7

RECURRENCE RELATIONS

7:1 BEGINNINGS OF SEQUENCES

A few years ago an advertisement on the London subway (or tube) system read as follows:

"If you can determine the next number in each of the following lists before you arrive at your stop, come in and we'll offer you a job!

1	2	4	8	16	32	_
0	1	3	6	10	15	_
1	1	2	3	5	8	_
2	9	28	65	126	_	
2	3	5	7	11	13	_"

Question 1.1. Find a plausible next entry in as many of the above lists as you can.

This chapter explores intrinsic properties of lists or sequences of numbers. Given a partial list, we would like to determine its next entry. More generally, we would like to find a formula for the nth number on the list. From this we can determine the growth rate of the sequence, that is, we can discover whether the nth number in the sequence grows like a polynomial or an exponential function of n. Many of the sequences we shall study have appeared before in the course; others are important in combinatorics and algorithms.

Definition. A **sequence** is a function whose domain is the positive integers and whose target is the real numbers. A sequence whose target is the integers is called an **integer sequence**.

In the London subway puzzle we see initial segments of five integer sequences.
We represent a sequence

$$S: a_1, a_2, a_3, \ldots, a_n, \ldots$$

by listing the values of the sequence at the integers $1, 2, 3, \ldots$. Even though a sequence is a function, it is common and convenient to use the subscript notation. We call a_1 the first term of S, a_2 the second term, and in general a_n the ***n*th term of the sequence**. Occasionally, we shall extend the domain of a sequence to include zero (e.g., Fibonacci numbers). In this case we talk about the 0th term of a sequence.

Example 1.1. Here are two common sequences:

$$S_1: 1, 2, 3, 4, 5, 6, \ldots,$$
$$S_2: 2, 4, 8, 16, 32, 64, \ldots.$$

For these sequences a formula for the nth term is not hard to guess: for S_1, $a_n = n$ and for S_2, $a_n = 2^n$. In many situations a formula for a sequence may not be immediately apparent.

Example 1.2. In the London subway problem, you probably recognized the Fibonacci numbers; let S_3 be this sequence:

$$S_3: 1, 1, 2, 3, 5, 8, \ldots.$$

We know that each Fibonacci number (after the first two) is the sum of its two immediate predecessors. Knowing this pattern, we can in principle calculate any Fibonacci number. However, the formula for the nth term is not obvious. Recall that, in Chapter 4, we verified this formula by induction but deferred until this chapter how such a formula could be discovered.

Question 1.2. Here are the initial segments of some (possibly familiar) integer sequences:

$$S_4: 1, 2, 6, 24, 120, \ldots$$
$$S_5: 0, 1, 3, 6, 10, 15, \ldots$$
$$S_6: 1, 3, 7, 15, 31, \ldots$$
$$S_7: 2, 3, 5, 7, 11, 13, 17, \ldots$$
$$S_8: 1, 4, 9, 16, 25, 36, \ldots$$
$$S_9: 1, -3, 9, -27, 81, \ldots$$

For at least two of the sequences S_4 to S_9, find a formula that generates the initial segment of the sequence as listed above.

Notice that there might be ambiguity when we see only the initial segment of a sequence. We can't be sure about the numbers that appear in the ... until we have a precise description of the sequence.

Example 1.3. Here are three different functions that each generate the initial segment of S_4:

$$f_n = n!$$
$$g_n = (\tfrac{1}{24})[(n-2)(n-3)(n-4)(n-5) - 8(n-1)(n-3)(n-4)(n-5) + 36(n-1)(n-2)(n-4)(n-5) - 96(n-1)(n-2)(n-3)(n-5) + 120(n-1)(n-2)(n-3)(n-4)] \qquad \{\text{try it!}\}$$
$$h_n = \begin{cases} n! & \text{if } n \leq 5 \\ 0 & \text{if } n > 5. \end{cases}$$

Question 1.3. Find two functions that each produce the first three values of the sequence: $5, 9, 17, \ldots$.

Formulas are not the only way to describe sequences. Look at S_7: A moment's reflection leads one to conjecture that this is the sequence of all prime numbers. Although this description is exact, we cannot write down a formula for a function that generates the primes. On the other hand, we could write down an algorithm using the Sieve of Eratosthenes (as described in Supplementary Exercise 4 of Chapter 4) to determine the next prime.

Another way that words can describe a sequence is by identifying patterns within the sequence. We know that each Fibonacci number is the sum of the two preceding ones:

$$F_n = F_{n-1} + F_{n-2},$$

and this pattern completely specifies the Fibonacci numbers once we know the first two values. Such a pattern is known as a recurrence relation.

Example 1.4. Notice that in S_5: $0, 1, 3, 6, 10, 15, \ldots$, the difference of successive terms is 1, 2, 3, and so on. In symbols,

$$a_n = a_{n-1} + (n-1) \qquad \text{for } n > 1. \tag{A}$$

Thus, like the Fibonacci numbers, the sequence S_5 is completely determined once we specify this pattern and the fact that $a_1 = 0$.

Question 1.4. For the sequence S_1 find an equation that relates a_n and a_{n-1}. Then do the same for S_2.

Example 1.5. Suppose that an algorithm SECRET performs M_n steps upon input of a positive integer n. Trial runs show that $M_1 = 1$, $M_5 = 3$, and $M_8 = 4$. Suppose that by analyzing SECRET we find the following pattern or recurrence relation:

$$M_n = M_{\lfloor n/2 \rfloor} + 1 \quad \text{if } n > 1; \qquad M_1 = 1. \tag{B}$$

Is SECRET good or exponential? The known values of M_n together with others we could calculate specify the start of a sequence; however, we don't know a formula for M_n. In Section 5 we shall discover such a formula and deduce that SECRET is a good algorithm.

Question 1.5. Starting with $a_1 = 0$, use equation (A) to calculate $a_2, \ldots, a_6$. Then use equation (B) to calculate $M_1, \ldots, M_8$.

Question 1.6. Here are two different recurrence relations for a sequence known as the **harmonic numbers**.

Let $H'_1 = 1$, and for $n > 1$ let $H'_n = H'_{n-1} + \dfrac{1}{n}$.

Let $H''_1 = 1$, and for $n > 1$ let

$$H''_n = \frac{1}{n}[H''_{n-1} + H''_{n-2} + \cdots + H''_1] + 1.$$

Determine the first five values of H'_n and H''_n.

Question 1.7. Let $C_1 = 1$, and for $n > 1$ let

$$C_n = C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1.$$

Determine the first five values of C_n. These are known as the **Catalan numbers**.

This chapter will present techniques to discover and verify formulas for the nth term of a sequence given an initial segment and a recurrence relation.

EXERCISES FOR SECTION 1

1. Give an example of a function that is not a sequence. Give an example of a sequence that is not an integer sequence.
2. Which of the following prescribes a sequence?
   (*a*) $2, 4, 6, 8, \ldots, 2n, \ldots$
   (*b*) $f_n = 3n - 1$.

(*c*) $g_n = 1/n$.

(*d*) $h_n = \pm\sqrt{n}$.

(*e*) $f_n = \dfrac{1}{n^3 - 14n^2 + 64n - 90}$.

(*f*) $G_1 = 1$, $G_2 = 2$ and $G_n = G_{n-1} + 2G_{n-2}$ for $n \geq 3$.

(*g*) $H_n = H_{n-1} + H_{n-2}$.

(*h*) A list of all positive integers that are perfect cubes.

(*i*) The set of all real numbers.

3. Which of the sequences in the preceding problem are integer sequences?

4. Match the following initial segments of sequences and formulas:

(*a*) 1 3 12 60 360 ⋯	(i) $f_n = n^2 - 6n + 8$
(*b*) 3 3 6 9 15 24 ⋯	(ii) $f_n = n^2 - 6n + 10$
(*c*) 2 4 8 14 22 32 ⋯	(iii) $f_n = n^2 + n - 2$
(*d*) 4 1 0 1 4 ⋯	(iv) $f_n = n^2 - n + 2$
(*e*) 1 4 10 20 35 ⋯	(v) $f_n = 2F_n$, where F_n is the nth Fibonacci number
(*f*) 0 2 6 14 30 ⋯	(vi) $f_n = 3F_n$
(*g*) 1 8 27 64 125 ⋯	(vii) $f_n = 2\binom{n}{2}$
(*h*) 5 2 1 2 5 ⋯	(viii) $f_n = 2^n - 1$
	(ix) $f_n = \binom{n+2}{3}$
	(x) $f_n = n^2$
	(xi) $f_n = (n+1)!/2$
	(xii) $f_n = 2n!$
	(xiii) $f_n = n^2$
	(xiv) $f_n = (n-3)^2$
	(xv) $f_n = 3n^2$
	(xvi) $f_n = n!$
	(xvii) $f_n = n^3$
	(xviii) $f_n = 2^{n-2}$
	(xix) $f_n = 2^n - 2$
	(xx) $f_n = (-1)^{n-1} \cdot 3^{n-1}$
	(xxi) $f_n = (n^2 - n)/2$

5. For each of S_4, S_5, S_6, S_8, and S_9 in Question 1.2 find a formula among those listed in Exercise 4 that generates the sequence.

6. For how many of their first entries do $f_n = n^3 - 3n^2 + 2n$ and $g_n = n^4 - 13n^3 + 56n^2 - 92n + 48$ agree?

7. The recurrence relation $F_n = F_{n-1} + F_{n-2}$ produces the Fibonacci numbers with the initial values $F_1 = F_2 = 1$. Give an example of initial values that produce a different sequence. Then find initial values that produce some Fibonacci numbers, but not all of them. Characterize the initial values that produce a sequence whose entries are each a Fibonacci number.

8. Define the extended Fibonacci numbers by $G_1 = 1$, $G_2 = 1$, $G_3 = 2$, and for $n > 3$

$$G_n = G_{n-1} + G_{n-2} + G_{n-3}.$$

Are the following statements true or false about G_n?

(i) For $n \geq 2$, $G_n = \binom{n}{2} + 1$.
(ii) For $n > 3$, $G_n \neq F_n$.
(iii) G_n is a Fibonacci number, but not necessarily F_n.
(iv) $G_n = F_n + G_{n-3}$ for $n > 3$.
(v) $G_n = O(2^n)$.

9. Here is a famous sequence of letters; identify the pattern:

$$o \quad t \quad t \quad f \quad f \quad s \quad s \quad e \quad n \quad t \quad e \quad \cdots.$$

10. Find two functions f and g such that $f_1 = g_1 = 3$ and $f_3 = g_3 = 5$, but $f_2 \neq g_2$.

11. For each of the following, find a formula that expresses the nth term f_n as a function of n:
(i) $f_1 = 1$, $f_n = f_{n-1} + 2$ for $n > 1$.
(ii) $f_1 = 2$, $f_n = f_{n-1} + 2n - 1$ for $n > 1$.
(iii) $f_1 = 1$, $f_n = f_{n-1} + 2n$ for $n > 1$.

12. Verify that for $n = 1, 2, \ldots, 5$, $H_n = 1 + (1/2) + \cdots + (1/n)$ satisfies both recurrence relations in Question 1.6. Use induction to show that this formula works in general.

13. If H_n is the nth harmonic number, then show that

$$H_{2^m} \geq 1 + \frac{m}{2}.$$

Explain why for every positive integer M there is an integer N such that $H_n \geq M$ for all $n \geq N$. Show also that

$$H_{2^m - 1} \leq m.$$

14. Let H_n be the nth harmonic number. Show that

$$H_n \leq \frac{(n+1)}{2}$$

in two different ways: (i) by induction using the recurrence relation for H_n'' and (ii) using the formula for H_n given in Exercise 12.

15. Show that $1 + \frac{\lfloor \log(n) \rfloor}{2} \leq H_n \leq 1 + \lfloor \log(n) \rfloor$ for $n > 0$.

16. The Catalan numbers of Question 1.7 satisfy the formula

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Verify this for $n \leq 5$.

17. Show that $\frac{1}{n}\binom{2n-2}{n-1}$ is always an integer.

18. The **Bernoulli numbers** are defined by $B_0 = 1$ and for $n > 0$,

$$B_n = -\frac{1}{n+1}\left[\binom{n+1}{n-1}B_{n-1} + \binom{n+1}{n-2}B_{n-2} + \cdots + \binom{n+1}{0}B_0\right].$$

Determine the next five values of B_n.

19. Check that $B_7 = B_9 = 0$.

20. Find the first Bernoulli number that is greater than 1.

21. Although the Bernoulli numbers start out small, the even ones grow very quickly. It can be shown that

$$|B_{2n}| = \frac{O((2n)!)}{(2\pi)^{2n}},$$

and that

$$\frac{(2n)!}{(2\pi)^{2n}} = O(|B_{2n}|).$$

Use the bounds on $n!$ derived from Stirling's formula (in Section 3.4) to obtain bounds on the growth rate of $|B_{2n}|$.

