CLASSICAL RELATIONS AND FUZZY RELATIONS

RELATIONS

- Relations represent mappings between sets and connectives in logic.

- A classical binary relation represents the presence or absence of a connection or interaction or association between the elements of two sets.

- Fuzzy binary relations are a generalization of crisp binary relations, and they allow various degrees of relationship (association) between elements.

CRISP CARTESIAN PRODUCT

Lets consider properties of crisp relations first and then extend the mechanism to fuzzy sets.

Definition of (crisp) Product set: Let A and B be two non-empty sets, the product set or Cartesian product $A \times B$ is defined as follows,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

(a set of ordered pairs a,b)

CRISP RELATIONS

Cartesian product of *n* sets

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \| a_1 \in A_1, \ldots, a_n \in A_n\}$$

Definition of Binary Relation

If A and B are two sets and there is a specific property between elements x of A and y of B, this property can be described using the ordered pair (x, y). A set of such (x, y) pairs, $x \in A$ and $y \in B$, is called a relation R.

$$R = \{ (x,y) \mid x \in A, y \in B \}$$

R (relation over A and B) is a subset of the Cartesian product of A×B

Crisp relation

Definition of *n*-ary relation

For sets $A_1, A_2, A_3, \ldots, A_n$, the relation among elements $x_1 \in A_1, x_2 \in A_2, x_3 \in A_3, \ldots, x_n \in A_n$ can be described by n-tuple $(x_1, x_2, \ldots, x_n)$. A collection of such n-tuples $(x_1, x_2, x_3, \ldots, x_n)$ is a relation R among $A_1, A_2, A_3, \ldots, A_n$.

$$(x_1, x_2, x_3, \ldots, x_n) \in R,$$
$$R \subseteq A_1 \times A_2 \times A_3 \times \ldots \times A_n$$

- Using the characteristic function defines the crisp relation R :

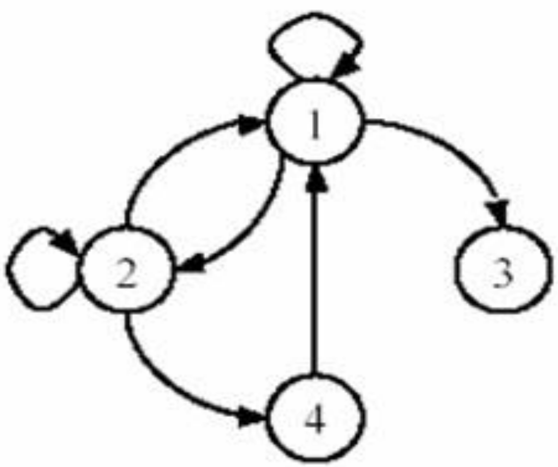
$$\mu_R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{iff } (x_1, x_2, \ldots, x_n) \in R, \\ 0 & \text{otherwise} \end{cases}$$

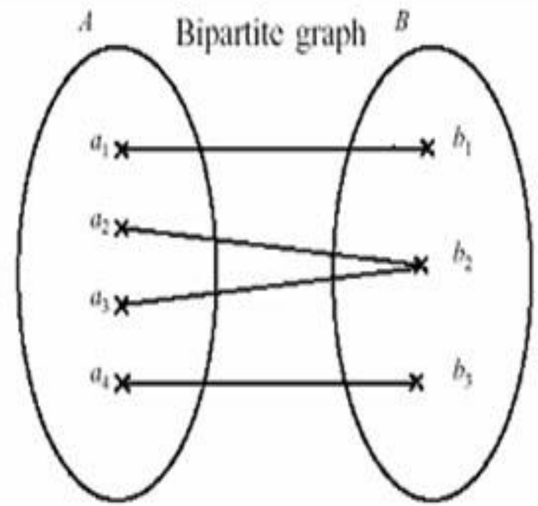
where $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$

CRISP BINARY RELATIONS

Examples of binary relations

Binary relation from A to B





Relation matrix

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	0	1	0
a_4	0	0	1

OPERATIONS ON CRISP RELATIONS

Function-theoretic operations for the two crisp relations (R, S) are defined as follows:

1. Union

$$R \cup S \rightarrow \chi_{R\cup S}(x, y) : \chi_{R\cup S}(x, y) = \max[\chi_R(x, y), \chi_S(x, y)]$$

2. Intersection

$$R \cap S \rightarrow \chi_{R\cap S}(x, y) : \chi_{R\cap S}(x, y) = \min[\chi_R(x, y), \chi_S(x, y)]$$

3. Complement

$$\overline{R} \rightarrow \chi_{\overline{R}}(x, y) : \chi_{\overline{R}}(x, y) = 1 - \chi_{\overline{R}}(x, y)$$

4. Containment

$$R \subset S \rightarrow \chi_R(x, y) : \chi_R(x, y) \leq \chi_S(x, y)$$

5. Identity

$$\phi \rightarrow \phi_R \text{ and } X \rightarrow E_R$$

ϕ_R Null relation , all matrix elements value is 0 , analogues to empty set

E_R Complete relation, all matrix elements value is 1 , analogues to universe set

PROPERTIES OF CRISP RELATIONS

The properties of crisp sets (given below) hold good for crisp relations as well.

- Commutativity,
- Associativity,
- Distributivity,
- Involution,
- Idempotency,
- DeMorgan’s Law,
- Excluded Middle Laws.

