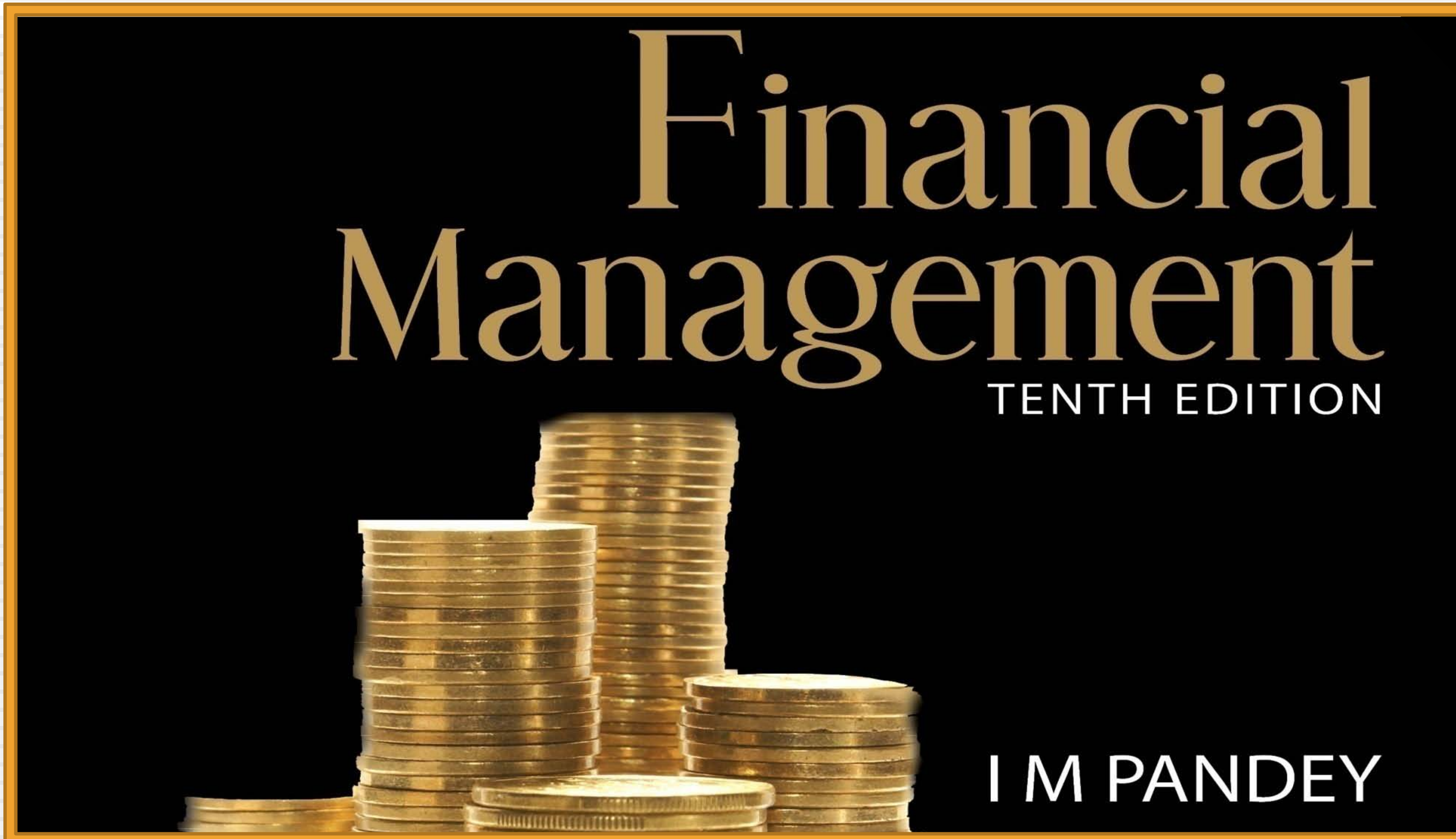

CHAPTER 5

Risk and Return: Portfolio Theory and Assets Pricing Models

LEARNING OBJECTIVES

- Discuss the concepts of portfolio risk and return
- Determine the relationship between risk and return of portfolios
- Highlight the difference between systematic and unsystematic risks
- Examine the logic of portfolio theory
- Show the use of capital asset pricing model (CAPM) in the valuation of securities
- Explain the features and modus operandi of the arbitrage pricing theory (APT)