7:2 ITERATION AND INDUCTION

In this section we begin to explore ways to deduce a formula for a sequence given a recurrence relation that the terms of the sequence satisfy.

When we look at the London subway puzzles or the sequences S_1 to S_9, listed in the previous section, we can, without too much difficulty, find patterns in the sequence entries. We found that the sequence S_5 satisfies the recurrence relation,

$$a_n = a_{n-1} + (n-1), \qquad n > 1. \tag{A}$$

In Question 1.4 you determined that for S_1

$$a_n = a_{n-1} + 1,$$

and for S_2

$$a_n = 2a_{n-1}.$$

Here is more precisely what we are looking for.

Definition. Suppose that S is the sequence

$$S: a_1, a_2, a_3, \ldots, a_n, \ldots.$$

If the nth term of S can be expressed as a function of previous terms in the sequence:

$$a_n = f(a_1, a_2, \ldots, a_{n-1}), \tag{B}$$

then equation (B) is called a **recurrence relation**, and we say that the sequence S **satisfies** that recurrence relation.

The function in (B) may depend on only some of the previous entries or it may depend upon all of them. The former happens frequently, but Questions 1.6 and 1.7 illustrate the latter possibility.

In a sense, once we have found a recurrence relation underlying a sequence we are done. We can use this relation to find the next (or any subsequent) term. However, it might get tedious to calculate a_{100}. It is important to look for a formula that would give us a_n directly.

Example 2.1. Let S_{10} be the sequence

$$S_{10}: 1, 3, 4, 7, 11, 18, \ldots,$$

where, like the Fibonacci numbers, each term is the sum of the preceding two terms. Thus a sequence is not completely specified by its recurrence relation.

Question 2.1. Find recurrence relations for the sequences $S_6: 1, 3, 7, 15, 31, \ldots,$ and $S_9: 1, -3, 9, -27, 81, \ldots.$ In each case, find a different sequence satisfying the same recurrence relation.

Once we have enough initial values of a sequence together with the recurrence relation, the sequence is determined. For example, if a sequence begins with $a_1 = 1$ and $a_2 = 1$ and then obeys

$$a_n = a_{n-1} + a_{n-2}, \tag{C}$$

we get the Fibonacci numbers; however, if $a_1 = 1$ and $a_2 = 3$ then the sequence S_{10} results.

Definition. Let k be the least integer such that once values are assigned to $a_1, a_2, \ldots, a_k$, then (B) prescribes a unique value for each a_n with $n > k$. Then the values of $a_1, a_2, \ldots, a_k$ are called the **initial conditions** of the recurrence relation. We say that the recurrence relation together with its initial conditions **generates** the sequence

$$S: a_1, a_2, \ldots, a_n, \ldots.$$

Typically, a recurrence relation will be given in the form

$$a_n = f(a_1, \ldots, a_{n-1}) \qquad \text{for } n > k,$$

where k is some fixed integer. The bound "$n > k$" specifies the range over which the recurrence relation holds, and the initial conditions that must be assigned are the values of $a_1, a_2, \ldots,$ and a_k.

Example 2.2. The recurrence relation (A) requires only one initial condition, the value of a_1. If $a_1 = 0$, the sequence generated is that given in S_5. If $a_1 = 2$, the sequence generated is

$$2, 3, 5, 8, 12, \ldots$$

(which is not a subsequence of Fibonacci numbers!). The recurrence relation

$$a_n = a_{n-4} + a_{n-2} \qquad \text{for } n > 4$$

expresses a_n in terms of two previous values but requires four initial conditions before all values of the sequence are uniquely defined.

Question 2.2. In the recurrence relation $a_n = a_{n-4} + a_{n-2}$, why are fewer than four initial conditions not enough to define a_n for all larger n? If $a_1 = a_2 = a_3 = a_4 = 1$, find the first 10 terms of the sequence determined by this recurrence relation. Describe the resulting sequence. Then determine an explicit formula for a_n.

Question 2.3. For each of the following, determine the number of initial conditions that must be assigned so that a unique sequence is generated:
(i) $a_n = na_{n-2}$
(ii) $a_n = a_{n-1} + a_{n-3}$
(iii) $a_n = 2a_{\lfloor n/2 \rfloor}$.

Example 2.3. Consider the recurrence relation (A) with initial condition $a_1 = 0$. Repeated application of (A) will lead to a formula. Since $a_{n-1} = a_{n-2} + (n-2)$, substitution in (A) yields

$$a_n = a_{n-2} + (n-2) + (n-1). \tag{D}$$

Since $a_{n-2} = a_{n-3} + (n-3)$, substitution in (D) yields

$$a_n = a_{n-3} + (n-3) + (n-2) + (n-1).$$

Continuing until we reach $a_1 = 0$, we get

$$\begin{aligned} a_n &= a_1 + 1 + 2 + \cdots + (n-2) + (n-1) \\ &= 0 + 1 + 2 + \cdots + (n-2) + (n-1) \\ &= \frac{n(n-1)}{2}, \qquad \text{(see Example 2.3.2)} \end{aligned}$$

a formula for the nth term of the sequence S_5.

The process used in Example 2.3 is known as **iteration** and in straightforward cases will lead to a formula for the sequence.

Question 2.4. Use iteration on each of the following recurrence relations and initial conditions to obtain a formula for the sequence they generate:
(i) $a_n = na_{n-1}$ for $n > 1$, $a_1 = 1$.
(ii) $b_n = b_{n-1} + 2$ for $n > 1$, $b_1 = 1$.

It seems clear that the formulas we come up with using iteration are correct, but to be certain we need to use induction. In Example 2.3 we decided that $a_n = n(n-1)/2$; now we prove that this formula is correct.

Example 2.3 (continued). Here is an inductive proof.

Theorem. $a_n = n(n-1)/2$ satisfies (A) with initial condition $a_1 = 0$.

First the base case: $a_1 = 1 \cdot 0/2 = 0$. We want to use the assumption that $a_k = k(k-1)/2$ to prove that

$$a_{k+1} = \frac{(k+1)(k+1-1)}{2} = \frac{(k+1)k}{2}.$$

To accomplish this, we begin with the recurrence relation:

$$\begin{aligned} a_{k+1} &= a_k + k \\ &= \frac{k(k-1)}{2} + k && \text{by the inductive hypothesis} \\ &= k\left[\frac{(k-1)}{2} + 1\right] && \text{by arithmetic} \\ &= \frac{k(k+1)}{2} \end{aligned}$$

just as we wanted. □

Question 2.5. Prove by induction that the formulas you obtained in Question 2.4 are correct.

Definition. A recurrence relation is called **homogeneous** if it is satisfied by the sequence that is identically zero (i.e., $a_n = 0$ for all n). Otherwise, it is called **inhomogeneous**.

Example 2.4. To test whether a recurrence relation is homogeneous, replace every a_j with zero and see if, for all n, a valid identity remains. For example, $a_n = a_{n-1} + a_{n-2}$ becomes $0 = 0 + 0$ and so is homogeneous, but $a_n = a_{n-1} + (n-1)$ becomes $0 = 0 + n - 1$ and is consequently inhomogeneous.

Question 2.6. Which of the recurrence relations of Questions 2.3 and 2.4 are homogeneous and which inhomogeneous?

Our only suggestion for solving inhomogeneous recurrence relations is the method of iteration and induction. If that fails, then it is time to consult a book specializing in recurrence relations.

How well does iteration and induction work on homogeneous recurrence relations? Let's try to use it to obtain the formula for the Fibonacci numbers knowing that they satisfy (C) with initial conditions $a_1 = a_2 = 1$. Since $a_{n-1} = a_{n-2} + a_{n-3}$, and $a_{n-2} = a_{n-3} + a_{n-4}$, we substitute these into (C) to obtain

$$a_n = a_{n-2} + 2a_{n-3} + a_{n-4}. \tag{E}$$

No formula is yet apparent, so let's keep substituting in the right-hand side of (E) using

$$a_{n-2} = a_{n-3} + a_{n-4},$$
$$a_{n-3} = a_{n-4} + a_{n-5},$$

and

$$a_{n-4} = a_{n-5} + a_{n-6}.$$

We get

$$\begin{aligned} a_n &= a_{n-3} + a_{n-4} + 2(a_{n-4} + a_{n-5}) + a_{n-5} + a_{n-6} \\ &= a_{n-3} + 3a_{n-4} + 3a_{n-5} + a_{n-6}. \end{aligned}$$

Still no formula has emerged, although the coefficients seem familiar. In fact, continuing in this vein never leads to the correct formula.

In summary, we have learned one technique that sometimes obtains a formula for a sequence. Given a sequence we first find a recurrence relation that it satisfies. Then we try iteration and induction to derive a formula for the sequence and to prove it correct. This technique is most likely to work for inhomogeneous recurrence relations. In Section 5 we shall use this technique to solve recurrence relations related to algorithms from Chapter 6. If iteration does not work (easily!) on a homogeneous recurrence relation, then we can use the techniques of the next sections.

EXERCISES FOR SECTION 2

1. Write down three new recurrence relations and specify the number of initial conditions. Which are homogeneous?

2. Here are two recurrence relations and initial conditions:
(i) $c_n = 2c_{n-1}$ for $n > 1$, $c_1 = 1$.
(ii) $d_n = d_{n-1} - d_{n-2}$ for $n > 2$, $d_1 = 1$, $d_2 = 2$.

For each, find different initial conditions that produce a sequence that is a subset of the original sequence. Then find initial conditions that produce a sequence that has no number in common with the original sequence.

3. For each of the following, determine the number of initial conditions:
   (i) $a_n = 2a_{n-1}a_{n-2}$
   (ii) $a_n = a_{n-2} - a_{n-3}$
   (iii) $a_n = a_1 + 2^n$
   (iv) $a_n = \begin{cases} 3a_{n/3} & \text{if 3 divides } n \\ 2a_{(n-1)/3} & \text{if 3 divides } (n-1) \\ a_{(n-2)/3} & \text{if 3 divides } (n-2). \end{cases}$
4. Use iteration on each of the following recurrence relations to obtain a formula for the sequence they generate:
   (i) $b_n = 2b_{n-1}$ for $n > 1$, $b_1 = 1$.
   (ii) $c_n = c_{n-1} + (2n - 2)$ for $n > 1$, $c_1 = 0$.
   (iii) $d_n = 2d_{n-1} + 2$ for $n > 1$, $d_1 = 1$.
   (iv) $e_n = e_{n-1} + (2n - 1)$ for $n > 1$, $e_1 = 1$.
   (v) $f_n = f_{n-1} + 3^n$ for $n > 1$, $f_1 = 3$.
   (vi) $g_n = g_{n-1} + k$ for $n > 0$ and k constant, $g_0 = 1$.
   (vii) $h_n = h_{n-1} + (-1)^{n+1}n$ for $n > 1$, $h_1 = 1$.
   (viii) $j_n = (n - 2)j_{n-1}$ for $j > 2$, $j_1 = 5$, $j_2 = 10$.
   (ix) $k_n = (4n^2 - 2n)k_{n-1}$ for $n > 0$, $k_0 = 1$.
   (x) $l_n = l_{n-1}l_{n-2}$ for $n > 2$, $l_1 = l_2 = 2$.
   (xi) $m_n = m_{n-1} + (n - 1)^2$ for $n > 1$, $m_1 = 0$.
5. Prove by induction that your formulas in the preceding exercise are correct.
6. Notice that if the formula for the nth term of a sequence is known, then it is easy to detect recurrence relations for the sequence. For example if $a_n = n!$, then $a_n = na_{n-1}$. Explain why the following equations give a means of finding recurrence relations from formulas:

$$a_n = \frac{a_n}{a_{n-1}} a_{n-1},$$

$$a_n = a_{n-1} + (a_n - a_{n-1}).$$

7. Each of the following formulas generates an integer sequence. For each find a recurrence relation that is satisfied by the sequence.
   (i) $a_n = n(n - 1)$.
   (ii) $a_n = 2n - 1$.
   (iii) $a_n = 2^n + 3^n$.
   (iv) $a_n = 2^n - 1$.

8. Find recurrence relations that are satisfied by the sequences formed from the following functions:
(i) $a_n = n^2 - 6n + 8$.
(ii) $a_n = n!/15!$.
(iii) $a_n = n!/[15!(n - 15)!]$ for $n > 14$.
(iv) $a_n = \binom{n}{j}$, where j is a fixed integer between 0 and n.
(v) $a_n = n^3 + 3n^2 + 3n + 1$.