Fuzzy relation

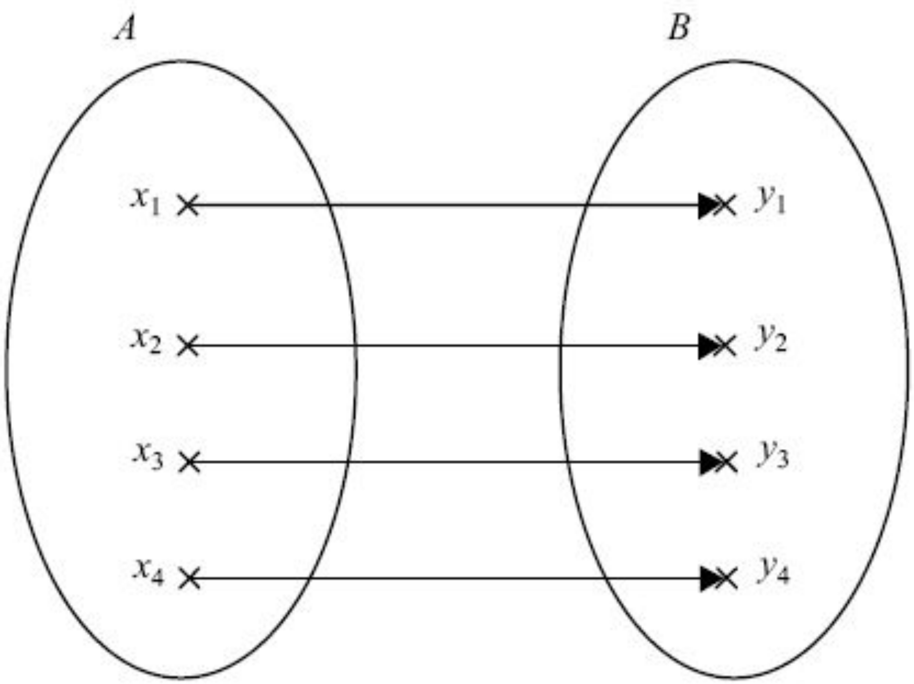
- A *fuzzy relation* is a fuzzy set defined on the Cartesian product of crisp sets A_1, A_2, ..., A_n where tuples (x_1, x_2, ..., x_n) may have varying degrees of membership within the relation.
- The membership grade indicates the strength of the relation present between the elements of the tuple.

$$\mu_R : A_1 \times A_2 \times \ldots \times A_n \rightarrow [0,1]$$

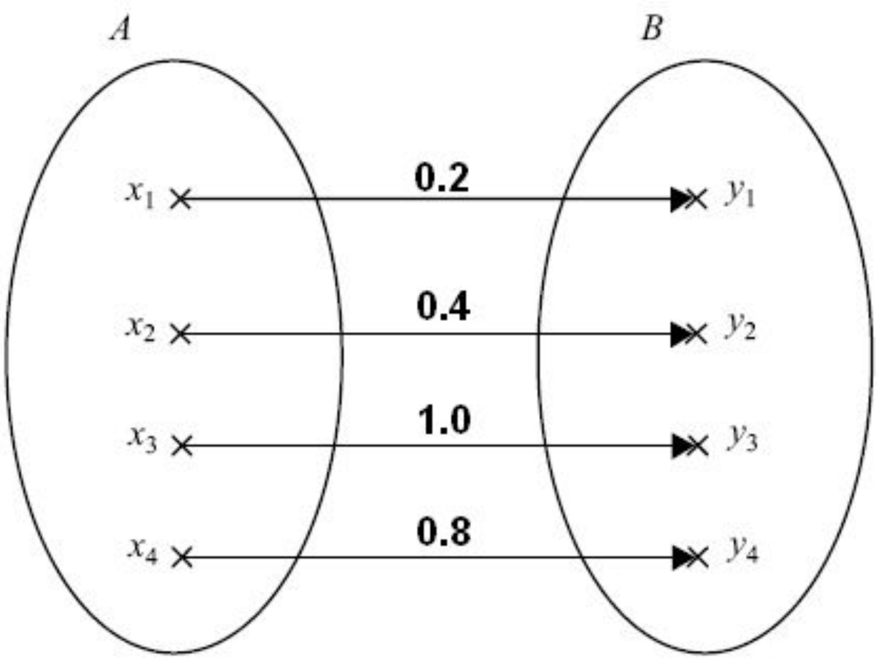
$$R = \left\{((x_1, x_2, \ldots, x_n), \mu_R) \mid \mu_R(x_1, x_2, \ldots, x_n) \geq 0, x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\right\}$$

Representation methods

- Bipartigraph : undirected binary graph, A ≠ B



(Crisp)



(Fuzzy)

Representation methods

- Matrix

A \ B	y_1	y_2	y_3	y_4
x_1	0	0	1	0
x_2	1	0	0	0
x_3	0	1	0	1
x_4	0	0	0	0

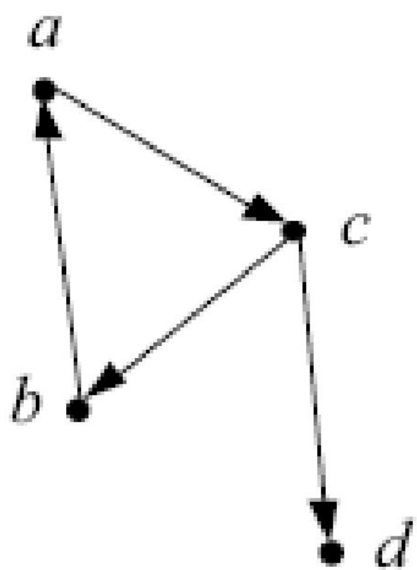
(Crisp)

