SOLUTIONS TO QUESTIONS IN CHAPTER 1

SECTION 1

1.1 The only number that appears on *A*, C, and *D* but not on *B* is 11. This can be verified by trial and error for each of 0,1,2,3,4,...,15.

1.2 This trick will always work if no two numbers appear on exactly the same set of cards. For example, if both 11 and 5 appeared on cards *A*, *C*, and *D* but not on *B*, then player 1 could not know which answer, 11 or 5, was correct.

1.3 There is more than one way to design such a pair of cards. Two (of many) examples follow

(a) $\boxed{2\quad 3}$ $\boxed{1\quad 3}$ **(b)** $\boxed{0\quad 3}$ $\boxed{1\quad 3}$

Two cards cannot be used to distinguish the numbers 0,1,2,3, and 4. The reasoning is as follows: Write down all possible yes/no responses to a set of two cards:

Card *1*	*Card 2*
no	no
no	*yes*
yes	no
yes	*yes*

To make the trick work, we must assign to each of these four responses one of the numbers O, 1,2,3, and 4. There are five numbers to assign to four responses, and so two numbers will elicit the same response.

1.4 10,000. There are 4 digits to be assigned with 10 choices for each digit (0,1,..., 9). We apply the Multiplication Principle: There are $10 \cdot 10 = 100$ choices for the first two digits. Regardless of these choices there are 100 choices for the last two digits. Using the Multiplication Principle a second time, there are 100 100 = 10,000 choices for all 4 digits.

1.5

	Responses			
Player 2's Number	*Card A*	*Card B*	*Card C*	*Card D*
0	no	no	no	no
1	no	no	no	yes
2	no	no	yes	no
3	no	no	yes	yes
4	no	yes	no	no
5	no	yes	no	yes
6	no	yes	yes	no
7	no	yes	yes	yes
8	yes	no	no	no
9	yes	no	no	yes
10	***yes***	no	yes	no
11	yes	no	yes	yes
12	yes	yes	no	no
13	yes	yes	no	yes
14	yes	yes	yes	no
15	yes	yes	yes	yes

SECTION 2

2.1 ***(a)*** 10101 = 1 + 4 + 16 = 21. ***(b)*** 100101 = 1 + 4 + 32 = 37, and (c) 11010 = 2 -t- 8 + 16= 26. An even number expressed in binary ends with a O and an odd number ends with a 1.

2.2

Decimal Number	*Binary Representation*
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

If we make each of the binary representations four digits long by adding 0s to the left and replace each O with a "no" and each 1 with a "yes," then this table would become the same as that in Question 1.5.

2.3 4 (= 100), 5 (= 101), 6 (= 110), 7 (= 111), 12 (= 1100), 13 (= 1101), 14 (= 1110), and 15 (= 1111).

2.4

Decimal Number	*Arithmetic*	*Binary Representation*
6	2 + 4	110
19	1 + 2 + 16	10011
52	4 + 16 + 32	110100
84	4 + 16 + 64	1010100
232	8 + 32 + 64 + 128	11101000

SECTION 3

3.1 Step 1. Place one cup of water in the bottom of a double boiler.
Step 2. Place one cup of water in the top of a double boiler.
Step 3. Place one cup of quick oatmeal in the top of the double boiler.
Step 4. Turn on stove burner to medium.
Step 5. Place double boiler on burner and heat for 10 minutes.
Step 6. Remove pot.
Step 7. Turn off burner.

3.2 The successive values assigned to z follow.
(a) 1. **(*b*)** 20, 10,5, 16,8,4,2, 1.
(c) 7,22,11,34,17,52,26, 13,40,20,10,5,16,8,4,2, 1.

SECTION 4

4.1 (a) Algorithm BtoD run on s = 10101.

Step No.	*j*	*m*	*Is there a jth entry in s?*	*Is the jth entry equal to 1?*
1	0	—		
2	0	0		
3	0	0	yes	
4	0	1		yes
5	1	1		
3	1	1	yes	
4	1	1		no
5	2	1		
3	2	1	yes	
4	2	5		yes
5	3	5		
3	3	5	yes	
4	3	5		no
5	4	5		
3	4	5	yes	
4	4	21		yes
5	5	21		
3	5	21	no STOP	

Result: $m = 21$.

(b) Algorithm **BtoD** run on $s = 11010$.

Step No.	*j*	*m*	*Is there a jth entry ins?*	*Is the jth entry Step equal to 1?*
1	0			
2	0	0		
3	0	0	yes	
4	0	0		no
5	1	0		
3	1	0	yes	
4	1	2		yes
5	2	2		
3	2	2	yes	
4	2	2		no
5	3	2		
3	3	2	yes	
4	3	10		yes
5	4	10		
3	4	10	yes	
4	4	26		yes
5	5	26		
3	5	26	no STOP	

Result: $m = 26$.

(c) Algorithm **BtoD** run on s = 100101.

Step No.	*j*	*m*	*Is there a jth entry ins?*	*Is the jth entry equal to 1?*
1	0	—		
2	0	0		
3	0	0	yes	
4	0	1		yes
5	1	1		
3	1	1	yes	
4	1	1		no
5	2	1		
3	2	1	yes	
4	2	5		yes
5	3	5		
3	3	5	yes	
4	3	5		no
5	4	5		
3	4	5	yes	
4	4	5		no
5	5	5		
3	5	5	yes	
4	5	37		yes
5	6	37		
3	6	37	no STOP	

Result: m = 37. These are the same answers as those of Question 2.1.

4.2 Response 1 is not an algorithm because the instruction to stop might not be reached in a finite number of steps, since the binary representation of m might never be written down in step 1.

4.3 Response 2 is an algorithm because (a) the instructions are clear; (*b*) after performing an instruction, there is no ambiguity about which instruction is to be performed next; and (c) the instruction to stop will be reached after a finite number of instructions. Unlike Response 1, Response 2 finds binary numbers in increasing order (as opposed to at random) so that the mth binary number produced will be the binary representation of the decimal number m. The algorithm is slow because it will consider all n-bit binary numbers, $n \leq 5$, before concluding that 10011 is 19 in binary.

4.4 The algorithm must stop because eventually m must equal zero. Response 3 run on m = 182:

Step No.	*m*	*Largest power of 2 that is* $\leq m$	*r*	*Is m* equal to O?	*Result*
1	182	$2^7 = 128$	7		1_______
2	54			no	
1	54	$2^5 = 32$	5		1_1_____
2	22			no	
1	22	$2^4 = 16$	4		1_11____
2	6			no	
1	6	22=4	2		1_11_1__
2	2			no	
1	2	$2^1 = 2$	1		1_11_11_
2 STOP	O			yes	10110110

4.5 The algorithm must stop because it repeatedly decreases the value of *m*. Therefore, the value of *m* must eventually be O.

