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Functional Analysis in Interdisciplinary Applications—II

ICAAM, Lefkosa, Cyprus, September 6–9, 2018

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Contents

Investigation of Finite-Difference Analogue of the Integral Geometry Problem with a Weight Function

Galitdin B. Bakanov

Abstract Finite—difference analogue of the two-dimensional problem of integral geometry with a weight function are studied. The stability estimate for the considered problem are obtained.

Keywords Integral geometry · Finite-difference problem · Stability estimate · Uniqueness of the solution

1 Introduction

The problems of integral geometry are to find the functions, which are determined on certain variety, through its integrals on certain set of subvarieties with lower dimension.

Additionally, the problems of integral geometry are correlated with various solutions (data interpretation objectives of exploration seismology, electro-exploration, acoustics, and inverse problems of kinetic equations, widely used in plasma physics and astrophysics). In recent years, the studies on problems of integral geometry have critical significance for tomography, which is intensively developing scientific—technic pillar that has several applications in medicine and industry. Therefore, development of various solution methods for the integral geometry problems is actual issue.

One of the stimuli for studying such problems is their connection with multidimensional inverse problems for differential equations [1]. In some inverse problems for hyperbolic equations were shown to reduce to integral geometry problems and, in particular, a problem of integral geometry was considered in the case of shift-invariant curves. Mukhometov [2] showed the uniqueness and estimated the stability of the solution of a two-dimensional integral geometry on the whole. His results were

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mainly based on the reduction of the two-dimensional integral geometry problem

$$V(\gamma, z) = \int_{K(\gamma,z)} U(x, y)\, \rho(x, y, z)\, ds, \quad \gamma \in [0, l], \quad z \in [0, l], \tag{1}$$

where $U \in C^2\left(\overline{D}\right)$, $\rho(x, y, z)$ is a known function to the boundary value problem

$$\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial x}\frac{\cos\theta}{\rho} + \frac{\partial W}{\partial y}\frac{\sin\theta}{\rho}\right) = 0, \quad (x, y, z) \in \Omega_1, \tag{2}$$

$$W(\xi(\gamma), \eta(\gamma), z) = V(\gamma, z), V(z, z) = 0, \gamma, z \in [0, l]. \tag{3}$$

Here, D is a plane bounded by a simply connected domain with a smooth boundary Γ :

$$x = \xi(z), \quad y = \eta(z), \quad z \in [0, l], \quad \xi(0) = \xi(l), \quad \eta(0) = \eta(l),$$

where a parameter z is the length of the curve Γ:

$$\Omega_1 = \Omega / \{(\xi(\gamma), \ \eta(\gamma), z) : z \in [0, l]\}, \ \Omega = \bar{D} \times [0, l];$$

$K(x, y, z)$ is the part of the curve $K(\gamma, z)$ included between the points $(x, y) \in \overline{D}$ and $(\xi(z), \eta(z))$, $z \in [0, l]$;

$$W(x, y, z) = \int_{K(x,y,z)} U(x, y)\, \rho(x, y, z)\, ds,$$

$\theta(x, y, z)$ is an angle between the tangent to $K(x, y, z)$ at the point (x, y) and the x—axis and the variable parameter s is the curve length.

2 Formulation of the Finite—Difference Problem

Suppose that the requirements on the family of curves $K(\gamma, z)$ and the domain D are necessary for the problem (1) to reduce to problem (2), (3) are met [2, 3]. Assume also that every line parallel to either the x— or the y—axis intersects boundary of D at no more than two points. Let

$$a_1 = \inf_{(x,y)\in D} x, \quad b_1 = \sup_{(x,y)\in D} x, \quad a_2 = \inf_{(x,y)\in D} y, \quad b_2 = \sup_{(x,y)\in D} y,$$

$$h_j = \left(b_j - a_j\right)/N_j, \quad j = 1, 2; \quad h_3 = l/N_3,$$

and the N_j, $j = 1, 2, 3$, are natural number. Suppose that

$$0 < \varepsilon < \min\left\{(b_1 - a_1)/3,\ (b_2 - a_2)/3\right\},$$

$$D^{\varepsilon} = \left\{(x, y) \in D : \min_{(\alpha,\beta)\in\Gamma} \rho\left((x, y), (\alpha, \beta)\right) > \varepsilon\right\},$$

$$R_h = \left\{\left(x_i, y_j\right) : x_i = a_1 + ih_1, y_j = a_2 + jh_2,\ i = 1, \ldots, N_1, j = 1, \ldots, N_2\right\}.$$