9. Which of the recurrence relations in Exercises 2, 3, and 4 are homogeneous and which inhomogeneous?

10. Sometimes iteration works on homogeneous recurrence relations. Use this technique to find formulas satisfying the following:
(i) $a_n = a_{n-1}$ for $n > 1$, $a_1 = 1$.
(ii) $b_n = 2b_{n-1}$ for $n > 1$, $b_1 = 2$.
Then prove that your formulas are correct.

11. At the end of this section we saw that the Fibonacci numbers satisfy all the following equations:

$$F_n = F_{n-1} + F_{n-2},$$
$$F_n = F_{n-2} + 2F_{n-3} + F_{n-4},$$

and

$$F_n = F_{n-3} + 3F_{n-4} + 3F_{n-5} + F_{n-6}.$$

Find a similar expression for F_n in terms of $F_{n-4}, F_{n-5}, \ldots, F_{n-8}$. Then for k, an arbitrary positive integer less than n, find and prove a formula that expresses F_n in terms of F_{n-k} and smaller Fibonacci numbers.

12. **(i)** Suppose that $T_n = T_{\lfloor n/2 \rfloor} + 2$ for $n > 1$, $T_1 = 1$. If n is a power of 2, use iteration to deduce a formula for T_n. Is this formula also valid for values of n that are not powers of 2? If so, prove your result; if not, find and prove a formula that is valid for these values of n.
(ii) Repeat for $S_n = 2S_{\lfloor n/2 \rfloor}$ for $n > 1$, $S_1 = 1$.
(iii) Repeat for $U_n = 2U_{\lfloor n/2 \rfloor} + 2$ for $n > 1$, $U_1 = 2$.

13. Given n lines in the plane no two of which are parallel and no three of which intersect in a point, how many regions do these lines create?

14. Let $H_n = H_{n-1} + H_{n-2} + 1$ for $n > 2$, $H_1 = H_2 = 1$. Find $H_3, \ldots, H_8$. Guess a relationship between H_n and F_n, then prove it by induction.

15. Let $Q_n = Q_{n-1} + Q_{n-2} + 2$ for $n > 2$, $Q_1 = Q_2 = 1$. Find $Q_3, \ldots, Q_8$. Guess a relationship between Q_n and F_n, then prove it by induction.

7:3 LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

The title of this section is a mouthful that describes the kind of recurrence relation that the Fibonacci numbers satisfy.

Definition. A recurrence relation of the form

$$a_n = k_1 a_{n-1} + k_2 a_{n-2} + \cdots + k_r a_{n-r} \qquad \text{for } n > r, \tag{A}$$

where $k_1, k_2, \ldots, k_r$ are constants is called a **linear homogeneous recurrence relation with constant coefficients**. We denote these by **LHRRWCC**. We assume that $k_r \neq 0$ and call r the **order** of the recurrence relation.

Here's what all these words mean. First *linear* refers to the fact that every term containing an a_i has exactly one such factor and it occurs to the first power. We introduced *homogeneous* in the previous section. The words *constant coefficients* mean that each of the k_is is a constant. In contrast, the recurrence relation $a_n = a_{n-1}^2$ is not linear although it is homogeneous, and the recurrence relation $b_n = nb_{n-1}$ does not have constant coefficients, but it is a first-order linear homogeneous recurrence relation.

The sequences S_2 and S_6 satisfy first- and second-order LHRRWCCs, respectively:

$$S_2: 2, 4, 8, 16, \ldots \qquad a_n = 2a_{n-1} \qquad \text{for } n > 1$$
$$S_6: 1, 3, 7, 15, 31, \ldots \qquad a_n = 3a_{n-1} - 2a_{n-2} \qquad \text{for } n > 2.$$

(By the way, can you now guess a formula for the nth term of S_6? If not, try comparing S_6 with S_2.)

Question 3.1. Which of the following are LHRRWCCs? For those that are not, explain why they fail to satisfy the definition.
(i) $a_n = a_{n-1} + 1$ for $n > 1$.
(ii) $a_n = a_{n-4}a_{n-2}$ for $n > 4$.
(iii) $a_n = a_{n-1} + n^2$ for $n > 1$.

Given a sequence that satisfies a LHRRWCC, we can find an explicit formula for the nth term of the sequence. The derivation seems magical, so we begin by working out the details for the sequences S_2 and S_6.

Example 3.1. The sequence $S_2: 2, 4, 8, 16, \ldots$ satisfies the recurrence relation

$$a_n = 2a_{n-1} \qquad \text{for } n > 1. \tag{B}$$

a first-order LHRRWCC, and also satisfies the formula $a_n = 2^n$. Note that 2 is the root of the equation

$$x - 2 = 0.$$

It is also a root of the equation obtained from the previous one by multiplying every term by x^{n-1}:

$$x^n - 2x^{n-1} = 0$$

or

$$x^n = 2x^{n-1}. \tag{C}$$

Notice the similarities between (B) and (C). One involves subscripts and as while the other involves superscripts and xs. Is this coincidence?

Example 3.2. The sequence $S_6: 1, 3, 7, 15, 31, \ldots$ satisfies the second-order LHRRWCC

$$a_n = 3a_{n-1} - 2a_{n-2}, \qquad n > 2, \tag{D}$$

and has nth term formula

$$a_n = 2^n - 1 = 2^n - 1^n,$$

a difference of two exponentials. Now 2 and 1 are roots of the equation

$$(x - 2)(x - 1) = 0$$

or

$$x^2 - 3x + 2 = 0.$$

Multiplying by x^{n-2}, we get

$$x^n - 3x^{n-1} + 2x^{n-2} = 0$$

or

$$x^n = 3x^{n-1} - 2x^{n-2}. \tag{E}$$

Again, notice the similarities between equations (D) and (E). These are not by chance. We turn now to the theory that connects LHRRWCCs, polynomial equations, and their roots.

Given a LHRRWCC (A) we create the corresponding equation

$$x^n - k_1 x^{n-1} - k_2 x^{n-2} - \cdots - k_r x^{n-r} = 0.$$

Next we divide through by the common factor x^{n-r} to get

$$x^r - k_1 x^{r-1} - \cdots - k_{r-1} x - k_r = 0. \tag{F}$$

Why do we do this? Because (F) will be helpful in solving (A).

Definition. Given a LHRRWCC (A), the equation (F) is called the **characteristic equation** of the recurrence relation. The left-hand side of the characteristic equation is a polynomial (often called the **characteristic polynomial**) whose degree equals the order of the recurrence relation. This polynomial has r roots $q_1, q_2, \ldots, q_r$ (either real or complex numbers) called the **characteristic roots** of the recurrence relation.

Notice that no characteristic root q_i is zero. This is because we assume in (A) that k_r is not zero. Thus $x = 0$ is not a root of (F).

Example 3.2 (continued). Using the recurrence relation for S_6:

$$a_n = 3a_{n-1} - 2a_{n-2},$$

we form the characteristic equation as follows:

$$x^n = 3x^{n-1} - 2x^{n-2}$$
$$x^n - 3x^{n-1} + 2x^{n-2} = 0$$
$$x^2 - 3x + 2 = 0,$$

which is the desired equation. In this case we easily find the characteristic roots, since (as we saw before)

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

Thus the characteristics roots are $q_1 = 1$ and $q_2 = 2$.

Question 3.2. For each of the following LHRRWCCs find the characteristic equation and the characteristic root or roots:

(i) $a_n = 2a_{n-1}$
(ii) $a_n = a_{n-1} + 6a_{n-2}$
(iii) $a_n = 2a_{n-1} - a_{n-2}$.

The next theorem demonstrates that the results of Examples 3.1 and 3.2 were not just coincidence.

Theorem 3.1. Let q be a nonzero real or complex number. Then

$$a_n = q^n$$

is a solution (also called a **basic solution**) to the recurrence relation (A) if and only if q is a characteristic root of the recurrence relation (i.e., a root of the characteristic polynomial).

Proof. The sequence $a_n = q^n$ is a solution to (A)

$$a_n = k_1 a_{n-1} + \cdots + k_r a_{n-r}$$

if and only if

$$q^n = k_1 q^{n-1} + \cdots + k_r q^{n-r}$$

if and only if

$$q^{n-r}[q^r - k_1 q^{r-1} - \cdots - k_r] = 0.$$

Since q is not zero, the last equation holds if and only if

$$q^r - k_1 q^{r-1} - \cdots - k_r = 0.$$

This last equation is true if and only if q is a characteristic root of the recurrence relation. □

Example 3.1 (continued). S_2:2, 4, 8, 16, . . . satisfies the LHRRWCC

$$a_n = 2a_{n-1}$$

which has (from Question 3.2) the characteristic equation

$$x - 2 = 0$$

and characteristic root $q_1 = 2$. By Theorem 3.1, $a_n = 2^n$ is a basic solution to this recurrence relation, and that's just what we've known all along!

Example 3.2 (continued). We can now find formulas that satisfy the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2}. \tag{D}$$

By the previous version of Example 3.2 we know that the characteristic roots of this recurrence relation are $q_1 = 1$ and $q_2 = 2$. By Theorem 3.1 both $a_n = 1^n = 1$ and $a_n = 2^n$ are basic solutions to the recurrence relation. But neither of these solutions gives a formula that generates the sequence given by S_6. Notice, however, that if we were to use (D) with initial condition $a_1 = a_2 = 1$, then we would get the sequence $1, 1, 1, \ldots$, and the formula for this is clearly $a_n = 1^n$. If we were to use the initial conditions $a_1 = 2$ and $a_2 = 4$, then we would get $2, 4, 8, \ldots$ and the formula for this is $a_n = 2^n$, the other formula uncovered by Theorem 3.1. The point is that had the initial conditions been different than they are in S_6, we might have found the generating formula. We have more work to do in this example.

Our goal is to find a formula for a sequence generated by a given recurrence relation with any set of initial conditions. The technique will be to combine basic solutions.

Theorem 3.2. If f_n and g_n both satisfy the recurrence relation (A), then for any constants c and d so does

$$a_n = c f_n + d g_n.$$

Proof. Since f and g are each solutions to (A), we have that

$$f_n = k_1 f_{n-1} + \cdots + k_r f_{n-r}, \text{ and}$$
$$g_n = k_1 g_{n-1} + \cdots + k_r g_{n-r}.$$

If we multiply the first equation by c and the second by d and then add them, we get

$$\begin{aligned} c f_n + d g_n &= c[k_1 f_{n-1} + \cdots + k_r f_{n-r}] \\ &\quad + d[k_1 g_{n-1} + \cdots + k_r g_{n-r}] \\ &= k_1[c f_{n-1} + d g_{n-1}] + \cdots + k_r[c f_{n-r} + d g_{n-r}], \end{aligned}$$

and this shows that $a_n = c f_n + d g_n$ is a solution to (A). □

More generally, it can be proved by induction that if $f_n^1, f_n^2, \ldots, f_n^s$ are all solutions of (A), then so is

$$a_n = c_1 f_n^1 + c_2 f_n^2 + \cdots + c_s f_n^s$$

for any constants $c_1, \ldots, c_s$. (See Exercises 14 and 15.) In this event a_n is said to be a **linear combination** of the fs. Usually, the fs will be basic solutions.

Example 3.2 (continued again). Theorem 3.2 says that the basic solutions $a_n = 1$ and $a_n = 2^n$ to the recurrence (D) can be combined so that $a_n = c1 + d2^n$ is also a solution for any constants c and d. To produce a formula that yields the specific sequence

$$S_6: 1, 3, 7, 15, 31, \ldots,$$

we need to find the correct values of c and d. The sequence S_6 comes with the initial conditions $a_1 = 1$ and $a_2 = 3$. We use this information to find c and d. If the correct formula for this sequence is given by $a_n = c1 + d2^n$, then we must have

$$1 = a_1 = c + d2^1 = c + 2d,$$

and

$$3 = a_2 = c + d2^2 = c + 4d.$$

If we subtract the first equation from the second, we get $2 = 2d$. From this $d = 1$ and then $c = -1$. Thus $a_n = 2^n - 1$ is a formula that meets the initial conditions and gives a solution to the recurrence relation.

Question 3.3. Prove by induction that $a_n = 2^n - 1$ satisfies the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_1 = 1$ and $a_2 = 3$.

Here is a summary of when the procedure followed in Example 3.2 works. It does not succeed in all cases. Given a recurrence relation as in (A), we find the characteristic equation

$$x^r - k_1 x^{r-1} - \cdots - k_r = 0.$$

The general theory of equations tells us that this equation has r real or complex roots, and so the equation can be factored into

$$(x - q_1)(x - q_2) \cdots (x - q_r) = 0,$$

where $q_1, q_2, \ldots, q_r$ are the roots. For example, we might get,

$$(x - 1)(x - 2) = 0, \qquad (x - 3)(x - 3)(x - 4) = 0, \qquad \text{or} \qquad (x - i)(x + i) = 0,$$

where i is the **imaginary number**, $\sqrt{-1}$. In the first and third examples the roots are distinct, but in the second example the root 3 appears twice. It is then called a **multiple root** and the root 3 is said to have **multiplicity** 2. It turns out that we must treat the two cases differently.