Here is algorithm **DtoB** run on m = 395; the values shown are those assigned to the variables after the execution of the given step:

Step No.	*j*	*m*	*q*	*r*	*Answer*
1	0	395	—	—	.
2	0	395	197	1	1
4,5	1	197	197	1	
2	1	197	98	1	11
4,5	2	98	98	1	
2	2	98	49	0	011
4,5	3	49	49	0	
2	3	49	24	1	1011
4,5	4	24	24	1	
2	4	24	12	0	01011
4,5	5	12	12	0	
2	5	12	6	0	001011
4,5	6	6	6	0	
2	6	6	3	0	0001011
4,5	7	3	3	0	
2	7	3	1	1	10001011
4, 5	8	1	1	1	
2	8	1	0	1	11 0001011
3 STOP			0		

Result: The binary representation of *m* = 395 is 110001011. Response 4 (**DtoB**) is easier to use than Response 3 because the user does less and easier arithmetic.

SECTION 5

5.1 (a) $A = \{1,4,6, 8,9,10,12,14,15,16, 18,20,21,22,24,25,26,27,28\}$, **(b)** $B = \{1,4,9, 16,25\}$, and (c) $C = \{4,8,9, 12,16, 18,20,24,25,27,28\}$.

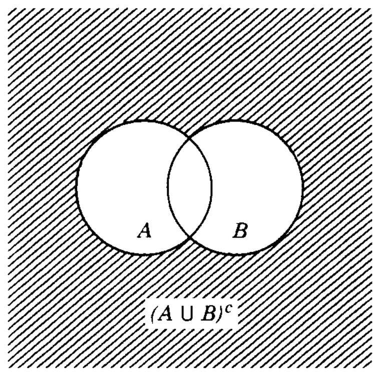
5.2 **(a)** $A^c = \{2,3, 5,7,11,13,17, 19,23,29\}$
(b) $B^c = \{2,3,5,6,7,8, 10,11,12,13,14,15,17, 18, 19,20,21,22,23,24,26,27, 28, 29\}$
(c) $C^c = \{1,2,3,5,6,7,10, 11,13,14,15,17, 19,21,22,23,26,29\}$

5.3 Every set is a subset of itself. In addition $B \subseteq A$, $C \subseteq A$, $A^c \subseteq B^c$, $A^c \subseteq C^c$.

5.4 Since $B \subseteq A$, $A \cup B = A$ and $A \cap B = B$. Similarly, $A \cup C = A$ and $A \cap C = C$. $B \cup C = \{1,4,8,9, 12,16, 18,20,24,25,27,28\}$ and $B \cap C = \{4,9, 16,25\}$.

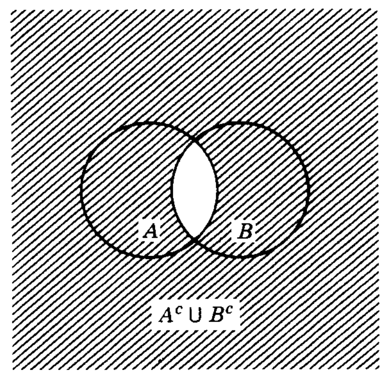
SECTION 6

6.1 *(a)*



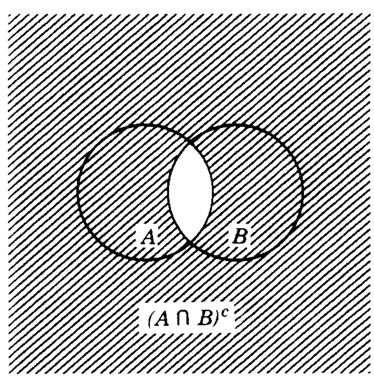
(a)

(b)



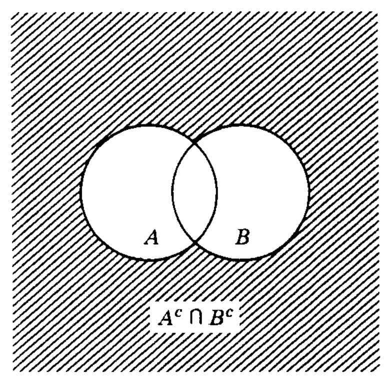
(b)

(c)



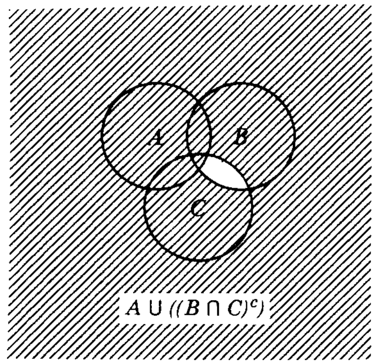
(c)

(d)



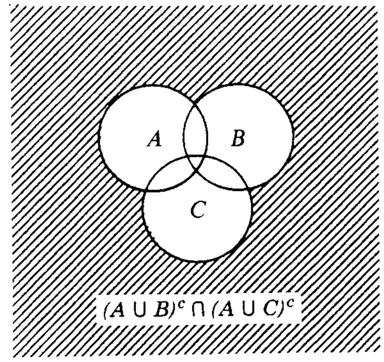
(d)

(e)



(e)

(f)



(f)

6.2 (i) $(A \cup B)^c = A^c \cap B^c$

Proof. Let x be in $(A \cup B)^c$. Then by the definition of complement, x is not in $A \cup B$. Thus x is not in A and x is not in B. If x is not in A, then x is in A'. Similarly, if x is not in B, then x is in B^c. Consequently, x is in $A^c \cap B^c$.

Conversely, if x is in $A^c \cap B'$, then x is in A^c and x is in l?'. In other words, x is not in A and x is not in B. Thus x is not in $A \cup B$. Consequently, x is in $(A \cup B)^c$.

We have shown that $(A \cup B)^c \subseteq A' \cap B'$, and that $A^c \cap B^c \subseteq (A \cup B)^c$. Thus $(A \cup B)^c = A^c \cap B'$, as desired. ❑

(ii) $(A \cap B)^c = A' \cup B'$

Proof. Let x be in $(A \cap B)^c$. Then x is not in $A \cap B$, which means that x is not in A or x is not in B (or both). Then x is in A' or x is in B' (or both). Thus x is in $A^c \cup B'$.

Conversely, let x be in $A' \cup B'$. Then x is not in A or x is not in B (or both), which means that x is not in $A \cap B$. Thus x is in $(A \cap B)^c$.

