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MCSC 201

2.) Data matrix $X_{1000 \times 10}$. Here, $N = 1000$ (no. of instances)
$d = 10$ (dimensionality)

Variance - Covariance matrix Σ..

First four PCs : w_1, w_2, w_3, w_4.

Let w_5 be the fifth principal component. By definition, it should maximize variance, be of unit length, and be orthogonal to the first four PCs so that after projection, $z_5 = w_5 x$ is uncorrelated with z_1, z_2, z_3, z_4, which correspond to each of the first four PCs respectively.

Thus, our constraints are:

(i) maximize $var(z_5)$, where $z_5 = w_5 x$;
$Var(z_5) = w_5^T \Sigma w_5$

(ii) w_5 be of unit length, i.e.
$w_5^T w_5 = 1$

and
~~(iii)~~ w_5 be orthogonal to each of w_1, w_2, w_3, w_4.

(iv) $w_5^T w_1 = 0$ $(w_5 \perp w_1)$

(v) $w_5^T w_2 = 0$ $(w_5 \perp w_2)$

(vi) $w_5^T w_3 = 0$ $(w_5 \perp w_3)$

(vii) $w_5^T w_4 = 0$ $(w_5 \perp w_4)$

Now, we write all these constraints in the Lagrangian function.

$$L(w_5, x, \alpha, \beta, \gamma, \delta, \varepsilon)$$
$$= w_5^T \Sigma w_5 - \alpha(w_5^T w_5 - 1) - \beta(w_5^T w_1 - 0) - \gamma(w_5^T w_2 - 0) - \delta(w_5^T w_3 - 0) - \varepsilon(w_5^T w_4 - 0)$$

Then, we differentiate L wrt w_5 and set it to 0,

i.e. $$\frac{\partial L}{\partial w_5} = 0.$$

After this work, we will see that w_5 should be the eigenvector of Σ with the fifth largest eigenvalue, λ_5.

5) Given dataset has two classes, where each instance is of the form ~~x~~ $\{x^t, r^t\}$, where x^t : a 2-dimensional vector attributes

r^t : class label

The entire dataset can be written as

$X = \{(x^t, r^t), t = 1, 2, 3, 4, 5\}$

Class means before projection

Class $C = 1$

$$m_1^b = \frac{\sum_t x^t r^t}{\sum r^t}, \text{ where } r^t = 1 \text{ if } C = 1$$

Column wise means

for x_1: $$\frac{1(1) + 10(0) + 6(1) + 2(1) + 16(0)}{3}$$

$$= \frac{1+6+2}{3} = \frac{9}{3} = 3$$

for x_2: $$\frac{5(1) + 2(0) + 12(1) + 10(1) + 4(0)}{3}$$

$$= \frac{5+12+10}{3} = \frac{27}{3} = 9$$

$$\therefore, m_1^b = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

Class $C=0$

$$m_2^b = \frac{\sum_t x^t (1-r^t)}{\sum (1-r^t)} = \begin{bmatrix} \frac{10+16}{2} \\ \frac{2+4}{2} \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$$

Class Means after projection $w = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$$m_1^a = \frac{\sum_t w^T x^t r^t}{\sum_t r^t} = w^T m_1^b$$

class 1

$$= \begin{bmatrix} 3 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = 9 + 63 = 72$$

$$m_2^a = \frac{\sum_t w^T x^t (1 - r^t)}{\sum_t (1 - r^t)} = w^T m_2^b$$

class 0

$$= \begin{bmatrix} 3 & 7 \end{bmatrix} \begin{bmatrix} 13 \\ 3 \end{bmatrix} = 39 + 21 = 60$$

Between class scatter matrix, $S_B = (m_1^b - m_2^b)(m_1^b - m_2^b)^T$

Now, $m_1 b - m_2 b$

$$= \begin{bmatrix} 3 \\ 9 \end{bmatrix} - \begin{bmatrix} 13 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$$

$$\therefore S_B = \begin{bmatrix} -10 \\ 6 \end{bmatrix} \begin{bmatrix} -10 & 6 \end{bmatrix}$$

$$S_B = \begin{bmatrix} 100 & -60 \\ -60 & 36 \end{bmatrix}$$

~~Class~~ within class scatter matrix for $C = 1$,

$$S_1 = \sum_t r^t (x^t - m_1^b)(x^t - m_1^b)^T$$

$$= \begin{bmatrix} 1-3 \\ 5-9 \end{bmatrix} \begin{bmatrix} 1-3 & 5-9 \end{bmatrix} + \begin{bmatrix} 6-3 \\ 12-9 \end{bmatrix} \begin{bmatrix} 6-3 & 12-9 \end{bmatrix} + \begin{bmatrix} 2-3 \\ 10-9 \end{bmatrix} \begin{bmatrix} 2-3 & 10-9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} + \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} + \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 14 & -2 \\ -2 & 26 \end{bmatrix}$$

Within Class Scatter matrix for C_2

$$S_2 = \sum_t (1-z^t)(x^t - m_2 b)(x^t - m_2 b)^T$$

$$= \begin{bmatrix} 10-13 \\ 2-3 \end{bmatrix} \begin{bmatrix} 10-13 & 2-3 \end{bmatrix} + \begin{bmatrix} 16-13 \\ 4-3 \end{bmatrix} \begin{bmatrix} 16-13 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 18 & 6 \\ 6 & 2 \end{bmatrix}$$

—x—x—x—