Theorem 3.3. Suppose that the LHRRWCC as shown in (A) has r distinct characteristic roots $q_1, q_2, \ldots, q_r$. Then every solution to (A) is a linear combination of the basic solutions:

$$a_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_r q_r^n \tag{G}$$

where $c_1, c_2, \ldots, c_r$ are constants.

The proof of this theorem essentially requires knowing that if all the roots of (F) are distinct, then it is possible to solve r equations in r unknowns to find the constants $c_1, c_2, \ldots, c_r$. The equations are determined by substituting $n = 1, 2, \ldots, r$ into (G). We see

$$\begin{aligned} a_1 &= c_1 q_1 + c_2 q_2 + \cdots + c_r q_r \\ a_2 &= c_1 q_1^2 + c_2 q_2^2 + \cdots + c_r q_r^2 \\ &\cdots \\ a_r &= c_1 q_1^r + c_2 q_2^r + \cdots + c_r q_r^r. \end{aligned}$$

In these equations the unknowns are $c_1, \ldots, c_r$. A complete proof of Theorem 3.3 requires knowledge of linear algebra and so is omitted.

Sometimes (like now) the arithmetic involved in solving for the constants $c_1, c_2, \ldots, c_r$ will be simplified if we consider sequences that begin with a zeroth term,

$$S: a_0, a_1, \ldots, a_n, \ldots$$

Any sequence can be transformed into this type by working backward with the recurrence relation to find a value for a_0 that is consistent. For example, look at Example 3.2. Since

$$a_n = 3a_{n-1} - 2a_{n-2},$$

we want the value of a_0 to be such that

$$a_2 = 3a_1 - 2a_0 \qquad \text{or} \qquad 3 = 3 \cdot 1 - 2a_0.$$

If we give a_0 the value of 0, then the sequence

$$S_6': 0, 1, 3, 7, 15, 31, \ldots$$

satisfies the same recurrence relation but with initial conditions $a_0 = 0$ and $a_1 = 1$. It also has the same formula $a_n = 2^n - 1$. By beginning at 0, the arithmetic in solving r equations in r unknowns might be easier. This can be especially convenient when doing small examples by hand.

Question 3.4. Suppose the recurrence relations in Question 3.1 have the following initial conditions:
(i) $a_1 = 1$.
(ii) $a_i = 2$ for all $i \leq 4$.
(iii) $a_1 = 1$.
In each case determine a value for a_0 that satisfies the same recurrence relation.

Example 3.3. The Fibonacci formula (at last!). Let's use the machine we've just built to discover the formula for the Fibonacci numbers that appeared out of the blue in Section 4.4. These numbers satisfy the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n > 1$$

with initial conditions $a_0 = 0$ and $a_1 = 1$. The characteristic equation of the LHRRWCC is

$$x^2 - x - 1 = 0,$$

and this has distinct characteristic roots $q_1 = (1 + \sqrt{5})/2$, which we called ϕ, and $q_2 = (1 - \sqrt{5})/2$, called ϕ'. Thus the general formula that solves this recurrence relation is

$$\begin{aligned} a_n &= c\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= c\phi^n + d(\phi')^n \end{aligned}$$

for some constants c and d. Notice that this looks like the Fibonacci formula we had earlier, but we need to determine the constants c and d from the initial conditions. (Here's where beginning at 0 makes life easier.)

$$\begin{aligned} 0 = a_0 &= c\phi^0 + d(\phi')^0 = c + d \\ 1 = a_1 &= c\phi^1 + d(\phi')^1 \\ &= c\,\frac{1+\sqrt{5}}{2} + d\,\frac{1-\sqrt{5}}{2}. \end{aligned}$$

From the first equation we get that $c = -d$. Substituting into the second equation, we get that

$$c = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad d = \frac{-1}{\sqrt{5}}.$$

In conclusion we have the formula for the Fibonacci numbers as

$$a_n = \frac{\phi^n - (\phi')^n}{\sqrt{5}}.$$

Question 3.5. For each of the following recurrence relations find the formula for the sequence of numbers generated if the characteristic equation has distinct roots. (These are recurrence relations from Question 3.2.)

(i) $a_n = a_{n-1} + 6a_{n-2}$ for $n > 1$, $a_0 = 2$, $a_1 = 1$.
(ii) $a_n = a_{n-1} + 6a_{n-2}$ for $n > 1$, $a_0 = 1$, $a_1 = 3$.
(iii) $a_n = 2a_{n-1} - a_{n-2}$ for $n > 1$, $a_0 = 2$, $a_1 = -1$.

In the next section we consider LHRRWCCs whose characteristic equations have multiple roots. The techniques will be similar.

EXERCISES FOR SECTION 3

1. **(i)** Give an example of a LHRRWCC.
(ii) Give an example of a linear homogeneous recurrence relation with coefficients that are not constant.
(iii) Give an example of a linear recurrence relation with constant coefficients that is inhomogeneous.
(iv) Give an example of a homogeneous recurrence relation with constant coefficients that is not linear.

2. Which of the following are LHRRWCCs? For those that are not, explain why they fail to satisfy the definition.
(i) $a_n = 2a_{n-1}$ for $n > 1$.
(ii) $a_n = 2a_{n-1} + 1$ for $n > 1$.
(iii) $a_n = a_{n-4} - a_{n-2}$ for $n > 4$.
(iv) $a_n = a_{n-1}^3 + 3a_{n-2}$ for $n > 2$.
(v) $a_n = a_{n-1} + a_1$ for $n > 1$ and a_1 a constant.
(vi) $a_n = a_{n-4} - a_{n-3} + a_{n-2} - a_{n-1}$ for $n > 4$.
(vii) $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ for $n > 3$.
For each LHRRWCC determine its order and find its characteristic equation.

3. Let $f(x)$ be the fourth degree polynomial

$$f(x) = (x-1)(x+2)(x-3)(x+4).$$

What are the roots of f? Are they distinct?

4. Find LHRRWCCs with each of the following characteristic equations:
   (i) $f(x) = (x - 1)(x + 2)(x - 3)$.
   (ii) $g(x) = (x + 3)(x - 3)$.
   (iii) $h(x) = (x - \frac{1}{2})(x + \frac{3}{2})$.
5. Find a cubic polynomial whose roots are 5, -1, and 3. Then find a LHRRWCC with this characteristic polynomial.
6. (i) For the LHRRWCC of the preceding exercise find initial conditions such that $a_n = 5^n$ is the formula for the sequence produced by the recurrence relation.
   (ii) Repeat part (i) finding initial conditions for the formula $a_n = 5^n + (-1)^n$.
   (iii) Repeat part (i) finding initial conditions for the formula $a_n = 5^n + 2(-1)^n - 3^n$.
7. Prove that the quadratic equation

$$x^2 + bx + c = 0$$

has roots $\dfrac{-b + \sqrt{b^2 - 4c}}{2}$ and $\dfrac{-b - \sqrt{b^2 - 4c}}{2}$.

8. Find a formula for the roots of the equation

$$x^4 + bx^2 + c = 0.$$

9. Suppose that the recurrence relations in Exercise 2 have the following initial conditions. Find values of a_0 that satisfy the same recurrence relation.
   (i) $a_1 = 0$.
   (ii) $a_1 = 1$.
   (iii) $a_1 = a_2 = 1$, $a_3 = a_4 = 2$.
   (iv) $a_1 = -1$, $a_2 = -2$.
   (v) $a_1 = 3$.
   (vi) $a_1 = \frac{1}{2}$, $a_2 = -1$, $a_3 = -\frac{1}{2}$, $a_4 = 1$.
   (vii) $a_1 = 1$, $a_2 = 2$, $a_3 = 4$.
10. Find the characteristic equation and characteristic roots of the following relations:
    (i) $a_n = 4a_{n-1} - 4a_{n-2}$.
    (ii) $a_n = -a_{n-1}$.
    (iii) $a_n = 5a_{n-2}$.
    (iv) $a_n = -8a_{n-3}$.
    (v) $a_n = 2a_{n-2} - a_{n-4}$.

    Which of the above relations have distinct roots and which have multiple roots?

11. Is the following statement true or false?

If both $a_n = f_n$ and $a_n = g_n$ are formulas that satisfy a given linear inhomogeneous recurrence relation with constant coefficients, then so is $a_n = c f_n + d g_n$ for every choice of constants c and d.

Either explain why this is true or find a counterexample.

12. If possible, for each situation listed give an example of a fourth-degree polynomial with

(i) Four distinct roots.
(ii) Two distinct roots, one of which has multiplicity one and the other multiplicity three.
(iii) Exactly two distinct roots, one of multiplicity one and one of multiplicity two.
(iv) Two distinct roots, each of multiplicity two.
(v) One root of multiplicity four.
(vi) One root of multiplicity two and one root of multiplicity three.

13. Find a recurrence relation that is satisfied by both

(i) $a_n = 1$ and $a_n = 3^n$,
(ii) $a_n = (-1)^n$ and $a_n = 2^n$,
(iii) $a_n = 2^n$ and $a_n = 3 \cdot 2^n + 2 \cdot 3^n$.

14. Prove that if f_n, g_n and h_n are three functions that each satisfy

$$a_n = k_1 a_{n-1} + k_2 a_{n-2} + \cdots + k_r a_{n-r},$$

then for any constants c, d, and e, the function

$$s_n = c f_n + d g_n + e h_n$$

also satisfies this recurrence relation.

15. Prove by induction (on j) that if $f_n^1, f_n^2, \ldots, f_n^j$ are j functions that each satisfy

$$a_n = k_1 a_{n-1} + k_2 a_{n-2} + \cdots + k_r a_{n-r},$$

then for any constants $c_1, c_2, \ldots, c_j$, the function

$$g_n = c_1 f_n^1 + c_2 f_n^2 + \cdots + c_j f_n^j$$

also satisfies this recurrence relation.

16. Find a formula for a function a_n that satisfies the following recurrence relation with given initial conditions:

(i) $a_n = -a_{n-1}$ for $n > 0$, $a_0 = 1$.
(ii) $a_n = 4a_{n-2}$ for $n > 1$, $a_0 = 0$, $a_1 = 1$.
(iii) $a_n = -8a_{n-3}$ for $n > 2$, $a_0 = a_1 = a_2 = 1$.

17. The recurrence relation $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ with $a_0 = 2$, $a_1 = 1$, and $a_2 = 4$ has characteristic roots 1 and -1. Show that the generating formula is not of the form

$$f_n = c1^n + d(-1)^n.$$

18. The **Lucas numbers** are defined by $L_1 = 1$, and

$$L_n = F_{n+1} + F_{n-1} \qquad \text{for } n > 1,$$

where F_n is the nth Fibonacci number. Find the first eight Lucas numbers. Find a recurrence relation for the Lucas numbers and then find a formula for L_n.

19. Is the following proof correct? Explain your answer.

Theorem. For all positive n, $L_n = F_n$.

Proof (by induction): $L_1 = F_1 = 1$. Assuming that the result is true for $n \le k$, we examine L_{k+1}:

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} && \text{from the recurrence relation found in Exercise 18} \\ &= F_k + F_{k-1} && \text{by induction} \\ &= F_{k+1} && \text{by definition.} \end{aligned}$$

20. Suppose that $T_n = 12T_{n-1} - 35T_{n-2}$ for $n > 1$ while $T_0 = 0$ and $T_1 = 2$. Find a formula for T_n.

21. How many n-bit binary sequences have no two consecutive zeros?

22. Suppose that the second floor of the firehouse has two poles to the first floor. Suppose that every higher floor of the firehouse has five poles. Two of these poles go down one floor while the remaining three poles go down two floors. If you slide down a pole that goes down two floors, you cannot get off at the intermediate floor. How many different ways are there to get from the nth floor to the first floor?

7:4 LHRRWCCS WITH MULTIPLE ROOTS: MORE ABOUT RABBITS

We reconsider the rabbit breeding model from Chapter 4 that led to the Fibonacci numbers. Suppose that each pair of newborn rabbits produces exactly one pair of bunnies after one month and this is all of their offspring. If rabbits are still assumed to be immortal, how many pairs of rabbits are there at the end of each month? Let the number of pairs of rabbits at the end of n months be denoted by b_n. Thus $b_1 = 1$, $b_2 = 2$, $b_3 = 3$ (since only the younger pair produces a new pair of bunnies), and $b_4 = 3 + (3 - 2) = 4$. In general, the number of rabbit pairs at time n equals the number of pairs at time $(n - 1)$ plus the number of new bunny pairs produced in the year $(n - 1)$. Thus

$$\begin{aligned} b_n &= b_{n-1} + (b_{n-1} - b_{n-2}) \\ &= 2b_{n-1} - b_{n-2} \qquad \text{for } n > 2, \end{aligned} \tag{A}$$

a second-order LHRRWCC with initial conditions $b_1 = 1$ and $b_2 = 2$.