A neighborhood $B(ih_1, jh_2)$ of the point $(a_1 + ih_1, a_2 + jh_2)$ is defined as the five-point set

$$\left\{(a_1 + ih_1, a_2 + jh_2), (a_1 + (i \pm 1)h_1, a_2 + (j \pm 1)h_2)\right\}.$$

A set D_h^{ε} consists of all points $(a_1 + ih_1, a_2 + jh_2)$ such that their neighborhoods $B(ih_1, jh_2)$ are contained in $D^{\varepsilon} \cap R_h$. A set Γ_h^{ε} is made of all points $(a_1 + ih_1, a_2 + jh_2) \in D_h^{\varepsilon}$ such that $B(ih_1, jh_2) \cap (D^{\varepsilon} \cap R_h) \setminus D_h^{\varepsilon} \neq 0$. Finally,

$$\Delta_h^{\varepsilon} = \bigcup_{\Gamma_h^{\varepsilon}} B(ih_1, jh_2),\ D_h = R_h \cap D,$$

$$\Omega_h^{\varepsilon} = \{(a_1 + ih_1, a_2 + jh_2, kh_3) : (a_1 + ih_1, a_2 + jh_2) \in D_h^{\varepsilon},\ k = 0, 1, \ldots, N_3 - 1\}.$$

From here on we suppose that the coefficients and the solution of problem (2), (3) have the following properties:

$$W(x, y, z) \in C^3(\Omega^{\varepsilon}), \theta(x, y, z) \in C^2(\Omega^{\varepsilon}), \Omega^{\varepsilon} = \overline{D^{\varepsilon}} \times [0, l], \tag{4}$$

$$\rho(x, y, z) \in C^2(\Omega), \rho(x, y, z) > c_1 > 0,\ \frac{\partial\theta}{\partial z} > \left|\frac{\partial\rho}{\partial z}\frac{1}{\rho}\right|. \tag{5}$$

We consider the finite-difference problem of finding the functions $\Phi_{i,j}^k$, which satisfy the equation

$$\left[\Phi_{\overset{\circ}{x}}\frac{A}{C} + \Phi_{\overset{\circ}{y}}\frac{B}{C}\right]_z = 0,\quad (a_1 + ih_1, a_2 + jh_2, kh_3) \in \Omega_h^{\varepsilon}, \tag{6}$$

and the boundary condition

$$\Phi_{i,j}^k = F_{i,j}^k,\ (a_1 + ih_1, a_2 + jh_2) \in \Delta_h^{\varepsilon},\ k = 1, \ldots, N_3 - 1, \tag{7}$$

$$\Phi_{i,j}^0 = \Phi_{i,j}^{N_3},\quad (a_1 + ih_1, a_2 + jh_2) \in D_h^{\varepsilon}, \tag{8}$$

where

$$\Phi_{i,j}^k = \Phi(x_i, y_j, z_k) = \Phi(a_1 + ih_1, a_2 + jh_2, kh_3),$$

$$\Phi_{\overset{\circ}{x}} = (\Phi_{i+1,j} - \Phi_{i-1,j})/2h_1,$$

$$\Phi_{\overset{\circ}{y}} = (\Phi_{i,j+1} - \Phi_{i,j-1})/2h_2,$$

$$f_z = \frac{\left(f_{i,j}^{k+1} - f_{i,j}^{k}\right)}{h_3}, \ A = \cos\theta_{i,j}^k, \ B = \sin\theta_{i,j}^k,$$

$$\theta_{i,j}^k = \theta\left(a_1 + ih_1, a_2 + jh_2, kh_3\right), \ C = \rho_{i,j}^k = \rho\left(a_1 + ih_1, a_2 + jh_2, kh_3\right).$$