We have shown that $(A \cap B)^c \subseteq A' \cup B^c$, and that $A^c \cup B^c \subseteq (A \cap B)^c$. Thus $(A \cap B)^c = A' \cup B^c$, as desired. ‘ ☐

SECTION 7

7.1 $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

7.2 Algorithm SUBSET run on $A = \{a_1, a_2, a_3\}$ with $n = 3$.

Step No.	*j*	*List of Subsets*
1		$\varnothing$
2	1	
3	1	{a,}
4	2	
3	2	$\{a_2\}, \{a_1, a_2\}$
4	3	
3	3	$\{a_3\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}$
5	3 STOP	

7.3 *A x A = {(O, O), (O, 1),. ... (O, 9),* (1, O),. ... *(9,9)}*. We can associate each of these ordered pairs with a unique decimal integer from O to 99. Similarly, with each element of *A* x *A x A we* can associate the decimal numbers from O to 999. In general, the set *An is* the set of all ordered *n*-tuples with entries from A= {0,1,2,.. ., 9}. Each element corresponds with a unique number from O to 10” – 1.

7.4 *(a) A* **x** *A = {(a, a), (a, b), (a, c),@, a), (b, b),@, c), (c, a), (c, b), (c, c)},* ***(b)*** r^2, *(c)* r^3, and *(d)* r^n.

7.5 **(a)** The bit vector of *T* is 110010, since the elements of *T* are the first, second, arid fifth elements in the list of *S*, ***(b)*** 101001 and 000111, (c) z is the sixth element in the list of S, and *(d) {y}* and {u, v, w, x}.

SECTION 8

8.1 $|A \cup C| = 17$ and $|B \cup C| = 13$.

8.2 The cardinality of each union of two cards is 12.

8.3 *(a)* $|A\ u\ B|$ never equals 4, since $A \subseteq A\ u\ R$ For ***b)*** to e) set A = {1,2,3,4,5} and ***(b)*** B = {1,2,3}, (c) B = {1,2,6}, ***(d)*** B= {1,6,7}, and (e) B = {6,7, 8}. ***(f)*** $|A\ u\ B|$ never equals 9, since $|A \mathbf{u} B|$ is the largest when *A* and *B* are disjoint and then $|A\ u\ B| = |A| + |B| = 8$.

8.4 Let *A* = {students enrolled in Discrete Mathematics} and *B* = {students enrolled in Computer Science}. We are given $|A|$ = 146, $|B|$ = *205,* and $|A\ u\ B|$ = *232.* From Theorem 8.1

$$
\begin{aligned}
|A \cap B| &= |A| + |B| - |A \cup B| \\
&= 146 + 205 - 232 \\
&= 119 \text{ students in both courses.}
\end{aligned}
$$

SECTION 9

9.1 The range of b is all of B because if q is any binary number and t its decimal equivalent, then $b(t) = q$.

9.2 The map f_1 is not a function from N to B because its range is not contained in B. The map f_2 is a function, since a binary number is either even or odd but never both. The range of f_2 is {O, 1}. The map f_3 is a function, since for each natural number r there is precisely one string with r 1s. The range of f_3 is all binary numbers that contain no zero. The map f_4 is not a function, since, for example, $f_4(6)$ should equal O because 2 divides 6, and yet $f_4(6)$ should equal 1 because 3 divides 6.

9.3 The function b is onto, since the range of b is all of B (see Question 9.1). The function f_2 is not onto, since {0,1} does not include all binary numbers. The function f_3 is not onto, since the range does not include binary numbers containing zeros.

9.4 The function b is one-to-one because if n # n' are two different numbers, then their binary representations differ and so $b(n)$ # $b(n')$. The function f_2 is not one-to-one because, for example, $f_2(2) = f_2(4)$ = O. The function f_3 is one-to-one because if r # r', then $f_3(r)$ and $f_3(r')$ are strings of ones of different length.

9.5 Yes, since A #A' implies that c(A) # $c(A')$.

9.6 If X is in $P(U)$, then $c \circ c(X) = c(X^c) = (X^c)^c = X$. Thus $c \circ c = i$ and c is its own inverse.

SECTION 10

10.1 (a)

x	y	$x \vee y$	$\sim(x \vee y)$	$-x$	$-Y$	$(-x) \wedge (\sim y)$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

(b) Note that $(x \wedge y) \wedge z = 1$ if and only if all of x, y, and z equal 1. Similarly, $x \wedge (y \wedge z) = 1$ if and only if all of x, y, and z equal 1.

10.2

x	y	$x \oplus y$	$x \vee y$	$\sim(x \wedge y)$	$(x \vee y) \wedge (\sim(x \wedge y))$	$(x \wedge \sim y) \vee (\sim x \wedge y)$
0	0	0	0	1	0	0
0	1	1	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	0

10.3 (a) $x \wedge y = 1$ if and only if both x and y equal 1 if and only if $y \wedge x = 1$.
(b) $(x \vee y) \vee z = 0$ if and only if all of x, y, and z equal 0 if and only if $x \vee (y \vee z) = 0$.

10.4 None.

10.5 *(a) is* a contradiction while both ***(b)*** and (c) are tautologies.

SOLUTIONS TO QUESTIONS IN CHAPTER 2

SECTION 1

1.1

	Value Assigned to x	Value Assigned to y	Value Assigned to $x$old	Value Assigned to $y$old
Before step 1	5	2	?	?
After step 1	5	2	5	?
After step 2	5	2	5	2
After step 3	5	5	5	2
After step 4	2	5	5	2

SECTION 2

2.1 Algorithm EXPONENT run with $x = 3$, $n = 4$.

Step No.	i	ans
2	0	1
4	0	3
5	1	3
4	1	9
5	2	9
4	2	27
5	3	27
4	3	81
5	4	81
6 STOP	4	81

SECTION 3

3.1 P_n: $2 + 4 + \cdots + 2n = n(n + 1)$ for all positive integers n. Proof by induction on n.

Step 1 (the base case): P_1 *is* the statement $2 \cdot 1 = 2 = 1(1 + 1)$.

Step 2 (the inductive hypothesis): Assume that P_k *is* true. P_k *is* the statement $2 + 4 + \cdots + 2k = k(k + 1)$.