Question 4.1. Give an inductive proof that $b_n = n$.

Although we have the solution to this rabbit problem, we continue the example, since it illustrates the case of LHRRWCCs with multiple roots. The recurrence relation (A) is a LHRRWCC whose characteristic equation is

$$x^2 - 2x + 1 = 0 \qquad \text{or} \qquad (x - 1)^2 = 0.$$

Thus 1 is a root of multiplicity 2, and from Theorem 3.1 we know that $b_n = 1^n = 1$ is a basic solution to (A). This does not satisfy the initial conditions of the current problem. Since there are no other roots to the characteristic equation, it must be the case that the solution to the recurrence takes a form different from that of the previous section. Here is the pertinent result for characteristic equations with multiple roots.

Theorem 4.1. Suppose that the following LHRRWCC has a characteristic root q of multiplicity $m > 1$:

$$a_n = k_1 a_{n-1} + k_2 a_{n-2} + \cdots + k_r a_{n-r} \tag{B}$$

for $n > r$. Then the following are (basic) solutions to (B):

$$\begin{aligned} a_n &= q^n \\ a_n &= nq^n \\ a_n &= n^2 q^n \\ &\cdots \\ a_n &= n^{m-1} q^n. \end{aligned}$$

Note that each root of (B) supplies as many basic solutions as its multiplicity. Thus there will be a total of r basic solutions to (B). Once again the number of basic solutions equals the order of the recurrence relation.

Proof. We prove the result for $m = 2$. Theorem 3.1 gives that $a_n = q^n$ is a solution to (B). Thus we must show that

$$a_n = nq^n$$

is also a solution. If $t(x)$ denotes the characteristic polynomial of the recurrence relation (B), set $p(x) = t(x)x^{n-r}$. Thus

$$p(x) = x^n - [k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_r x^{n-r}].$$

Since q is a root of multiplicity 2 for $t(x)$, the characteristic polynomial, q is also a root of multiplicity 2 for $p(x)$. Set

$$D(x) = \frac{p(x) - p(q)}{x - q}.$$

(For those of you with a calculus background, $D(x)$ is the **difference quotient** that leads to the derivative.) Since q is a root of $p(x)$, the quantity $p(q)$ is just a fancy way to write zero. Furthermore, since q is a multiple root of $p(x)$, when we divide $(x - q)$ into $p(x)$ we are left with a polynomial that still has q as a root. Thus $D(q) = 0$. This is the heart of the proof. What is left is an algebraic rearrangement of $D(x)$ after which we substitute $x = q$ and find that $a_n = n \cdot q^n$ is a solution of (B).

First we collect the terms from $D(x)$ that have the same exponent to get:

$$\begin{aligned} D(x) = {} & \frac{x^n - q^n}{x - q} - k_1 \frac{x^{n-1} - q^{n-1}}{x - q} - \cdots \\ & - k_j \frac{x^{n-j} - q^{n-j}}{x - q} - \cdots . \end{aligned} \tag{C}$$

Question 4.2. Construct $D(x)$ if $p(x) = x^2 - 2x + 1$. Be sure to leave the characteristic root as q (rather than substitute its value). Simplify $D(x)$ by dividing $x - q$ into each term. (Here $n = r = 2$.)

We simplify (C) term by term. From Exercise 5 in the Supplementary Exercises for Chapter 2 (or by multiplying out the right-hand side), we note that

$$x^n - q^n = (x - q)[x^{n-1} + \cdots + x^{n-1-i}q^i + \cdots + q^{n-1}].$$

Notice that there is a convenient factor of $x - q$ in the above expression and that the exponents of each term sum to $n - 1$. Thus

$$\frac{x^n - q^n}{x - q} = x^{n-1} + \cdots + x^{n-1-i}q^i + \cdots + q^{n-1}. \tag{E}$$

The right-hand side of (E) is just one term in the expansion of $D(x)$. We eventually want to compute $D(q)$, so we substitute $x = q$ into the right-hand side of (E) just to see what happens. Every term becomes q^{n-1}. Since the exponent on x decreases from $n - 1$ to 0 in steps of 1, there are n terms and so the total contribution from (E) will be nq^{n-1}.

Next we simplify the general term in (C):

$$-k_j \frac{x^{n-j} - q^{n-j}}{x - q} = \frac{-k_j}{x - q}[x^{n-j} - q^{n-j}]$$
$$= \frac{-k_j}{x - q}(x - q)[x^{n-j-1} + \cdots + x^{n-j-1-i}q^i + \cdots]. \quad \text{(F)}$$

Notice that there is a convenient factor of $x - q$ in this expression and that the exponents of each term sum to $n - j - 1$. As above we substitute $x = q$ into the right-hand side of (F). Every term becomes q^{n-j-1}. Since the exponent on x decreases from $n - j - 1$ to 0 in steps of 1, there are $n - j$ terms and so the total contribution from (F) will be

$$-k_j(n - j)q^{n-j-1}. \quad \text{(G)}$$

Thus $D(q)$ contains a term (G) for each j with $0 \leq j \leq r$. Since $D(q) = 0$, we get

$$0 = nq^{n-1} - [k_1(n - 1)q^{n-2} + \cdots + k_j(n - j)q^{n-j-1} + \cdots]. \quad \text{(H)}$$

We multiply both sides of (H) by q and rearrange the terms to obtain

$$nq^n = k_1(n - 1)q^{n-1} + \cdots + k_j(n - j)q^{n-j} + \cdots. \quad \text{(I)}$$

Finally, (I) is the same as (B) after substituting

$$a_{n-j} = (n - j)q^{n-j}$$

for $j = 0, \ldots, r$. Thus $a_n = nq^n$ is a solution to the original recurrence relation. $\square$

We stop the proof of Theorem 4.1 with the completion of the case $m = 2$. For $m > 2$ a proof using calculus is outlined in Supplementary Exercises 19 and 20.

Example 4.1. We analyze the rabbit recurrence (A). The characteristic equation is

$$(x - 1)^2 = 0,$$

so by Theorem 4.1 both $b_n = 1$ and $b_n = n1^n = n$ are basic solutions to this recurrence relation. The initial conditions $b_1 = 1$ and $b_2 = 2$ show that the second solution, $b_n = n$, is exactly the one we want.

Question 4.3. Find the characteristic equation and root or roots of the second-order recurrence relation

$$b_n = 4b_{n-1} - 4b_{n-2} \qquad \text{for } n \geq 2.$$

Check that both $b_n = 2^n$ and $b_n = n2^n$ are solutions to this.

Question 4.4. Show that the characteristic equation of

$$c_n = -3c_{n-1} - 3c_{n-2} - c_{n-3} \qquad \text{for } n \geq 3$$

has -1 as a characteristic root of multiplicity 3. Check that each of $c_n = (-1)^n$, $c_n = n(-1)^n$ and $c_n = n^2(-1)^n$ is a solution to this recurrence relation.

Example 4.1 (altered). Suppose that we want to solve the recurrence relation given in (A) with initial conditions $b_1 = 1$ and $b_2 = 3$. As before, both $b_n = 1$ and $b_n = n$ are basic solutions. Theorem 3.2 (which isn't restricted to recurrence relations whose characteristic roots are distinct) tells us that

$$b_n = c1 + dn$$

is also a solution for all constants c and d. Using the initial conditions, we find that

$$1 = b_1 = c1 + d1$$
$$3 = b_2 = c1 + d2.$$

Subtracting the first equation from the second, we deduce that

$$c = -1, \qquad d = 2, \qquad \text{and} \qquad b_n = 2n - 1.$$

Question 4.5. Find a solution to the recurrence relation of Question 4.4 with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = 1$.

Finally, we reach the generalization of Theorem 3.3 (which we also do not prove).

Theorem 4.2. Given a sequence $S: a_1, \ldots, a_n, \ldots$ whose terms satisfy a LHRRWCC of order r, then a_n is a linear combination of the r basic solutions given by Theorems 3.1 and 4.1.

Example 4.2. Suppose that we have a recurrence relation of order 6 (so $r = 6$ and the characteristic equation has degree 6) and suppose that we (luckily) find that the characteristic equation factors as

$$f(x) = (x - 1)^3(x + 1)^2(x - 17).$$

Then by Theorem 4.1 we have the following basic solutions to the recurrence relation:

$$a_n = 1^n,\ a_n = n1^n,\ a_n = n^2 1^n,$$
$$a_n = (-1)^n,\ a_n = n(-1)^n,$$

and

$$a_n = 17^n.$$

Using Theorem 4.2, we see that every solution is of the form

$$a_n = c_1 1 + c_2 n + c_3 n^2 + c_4(-1)^n + c_5 n(-1)^n + c_6 17^n.$$

In a concrete situation we would use the six initial conditions and solve six equations to find $c_1, \ldots, c_6$.

Exercises 9 through 12 present other variations of the rabbit-breeding model. However, the study of LHRRWCCs has not been developed for the interest of rabbit breeders. There are important uses of recurrence relations in combinatorics and in computer science. Often the complexity analyses of recursive algorithms lead to recurrence relations that must be solved. In the next section we'll meet a type of recurrence relation that occurs frequently in the "divide-and-conquer" algorithms.

EXERCISES FOR SECTION 4

1. Write down equations of degree 2, 3, and 4, each with a multiple root. Specify the root and its multiplicity. Then write down an equation that has one root of multiplicity 1, one root of multiplicity 2, one root of multiplicity 3, and no other roots.
2. Which of the following equations have multiple roots?
   (i) $f(x) = x^2 - 1$.
   (ii) $f(x) = x^2 + 2x + 1$.
   (iii) $f(x) = x^2 + x - 12$.
   (iv) $f(x) = x^2 - 6x + 9$.
   (v) $f(x) = x^4 - 2x^2 + 1$.
   (vi) $f(x) = x^4 + 2x^2 + 1$.
   (vii) $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$.

3. For each of the functions in Exercise 2 write down a recurrence relation with characteristic equation $f(x) = 0$. Then find a formula that satisfies the recurrence relation.

4. Show that the recurrence relation

$$a_n = 10a_{n-1} - 40a_{n-2} + 80a_{n-3} - 80a_{n-4} + 32a_{n-5}$$

has the characteristic equation $(x - 2)^5 = 0$. Then check that $a_n = 2^n$, $a_n = n2^n$, $a_n = n^2 2^n$, $a_n = n^3 2^n$ and $a_n = n^4 2^n$ all satisfy the recurrence relation.

5. For each of the following, find a recurrence relation with initial conditions that has this as a solution:

(i) $a_n = 3^n + n3^n$.
(ii) $a_n = 3^n + 2n3^n$.
(iii) $a_n = 2n - 1 + 2^n$.
(iv) $a_n = 1 + (-1)^n + 2^n$.
(v) $a_n = 4(\frac{1}{2})^n + 8n(\frac{1}{2})^n$.

6. For each recurrence relation in list A find its characteristic equation in list B:

List A

(i) $a_n = 5a_{n-2} + 4a_{n-4}$.
(ii) $a_n = 7a_{n-1} - 17a_{n-2} + 17a_{n-3} - 6a_{n-4}$.
(iii) $a_n = 8a_{n-1} - 23a_{n-2} + 28a_{n-3} - 12a_{n-4}$.
(iv) $a_n = 9a_{n-1} - 29a_{n-2} + 39a_{n-3} - 18a_{n-4}$.
(v) $a_n = 6a_{n-1} - 13a_{n-2} + 12a_{n-3} - 4a_{n-4}$.
(vi) $a_n = 5a_{n-1} - 9a_{n-2} + 7a_{n-3} - 2a_{n-4}$.
(vii) $a_n = 7a_{n-1} - 18a_{n-2} + 20a_{n-3} - 8a_{n-4}$.

List B

(a) $f(x) = x^4 - x^3 + x^2 - x + 1$.
(b) $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$.
(c) $f(x) = x^4 + 10x^3 + 25x^2 + 20x + 4$.
(d) $f(x) = x^4 - 8x^3 - 23x^2 - 28x - 12$.
(e) $f(x) = x^4 - 8x^3 + 23x^2 - 28x + 12$.
(f) $f(x) = (x - 1)^2(x - 2)^2(x - 3)$.
(g) $f(x) = (x + 1)^2(x + 2)^2$.
(h) $f(x) = (x + 1)^2(x - 2)^2$.
(i) $f(x) = (x - 1)^2(x - 2)^2$.
(j) $f(x) = x^4 + 7x^3 + 18x^2 + 20x + 8$.
(k) $f(x) = (x - 1)(x - 2)^3$.
(l) $f(x) = (x - 1)(x - 3)^2$.
(m) $f(x) = (x - 1)^3(x - 2)$.
(n) $f(x) = x^4 + 7x^3 - 17x^2 + 17x - 6$.

