相关理论与二阶矩过程 (1)　　by waiter for W22, 23, 24.

1 (1) $E\{X(t)\} = E\{a\cos wt + b\sin wt\} = E\{a\}\cdot\cos wt + E\{b\}\cdot\sin wt = 0.$

$R_{XX}(t_1,t_2) = E\{X(t_1)\cdot X^*(t_2)\} = E\{(a\cos wt_1 + b\sin wt_1)(a\cos wt_2 + b\sin wt_2)\}$

$= E\{a^2\cdot\cos wt_1\cos wt_2 + ab(\cos wt_1\sin wt_2 + \sin wt_1\cos wt_2) + b^2\sin wt_1\sin wt_2\}$

$= E\{a^2\}\cdot\cos wt_1\cos wt_2 + E\{a\}\cdot E\{b\}(\cdots) + E\{b^2\}\cdot\sin wt_1\sin wt_2$

$Var\{X\} = E\{X^2\} - (E\{X\})^2$
$\Rightarrow E\{a^2\} = E\{b^2\} = 1 + 0^2 = 1$

$= \cos wt_1\cos wt_2 + \sin wt_1\sin wt_2 = \cos w(t_1 - t_2)$

∴是宽平稳过程.　　对于实Gauss过程 $X(t)$，$X(t)$ 宽平稳 ⇒ $X(t)$ 严平稳

∵ $X(t)$ 是Gauss过程，$X(t)$ 是宽平稳过程　∴ $X(t)$ 是严平稳过程.

$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}(a\cos wt + b\sin wt)\,dt = \lim_{T\to+\infty}\frac{1}{2T}\left(\frac{a}{w}\sin wt\Big|_{-T}^{T} - \frac{b}{w}\cos wt\Big|_{-T}^{T}\right)$

$= \lim_{T\to+\infty}\frac{a}{wT}\sin wT = 0 = m_X$　　∴是均值遍历.

Gauss分布: $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(2) $f_a(x) = f_b(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) \Rightarrow f_{a,b}(a,b) = \frac{1}{2\pi}\exp(-\frac{a^2+b^2}{2})$

Jacobi行列式 $\frac{D(a,b)}{D(\rho,\theta)}$

$X(t) = a\cos wt + b\sin wt$
$= \rho\cos(wt+\theta)$
$\Rightarrow \begin{cases} a = \rho\cos\theta \\ b = -\rho\sin\theta \end{cases} \Rightarrow \left|\frac{\partial(a,b)}{\partial(\rho,\theta)}\right| = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ -\sin\theta & -\rho\cos\theta \end{vmatrix} = -\rho$

$\therefore f_{\rho,\theta}(\rho,\theta) = f_{a,b}(a,b)\cdot\left|\frac{D(a,b)}{D(\rho,\theta)}\right| = \frac{\rho}{2\pi}\exp(-\frac{\rho^2}{2})$

$\therefore f_\rho(\rho) = \int_0^{2\pi} f_{\rho,\theta}(\rho,\theta)\,d\theta = \rho\exp(-\frac{\rho^2}{2})$　$(\rho \ge 0)$

$f_\theta(\theta) = \int_0^{+\infty} f_{\rho,\theta}(\rho,\theta)\,d\rho = \frac{1}{2\pi}$　$(0 \le \theta \le 2\pi)$

∵ $f_{\rho,\theta}(\rho,\theta) = f_\rho(\rho)\cdot f_\theta(\theta)$　∴ ρ,θ 统计独立.

2. $S(w) \xrightarrow{F^{-1}} R_X(\tau) = R_Y(\tau)$，$\rho(w) = S_{XY}(w) \xrightarrow{F^{-1}} R_{XY}(\tau)$.

$R_Z(t_1,t_2) = E\{Z(t_1)\cdot Z^*(t_2)\} = E\{[X(t_1)\cos(W_ct_1+\theta) + Y(t_1)\sin(W_ct_1+\theta)][X(t_2)\cos(W_ct_2+\theta) + Y(t_2)\sin(W_ct_2+\theta)]\}$.

$= E\{X(t_1)X(t_2)\cos(W_ct_1+\theta)\cos(W_ct_2+\theta) + X(t_1)Y(t_2)\cos(W_ct_1+\theta)\sin(W_ct_2+\theta)$
$+ Y(t_1)X(t_2)\sin(W_ct_1+\theta)\cos(W_ct_2+\theta) + Y(t_1)Y(t_2)\sin(W_ct_1+\theta)\sin(W_ct_2+\theta)\}$

$= E\{X(t_1)X(t_2)\}\cos(\cdots)\cdot\cos(\cdots) + E\{Y(t_1)Y(t_2)\}\sin(\cdots)\sin(\cdots) + E\{X(t_1)Y(t_2)\}\cos(\cdots)\sin(\cdots)$
$+ E\{X(t_2)Y(t_1)\}\sin(\cdots)\cos(\cdots)$

$= R_X(t_2-t_1)\cdot\cos(W_ct_2+\theta-W_ct_2-\theta) + R_{XY}(t_2-t_1)\cdot\sin(W_ct_2+\theta-W_ct_1-\theta)$

$= R_X(\tau)\cdot\cos W_c\tau + R_{XY}(\tau)\sin W_c\tau$.　　$R_{XY}(\tau) = R_{YX}(-\tau)$

$\downarrow F$

$\therefore S_Z(w) = S(w) * \pi[\delta(w-w_c) + \delta(w+w_c)] + S_{XY}(w) * \pi[\delta(w-w_c) - \delta(w+w_c)]$

$= \pi[S(w-w_c) + S(w+w_c)] + \pi[S_{XY}(w-w_c) - S_{XY}(w+w_c)]$　　即频谱搬移.

