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Linear Regression with Multiple Variables

Multiple Features

Multiple features (variables)

one feature

Size in feet² (x)	Price ($) in 1000's ($y$)
2104	400
1416	232
1534	315
852	178
…	…

$$f_{w,b}(x) = wx + b$$

Multiple features (variables)

	Size in feet²	Number of bedrooms	Number of floors	Age of home in years	Price ($) in $1000's
	x_1	x_2	x_3	x_4	
	2104	5	1	45	460
$i=2$	1416	3	2	40	232
	1534	3	2	30	315
	852	2	1	36	178
	...	...	...	...	...

$j=1 \ldots 4$

$n=4$

x_j = j^{th} feature

n = number of features

$\vec{x}^{(i)}$ = features of i^{th} training example

$x_j^{(i)}$ = value of feature j in i^{th} training example

$\vec{x}^{(2)} = [1416 \;\; 3 \;\; 2 \;\; 40]$

$x_3^{(2)} = 2$

Model:

Previously: $f_{w,b}(x) = wx + b$

$f_{w,b}(x) = w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + b$

example

$f_{w,b}(x) = 0.1 x_1 + 4 x_2 + 10 x_3 + -2 x_4 + 80$

size #bedrooms #floors years base price

$$f_{w,b}(\mathrm{x}) = w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + b$$

$$f_{\vec{w},b}(\vec{x}) = w_1x_1 + w_2x_2 + \cdots + w_nx_n + b$$

$\vec{w} = [w_1 \ w_2 \ w_3 \ \ldots \ w_n]$ parameters of the model

b is a number

vector $\vec{x} = [x_1 \ x_2 \ x_3 \ \ldots \ x_n]$

$$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b = w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_nx_n + b$$

dot product

multiple linear regression

(not multivariate regression)

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Linear Regression with Multiple Variables

Vectorization
Part 1

Parameters and features

$\vec{w} = [w_1 \quad w_2 \quad w_3]$ n=3

b is a number

$\vec{x} = [x_1 \quad x_2 \quad x_3]$

NumPy

linear algebra: count from 1

w[0] w[1] w[2]

```
w = np.array([1.0,2.5,-3.3])
b = 4
x = np.array([10,20,30])
```

x[0] x[1] x[2]

code: count from 0

Without vectorization

$$f_{\vec{w},b}(\vec{x}) = \left(\sum_{j=1}^{n} w_j x_j\right) + b$$

$\sum_{j=1}^{n} \rightarrow j=1\ldots n$

1, 2, 3

range(0,n) → j = 0...n-1

```
f = 0
for j in range(0,n):
    f = f + w[j] * x[j]
f = f + b
```

range(n)

Without vectorization n = 100,000

$$f_{\vec{w},b}(\vec{x}) = w_1x_1 + w_2x_2 + w_3x_3 + b$$

```
f = w[0] * x[0] +
    w[1] * x[1] +
    w[2] * x[2] + b
```

Vectorization

$$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$$

```
f = np.dot(w,x) + b
```

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Linear Regression with Multiple Variables

Vectorization Part 2

Without vectorization

```
for j in range(0,16):
    f = f + w[j] * x[j]
```

t_0

f + w[0] * x[0]

t_1

f + w[1] * x[1]

…

t_{15}

f + w[15] * x[15]

Vectorization

```
np.dot(w,x)
```

t_0

w[0]	w[1]	…	w[15]
*	*	…	*
x[0]	x[1]	…	x[15]

in parallel

t_1

w[0]*x[0] + w[1]*x[1] +…+ w[15]*x[15]

efficient → scale to large datasets

Gradient descent

$\vec{w} = (w_1 \quad w_2 \quad \cdots \quad w_{16})$ ~~b~~ parameters

derivatives $\vec{d} = (d_1 \quad d_2 \quad \cdots \quad d_{16})$

```
w = np.array([0.5, 1.3, … 3.4])
d = np.array([0.3, 0.2, … 0.4])
```

compute $w_j = w_j - \overbrace{0.1}^{\text{learning rate } \alpha} d_j$ for $j = 1 \ldots 16$

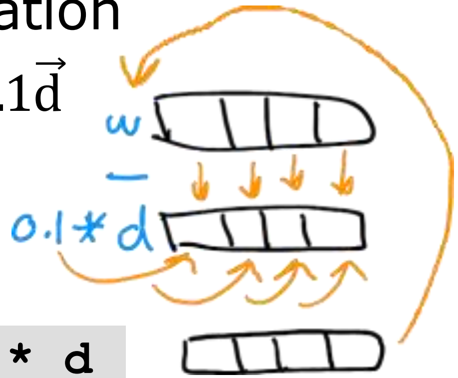
Without vectorization

$$w_1 = w_1 - 0.1 d_1$$
$$w_2 = w_2 - 0.1 d_2$$
$$\vdots$$
$$w_{16} = w_{16} - 0.1 d_{16}$$

```
for j in range(0,16):
    w[j] = w[j] - 0.1 * d[j]
```