(*o*) $f(x) = (x - 1)(x + 1)(x - 2)(x + 2)$.
(*p*) $f(x) = (x - 1)^2(x - 2)(x - 3)$.
(*q*) $f(x) = (x - 1)(x - 2)(x - 4)^2$.
(*r*) $f(x) = (x - 2)(x - 3)(x - 4)^2$.
(*s*) $f(x) = (x - 2)(x - 3)^2(x - 4)$.
(*t*) $f(x) = (x - 2)^2(x - 3)(x - 4)$.
(*u*) $f(x) = (x - 1)(x - 2)(x - 3)^2$.

Then for each recurrence relation in list A find the most general form of a solution to it.

7. Use Theorems 4.1 and 4.2 to find solutions as general as possible to the following recurrence relations:
(i) $a_n = 2a_{n-1} - a_{n-2}$.
(ii) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$.
(iii) $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$.

8. Find a formula for the solution of the following recurrence relations:
(i) $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n > 4$ with $a_0 = a_1 = a_2 = 1$.
(ii) $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ for $n > 4$ with $a_0 = 3$, $a_1 = 4$ and $a_2 = 7$.

9. Suppose that at the end of each month a rabbit pair produces a pair of bunnies, but that after two sets of offspring they produce no more. Write down the recurrence relation with initial conditions that describes this model, beginning with one pair.

10. Suppose that at the end of one month a rabbit pair produces one pair of bunnies, but that during the next month the (older) rabbit pair dies. Beginning with one pair, write down the number of rabbit pairs at the end of each month for the first five months. Then write down a recurrence relation for the number of rabbit pairs at the end of each month.

11. Suppose that at the end of each month a pair of rabbits produces one new bunny pair, but that rabbits die during their third month after having produced bunnies twice. Write down the recurrence relation that decribes this model and the initial conditions assuming that we begin with one pair.

12. We return to the original Fibonacci model of rabbit breeding: A pair of rabbits requires a month to mature to the age of reproduction and then they mate and produce two bunnies. We now do not assume that these are one male and one female, and furthermore we assume that whatever sex they are, a mate is found for each from another warren of rabbits. Thus at the beginning of the first and second months we have one pair, but at the beginning of the third month we have three pairs of rabbits, one old and two new young pairs. How many pairs do we have at the beginning of the fourth and fifth months? Write down a recurrence relation with initial conditions that describes this model.

13. For each of the following determine whether $a_n = O(2^n)$ and whether $2^n = O(a_n)$:

(i) $a_n = 2a_{n-1} - a_{n-2}$, $a_1 = 1$, $a_2 = 2$.

(ii) $a_n = 2a_{n-1}$, $a_1 = 1$.

(iii) $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = 0$, $a_1 = 2$.

(iv) $a_n = a_{n-1} + a_{n-2}$, $a_1 = a_2 = 1$.

14. Is the following true or false? Explain.

A formula a_n that satisfies a LHRRWCC will always be exponential in n; that is, there will always be constants $1 < r < s$ such that $r^n = O(a_n)$ and $a_n = O(s^n)$.

15. Find solutions to the following recurrence relations:

$$a_n = 4a_{n-1} - a_{n-2} \text{ for } n > 1 \text{ with } a_0 = 0 \text{ and } a_1 = 1.$$

$$b_n = 4b_{n-1} - b_{n-2} \text{ for } n > 1 \text{ with } b_0 = 2 \text{ and } b_1 = 4.$$

16. Prove by induction that the sequences a_n and b_n of Exercise 15 satisfy

$$b_n = a_{n+1} - a_{n-1},$$

and

$$a_{m+n} = a_m a_{n+1} - a_{m-1} a_n.$$ [1]

7:5 DIVIDE-AND-CONQUER RECURRENCE RELATIONS

The goal of this section is to formulate and solve recurrence relations that generate the complexity functions of divide-and-conquer algorithms like the searching and sorting procedures from Chapter 6.

Example 5.1. In the algorithm BINARYSEARCH we are given an ordered array of length n and an element S to search for. We begin by comparing S with the middle entry of the array. If these are not equal, we search half of the original array. This leads to the recurrence relation

$$B_n = 3 + B_{\lfloor n/2 \rfloor} \text{ for } n > 1, \qquad B_1 = 4, \tag{A}$$

where B_n denotes the maximum number of comparisons needed in BINARYSEARCH with input an array of n elements. (Reread Section 6.2.)

[1] For an application of the results of Exercises 15 and 16 to the Lucas–Lehmer test for Mersenne primes, see D. E. Knuth, *Seminumerical Algorithms*, Volume 2 of *The Art of Computer Programming*, Addison-Wesley, Reading, Mass., 1973, pp. 356–359.

Question 5.1. Explain why (A) is a recurrence relation for B_n. Use (A) to obtain B_n for $n = 2, 3, 4$, and 5. Compare these numbers with the derived complexity result, $3\lfloor \log(n) \rfloor + 4$.

Example 5.2. The idea behind BININSERT is similar to that of BINARY-SEARCH. The input to the procedure is an array of $n + 1$ entries with the first n in increasing order. The goal is to insert the $(n + 1)$st entry into the correct position of the array. Again we compare with the middle entry and then search half of the array. We repeat this process until we find the correct position. After shifting elements, the $(n + 1)$st entry is inserted in the correct position.

This leads to the recurrence relation

$$C_n = C_{\lfloor n/2 \rfloor} + 2 \text{ for } n > 1, \qquad C_1 = 4, \tag{B}$$

where C_n is the number of comparisons performed by BININSERT on an array of length n. [Reread Section 6.3 to remind yourself why (B) is the recurrence relation for C_n.]

Example 5.3. In the algorithm MERGESORT we begin with an unsorted list of n elements, divide the list in half, sort each half, and then merge the two parts. Thus if M_n denotes the number of comparisons performed in the worst case of MERGESORT, then in the case that $n = 2^k$

$$M_n = 2M_{n/2} + (3n + 1) \text{ for } n > 1, \qquad M_1 = 1, \tag{C}$$

gives the recurrence relation for M_n.

Question 5.2. Explain why (C) is a recurrence relation for M_n. Use (C) to obtain M_n for $n = 2, 4$, and 8. Compare these numbers with the complexity bound derived for the case $n = 2^k$, namely $3n \log(n) + 2n - 1$.

Each of the above algorithmic problems is solved by dividing it into smaller problems, solving the smaller problems and then combining these solutions; we have called these divide-and-conquer algorithms. Suppose that a_n is the number of steps in the worst case of a divide-and-conquer algorithm. Then $a_{\lfloor n/2 \rfloor}$ or $a_{\lfloor n/d \rfloor}$ gives the maximum number of steps needed to solve a problem of half or one dth the size. The number of steps needed to solve some or all of the smaller problems plus the number needed to combine these solutions into a final one is given by a so-called **divide-and-conquer recurrence relation** like

$$a_n = k\, a_{\lfloor n/d \rfloor} + cn + e, \tag{D}$$

where c, d, e, and k are constants. We shall not solve the most general version of (D); however, the text and the exercises contain the most important cases.

Question 5.3. Find constants to show that (A), (B), and (C) are special cases of (D).

Question 5.4. Suppose that we have the recurrence relation

$$a_n = a_{\lfloor n/3 \rfloor} + 1.$$

How many initial conditions must be specified before this relation gives a value for all positive values of n? Then using the recurrence for a_n, specify a set of initial conditions and determine the resulting values of a_n for all $n \leq 7$.

Notice that the presence of the floor function in these recurrence relations could lead to some computational awkwardness. For instance, if we want to show that $B_n = 3\lfloor \log(n) \rfloor + 4$, then working with $B_{\lfloor n/2 \rfloor}$ would require consideration of two cases depending on the parity of n. One way to avoid this problem is to consider the special case of $n = 2^i$ (or $n = d^i$) and then to try to generalize the solution to the arbitrary case.

Example 5.4. Consider a recurrence relation of the form

$$a_n = a_{\lfloor n/2 \rfloor} + c,$$

where c is a constant. We'll try iteration and induction, since this is an inhomogeneous recurrence relation, and we'll experiment with the special case when $n = 2^i$.

$$\begin{aligned}
a_n &= a_{\lfloor n/2 \rfloor} + c \\
&= a_{n/2} + c && \text{since } n/2 \text{ is an integer} \\
&= a_{n/(2^2)} + 2c && \text{since } a_{n/2} = a_{\lfloor n/(2^2) \rfloor} + c = a_{n/(2^2)} + c \\
&\quad \cdots \\
&= a_1 + ic \\
&= a_1 + \log(n)c.
\end{aligned}$$

Question 5.5. Prove by induction that $a_n = a_1 + \log(n)c$ is a solution of the recurrence relation $a_n = a_{\lfloor n/2 \rfloor} + c$ if $n = 2^i$.

Example 5.4 worked out nicely using iteration because we assumed that $n = 2^i$, and it seems reasonable to conjecture that this bound is correct for all values of n. Thus we attempt to prove the same result for arbitrary n by induction. To avoid problems with the floor function, we'll shift now to inequalities. That is, we use the fact that $\lfloor x \rfloor \leq x$. This will lead to upper bounds on the solution function, like $B_n \leq 3\lfloor \log(n) \rfloor + 4$; however, such an upper bound is often satisfactory, since it leads to big oh results, like $B_n = O(\log(n))$.

Theorem 5.1. If a_n is the nth term of an integer sequence that satisfies

$$a_n = a_{\lfloor n/2 \rfloor} + c,$$

where c is a constant, then

$$a_n \leq c\lfloor \log(n) \rfloor + a_1.$$

Proof. (We do not assume that n is a power of 2.) The base case holds with $n = 1$:

$$a_1 \leq c\lfloor \log(1) \rfloor + a_1 = a_1.$$

The inductive hypothesis is that

$$a_n \leq c\lfloor \log(n) \rfloor + a_1$$

for all $n < k$, and we try to obtain the same bound for a_k. We know that

$$\begin{aligned}
a_k &= a_{\lfloor k/2 \rfloor} + c \\
&\leq c\lfloor \log(\lfloor k/2 \rfloor) \rfloor + a_1 + c && \text{by induction} \\
&\leq c\lfloor \log(k/2) \rfloor + a_1 + c && \text{since } \lfloor k/2 \rfloor \leq k/2 \\
&= c\lfloor \log(k) - 1 \rfloor + a_1 + c \\
&= c\lfloor \log(k) \rfloor - c + a_1 + c \\
&= c\lfloor \log(k) \rfloor + a_1.
\end{aligned}$$

□

Notice that we obtain a slightly smaller bound by using $\lfloor \log(k) \rfloor$ in place of $\log(k)$.

Example 5.1 (concluded). When the results of Theorem 5.1 are applied to the recurrence relation for B_n with $c = 3$, and $B_1 = 4$, we have that $B_n \leq 3\lfloor \log(n) \rfloor + 4 = O(\log(n))$, just as in Theorem 2.1 of Chapter 6.

Question 5.6. Apply Theorem 5.1 to (B) and compare the result with that of Theorem 3.1 in Chapter 6.

The exercises ask you to solve a number of special cases of the generic divide-and-conquer recurrence relation (D). Here is one more case that will yield an alternative analysis of MERGESORT.

Example 5.3 (varied). Suppose that $n = 2^k$ for some integer k. Then the recurrence relation in (C) holds for MERGESORT:

$$M_n = 2M_{n/2} + (3n + 1), \qquad \text{with } M_1 = 1.$$

Instead of solving the above recurrence, we consider an inequality version:

$$M_n \leq 2M_{n/2} + 4n, \tag{C'}$$

since $(3n + 1) \leq 4n$ for $n \geq 1$. This will be easier to solve and will lead to an upper bound on M_n. In the next theorem we solve a more general form of recurrence relation of which this is a special case.

Question 5.7. Use iteration and induction to verify that for the case $n = 2^k$, $M_n \leq 4n \log(n) + M_1 n$ satisfies (C').

More generally, if we use a divide-and-conquer algorithm that solves d smaller problems each of which is $(1/d\text{th})$ of the original, then the complexity analysis might involve a recurrence relation of the form

$$a_n = d\, a_{\lfloor n/d \rfloor} + cn,$$

or

$$a_n \leq d\, a_{\lfloor n/d \rfloor} + cn, \tag{E}$$

where c and d are constants, $d > 1$. Manipulation of (E) is simplified if we use the logarithm to the base d, denoted by $\log_d$.