3. $X(t)$ 宽平稳 $\Rightarrow E\{X(t)\}=\mu_X,\ R_X(t_1,t_2)=R_X(\tau)$.

$E\{Y(t)\}=E\{\int_t^{t+\theta}X(s)ds\}=E_\theta\{E\{\int_t^{t+\theta}X(s)ds\,|\,\theta\}\}$　　条件期望, $E\{X\}=E\{E[X|Y]\}$.

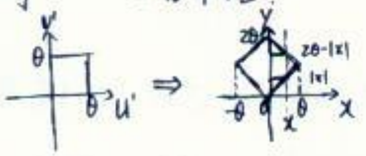
$=E_\theta\{\int_t^{t+\theta}E\{X(s)\}ds\,|\,\theta\}=E_\theta\{\theta\cdot\mu_X\}=1.5\mu_X$.

$E\{Y(t)Y(s)\}=E\{\int_t^{t+\theta}X(u)du\int_s^{s+\theta}X(v)dv\}\overset{同理}{=}E_\theta\{E\{\int_t^{t+\theta}X(u)du\int_s^{s+\theta}X(v)dv\,|\,\theta\}\}$

$=E_\theta\{\int_t^{t+\theta}\int_s^{s+\theta}E\{X(u)X(v)\}dudv\,|\,\theta\}=E_\theta\{\int_t^{t+\theta}\int_s^{s+\theta}R(u-v)dudv\,|\,\theta\}$

$\overset{u'=u-t,\ v'=v-s}{=}E\{\int_0^\theta\int_0^\theta R(u'-v'+\underbrace{t-s}_{\tau})du'dv'\}$　　$\therefore$ 宽平稳.

用积分换元来处理二重积分. 设 $x=u'-v'$, $y=u'+v'$



则有: $\int_0^\theta\int_0^\theta R(u'-v'+\tau)du'dv'=\int_{-\theta}^{\theta}\int_{|x|}^{2\theta-|x|}\frac{1}{2}R(x+\tau)dydx$

$=\int_{-\theta}^{\theta}(\theta-|x|)\cdot R(x+\tau)dx$　　$\therefore R_Y(\tau)=E\{\int_{-\theta}^{\theta}(\theta-|x|)R(x+\tau)dx\}$.

$\therefore S_Y(w)=\int_{-\infty}^{+\infty}R_Y(\tau)\cdot e^{-jw\tau}d\tau=\int_{-\infty}^{+\infty}E\{\int_{-\theta}^{+\theta}(\theta-|x|)R(x+\tau)dx\}\cdot e^{-jw\tau}d\tau$.

$=E\{\int_{-\theta}^{\theta}(\theta-|x|)dx\int_{-\infty}^{+\infty}R(x+\tau)\cdot e^{-jw(x+\tau)}\cdot e^{jwx}d\tau\}=E\{\int_{-\theta}^{\theta}(\theta-|x|)\cdot S_x(w)\cdot e^{jwx}dx\}$

$=S_x(w)\cdot E\{\int_{-\theta}^{\theta}(\theta-|x|)\cdot e^{jwx}dx\}=S_x(w)\cdot E\{\int_{-\theta}^{0}(\theta+x)e^{jwx}dx+\int_0^\theta(\theta-x)e^{jwx}dx\}$

$=S_x(w)\cdot E\{\frac{1-i\theta w-e^{-i\theta w}}{w^2}+\frac{1+i\theta w-e^{i\theta w}}{w^2}\}=S_x(w)\cdot E\{\frac{2}{w^2}-\frac{2\cos w\theta}{w^2}\}$

$=S_x(w)\cdot(\frac{2}{w^2}-\int_1^2\frac{2\cos w\theta}{w^2}d\theta)=S_x(w)\cdot(\frac{2}{w^2}-\frac{2}{w^3}(\sin 2w-\sin w))$

小结: 相关函数的计算 ⇒ 平稳性, 遍历性的判定.

① 普通随机过程 $X(t)$　例如第1,2题.

方法: 按定义计算即可, 可能会用到和差角公式.

参考例题: 《习题集》第6章1,2,3,4,5,8

教材例2.1,2.3,2.4、《习题集第6章19,22,

② 随机过程 $X(t)$ 的积分　例如第3题

方法: 需要使用积分换元来处理二重积分.

若积分限中含随机变量, 则还需要使用条件期望公式.

参考例题: 教材例2.5

③ 含possion过程的2阶随机过程.

④ 含Gauss过程的2阶随机过程　> 未完待续.