A \ B	y_1	y_2	y_3	y_4
x_1	0.0	0.0	0.8	0.0
x_2	1.0	0.0	0.0	0.0
x_3	0.0	0.9	0.0	1.0
x_4	0.0	0.0	0.0	0.0

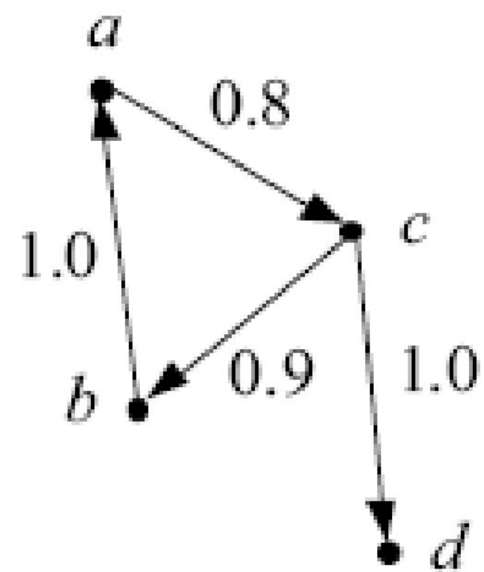
(Fuzzy)

Representation methods

- Digraph : directed binary graph, A = B



(Crisp)

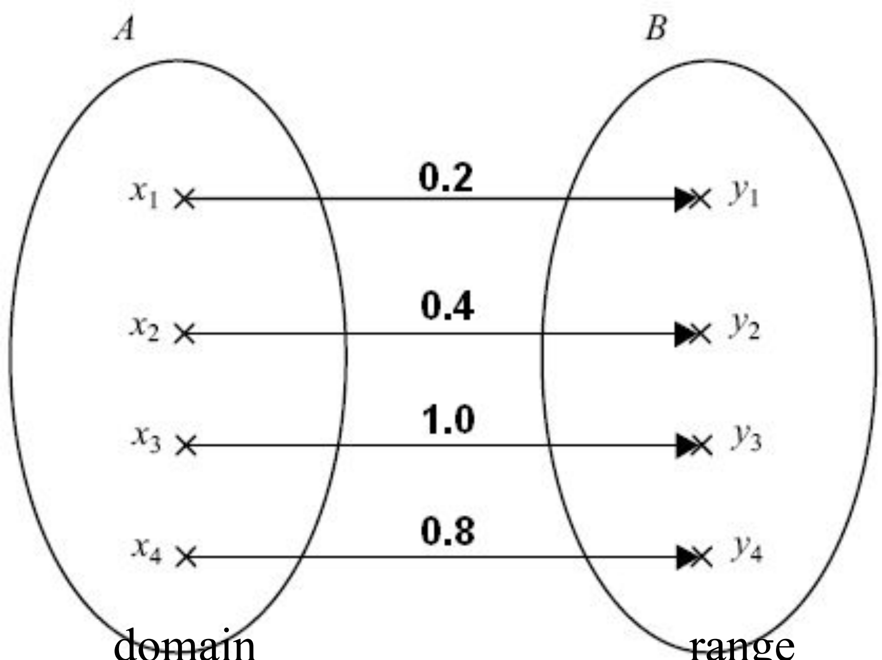


(Fuzzy)

Domain and range of fuzzy relation

- Domain: fuzzy set having the membership fn as $\mu_{dom(R)}(x) = \max_{y \in B} \mu_R(x,\ y)$
- Range :

$$\mu_{ran(R)}(y) = \max_{x \in A} \mu_R(x,\ y)$$



Domain and range of fuzzy relation

- Fuzzy matrix

$$
\mathbf{R} = \begin{array}{c|ccccc}
 & y_1 & y_2 & y_3 & y_4 & y_5 \\
\hline
x_1 & .9 & 1 & 0 & 0 & 0 \\
x_2 & 0 & .4 & 0 & 0 & 0 \\
x_3 & 0 & .5 & 1 & .2 & 0 \\
x_4 & 0 & 0 & 0 & 1 & .4 \\
x_5 & 0 & 0 & 0 & 0 & .5 \\
x_6 & 0 & 0 & 0 & 0 & .2
\end{array}
$$

$$\mu_{dom(R)}(x_1) = 1.0$$

$$\mu_{dom(R)}(x_2) = 0.4$$

$$\mu_{dom(R)}(x_3) = 1.0$$

$$\mu_{dom(R)}(x_4) = 1.0$$

$$\mu_{dom(R)}(x_5) = 0.5$$

$$\mu_{dom(R)}(x_6) = 0.2$$

Composition of classical Relations

Let R be a relation from a set A to a set B and S a relation from B to a set C.
The composite of R and S is the relation consisting of ordered pairs (a, c), where a $\in$ A, c $\in$ C, and for which there exists an element b $\in$ B such that (a, b) $\in$ R and (b, c) $\in$ S. We denote the composite of R and S by R ° S.

