that $v(x) \in C^4(\Omega)$ and

$$\Delta^2 v(x) = \Delta^2 I_L u = I_L \Delta^2 u = Lu = 0, \quad x \in \Omega,$$

i.e., the function $v(x)$ is biharmonic in $\Omega$. Due to Corollary 1, functions $v(S^k x)$ are also biharmonic in the domain $\Omega$. On the other hand, based on (9) we can get the equalities

$$\begin{gathered} v(Sx) = a_l u(x) + a_1 u(Sx) + \ldots + a_{l-1} u(S^{l-1}x) \\ v(S^2x) = a_{l-1} u(x) + a_l u(Sx) + \ldots + a_{l-2} u(S^{l-1}x) \\ \cdots \quad \cdots \quad \cdots \\ v(S^{l-1}x) = a_2 u(x) + a_3 u(Sx) + \ldots + a_1 u(S^{l-1}x), \end{gathered} \tag{10}$$

where the condition $S^l = I$ is used. Hence, the system of algebraic Equations (9) and (10) for the functions $u(x), u(Sx), \dots, u(S^{l-1}x)$ can be rewritten in matrix form using the matrix $A$

$$\begin{pmatrix} v(x) \\ v(Sx) \\ \vdots \\ v(S^{l-1}x) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_l \\ a_l & a_1 & \dots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{pmatrix} \begin{pmatrix} u(x) \\ u(Sx) \\ \vdots \\ u(S^{l-1}x) \end{pmatrix}.$$

By virtue of the lemma's conditions, the determinant of this system is nonzero, $\det A \neq 0$. If we use Theorem 1 for

$$b = \left(u(x), u(Sx), \dots, u(S^{l-1}x)\right)^T, \quad g = \left(v(x), v(Sx), \dots, v(S^{l-1}x)\right)^T,$$

then we obtain

$$u(x) = b_1 = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_k^{1-j}g_j = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k}\sum_{j=1}^{l}\lambda_k^{1-j}v(S^{j-1}x)$$
$$= \sum_{j=1}^{l}v(S^{j-1}x)\frac{1}{l}\sum_{k=1}^{l}\frac{1}{\lambda_k^{j-1}\mu_k} = \sum_{j=1}^{l}c_j v(S^{j-1}x).$$

Consequently, remembering the definition of the operator $J_L$ from (8), we get

$$u(x) = \sum_{j=1}^{l}c_j v(S^{j-1}x) = c_1 v(x) + c_2 v(Sx) + \ldots + c_l v(S^{l-1}x) = J_L v(x). \tag{11}$$

As noted above, the functions $v(S^k x)$ are biharmonic in $\Omega$ for $k = 0, 1, \ldots, l-1$, and therefore, the function $u(x)$ from (11) is also biharmonic in $\Omega$. The lemma is proved. □

By virtue of Lemma 4, we are able to prove the statement.

**Theorem 2.** *Suppose that for all $k = 1, \ldots, l$ the inequalities $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ hold true and solutions to the Dirichlet, Neumann, Navier and Riquier–Neumann boundary value problems exist. Then,*

*(1) The solution to the Dirichlet problem (1), (2) is unique;*
*(2) The solution to the Neumann problem (1), (3) is unique up to a constant term;*
*(3) The solution to the Navier problem (1), (4) is unique;*
*(4) The solution to the Riquier–Neumann problem (1), (5) is unique up to a constant term.*

**Proof.** (1) Let us prove that the homogenous problem (1), (2) has only a zero solution, and then that the solution of the non-homogenous problem (1), (2) is unique. Let $u(x)$ be a solution of the homogenous problem (1), (2). If $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ for $k = 1, \ldots, l$, then by Lemma 1 $\det A \neq 0$. By Lemma 4 the function $u(x)$ is biharmonic in $\Omega$ and satisfies the homogeneous conditions (2). Hence, the function $u(x)$ is a solution to the Dirichlet problem

$$\Delta^2 u(x) = 0,\ x \in \Omega; \quad u(x)|_{\partial\Omega} = \frac{\partial u(x)}{\partial\nu}|_{\partial\Omega} = 0.$$

By virtue of the uniqueness of the solution to the Dirichlet problem, we have $u(x) = 0$.

(2) Let $u(x)$ be a solution to the homogenous problem (1), (3), then by Lemma 4, the function $u(x)$ is biharmonic in $\Omega$ and therefore satisfies the boundary conditions of the Neumann problem

$$\Delta^2 u(x) = 0,\ x \in \Omega; \quad \frac{\partial u(x)}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial^2 u(x)}{\partial\nu^2}\Big|_{\partial\Omega} = 0.$$

The solution to this problem is only the function $u(x) = const$ [22].

(3) Let a function $u(x)$ satisfy the homogeneous boundary conditions of the Equations (1) and (4), then by Lemma 4

$$\Delta^2 u(x) = 0,\ x \in \Omega; \quad u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,\ x \in \partial\Omega.$$

The solution to this problem is only the function $u(x) = 0$ [25].

(4) Let the function $u(x)$ satisfy the homogeneous boundary conditions of the Equations (1) and (5), then by Lemma 4

$$\Delta^2 u(x) = 0,\ x \in \Omega; \quad \frac{\partial u(x)}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial\Delta u}{\partial\nu}\Big|_{\partial\Omega} = 0,\ x \in \partial\Omega.$$

The solution to this problem is only the function $u(x) = const$ [23]. $\square$

## 5. Dirichlet Boundary Value Problem

In [29], an elementary solution to the biharmonic equation was defined as

$$E_4(x,\xi) = \begin{cases} \dfrac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n > 4,\, n = 3 \\ -\dfrac{1}{4}\ln|x-\xi|, & n = 4 \\ \dfrac{|x-\xi|^2}{4}\big(\ln|x-\xi| - 1\big), & n = 2 \end{cases}, \tag{12}$$

and it was proved that for $n \geq 3$, the function of the form

$$G_4(x,\xi) = E_4(x,\xi) - E_4\Big(\frac{x}{|x|}, |x|\xi\Big) - \frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E\Big(\frac{x}{|x|}, |x|\xi\Big), \tag{13}$$

where $E(x,\xi) = |x-\xi|^{2-n}/(n-2)$, was the Green's function of the Dirichlet problem for the biharmonic equation in the unit ball. Then, in [25], the following statement was established.

**Theorem 3** ([25])**.** *Let $\varphi_0 \in C^{2+\varepsilon}(\partial S)$, $\varphi_1 \in C^{1+\varepsilon}(\partial S)$ $(\varepsilon > 0)$ and $f \in C^1(\bar{S})$, then the solution to the Dirichlet problem for the biharmonic equation for $n > 4$ or $n = 3$ can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial\Omega} g_0(\xi)\frac{\partial}{\partial \nu}\Delta_\xi G_4(x,\xi)\, ds_\xi - \frac{1}{\omega_n}\int_{\partial\Omega} g_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n}\int_\Omega G_4(x,\xi) f(\xi)\, d\xi, \tag{14}$$

*where $\omega_n$ is the area of the unit sphere in $\mathbb{R}^n$.*

In what follows, we need the following statement.

**Lemma 5** ([27])**.** *Let the function $g(x)$ be continuous on $\partial\Omega$ or $\Omega$. Then, for any $k \in \mathbb{N}$*

$$\int_{\partial\Omega} g(S^k y)\, ds_y = \int_{\partial\Omega} g(y)\, ds_y, \quad \int_\Omega g(S^k y)\, dy = \int_\Omega g(y)\, dy.$$

Let us prove the existence theorem for the solution to the problems (1) and (2).