Theorem 5.2. If c and d are constants with $d > 1$ and

$$a_n \leq d\, a_{\lfloor n/d \rfloor} + cn,$$

then

$$a_n \leq cn \log_d(n) + a_1 n.$$

Proof. We prove this by induction on n. For the base case we have that $a_1 \leq 0 + a_1 1$. We assume that the theorem is true for all $n < k$ and we examine a_k.

$$
\begin{aligned}
a_k &\le d\, a_{\lfloor k/d \rfloor} + ck \\
&\le d\left[c\left\lfloor\frac{k}{d}\right\rfloor \log_d\left(\left\lfloor\frac{k}{d}\right\rfloor\right) + a_1\left\lfloor\frac{k}{d}\right\rfloor\right] + ck && \text{by induction} \\
&\le d\left[c\left(\frac{k}{d}\right)\log_d\left(\frac{k}{d}\right) + a_1\left(\frac{k}{d}\right)\right] + ck && \text{since } \left\lfloor\frac{k}{d}\right\rfloor \le \frac{k}{d} \\
&= ck\log_d\left(\frac{k}{d}\right) + a_1 k + ck && \text{by algebra} \\
&= ck\left[\log_d(k) - 1\right] + a_1 k + ck && \text{by properties of } \log_d \\
&= ck\log_d(k) + a_1 k. && \text{by algebra} \quad \square
\end{aligned}
$$

Example 5.3 (last thoughts). If $d = 2$ and $c = 4$, Theorem 5.2 gives the following bound on the complexity of MERGESORT.

$$M_n \le 4n\log(n) + n.$$

Note that the recurrence relations (C) and (C′) and hence this bound hold only when $n = 2^i$. You should check that this is a larger upper bound than that of Theorem 7.1 from Chapter 6.

In the most general divide-and-conquer recurrence relation

$$a_n = k\, a_{\lfloor n/d \rfloor} + cn + e,$$

we have just seen that if $k = d$ and $e = 0$, then $a_n = O(n\log(n))$. Exercise 11 demonstrates that if $k < d$, then $a_n = O(n)$. In contrast Exercise 12 shows that if $k > d$, then $a_n = O(n^q)$, where $q = \log_d(k)$. Thus the complexity of a recursive procedure is quite sensitive to small changes in the constants.

EXERCISES FOR SECTION 5

1. Suppose that $a_0 = 1$ is the initial condition for each of the following recurrence relations. Then list the first five terms of the sequence generated.

- **(i)** $a_n = a_{\lfloor n/2 \rfloor} + 1$.
- **(ii)** $a_n = a_{\lfloor n/3 \rfloor} - 1$.
- **(iii)** $a_n = a_{\lfloor n/5 \rfloor}$.
- **(iv)** $a_n = 2a_{\lfloor n/2 \rfloor}$.
- **(v)** $a_n = 3a_{\lfloor n/3 \rfloor} + 1$.
- **(vi)** $a_n = a_{\lfloor n/4 \rfloor} + 1$.
- **(vii)** $a_n = 5a_{\lfloor n/5 \rfloor} - n$.
- **(viii)** $a_n = a_{\lfloor n/3 \rfloor} + 3n$.

2. How many initial conditions are needed for each of the following recurrence relations?

(i) $a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor}$.
(ii) $a_n = a_{\lfloor n/4 \rfloor} + 1$.
(iii) $a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$.
(iv) $a_n = a_{n-2} + a_{\lfloor n/2 \rfloor}$.

3. **(i)** Suppose that $h_n = 2h_{\lfloor n/2 \rfloor} + s$, where s is a constant. Use iteration and induction to solve this recurrence relation in the case that $n = 2^i$.
(ii) For arbitrary n find an upper bound on h_n.

4. Consider the recurrence relation $a_n = a_{\lfloor n/d \rfloor} + c$, where c and d are constants, $d > 1$. Show that if $n = d^i$, then $a_n = a_1 + c \log_d(n)$. What happens for arbitrary n?

5. Suppose that $n = d^i$, $d > 1$. Use iteration and induction to deduce that the recurrence relation

$$a_n = d\, a_{\lfloor n/d \rfloor} + c$$

is satisfied by

$$a_n = d^i a_1 + d^{i-1} c + \cdots + dc + c.$$

Then explain why

$$a_n = \left(a_1 + \frac{c}{d-1} \right) n - \frac{c}{d-1}.$$

6. Given the recurrence $z_n = k\, z_{\lfloor n/d \rfloor}$ with $d > 1$, solve for z_n.

7. Explain why Theorem 5.2 and the preceding exercises contain the condition that $d > 1$.

8. Why is the following not a valid proof?

Theorem. If a_n is the nth term of a sequence that satisfies

$$a_n = d\, a_{\lfloor n/d \rfloor} + c$$

for some constants c and d with $d > 1$, then

$$a_n = O(n).$$

Proof. We must show that $a_n \le sn$ for some constant s. Let $s = a_1$ so that the base case is met: $a_1 \le s1 = a_1$. Then assume that for all $n < k$, $a_n \le sn$.

From the recurrence relation we have

$$\begin{aligned}
a_n &= d\,a_{\lfloor n/d \rfloor} + c \\
&\leq d\left(s\left\lfloor \frac{n}{d} \right\rfloor\right) + c \qquad \text{by induction} \\
&\leq \frac{dsn}{d} + c \\
&= sn + c \\
&= O(n).
\end{aligned}$$

9. Prove the Theorem of the preceding exercise (correctly). (*Hint:* Use the result from Exercise 5.)

10. Use iteration and induction to find a function f_n such that $a_n = O(f_n)$ for each of the following:

(i) $a_n = k\,a_{\lfloor n/d \rfloor} + c$, where c, d, and k are constants such that $k \neq d$ and $1 < d$.

(ii) $a_n = a_{\lfloor n/d \rfloor} + \log(n)$, where d is a constant greater than 1.

(iii) $a_n = d\,a_{\lfloor n/d \rfloor} + n^2$, where $1 < d$.

11. Show that if c, d, and k are constants such that $k \neq d$, and

$$a_n = k\,a_{\lfloor n/d \rfloor} + cn,$$

then

$$a_n \leq sn^{\log_d(k)} + \left(\frac{dc}{d-k}\right)n \qquad \text{where } s \text{ is a constant.}$$

12. Show that if a_n is as given in the previous problem and $k < d$, then

$$a_n = O(n).$$

13. For each of the recurrence relations in Exercise 1 find a function f_n such that $a_n = O(f_n)$.

14. For each of the following recurrence relations decide whether $a_n = O(1)$, $O(\log^n(n))$, $O(n)$, $O(n \log(n))$, $O(n^2)$ or $O(2^n)$:

(i) $a_n = a_{\lfloor n/3 \rfloor}$, $a_0 = 1$.

(ii) $a_n = a_{\lfloor n/4 \rfloor} + 1$, $a_0 = 1$.

(iii) $a_n = a_{\lfloor n/d \rfloor} - 1$ for some constant $d > 1$, $a_0 = 1$.

(iv) $a_n = 3a_{\lfloor n/3 \rfloor}$, $a_0 = 3$.

(v) $a_n = 3a_{\lfloor n/3 \rfloor} + 3$, $a_0 = 3$.

(vi) $a_n = a_{\lfloor n/3 \rfloor} + 3n$, $a_0 = 1$.
(vii) $a_n = 3a_{\lfloor n/3 \rfloor}$, $a_0 = 2$.
(viii) $a_n = 3a_{\lfloor n/3 \rfloor} + 1$, $a_0 = 0$.
(ix) $a_n = 3a_{\lfloor n/3 \rfloor} + n$, $a_0 = 1$.
(x) $a_n = 2a_{\lfloor n/3 \rfloor}$, $a_0 = 1$.
(xi) $a_n = 2a_{\lfloor n/3 \rfloor} + 1$, $a_0 = 1$.
(xii) $a_n = 2a_{\lfloor n/3 \rfloor} - 1$, $a_0 = 1$.
(xiii) $a_n = 4a_{\lfloor n/3 \rfloor}$, $a_0 = 1$.
(xiv) $a_n = 4a_{\lfloor n/3 \rfloor} + 1$, $a_0 = 1$.

15. Reread Example 1.5 and explain why SECRET is a good algorithm.

16. Let the recurrence relation for P_n be defined by

$$P_n = P_{n-1} + P_{n-2} + \cdots + P_1$$

for $n > 1$ with $P_1 = c$, some constant. Is this a LHRRWCC? If so, write down its characteristic equation. In any case, determine the first eight values of P_n, in terms of c. Then guess and prove a formula for P_n as a function of n and c.

17. Here is the algorithm MAX from Exercise 4.12 of Chapter 2. Given an array of n real numbers, it finds the maximum number and stores it in the variable max.

Algorithm MAX

STEP 1. Input n, a positive integer, and $x_1, \ldots, x_j, \ldots, x_n$, real numbers
STEP 2. Set max $:= x_1$
STEP 3. For $j = 2$ to n do
  STEP 4. If $x_j >$ max then max $:= x_j$
STEP 5. Output max and stop.

Explain why MAX always make $(n - 1) = O(n)$ comparisons.

18. In comparison with MAX, here is the idea for a recursive divide-and-conquer algorithm to find the maximum entry in an array of n numbers. If the list has one element, then max equals this entry. Otherwise, we divide the list in half:

$$L_1 = x_1, \ldots, x_{\lfloor n/2 \rfloor}$$
$$L_2 = x_{\lfloor n/2 \rfloor + 1}, \ldots, x_n.$$

Let m_1 be the maximum entry in L_1 and m_2 the maximum in L_2. Then we compare m_1 and m_2, and the larger is the overall maximum. If M_n is the number of comparisons performed using this idea on a list of length n, then find a divide-and-conquer recurrence relation that M_n satisfies. What are the values of $M_1, \ldots, M_8$?

19. Solve the recurrence relation of the preceding exercise. (*Hint:* first let $n = 2^k$. Otherwise, use the trick of MERGESORT to extend the array to one with 2^k entries.) Is this algorithm more efficient than MAX?

7:6 RECURRING THOUGHTS

In this chapter recurrence relations have come up in the definitions of integer sequences, in mathematical models, and in the complexity analysis of algorithms. With naturally specified initial conditions, the goal is to find a formula (or at least an upper bound) for the nth term of the sequence. Once we have found such a formula, then it is not difficult to prove this result by induction. In fact, recurrence relations are ideally suited to inductive proofs, using either ordinary or complete induction, because they give the nth term as a function of preceding terms. Thus the hard question is generally to find the solution to the recurrence relation.

The first commonsense approach is to use iteration. This technique works well on inhomogeneous recurrence relations, especially on the divide-and-conquer recurrence relations. In general, it does not work so well on homogeneous recurrence relations.

For the special case of linear homogeneous recurrence relations with constant coefficients, we have presented a complete solution using characteristic equations and their roots. In theory, we can find the solution of any LHRRWCC.

The general technique used to solve LHRRWCCs is an important one with wider application in mathematics. We look for "basic" or "linearly independent" solutions and combine them in "linear combinations" to derive all possible solutions. This technique is used whenever the underlying mathematical structure is a "linear space." For example, the field of linear algebra deals with the solution of homogeneous systems of linear equations, and the field of differential equations studies the solution of linear homogeneous differential equations. It is not by chance (or bad planning) that the same words appear repeatedly in different fields; the underlying ideas and solution techniques are really the same.

Iteration and induction is the technique of choice for the divide-and-conquer recurrence relations. Typically, we use iteration on a simplified case, as when $n = 2^k$ or $n = d^k$, and then find that the resulting formula gives a bound for a solution of the general recurrence relation. We can solve or get tight upper bounds on essentially all recurrence relations of the form

$$a_n = k\, a_{\lfloor n/d \rfloor} + cn + e,$$

where c, d, e, and k are constants. These techniques will also work on other, more irregular recurrence relations.

SUPPLEMENTARY EXERCISES FOR CHAPTER 7

1. Let L be the list of all positive integers that begin with a 7, listed in increasing order. Write down the first 12 entries of L. Can you (within, say, 5 minutes) find a formula L_n that give the nth entry of L as a function of n?

2. Suppose that n points are placed around a circle and that every pair of points is joined by a line, either straight or curved, but drawn so that at most two lines cross each other at the same point. Into how many regions is the interior of the circle divided? Call this number R_n. Does $R_n = 2^{n-1}$ for all positive n?

3. Let S_n be defined by $S_n = S_{n-1} + 1/n^2$ for $n > 1$, with $S_1 = 1$. Use iteration and induction to find a formula for S_n.

4. Let SR_n be defined by $SR_n = SR_{n-1} + 1/\sqrt{n}$ for $n > 1$, with $SR_1 = 1$. Find and justify a formula for SR_n.

