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On Solvability of Some Boundary Value Problems with Involution for the Biharmonic Equation

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On Solvability of Some Boundary Value Problems with Involution for the Biharmonic Equation

Valery V. Karachik and Batirkhan Kh. Turmetov

Abstract In this paper we study new classes of well-posed boundary-value problems for the biharmonic equation. The considered problems are Bitsadze–Samarskii type nonlocal boundary value problems. The investigated problems are solved by reducing them to the Neumann and Dirichlet type problems. In this paper, theorems on existence and uniqueness of the solution are proved, and exact conditions for solvability of the problems are found. In addition, integral representations of the solution are obtained.

Keywords Biharmonic equation · Nonlocal problem · Involution · Neumann type problem

1 Introduction

Significant number of mathematical models in physics and engineering lead to partial differential equations. The steady processes of various physical nature are described by the partial differential equations of elliptic type. One of the important special cases of fourth order elliptic equations is the biharmonic equation $\Delta^2 u(x) = f(x)$. Investigation of mathematical models of the plane deformation of the elasticity theory in many cases is reduced to integration of the biharmonic equation with the appropriate boundary conditions and under some uniqueness conditions for the unknown function.

Moreover investigation of many mathematical models of continuum mechanics are reduced to solving the harmonic and biharmonic equations. Application of biharmonic problems in mathematical models of mechanics and physics can be found in

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the numerous scientific investigations (see, for example, [2, 9, 23]). Multiple applications of boundary value problems for the biharmonic equation in mathematical models of mechanics and physics encourages investigation of various formulations of boundary value problems for the biharmonic equation. There is a big interest in studying of boundary value problems (see [7, 24, 33]) for the biharmonic equation.

The Dirichlet problem (see, for example, [3]) is the well known boundary value problem for biharmonic equation. In recent years other types of boundary value problems for the biharmonic equation, such as the problems by Riquier (see [11]), by Neumann (see [12]), by Robin (see [17]) and etc. [1] are actively studied.

Nonlocal boundary value problems for elliptic equations in which boundary conditions are given in the form of a connection between the values of the unknown function and its derivatives at various points of the boundary are called the problems of Bitsadze–Samarskii type [6]. Numerous applications of nonlocal boundary value problems for elliptic equations in problems of physics, engineering, and other branches of science are described in detail in [28, 29]. Solvability of nonlocal boundary value problems for elliptic equations is discussed in [4, 8, 10, 20, 21, 25]. Boundary problems with involution for elliptic equations of the second and fourth order, as a special case of nonlocal problems, are considered in [18, 22, 26, 27, 32].

Let $\Omega = \{x \in R^n : |x| < 1\}$ be the unit ball, $n \geq 2$, and $\partial\Omega$ be the unit sphere. For any point $x = (x_1, x_2, \ldots, x_n) \in \Omega$ we consider the point $x^* = Sx$, where S is a real orthogonal matrix $S \cdot S^T = E$. Suppose also that $S^2 = E$.

In this paper we study the following nonlocal boundary value problem

$$\Delta^2 u(x) = f(x),\ x \in \Omega, \tag{1}$$

$$D_\nu^m u(x) + \alpha D_\nu^m u(x^*) = g_1(x),\ x \in \partial\Omega, \tag{2}$$

$$D_\nu^{m+1} u(x) + \beta D_\nu^{m+1} u(x^*) = g_2(x),\ x \in \partial\Omega, \tag{3}$$

where $0 \leq m \leq 2$, a, b are real numbers, $D_\nu^m = \frac{\partial^m}{\partial \nu^m}$, ν is a unit vector of the outward normal to $\partial\Omega$, $D_\nu^0 = I$ is a unit operator.

By a solution of the problem (1)–(3) we mean a function $u(x) \in C^4(\Omega) \cap C^{m+1}(\bar{\Omega})$ satisfying conditions (1)–(3) in the classical sense. In the case $\alpha = \beta = 0$ when $m = 0$ we obtain the well-known Dirichlet problem and when a Neumann type problem [1, 12, 30, 31].

Content is constructed as follows. In Sect. 1, in Lemmas 1 and 2, we give some auxiliary propositions. In Sect. 2, in Theorems 1 and 2, we give results on solvability of Neumann type problems. In Sect. 3, we prove Theorem 3 on uniqueness of the solution of problem (1)–(3). In Sect. 4 we prove Theorem 4 on existence of the solution of problem (1)–(3), and in Theorem 5 we give a method for constructing a solution of problem (1)–(3) with homogeneous boundary conditions.