With vectorization

$$\vec{w} = \vec{w} - 0.1\vec{d}$$




```
w = w - 0.1 * d
```

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Linear Regression with Multiple Variables

Gradient Descent for Multiple Regression

	Previous notation	Vector notation
Parameters	$w_1, \cdots, w_n$	$\vec{w} = [w_1 \quad \cdots \quad w_n]$ vector of length n
	b	b still a number
Model	$f_{\vec{w},b}(\vec{x}) = w_1x_1 + \cdots + w_nx_n + b$	$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$ dot product
Cost function	$J(\underbrace{w_1, \cdots, w_n}, b)$	$J(\vec{w}, b)$

Gradient descent

repeat {

$$w_j = w_j - \alpha \frac{\partial}{\partial w_j} J(\underbrace{w_1, \cdots, w_n}, b)$$

$$b = b - \alpha \frac{\partial}{\partial b} J(\underbrace{w_1, \cdots, w_n}, b)$$

}

repeat {

$$w_j = w_j - \alpha \frac{\partial}{\partial w_j} J(\vec{w}, b)$$

$$b = b - \alpha \frac{\partial}{\partial b} J(\vec{w}, b)$$

}

Gradient descent

One feature

repeat {

$$w = w - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{w,b}\left(x^{(i)}\right) - y^{(i)} \right) x^{(i)}$$

$\hookrightarrow \frac{\partial}{\partial w} J(w,b)$

$$b = b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{w,b}\left(x^{(i)}\right) - y^{(i)} \right)$$

simultaneously update w, b

}

n features ($n \geq 2$)

repeat {

$j=1$

$$w_1 = w_1 - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right) - y^{(i)} \right) x_1^{(i)}$$

$\hookrightarrow \frac{\partial}{\partial w_1} J(\vec{w},b)$

$\vdots$

$j=n$

$$w_n = w_n - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right) - y^{(i)} \right) x_n^{(i)}$$

$$b = b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right) - y^{(i)} \right)$$

simultaneously update

w_j (for $j = 1, \cdots, n$) and b

}

An alternative to gradient descent

Normal equation
- Only for linear regression
- Solve for w, b without iterations

Disadvantages
- Doesn't generalize to other learning algorithms.
- Slow when number of features is large (> 10,000)

What you need to know
- Normal equation method may be used in machine learning libraries that implement linear regression.
- Gradient descent is the recommended method for finding parameters w,b

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Practical Tips for Linear Regression

Feature Scaling
Part 1

Feature and parameter values

$\widehat{price} = w_1 x_1 + w_2 x_2 + b$

size #bedrooms

x_1: size (feet^2)
range: $300 - 2{,}000$
large

x_2: # bedrooms
range: $0 - 5$
small

House: $x_1 = 2000$, $x_2 = 5$, $price = \$500k$ one training example

size of the parameters w_1, w_2?

$w_1 = 50, \quad w_2 = 0.1, \quad b = 50$

$\widehat{price} = \underbrace{50 * 2000}_{100{,}000K} + \underbrace{0.1 * 5}_{0.5K} + \underbrace{50}_{50K}$

$\widehat{price} = \$100{,}050.5k = \$100{,}050{,}500$

$w_1 = 0.1, \quad w_2 = 50, \quad b = 50$
small large

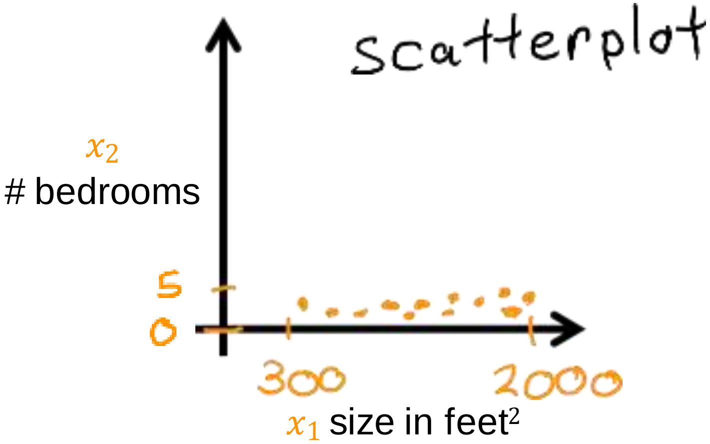
$\widehat{price} = \underbrace{0.1 * 2000k}_{200K} + \underbrace{50 * 5}_{250K} + \underbrace{50}_{50K}$