**Theorem 4.** *Let the coefficients $\{a_k : k = 1, \ldots, l\}$ of the operator L be such that $\mu_k \neq 0$ for $k = 1, \ldots, l$ and $g_0 \in C^{2+\varepsilon}(\partial\Omega)$, $g_1 \in C^{1+\varepsilon}(\partial\Omega)$ $(\varepsilon > 0)$, and $f \in C^1(\bar{\Omega})$. Then, the solution to the problems (1) and (2) exists and can be represented as*

$$u(x) = \frac{1}{\omega_n}\int_{\partial\Omega} g_0(\xi)\Lambda\Delta_\xi G_4(x,\xi)\, ds_\xi - \frac{1}{\omega_n}\int_{\partial\Omega} g_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n}\int_\Omega G_4(x,\xi) J_L f(\xi)\, d\xi, \tag{15}$$

*where the operator $J_L$ is defined in (8).*

**Proof.** Let the function $u(x)$ be a solution to the problems (1) and (2). Denote $v(x) = I_L u(x)$, then using (11), we get $u(x) = J_L v(x)$. Due to Lemma 3 on the commutativity of the operators $I_S$ and $\Delta$ and $I_S$ and $\Lambda$, taking into account the equalities $\frac{\partial}{\partial \nu} u|_{\partial\Omega} = \Lambda u|_{\partial\Omega}$ and $Lu = \Delta^2 v$, we get the following problem for the function $v(x)$

$$\Delta^2 v(x) = f(x),\ x \in \Omega; \quad v\big|_{\partial\Omega} = I_L g_0(s),\ \Lambda u\big|_{\partial\Omega} = I_L g_1(s),\ s \in \partial\Omega. \tag{16}$$

It is clear that $g_0 \in C^{2+\varepsilon}(\partial\Omega) \Rightarrow I_L g_0 \in C^{2+\varepsilon}(\partial\Omega)$, $g_1 \in C^{1+\varepsilon}(\partial\Omega) \Rightarrow I_L g_1 \in C^{1+\varepsilon}(\partial\Omega)$ and hence the solution to the Dirichlet problem (16) exists and is unique.

Next, let the function $v(x)$ be a solution to the problem (16). Then, for the function $u(x) = J_L v(x)$, we obtain Equation (1) and after applying the operator $J_L$ to the boundary conditions of the problem (16), we obtain the boundary conditions (2).

By virtue of Theorem 3, the solution to the problem (16) can be represented as

$$\begin{aligned} v(x) = {} & \frac{1}{\omega_n} \int_{\partial\Omega} I_L g_0(\xi) \Lambda \Delta_\xi G_4(x,\xi)\, ds_\xi \\ & - \frac{1}{\omega_n} \int_{\partial\Omega} I_L g_1(\xi) \Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_\Omega G_4(x,\xi) f(\xi)\, d\xi. \end{aligned}$$

Applying the operator $J_L$ to both sides of the equality and taking (11) into account, we obtain

$$\begin{aligned} u(x) = {} & \frac{1}{\omega_n} J_L \int_{\partial\Omega} I_L g_0(\xi) \Lambda \Delta_\xi G_4(x,\xi)\, ds_\xi \\ & - \frac{1}{\omega_n} J_L \int_{\partial\Omega} I_L g_1(\xi) \Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n} J_L \int_\Omega G_4(x,\xi) f(\xi)\, d\xi. \end{aligned} \tag{17}$$

In the resulting expression, we first transform the last integral. It is easy to see that $|S^k x - S^k \xi| = |S^k(x - \xi)| = |x - \xi|$, which means that according to (12), $E_4(S^k x, S^k \xi) = E_4(x,\xi)$ and hence, given (13), we find $G_4(S^k x, S^k \xi) = G_4(x,\xi)$. Further, by virtue of Lemma 5

$$I_{S^k} \int_\Omega G_4(x,\xi) f(\xi)\, d\xi = \int_\Omega G_4(S^k x, S^k \xi) f(S^k \xi)\, d\xi = \int_\Omega G_4(x,\xi) I_{S^k} f(\xi)\, d\xi.$$

Therefore, according to (8), we have

$$J_L \int_\Omega G_4(x,\xi) f(\xi)\, d\xi = \sum_{k=1}^{l} c_k I_{S^{k-1}} \int_\Omega G_4(x,\xi) f(\xi)\, d\xi = \int_\Omega G_4(x,\xi) J_L f(\xi)\, d\xi.$$

Similarly, by Lemma 5, we get

$$\begin{aligned} J_L \int_{\partial\Omega} \hat{g}_0(\xi) \frac{\partial}{\partial \nu} \Delta_\xi G_4(x,\xi)\, ds_\xi &= \int_{\partial\Omega} J_L \hat{g}_0(\xi) \frac{\partial}{\partial \nu} \Delta_\xi G_4(x,\xi)\, ds_\xi, \\ J_L \int_{\partial\Omega} \hat{g}_1(\xi) \Delta_\xi G_4(x,\xi)\, ds_\xi &= \int_{\partial\Omega} J_L \hat{g}_1(\xi) \Delta_\xi G_4(x,\xi)\, ds_\xi, \end{aligned}$$

where $\hat{g}_0 = I_L g_0$, $\hat{g}_1 = I_L g_1$. Thus, the function $u(x)$ from (17) can be rewritten as

$$\begin{aligned} u(x) = {} & \frac{1}{\omega_n} \int_{\partial\Omega} J_L I_L g_0(\xi) \Lambda \Delta_\xi G_4(x,\xi)\, ds_\xi \\ & - \frac{1}{\omega_n} \int_{\partial\Omega} J_L I_L g_1(\xi) \Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_\Omega G_4(x,\xi) J_L f(\xi)\, d\xi. \end{aligned} \tag{18}$$

By virtue of (9) and (11), the following equalities are also true

$$\hat{g}_i(\xi) = I_L g_i(\xi), \quad g_i(\xi) = J_L \hat{g}_i(\xi), \quad i = 1,2,$$

from which it follows that $J_L I_L g_i(\xi) = g_i(\xi)$ for an arbitrary function $g_i(\xi)$ on $\partial\Omega$. So (18) is converted to (15). This proves the theorem. $\square$

**Corollary 3.** *Let $v_0(x)$ and $v_1(x)$ be functions harmonic in $\Omega$ such that $v_0(x)|_{\partial\Omega} = g_0$ and $v_1(x)|_{\partial\Omega} = g_1$. Then, the solution of the Dirichlet problem (1), (2) can be written as*

$$u(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2} v_1(x) + \frac{1}{\omega_n}\int_\Omega G_4(x,\xi) J_L f(\xi)\, d\xi. \tag{19}$$

The representation (19) of the function $u(x)$ follows from Theorem 4 and from the representation of the solution to the Dirichlet problem for the homogeneous biharmonic equation

$$\Delta^2 u_0(x) = 0,\ x \in \Omega;\quad u_0|_{\partial\Omega} = g_0(s),\ \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = g_1(s),\ s \in \partial\Omega \tag{20}$$

in the form [30]

$$u_0(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2} v_1(x).$$

In [31], a method for representing the solution of an inhomogeneous equations without using the Green's function was given.