INTRODUCTION

- A portfolio is a bundle or a combination of individual assets or securities.
- Portfolio theory provides a normative approach to investors to make decisions to invest their wealth in assets or securities under risk.
- Extend the portfolio theory to derive a framework for valuing risky assets. This framework is referred to as the capital asset pricing model (CAPM). An alternative model for the valuation of risky assets is the arbitrage pricing theory (APT).
- The return of a portfolio is equal to the weighted average of the returns of individual assets (or securities).

PORTFOLIO RETURN: TWO-ASSET CASE

- The return of a portfolio is equal to the weighted average of the returns of individual assets (or securities) in the portfolio with weights being equal to the proportion of investment value in each asset.
- We can use the following equation to calculate the expected rate of return of individual asset:

$$E(R_x) = (R_1 \times P_1) + (R_2 \times P)_2 + (R_3 \times P_3) + ... + (R_n \times P_n)$$

$$E(R_x) = \sum_{i=1}^{n} R_i P_i$$

Expected Rate of Return: Example

- Suppose you have an opportunity of investing your wealth either in asset X or asset Y. The possible outcomes of two assets in different states of economy are as follows:

Possible Outcomes of two Assets, *X and Y*

State of Economy	*Probability*	*Return (%)*	
		X	*Y*
A	0.10	– 8	14
B	0.20	10	– 4
C	0.40	8	6
D	0.20	5	15
E	0.10	– 4	20

Cont...

The expected rate of return of X is the sum of the product of outcomes and their respective probability. That is:

$$E(R_x) = (-8 \times 0.1) + (10 \times 0.2) + (8 \times 0.4) + (5 \times 0.2)$$
$$+ (-\ 4 \times 0.1) = 5\%$$

Similarly, the expected rate of return of Y is:

$$E(R_y) = (14 \times 0.1) + (-\ 4 \times 0.2) + (6 \times 0.4) + (15 \times 0.2)$$
$$+\ (20 \times 0.1) = 8\%$$

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PORTFOLIO RISK: TWO-ASSET CASE

- Risk of individual assets is measured by their variance or standard deviation.

- We can use variance or standard deviation to measure the risk of the portfolio of assets as well.

- The risk of portfolio would be less than the risk of individual securities, and that the risk of a security should be judged by its contribution to the portfolio risk.

Measuring Portfolio Risk for Two Assets

- The portfolio variance or standard deviation depends on the co-movement of returns on two assets.
- Covariance of returns on two assets measures their co-movement.
- Three steps are involved in the calculation of covariance between two assets:

Determining the expected returns on assets.

Determining the deviation of possible returns from the expected return for each asset.

Determining the sum of the product of each deviation of returns of two assets and respective probability.

Covariance of Returns of Securities X and Y

State of Economy	*Probability*	*Returns*		*Deviation from Expected Returns*		*Product of Deviation & Probability (Covariance)*
		X	*Y*	*X*	*Y*	
A	0.1	– 8	14	– 13	6	– 7.8
B	0.2	10	– 4	5	– 12	– 12.0
C	0.4	8	6	3	– 2	– 2.4
D	0.2	5	15	0	7	0.0
E	0.1	– 4	20	– 9	12	– 10.8
		$E(R_X) = 5$	$E(R_Y) = 8$			Covar = – 33.0