We note that in the finite-difference formulation information on the solution is given not only on the boundary Γ but also in its ε—neighborhood, because the partial derivatives θ_z, W_{xz}, W_{yz}, W_{xy} have singularities of the type

$$\left[(x - \xi(z))^2 + (y - \eta(z))^2\right]^{-\frac{1}{2}}$$

in a neighborhood of an arbitrary point $(\xi(z), \eta(z), z)$ (see [2]).

3 Main Results

It is not difficult to verify the following assertion.

Lemma 1 *If u and v are mesh functions, then*

$$\left(\frac{u}{v}\right)_z = \frac{u_z v^k - u^k v_z}{v^k v^{k+1}}, \tag{9}$$

$$(uv)_z = u^k v_z + u_z v^k + h_3 u_z v_z, \tag{10}$$

$$(uv)_{\overset{\circ}{x}} = u_{\overset{\circ}{x}} v_i + u_i v_{\overset{\circ}{x}} + \frac{h_1^2}{2}[u_x v_x]_{\bar{x}}. \tag{11}$$

Theorem 1 *Suppose that the solution of problem (6)–(8) exists on Ω_h^ε and $\left|\Phi_{\overset{\circ}{x}z}\right| \le c_2$, $\left|\Phi_{\overset{\circ}{y}z}\right| \le c_2$, where c_2 is constant, and that*

$$(AB_z - A_z B) - \left|\frac{C_z}{C}\right| \ge \alpha > 0$$

for all N_j, $j = 1, 2, 3$. Then, there exists a positive constant N^{**} such that, for all $N_j > N^{**}$, $j = 1, 2, 3$,

$$\sum_{\Omega_h^\varepsilon}\left(\Phi_{\overset{\circ}{x}}^2+\Phi_{\overset{\circ}{y}}^2\right)h_1\,h_2\,h_3\le c_3\sum_{\Delta_h^\varepsilon}\left(F_{\overset{\circ}{x}}^2h_1h_3+F_{\overset{\circ}{y}}^2h_2h_3+F_z^2(h_2+h_1)h_3\right)+c_2h_3^2. \tag{12}$$

Here, c_3 is a constant dependent on the function $\rho(x,y,z)$ and on the family of curves $K(\gamma,z)$.

Proof Multiplying both sides of (6) by $2C\left(-B\Phi_{\overset{\circ}{x}}+A\Phi_{\overset{\circ}{y}}\right)$ [4–7], we get $J_1+J_2=0$, where

$$J_1=J_2=C\left[-\Phi_{\overset{\circ}{x}}B+\Phi_{\overset{\circ}{y}}A\right]\left[\Phi_{\overset{\circ}{x}}\frac{A}{C}+\Phi_{\overset{\circ}{y}}\frac{B}{C}\right]_z.$$

Using the product differentiation formula (10), we obtain

$$J_1=\left[C\left(-\Phi_{\overset{\circ}{x}}B+\Phi_{\overset{\circ}{y}}A\right)\left(\Phi_{\overset{\circ}{x}}\frac{A}{C}+\Phi_{\overset{\circ}{y}}\frac{B}{C}\right)\right]_z-$$

$$-\left[C\left(-\Phi_{\overset{\circ}{x}}B+\Phi_{\overset{\circ}{y}}A\right)\right]_z\left(\Phi_{\overset{\circ}{x}}\frac{A}{C}+\Phi_{\overset{\circ}{y}}\frac{B}{C}\right)-$$

$$-h_3\left[C\left(-\Phi_{\overset{\circ}{x}}B+\Phi_{\overset{\circ}{y}}A\right)\right]_z\left[\Phi_{\overset{\circ}{x}}\frac{A}{C}+\Phi_{\overset{\circ}{y}}\frac{B}{C}\right]_z=0.$$