**Example 3.** *Let $S$ be a real orthogonal matrix such that $S^2 = I$. The problem (1), (2) in this case takes the form*

$$a_1\Delta^2 u(x) + a_2 \Delta^2 u(Sx) = f(x),\ x \in \Omega;\quad u|_{\partial\Omega} = g_0(s),\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = g_1(s),\ s \in \partial\Omega. \tag{21}$$

*Then, $l = 2$, $\lambda_1 = e^{i\pi} = -1$, $\lambda_2 = e^{2i\pi} = 1$, $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$ and*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix},\quad \det A = \mu_1 \cdot \mu_2 = a_1^2 - a_2^2.$$

*Let $a_1^2 - a_2^2 \neq 0 \Leftrightarrow a_1 \neq \pm a_2$. Then, by (7), we have*

$$c_1 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^0 \mu_k} = \frac{1}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{a_1}{a_1^2 - a_2^2},$$
$$c_2 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^1 \mu_k} = \frac{1}{2}\left(\frac{1}{-\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(-\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{-a_2}{a_1^2 - a_2^2}$$

*and that means*

$$J_L f = c_1 f(x) + c_2 f(Sx) = \frac{a_1 f(x) - a_2 f(Sx)}{a_1^2 - a_2^2}.$$

*In accordance with Corollary 3, the solution to the problem (21) can be written as*

$$u(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2} v_1(x) + \frac{1}{\omega_n}\int_\Omega G_4(x,\xi)\frac{a_1 f(\xi) - a_2 f(S\xi)}{a_1^2 - a_2^2}\, d\xi,$$

*where $v_0(x)$ and $v_1(x)$ are harmonic functions in $\Omega$ such that $v_0|_{\partial\Omega} = g_0$, $v_1|_{\partial\Omega} = g_1$.*

## 6. Neumann Boundary Value Problem

Let us investigate the necessary and sufficient conditions for solvability of the Neumann boundary value problem (1), (3). Let $G_4(x,\xi)$ be the Green's function (13) of the classical Dirichlet problem for a biharmonic equation in the unit ball.

**Theorem 5.** *Let the coefficients $\{a_k : k = 1, \dots, l\}$ of the operator $L$ be such that $\mu_k = a_1\lambda_0^k + \ldots + a_l \lambda_{l-1}^k \neq 0$, for $k = 1, \dots, l$ and $f \in C^2(\bar{\Omega})$, $g_0(x) \in C^{3+\varepsilon}(\partial\Omega)$,*

$g_1(x) \in C^{2+\varepsilon}(\partial\Omega)$, $\varepsilon > 0$. *For solvability of the problems* (1) *and* (3), *the following condition is necessary and sufficient*

$$\int_{\partial\Omega} (g_0(x) - g_1(x))\, ds_x + \frac{1}{2\mu_l} \int_\Omega (1 - |x|^2) f(x)\, dx = 0. \tag{22}$$

*If a solution of the problem exists, then it is unique up to constant term and it can be represented in the form*

$$u(x) = \int_0^1 w(tx) \frac{dt}{t}, \tag{23}$$

*where*

$$w(x) = \frac{1}{\omega_n} \int_{\partial\Omega} g_0(\xi)(\Lambda - 1)\Delta_\xi G_4(x,\xi)\, ds_\xi \\ - \frac{1}{\omega_n} \int_{\partial\Omega} g_1(\xi)\Delta_\xi G_4(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_\Omega G_4(x,\xi) J_L(\Lambda + 4) f(\xi)\, d\xi,$$

*the operator $J_L$ is defined in* (8), *and the function $G_4(x,\xi)$ is given in* (13).

**Proof.** Suppose that a solution $u(x)$ to the problem (1), (3) exists, and the operator $\Lambda$ is defined in Lemma 3. For convenience, assume that the function $u(x)$ belongs to the class $C^4(\Omega) \cap C^2(\bar{\Omega})$. For this, it is enough to require $f(x) \in C^2(\bar{\Omega})$, $g_0(x) \in C^{3+\varepsilon}(\partial\Omega)$, $g_1(x) \in C^{2+\varepsilon}(\partial\Omega)$, $\varepsilon > 0$. Apply the operator $\Lambda$ to the function $u(x)$ and denote $w(x) = \Lambda u(x)$. Then, taking into account the equalities $\Delta^2 \Lambda u(x) = \Delta(\Lambda + 2)\Delta u(x) = (\Lambda + 4)\Delta^2 u(x)$, where $x \in \Omega$, and the commutability of the operators $\Lambda$ and $I_S$ (see Lemma 3), we have

$$Lw(x) = I_L \Delta\Lambda u(x) = (\Lambda + 4) I_L \Delta u(x) = (\Lambda + 4) Lu(x) = (\Lambda + 4) f(x).$$

From the boundary conditions (3), by using the property of the operator $\Lambda$, it follows that

$$w(x)|_{\partial\Omega} = \Lambda u(x)|_{\partial\Omega} = \frac{\partial u(x)}{\partial\nu}\Big|_{\partial\Omega} = g_0(x),$$
$$\Lambda w(x)|_{\partial\Omega} - w(x)|_{\partial\Omega} = (\Lambda - 1) w(x)|_{\partial\Omega} = \Lambda(\Lambda - 1) u(x)|_{\partial\Omega} = \frac{\partial^2 u(x)}{\partial\nu^2}\Big|_{\partial\Omega} = g_1(x).$$

Thus, if $u(x)$ is a solution to the problem (1), (3), then, for the function $w(x) = \Lambda u(x)$, we obtain the Dirichlet-type boundary value problem (1), (2)

$$Lw(x) = (\Lambda + 4) f(x),\ x \in \Omega; \quad w(x)|_{\partial\Omega} = g_0(x),\ \frac{\partial w(x)}{\partial\nu}\Big|_{\partial\Omega} = g_0(x) + g_1(x). \tag{24}$$

Moreover, the equality $w(x) = \Lambda u(x)$ implies the necessity of the condition $w(0) = 0$ because $(\Lambda u)(0) = 0$. Further, the functions $F(x) = (\Lambda + 4) f(x) \in C^1(\bar{\Omega})$ and $g_0(x) \in C^{3+\varepsilon}(\partial\Omega)$, $g_1(x) \in C^{2+\varepsilon}$ satisfy the conditions of Theorem 4, which means that the solution to problem (24) exists and is unique and can be represented in the form (19)

$$w(x) = w_0(x) + \frac{1}{\omega_n} \int_\Omega G_4(x,\xi)(\Lambda_\xi + 4) J_L f(\xi)\, d\xi, \tag{25}$$

where the function $w_0(x)$ is a solution to the Dirichlet problem (20) with boundary functions $g_0(x)$ and $g_0(x) + g_1(x)$, as in problem (24), and the operator $J_L$ is defined in (8). Here, under the integral, the commutativity of the operators $\Lambda_\xi$ and $J_L$ was used.

