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Содержание

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An explicit solution formula for a multidimensional time-fractional differential equation

Abstract. *This paper intends on the obtaining the explicit solution of n-dimensional differential equation with Caputo derivative in the infinite domain with non-zero initial conditions and vanishing conditions at infinity. It is shown that this equation can be derived from the integro- differential wave equation with memory. Based on Laplace and Fourier transforms the properties of the Fox H - function and convolution theorem, explicit solution for the considered equation is obtained.*

Keywords. *time-fractional differential equation, Laplace transform, Fourier transform, convolution theorem, explicit solution*

In this work we have obtained the explicit solution of $n-$ dimensional differential equation with Caputo derivative in the infinite domain with non-zero initial conditions and vanishing conditions at infinity. It is shown that this equation can be derived from the integro-differential wave equation with memory in which the kernel is $t^{m-1-\alpha}E_{m-\alpha,m-\alpha}\left(-t^{m-\alpha}\right)$, $m-1<\alpha\le m$ (m is a positive integer), where $E_{\alpha,\beta}$ is the two parametric Mittag-Leffler function.

We consider the following $n-$ dimensional differential equation with Caputo fractional derivative of order $\alpha>0$

$$\rho\frac{\partial^m u(x,t)}{\partial t^m}+{}_0^C D_t^\alpha u-\Delta u(x,t)=f(x,t), \tag{1}$$

which satisfies the initial and boundary conditions

$$\frac{\partial^j u(x,0)}{\partial t^j}=\varphi_j(x),\ j=0,1,..,m-1,\ \lim_{|x|\to\infty}(u,\nabla u)(x,t)=0,\ t>0, x\in\mathrm{R}^n, \tag{2}$$

where $\rho-$positive constant, ${}_0^C D_t^\alpha$ is defined for $m-1<\alpha\le m$ (m is a positive integer) as [1]

$${}_0^C D_t^\alpha u(x,t):=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{\partial^m u(x,\tau)}{\partial\tau^m}\frac{d\tau}{(t-\tau)^{\alpha+1-m}},\ m-1<\alpha\le m,$$

$${}_0^C D_t^\alpha u(x,t):=\frac{\partial^m u(x,\tau)}{\partial\tau^m},\ \alpha=m;$$

Δ is the $n-$ dimensional Laplace operator with respect to x and $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$.

To solve problem (1), (2), we apply the Laplace and Fourier transforms of the function $u(x,t)$, respectively, with respect to $t>0$ and $x \in \mathrm{R}^n$

$$\hat{u}(x,s) = \int_0^\infty u(x,t) e^{-st} dt, \quad x \in \mathrm{R}^n, s \in \mathrm{C},$$

$$\tilde{u}(\xi,t) := \int_{\mathrm{R}^n} u(x,t) e^{-i\xi \cdot x} dx, \quad \xi \in \mathrm{R}^n, t > 0$$

and their inverse transformations with respect to $s \in \mathrm{C}$ and $\xi \in \mathrm{R}^n$

$$u(x,t) = L^{-1}[\hat{u}](x,s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{u}(x,s) e^{st} ds, \quad x \in \mathrm{R}^n, t > 0,$$

$$u(x,t) = F^{-1}[\tilde{u}](\xi,t) := \frac{1}{(2\pi)^n} \int_{\mathrm{R}^n} \tilde{u}(\xi,t) e^{i\xi \cdot x} d\xi, \quad \xi \in \mathrm{R}^n, t > 0.$$

Here

$$x = (x_1, \dots, x_n) \in \mathrm{R}^n, \xi = (\xi_1, \dots, \xi_n) \in \mathrm{R}^n, \ \xi \cdot x = \sum_{j=1}^n \xi_j \cdot x_j, \ dx = dx_1 dx_2 \dots dx_n, \ d\xi = d\xi_1 d\xi_2 \dots d\xi_n$$

and $\gamma \in \mathrm{R}$ is a fixed number.

The following main results are obtained.

Theorem 1. *The explicit solution of the problem (1) and (2) can be expressed by formula*

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathrm{R}^n} \sum_{k=0}^\infty \frac{(-1)^k}{\rho^{k+1}} \left(\varepsilon^{k+1}_{m,(m-\alpha)k+m,-\frac{|\xi|^2}{\rho};0+} \tilde{f} \right)(\xi,t) e^{i\xi \cdot x} d\xi + \int_{\mathrm{R}^n} \sum_{j=0}^{m-1} G_j(x-\xi,t) \varphi_j(\xi) d\xi, \quad (3)$$

provided that the integrals and series in the ride-hand side of (3) are convergent, where $\varepsilon^{\gamma}_{\alpha,\beta,\omega,a+}$ is Prabhakar fractional integral operator (see [2]) and the Green functions $G_j(x,t)$, $j = 0, \dots, m-1$ are given by

$$G_j(x,t) := G_{1_j}(x,t) + G_{2_j}(x,t),$$

where

$$G_{1_j}(x,t) = \frac{1}{2\pi^{n/2} |x|^n} \times$$

$$\times \sum_{k=0}^\infty \frac{(-1)^k}{\rho^k k!} t^{(m-\alpha)k+j} H^{2,0}_{1,2} \left[\frac{\sqrt{\rho}}{2} \frac{|x|}{t^{m/2}} \middle| \begin{matrix} (1+(m-\alpha)k+j, m/2) \\ (n/2, 1/2), (1+k, 1/2 \end{matrix} \right],$$

$$G_{2_j}(x,t) = \frac{1}{2\pi^{n/2} |x|^n} \times$$

$$\times \sum_{k=0}^\infty \frac{(-1)^k}{\rho^{k+1} k!} t^{(m-\alpha)(k+1)+j} H^{2,0}_{1,2} \left[\frac{\sqrt{\rho}}{2} \frac{|x|}{t^{m/2}} \middle| \begin{matrix} (1+(m-\alpha)(k+1)+j, m/2) \\ (n/2, 1/2), (1+k, 1/2 \end{matrix} \right],$$

where $x \in \mathrm{R}^n, \xi \in \mathrm{R}^n, d\xi = d\xi_1 d\xi_2 ... d\xi_n$ *and* $H_{p,q}^{m,n}(z)$ *is Fox's* $H-$ *function (see [3]).*

Theorem 2. *The integro-differential wave equation*

$$\rho u_{tt} - \Delta u + \int_0^t k(t-\tau)\Delta u(x,\tau)d\tau = 0, \quad x \in \mathrm{R}^n, t > 0 \tag{4}$$

with memory $k(t) = t^{1-\alpha}E_{2-\alpha,2-\alpha}\left(-t^{2-\alpha}\right)$, $1 < \alpha \le 2$, *is equivalent to the time-fractional wave equation*

$$u_{tt} + {}_0^C D_t^\alpha u - \frac{1}{\rho}\Delta u(x,t) = 0.$$

Remark. *The equation (4) with memory kernel* $k(t) = t^{1-\alpha}E_{2-\alpha,2-\alpha}\left(-t^{2-\alpha}\right)$ *describes the time-fractional wave equation (1) for* $m = 2$.

From this remark it follows that the solution of equation (4) with conditions (2) (for $m = 2$) can be given by formula (3) for $f(x,t) = 0$ and $m = 2$.

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Структура существенного спектра и дискретный спектр оператора энергии четырехэлектронных систем в примесной модели хаббарда. Триплетное состояние

Structutre of essential spectra and discrete spectrum of the energy operator of four-electron systems in the impurity hubbard model. Triplet state

Аннотация. *Рассматривается оператор энергии четырех-электронных систем в примесном модели Хаббарда и исследуется струк-*