Example

$$\sigma_x^2 = 0.1(-8-5)^2 + 0.2(10-5)^2 + 0.4(8-5)^2$$
$$+0.2(5-5)^2 + 0.1(-4-5)^2$$
$$= 16.9 + 3.6 + 0 + 8.1 = 33.6$$
$$\sigma_x = \sqrt{33.6} = 5.80\%$$
$$\sigma_y^2 = 0.1(14-8)^2 + 0.2(-4-8)^2 + 0.4(6-8)^2$$
$$+0.2(15-8)^2 + 0.1(20-8)^2$$
$$= 3.6 + 28.8 + 1.6 + 9.8 + 14.4 = 58.2$$
$$\sigma_y = \sqrt{58.2} = 7.63\%$$

The correlation of the two securities X and Y is as follows:

$$\text{Cor}_{xy} = \frac{-33.0}{5.80 \times 7.63} = \frac{-33.0}{44.25} = -0.746$$

Securities *X* and *Y* are negatively correlated. The correlation coefficient of – 0.746 indicates a high negative relationship.

Measuring Portfolio Risk for Two Assets

- The formula for calculating covariance of returns of the two securities *X and Y is as follows:*

$$\text{Cov}_{xy} = \sum_{i=1}^{n} [R_x - E(R_x)][R_y - E(R_y)] \times P_i$$

- The relationship between the returns of securities X and Y have following possibilities:
- Positive covariance
- Negative covariance
- Zero covariance

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Correlation

- Correlation is a measure of the linear relationship between two variables (say, returns of two securities, X and Y in our case).
- The formula for covariance of returns on X and Y can be expressed as follows:

$$\text{Covariance } XY = \text{Standard deviation } X \times \text{Standard deviation } Y \times \text{Correlation } XY$$

$$\text{Cov}_{xy} = \sigma_x \sigma_y \text{Cor}_{xy}$$

- We can determine the correlation by dividing covariance by the standard deviations of returns on securities *X and Y:*

$$\text{Correlation } X,Y = \frac{\text{Covariance } XY}{\text{Standard deviation } X \times \text{Standard deviation } Y}$$

$$\text{Cor}_{xy} = \frac{\text{Cov}_{xy}}{\sigma_x \sigma_y}$$

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Correlation

- The value of correlation, called the correlation coefficient, could be positive, negative or zero.
- It depends on the sign of covariance since standard deviations are always positive numbers.
- The correlation coefficient always ranges between –1.0 and +1.0.
- A correlation coefficient of +1.0 implies a perfectly positive correlation while a correlation coefficient of –1.0 indicates a perfectly negative correlation.

Variance and Standard Deviation of a Two-Asset Portfolio

- The variance of a two-asset portfolio is not the weighted average of the variances of assets since they co-vary as well.

- The variance of two-security portfolio is given by the following equation:

$$\sigma_p^2 = \sigma_x^2 w_x^2 + \sigma_y^2 w_y^2 + 2 w_x w_y \,\mathrm{Cov}_{xy}$$
$$= \sigma_x^2 w_x^2 + \sigma_y^2 w_y^2 + 2 w_x w_y (\sigma_x \sigma_y \,\mathrm{Cor}_{xy})$$

Covariance Calculation Matrix

I		II		III	
σ_x^2	Cov_{xy}	w_x^2	$w_x w_y$	$\sigma_x^2 w_x^2$	$w_x w_y Cov_{xy}$
Cov_{xy}	σ_y^2	$w_x w_y$	w_y^2	$w_x w_y Cov_{xy}$	$\sigma_y^2 w_y^2$

Minimum Variance Portfolio

- The minimum variance portfolio is also called the **optimum portfolio**.
- We can use the following general formula for estimating optimum weights of two securities *X and Y so* that the portfolio variance is minimum:

$$w^* = \frac{\sigma_y^2 - \text{Cov}_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\text{Cov}_{xy}}$$

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Portfolio Risk Depends on Correlation between Assets

- Investing wealth in more than one security reduces portfolio risk.
- This is attributed to diversification effect.
- However, the extent of the benefits of portfolio diversification depends on the correlation between returns on securities.
- When correlation coefficient of the returns on individual securities is perfectly positive then there is no advantage of diversification. The weighted standard deviation of returns on individual securities is equal to the standard deviation of the portfolio.
- Diversification always reduces risk provided the correlation coefficient is less than 1.