Now, let $w(x)$ be a solution to the problem (24) and $w(0) = 0$. Then, the function from (23)

$$u(x) = \int_0^1 w(tx) \frac{dt}{t},$$

satisfies the equality $\Lambda u(x) = w(x)$ since

$$\Lambda u(x) = \int_0^1 \sum_{i=1}^n x_i w_{x_i}(tx)\,dt = \int_0^1 \frac{dw(tx)}{dt}\,dt = w(x) - w(0) = w(x).$$

As noted above, $Lw(x) = (\Lambda + 4)Lu(x)$, which means that the equation from (24) implies

$$0 = (\Lambda + 4)Lu(x) - (\Lambda + 4)f(x) = (\Lambda + 4)(Lu(x) - f(x)). \tag{26}$$

**Lemma 6.** *The general solution to the equation* $(\Lambda + 4)v(x) = 0$ *has the form*

$$v(x) = C\big(\ln(x_2/x_1), \ldots, \ln(x_n/x_1)\big)x_1^{-4},$$

*where* $C(\tau_2, \ldots, \tau_n)$ *is an arbitrary differentiable function.*

**Proof.** Making the change of variables $t_i = \ln x_i$, $i = 1, \ldots, n$ in the equation under consideration, we obtain the equation of the form $(v_{t_1}, \ldots, v_{t_n})^T \cdot (1, \ldots, 1)^T + 4v = 0$, and then again, after the replacement $t = B\tau$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \ldots & & & & \\ 1 & 0 & 0 & \ldots & -1 \end{pmatrix},$$

the differential part of the equation takes the form

$$\big(B^T(v_{\tau_1}, \ldots, v_{\tau_n})^T\big) \cdot (1, \ldots, 1)^T = B(1, \ldots, 1)^T \cdot (v_{\tau_1}, \ldots, v_{\tau_n})^T = v_{\tau_1}.$$

Therefore, the original equation is reduced to the form $v_{\tau_1} + 4v = 0$. The general solution to this equation has the form $v = C(\tau_2, \ldots, \tau_n)e^{-4\tau_1}$, where $C(\tau_2, \ldots, \tau_n)$ is some differentiable function. Returning to the old variables and taking into account that $B^{-1} = B$, we find

$$v(x) = C(t_2 - t_1, \ldots, t_n - t_1)e^{-4t_1} = C\big(\ln(x_2/x_1), \ldots, \ln(x_n/x_1)\big)x_1^{-4}.$$

The lemma is proved. □

According to Lemma 6, equality (26) implies the equality $Lu(x) - f(x) = v(x)$. Since the left-hand side of this equality is a function differentiable at the origin, and the function $v(x)$ has no limit at the point $x = 0$ if $C \neq const$ and has an infinite limit if $C = const \neq 0$, then this equality is possible only if $v(x) = 0$.

Therefore, the function $u(x)$ satisfies Equation (1). Substituting $w(x) = \Lambda u(x)$ into the boundary conditions (24), we obtain the boundary conditions (3) and hence the function $u(x)$ is a solution to problems (1) and (3).

Thus, the equality $w(0) = 0$ is the necessary and sufficient condition for the solvability of problems (1) and (3).

Let us find the conditions under which the equality $w(0) = 0$ is satisfied. From the representation (25) we find

$$w(0) = w_0(0) + \frac{1}{\omega_n}\int_\Omega G_4(0, \xi)(\Lambda + 4)J_L f(\xi)\,d\xi. \tag{27}$$

In the obtained equality, the index $\xi$ of the operator $\Lambda_\xi$ is omitted, since all functions in it depend only on $\xi$. Let us calculate the value of $w_0(0)$. The paper [22] states that

$$w_0(x) = v_0(x) + \frac{1-|x|^2}{2}\Lambda v_0(x) - \frac{1-|x|^2}{2}v_1(x),$$

where $v_0(x)$ and $v_1(x)$ are harmonic functions in $\Omega$ such that $v_0(x)|_{\partial\Omega} = g_0$ and $v_1(x)|_{\partial\Omega} = g_0 + g_1$. It is known [32] that

$$v_0(0) = \frac{1}{\omega_n}\int_{\partial\Omega} g_0(\xi)\,ds_\xi, \quad v_1(0) = \frac{1}{\omega_n}\int_{\partial\Omega}\big(g_0(\xi)+g_1(\xi)\big)\,ds_\xi,$$

and therefore, since $\Lambda v_0(x)|_{x=0} = 0$, we obtain

$$w_0(0) = v_0(0) - \frac{1}{2}v_1(0) = \frac{1}{2\omega_n}\int_{\partial\Omega}\big(g_0(\xi)-g_1(\xi)\big)\,ds_\xi. \tag{28}$$

Let us move on to the second term in (27). Denote $\hat{f} = J_L f$.

**Lemma 7.** *For $n \geq 3$, the equality*

$$\int_\Omega G_4(0,\xi)(\Lambda+4)\hat{f}(\xi)\,d\xi = \int_\Omega \frac{1-|\xi|^2}{4}\hat{f}(\xi)\,d\xi \tag{29}$$

*holds true, where $G_4(x,\xi)$ is the Green's function of the Dirichlet problem for the biharmonic equation in the unit ball (13).*

**Proof.** Let $n > 4$ or $n = 3$. From (13), using the symmetry of the functions $E_4(x,\xi)$ and $E(x,\xi)$, we get

$$G_4(0,\xi) = E_4(\xi,0) - E_4(\xi/|\xi|,0) + \frac{|\xi|^2-1}{4}E(\xi/|\xi|,0) = \frac{|\xi|^{4-n}-1}{2(n-2)(n-4)} + \frac{|\xi|^2-1}{4(n-2)}.$$

To prove (29), based on the obtained equality, we calculate successively two integrals

$$I_1 = \frac{1}{2(n-2)(n-4)}\int_\Omega (|\xi|^{4-n}-1)(\Lambda+4)\hat{f}(\xi)\,d\xi,$$
$$I_2 = \frac{1}{4(n-2)}\int_\Omega (|\xi|^2-1)(\Lambda+4)\hat{f}(\xi)\,d\xi.$$

Let us find the auxiliary integral

$$\begin{aligned}J_k &= \int_\Omega (|\xi|^k-1)\Lambda\hat{f}(\xi)\,d\xi \\ &= \int_\Omega (|\xi|^k-1)\sum_{i=1}^n \xi_i\frac{\partial}{\partial\xi_i}\hat{f}(\xi)\,d\xi = \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial\xi_i}\Big(\xi_i(|\xi|^k-1)\hat{f}(\xi)\Big)\,d\xi \\ &\quad - \int_\Omega \sum_{i=1}^n (|\xi|^k-1)\hat{f}(\xi)\,d\xi - \int_\Omega \hat{f}(\xi)\sum_{i=1}^n \xi_i\frac{\partial}{\partial\xi_i}(|\xi|^k-1)\,d\xi.\end{aligned}$$

Applying the divergence theorem to the first integral above, we get

$$\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial\xi_i}\Big(\xi_i(|\xi|^k-1)\hat{f}(\xi)\Big)\,d\xi = \int_{\partial\Omega} |\xi|^2(|\xi|^k-1)\hat{f}(\xi)\,d\xi = 0.$$

Therefore,

$$J_k = -n \int_\Omega (|\xi|^k - 1)\hat{f}(\xi)\, d\xi - k \int_\Omega |\xi|^k \hat{f}(\xi)\, d\xi$$
$$= -(n+k) \int_\Omega |\xi|^k \hat{f}(\xi)\, d\xi + n \int_\Omega \hat{f}(\xi)\, d\xi.$$