5. Refer to the definitions of S_n and SR_n in the preceding exercises. Which of the following are true and which false? Justify.
   (i) $S_n = O(1)$
   (ii) $SR_n = O(1)$
   (iii) $1 = O(S_n)$
   (iv) $1 = O(SR_n)$
   (v) $S_n = O(\log(n))$
   (vi) $SR_n = O(\log(n))$
   (vii) $\log(n) = O(S_n)$
   (viii) $\log(n) = O(SR_n)$
   (ix) $S_n = O(n)$
   (x) $SR_n = O(n)$
   (xi) $n = O(S_n)$
   (xii) $n = O(SR_n)$

6. The **Towers of Hanoi** puzzle consists of a board with three pegs rising from the base. On one peg there are six circular disks of differing size. The largest disk is on the bottom and the others are stacked above it in order of decreasing size. These disks are to be transferred, one at a time, onto another peg so that at no time is a larger disk placed above a smaller one. What is the minimum number of moves needed to move the six disks?

7. We consider the abstract n-fold Tower of Hanoi puzzle in which we suppose that n disks are stacked on one peg and must be moved to another peg, as described in Exercise 6. Let H_n denote the minimum number of moves required to transfer the n disks. Then $H_1 = 1$ and $H_2 = 3$. Find a recurrence relation that expresses H_n in terms of H_{n-1}, the number of moves needed to move the top $(n - 1)$ disks. Then find a formula for H_n and prove that it is correct.

8. Suppose that we consider a variant on the Tower of Hanoi puzzle in which there are four pegs with n disks stacked on one peg. Let M_n denote the minimum number of moves needed to move the stack of n disks to another peg. Calculate M_n for $n = 2, 3, 4$, and 5. Do these values agree with those of H_n? Find a recurrence relation for M_n and find as small a function f_n as possible such that $M_n = O(f_n)$.

9. Explain why the polynomial

$$p(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0$$

has a root s if and only if $p(x)$ can be factored as

$$p(x) = (x - s)q(x),$$

where $q(x)$ is a polynomial of degree $(r - 1)$. [*Hint:* Suppose that when $p(x)$ is divided by $(x - s)$, $q(x)$ is the quotient and $r(x)$ the remainder. In other words, $p(x) = (x - s)q(x) + r(x)$.]

10. The complex (or imaginary) number i has the property that

$$i^2 = -1.$$

Explain why $(-i)^2 = -1$ and $(-i)i = +1$. What is the value of i^3, $(-i)^3$, i^4 and $(-i)^4$? In general, what is the value of i^{2n-1}, $(-i)^{2n-1}$, i^{2n} and $(-i)^{2n}$?

11. **(i)** The sequence $2, 0, -2, 0, 2, 0, -2, 0, \ldots$ satisfies

$$a_n = -a_{n-2} \text{ for } n \geq 2, \qquad a_0 = 2, a_1 = 0.$$

Use the methods of Section 3 to find a formula for the nth term of this sequence. By inspection we can see that the following is also a formula for the nth term of the sequence:

$$f_n = \begin{cases} 2 & \text{if 4 divides } n \\ 0 & \text{if 4 divides } (n-1) \text{ or } (n-3) \\ -2 & \text{if 4 divides } (n-2). \end{cases}$$

Do these agree?

(ii) Repeat the problem in part (i) with the sequence

$$1, 3i, -1, -3i, 1, 3i, -1, -3i, \ldots$$

What formula can you derive for this by inspection? Does it agree with your formula obtained through a recurrence relation?

12. A generalized Fibonacci number is defined as follows: For k a fixed integer greater than 2,

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k \qquad \text{for } n \geq k$$

with initial conditions $F_0^k = F_1^k = \cdots = F_{k-2}^k = 0$ and $F_{k-1}^k = 1$.

(i) For $k = 3, 4,$ and 5 write out the first 10 generalized Fibonacci numbers.

(ii) For $k = 3, 4,$ and 5 find the characteristic equation of F_n^k.

(iii) For $k = 3$ find the approximate values of the characteristic roots.

13. **(i)** Let Y_n denote the number of strings of length n, containing 0s, 1s and (-1)s with no two consecutive 1s and no two consecutive (-1)s. Determine Y_1, Y_2, and Y_3 by listing all such strings.

(ii) Find a recurrence relation for Y_n. Then, if possible, solve it using the initial conditions found in part (i).

14. Let a_n be the recurrence relation defined by

$$a_n = a_{n-1} + a_{n-2} - n + 3 \qquad \text{for } n \geq 2$$

with initial conditions $a_0 = 0$ and $a_1 = 2$. Find a formula for a_n expressed in terms of F_n, the nth Fibonacci number.

15. For m a fixed positive integer, consider

$$a_n = a_{n-1} + a_{n-2} + \binom{n}{m} \qquad \text{for } n \geq 2$$

with initial conditions $a_0 = 0$ and $a_1 = 1$.

(i) For $m = 5$, find the first 10 entries of the sequence a_n. Express each entry in terms of Fibonacci numbers. (Recall that $\binom{n}{m}$ is equal to 0 when $m > n$.)

(ii) For arbitrary m, find a formula for a_n, expressed in terms of F_n, the nth Fibonacci number.

16. Suppose that $q = r/s$ is a rational number that is a root of

$$x^3 + bx^2 + cx + d = 0,$$

where $\gcd(r, s) = 1$ and where b, c, and d are all integers. Explain why $s = 1$ and r is a divisor of d. Then explain why when searching for a rational root of a cubic equation of the form above, one needs to check only the divisors of d.

17. Find all rational roots of the following equations.

(i) $x^3 - 2x^2 + x - 2 = 0$.

(ii) $x^3 + x^2 + x - 3 = 0$.

(iii) $x^3 - 3x^2 + 2x = 0$.
(iv) $x^3 + x^2 + x + 1 = 0$.
(v) $x^3 - 4x^2 + x + 6 = 0$.
(vi) $x^3 + 2x^2 - 3x + 7 = 0$.

18. Explain why every cubic polynomial has some real number as a root.

CAVEAT. The following problems, 19 and 20, require some knowledge of calculus, specifically knowing how to find the derivative of a polynomial and the product rule.

19. By definition we know that if s is a root of a polynomial $p(x)$ of multiplicity $m > 1$, then $p(x)$ can be factored as

$$p(x) = (x - s)^m q(x),$$

where $q(x)$ is a polynomial. Prove that if s is a root of $p(x)$ of multiplicity $m > 1$, then s is also a root of $p'(x)$ of multiplicity $(m - 1)$, where $p'(x)$ is the derivative of $p(x)$.

20. This exercise is a general proof of Theorem 4.1. If the LHRRWCC

$$a_n = k_1 a_{n-1} + k_2 a_{n-2} + \cdots + k_r a_{n-r} \qquad \text{for } n > r \qquad \text{(A)}$$

has a characteristic root q of multiplicity $m > 1$, then the following are all solutions to the recurrence relation:

$$\begin{aligned} a_n &= q^n \\ a_n &= nq^n \\ a_n &= n^2 q^n \\ &\cdots \\ a_n &= n^{m-1} q^n. \end{aligned}$$

From the text we know that $a_n = q^n$ and $a_n = nq^n$ are solutions.

CASE 1. We repeat the case where $m = 2$, since the technique here generalizes more readily than the one given in the text. Let $p(x)$ be the characteristic polynomial of (A). Calculate the function $xp'(x)$, where $p'(x)$ is the derivative of $p(x)$. Show that q is a root of $xp'(x)$ and determine its multiplicity. From this deduce that $a_n = nq^n$ is a solution to (A).

CASE 2 ($m > 2$). Calculate the function $x(xp'(x))'$. Using the results of Case 1, show that q is a root of this function and find its multiplicity. Then deduce that $a_n = n^2q^n$ is a solution of (A).

CASE 3 ($m > 3$). Let $P_i(x)$ be the polynomial obtained from $p(x)$ by i times repeating the process of taking the derivative of $p(x)$ and multiplying by x, then taking the derivative of this new function and multiplying by x:

$$P_1(x) = xp'(x),$$
$$P_2(x) = x(xp'(x))',$$

and so on. Calculate $P_3(x)$ and $P_4(x)$ starting with $p(x)$ the characteristic equation of (A). Then write out the general form of $P_i(x)$. Prove by induction on i that if q is a root of the characteristic equation of (A) of multiplicity $m \geq i \geq 1$, then q is a root of multiplicity $(m - i)$ of $P_i(x)$. From this deduce that for $i = 1, \ldots, m - 1$, $a_n = n^i q^n$ is a solution of (A).

21. Reread Exercises 6 to 8 about the Tower of Hanoi puzzle. Suppose as in Exercise 8 that the puzzle has four pegs with the disks on the first peg, and suppose that we move the disks as follows:
(*a*) Move the top $\lfloor n/2 \rfloor$ disks to the second peg, one by one following the rules of the puzzle.
(*b*) Move all but the last of the remaining disks to the third peg by a legal series of moves.
(*c*) Move the largest disk to the fourth peg.
(*d*) Move the bottom half from the third peg to the fourth peg.
(*e*) Move the top half from the second peg to the fourth peg.
If H'_n denotes the minimum number of moves needed to transfer n disks in this version of the Tower of Hanoi puzzle, then find a recurrence relation for H'_n and solve for H'_n.

NOTE. From Section 1 it seems that the odd-indexed Bernoulli numbers are zero, starting with B_3. Here is a sequence of exercises that shows why $B_{2k+1} = 0$ for $k \geq 1$.

22. A function is called **even** if $f(x) = f(-x)$ for all values of x. Which of the following functions are even?
(i) $f(x) = c$, c a constant.
(ii) $f(x) = x$.
(iii) $f(x) = x^2$.
(iv) $f(x) = x^3$.
(v) $f(x) = x^4$.
(vi) $f(x) = x^5$.
(vii) $f(x) = x^{2n}$.
(viii) $f(x) = x^{2n+1}$.
(ix) $f(x) = 2^x$.
(x) $f(x) = \sqrt{x}$.

23. Prove that the function

$$f(x) = \frac{x}{(e^x - 1)} + \frac{x}{2}$$

is even.

24. An infinite polynomial of the form

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n + \cdots,$$

where $c_0, c_1, \ldots, c_n, \ldots$ are constants, is said to be even if $p(x) = p(-x)$. Explain why $p(x)$ is even if and only if all the odd-indexed terms $c_1, c_3, \ldots, c_{2k+1}$ are zero.

25. Use the results of the preceding two exercises to explain why every other Bernoulli number starting with B_3 is zero.

26. One form of the so-called "**Ballot problem**" asks what the probability is that in an election between candidates A and B the number of votes for A always exceeds that for B until the last ballot is cast when the votes are tied. Suppose that a vote for A is denoted by $+1$ and a vote for B by -1. Then there must be an even number of voters, say $2m$. We want to determine the number of strings of m $+1$s and m -1s such that every partial sum, from 1 to $i < 2m$ is positive. Write down all such strings for $m = 1$, 2, and 3. Then check that the number with a final tie is given by the mth Catalan number as defined in Question 1.7.

NOTE. The Catalan numbers arise in a number of fundamental problems of computer science including the problem of having a computer evaluate an arithmetic expression; Exercises 27–29 explore these connections. To a computer each of the operations +, −, *, /, and ^ is a "binary operation." Each operation requires two numbers upon which to act. Parentheses tell us exactly which two numbers are to be combined into one by each operation.

27. Explain why the addition of parentheses makes a difference in the following expressions.

(i) $a - b - c$.

(ii) $x/y/z$.

(iii) r^{s^t}.

28. Calculate the number of ways to parenthesize expressions with three, four, and five variables by listing all possibilities. Show that these numbers are given by the corresponding Catalan numbers.

29. Suppose that we have an expression combining n variables, like

$$x_1 \mathbin{?} x_2 \mathbin{?} x_3 \mathbin{?} \cdots \mathbin{?} x_n$$

where ? stands for one of the usual arithmetic operations. Show that the number of different parenthesizations of an expression with n variables satisfies the Catalan recurrence relation.

30. Use the fact that

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}$$

to derive a recurrence relation for C_n in terms of C_{n-1}, using no other Catalan number.

31. Show that

$$C_n \geq \frac{1}{n} 2^{n-1}$$

for all positive n. Is there a positive integer k such that $C_n = O(n^k)$?

32. A tree is called a **planted planar** tree if one vertex of degree 1 is designated as the root r and then the tree is drawn in the plane. For example, the following diagram shows all different planted planar trees with 1, 2, and 3 edges. Let PT_n denote the number of different planted planar trees with n edges. Determine PT_4 and PT_5, and for $n = 1, 2, \ldots, 5$ show that $PT_n = C_n$.