2 Auxiliary Statements

First we note that if $x \in \Omega$, or $x \in \partial\Omega$, then $x^* = Sx \in \Omega$, or $x^* = Sx \in \partial\Omega$, respectively, since the transformation of the space R^n by the matrix S preserves the norm $|x^*|^2 = |Sx|^2 = (Sx, Sx) = (S^T Sx, x) = |x|^2$.

The case $x^* = -x$ investigated in [18, 22, 26, 27, 32] is a particular case of the situation considered here since for $S = -E$ we have $S \cdot S^T = -E(-E) = E$.

Consider the operator

$$I_S u(x) = u(Sx) = u(x^*).$$

In view of what has been said above, this operator is defined on functions $u(x)$, $x \in \Omega$. We also consider the operator $\Lambda u = \sum_{i=1}^{n} x_i u_{x_i}(x)$ that is homogeneous, preserves the biharmonicity of function $u(x)$, and has the property $D_\nu^m u|_{\partial\Omega} = \Lambda^{[m]} u|_{\partial\Omega}$, where $\Lambda^{[m]} = \Lambda(\Lambda - 1)\dots(\Lambda - m + 1)$ [14]. Let S^i_{col} and S^i_{row} be the ith column and ith row of the matrix S, respectively.

We prove two simple lemmas. Let $u(x)$ be a twice continuously differentiable function in Ω.

Lemma 1 *Operators Λ and I_S are commutative $\Lambda I_S u(x) = I_S \Lambda u(x)$, and also the equality $\nabla I_S = I_S S^T \nabla$ holds, and operators Δ and I_S are also commutative.*

Proof We can write the operator Λ in the form $\Lambda u = (x, \nabla)u$. Since

$$\frac{\partial}{\partial x_i} I_S u(x) = \frac{\partial}{\partial x_i} u(x) = \frac{\partial}{\partial x_i} u((S^1_{row}, x), \dots, (S^n_{row}, x)) = \sum_{j=1}^{n} s_{ji} I_S u_{x_j}(x)$$

$$= (S^i_{col}, I_S \nabla u(x)) = I_S (S^i_{col}, \nabla) u(x). \tag{4}$$

then

$$\Lambda I_S u(x) = \Lambda u(Sx) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} u(Sx) = \sum_{i=1}^{n} x_i \left(S^i_{col}, I_S \nabla u(x)\right)$$

$$= \left(\sum_{i=1}^{n} x_i S^i_{col}, I_S \nabla u(x)\right) = (Sx, I_S \nabla u(x)) = I_S (x, \nabla u(x)) = I_S \Lambda u(x).$$

Further, due to the formula (4), we find

$$\frac{\partial^2}{\partial x_i^2} I_S u(x) = \frac{\partial}{\partial x_i} I_S (C^i_{col}, \nabla) u(x) = I_S (C^i_{col}, \nabla)^2 u(x)$$

and therefore

$$\Delta I_S u(x) = \sum_{i=1}^{n} I_S (S_{col}^i, \nabla)^2 u(x) = I_S \left| \left((S_{col}^1, \nabla), \ldots, (S_{col}^n, \nabla) \right) \right|^2 u(x)$$

$$= I_S \left| S^T \nabla \right|^2 u(x) = I_S (S^T \nabla, S^T \nabla) u(x) = I_S (S S^T \nabla, \nabla) u(x) = I_S \Delta u(x).$$

At last,

$$\nabla I_S u(x) = I_S ((S_{col}^1, \nabla), \ldots, (S_{col}^n, \nabla)) u(x) = I_S (S^T \nabla) u(x).$$

Lemma is proved.

Corollary 1 *If the function $u(x)$ is biharmonic in Ω, then the function $u(x^*) = I_S u(x)$ is also biharmonic in Ω.*

Indeed, due to Lemma 1, $\Delta^2 u(x) = 0 \Rightarrow \Delta^2 I_S u(x) = I_S \Delta^2 u(x) = 0$.

Lemma 2 *The operator $1 + \alpha I_S$, when $\alpha \neq \pm 1$ is invertible and the operator*

$$J_\alpha = \frac{1}{1 - \alpha^2} (1 - \alpha I_S) \tag{5}$$

is inverse to $1 + \alpha I_S$.

Proof It is easy to see that

$$J_\alpha (1 + \alpha I_S) u(x) = \left(\frac{1 - \alpha I_S}{1 - \alpha^2} \right) (1 + \alpha I_S) u(x) = \frac{1}{1 - \alpha^2} (u(x) - \alpha^2 {I_S}^2 u(x))$$

$$= \frac{1}{1 - \alpha^2} (1 - \alpha^2) u(x).$$

Thus, if $\alpha^2 \neq 1$ then we can divide both sides of the equality by $1 - \alpha^2$ and hence the operator J_α is inverse to $1 + \alpha I_S$. Lemma is proved.

