MCAC 201: Design and Analysis of Algorithms

Neelima Gupta

ngupta@cs.du.ac.in

April 17, 2023

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3. $f(n) = 2^n$, $g(n) = 3^n$.

Practice Questions

Use the equivalent definitions (limits) to prove the following:

1. Show that a polynomial of degree d, with positive leading coefficient is $\Theta(n^d)$.
2. For $g(n) = f(n) + o(f(n))$, show that $g(n) = \Theta(f(n))$.
3. Show that
   a. $\log^M n = o(n^\epsilon)$ where M and ϵ are positive constants.
   b. $\log n = o(n)$.
4. $a^n = o(b^n)$ for all $a < b$.