Stochastic Process: Problem Set

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1 Markov Process

1. ('14) Suppose there are 6 dies and they are all *unmarked* initially. In each turn, we toss all *unmarked* dies and *mark* those whose number appears most often (if there are more than one number with the most frequent appearences, we pick one at random). For example, suppose the numbers in the first turn is 2,4,5,6,4,3, we *mark* two dies with number 4 and toss the remaining four dies in the second turn. Denote by X_n the number of *unmarked* dies at the beginning of the n-th turn, prove that $\{X_n\}$ forms a Markov chain and derive the corresponding transition matrix. Please state that whether this Markov chain is reducible, periodic, recurrent or not.

2. ('07, '14) An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella, if one is there. Otherwise, she gets wet. Assume that independent of the past, it rains on each trip with probability p. To formulate a Markov chain, let X_n be the number of umbrellas at her current location.

   (a) Find the transition probability for this Markov chain.

   (b) Calculate the limiting fraction of time she gets wet.

3. ('10) Let (V, E) be a simple undirected graph (no self cycles or multiple edges) with $|V| < \infty$ and $|E| = m$. Assume that this graph is connected and consider the following random walk: when the individual is at vertex u at time n, she will move to one of all $d(u)$ adjacent vertices with equal probability at time $n + 1$.

   (a) Derive the stationary distribution of her location.

   (b) Let $T_{u\to v}$ be the time it takes her to first arrive at vertex v when starting at vertex u. Show that $\mathbb{E}T_{u\to u} = 2m/d(u)$.

   (c) We call an edge (u, v) a *bridge* if the graph becomes *unconnected* without this edge. Prove that, if (u, v) is a bridge, we have $\mathbb{E}T_{u\to v} + \mathbb{E}T_{v\to u} = 2m$.

2 Poisson Process

1. ('07, '14) Consider a Poisson process $N(t)$ with rate λ, and define $T_n = \min\{t : N(t) = n\}$ as the n-th hitting time. Calculate $\mathbb{E}[T_{N(t)+1} - T_{N(t)}]$ and its limit as $t \to \infty$.

2. ('10) Consider two independent Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. What is the probability that the two-dimensional process $(N_1(t), N_2(t))$ ever visits the point (i, j)?

3. ('08, '10) Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that two tellers have exponential service times with rates λ and μ.

   (a) What is the expected total amount of time for Carol to complete her businesses?

   (b) What is the expected total time until the last of the three customers leaves?

   (c) What is the probability that Carol is the last one to leave?

4. Starting at some fixed time, which we will call $t = 0$ for convenience, satellites are launched at times of a Poisson process with rate λ. After an independent amount of time having distribution function F and mean μ, the satellite stops working. Let $X(t)$ be the number of working satellites at time t.

   (a) Find the distribution of $X(t)$.

   (b) Let $t \to \infty$, show that the limiting distribution of $X(t)$ is $\mathsf{Poisson}(\lambda\mu)$.

3 $\mathcal{L}_2$ Process

1. ('10, '14) Let $X(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$ with $a, b \sim \mathcal{N}(0,1)$. Furthermore, a, b are independent.

   (a) Find the mean and the autocorrelation function of $X(t)$. Is $\{X(t)\}$ weak sense stationary? Is $\{X(t)\}$ strict sense stationary?

   (b) Rewrite $X(t)$ as $X(t) = \rho\cos(\omega_0 t + \varphi), \rho \geq 0, \varphi \in [0, 2\pi)$. Find the joint distribution of (ρ, φ). Is ρ and φ statistically independent?

   (c) Find the power spectrum density of $X(t)$ and $\frac{dX(t)}{dt}$.

2. ('08) Let $\theta \sim \mathcal{U}(0, 2\pi)$. Compute the following limit in terms of the *quadratic mean*:

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \cos(2\pi t + \theta)dt$$

3. ('07) Consider the stochastic process $Y(t) = X_{N(t)}$, where $N(t)$ is a Poisson process with rate λ, $\{X_n\}_{n=1}^{\infty}$ are i.i.d. random variables with mean μ and variance σ^2. Derive the power spectrum density $S_Y(\omega)$ of $Y(t)$.

4. Consider a deterministic bandlimited signal $x(t)$ with $X(\omega) = 0$ for $||\omega| - \omega_c| > W$. Denote the Hilbert transform of $x(t)$ by $\hat{x}(t)$, and $T_s = \pi/W$. Show that

$$x(t) = \sum_{n=-\infty}^{\infty} \left[x(nT_s)\cos\left(\omega_c(t - nT_s)\right) - \hat{x}(nT_s)\sin\left(\omega_c(t - nT_s)\right)\right] \cdot \mathsf{sinc}\left(\frac{\pi}{T_s}(t - nT_s)\right)$$

   where $\mathsf{sinc}(x) \triangleq \sin(x)/x$.

4 Gaussian Process

1. Let (X_1, X_2, X_3) be random vector with

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma^2 \end{pmatrix}\right).$$

   Find $\mathbb{E}[X_1X_2X_3]$ and $\mathbb{E}[X_1^2X_2^2X_3^2]$.

2. ('14) Let (X, Y) be independent random variables with $X, Y \sim \mathcal{N}(0,1)$.

   (a) Find $\mathbb{E}[(X - 3Y)^2|X + 2Y = 3]$.

   (b) Find $\mathbb{E}[(X - 3Y)^3|X + 2Y = 3]$.

3. ('09) Suppose that $X(t)$ is zero-mean Gaussian white noise with power spectral density $N_0/2$. Please design a linear time-invariant filter whose output $Y(t)$ satisfies that

$$\mathbb{E}\left(Y(1)Y(3)|Y(2)\right) = CY^2(2)$$

   where C is a known deterministic constant.

4. ('14) Consider the Gaussian white noise $X(t)$ with $\mathbb{E}[X(t)] = 0$ and $\mathbb{E}[X(t)X(s)] = \sigma^2\delta(t-s)$, where $\delta(\cdot)$ is the Dirac function. Find the autocorrelation function of the following process

$$Y(t) = \sin^3\left(\int_0^t X(\tau)d\tau\right)$$