Composition of classical Relations

- Example:
- Let R is the relation from {1,2,3} to {1,2,3,4}
- with R = {(1,1), (1,4), (2,3), (3,1), (3,4)}
- S is the relation from {1,2,3,4} to {0,1,2}
- with S = {(1,0), (2,0), (3,1), (3,2), (4,1)}
- Find R ° S

For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in R ° S. Computing all the ordered pairs in the composite, we find

R ° S = ((1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}

COMPOSITION ON CRISP RELATIONS

The composition operations are of two types:

1. Max-min composition
2. Max-product composition.

The max-min composition is defined by the function theoretic expression as

$$T = R \circ S$$

$$\chi_T(x, z) = \bigvee_{y \in Y} [\chi_R(x, y) \wedge \chi_S(y, z)]$$

The max-product composition is defined by the function theoretic expression as

$$T = R \circ S$$

$$\chi_T(x, z) = \bigvee_{y \in Y} [\chi_R(x, y) \cdot \chi_S(y, z)]$$

FUZZY RELATION

A fuzzy relation R is a mapping from the Cartesian space X x Y to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation $\mu_R(x,y)$

$$\mu_R : A \times B \rightarrow [0, 1]$$

$$R = \{((x, y), \mu_R(x, y)) |\ \mu_R(x, y) \geq 0\ ,\ \ x \in A,\ \ y \in B\}$$

Example of Fuzzy Relations

Any fuzzy set R on U= U1× U2 × … × Un is called fuzzy relation on U

Example: Fuzzy Relation R [LESS_THAN] on U1 × U2,
where U1=U2={0,10,20,…}

	0	10	20	30	40	50	60	70	80
0	0	0.1	0.2	0.3	0.4	0.5	0.7	0.9	1
10	0	0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
20	0	0	0	0.1	0.2	0.3	0.4	0.5	0.7
30	0	0	0	0	0.1	0.2	0.3	0.4	0.5
40	0	0	0	0	0	0.1	0.2	0.3	0.4
50	0	0	0	0	0	0	0.1	0.2	0.3
60	0	0	0	0	0	0	0	0.1	0.2
70	0	0	0	0	0	0	0	0	0.1
80	0	0	0	0	0	0	0	0	0

FUZZY CARTESIAN PRODUCT

Let R be a fuzzy subset of M and S be a fuzzy subset of N. Then the Cartesian product R × S is a fuzzy subset of N × M such that

$$\forall \vec{x} \in M, \vec{y} \in N \ \mu_{R\times S}(\vec{x}, \vec{y}) = \min\left(\mu_R(\vec{x}), \mu_S(\vec{y})\right)$$

Example:

Let R be a fuzzy subset of {a, b, c} such that R = a/1 + b/0.8 + c/0.2 and S be a fuzzy subset of {1, 2, 3} such that S = 1/1 + 3/0.9 + 2/0.5. Then R x S is given by

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0.5 & 0.9 \\ 0.8 & 0.5 & 0.8 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

OPERATIONS ON FUZZY RELATION

The basic operation on fuzzy sets also apply on fuzzy relations.

1. Union:

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$$

2. Intersection:

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min\left[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)\right]$$

3. Complement:

$$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

4. Containment:

$$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$$

5. Inverse:

The inverse of a fuzzy relation R on $X \times Y$ is denoted by R^{-1}. It is a relation on $Y \times X$ defined by $R^{-1}(y, x) = R(x, y)$ for all pairs $(y, x) \in Y \times X$.

6. Projection:

For a fuzzy relation $R(X, Y)$, let $[R \downarrow Y]$ denote the projection of R onto Y. Then $[R \downarrow Y]$ is a fuzzy relation in Y whose membership function is defined by

$$\mu_{[R\downarrow Y]}(x, y) = \max_{x} \mu_{\underset{\sim}{R}}(x, y)$$

The projection concept can be extended to an n-ary relation $R(x_1, x_2, \ldots, x_n)$.

Example of Fuzzy Union relation

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

Example of Fuzzy Intersection relation

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

Example of Fuzzy complement relations

- Complement relation:

$$\forall (x, y) \in A \times B$$
$$\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$$

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$M_{\overline{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Projection

Example 3.9 There is a relation $R \subseteq A \times B$. The projection with respect to A or B shall be,

$M_R =$

$A \backslash B$	b_1	b_2	b_3
a_1	0.1	0.2	1.0
a_2	0.6	0.8	0.0
a_3	0.0	1.0	0.3

$M_{R_A} =$

a_1	1.0
a_2	0.8
a_3	1.0

$M_{R_B} =$

b_1	b_2	b_3
0.6	1.0	1.0

PROPERTIES OF FUZZY RELATIONS

The properties of fuzzy sets (given below) hold good for fuzzy relations as well.

- Commutativity,
- Associativity,
- Distributivity,
- Involution,
- Idempotency,
- DeMorgan’s Law,
- Excluded Middle Laws.

