Tutorial 3

Exercise 1

Let $T_j(n)$, such that $1 \leq j \leq k$, be the maximum of $T_1(n), \ldots, T_k(n)$ and recall that the probabilities p_i satisfy $p_1 + p_2 + \cdots + p_k = 1$. Then

$$A(n) = \sum_{i=1}^{k} p_i \cdot T_i(n) \leq \sum_{i=1}^{k} p_i \cdot T_j(n) \leq T_j(n) \sum_{i=1}^{k} p_i = T_j(n) = \max_{1 \leq i \leq k} T_i(n) = W(n).$$

Exercise 2

We shall solve the following recurrence equation

$$\begin{array}{rcl} T(1) & = & 7 \\ T(n) & = & 2 + T(n-1) \text{ for } n > 1 \end{array}$$

by repeated substitutions. Let us unfold $T(n)$.

$$T(n) = 2 + T(n-1) = 2 + 2 + T(n-2) = 2 + 2 + 2 + T(n-3) = \ldots$$

Observe the pattern $T(n) = \underbrace{2 + 2 + \cdots + 2}_{i\times} + T(n-i)$ for $i < n$. So if $i = n-1$ then

$$T(n) = \underbrace{2 + 2 + \cdots + 2}_{(n-1)\times} + T(n-n+1) = 2(n-1) + T(1) = 2n - 2 + 7 = 2n + 5.$$

Proof: by induction on n we show that $T(n) = 2n + 5$. In the base case $n = 1$ and $T(1) = 2 \cdot 1 + 5 = 7$ as required. Let $n > 1$. By IH (induction hypothesis) assume that $T(j) = 2j + 5$ for all $1 \leq j < n$. Now

$$T(n) = \text{(by definition) } 2 + T(n-1) = \text{(by IH) } 2 + 2(n-1) + 5 = 2 + 2n - 2 + 5 = 2n + 5.$$

Exercise 3

We shall solve the following recurrence equation

$$\begin{array}{rcl} T(0) & = & 1 \\ T(n) & = & nT(n-1) \text{ for } n > 1 \end{array}$$

by repeated substitutions. Let us unfold $T(n)$.

$$T(n) = nT(n-1) = n(n-1)T(n-2) = n(n-1)(n-2)T(n-3) = \ldots$$

Observe the pattern $T(n) = n(n-1)(n-2)\cdots(n-i+1)T(n-i)$ for $i \leq n$. So if $i = n$ then

$$T(n) = n(n-1)(n-2)\cdots(n-n+1)T(0) = n(n-1)(n-2)\cdots 2 \cdot 1 \cdots T(0) = n!.$$

Proof: by induction on n we show that $T(n) = n!$. In the base case $n = 0$ and $T(0) = 1 = 0!$ as required. Let $n > 0$. By IH (induction hypothesis) assume that $T(j) = j!$ for all $0 \leq j < n$. Now

$$T(n) = \text{(by definition) } nT(n-1) = \text{(by IH) } n \cdot (n-1)! = n!.$$

Exercise 4

Let n be the size of the array, i.e., $n = b-a+1$ and assume that $a \leq b$. Then $W(1) = 1$ and $W(n) = 1+W(\lfloor(n+1)/2\rfloor)$ for $n > 1$. Let us reformulate the recurrence equation such that we consider only the points where $n = 2^k$ for some integer $k \geq 0$.

Let $n = 2^k$. Observe that $\lfloor(n+1)/2\rfloor = \lfloor(2^k+1)/2\rfloor = 2^{k-1}$. Hence we shall solve the following recurrence equation

$$\begin{array}{rcl} W(2^0) & = & 1 \\ W(2^k) & = & 1 + W(2^{k-1}) \text{ for } k > 0. \end{array}$$

by repeated substitutions. Let us unfold $W(2^k)$.

$$W(2^k) = 1 + W(2^{k-1}) = 1 + 1 + W(2^{k-2}) = 1 + 1 + 1 + W(2^{k-3}) = \ldots$$

Observe the pattern $W(2^k) = i + W(2^{k-i})$ for $i \leq k$. So if $i = k$ then

$$W(2^k) = k + W(2^0) = k + 1.$$

However, $2^k = n$ implies that $k = \log_2(n)$. So $W(n) = \log_2 n + 1$.