Exposing brackets in expression J_1 and J_2 taking into account formulas (9), (10) and equality (6), taking into account that

$$\left(1-\frac{C^k}{C^{k+1}}\right)\approx o(h_3),\ \left(\frac{1}{C^{k+1}}-\frac{1}{C^k}\right)\approx o(h_3),\ \left(\frac{1}{C^{k+1}}+\frac{1}{C^k}\right)\approx\frac{2}{C^k}+o(h_3),$$

$$D=2AB=2\cos\theta\sin\theta=\sin 2\theta,\ \mathrm{E}=A^2-B^2=\cos^2\theta-\sin^2\theta=\cos 2\theta,$$

using the formulas

$$\Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{x}z}=\frac{1}{2}\left(\Phi_{\overset{\circ}{x}}^2\right)_z-\frac{h_3}{2}\Phi_{\overset{\circ}{x}z}^2,\ \Phi_{\overset{\circ}{y}}\Phi_{\overset{\circ}{y}z}=\frac{1}{2}\left(\Phi_{\overset{\circ}{y}}^2\right)_z-\frac{h_3}{2}\Phi_{\overset{\circ}{y}z}^2,$$

we get

$$J_3+J_4+J_5+J_6+J_7+J_8=0, \tag{13}$$

where

$$J_3=\frac{1}{2}\left\{\Phi_{\overset{\circ}{x}}^2\left[(AB_z-A_zB)+D\frac{C_z}{C}\right]-2\Phi_{\overset{\circ}{y}}^{k+1}\mathrm{E}\frac{C_z}{C}+\right.$$

$$\left.+\left(\Phi_{\overset{\circ}{y}}^{k+1}\right)^2\left[(AB_z-A_zB)-D\frac{C_z}{C}\right]\right\},$$

$$J_4 = \frac{1}{2}\left\{\left(\Phi^{k+1}_{\mathring{y}}\right)^2\left[(AB_z - A_zB) + D\frac{C_z}{C}\right] - 2\Phi^{k+1}_{\mathring{x}}\Phi_{\mathring{y}}\mathrm{E}\frac{C_z}{C} + \right.$$

$$\left. + \Phi^2_{\mathring{y}}\left[(AB_z - A_zB) - D\frac{C_z}{C}\right]\right\},$$

$$J_5 = -\frac{h_3^2}{2}\Phi^2_{\mathring{xz}}\left[(\mathrm{AB}_z - \mathrm{A}_z\mathrm{B}) + D\frac{C_z}{C}\right] - \frac{h_3^2}{2}\Phi^2_{\mathring{yz}}\left[(\mathrm{AB}_z - \mathrm{A}_z\mathrm{B}) - D\frac{C_z}{C}\right] +$$

$$+ \Phi^2_{\mathring{x}}(\mathrm{A}_z\mathrm{B} + \mathrm{AB}C_z)\, o(h_3) + \Phi^2_{\mathring{y}}(\mathrm{AB}_z - \mathrm{AB}C_z)\, o(h_3) +$$

$$+ \Phi_{\mathring{x}}\Phi^{k+1}_{\mathring{y}}B^2C_z o(h_3) - \Phi_{\mathring{y}}\Phi^{k+1}_{\mathring{x}}A^2C_z o(h_3).$$

$$J_6 = \Phi_{\mathring{x}}\Phi_{\mathring{y}}BB_z\left(1 - \frac{C^k}{C^{k+1}}\right) + \Phi_{\mathring{x}}\Phi_{\mathring{y}}AA_z\left(\frac{C^k}{C^{k+1}} - 1\right) -$$

$$- h_3\Phi_{\mathring{x}}\Phi_{\mathring{yz}}\left(AA_z + BB_z\frac{C^k}{C^{k+1}}\right) + h_3\Phi_{\mathring{y}}\Phi_{\mathring{xz}}\left(AA_z\frac{C^k}{C^{k+1}} + BB_z\right),$$

