

Eigenvalues and Eigenvectors of a 3 by 3 matrix

Just as 2 by 2 matrices can represent transformations of the plane, 3 by 3 matrices can represent transformations of 3D space. The picture is more complicated, but as in the 2 by 2 case, our best insights come from finding the matrix's **eigenvectors**: that is, those vectors whose direction the transformation leaves unchanged.

If non-zero \mathbf{e} is an eigenvector of the 3 by 3 matrix A , then

$$A \mathbf{e} = \lambda \mathbf{e}$$

for some scalar λ . This scalar is called an **eigenvalue** of A .

This may be rewritten

$$A \mathbf{e} = \lambda I \mathbf{e},$$

and in turn as

$$(A - \lambda I) \mathbf{e} = \mathbf{0}.$$

As in the 2 by 2 case, the matrix $A - \lambda I$ must be singular. Once again, then, we ask: which are the values of λ for which $A - \lambda I$ is singular? That is, the values that satisfy the **characteristic equation**

$$\det(A - \lambda I) = 0?$$

Consider the example

$$\begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}.$$

The characteristic equation is

$$\det \begin{pmatrix} -2 - \lambda & -4 & 2 \\ -2 & 1 - \lambda & 2 \\ 4 & 2 & 5 - \lambda \end{pmatrix} = 0.$$

Expanding the determinant,

$$(-2 - \lambda) [(1 - \lambda)(5 - \lambda) - 2 \times 2] + 4[(-2) \times (5 - \lambda) - 4 \times 2] + 2[(-2) \times 2 - 4(1 - \lambda)] = 0.$$

Expanding the brackets and simplifying:

$$-\lambda^3 + 4\lambda^2 + 27\lambda - 90 = 0,$$

or, equivalently

$$\lambda^3 - 4\lambda^2 - 27\lambda + 90 = 0.$$

By trial and error, we find that

$$3^3 - 4 \times 3^2 - 27 \times 3 + 90 = 0,$$

and it follows from the Factor Theorem that $(\lambda - 3)$ is a factor. Indeed,

$$\lambda^3 - 4\lambda^2 - 27\lambda + 90 = (\lambda - 3)(\lambda^2 - \lambda - 30)$$

and

$$(\lambda - 3)(\lambda^2 - \lambda - 30) = (\lambda - 3)(\lambda + 5)(\lambda - 6),$$

meaning that the eigenvalues are 3, -5 and 6.

We now go on to solve

$$\begin{pmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for each eigenvalue λ . Now, every such system will have infinitely many solutions, because if \mathbf{e} is an eigenvector, so is any multiple of \mathbf{e} . So our strategy will be to try to find the eigenvector with $X = 1$, and then if necessary scale up. (If there is no such eigenvector, we know that X must in fact be zero, and we instead look for the eigenvector with $Y = 1$, and so on.)

Eigenvector corresponding to eigenvalue 3

In the case $\lambda = 3$, we have

$$\begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$-5 - 4Y + 2Z = 0,$$

$$-2 - 2Y + 2Z = 0.$$

Subtracting the first from the second:

$$3 + 2Y = 0,$$

and thus $Y = -\frac{3}{2}$.

Substituting back into the second equation,

$$-2 + 3 + 2Z = 0,$$

giving $Z = -\frac{1}{2}$.

Checking in the third equation,

$$4 - 3 - 1 = 0,$$

which works. This gives us the eigenvector

$$\left(1, -\frac{3}{2}, -\frac{1}{2}\right).$$

For convenience, we can scale up by a factor of 2, to get

$$(2, -3, -1).$$

Eigenvector corresponding to eigenvalue -5

In the case $\lambda = -5$, we have

$$\begin{pmatrix} 3 & -4 & 2 \\ -2 & 6 & 2 \\ 4 & 2 & 10 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$3 - 4Y + 2Z = 0,$$

$$-2 + 6Y + 2Z = 0.$$

Subtracting the first from the second:

$$-5 + 10Y = 0,$$

and thus $Y = \frac{1}{2}$.

Substituting back into the second equation,

$$-2 + 3 + 2Z = 0,$$

giving $Z = -\frac{1}{2}$.

Checking in the third equation,

$$4 + 1 - 5 = 0,$$

which works. This gives us the eigenvector

$$\left(1, -\frac{1}{2}, \frac{1}{2}\right).$$

Once again, we can scale up by a factor of 2, to get

$$(2, -1, 1).$$

Eigenvector corresponding to eigenvalue 6

In the case $\lambda = 6$, we have

$$\begin{pmatrix} -8 & -4 & 2 \\ -2 & -5 & 2 \\ 4 & 2 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $X = 1$ gives, as our first two equations,

$$-8 - 4Y + 2Z = 0,$$

$$-2 - 5Y + 2Z = 0.$$

Subtracting the first from the second:

$$6 - Y = 0,$$

and thus $Y = 6$.

Substituting back into the second equation,

$$-2 - 30 + 2Z = 0,$$

giving $Z = 16$.

Checking in the third equation,

$$4 + 12 - 16 = 0,$$

which works. This gives us the eigenvector

$$(1, 6, 16).$$

General considerations

In general, the eigenvalues of a real 3 by 3 matrix can be

- (i) three distinct real numbers, as here;
- (ii) three real numbers with repetitions;
- (iii) one real number and two conjugate non-real numbers.

The geometric interpretation of the transformation depends on which of the above is true: the first will involve stretches in the three eigenvector directions, the third will involve a rotation and a stretch along its axis and the second will usually involve one of several types of 3D shear. We leave these details for the moment, however.

Another important use of eigenvalues and eigenvectors is *diagonalisation*, and it is to this that we now turn.