Hence, using the value of $J_{4-n}$, we have

$$I_1 = \frac{1}{2(n-2)(n-4)} \int_\Omega (4|\xi|^{4-n} - 4 - 4|\xi|^{4-n} + n)\hat{f}(\xi)\, d\xi = \frac{1}{2(n-2)} \int_\Omega \hat{f}(\xi)\, d\xi.$$

Find $I_2$. Using the value of $J_2$, we get

$$I_2 = \frac{1}{4(n-2)} \int_\Omega (4|\xi|^2 - 4 - (n+2)|\xi|^2 + n)\hat{f}(\xi)\, d\xi$$
$$= \frac{1}{4(n-2)} \int_\Omega ((2-n)(|\xi|^2 - 1) - 2)\hat{f}(\xi)\, d\xi$$
$$= \frac{1}{4} \int_\Omega (1 - |\xi|^2)\hat{f}(\xi)\, d\xi - \frac{1}{2(n-2)} \int_\Omega \hat{f}(\xi)\, d\xi.$$

Thus,

$$\int_\Omega G_4(0,\xi)(\Lambda + 4)\hat{f}(\xi)\, d\xi = (I_1 + I_2) = \frac{1}{2(n-2)} \int_\Omega \hat{f}(\xi)\, d\xi$$
$$+ \frac{1}{4} \int_\Omega (1 - |\xi|^2)\hat{f}(\xi)\, d\xi - \frac{1}{2(n-2)} \int_\Omega \hat{f}(\xi)\, d\xi = \int_\Omega \frac{1 - |\xi|^2}{4} \hat{f}(\xi)\, d\xi.$$

Let $n = 4$. Then,

$$G_4(0,\xi) = -\frac{1}{4} \ln|\xi| + \frac{|\xi|^2 - 1}{4}.$$

As in the previous case, we write

$$\int_\Omega \ln|\xi| \Lambda \hat{f}(\xi)\, d\xi = -4 \int_\Omega \ln|\xi| \hat{f}(\xi)\, d\xi - \int_\Omega \hat{f}(\xi)\, d\xi,$$

and hence,

$$-\frac{1}{4} \int_\Omega \ln|\xi| (\Lambda + 4)\hat{f}(\xi)\, d\xi$$
$$= \int_\Omega \ln|\xi| \hat{f}(\xi)\, d\xi + \frac{1}{4} \int_\Omega \hat{f}(\xi)\, d\xi - \int_\Omega \text{ł}|\xi| \hat{f}(\xi)\, d\xi = \frac{1}{4} \int_\Omega \hat{f}(\xi)\, d\xi.$$

Using the value of $I_2$ for $n = 4$, we get

$$\int_\Omega G_4(0,\xi)(\Lambda + 4)\hat{f}(\xi)\, d\xi = \frac{1}{4} \int_\Omega \hat{f}(\xi)\, d\xi + \frac{1}{4} \int_\Omega (1 - |\xi|^2)\hat{f}(\xi)\, d\xi$$
$$- \frac{1}{4} \int_\Omega \hat{f}(\xi)\, d\xi = \frac{1}{4} \int_\Omega (1 - |\xi|^2)\hat{f}(\xi)\, d\xi,$$

which is the same as (29). This proves the lemma. □

Using (28) and (29) obtained above and remembering the notation $\hat{f} = J_L f$ from (27), we find

$$w(0) = \frac{1}{2\omega_n} \int_{\partial\Omega} (g_0(\xi) - g_1(\xi))\, ds_\xi + \frac{1}{4\omega_n} \int_\Omega (1 - |\xi|^2) J_L f(\xi)\, d\xi.$$

This implies that the condition $w(0) = 0$ is equivalent to the condition

$$\int_{\partial\Omega} (g_0(\xi) - g_1(\xi))\, ds_\xi + \int_\Omega \frac{1-|\xi|^2}{2} J_L f(\xi)\, d\xi = 0. \tag{30}$$

Let us transform the obtained condition a little more. First, note that, according to the properties of the matrix $S$ and by virtue of Lemma 5

$$\int_\Omega \frac{1-|\xi|^2}{2} f(S\xi)\, d\xi = \int_\Omega \frac{1-|S\xi|^2}{2} f(S\xi)\, d\xi = \int_\Omega \frac{1-|\xi|^2}{2} f(\xi)\, d\xi.$$

According to the definition of the operator $J_L$ (11), we get

$$\int_\Omega \frac{1-|\xi|^2}{2} J_L f(\xi)\, d\xi = \sum_{k=1}^{l} c_k \int_\Omega \frac{1-|\xi|^2}{2} f(S^{k-1}\xi)\, d\xi = \int_\Omega \frac{1-|\xi|^2}{2} f(\xi)\, d\xi \sum_{k=1}^{l} c_k.$$

In [27], Lemma 4.2, it is established that $\sum_{k=1}^n a_k \sum_{k=1}^n c_k = 1$ and hence $\sum_{k=1}^n c_k = 1/\mu_l$. Therefore, the condition (30) is reduced to the form (22). The theorem is proved. □

**Example 4.** *Let $S = -I$. Consider the homogeneous problem (1), (3) of the form*

$$a_1 \Delta^2 u(x) + a_2 \Delta^2 u(-x) = x_i,\ x \in \Omega; \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial^2 u}{\partial \nu^2}\Big|_{\partial\Omega} = 0, \tag{31}$$

*where $1 \le i \le n$. Similar to Example 3, we have $l = 2$, $\lambda_1 = -1$, $\lambda_2 = 1$, $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$ and*

$$c_1 = \frac{a_1}{a_1^2 - a_2^2}, \quad c_2 = \frac{-a_2}{a_1^2 - a_2^2},$$

*if $a_1^2 - a_2^2 \neq 0$. Therefore,*

$$J_L f = c_1 f(x) + c_2 f(-x) = \frac{a_1}{a_1^2 - a_2^2} f(x) - \frac{a_2}{a_1^2 - a_2^2} f(-x).$$

*The solvability condition (22) is satisfied because $g_0 = g_1 = 0$ and $\int_\Omega (1 - |x|^2) x_i\, dx = 0$. Compute the function*

$$w(x) = \frac{1}{\omega_n} \int_\Omega G_4(x,\xi) J_L(\Lambda + 4)\xi_i\, d\xi.$$

*It is easy to see that*

$$J_L(\Lambda + 4)\xi_i = 5 J_L \xi_i = \frac{5a_1}{a_1^2 - a_2^2}\xi_i + \frac{5a_2}{a_1^2 - a_2^2}\xi_i = \frac{5}{a_1 - a_2}\xi_i.$$

*In [29], Theorem 6.1, it is proved that*

$$\frac{1}{\omega_n} \int_\Omega G_4(x,\xi) |\xi|^{2k} H_m(\xi)\, d\xi = \frac{|x|^{2k+4} - 1 - (k+2)(|x|^2 - 1)}{(2k+2)(2k+4)(2k+2m+n)(2k+2m+n+2)} H_m(x),$$

*where $H_m(x)$ is a homogeneous harmonic polynomial of degree $m$. Hence it follows that*

$$\frac{1}{\omega_n} \int_\Omega G_4(x,\xi)\xi_i\, d\xi = \frac{|x|^4 - 1 - 2(|x|^2 - 1)}{2 \cdot 4(n+2)(n+4)} x_i,$$