Step 3 (the inductive step): Verify that P_{k+1} *is* true. P_{k+1} *is* the statement $2 + 4 + \cdots + 2k + 2(k + 1) = (k + 1)(k + 2)$.

$$\begin{aligned} 2 + 4 + \cdots + 2k + 2(k + 1) & \\ &= (2 + \cdots + 2k) + 2(k + 1) && \text{by associativity} \\ &= k(k + 1) + 2(k + 1) && \text{by inductive hypothesis} \\ &= (k + 1)(k + 2) && \text{by factoring.} \end{aligned}$$

Therefore, P_{k+1} *is* true, and P_n is true for all positive n. ❑

3.2 Suppose that $n = 2$. If $x = -1$, then $1 - x + x^2 = 3$. If $x \neq -1$, then

$$1 - x + x^2 = \frac{(1 - x + x^2)(1 + x)}{1 + x} = \frac{1 + x^3}{1 + x}.$$

Suppose that $n = 3$. If $x = -1$, then $1 - x + x^2 - x^3 = 4$. If $x \neq -1$, then

$$1 - x + x^2 - x^3 = \frac{(1 - x + x^2 - x^3)(1 + x)}{1 + x} = \frac{1 - x^4}{1 + x}.$$

3.3 P_n: $1 + x + x^2 + \cdots + x^n = \begin{cases} (1 - x^{n+1})/(1 - x) & \text{if } x \neq 1 \\ n + 1 & \text{if } x = 1 \end{cases}$

First notice that when $x = 1$, the left-hand side of the equation of P_n is the sum of $(n + 1)$ terms, each equal to 1, and so the equation is valid. Now we focus on the case when $x \neq 1$. Proof by induction on n.

Step 1 (the base case):

$$P_0 \text{ is the statement } 1 = \frac{1 - x^{0+1}}{1 - x} = \frac{1 - x}{1 - x}$$

which is true, since $x \neq 1$.

$$P_1 \text{ is the statement } 1 + x = \frac{1 - x^2}{1 - x}\text{"}$$

This statement is also true since

$$\frac{1 - x^2}{1 - x} = \frac{(1 - x)(1 + x)}{1 - x} = 1 + x, \qquad \text{for } x \neq 1.$$

Step 2 (the inductive hypothesis) Assume that P_k *is* true. P_k is the statement

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x} \qquad \text{if } x \neq 1.$$

Step 3 (the inductive step) Show that P_{k+1} is true. P_{k+1} is the statement

$$1 + x + x^2 + \cdots + x^k + x^{k+1} = \frac{1 - x^{k+2}}{1 - x} \qquad \text{if } x \neq 1,$$

$$\begin{aligned}
&1 + x + \cdots + x^k + x^{k+1} \\
&\quad = (1 + \cdots + x^k) + x^{k+1} && \text{by associativity} \\
&\quad = \frac{1 - x^{k+1}}{1 - x} + x^{k+1} && \text{by inductive hypothesis} \\
&\quad = \frac{1 - x^{k+1} + (1 - x)x^{k+1}}{1 - x} && \text{with common denominator} \\
&\quad = \frac{1 - x^{k+1} + x^{k+1} - xx^{k+1}}{1 - x} && \text{by algebra} \\
&\quad = \frac{1 - x^{k+2}}{1 - x} && \text{by more algebra.}
\end{aligned}$$

Hence P_{k+1} is true, and P_n is true for all positive integers n. □

SECTION 4

4.1 If A contains one element, then it has one even subset, the empty set. If A contains two elements, then it has two even subsets, the empty set and the whole set ,4. You should check that a 3-set has four even subsets. Thus it seems as if whenever A is a set with n elements, then the number of even subsets of A *is* 2^{n-1}. We prove this by induction on n.

Step 1 (the base case): P_1 is the statement that a set A with one element has exactly $2^{1-1} = 2^0 = 1$ even subset. We just checked that this is true.

Step 2 (the inductive hypothesis) We assume that P_k is true. P_k *is* the statement that a set A with k elements has exactly 2^{k-1} even subsets.

Step 3 (the inductive step): We must verify that P_{k+1} is true. P_{k+1} is the statement that a set A with $k+1$ elements has exactly $2^{k+1-1} = 2^k$ even subsets.

Consider a set A with $k+1$ elements. We must show that A has exactly 2^k even subsets. Let x be an element in A and define B to be $A - \{x\}$. By Example 4.1 we know that B has exactly 2^k subsets. We build upon these subsets to obtain all even subsets of A. Namely, let S be an even subset of A. If S does not contain x, then S is an even subset of B. If S does contain x, then S — {x} is an odd subset of B, where by an odd subset we mean one containing an odd number of elements. Furthermore, every subset of B is either even or odd. An even subset of B is also an even subset of A, and an odd subset T of B turns into an even subset of A by forming $T \cup \{x\}$. The number of even subsets of B *is* 2^{k-1} by the inductive hypothesis. The number of odd subsets of B *is* the total number of subsets (2^k) minus the number of even subsets (2^{k-1}), or $2^k - 2^{k-1}$. Thus the number of odd subsets of B *is*

$$\begin{aligned} 2^k - 2^{k-1} &= 2^{k-1}(2 - 1) \\ &= 2^{k-1} \end{aligned}$$

Thus #(even subsets of A) = #(even subsets not containing x)

$$\begin{aligned} &\quad + \#(\text{even subsets containing x}) \\ &= \#(\text{even subsets of } B) + \#(\text{odd subsets of } B) \\ &= 2^{k-1} + 2^{k-1} \qquad \text{by inductive hypothesis and the argument given above} \\ &= 2 \cdot 2^{k-1} \\ &\cdot\ 2^k. \end{aligned}$$

Thus P_{k+1} is established, and P_n is true for every positive integer n. □

4.2 P_n is the proposition that the nth time the comment in algorithm SUM is encountered, it is correct. The last time the comment is encountered j will have the value r, and if the comment is correct, then ans will have the value $r(r+1)/2$ and the output will be as claimed.

Step 1 (the base case): We check *PI*. The variable ans is initially equal to O, but the first time the comment is encountered, ans has been incremented by j = 1 so ans equals 1. The comment asserts that the value of ans is $1 \cdot 2/2 = 1$ and so the comment is correct. You might check also that P_2 *is* valid.

Step 2 (the inductive hypothesis): We assume P_k, which states that the kth time the comment is encountered, it is true.

Step 3 (the inductive step): We must prove P_{k+1}, which states that the $(k+1)$st time the comment is encountered it is valid. Now the kth time that the comment is reached j has the value of k and by the inductive hypothesis the value in ans is $k(k+1)/2$. The next time j has the value $(k+1)$ and ans has been increased by this value of j:

$$\begin{aligned}\text{ans \{after } k+1 \text{ encounters\}} &= j + \text{ans \{after } k \text{ encounters\}}\\ &= (k+1) + \frac{k(k+1)}{2}\\ &= (k+1)(1+k/2)\\ &= \frac{(k+1)(k+2)}{2},\end{aligned}$$

which is the assertion of P_{k+1}. Thus P_n is true for all positive n. ❑

4.3 *(a)* 4, (b) 3, (c) 4, and **(*d*)** 3. The binary representation of 14 can be obtained from that of 7 by adding a O at the right. The binary representation of 13 can be obtained from that of 6 by adding a 1 at the right.