3 Neumann Type Problems

In this section we study the following problem

$$\Delta^2 v(x) = \varphi(x), \quad x \in \Omega, \tag{6}$$

$$D_\nu^m v(x)|_{\partial\Omega} = \psi_1(x), \quad x \in \partial\Omega, \tag{7}$$

$$D_\nu^{m+1} v(x)|_{\partial\Omega} = \psi_2(x), \quad x \in \partial\Omega, \tag{8}$$

where $0 \le m \le 2$.

Problem (6)–(8) in the case $m = 0$ is the Dirichlet problem, and in the cases $m = 1$ or $m = 2$ is a problem of Neumann type. It is known (see, e.g. [1]) that the Dirichlet problem is unconditionally solvable, but for the solvability of the Neumann type problems it is necessary to fulfill so called "orthogonality conditions" for the given functions [5].

Bitsadze A. V. in his work [5] established that for the solvability of the problem (6)–(8) in the case $\varphi(x) = 0$ and $m = 1$ the following condition is necessary and sufficient to be fulfilled

$$\int\limits_{\partial\Omega} [\psi_2(x) - \psi_1(x)]\, dS_x = 0.$$

Furthermore, in [19] this result was improved for the inhomogeneous equation. The following proposition is true.

Theorem 1 *Let $m = 1$, $\varphi(x) \in C(\bar{\Omega})$, $\psi_1(x) \in C^1(\partial\Omega)$, $\psi_2(x) \in C(\partial\Omega)$. Then for the solvability of the problem (6)–(8) the following condition is necessary and sufficient*

$$\int\limits_{\partial\Omega} [\psi_2(x) - \psi_1(x)]\, dS_x = \frac{1}{2}\int\limits_{\partial\Omega} \left(1 - |x|^2\right) \psi(x)\, dx. \tag{9}$$

If solution of the problem exists, then it is unique up to a constant.

The case $m = 2$ is considered in [30] and the following proposition is established.

Theorem 2 *Let $m = 2$, $0 < \lambda < 1$, $n \ge 3$, $\varphi(x) \in C^{\lambda+1}(\bar{\Omega})$, $\psi_1(x) \in C^{\lambda+1}(\partial\Omega)$, $\psi_2(x) \in C^{\lambda}(\partial\Omega)$. Then for the solvability of the problem (6)–(8) the following conditions are necessary and sufficient*

$$\int\limits_{\partial\Omega} \psi_2(x)\, dS_x = \frac{n-1}{2}\int\limits_{\Omega} |x|^2\varphi(x)\, dx - \frac{n-3}{2}\int\limits_{\Omega} \varphi(x)\, dx, \tag{10}$$

$$\int\limits_{\partial\Omega} x_j[\psi_2(x) - \psi_1(x)]\, dS_x = \frac{n-1}{2}\int\limits_{\Omega} x_j|x|^2\varphi(x)dx - \frac{n-3}{2}\int\limits_{\Omega} x_j\varphi(x)\, dx. \tag{11}$$

If solution of the problem exists, then it is unique up to a first order polynomial.

Note that the case when the boundary conditions of the problem have the form $D_\nu v(x)|_{\partial\Omega} = \psi_1(x)$ and $D_\nu^3 v(x)|_{\partial\Omega} = \psi_2(x)$, where $x \in \partial\Omega$ is considered in [15].

4 Uniqueness

In this section we investigate the uniqueness of the solution of the problem (1)–(3). The following proposition is true.

Theorem 3 *Let $\alpha^2 \neq 1$, $\beta^2 \neq 1$, and solution of the problem (1)–(3) exists. Then*
(1) if $m = 0$, then solution of the problem is unique;
(2) if $m = 1$, then solution of the problem is unique up to a constant;
(3) if $m = 2$, then solution of the homogeneous problem (1)–(3) is a function of the form

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j,$$

where c_j for $j = 0, \dots, n$ are arbitrary constants.

Proof To prove the uniqueness of the solution of problem (1)–(3), consider a function $u(x)$—a solution of the homogeneous problem (1)–(3) (all right-hand sides in the problem are zero). If the problem (1)–(3) has at least two solutions such a function exists. It is clear that $u(x)$ is a biharmonic function, satisfying the following homogeneous conditions

$$D_\nu^m u(x) + \alpha D_\nu^m u(x^*)\Big|_{\partial\Omega} = \Lambda^{[m]}(1 + \alpha I_S)u(x)\Big|_{\partial\Omega} = 0,\ D_\nu^{m+1} u(x) + \beta D_\nu^{m+1} u(x^*)\Big|_{\partial\Omega}$$