$\widehat{price} = \$500k$ more reasonable

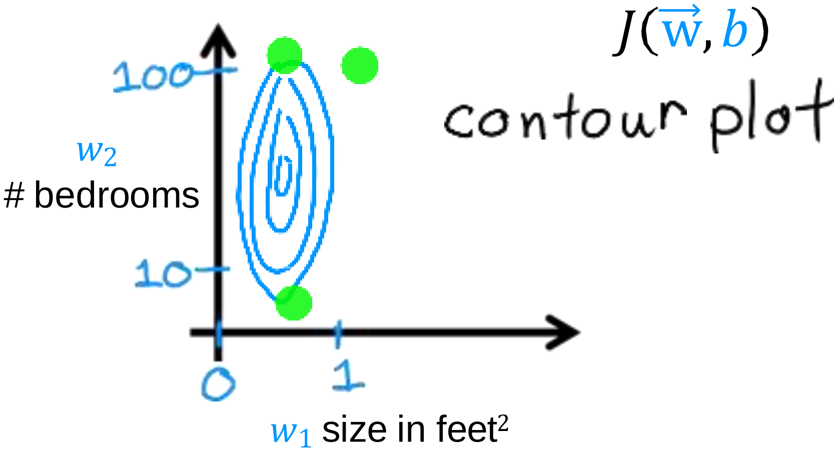
Feature size and parameter size

	size of feature x_j	size of parameter w_j
size in feet²	⟷	⟷
#bedrooms	⟷	⟷

Features

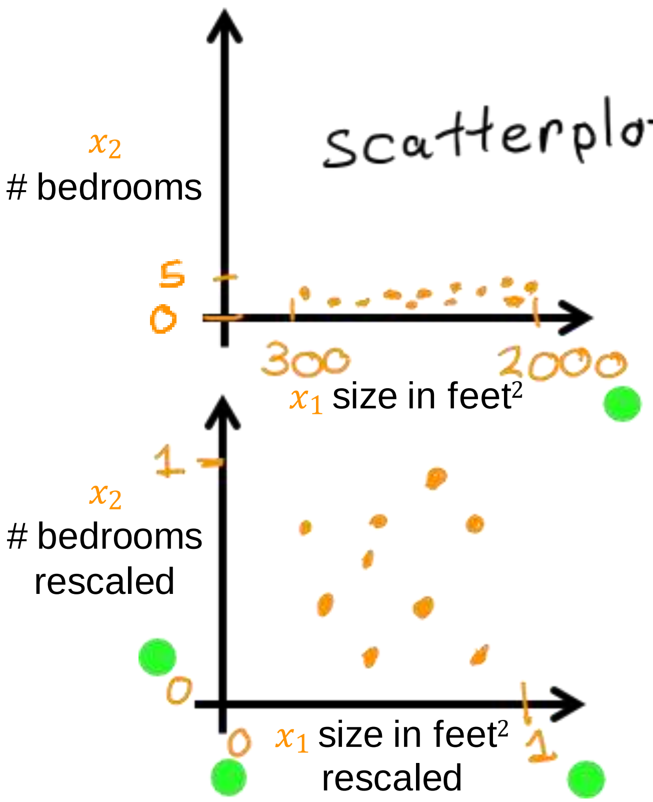


Parameters

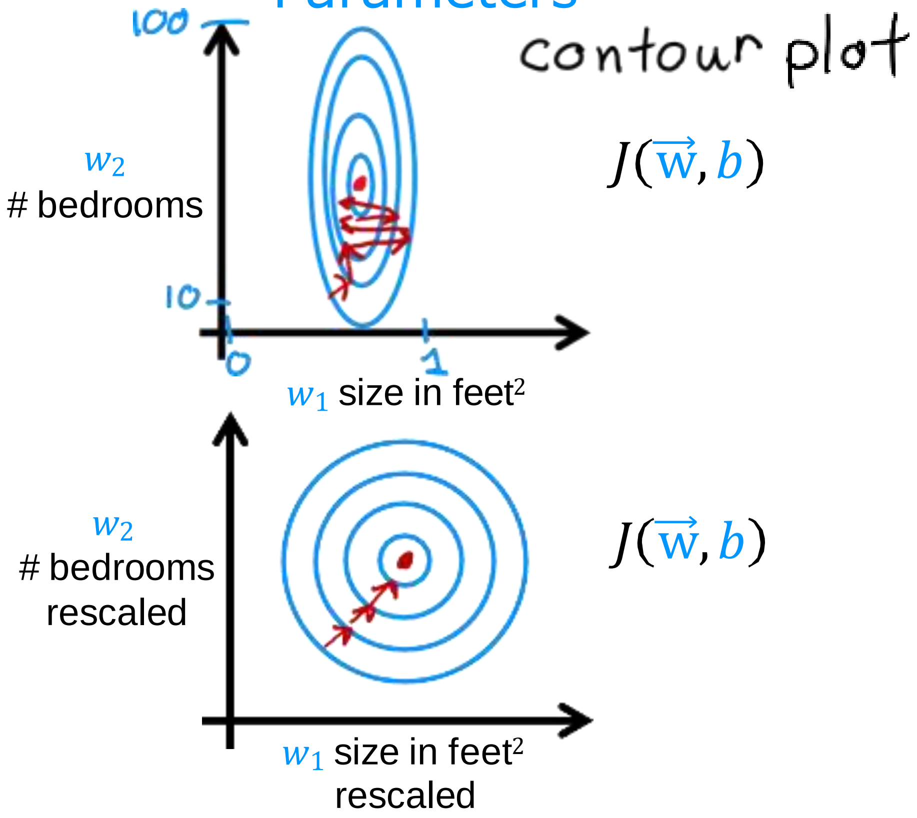


Feature size and gradient descent