$$J_7 = \left[\left(-\Phi_{\mathring{x}}B + \Phi_{\mathring{y}}A\right)\left(\Phi_{\mathring{x}}A + \Phi_{\mathring{y}}B\right)\right]_z - \Phi_{\mathring{x}}\Phi_{\mathring{yz}} + \Phi_{\mathring{y}}\Phi_{\mathring{xz}},$$

$$J_8 = h_3\Phi^2_{\mathring{x}}AB_z\frac{C_z}{C} + h_3^2\Phi_{\mathring{x}}\Phi_{\mathring{xz}}AB_z\frac{C_z}{C} + h_3\Phi_{\mathring{x}}\Phi_{\mathring{y}}AA_z\frac{C_z}{C} + h_3\Phi_{\mathring{x}}\Phi_{\mathring{y}}BB_z\frac{C_z}{C} -$$

$$- h_3^2\Phi_{\mathring{x}}\Phi_{\mathring{yz}}AA_z\frac{C_z}{C} + h_3^2\Phi_{\mathring{y}}\Phi_{\mathring{xz}}BB_z\frac{C_z}{C} - h_3\Phi^2_{\mathring{y}}A_zB\frac{C_z}{C} - h_3^2\Phi_{\mathring{y}}\Phi_{\mathring{yz}}A_zB\frac{C_z}{C}.$$

Now we will transform and we estimate each of these elements.

The expression for J_3 and J_4 is a quadratic form with respect to $\Phi_{\mathring{x}}$ and $\Phi^{k+1}_{\mathring{y}}$, to $\Phi^{k+1}_{\mathring{x}}$ and $\Phi_{\mathring{y}}$, whose determinant

$$\begin{vmatrix} (AB_z - A_zB) + D\frac{C_z}{C} & -\mathrm{E}\frac{C_z}{C} \\ -\mathrm{E}\frac{C_z}{C} & (AB_z - A_zB) - D\frac{C_z}{C} \end{vmatrix} = (AB_z - A_zB)^2 - D^2\left|\frac{C_z}{C}\right|^2 - \mathrm{E}^2\left|\frac{C_z}{C}\right|^2 =$$

$$(AB_z - A_zB)^2 - \left|\frac{C_z}{C}\right|^2,$$

because $\mathrm{E}^2 + D^2 = 1$, where $\mathrm{E} = \cos 2\theta$, $D = sin 2\theta$. Then, from the condition $(AB_z - A_zB) - \left|\frac{C_z}{C}\right| \geq \alpha > 0$ the positive definiteness of the quadratic form J_3 and J_4 follows. Using the inequality

$$ax^2 + 2bxy + cy^2 \ge \frac{2\left(ac - b^2\right)}{a + c + \sqrt{(a-c)^2 + 4b^2}}\left(x^2 + y^2\right)$$

for a positively definite quadratic form $ax^2 + 2bxy + cy^2$, we obtain

$$J_3 \ge \frac{1}{2}\left[(AB_z - A_zB) - \left|\frac{C_z}{C}\right|\right]\left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{y}}\right)^2\right], \tag{14}$$

$$J_4 \ge \frac{1}{2}\left[AB_z - A_zB - \left|\frac{C_z}{C}\right|\right]\left[\left(\Phi^{k+1}_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2\right]. \tag{15}$$