$$= \Lambda^{[m+1]}(1 + \beta I_S)u(x)\Big|_{\partial\Omega} = 0. \tag{12}$$

Since $\alpha^2 \neq 1$ and $\beta^2 \neq 1$, then applying to the equality (12) the operators J_α and J_β from (5) and using Lemma 1, we get

$$0 = J_\alpha \Lambda^{[m]}(1 + \alpha I_S)u(x) = \Lambda^{[m]} J_\alpha (1 + \alpha I_S)u(x) = \Lambda^{[m]} u(x) = D_\nu^m u(x),$$

$$0 = J_\beta \Lambda^{[m+1]}(1 + \alpha I_S)u(x) = \Lambda^{[m+1]} J_\beta (1 + \beta I_S)u(x) = \Lambda^{[m+1]} u(x) = D_\nu^{m+1} u(x),$$

where $x \in \partial\Omega$, or

$$D_\nu^m u(x)\big|_{\partial\Omega} = D_\nu^{m+1} u(x)\big|_{\partial\Omega} = 0.$$

Therefore, if $u(x)$ is a solution of the homogenous problem (1)–(3), then it is also a solution of the homogeneous problem (6)–(8). Then, due to uniqueness of the solution of the Dirichlet problem (the case $m = 0$), we obtain the uniqueness of the solution of the problem (1)–(3). Similarly, by the statements of Theorems 1 and 2, we obtain the remaining statements of this theorem. Theorem is proved.

Remark 1 If $\alpha^2 = 1$ and $\beta^2 = 1$, then the homogeneous problem (1)–(3) can have an infinite number of linearly independent solutions. For example, suppose that $m =$

$0, \alpha = \beta = 1, S = -E$. Consider the function $u_k(x) = |x|^2 H_{2k-1}(x)$, where $k \in N$ and $H_{2k-1}(x)$ is a homogeneous harmonic polynomial of degree $2k-1$. It is obvious that $\Delta^2 u_k(x) = 0,\ x \in \Omega$. Further, since $u_k(x^*) = (-1)^{2k-1} u_k(x) = -u_k(x)$ and $\Lambda u_k(x) = (2k+1) u_k(x)$, then the function $u_k(x)$ for all $k \in N$ satisfies the homogeneous conditions of the problem (1)–(3)

$$u_k(x) + u_k(x^*) = u_k(x) - u_k(x) = 0,$$

$$\Lambda u_k(x) + \Lambda u_k(x^*) = \Lambda u_k(x) + (\Lambda u_k)(x^*) = (2k+1)(u_k(x) + u_k(x^*)) = 0.$$

5 Existence

In this section we present a statement on the existence of a solution of the problem (1)–(3).

Theorem 4 *Let $\alpha \neq \pm 1$, $\beta \neq \pm 1$ and $f(x)$, $g_1(x)$, $g_2(x)$ be smooth enough functions. Then*

(1) if $m = 0$, then solution of the problem (1)–(3) exists and is unique;

(2) if $m = 1$, and $\alpha \neq -1$, $\beta \neq -1$, then the necessary and sufficient condition for solvability of the problem (1)–(3) has the form

$$\frac{1}{2} \int\limits_{\Omega} \left(1 - |x|^2\right) f(x)\, dx = \int\limits_{\partial\Omega} \left[\frac{g_2(x)}{1+b} - \frac{g_1(x)}{1+a} \right] dS_x. \tag{13}$$

If the solution exists, then it is unique up to a constant;

(3) if $m = 2$, and $\alpha \neq -1$, $\beta \neq -1$, then the necessary and sufficient condition for solvability of the problem (1)–(3) has the form

$$\frac{1}{1+b} \int\limits_{\partial\Omega} g_2(x)\, dS_x = \frac{n-1}{2} \int\limits_{\Omega} |x|^2 f(x)\, dx - \frac{n-3}{2} \int\limits_{\Omega} f(x)\, dx, \tag{14}$$

and

$$\int\limits_{\partial\Omega} \left(\left(E + \beta S^T\right)^{-1} x \right)_j g_2(x) - \left(\left(E + \alpha S^T\right)^{-1} x \right)_j g_1(x)\, dS_y$$

$$= \frac{n-1}{2} \int\limits_{\Omega} x_j |x|^2 f(x)\, dx - \frac{n-3}{2} \int\limits_{\Omega} x_j f(x)\, dx,\ j = 1, \ldots, n, \tag{15}$$

where $(x)_j = x_j$ is the jth element of a vector x. If a solution exists, then it is unique up to a first order polynomial.