Features



Parameters



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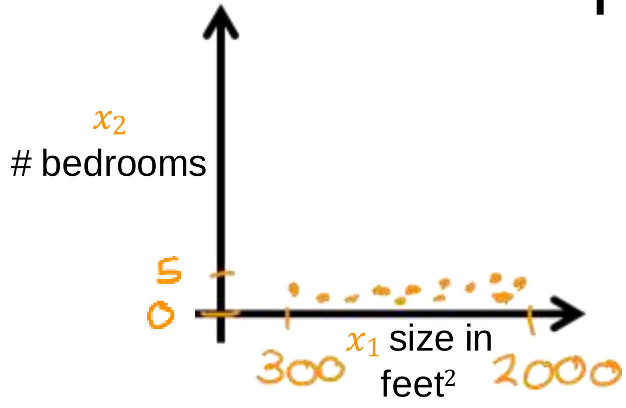
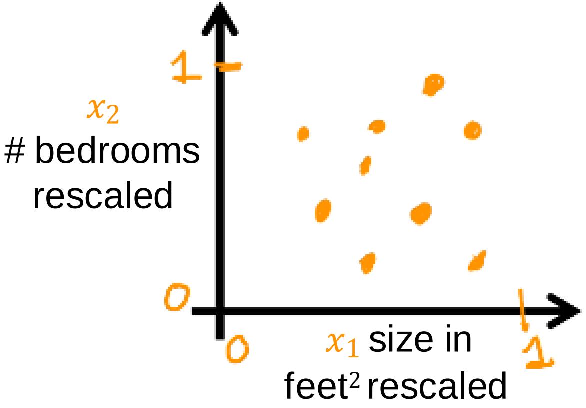
Feature Scaling Part 2

Feature scaling



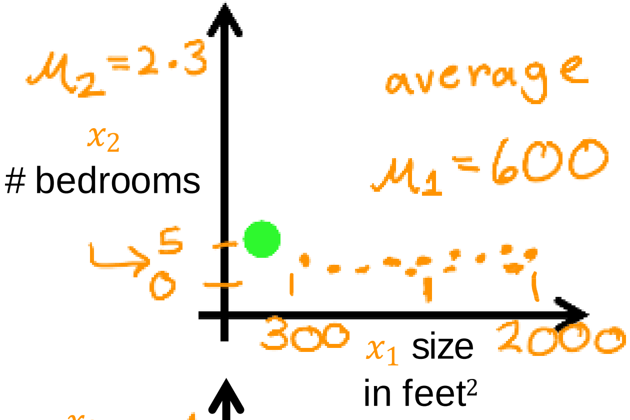


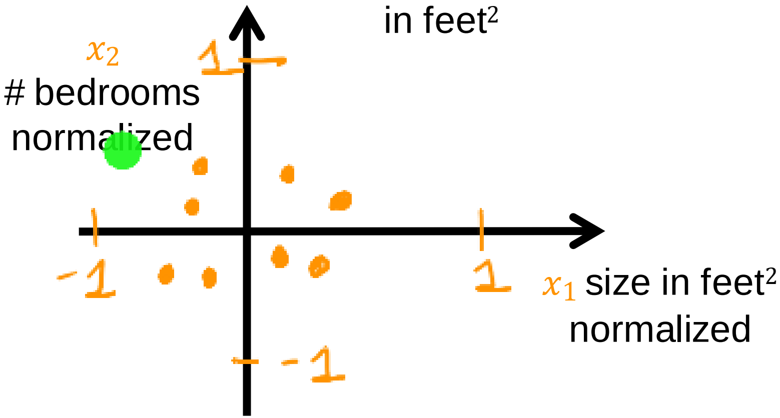
$$300 \le x_1 \le 2000 \qquad 0 \le x_2 \le 5$$