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PORTFOLIO RISK-RETURN ANALYSIS: TWO-ASSET CASE

- Perfectly positive correlation (+1.0);
- Perfectly negative correlation (–1.0);
- No correlation (0.0),
- Positive correlation (0.5), and
- Negative correlation (– 0.25).

Special situations

Perfect Positive Correlation

- When correlation is +1.0, the portfolio risk (standard deviation) is simply given by the following formula:

$$\sigma_p = \sqrt{\sigma_x^2 w_x^2 + \sigma_y^2 w_y^2 + 2 w_x w_y \sigma_x \sigma_y}$$

$$= \sigma_x w_x + \sigma_y w_y$$

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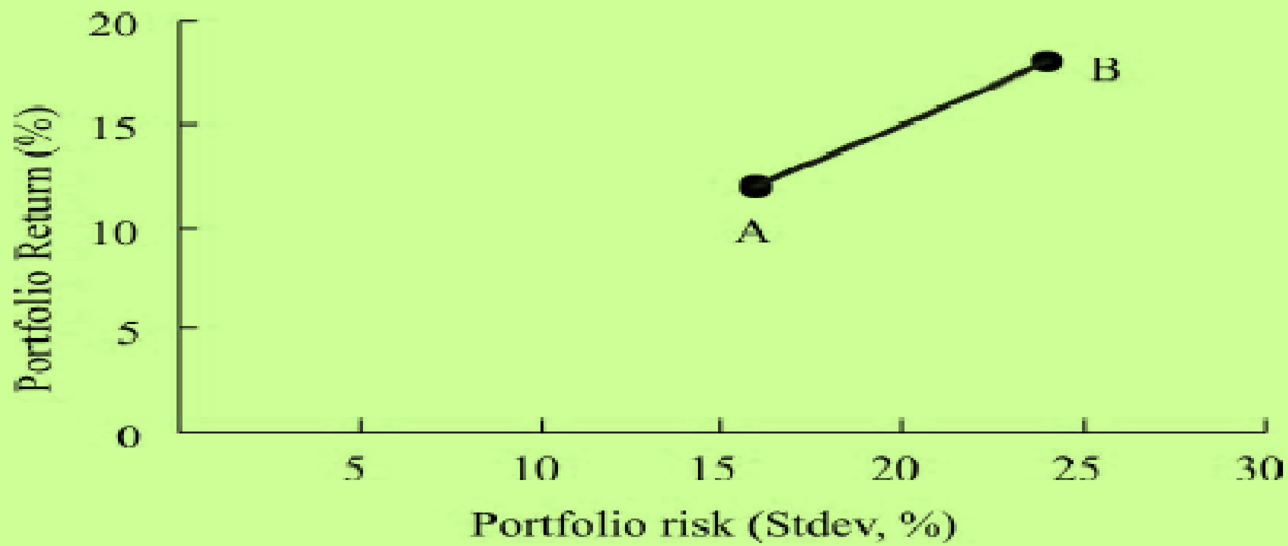
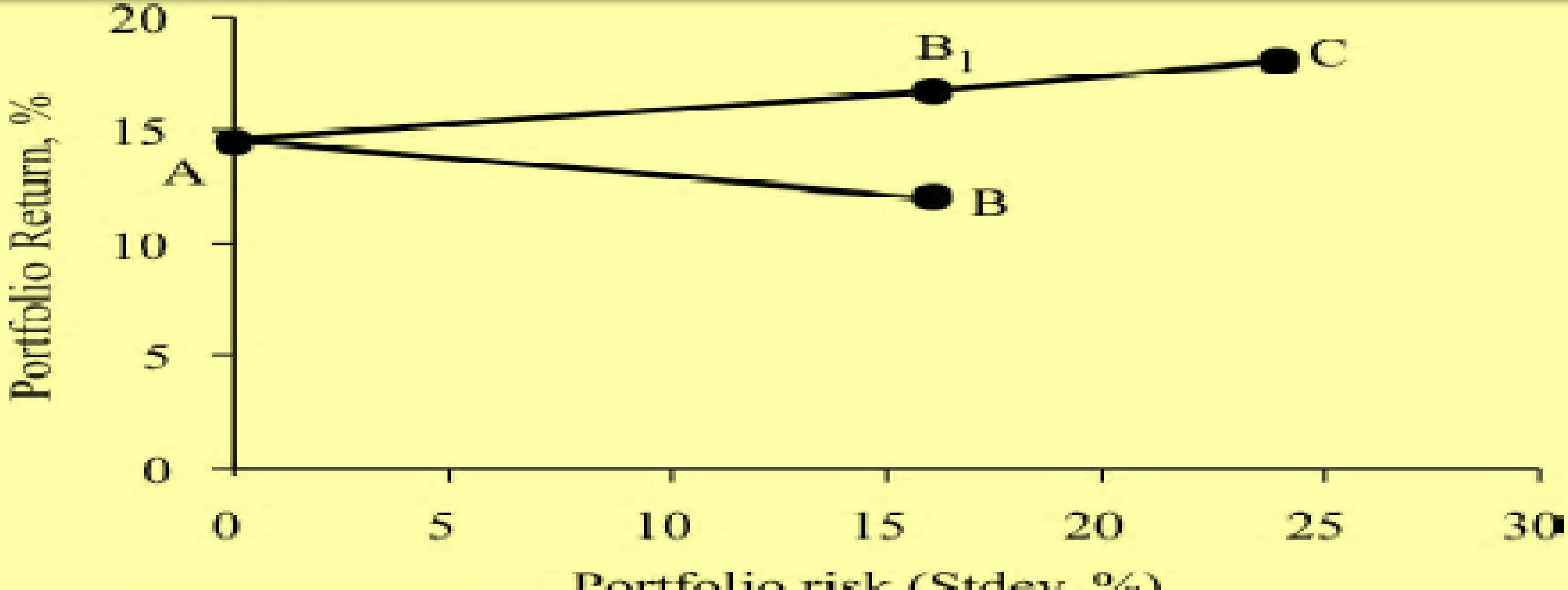