Proof Consider the auxiliary Dirichlet problem

$$\Delta^2 v(x) = f(x), \quad x \in \Omega, \tag{16}$$

$$D_\nu^m v(x) = J_\alpha g_1(x), \quad x \in \partial\Omega, \tag{17}$$

$$D_\nu^{m+1} v(x) = J_\beta g_2(x), \quad x \in \partial\Omega, \tag{18}$$

where the operator J_α is defined in (5). We check that its solution $v(x)$ is also a solution of the considered problem (1)–(3). Indeed, the function $v(x)$ satisfies the Eq. (1). Applying the operator $1 + \alpha I_S$ to the condition (17) and using Lemmas 1 and 2, we get

$$g_1(x) = (1 + \alpha I_S) J_\alpha g_1(x) = (1 + \alpha I_S) D_\nu^m v(x)|_{\partial\Omega} = (1 + \alpha I_S) \Lambda^{[m]} v(x)|_{\partial\Omega}$$

$$= \Lambda^{[m]} (1 + \alpha I_S) v(x)|_{\partial\Omega} = D_\nu^m (1 + \alpha I_S) v(x)|_{\partial\Omega} = D_\nu^m v(x) + \alpha D_\nu^m v(x^*)|_{\partial\Omega},$$

where $x \in \partial\Omega$, i.e. condition (2) holds. Similarly, applying the operator $1 + \beta I_S$ to the condition (18) and using Lemmas 1 and 2, we get

$$g_2(x) = (1 + \beta I_S) J_\beta g_2(x) = (1 + \beta I_S) D_\nu^{m+1} v(x)|_{\partial\Omega} = (1 + \beta I_S) \Lambda^{[m+1]} v(x)|_{\partial\Omega}$$

$$= \Lambda^{[m+1]} (1 + \beta I_S) v(x)|_{\partial\Omega} = D_\nu^{m+1} (1 + \beta I_S) v(x)|_{\partial\Omega} = D_\nu^{m+1} v(x) + \beta D_\nu^{m+1} v(x^*)|_{\partial\Omega},$$

where $x \in \partial\Omega$, i.e. condition (3) holds also. So, the function $v(x)$ is a solution of the problem (1)–(3), and if $v(x)$ exists, then the problem (1)–(3) is solvable.

The case when the solution of the problem (16)–(18) does not exist, but $u(x)$ exists, is impossible. Indeed, let $u(x)$ be a solution of the Eq. (16). Applying the operator J_α to the condition (2) and using Lemmas 1 and 2, we have

$$J_\alpha g_1(x) = J_\alpha (D_\nu^m u(x) + \alpha D_\nu^m u(x^*))|_{\partial\Omega} = J_\alpha D_\nu^m (1 + \alpha I_S) u(x)|_{\partial\Omega}$$

$$= J_\alpha \Lambda^{[m]} (1 + \alpha I_S) u(x)|_{\partial\Omega} = \Lambda^{[m]} J_\alpha (1 + \alpha I_S) u(x)|_{\partial\Omega} = \Lambda^{[m]} u(x) = D_\nu^m u(x)|_{\partial\Omega},$$

where $x \in \partial\Omega$, i.e. condition (17) holds. Similarly, from (3) we get

$$J_\beta g_2(x) = J_\beta (D_\nu^{m+1} u(x) + \beta D_\nu^{m+1} u(x^*))|_{\partial\Omega} = J_\beta D_\nu^{m+1} (1 + \beta I_S) u(x)|_{\partial\Omega}$$

$$= J_\beta \Lambda^{[m+1]} (1 + \beta I_S) u(x)|_{\partial\Omega} = \Lambda^{[m+1]} J_\beta (1 + \beta I_S) u(x)|_{\partial\Omega} = \Lambda^{[m+1]} u(x)|_{\partial\Omega}$$

$$= D_\nu^{m+1} u(x)|_{\partial\Omega},$$

i.e. condition (18) holds also. Hence, $u(x)$ is a solution of the problem (16)–(18), which contradicts to the assumption. Problems (1)–(3) and (16)–(18) are solvable simultaneously. Smoothness of the functions $J_\alpha g_1(x)$ and $g_1(x)$ as well as $J_\beta g_2(x)$ and $g_2(x)$ are the same.

Using Theorems 1 and 2, we can find solvability conditions of the problem (16)–(18). Obviously these conditions will be the solvability conditions of the problem (1)–(3).

(1) Let $m = 0$. Then the problem (16)–(18) is the Dirichlet problem. For any functions on the right-hand sides of the problem with a given smoothness its solution exists and is unique.

(2) Let $m = 1$. In this case, by Theorem 1, the necessary and sufficient solvability condition of the problem (16)–18) is the integral equality

$$\frac{1}{2}\int\limits_{\Omega}(1-|x|^2)f(x)\,dx = \int\limits_{\partial\Omega}\left[J_\beta g_2(x) - J_\alpha g_1(x)\right]dS_x. \tag{19}$$

Let us transform the integral on the right hand side of (19).