$$x_{1,scaled} = \frac{x_1}{\underset{max}{2000}} \qquad x_{2,scaled} = \frac{x_2}{\underset{max}{5}}$$

$$0.15 \le x_{1,scaled} \le 1 \qquad 0 \le x_{2,scaled} \le 1$$

Mean normalization





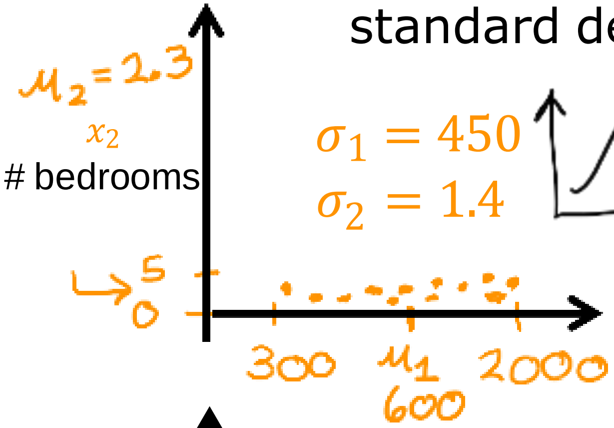
$$300 \le x_1 \le 2000 \qquad 0 \le x_2 \le 5$$

$$x_1 = \frac{x_1 - \mu_1}{\underbrace{2000-300}_{\text{max-min}}} \qquad x_2 = \frac{x_2 - \mu_2}{\underbrace{5-0}_{\text{max-min}}}$$

$$-0.18 \le x_1 \le 0.82 \qquad -0.46 \le x_2 \le 0.54$$

Z-score normalization

standard deviation σ

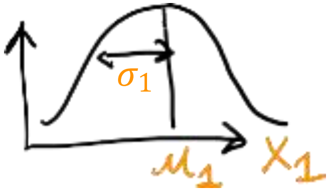


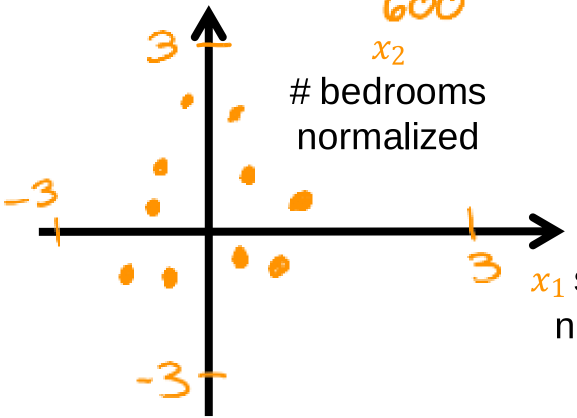
x_2
\# bedrooms

x_1 size in feet^2

$\sigma_1 = 450$

$\sigma_2 = 1.4$





x_2
\# bedrooms normalized

x_1 size in feet^2 normalized

$300 \leq x_1 \leq 2000$ $\quad$ $0 \leq x_2 \leq 5$

$$x_1 = \frac{x_1 - \mu_1}{\sigma_1} \qquad x_2 = \frac{x_2 - \mu_2}{\sigma_2}$$

$-0.67 \leq x_1 \leq 3.1$ $\quad$ $-1.6 \leq x_2 \leq 1.9$

Feature scaling

aim for about $-1 \leq x_j \leq 1$ for each feature x_j

$-3 \leq x_j \leq 3$

$-0.3 \leq x_j \leq 0.3$

acceptable ranges

$0 \leq x_1 \leq 3$ okay, no rescaling

$-2 \leq x_2 \leq 0.5$ okay, no rescaling

$-100 \leq x_3 \leq 100$ too large → rescale

$-0.001 \leq x_4 \leq 0.001$ too small → rescale

$98.6 \leq x_5 \leq 105$ too large → rescale

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Checking Gradient Descent for Convergence