Taking into account that $\left(1 - \frac{C^k}{C^{k+1}}\right) \approx o\left(h_3\right)$, we have

$$J_6 = \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{y}}BB_z o\left(h_3\right) + \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{y}}AA_z o\left(h_3\right) - h_3\Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{yz}}\left(AA_z + \frac{BB_zC^k}{C^{k+1}}\right) +$$

$$+h_3\Phi_{\overset{\circ}{y}}\Phi_{\overset{\circ}{xz}}\left(\frac{AA_zC^k}{C^{k+1}} + BB_z\right) = \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{y}}BB_z o\left(h_3\right) + \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{y}}AA_z o\left(h_3\right)\left(AA_z + BB_z\right) \times$$

$$\times\left(\Phi_{\overset{\circ}{y}}\Phi_{\overset{\circ}{xz}} - \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{yz}}\right) - h_3\Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{yz}}BB_z o\left(h_3\right) + h_3\Phi_{\overset{\circ}{y}}\Phi_{\overset{\circ}{xz}}AA_z o\left(h_3\right) =$$

$$= \Phi_{\overset{\circ}{x}}\Phi_{\overset{\circ}{y}}\left(AA_z + BB_z\right)o\left(h_3\right) + \left(AA_z + BB_z\right)\left(\Phi^k_{\overset{\circ}{y}}\Phi^{k+1}_{\overset{\circ}{x}} - \Phi^k_{\overset{\circ}{x}}\Phi^{k+1}_{\overset{\circ}{y}}\right) -$$

$$-\left(\Phi^k_{\overset{\circ}{x}}\Phi^{k+1}_{\overset{\circ}{y}} + \Phi^k_{\overset{\circ}{x}}\Phi^k_{\overset{\circ}{y}}\right)BB_z o\left(h_3\right) + \left(\Phi^k_{\overset{\circ}{y}}\Phi^{k+1}_{\overset{\circ}{x}} - \Phi^k_{\overset{\circ}{y}}\Phi^k_{\overset{\circ}{x}}\right)AA_z o\left(h_3\right).$$

Using the inequality $|ab| \le (a^2 + b^2)/2$, we get

$$\begin{aligned} J_6 \le \tfrac{1}{2}\Bigg\{&\left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2\right]\left(AA_z + BB_z\right)o\left(h_3\right) + \\ &+\left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{y}}\right)^2\right]\left(AA_z + BB_z\right) + \\ +&\left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{y}}\right)^2 + \left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2\right]BB_z o\left(h_3\right) + \\ +&\left[\left(\Phi^k_{\overset{\circ}{y}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2 + \left(\Phi^k_{\overset{\circ}{x}}\right)^2\right]AA_z o\left(h_3\right)\Bigg\}. \end{aligned} \tag{16}$$

Taking into account that $(AB_z - A_zB) - \left|\frac{C_z}{C}\right| \geq \alpha > 0$, conditions $\left|\Phi_{\mathring{x}z}\right| \leq c_2$, $\left|\Phi_{\mathring{y}z}\right| \leq c_2$ and the inequality $|ab| \leq (a^2+b^2)/2$, we obtain

$$
\begin{aligned}
J_5 \leq\ & \Phi^2_{\mathring{x}}\left(A_zB + ABC_z\right) o\left(h_3\right) + \Phi^2_{\mathring{y}}\left(AB_z - ABC_z\right) o\left(h_3\right) + \\
&+ \tfrac{1}{2}\left\{\left[\left(\Phi^k_{\mathring{x}}\right)^2 + \left(\Phi^{k+1}_{\mathring{x}}\right)^2\right]\left(A_zB + ABC_z\right) o\left(h_3\right) + \right. \\
&+ \left[\left(\Phi^k_{\mathring{y}}\right)^2 + \left(\Phi^{k+1}_{\mathring{y}}\right)^2\right] \times \left(AB_z + ABC_z\right) o\left(h_3\right) + \\
&+ \left[\left(\Phi^k_{\mathring{x}}\right)^2 + \left(\Phi^{k+1}_{\mathring{y}}\right)^2\right] B^2C_z o\left(h_3\right) + \\
&+ \left.\left[\left(\Phi^k_{\mathring{y}}\right)^2 + \left(\Phi^{k+1}_{\mathring{x}}\right)^2\right] A^2C_z o\left(h_3\right)\right\} + c_2h_3^2.
\end{aligned}
\tag{17}
$$