Lemma 3 *Let the function $\varphi(x)$ be continuous on $\partial\Omega$ and S be an orthogonal matrix, then*

$$\int\limits_{\partial\Omega}\varphi(Sx)\,dS_x = \int\limits_{\partial\Omega}\varphi(x)\,dS_x.$$

Proof Let the function $w(x)$ be a solution of the Dirichlet problem for the Laplace equation in Ω with condition $w(x)|_{\partial\Omega} = \varphi(x)$, $x \in \partial\Omega$. Then the function $w(Sx)$ is a solution of the Dirichlet problem for the Laplace equation in Ω with the condition $w(Sx)|_{\partial\Omega} = \varphi(Sx)$, $x \in \partial\Omega$. Therefore, due to the Poisson's formula, we have

$$\int\limits_{\partial\Omega}\varphi(Sx)\,dS_x = \int\limits_{\partial\Omega}w(Sx)\,dS_x = \omega_n w(0) = \int\limits_{\partial\Omega}\varphi(x)\,dS_x,$$

where ω_n is the area of the unit sphere. Lemma is proved.

Using the proved Lemma 3, the condition $\alpha \neq -1$, we find

$$\int\limits_{\partial\Omega}J_\alpha g_1(x)\,dS_x = \frac{1}{1-\alpha^2}\left(\int\limits_{\partial\Omega}g_1(x)\,dS_x - \alpha\int\limits_{\partial\Omega}I_S g_1(x)\,dS_x\right)$$

$$= \frac{1}{1-\alpha^2}\left[\int\limits_{\partial\Omega}g_1(x)\,dS_x - \alpha\int\limits_{\partial\Omega}g_1(Sx)\,dS_x\right] = \int\limits_{\partial\Omega}\frac{g_1(x)}{1+\alpha}\,dS_x. \tag{20}$$

This implies that condition (19) can be transformed to the form (13)

$$\frac{1}{2}\int\limits_{\Omega}(1-|x|^2)f(x)\,dx=\int\limits_{\partial\Omega}\left[\frac{g_2(x)}{1+\beta}-\frac{g_1(x)}{1+\alpha}\right]dS_x.$$

(3) Let $m=2$. By Theorem 2 the necessary and sufficient solvability conditions of the problem (16)–(18) take the form

$$\int\limits_{\partial\Omega}J_\beta g_2(x)\,dS_x=\frac{n-1}{2}\int\limits_{\Omega}|x|^2f(x)\,dx-\frac{n-3}{2}\int\limits_{\Omega}f(x)\,dx,$$

$$\int\limits_{\partial\Omega}x_j[J_\beta g_2(x)-J_\alpha g_1(x)]\,dS_x=\frac{n-1}{2}\int\limits_{\Omega}x_j|x|^2f(x)dx-\frac{n-3}{2}\int\limits_{\Omega}x_jf(x)dx,$$

where $j=1,\dots,n$. Using (20) the first condition can be easily transformed to the form (14). Further, taking into account orthogonality of powers of the matrices S, S^T, and using Lemma 3 we can write

$$\int\limits_{\partial\Omega}x_jJ_\alpha g_1(x)\,dS_x=\frac{1}{1-\alpha^2}\left(\int\limits_{\partial\Omega}x_jg_1(x)\,dS_x-\alpha\int\limits_{\partial\Omega}S^T\,(Sx)_j\,g_1(Sx)\,dS_x\right)$$

$$=\frac{1}{1-\alpha^2}\left(\int\limits_{\partial\Omega}y_jg_1(y)\,dS_y-\alpha\int\limits_{\partial\Omega}\left(S^Ty\right)_j g_1(y)\,dS_y\right)$$

$$=\frac{1}{1-\alpha^2}\int\limits_{\partial\Omega}\left(\sum_{k=0}^{1}\left(-\alpha S^T\right)^ky\right)_j g_1(y)\,dS_y=\frac{1-\alpha^2}{1-\alpha^2}\int\limits_{\partial\Omega}\left(E+\alpha S^T\right)^{-1}g_1(y)\,dS_y$$

$$=\int\limits_{\partial\Omega}\left(\left(E+\alpha S^T\right)^{-1}y\right)_j g_1(y)\,dS_y,$$

where $j=1,\dots,n$. Thus, the second condition can be rewritten in the form (15)

$$\int\limits_{\partial\Omega}\left(\left(E+\beta S^T\right)^{-1}x\right)_j g_2(x)-\left(\left(E+\alpha S^T\right)^{-1}x\right)_j g_1(x)\,dS_y$$

$$=\frac{n-1}{2}\int\limits_{\Omega}x_j|x|^2f(x)\,dx-\frac{n-3}{2}\int\limits_{\Omega}x_jf(x)\,dx,$$

for $j=1,\dots,n$. Theorem is proved.