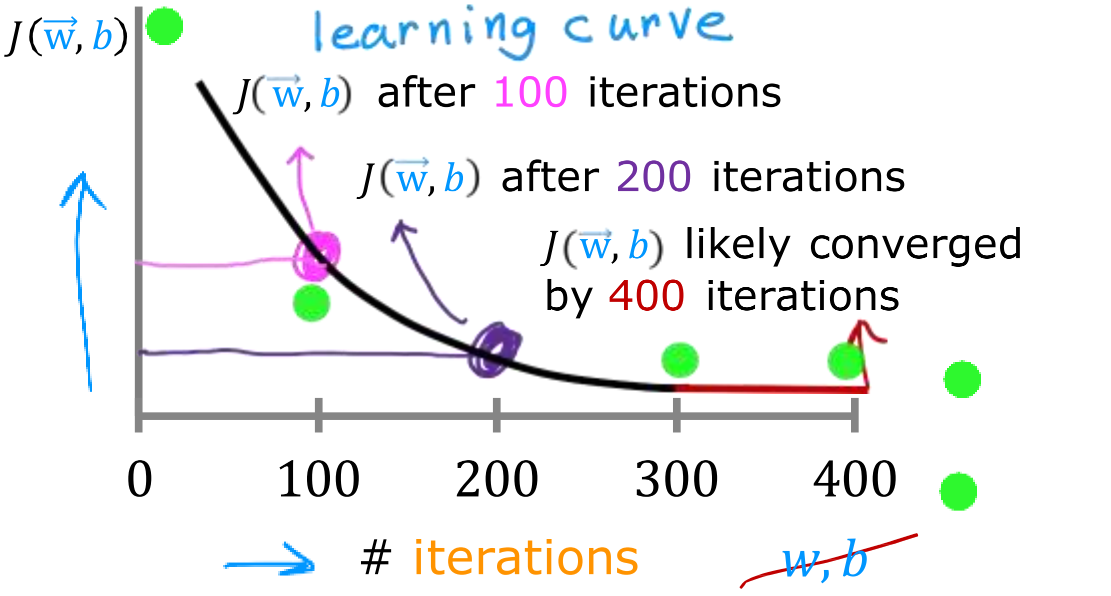
Gradient descent

$$\begin{cases} w_j = w_j - \alpha \dfrac{\partial}{\partial w_j} J(\vec{\mathrm{w}}, b) \\ b = b - \alpha \dfrac{\partial}{\partial b} J(\vec{\mathrm{w}}, b) \end{cases}$$

Make sure gradient descent is working correctly

objective: $\min_{\vec{w},b} J(\vec{w}, b)$

$J(\vec{w}, b)$ should decrease after every iteration



iterations needed varies 30 1,000 100,000

Automatic convergence test

Let ε "epsilon" be 10^{-3}.
0.001

If $J(\vec{w}, b)$ decreases by $\leq \varepsilon$ in one iteration, declare convergence.

(found parameters $\vec{w}, b$ to get close to global minimum)

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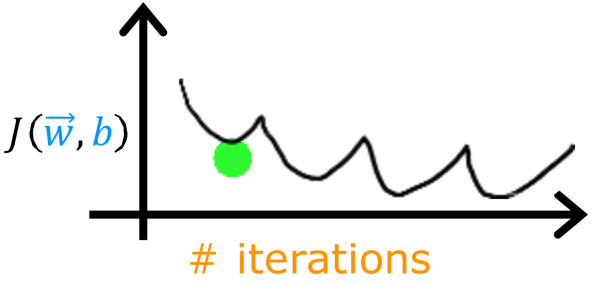
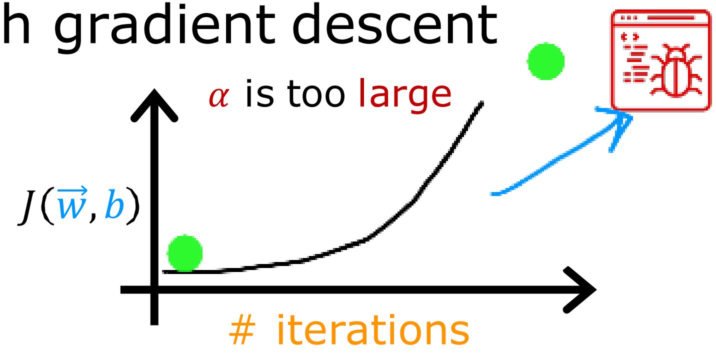
Choosing the Learning Rate