Due to formula (11), we have

$$
\begin{aligned}
\Phi_{\mathring{y}}\Phi_{\mathring{x}z} &= \left[\Phi_{\mathring{y}}\Phi_z\right]_{\mathring{x}} - \Phi_{\mathring{x}\mathring{y}}\Phi_z - \tfrac{h_1^2}{2}\left[\Phi_{\mathring{y}x}\Phi_{zx}\right]_{\bar{x}}, \\
-\Phi_{\mathring{x}}\Phi_{\mathring{y}z} &= -\left[\Phi_{\mathring{x}}\Phi_z\right]_{\mathring{y}} + \Phi_{\mathring{x}\mathring{y}}\Phi_z + \tfrac{h_2^2}{2}\left[\Phi_{\mathring{x}y}\Phi_{zy}\right]_{\bar{y}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
J_7 &= \left[\left(-\Phi_{\mathring{x}}B + \Phi_{\mathring{y}}A\right)\left(\Phi_{\mathring{x}}A + \Phi_{\mathring{y}}B\right)\right]_z - \Phi_{\mathring{x}}\Phi_{\mathring{y}z} + \Phi_{\mathring{y}}\Phi_{\mathring{x}z} = \\
&= \left[\left(-\Phi_{\mathring{x}}B + \Phi_{\mathring{y}}A\right)\left(\Phi_{\mathring{x}}A + \Phi_{\mathring{y}}B\right)\right]_z + \left[\Phi_{\mathring{y}}\Phi_z\right]_{\mathring{x}} - \\
&\quad - \left[\Phi_{\mathring{x}}\Phi_z\right]_{\mathring{y}} - \tfrac{h_1^2}{2}\left[\Phi_{\mathring{y}x}\Phi_{zx}\right]_{\bar{x}} + \tfrac{h_2^2}{2}\left[\Phi_{\mathring{x}y}\Phi_{zy}\right]_{\bar{y}}.
\end{aligned}
\tag{18}
$$

Now we will transform and estimate J_8:

$$
\begin{aligned}
J_8 =\ & h_3\Phi^2_{\mathring{x}}AB_z\frac{C_z}{C} + h_3\Phi^k_{\mathring{x}}\Phi^{k+1}_{\mathring{x}}AB_z\frac{C_z}{C} - h_3\Phi^2_{\mathring{x}}AB_z\frac{C_z}{C} - h_3\Phi_{\mathring{x}}\Phi_{\mathring{y}}AA_z\frac{C_z}{C} + \\
&+ h_3\Phi_{\mathring{x}}\Phi_{\mathring{y}}BB_z\frac{C_z}{C} - h_3\Phi^k_{\mathring{x}}\Phi^{k+1}_{\mathring{y}}AA_z\frac{C_z}{C} + h_3\Phi_{\mathring{x}}\Phi_{\mathring{y}}AA_z\frac{C_z}{C} + h_3\Phi^k_{\mathring{y}}\Phi^{k+1}_{\mathring{x}}BB_z\frac{C_z}{C} - \\
&- h_3\Phi_{\mathring{y}}\Phi_{\mathring{x}}BB_z\frac{C_z}{C} - h_3\Phi^2_{\mathring{y}}A_zB\frac{C_z}{C} - h_3\Phi^k_{\mathring{y}}\Phi^{k+1}_{\mathring{y}}A_zB\frac{C_z}{C} + h_3\Phi^2_{\mathring{y}}A_zB\frac{C_z}{C}.
\end{aligned}
$$

Using the inequality $|ab| \leq (a^2+b^2)/2$, we obtain

$$
\begin{aligned}
J_8 \leq \tfrac{h_3}{2}&\left\{\left[\left(\Phi^k_{\mathring{x}}\right)^2 + \left(\Phi^{k+1}_{\mathring{x}}\right)^2\right]AB_z\tfrac{C_z}{C} + \left[\left(\Phi^k_{\mathring{x}}\right)^2 + \left(\Phi^{k+1}_{\mathring{y}}\right)^2\right]AA_z\tfrac{C_z}{C} + \right. \\
&+ \left.\left[\left(\Phi^k_{\mathring{y}}\right)^2 + \left(\Phi^{k+1}_{\mathring{x}}\right)^2\right]BB_z\tfrac{C_z}{C} + \left[\left(\Phi^k_{\mathring{y}}\right)^2 + \left(\Phi^{k+1}_{\mathring{y}}\right)^2\right]A_zB\tfrac{C_z}{C}\right\}.
\end{aligned}
\tag{19}
$$