Proof: by induction on k we show that $W(2^k) = k + 1$. In the base case $k = 0$ and $W(2^0) =$ (by definition) $1 = 0 + 1$ as required. Let $k > 0$. By IH (induction hypothesis) assume that $W(2^j) = j + 1$ for all $0 \leq j < k$. Now

$$W(2^k) = \text{(by definition) } 1 + W(2^{k-1}) = \text{(by IH) } 1 + (k-1) + 1 = k + 1.$$

Exercise 5

$$T(n) = n^3$$

Exercise 6

a) Let us fix $n_0 = 1$ and $M = 5$, then $n+4 \leq 5n$ for all $n \geq 1$. Another possibility is e.g. $n_0 = 4$ and $M = 2$ because $n + 4 \leq 2n$ for all $n \geq 4$. The validity of the second case can be proved by induction. In the basic step we have to check that $4 + 4 \leq 2 \cdot 4$ which is true. For the inductive step let $n > 4$ and by IH we assume that $(n-1) + 4 \leq 2(n-1)$. We want to show that $n + 4 \leq 2n$, which is true because

$$n + 4 = (n-1) + 4 + 1 \leq \text{ (by IH) } 2(n-1) + 1 = 2n - 2 + 1 = 2n - 1 \leq 2n.$$

b) Let us fix e.g. $n_0 = 1$ and $M = 5$, then $3n^5 + 2n^3 \leq M \cdot n^5$ for every $n \geq n_0$. Why and how did we find it?

$$3n^5 + 2n^3 \leq 3n^5 + 2n^5 = 5n^5$$

c) Let $n_0 = 100$ and $M = 1$. Then $n^7 \leq 2^n$ for all $n \geq 100$. The argument for "why" is a little bit more difficult this time and we will prove it by induction on n.

For the base case (n=100) we have $100^7 \leq 2^{100}$ which is true (check this on a calculator if you want to :-)).

Let $n > 100$ and we assume by IH that $(n-1)^7 \leq 2^{n-1}$. We aim to show that

$$n^7 \leq 2^n. \tag{1}$$

In order to do that we will in fact prove a stronger claim

$$n^7 \leq (n-1)^7 \cdot 2. \tag{2}$$

If we succeed to show this then by IH $(n-1)^7 \cdot 2 \leq 2^{n-1} \cdot 2 = 2^n$, and the inequality (1) is also proven.

To finish the proof we will now argue that (2) is true for all $n > 100$. To prove $n^7 \leq (n-1)^7 \cdot 2$ is the same as to prove $\sqrt[7]{n^7} \leq \sqrt[7]{(n-1)^7 \cdot 2}$ (both sides are non-negative), which is the same as $n \leq (n-1) \cdot \sqrt[7]{2}$, which is the same as $n \leq n \cdot \sqrt[7]{2} - \sqrt[7]{2}$, which is the same as $\sqrt[7]{2} \leq n \cdot (\sqrt[7]{2} - 1)$, which is the same as $n \geq \frac{\sqrt[7]{2}}{\sqrt[7]{2}-1}$, which is obviously true for all $n > 100$ because $\frac{\sqrt[7]{2}}{\sqrt[7]{2}-1} \simeq 11$.

d) Let us fix. e.g. $n_0 = 2$ and $M = 1/2$, then $n^2 \cdot \log n \geq n^2 = (1/2) \cdot 2n^2$ for all $n \geq 2$.

e) First we show that $5n^2 + 2n + 4$ is $O(n^2)$. Let $n_0 = 1$ and $M = 11$. Then

$$5n^2 + 2n + 4 \leq 5n^2 + 2n^2 + 4n^2 = 11n^2 = M \cdot n^2$$

for all $n \geq 1$.
Next we have to show that $5n^2 + 2n + 4$ is $\Omega(n^2)$. Let $n_0 = 1$ and $M = 1$ which immediately implies that $5n^2 + 2n + 4 \geq n^2$ for all $n \geq 1$.

Exercise 7

- $20n^2 = O(n^2)$
  (!!! **but also** e.g. $20n^2 = O(n^3)$, $20n^2 = O(n^8)$ and $20n^2 = O(2^n)$ even if these are not optimal estimates)
- $10n^3 + 6n = O(n^3)$
- $2^n + n^{1691} = O(2^n)$
- $1000n + n^2 = O(n^2)$
- $\log_5(3n^4) = O(\log_2 n)$
  (and we usually write only $O(\log n)$ since the base of the logarithm is irrelevant for the O-notation as proved during the lecture)
  The argument for this is the following: $\log_5(3n^4) = \log_5 3 + \log_5 n^4 = \log_5 3 + 4 \cdot \log_5 n = O(\log_5 n) = O(\log_2 n)$

Exercise 8

It is true that 2^n is $O(3^n)$ but 2^n is not $\Omega(3^n)$. Hence 2^n is not $\Theta(3^n)$.

Exercise 9

Assume that $f(n) = O(g(n))$ and $g(n) = O(h(n))$ which means that there are natural numbers n_0 and n_1, and real numbers M_0 and M_1 such that

- $f(n) \leq M_0 \cdot g(n)$ for all $n \geq n_0$ and
- $g(n) \leq M_1 \cdot h(n)$ for all $n \geq n_1$.

Let us define $n_2 = \max\{n_0, n_1\}$ and $M_2 = M_0 \cdot M_1$, which implies that

$$f(n) \leq M_0 \cdot g(n) \leq M_0 \cdot (M_1 \cdot h(n)) = (M_0 \cdot M_1) \cdot h(n) = M_2 \cdot h(n)$$

for all $n \geq n_2$ and hence $f(n) = O(h(n))$.