Risk-return relationships under perfect positive correlation

There is no advantage of diversification when the returns of securities have perfect positive correlation.

Perfect Negative Correlation

- In this the portfolio return increases and the portfolio risk declines.
- It results in **risk-less portfolio**.
- The correlation is -1.0.



Risk-return relationships under perfect negative correlation

Zero-variance portfolio

- When correlation is – 1.0, the portfolio risk (standard deviation) is simply given by the following formula:

$$\sigma_p = \sqrt{\sigma_x^2 w_x^2 + \sigma_y^2 w_y^2 - 2 w_x w_y \sigma_x \sigma_y}$$

$$= ABS\left[\sigma_x w_x - \sigma_y w_y\right]$$

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Zero Correlation

- Zero correlation means that the returns of two securities are independent of each other.
- There is no possibility of the standard deviation reducing to zero and achieving a risk-less portfolio.
- When correlation is zero, we can determine the minimum variance portfolio as follows:

$$w_x = \frac{\sigma_y^2}{\sigma_x^2 + \sigma_y^2}$$

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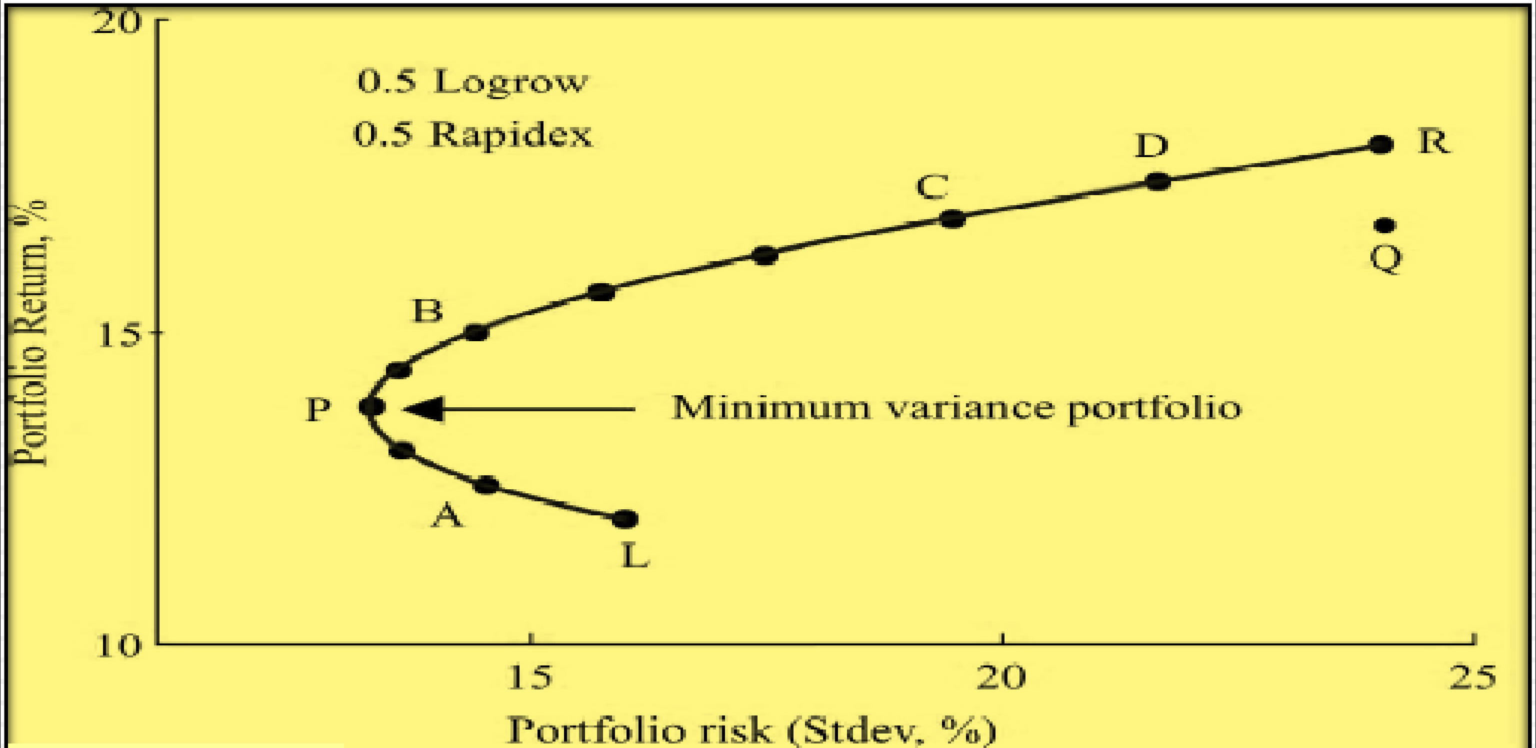
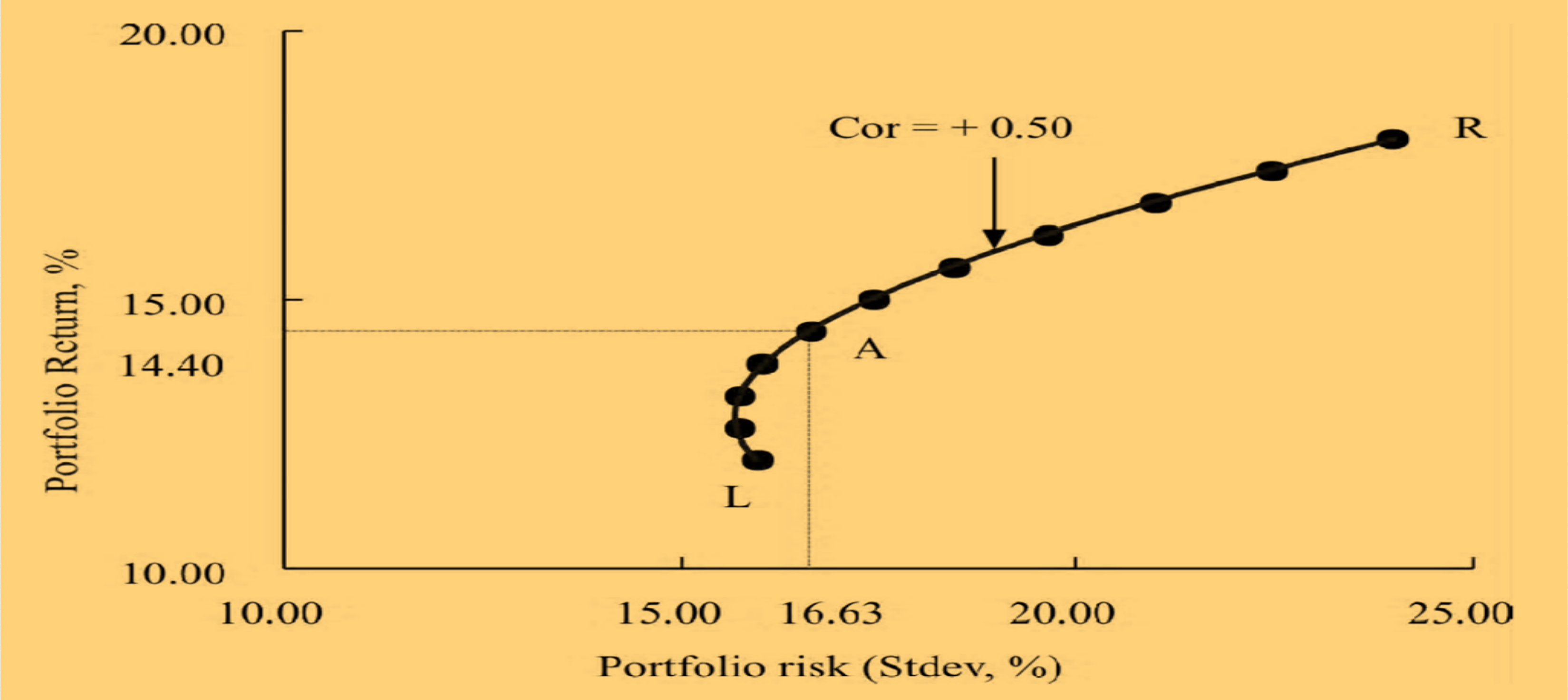