COMPOSITION OF FUZZY RELATIONS

Two fuzzy relations R and S are defined on sets A, B and C. That is, $R \subseteq A \times B$, $S \subseteq B \times C$. The composition $S \bullet R = SR$ of two relations R and S is expressed by the relation from A to C:

For $(x, y) \in A \times B$, $(y, z) \in B \times C$,

$$\mu_{S \bullet R}(x, z) = \max_{y} [\min (\mu_R(x, y), \mu_S(y, z))]$$

$$= \vee_{y} [\mu_R(x, y) \wedge \mu_S(y, z)]$$

$M_{S \bullet R} = M_R \bullet M_S$ (matrix notation)

(max-min composition)

Example:

$$X = \{x_1, x_2\}, Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-min composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_1, z_1) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_1, y) \wedge \mu_{\tilde{S}}(y, z_1)) \\ &= \max[\min(0.7, 0.9), \min(0.5, 0.1)] \end{aligned}\right\} \tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

$$= 0.7$$

Two fuzzy relations R and S are defined on sets A, B and C. That is, $R \subseteq A \times B$, $S \subseteq B \times C$. The composition $S \bullet R = SR$ of two relations R and S is expressed by the relation from A to C:

For $(x, y) \in A \times B$, $(y, z) \in B \times C$,

$$\mu_{S \bullet R}(x, z) = \max_y [\mu_R(x, y) \bullet \mu_S(y, z)]$$

$$= \vee_y [\mu_R(x, y) \bullet \mu_S(y, z)]$$

$M_{S \bullet R} = M_R \bullet M_S$ (matrix notation)

(max-product composition)

Max-product example:

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \text{ and } \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-product composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_2, z_2) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_2, y) \bullet \mu_{\tilde{S}}(y, z_2)) \\ &= \max[(0.8, 0.6), (0.4, 0.7)] \\ &= 0.48 \end{aligned}\right\} \quad \tilde{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} .63 & .42 & .35 \\ .72 & .48 & .30 \end{bmatrix} \end{array}$$

Max Min Composition Example

- Max-min composition

$$\forall (x, y) \in A \times B,\ \forall (y, z) \in B \times C$$

$$\mu_{S \circ R}(x,\ z) = \max_{y}[\min(\mu_R(x,\ y),\ \mu_S(y,\ z))]$$

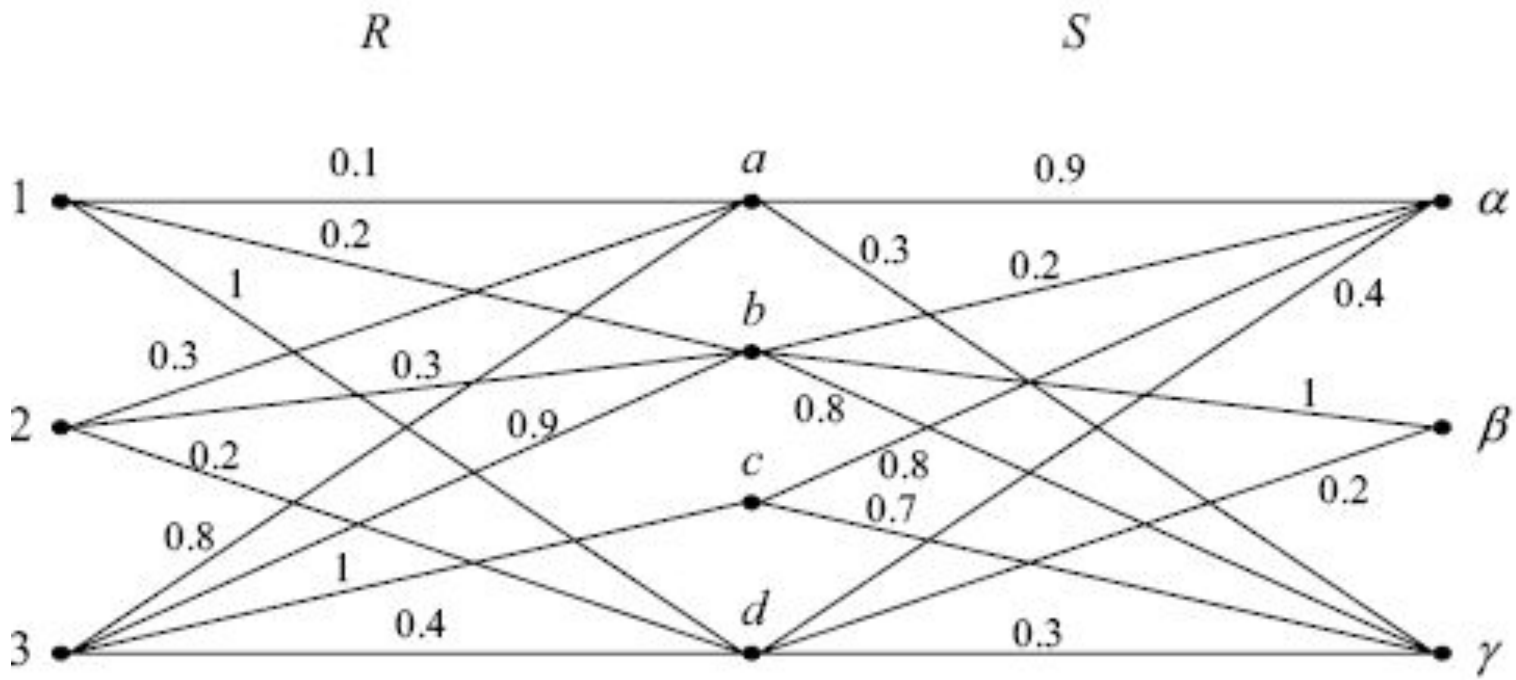
$$= \vee_{y}[\mu_R(x,\ y) \wedge \mu_S(y,\ z)]$$