*which means*

$$w(x) = \frac{5}{a_1 - a_2} \frac{|x|^4 - 1 - 2(|x|^2 - 1)}{8(n+2)(n+4)} x_i.$$

*According to (23), the solution to the problem (31) has the form*

$$u(x) = \int_0^1 w(tx)\frac{dt}{t} = \frac{5x_i}{8(a_1-a_2)(n+2)(n+4)}\int_0^1 (|tx|^4 - 1 - 2(|tx|^2-1))\,dt$$
$$= \frac{5x_i(\frac{1}{5}|x|^4 - 1 - 2(\frac{1}{3}|x|^2-1)}{8(a_1-a_2)(n+2)(n+4)} = \frac{|x|^4 - \frac{10}{3}|x|^2 + 5}{8(a_1-a_2)(n+2)(n+4)}x_i.$$

*It is not hard to check that*

$$\Delta^2 u(x) = \frac{\Delta^2|x|^4 x_i}{8(a_1-a_2)(n+2)(n+4)} = \frac{x_i}{a_1-a_2}, \quad \Delta^2 u(-x) = -\frac{x_i}{a_1-a_2},$$

*that is why Equation (31) is satisfied, $Lu(x) = x_i$. Boundary conditions are also satisfied*

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega} = \frac{x_i(5|x|^4 - 10|x|^2+5)}{8(a_1-a_2)(n+2)(n+4)}|_{\partial\Omega} = 0, \; \frac{\partial^2 u}{\partial \nu^2}|_{\partial\Omega} = \frac{x_i(25|x|^4 - 30|x|^2+5)}{8(a_1-a_2)(n+2)(n+4)}|_{\partial\Omega} = 0.$$

## 7. Navier Boundary Value Problem

Let $G_2(x,\xi)$ be the Green's function to the Dirichlet problem for the Poisson equation in the unit ball $\Omega$. It is well known (see, e.g., [32,33]) that this function has the form

$$G_2(x,\xi) = E(x,\xi) - E\Big(\frac{x}{|x|}, |x|\xi\Big). \tag{32}$$

Using the function $G_2(x,\xi)$, in [25], Theorem 4.1, it is shown that the Green's function $G_4^n(x,\xi)$ of the Navier boundary value problem for the biharmonic equation

$$\Delta^2 u(x) = f(x), \quad u|_{\partial\Omega} = r_0(x), \quad \Delta u|_{\partial\Omega} = r_1(x),\ x \in \partial\Omega \tag{33}$$

has the form

$$G_4^n(x,\xi) = \frac{1}{\omega_n}\int_\Omega G_2(x,y)G_2(y,\xi)\,dy. \tag{34}$$

Then, in [25], Corollary 4.2 shows that if the function $u \in C^4(\Omega)\cap C^3(\bar{\Omega})$ is a solution to the Navier problems (1) and (4) then it can be represented in the form

$$u(x) = u_0^n(x) + \frac{1}{\omega_n}\int_\Omega G_4^n(x,\xi)f(\xi)\,d\xi, \tag{35}$$

where $u_0^n(x)$ is a solution to the problem (33) when $f = 0$, which can be written as

$$u_0^n(x) = \frac{1}{\omega_n}\int_{\partial\Omega}\Lambda\Delta_\xi G_4^n(x,\xi)r_0(\xi)\,ds_\xi + \frac{1}{\omega_n}\int_{\partial\Omega}\Lambda G_4^n(x,\xi)r_1(\xi)\,ds_\xi. \tag{36}$$

If $r_0 \in C(\partial\Omega)$, $r_1 \in C^{1+\varepsilon}(\partial\Omega)$ and $f \in C^1(\bar{\Omega})$, then $u(x)$ from (35) is a solution to the Navier boundary value problem (33).

**Theorem 6.** *Let the coefficients $\{a_k : k = 1,\dots,l\}$ of the operator $L$ be such that $\mu_k \neq 0$ for $k = 1,\dots,l$ and $r_0 \in C(\partial\Omega)$, $r_1 \in C^{1+\varepsilon}(\partial\Omega)$ and $f \in C^1(\bar{\Omega})$. Then, the solution to the problem (1), (4) exists and can be represented as*

$$u(x) = u_0^n(x) + \frac{1}{\omega_n}\int_\Omega G_4^n(x,\xi)J_L f(\xi)\,d\xi, \tag{37}$$

*where the operator $J_L$ is defined in (8) and $u_0^n(x)$ is taken from (36).*

**Proof.** Similar to the proof of Theorem 4, using Lemma 3 on the commutativity of the operators $I_S$ and $\Delta$, it is easy to show that the problems (1) and (4) are equivalent to the Navier problem of type (33)

$$\Delta^2 v(x) = f(x),\ x \in \Omega; \quad v|_{\partial\Omega} = I_L r_0(x), \quad \Delta v|_{\partial\Omega} = I_L r_1(x),\ x \in \partial\Omega. \tag{38}$$

Moreover, if the function $v(x)$ is a solution to this problem, then the function $u(x) = J_L v(x)$ is a solution to the problems (1) and (4). The converse is also true. If the function $u(x)$ is a solution to the problems (1) and (4), then the function $v(x) = I_L u(x)$ is a solution to the problem (38).

It is clear that $r_0 \in C(\partial\Omega) \Rightarrow I_L r_0 \in C(\partial\Omega)$, $r_1 \in C^{1+\varepsilon}(\partial\Omega) \Rightarrow I_L r_1 \in C^{1+\varepsilon}(\partial\Omega)$ and, therefore, according to [25], Theorem 4.1, the solution to the Navier problem (38) exists, is unique, and can be written as

$$v(x) = \frac{1}{\omega_n} \int_{\partial\Omega} I_L r_0(\xi) \Lambda \Delta_\xi G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_{\partial\Omega} I_L r_1(\xi) \Lambda G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_\Omega G_4^n(x,\xi) f(\xi)\, d\xi,$$

where the equality $\frac{\partial}{\partial \nu} v|_{\partial\Omega} = \Lambda v|_{\partial\Omega}$ is used. Applying the operator $J_L$ to both parts of the resulting equality and taking into account that $u(x) = J_L v(x)$, we obtain

$$u(x) = \frac{1}{\omega_n} J_L \int_{\partial\Omega} I_L r_0(\xi) \Lambda \Delta_\xi G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} J_L \int_{\partial\Omega} I_L r_1(\xi) \Lambda G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} J_L \int_\Omega G_4^n(x,\xi) f(\xi)\, d\xi. \tag{39}$$

In the obtained expression, we transform the last integral. It is easy to see that $|S^k x - S^k \xi| = |S^k(x-\xi)| = |x-\xi|$, and hence, according to (32), $G_2(S^k x, S^k \xi) = G_2(x,\xi)$ and therefore, taking into account (34) and Lemma 5, we find

$$G_4^n(S^k x, S^k \xi) = \frac{1}{\omega_n} \int_\Omega G_2(S^k x, y) G_2(y, S^k \xi)\, dy = \frac{1}{\omega_n} \int_\Omega G_2(S^k x, S^k y) G_2(S^k y, S^k \xi)\, dy = G_4^n(x,\xi).$$

Further, again by virtue of Lemma 5

$$I_{S^k} \int_\Omega G_4^n(x,\xi) f(\xi)\, d\xi = \int_\Omega G_4^n(S^k x, S^k \xi) f(S^k \xi)\, d\xi = \int_\Omega G_4^n(x,\xi) I_{S^k} f(\xi)\, d\xi,$$

whence by (8) we find

$$J_L \int_\Omega G_4^n(x,\xi) f(\xi)\, d\xi = \sum_{k=1}^{l} c_k I_{S^{k-1}} \int_\Omega G_4^n(x,\xi) f(\xi)\, d\xi = \int_\Omega G_4^n(x,\xi) J_L f(\xi)\, d\xi.$$