SECTION 5

5.1 $n = 7$: four multiplications, since $x^7 = (x^4)(x^2)x$.
$n = 11$: five multiplications, since $x^{11} = (x^8)(x^2)x$
$n = 12$: four multiplications, since $x^{12} = x^8x^4$
$n = 16$: four multiplications, since $x^{16} = (x^8)(x^8)$

5.2 Revised Algorithm **DtoB** used to find X^{37}:

Variables	*j*	*m*	*q*	*r*	*x*	*ans*	*No. Multiplications and Divisions*
Values After Step 2.5	0	37	18	1	x	x	2
	1	18	9	0	x^2	x	2
	2	9	4	1	x^4	x^5	3
	3	4	2	0	x^8	X^5	2
	4	2	1	0	x^{16}	x^5	2
	5	1	0	1	x^{32}	x^{37}	3
							Total No. 14

Revised Algorithm DtoB used to find X^52.

Variables	*j*	*m*	*q*	*r*	*x*	*ans*	*No. Multiplications and Divisions*
Values	0	52	26	o	x	1,	I
After	1	26	13	0	X^2	1	2
Step 2.S	2	13	6	1	x^4	x^4	3
	3	6	3	0	X^8	X^4	2
	4	3	1	1	x^{16}	x^{20}	3
	5	1	0	1	x^{32}	x^{52}	3
							Total No. 14

SECTION 6

6.1 Steps 1 and 5 were bookkeeping steps in DtoB and are not needed in FASTEXP because of step 2.5.

6.2 $\log(2^2)$ = log (4)=2 log (2^3) = log (8)= 3
$\log(2^5)$ = log (32)= 5 $\log(2^{10})$ = 10
$2^{\log(2)} = 2^1 = 2$ $2^{\log(4)} = 2^2 = 4$
$2^{\log(6)} = 22.58A...$ = ? (Do the next question and then return to finish this.)
$2^{\log(8)} = 2 3 = 8$

63 By the definition of logarithm if log $(2^p) = h$, then 2^h = 2P. This implies that $h = p$. Thus log (2P) = p. If log(q) $= t$, then by definition $2^t = q$ and by substitution $2^{10s\,(q)} = 2^t = q$.

6.4 $\lfloor\frac{17}{3}\rfloor = 5$, $\lceil\frac{25}{7}\rceil = 4$, $\lfloor\log(8)\rfloor = 3$, $\lceil\log(13)\rceil = 4$, $\lfloor -\frac{14}{9}\rfloor = -2$, $\lceil\log(25)\rceil = 5$, and $\lfloor\log(13.73)\rfloor = 3$.

SECTION 7

7.1 N = 17: $\sqrt{17}$ = 4.1231...>4.0874. . . = log (17). This does not contradict Theorem 7.2, but says that more is true than is stated in the theorem. Namely, it is true that if $n \geq 17$ (see Exercise 7.5), then& > log(n). (The bound $n \geq 64$ was used for ease of calculation and proof argument.)

SECTION 8

8.1 Note that $f(n) = 12n^2 - 11 \leq 12n^2$, since subtraction makes things smaller. Thus letting C = 12, we have $f(n) = O(n^2)$. Similarly,

$$\begin{aligned} h(n) &= 3n^2 + 4n + 11 \\ &\leq 3n^2 + 4n^2 + 11n^2, \text{ since } n \leq n^2 \text{ and } 1 \leq n^2 \\ &= 18n^2. \end{aligned}$$

Thus letting $C = 18$, we have $h(n) = O(n^2)$.

8.2 An algorithm will be called cubic if there is a function, say $f(n)$, that counts the number of operations given a problem of size n and $f(n) = O(n^3)$. Both L and C ought to take about 16 minutes to solve a problem of size 200. On a problem of size 1000, L should take about 80 minutes and C should take about 2000 minutes.

8.3 $$\begin{aligned} f(n) &= 2n^7 - 6n^5 + 10n^2 - 5 \\ &\leq 2n^7 + 6n^5 + 10n^2 + 5 \\ &\leq 2n^7 + 6n^7 + 10n^7 + 5n^7 = 23n^7. \end{aligned}$$

Thus with C = 23, $f(n) = O(n^7)$.

8.4 By Theorem 8.1 $f = O(n^5)$ and g = $O(n^4)$. Thus by Theorem 8.2

$$f + g = 0(n^5 + n') = O(n^5),$$

and

$$f \cdot g = O(n^5 n^4) = O(n^9).$$

SECTION 9

9.1 (a) 353 is a prime number, *(b)* 238 is not an even integer (or 238 is an odd integer).

9.2 **(a)** There exists an integer greater than one that does not have a prime divisor. *(b)* There exists an integer of the form 4n + 1 that is not a prime. (c) There exists a prime greater than 2 that is not odd.

9.3 (a) For every integer n, $3n + 1$ is not a prime number. *(b)* For every integer n, $\log(n) \leq n$. *(c)* For every integer n, $n^2 \leq 2^n$.