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

Max Min Composition of fuzzy relations



Max Min Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \bullet R}(1,\ \alpha) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.0, 0.8), \min(1.0, 0.4)]$$
$$= \max[0.1, 0.2, 0.0, 0.4] = 0.4$$

Max Min Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

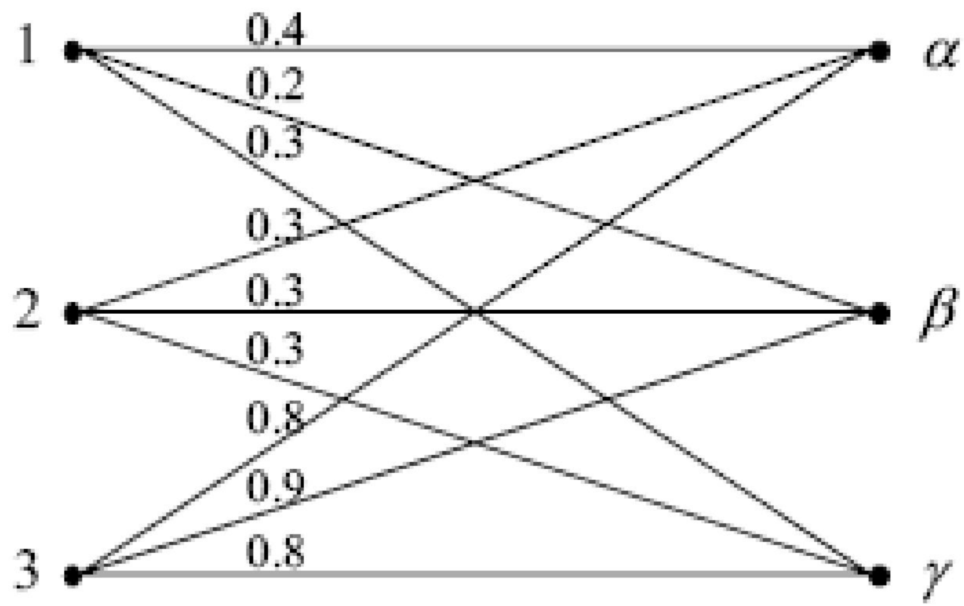
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \bullet R}(1,\ \beta) = \max[\min(0.1, 0.0), \min(0.2, 1.0), \min(0.0, 0.0), \min(1.0, 0.2)]$$
$$= \max[0.0, 0.2, 0.0, 0.2] = 0.2$$

Composition of fuzzy relations

S•R	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

S•R



CLASSICAL EQUIVALENCE RELATION

Let relation R on universe X be a relation from X to X. Relation R is an equivalence relation if the following three properties are satisfied.

1. Reflexivity
2. Symmetry
3. Transitivity

The function theoretic forms of representation of these properties are as follows:

1. Reflexivity:

$$\chi_R(x_i, x_i) = 1 \text{ or } (x_i, x_i) \in R$$

2. Symmetry:

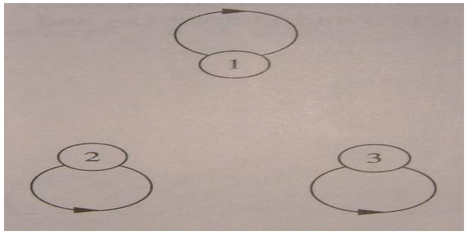
$$\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$$
$$\text{i.e., } (x_i, x_j) \in R \Rightarrow (x_j, x_i) \in R$$

3. Transitivity:

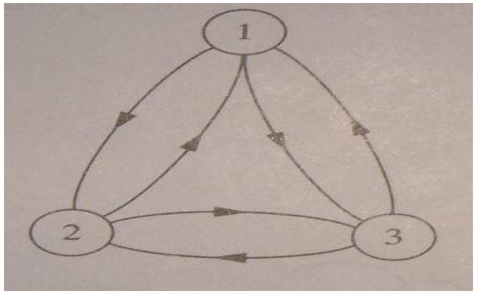
$$\chi_R(x_i, x_j) \text{ and } \chi_R(x_j, x_k) = 1, \text{ so } \chi_R(x_i, x_k) = 1$$
$$\text{i.e., } (x_i, x_j) \in R(x_j, x_k) \in R, \text{ so } (x_i, x_k) \in R$$

Classical Equivalence relation

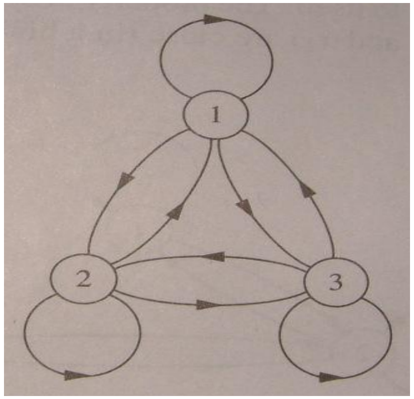
- reflexivity



- symmetry



- transitivity



FUZZY EQUIVALENCE RELATION

Let $\underset{\sim}{R}$ be a fuzzy relation on universe X, which maps elements from X to X. Relation $\underset{\sim}{R}$ will be a fuzzy equivalence relation if all the three properties – reflexive, symmetry and transitivity – are satisfied. The membership function theoretic forms for these properties are represented as follows:

1. Reflexivity:

$$\mu_{\underset{\sim}{R}}(x_i, x_i) = 1 \quad \forall x \in X$$

   If this is not the case for few $x \in X$, then $R(X, X)$ is said to be irreflexive.