Similarly, by Lemma 5, using the commutativity Lemma 3, we obtain

$$J_L \int_{\partial\Omega} \hat{r}_0(\xi) \Lambda \Delta_\xi G_4^n(x,\xi)\, ds_\xi = \int_{\partial\Omega} J_L \hat{r}_0(\xi) \Lambda \Delta_\xi G_4^n(x,\xi)\, ds_\xi,$$
$$J_L \int_{\partial\Omega} \hat{r}_1(\xi) \Lambda G_4^n(x,\xi)\, ds_\xi = \int_{\partial\Omega} J_L \hat{r}_1(\xi) \Lambda G_4^n(x,\xi)\, ds_\xi,$$

where $\hat{r}_0 = I_L r_0$, $\hat{r}_1 = I_L r_1$ is denoted. Thus, the solution $u(x)$ from (39) can be rewritten as

$$u(x) = \frac{1}{\omega_n} \int_{\partial\Omega} J_L I_L r_0(\xi) \Lambda \Delta_\xi G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_{\partial\Omega} J_L I_L r_1(\xi) \Lambda G_4^n(x,\xi)\, ds_\xi + \frac{1}{\omega_n} \int_{\Omega} G_4^n(x,\xi) J_L f(\xi)\, d\xi. \quad (40)$$

By virtue of (9) and (11), we have the equalities $\hat{r}_i(\xi) = I_L r_i(\xi), r_i(\xi) = J_L \hat{r}_i(\xi), i = 1,2$ from which it follows that $J_L I_L r_i(\xi) = r_i(\xi)$ for an arbitrary function $r_i(\xi)$ on $\partial\Omega$. Therefore (40) implies (37). The theorem is proved. □

**Example 5.** *Let us find the solution to the homogeneous problem* (1), (4) *if* $S = -I$ *and* $f(x) = x_i|x|^2, 1 \le i \le n$. *In this case, we have*

$$a_1 \Delta^2 u(x) + a_2 \Delta^2 u(-x) = x_i|x|^2,\ x \in \Omega; \quad u\big|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,\ s \in \partial\Omega. \quad (41)$$

*Similar to Example 4 for* $a_1^2 - a_2^2 \neq 0$, *we obtain*

$$J_L f = c_1 f(x) + c_2 f(-x) = \frac{x_i|x|^2}{a_1 - a_2}.$$

*In accordance with* (36), *the solution to the problem* (41) *has the form*

$$u(x) = \frac{1}{\omega_n} \int_\Omega G_4^n(x,\xi) J_L f(\xi)\, d\xi,$$

*where, by* (34),

$$G_4^n(x,\xi) = \frac{1}{\omega_n} \int_\Omega G_2(x,y) G_2(y,\xi)\, dy.$$

*Similar to Example 4,*

$$\frac{1}{\omega_n} \int_\Omega G_2(x,\xi) |\xi|^{2k} H_m(\xi)\, d\xi = \frac{(|x|^{2k+2} - 1) H_m(x)}{(2k+2)(2k+2m+n)},$$

*whence it follows*

$$\frac{1}{\omega_n} \int_\Omega G_2(y,\xi) |\xi|^2 \xi_i\, d\xi = \frac{|y|^4 - 1}{4(n+4)} y_i,$$

$$\frac{1}{\omega_n} \int_\Omega G_2(x,y) \frac{|y|^4 - 1}{4(n+4)} y_i\, dy = \frac{(|x|^6 - 1) x_i}{4 \cdot 6(n+4)(n+6)} - \frac{(|x|^2 - 1) x_i}{4 \cdot 2(n+4)(n+2)}.$$

*Hence,*

$$u(x) = \frac{x_i}{a_1 - a_2} \left( \frac{|x|^6 - 1}{24(n+4)(n+6)} - \frac{|x|^2 - 1}{8(n+4)(n+2)} \right).$$

*Using the equality* $\Delta(|x|^{2k} H_m(x)) = 2k(2k + 2m + n - 2)|x|^{2k-2} H_m(x)$, *it is not hard to check that the resulting function is the solution to* (41).

## 8. Riquier–Neumann Boundary Value Problem

Consider the Riquier–Neumann boundary value problem for the biharmonic equation in $\Omega$ corresponding to the problems (1) and (5)

$$\Delta^2 u(x) = f(x), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = p_0(x), \quad \frac{\partial \Delta u}{\partial \nu}\Big|_{\partial\Omega} = p_1(x),\ x \in \partial\Omega. \quad (42)$$

An explicit form of the Green's function to the Neumann problem for the Poisson equation in the unit ball $N(x,\xi)$ can be found in [34]

$$N(x,\xi) = E(x,\xi) - E_0(x,\xi),$$

where the harmonic function $E_0(x,\xi)$ is written in the form

$$E_0(x,\xi) = \int_0^1 \left(\hat{E}\left(\frac{x}{|x|}, t|x|\xi\right) + 1\right)\frac{dt}{t}$$

and $\hat{E}(x,\xi) = \Lambda_x E(x,\xi)$. For the cases $n = 2$ and $n = 3$ see [32]. Using the Green's function $N(x,\xi)$, in [25], Theorem 5.4, the Green's function of the Riquier–Neumann problem for the biharmonic Equation (42) is constructed

$$G_4^{rn}(x,\xi) = \frac{1}{\omega_n}\Big(\int_\Omega N(x,y)N(y,\xi)\,dy - \frac{1}{|\Omega|}\int_\Omega N(x,y)\,dy \cdot \int_\Omega N(y,\xi)\,dy\Big), \tag{43}$$

and in [25], Theorem 5.5, it is proved that if $p_0 \in C(\partial\Omega)$, $p_1 \in C^{1+\varepsilon}(\partial\Omega)$ and $f \in C^1(\bar{\Omega})$, then the solution to the Riquier–Neumann problem (42) exists and is unique up to a constant term under the condition

$$\int_{\partial\Omega} p_1(\xi)\,ds_\xi = \int_\Omega f(\xi)\,d\xi \tag{44}$$

and can be written as

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_\Omega G_4^{rn}(x,\xi)f(\xi)\,d\xi, \tag{45}$$

where the function $u_0(x)$ denotes the solution to the Riquier–Neumann problem for the homogeneous biharmonic equation

$$u_0(x) = -\frac{1}{\omega_n}\int_{\partial\Omega} \Delta_\xi G_4^{rn}(x,\xi)p_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial\Omega} G_4^{rn}(x,\xi)p_1(\xi)\,ds_\xi. \tag{46}$$

**Theorem 7.** *Let the coefficients $\{a_k : k = 1,\dots,l\}$ of the operator $L$ be such that $\mu_k \neq 0$ for $k = 1,\dots,l$ and $p_0 \in C(\partial\Omega)$, $p_1 \in C^{1+\varepsilon}(\partial\Omega)$ ($\varepsilon > 0$) and $f \in C^1(\bar{\Omega})$. Then, under the condition*

$$\int_{\partial\Omega} p_1(\xi)\,ds_\xi = \frac{1}{\mu_l}\int_\Omega f(\xi)\,d\xi, \tag{47}$$