Identify problem with gradient descent





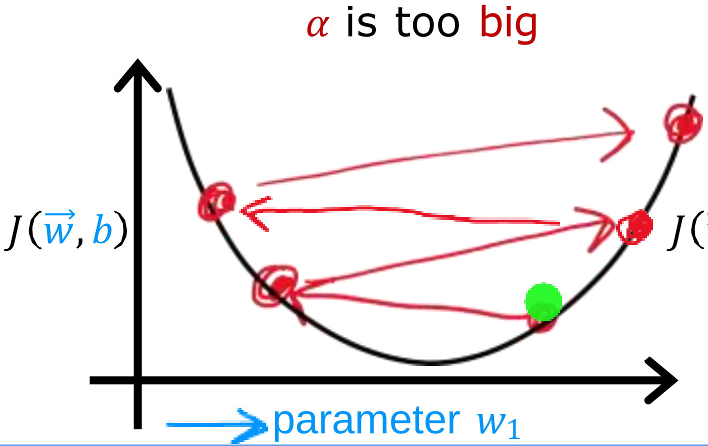
or learning rate is too large

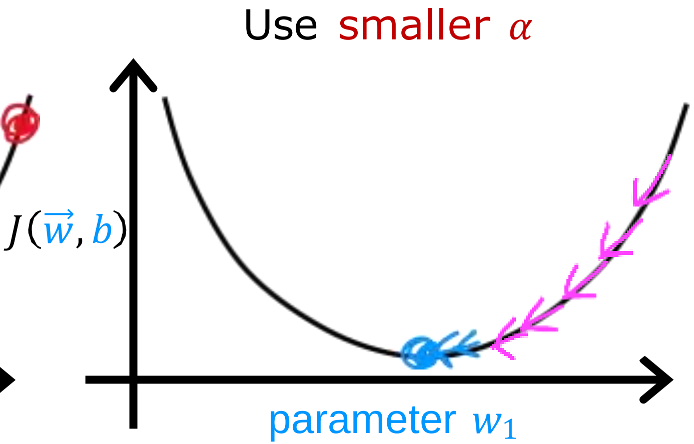
$w_1 = w_1 + \alpha d_1$

use a minus sign

$w_1 = w_1 - \alpha d_1$

Adjust learning rate





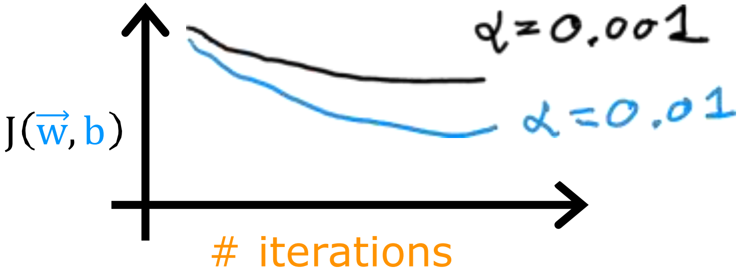
With a small enough α, $J(\vec{w}, b)$ should decrease on every iteration

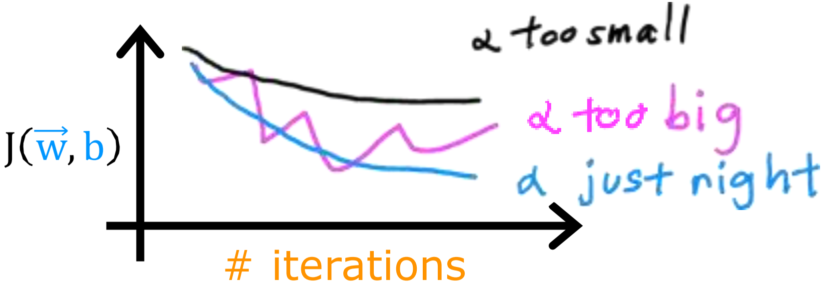
If α is too small, gradient descent takes a lot more iterations to converge

Values of α to try:

... 0.001 0.003 0.01 0.03 0.1 0.3 1 ...

3X ≈3X 3X ≈3X 3X ≈3X





Practical Tips for Linear Regression

Feature Engineering

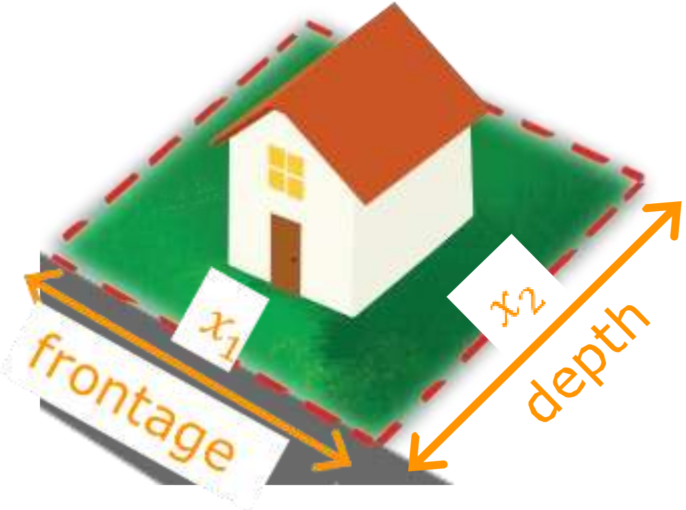
Feature engineering

$$f_{\vec{w},b}(\vec{x}) = w_1 \underbrace{x_1}_{\text{frontage}} + w_2 \underbrace{x_2}_{\text{depth}} + b$$

$$area = frontage \times depth$$

$$\underbrace{x_3 = x_1 x_2}_{\text{new feature}}$$

$$f_{\vec{w},b}(\vec{x}) = \underbrace{w_1}x_1 + \underbrace{w_2}x_2 + \underbrace{w_3}x_3 + b$$



Feature engineering:
Using intuition to design new features, by transforming or combining original features.