Remark 2 If we consider the case of involution $S = -E$, then the solvability condition given by (15) can be simplified using the equality

$$\left(E + \alpha S^T\right)^{-1} = (E - \alpha E)^{-1} = \frac{1}{1-\alpha} E.$$

6 Representation of the Solution

In this section we give a method of constructing solutions of the problem (1)–(3) with homogeneous boundary conditions.

Theorem 5 *Let* $0 \le m \le 2$, $g_1(x) = g_2(x) = 0$. *Then*

(1) if $m = 0$, *then the solution of the problem (1)–(3) can be represented in the form*

$$u(x) = \int\limits_{\Omega} G_D(x, y) f(y)\, dy,$$

where $G_D(x, y)$ *is the Green's function of the Dirichlet problem for the biharmonic equation (1) in* Ω.

(2) if $m = 1$ *and (13) holds, then solution of the problem (1)–(3) can be represented in the form*

$$u(x) = \int\limits_0^1 \frac{v(sx)}{s}\, ds + C, \tag{21}$$

where C *is arbitrary constant and* $v(x)$ *is a solution of the following the Dirichlet problem*

$$\Delta^2 v(x) = (\Lambda + 4) f(x),\ x \in \Omega;\ v(x)|_{\partial\Omega} = 0,\ v(0) = 0,\ D_\nu^1 v(x)\big|_{\partial\Omega} = 0. \tag{22}$$

(3) if $m = 2$ *and (14), (15) hold, then solution of the problem (1)–(3) can be represented in the form*

$$u(x) = \int\limits_0^1 (1 - s)\frac{v(sx)}{s^2}\, ds + \sum_{j=1}^{n} c_j x_j + c_0, \tag{23}$$

where c_j, $j = 0, \ldots, n$ *are arbitrary constants and* $v(x)$ *is a solution of the following Dirichlet problem*

$$\Delta^2 v(x) = (\Lambda + 4)\,(\Lambda + 3)\, f(x),\ x \in \Omega,\ v(x)|_{\partial\Omega} = 0,\ v(0) = 0,$$

$$D_\nu^1 v(x)\big|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial x_j}(0) = 0,\ j = 1, \dots, n. \tag{24}$$

Proof The auxiliary problem (16)–(18), whose solution coincides with the solution of the problem (1)–(3) (see the proof of Theorem 3), with the help of properties of the operator Λ takes the form

$$\Delta^2 v(x) = f(x),\ x \in \Omega,$$

$$\Lambda^{[m]} v(x)|_{\partial\Omega} = 0,\ \Lambda^{[m+1]} v(x)|_{\partial\Omega} = 0.$$

(1) Let $m = 0$, then in this case auxiliary problem is the Dirichlet problem and its solution coincides with a solution of the problem (1)–(3)

$$v(x) = u(x) = \int\limits_\Omega G_D(x, y) f(y)\, dy. \tag{25}$$

(2) Let $m = 1$. Boundary conditions for the auxiliary problem take the form

$$\Lambda^{[1]} v(x)|_{\partial\Omega} \equiv \Lambda v(x)|_{\partial\Omega} = 0,\ \Lambda^{[2]} v(x)|_{\partial\Omega} \equiv (\Lambda^2 - \Lambda) v(x)|_{\partial\Omega} = \Lambda^2 v(x)|_{\partial\Omega} = 0.$$

Let us apply the operator $\Lambda + 4$ to the biharmonic equation of the problem. Due to the equality $\Delta^k \Lambda u = (\Lambda + 2k)\Delta^k u$ (see [13]) and denoting $w = \Lambda v$, for $w(x)$ we get the following Dirichlet problem (22)

$$\Delta^2 w(x) = (\Lambda + 4) f(x),\ x \in \Omega,$$

$$w(x)|_{\partial\Omega} = 0,\ \Lambda w(x)|_{\partial\Omega} = 0.$$

By the formula (25) we find

$$w(x) = \int\limits_\Omega G_D(x, y)(\Lambda + 4) f(y)\, dy.$$

As in [13] equation $w = \Lambda v$ in the class of smooth functions $v(x)$ has a solution only if $w(0) = 0$, and this solution can be written in the form

$$u(x) = \int\limits_0^1 \frac{w(sx)}{s}\, ds + C.$$

(3) Let $m = 2$. Boundary conditions for the auxiliary problem take the form

$$\Lambda^{[1]}v(x)|_{\partial\Omega} \equiv (\Lambda^2-\Lambda)v(x)|_{\partial\Omega}=0,\ \Lambda^{[2]}v(x)|_{\partial\Omega} \equiv (\Lambda^2-\Lambda)(\Lambda-2)v(x)|_{\partial\Omega}$$