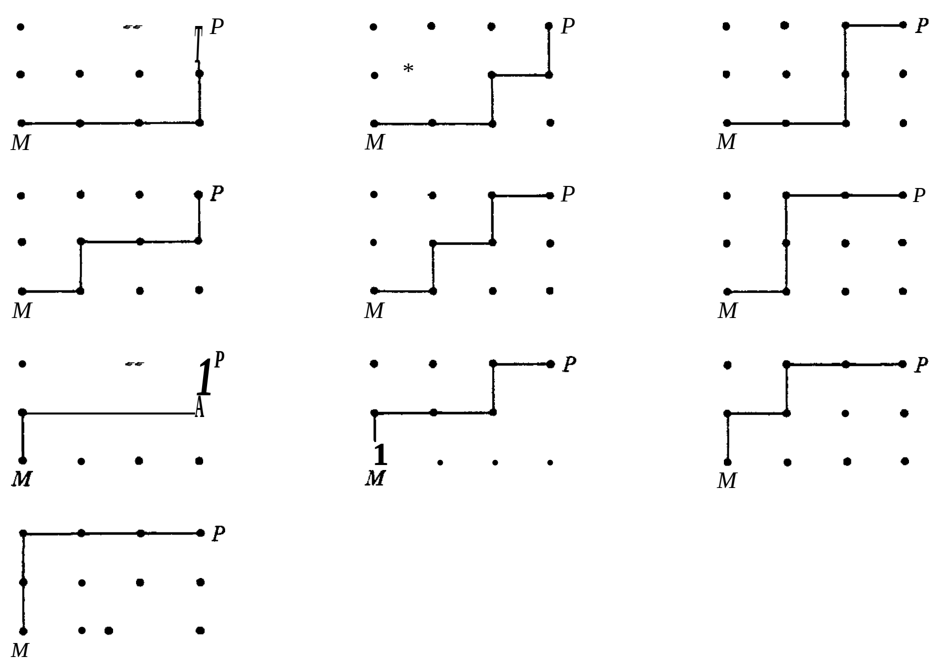
9.4 (a) The hypothesis is that n is even; the conclusion is that $n^2 + n + 1$ is prime. The negation is that there is some even integer n such that $n^2 + n + 1$ is not prime.
(*b*) The hypothesis is that $n^2 + n + 1$ is prime; the conclusion is that n is even. The negation is that there exists an integer n such that $n^2 + n + 1$ is prime and n is not even.
(c) The hypothesis is that n is divisible by 6; the conclusion is that n^2 is divisible by 4. The negation is that there is an integer n that is divisible by 6 but n^2 is not divisible by 4.

9.5 (*a*) The converse of 9.4(a) is 9.4(b). The contrapositive of 9.4(a) is that if $n^2 + n + 1$ is not prime, then n is not even.
(*b*) The converse of 9.4(b) is 9.4(a). The contrapositive of 9.4(b) is that if n is not even, then $n^2 + n + 1$ is not prime.
(c) The converse of 9.4(c) is that if n^2 is divisible by 4, then n is divisible by 6. The contrapositive of 9.4(c) is that if n^2 is not divisible by 4, then n is not divisible by 6.
(*d*) The converse of Lemma 7.1 is that if $2^r > (r + 1)^2$, then r is greater than 5. The contrapositive of Lemma 7.1 is that if $2^r \leq (r + 1)^2$, then r is no greater than 5.
(e) The converse of Theorem 7.2 is that if $\sqrt{n} > \log(n)$, then $n \geq 64$. The contrapositive of Theorem 7.2 is that if $\sqrt{n} \leq \log(n)$, then $n < 64$.

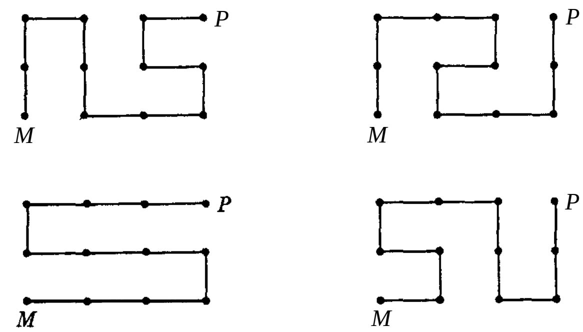
SOLUTIONS TO QUESTIONS IN CHAPTER 3

SECTION 1

1.1 (1) There are 10 such paths:



(2) There are 4 such paths:



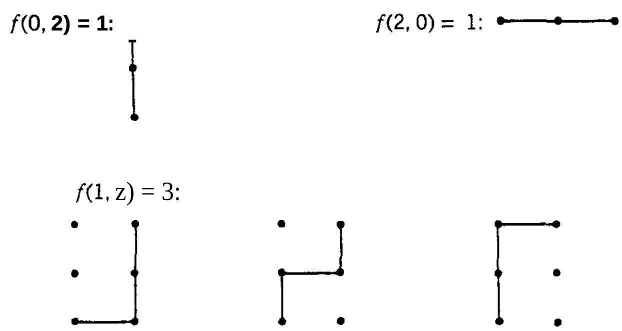
(3) There are no such paths.

1.2 To get outside the rectangle of Figure 3.1 would require either more than 5 *N*s or more than 6 *E*s (or both). Since *P is* 6 units to the right of *M* and 5 units above *M*, any sequence of 6 *Es* and 5 Ns will correspond with a path from *M* to *P*. Any sequence consisting of exactly 3 *Es* and 2 Ns corresponds with a trip from *M* to *P in* Figure 3.2 and any such trip corresponds with such a sequence.

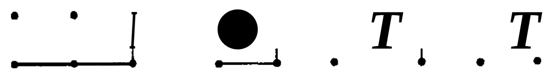
1.3 Read on to learn the true answer.

SECTION 2

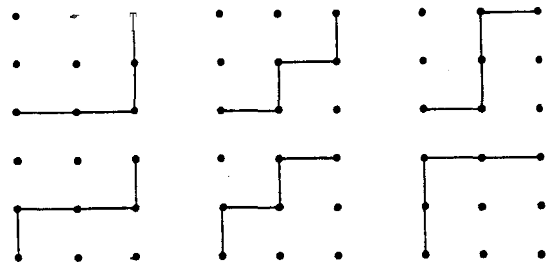
2.1 (a)



$f(2, 1) = 3$:



$f(2, 2) = 6$:



(b) $f(0, 3) = 1 = f(3, 0)$, $f(1, 3) = 4 = f(3, 1)$, and $f(2, 3) = 10 = f(3, 2)$.

2.2 There is just one shortest path from $(0, 0)$ to $(i, 0)$, a straight line consisting of a path of i *Es*. Thus $f(i, 0) = 1$. Similarly, $f(0, j) = 1$.

23 Use the fact that $f(i, j) = f(i - 1, j) + f(i, j - 1)$:

$$f(3, 3) = f(2, 3) + f(3, 2) = 10 + 10 = 20.$$
$$f(4, 2) = f(3, 2) + f(4, 1) = 10 + f(4, 1)$$
$$= 10 + f(3, 1) + f(4, 0) = 10 + 4 + 1 = 15.$$

2.4 The fifth row: 1 5 10 10 5 1. The coordinates of all points that end up on the fifth row of Pascal's triangle is $\{(i, j)\ i + j = 5\}$. The f values of these points are exactly the corresponding entries of Pascal's triangle.

2.5 $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, and $8! = 40{,}320$. The first n such that $n! > 1{,}000{,}000$ is $n = 10$. This can be determined by continuing to calculate $9! = 362{,}880$ and $10! = 3{,}628{,}800 > 1{,}000{,}000$. Any value greater than 10 would also do.