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Polynomial Regression

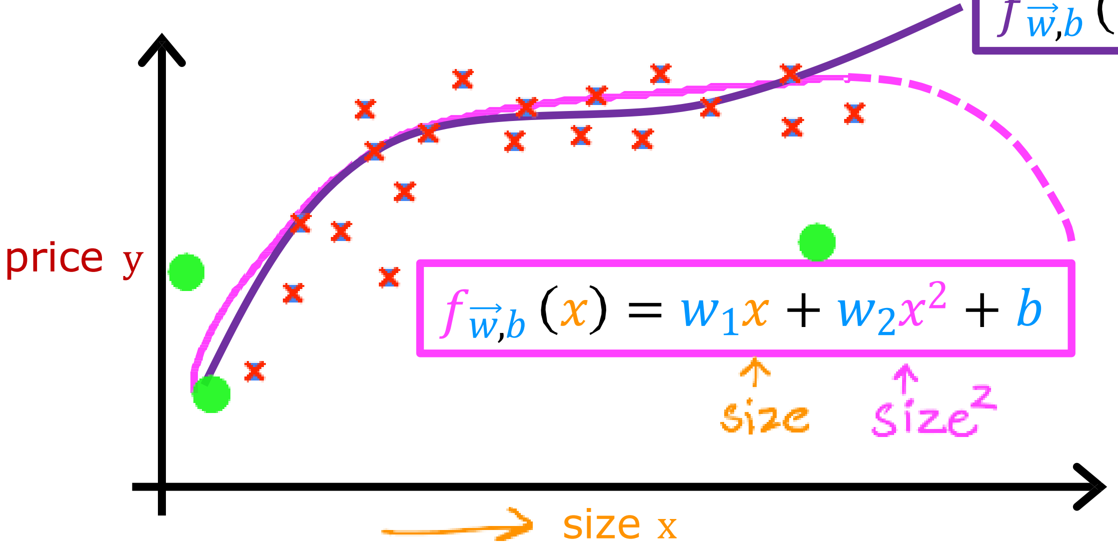
Polynomial regression

$$f_{\vec{w},b}(x) = w_1x + w_2x^2 + w_3x^3 + b$$

size $\quad$ size^2 $\quad$ size^3

$1-10^3$ $\quad$ $1-10^6$ $\quad$ $1-10^9$

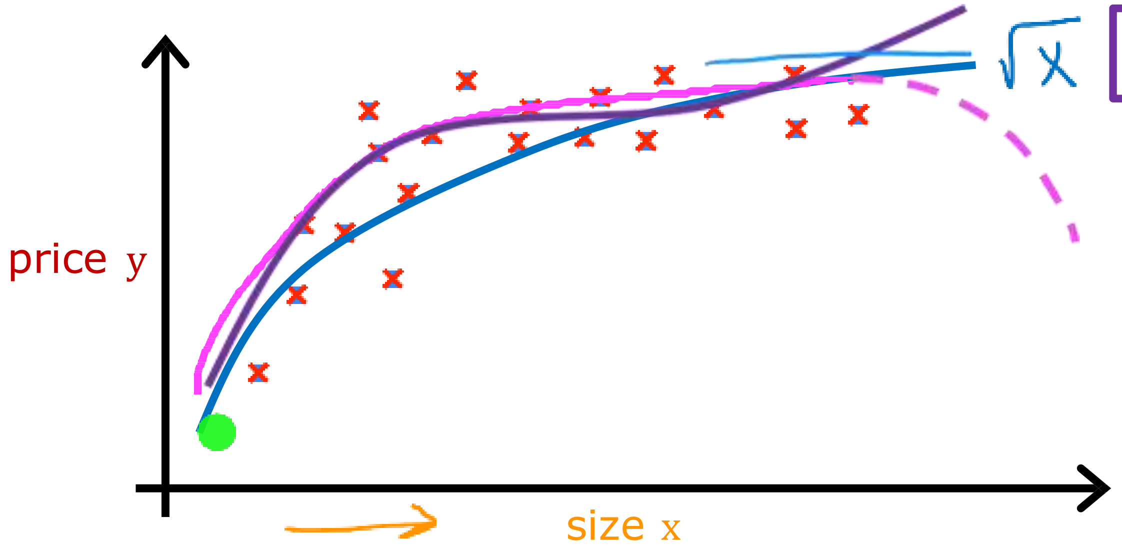
feature scaling



$$f_{\vec{w},b}(x) = w_1x + w_2x^2 + b$$

size $\quad$ size^2

Choice of features



$$f_{\vec{w},b}(x) = w_1 x + w_2\sqrt{x} + b$$

size $\sqrt{\text{size}}$

what features to use?
↳ course 2