$$=\Lambda(\Lambda^2-\Lambda)v(x)|_{\partial\Omega}=0.$$

Let us apply the operator $(\Lambda+4)(\Lambda+3)$ to the biharmonic equation of the problem. If we denote $w=(\Lambda^2-\Lambda)v$, then due to the equality $\Delta^k\Lambda u=(\Lambda+2k)\Delta^k u$ we get the Dirichlet problem (24)

$$\Delta^2 w(x)=(\Lambda+4)(\Lambda+3)f(x),\ x\in\Omega,$$

$$w(x)|_{\partial\Omega}=0,\ \Lambda w(x)|_{\partial\Omega}=0.$$

By the formula (25) we find

$$w(x)=\int_{\Omega} G_D(x,y)(\Lambda+4)(\Lambda+3)f(y)\,dy.$$

To find the function $v(x)=u(x)$ it is necessary to solve the equation $w=(\Lambda^2-\Lambda)v$. As in [17] equation $\hat{w}=(\Lambda-\lambda)\hat{v}$ in the class of smooth functions $\hat{v}(x)$ has a solution only if $\lim\limits_{s\to 0}\frac{\hat{w}(sx)}{s^\lambda}=0$, where $x\in\Omega$, and this solution can be written in the form

$$\hat{v}(x)=\int_0^1 \frac{\hat{w}(sx)}{s^{\lambda+1}}\,ds+H_\lambda(x),$$

where $H_\lambda(x)$ is arbitrary homogenous polynomial of order $\lambda\in N_0$. Using this result a smooth solution of the equation $w=(\Lambda^2-\Lambda)u$ can be written in the form

$$u(x)=\int_0^1\left(\int_0^1 \frac{w(stx)}{s^2}\,ds+H_1(tx)\right)\frac{dt}{t}+C=\int_0^1\left(\int_0^t \frac{w(\tau x)}{\tau^2}\,d\tau\right)dt$$

$$+H_1(x)+C=\int_0^1\frac{w(\tau x)}{\tau^2}\int_\tau^1 dt\,d\tau+H_1(x)+C=\int_0^1\frac{w(\tau x)}{\tau^2}(1-\tau)\,d\tau+H_1(x)+C,$$

which coincides with (23). Conditions, under which this solution exists, are equivalent to the existence conditions of the last singular integral. It is easy to see that the integral converges if $w(0)=0$ and $\frac{\partial w}{\partial x_j}(0)=0$ for $j=1,2,\ldots,n$. Theorem is proved.

Example 1 Consider the following boundary value problem of the type (1)–(3)

$$\Delta^2 u(x)=x_i,\ x\in\Omega,$$

$$u(x) + 2u(x^*)|_{\partial\Omega} = 3x_j^2, \quad D_\nu^1 u(x) - 2D_\nu^1 u(x^*)|_{\partial\Omega} = -x_k^2, \; x \in \partial\Omega,$$

where $x^* = -Ex$ and $1 \leq i, j, k \leq n$. In this case $\alpha = 2$, $\beta = -2$ and because

$$J_\alpha = J_2 = -\frac{1}{3}(E - 2I_{-E}), \;\; J_\beta = J_{-2} = -\frac{1}{3}(E + 2I_{-E})$$

we have

$$J_2(3x_j^2) = -(1-2)x_j^2 = x_j^2$$

and

$$J_{-2}(-x_k^2) = \frac{1}{3}(1+2)x_k^2 = x_k^2.$$

The auxiliary problem (16)–(18) takes the form

$$\Delta^2 u(x) = x_i, \; x \in \Omega,$$

$$u(x)|_{\partial\Omega} = x_j^2, \quad D_\nu^1 u(x)|_{\partial\Omega} = x_k^2, \; x \in \partial\Omega.$$

Solution of this problem and also of the considered problem using Examples 2–4 of [16] has the form

$$u(x) = x_j^2 + (|x|^2 - 1)\left(\frac{x_k^2}{2} - x_j^2\right) + (|x|^2 - 1)^2\left(\frac{x_i}{8(n+2)(n+4)} + \frac{1}{2n}\right).$$

Boundary conditions and equation (see [16]) are fulfilled

$$u(x) + 2u(x^*)|_{\partial\Omega} = x_j^2 + 2(-x_j)^2 = 3x_j^2, \quad \Lambda u(x) - 2\Lambda$$

$$u(x^*)|_{\partial\Omega} = -\frac{1}{2}\Lambda(|x|^2 - 1)x_k^2|_{\partial\Omega} = -x_k^2.$$

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