2.6 $\binom{3}{0} = \binom{3}{3} = 1$, $\binom{3}{1} = \binom{3}{2} = 3$. $\binom{4}{i}$ and $\binom{5}{j}$ turn out to be the fourth and fifth row of Pascal's triangle; see Figure 3.7.

2.7 Use the definition of $\binom{m}{i}$:

$$\binom{k}{0} = \frac{k!}{0!(k - 0)!} = \frac{k!}{k!} = 1 = \frac{k!}{k!} = \frac{k!}{k!(k - k)!} = \binom{k}{k}$$

Similarly,

$$\binom{k}{k-1} = \frac{k!}{(k-1)!(k-(k-1))!} = k = \frac{k!}{1!(k-1)!} = \binom{k}{1}.$$

2.8 We use the fact that x_2 is the sum of the two numbers above:

$$x_2 = \binom{k}{1} + \binom{k}{2} = k + \frac{k!}{2!(k-2)!} = k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2} = \binom{k+1}{2}$$

2.9 Use the formula $f(P) = \binom{m+n}{m} = \frac{(4+3)!}{4!3!} = \frac{7!}{4!3!} = 35$.

SECTION 3

3.1 The fifteen 4-subsets of A are

$\{a_1, a_2, a_3, a_4\}$ $\{a_1, a_2, a_3, a_5\}$ $\{a_1, a_2, a_3, a_6\}$
$\{a_1, a_2, a_4, a_5\}$ $\{a_1, a_2, a_4, a_6\}$ $\{a_1, a_2, a_5, a_6\}$
$\{a_1, a_3, a_4, a_5\}$ $\{a_1, a_3, a_4, a_6\}$ $\{a_1, a_3, a_5, a_6\}$
$\{a_1, a_4, a_5, a_6\}$ $\{a_2, a_3, a_4, a_5\}$ $\{a_2, a_3, a_4, a_6\}$
$\{a_2, a_3, a_5, a_6\}$ $\{a_2, a_4, a_5, a_6\}$ $\{a_3, a_4, a_5, a_6\}$

Note that

$$\binom{6}{4} = \frac{6!}{4!2!} = 15.$$

The six 5-subsets of A are

$\{a_1, a_2, a_3, a_4, a_5\}$ $\{a_1, a_2, a_3, a_4, a_6\}$
$\{a_1, a_2, a_3, a_5, a_6\}$ $\{a_1, a_2, a_4, a_5, a_6\}$
$\{a_1, a_3, a_4, a_5, a_6\}$ $\{a_2, a_3, a_4, a_5, a_6\}$

Note that

$$\binom{6}{5} = \frac{6!}{5!1!} = 6.$$

3.2

3-subsets of A containing a_6	Remaining 3-subsets of A
$\{a_1, a_2, a_6\}$	$\{a_1, a_2, a_3\}$
$\{a_1, a_3, a_6\}$	$\{a_1, a_2, a_4\}$
$\{a_1, a_4, a_6\}$	$\{a_1, a_2, a_5\}$
$\{a_1, a_5, a_6\}$	$\{a_1, a_3, a_4\}$
$\{a_2, a_3, a_6\}$	$\{a_1, a_3, a_5\}$
$\{a_2, a_4, a_6\}$	$\{a_1, a_4, a_5\}$
$\{a_2, a_5, a_6\}$	$\{a_2, a_3, a_4\}$
$\{a_3, a_4, a_6\}$	$\{a_2, a_3, a_5\}$
$\{a_3, a_5, a_6\}$	$\{a_2, a_4, a_5\}$
$\{a_4, a_5, a_6\}$	$\{a_3, a_4, a_5\}$

3.3 (a) $\binom{6}{3} = 6!/(3!3!) = 20$ (b) $\binom{11}{5} = 11!/(5!6!) = 462$

(c) $\binom{17}{9} = 17!/(9!8!) = 24{,}310$.

3.4 There are $\binom{11}{3} = 1651$ l-letter sequences of Ns and Es with exactly 3 Es.

There are $\binom{11}{4} = 3301$ l-letter sequences of Ns and Es with exactly 4 Es.

And there are $\binom{11}{7} = \binom{11}{4} = 330\ 1$ l-letter sequences of Ns and Es with exactly 7 Es. There are $2^{11} = 2048$ sequences of Es and Ns containing 11 letters.

3.5 We can transform a j-subset of A to a j-subset of I_n by renaming each element a_i of A as the integer i. Similarly, we can transform a j-subset of I_n to a j-subset of A by renaming the element i of I_n as a_i. The 4-subset of I_n that corresponds with $\{$al$, a_2, a_4, a_{n-1}\}$ is $\{1,2,4, n-1\}$. The 5-subset of A that corresponds with $\{1,2,3,5, 8\}$ is $\{a_1, a_2, a_3, a_5, a_8\}$.

3.6 {1, 3,4,7}, {1,3, 5,6}, {2, 3,4,7}, {2,3, 5,6}, {3,6, 7, 8}, {4,6,7,8}

3.7 {1,2,3}, {1,2,4}, {1,2,5}, {1,2,6}, {1,3,4}, {1,3,5}, {1,3,6}, {1,4,5}, {1,4,6}, {1, 5,6}, {2, 3,4}, {2, 3,5}, {2,3,6}, {2,4,5}, {2,4, 6}, {2, 5, 6}, {3,4,5}, {3,4,6}, {3, 5,6}, {4, 5,6}.

3.8 {1,2,3}, {1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, {1,3,4}, {1,3,5}, {1,3,6}, {1,3,7}, {1,4,5}, {1,4,6}, {1,4,7}, {1,5,6}, {1,5,7}, {1,6,7}, {2,3,4}, {2,3,5}, {2,3,6}, {2, 3,7}, {2,4,5}, {2,4,6}, {2,4,7}, {2,5,6}, {2, 5, 7}, {2,6, 7}, {3,4, 5}, {3,4,6}, {3,4,7}, {3, 5,6}, {3, 5,7}, {3,6, 7}, {4, 5,6}, {4, 5,7}, {4, 6, 7}, {5,6,7}.