Supposing that A, B, C enough smooth limit functions and taking into account the expressions (14)–(19), from (13) we get

$$\frac{1}{2}\left[(AB_z - A_zB) - \left|\frac{C_z}{C}\right|\right] \times \left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{y}}\right)^2\right] \leq$$
$$\leq \frac{h_3}{2} K \left[\left(\Phi^k_{\overset{\circ}{x}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{x}}\right)^2 + \left(\Phi^k_{\overset{\circ}{y}}\right)^2 + \left(\Phi^{k+1}_{\overset{\circ}{y}}\right)^2\right] + R^k_{i,j} + c_2 h_3^2, \tag{20}$$

where

$$R^k_{i,j} = \left[\Phi_{\overset{\circ}{x}}\Phi_z\right]_{\overset{\circ}{y}} - \left[\Phi_{\overset{\circ}{y}}\Phi_z\right]_{\overset{\circ}{x}} - \left[\left(\Phi_{\overset{\circ}{x}}A + \Phi_{\overset{\circ}{y}}B\right)\left(-\Phi_{\overset{\circ}{x}}B + \Phi_{\overset{\circ}{y}}A\right)\right]_z +$$
$$+\frac{h_1^2}{2}\left[\Phi_{zx}\Phi_{\overset{\circ}{yx}}\right]_{\bar{x}} - \frac{h_2^2}{2}\left[\Phi_{zy}\Phi_{\overset{\circ}{xy}}\right]_{\bar{y}}.$$

Let

$$\left[(AB_z - A_zB) - \left|\frac{C_z}{C}\right|\right] \geq \alpha > 0, \quad N_j > 9, \quad j = 1, 2, \quad \mathrm{K}h_3 < \frac{\alpha}{2},$$

i.e. $N_3 > \frac{2\mathrm{K}l}{\alpha}$, because $h_3 = \frac{l}{N_3}$, where α and K is a constant.

Then, from (20) we have

$$\sum_{\Omega_h^\varepsilon}\left(\Phi^2_{\overset{\circ}{x}} + \Phi^2_{\overset{\circ}{y}}\right)h_1 h_2 h_3 \leq \frac{2}{\alpha}\sum_{\Omega_h^\varepsilon} R^k_{i,j} + c_2 h_3^2. \tag{21}$$

Using conditions (7), (8) and inequality $|ab| \leq (a^2 + b^2)/2$, we can transform (21) so as to obtain the desired estimate (12). Thus, the proof of Theorem is completed.

References

1. Lavrent'ev, M.M., Romanov, V.G.: Three linearized inverse problems for hyperbolic equations. Sov. Math. Dokl. **7**, 1650–1652 (1966)
2. Mukhometov, R.G.: On a problem of integral geometry. Math. Probl. Geophys. **6**, 212–242 (1975). (in Russian)
3. Romanov, V.: Integral Geometry and Inverse Problems for Hyperbolic Equations. Springer, Berlin (1974)
4. Kabanikhin, S.I., Bakanov, G.B.: On the stability of a finite-difference analogue of a two dimensional problem of integral geometry. Sov. Math. Dokl. **35**, 16–19 (1987)
5. Kabanikhin, S.I., Bakanov, G.B.: Discrete analogy of Gel'fand-Levitan method. Dokl. Akad. Nauk. **356**, 157–160 (1997)
6. Bakanov, G.B.: On the stability estimation of differential- difference analogue of the integral geometry problem with a weight function. In: AIP Conference Proceedings, vol. 1759, p. 020067 (2016)