2. Symmetry:

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i) \text{ for all } x_i, x_j \in X$$

   If this is not satisfied for few $x_i, x_j \in X$, then $R(X, X)$ is called asymmetric.

3. Transitivity:

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \text{ and } \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2$$

$$\Rightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda$$

where

$$\lambda = \min\,[\lambda_1, \lambda_2]$$

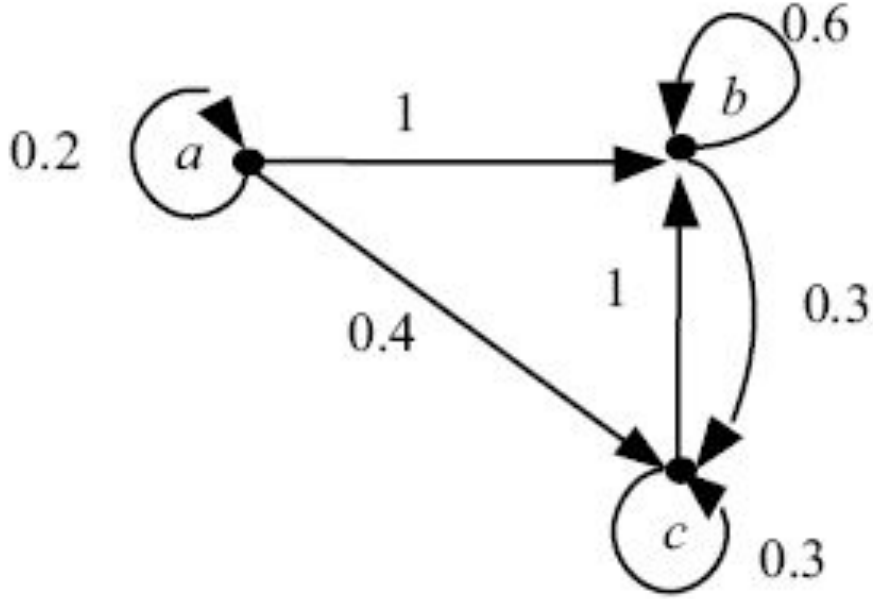
i.e., $\mu_{\underset{\sim}{R}}(x_i, x_k) \geq \max_{x_j \in X} \min[\mu_{\underset{\sim}{R}}(x_i, x_j), \mu_{\underset{\sim}{R}}(x_j, x_k)] \qquad \forall (x_i, x_k) \in X^2$

Not satisfied for few members – nontransitive ,

Not satisfied for all the members – antitransitive

Equivalence relation – also known as similarity relation

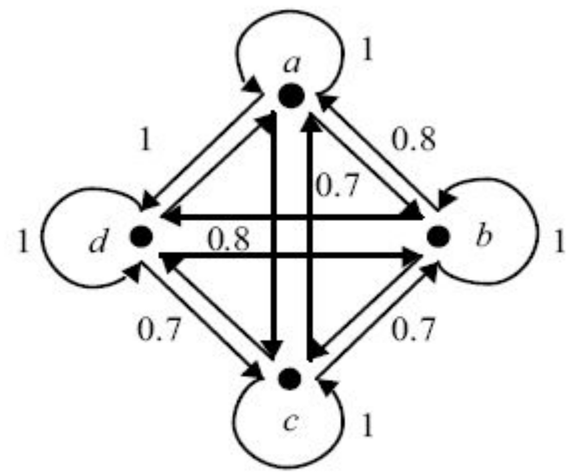
Example of fuzzy transitive relation



Fuzzy equivalence relation

- ✔ Reflexive relation
- ✔ Symmetric relation
- ✔ Transitive relation

	a	b	c	d
a	1.0	0.8	0.7	1.0
b	0.8	1.0	0.7	0.8
c	0.7	0.7	1.0	0.7
d	1.0	0.8	0.7	1.0



Test the equivalence relation

	a	b	c	d
a	1.0	0.8	0.4	0.1
b	0.8	1.0	0.0	0.0
c	0.4	0.0	1.0	0.5
d	0.1	0.0	0.5	1.0

(i) Reflexive relation √

(ii) Symmetric relation √

(iii) Transitive relation X

	a	b	c	d
a	1.0	0.8	0.4	0.4
b	0.8	1.0	0.4	0.1
c	0.4	0.4	1.0	0.5
d	0.4	0.1	0.5	1.0

Crisp Tolerance Relation

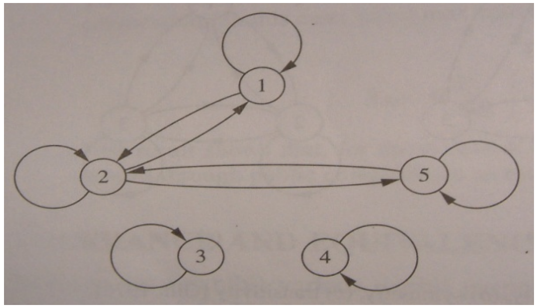
- Also called proximity relation
- Only the reflexivity and symmetry
- Can be reformed into an equivalence relation
  - By at most (n-1) compositions with itself

$$R_1^{n-1} = R_1 \circ R_1 \circ .... \circ R_1 = R$$

Crisp Tolerance Relation

- Example
  - $X=\{x_1,x_2,x_3,x_4,x_5\}$={Omaha, Chicago, Rome, London, Detroit}

$$\underset{\sim}{R}_1 = \begin{array}{c} \\ x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{array} \begin{array}{c} \begin{array}{ccccc} x1 & x2 & x3 & x4 & x5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