*the solution of the Riquier–Neumann problem (1), (5) exists up to a constant term and can be represented as*

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_\Omega G_4^{rn}(x,\xi)J_L f(\xi)\,d\xi, \tag{48}$$

*where the function $u_0(x)$ is defined in (46).*

**Proof.** Similar to the proofs of Theorems 4 and 6, using Lemma 3 on the commutativity of the operators $I_S$ and $\Delta$, $I_S$ and $\Lambda$ and the equality $\frac{\partial}{\partial\nu}v|_{\partial\Omega} = \Lambda v|_{\partial\Omega}$, it is not hard to see that the problem (1), (5) is equivalent to the Riquier–Neumann problem of type (42)

$$\Delta^2 v(x) = f(x),\ x \in \Omega; \quad \Lambda v\big|_{\partial\Omega} = I_L p_0(x), \quad \Lambda\Delta v\big|_{\partial\Omega} = I_L p_1(x),\ x \in \partial\Omega. \tag{49}$$

If the function $v(x)$ is a solution to this problem, then the function $u(x) = J_L v(x)$ is a solution to the problems (1) and (5) and vice versa, if the function $u(x)$ is a solution to the problems (1) and (5), then the function $v(x) = I_L u(x)$ is a solution to the problem (49).

It is clear that $p_0 \in C(\partial\Omega) \Rightarrow I_L p_0 \in C(\partial\Omega)$, $p_1 \in C^{1+\varepsilon}(\partial\Omega) \Rightarrow I_L p_1 \in C^{1+\varepsilon}(\partial\Omega)$ and hence, according to [25], Theorem 5.5, the solution of the Navier problem (49) exists under the condition (44)

$$\int_{\partial\Omega} I_L p_1(\xi)\,ds_\xi = \int_\Omega f(\xi)\,d\xi \tag{50}$$

and can be written as (45)

$$v(x) = v_0(x) + \frac{1}{\omega_n}\int_\Omega G_4^{rn}(x,\xi)f(\xi)\,d\xi,$$

where according to (46)

$$v_0(x) = -\frac{1}{\omega_n}\int_{\partial\Omega}\Delta_\xi G_4^{rn}(x,\xi) I_L p_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial\Omega} G_4^{rn}(x,\xi) I_L p_1(\xi)\,ds_\xi.$$

Applying the operator $J_L$ to the function $v(x)$ and taking into account that $u(x) = J_L v(x)$ we get

$$u(x) = J_L v_0(x) + \frac{1}{\omega_n} J_L \int_\Omega G_4^{rn}(x,\xi) f(\xi)\,d\xi. \tag{51}$$

Transforming the integral in the resulting expression, it is easy to see that, similarly to the equalities obtained in the proofs of Theorems 4 and 6 with respect to the function $G_2(x,\xi)$, we get

$$\begin{aligned} N(S^k x, S^k\xi) &= E(S^k x, S^k \xi) - E_0(S^k x, S^k\xi) = E(x,\xi) \\ &- \int_0^1 \left(\Lambda_1 E\left(S^k \frac{x}{|x|}, t|x| S^k\xi\right) + 1\right)\frac{dt}{t} = E(x,\xi) - E_0(x,\xi) = N(x,\xi) \end{aligned}$$

and hence, by virtue of Lemma 5, taking into account (43), we find

$$\begin{aligned} G_4^{rn}(S^k x, S^k\xi) &= \frac{1}{\omega_n}\Big(\int_\Omega N(S^k x, y) N(y, S^k\xi)\,dy - \frac{1}{|\Omega|}\int_\Omega N(S^k x, y)\,dy \\ &\cdot \int_\Omega N(y, S^k\xi)\,dy\Big) = \frac{1}{\omega_n}\Big(\int_\Omega N(S^k x, S^k y) N(S^k y, S^k \xi)\,dy \\ &- \frac{1}{|\Omega|}\int_\Omega N(S^k x, S^k y)\,dy \cdot \int_\Omega N(S^k y, S^k\xi)\,dy\Big) = G_4^{rn}(S^k x, S^k \xi). \end{aligned}$$

Further, again by virtue of Lemma 5

$$I_{S^k}\int_\Omega G_4^{rn}(x,\xi) f(\xi)\,d\xi = \int_\Omega G_4^{rn}(S^k x, S^k\xi) f(S^k\xi)\,d\xi = \int_\Omega G_4^{rn}(x,\xi) I_{S^k} f(\xi)\,d\xi,$$

whence, using (8), we get

$$J_L \int_\Omega G_4^{rn}(x,\xi) f(\xi)\,d\xi = \sum_{k=1}^{l} c_k I_{S^{k-1}} \int_\Omega G_4^{rn}(x,\xi) f(\xi)\,d\xi = \int_\Omega G_4^{rn}(x,\xi) J_L f(\xi)\,d\xi.$$

Now let us transform $J_L v_0(x)$ from (51). By Lemma 5, using the commutativity Lemma 3 and the equalities $J_L I_L p_i(\xi) = p_i(\xi)$, $i = 1,2$, we get

$$\begin{aligned} J_L v_0(x) &= -\frac{1}{\omega_n} J_L \int_{\partial\Omega}\Delta_\xi G_4^{rn}(x,\xi) I_L p_0(\xi)\,ds_\xi - \frac{1}{\omega_n} J_L \int_{\partial\Omega} G_4^{rn}(x,\xi) I_L p_1(\xi)\,ds_\xi \\ &= -\frac{1}{\omega_n}\int_{\partial\Omega}\Delta_\xi G_4^{rn}(x,\xi) J_L I_L p_0(\xi)\,ds_\xi - \frac{1}{\omega_n}\int_{\partial\Omega} G_4^{rn}(x,\xi) J_L I_L p_1(\xi)\,ds_\xi = u_0(x), \end{aligned}$$

where the function $u_0(x)$ is defined in (45). Thus, the solution $u(x)$ from (51) becomes

$$u(x) = u_0(x) + \frac{1}{\omega_n}\int_\Omega G_4^{rn}(x,\xi) J_L f(\xi)\,d\xi,$$

matching (48).

Finally, let us transform the condition (50) for the existence of the function $v(x)$. We have

$$\int_{\partial\Omega} I_L p_1(\xi)\,ds_\xi = \sum_{k=1}^{l} a_k \int_{\partial\Omega} p_1(S^{k-1}\xi)\,ds_\xi = \mu_l \int_{\partial\Omega} p_1(\xi)\,ds_\xi,$$

where the value of $\mu_k$ is defined in (6). Since $\mu_l \neq 0$, this implies the condition (47). The theorem is proved. □

## 9. Conclusions

Summarizing the investigation carried out, we note that due to the properties of the matrix $A$ studied in Lemmas 1–4 and Theorem 1, and also with the help of the Green's functions to the Dirichlet, Neumann, Navier and Riquier–Neumann boundary value problems for a biharmonic equation from [25,30,34] we are able to obtain solvability conditions and an integral representation of the solutions to the problems under consideration for a nonlocal biharmonic equation in the unit ball. The main results are contained in Theorems 4–7.

If we consider possible further applications of the proposed method, we hope that the proposed method can also be used to study firstly nonlocal boundary value problems for the nonlocal biharmonic equation and secondly for the boundary value problems for a nonlocal polyharmonic equation. The described problems are the subject of further work and we are going to consider them in our next articles.

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