3.9 Algorithm JSET run on $j = 4$ and $n = 6$:

Step	*No.*	h	b_h	$n+h-j$	*FOUND*	k	b_k	*SUBSET*
2								{1,2,3,4}
3		5			false			
4,5,6		4	4	6	true			
8		4	5					
9-12						5		{1,2,3,5}
3		5			false			
4,5,6		4	5	6	true			
8		4	6					
9-12						5		{1,2,3,6}
3		5			false			
4,5,6		4	6	6	false			
		3	3	5	true			
8		3	4					
9, 10						4	5	
11, 12								{1,2,4,5}
3		5			false			
4,5,6		4	5	6	true			
8		4	6					
9–12						5		{1,2,4,6}
3		5			false			
4, 5,6		4	6	6	false			
		3	5	5	true			
8		3	5					
9, 10						4	6	
11, 12								{1,2,5,6}
3		5			false			
4,5,6		4	6	6	false			
		3	5	5	false			
		2	2	4	true			
8		2	3					
9, 10						3	4	
						4	5	
11, 12								{1,3,4,5}
3		5			false			
4,5,6		4	5	6	true			
8		4	6					
9-12						5		{1,3,4,6}
3		5			false			
4, 5,6		4	6	6	false			
		3	4	5	true			
8		3	5					
9-12						4	6	{1,3,5,6}

3.9 (continued)

Step	*No.*	*h*	b_h	$n+h-j$	*FOUND*	*k*	b_k	*SUBSET*
3		5			false			
4, 5,6		4	6	6	false			
		3	5	5	false			
		2	3	4	true			
8		2	4					
9-12						3	5	{1,4,5,6}
3		5			false			
4, 5,6		4	6	6	false			
		3	5	5	false			
		2	4	4	false			
		1	1	3	true			
8		1	2					
9-12						2	3	{2, 3,4, 5}
3		5			false			
4, 5,6		4	5	6	true			
8		4	6					
9-12						5		{2,3,4,6}
3		5			false			
4, 5,6		4	6	6	false			
		3	4	5	true			
8		3	5					
9-12						4	6	{2,3, 5,6}
3		5			fake			
4,5,6		4	6	6	fake			
		3	5	5	false			
		2	3	4	true			
8		2	4					
9-12						3	5	{2, 4,5, 6}
3		5			false			
4,5,6		4	6	6	false			
		3	5	5	false			
		2	4	4	fake			
		1	2	3	true			
8		1	3					
9-12						2	4	{3,4, 5,6}
3		5			false			
4,5,6		4	6	6	false			
		3	5	5	false			
		2	4	4	false			
		1	3	3	false			
7					false	STOP		

3.10 The algorithm JSET produces $\binom{n}{3} = n(n-1)(n-2)/6 = O(n^3)$ subsets when $j = 3$.

3.11 Note $\binom{n}{j} = \dfrac{n(n-1)(n-2)\cdots(n-j+1)}{j!} \le \dfrac{n^j}{j!} = O(n^j)$.

SECTION 4

4.1 The 4! permutations of the set {1,2,3,4} are
(1234), (1243), (1423), (4123), (1324),
(1342), (1432), (4132), (3124), (3142),
(3412), (4312), (2134), (2143), (2413),
(4213), (2314), (2341), (2431), (4231),
(3214), (3241), (3421), (4321).

4.2 A 3-subset can be formed by filling in the three blanks {_, _,_} with distinct elements of the n-set. There are n choices for the first blank, $n-1$ choices for the second blank, and $n-2$ choices for the third blank. By the Multiplication Principle, there are a total of $n(n-1)(n-2)$ ways to fill in the blanks. However, once a subset is "filled in," then every permutation of the elements in that set will produce the same set. There are 3! permutations of each set and so in the total of $n(n-1)(n-2)$, each set is listed 3! times. Thus there are $n(n-1)(n-2)/6$ different 3-subsets of an n-set.

43 Trace of algorithm PERM run on S = {1,2,3,4}:

Values of j	*Permutations*
1	(1)
2	(1 2) (2 1)
3	(1 2 3) (1 3 2) (3 1 2)
	(2 1 3) (2 3 1) (3 2 1)
4	(1 2 3 4) (1 2 4 3) (1 4 2 3) (4 1 2 3)
	(1 3 2 4) (1 3 4 2) (1 4 3 2) (4 1 3 2)
	(3 1 2 4) (3 1 4 2) (3 4 1 2) (4 3 1 2)
	(2 1 3 4) (2 1 4 3) (2 4 1 3) (4 2 1 3)
	(2 3 1 4) (2 3 4 1) (2 4 3 1) (4 2 3 1)
	(3 2 1 4) (3 2 4 1) (3 4 2 1) (4 3 2 1)

4.4 To find an integer N so that for $n \geq N$, $n! > 10 \cdot 2^n$, proceed as in the proof of Theorem 4.2: Suppose that N = Cr^r = $10 \cdot 2^2$ = 40. Then for $n > 40$, we know that $n! > 10 \cdot 2^n$.

SECTION 5

5.1 The four colors of the code are *r, y, b,* and w. It is not possible to determine their order.

5.2 We may assume, by Question 5.1, that the colors in the code are *r*, *y*, *b*, and w. The secret code can be determined from the guesses given in Example 5.1 by the following reasoning

Deduction	*Guesses and Deductions Used*
1. *r* and y are not in the first and second positions, respectively	Guess 2
2. Either *b is* in the third position or w is in the fourth position (but not both)	Guess 1 and Guess 2
3. Either w is in the second position or y is in the third position (but not both)	Guess 2 and Guess 3
4. w is in the second position	Deduction 3 and Guess 4
5. *b* is in the third position	Deductions 2 and 4
6. *r is* in the fourth position	Deductions 1,4, and 5
7. y is in the first position **Result:** Code = y w *b r*	Deductions 1,4, 5, and 6

Note that this reasoning is only one of many ways to arrive at the above result.

5.3 Left to the reader.

5.4 The number of codes is 4! = 24. Use PERM or the results of Question 4.3.

5.5 *g g b p g g p b g b g p g p g b g p b g g b p g*
b g g p p g g b p g b g b g p g p b g g b p g g

5.6 We have seen that if the code has 4 colors, then there are 24 possible codes, and if the code has 3 colors, there are 12 codes (see Question 5.5). If there are just 2 colors, there are 4 ways to have 3 of one color and 1 of the other color, and there are $\binom{4}{2} = 6$ ways to have 2 colors occurring twice each. If there is one color, there is only one possible code. Thus the maximum number of possible codes and hence in step 2 of Algorithm is 24, when there are 4 different colors.

SECTION 6

6.1 $n = 3$: $\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0.$

$n = 4$: $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 1 - 4 + 6 - 4 + 1 = 0.$

$n = 5$:

$$\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 1 - 5 + .0 - .0 + 5 - 1 = 0.$$

6.2 $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 1 + 3 + 6 = 10 = \binom{5}{3}.$

$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 1 + 3 + 6 + 10 = 20 = \binom{6}{3}.$

$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} = 1 + 3 + 6 + 10 + 15 = 35 = \binom{7}{3}.$

From this evidence it seems reasonable to conjecture that

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{i}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

For a proof of this fact, read the rest of this section.