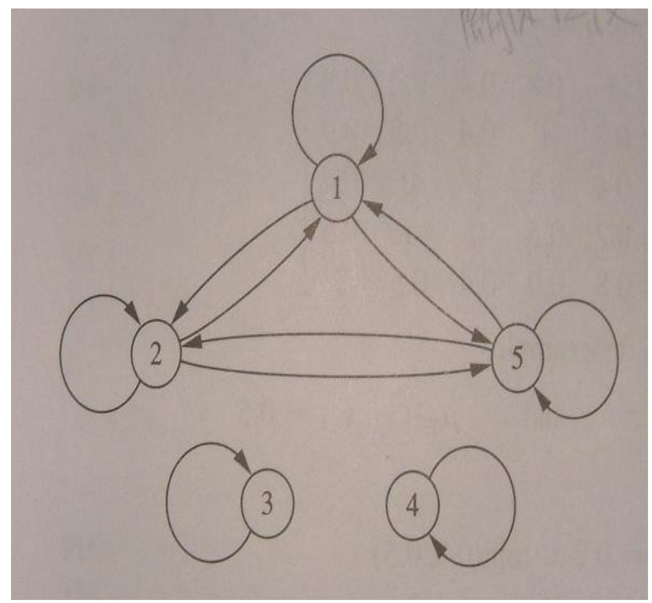
  - R1 does not properties of transitivity
    - e.g. (x1,x2) R1 (x2,x5) R1 but (x1,x5) R1

$\in$ $\in$ $\notin$

Crisp Tolerance Relation

– R1 can become an equivalence relation through two compositions

$$R_1 \circ R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = R$$



CLASSICAL TOLERANCE RELATION

A tolerance relation R_1 on universe X is one where the only the properties of reflexivity and symmetry are satisfied. The tolerance relation can also be called proximity relation. An equivalence relation can be formed from tolerance relation R_1 by $(n-1)$ compositions within itself, where n is the cardinality of the set that defines R_1, here it is X, i.e.

$$\underbrace{R_1^{n-1}}_{\text{Tolerance relation}} = R_1 \circ R_1 \circ \cdots \circ R_1 = \underbrace{R}_{\text{Equivalence relation}}$$

FUZZY TOLERANCE RELATION

A binary fuzzy relation that possesses the properties of reflexivity and symmetry is called fuzzy tolerance relation or resemblance relation.

The equivalence relations are a special case of the tolerance relation. The fuzzy tolerance relation can be reformed into fuzzy equivalence relation in the same way as a crisp tolerance relation is reformed into crisp equivalence relation, i.e.,

$$\underbrace{\underset{\sim}{R}_1^{n-1}}_{\text{Fuzzy tolerance relation}} = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 \circ \cdots \circ \underset{\sim}{R}_1 = \underbrace{\underset{\sim}{R}}_{\text{Fuzzy equivalence relation}}$$

where ‘n’ is the cardinality of the set that defines R1.

Fuzzy tolerance and equivalence relations

- Example 3.11
  -
$$\underset{\sim}{R}_1=\begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

Reflexive and symmetric

  - It is not transitive

$$\mu_{\underset{\sim}{R}}(x_1, x_2)=0.8 \quad \mu_{\underset{\sim}{R}}(x_2, x_5)=0.9 \text{ but } \mu_{\underset{\sim}{R}}(x_1, x_5)=0.2 \le \min(.8,.9)$$

  - One composition

$$\underset{\sim}{R}_1^2=\underset{\sim}{R}_1 \circ \underset{\sim}{R}_1=\begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0 & 0.4 \\ 0.2 & 0.5 & 0 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

Fuzzy tolerance and equivalence relations

- Example 3.11(續)

$\mu_{\underset{\sim}{R}^2}(x_1, x_2)=0.8 \quad \mu_{\underset{\sim}{R}^2}(x_2, x_4)=0.5$ but $\mu_{\underset{\sim}{R}}(x_1, x_4)=0.2 \le \min(.8,.5)$

$$\underset{\sim}{R}_1^3 = \underset{\sim}{R}_1^4 \circ \underset{\sim}{R} = \begin{matrix} 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{matrix}$$

$\underset{\sim}{R}_1^3(x_1, x_2)=0.8 \quad \underset{\sim}{R}_1^3(x_2, x_4)=0.5$ 則 $\underset{\sim}{R}_1^3(x_1, x_4)=0.5 \ge 0.5$

SUMMARY

The chapter provides the definition, properties, and operations of the following:

- Classical sets,
- Fuzzy sets,
- Classical relations,
- Fuzzy relations.

The chapter also discusses crisp and fuzzy equivalence and tolerance relations.