Risk-return relationships with zero correlation

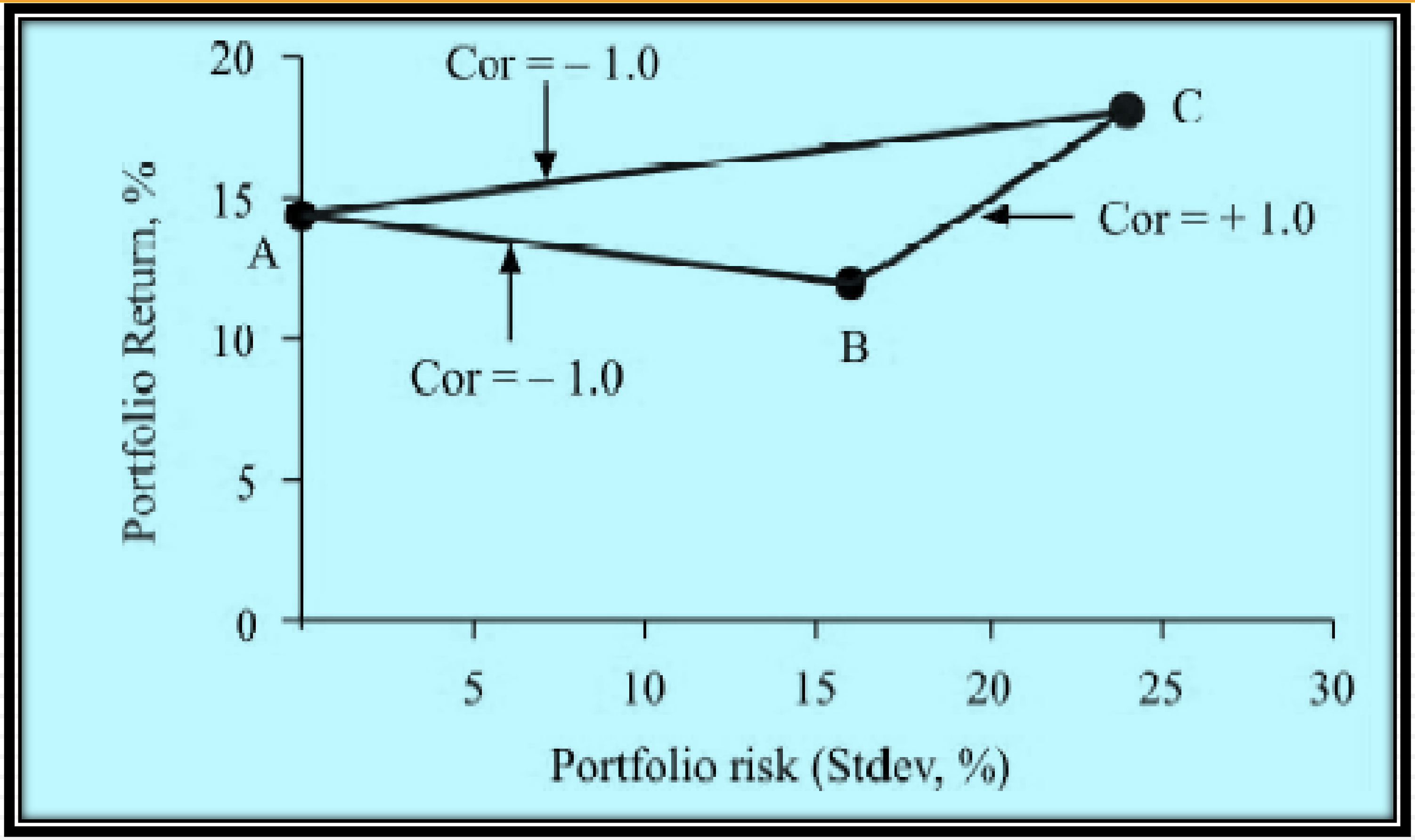
Positive Correlation

- In reality, returns of most assets have positive but less than 1.0 correlation.



Risk-return relation with positive correlation

Limits to diversification



Since any probable correlation of securities Logrow and Rapidex will range between – 1.0 and + 1.0, the triangle in the above figure specifies the limits to diversification. The risk-return curves for any correlations within the limits of – 1.0 and + 1.0, will fall within the triangle ABC.

Minimum variance portfolio

- When correlation is positive or negative, the minimum variance portfolio is given by the following formula:

$$w_x = \frac{\sigma_y^2 - \sigma_x \sigma_y \text{Cor}_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_x \sigma_y \text{Cor}_{xy}}$$

EFFICIENT PORTFOLIO AND MEAN-VARIANCE CRITERION

Investment Opportunity Set: Two-Asset Case

- The investment or portfolio opportunity set represents all possible combinations of risk and return resulting from portfolios formed by varying proportions of individual securities.

- It presents the investor with the risk-return trade-off.

Portfolio Return and Risk for Different Correlation Coefficients

Weight		Portfolio Return (%)	*Portfolio Risk, σ_p (%)*				
			Correlation				
			+1.00	-1.00	0.00	0.50	-0.25
Logrow	Rapidex	R_p	σ_p	σ_p	σ_p	σ_p	σ_p
1.00	0.00	12.00	16.00	16.00	16.00	16.00	16.00
0.90	0.10	12.60	16.80	12.00	14.60	15.74	13.99
0.80	0.20	13.20	17.60	8.00	13.67	15.76	12.50
0.70	0.30	13.80	18.40	4.00	13.31	16.06	11.70
0.60	0.40	14.40	19.20	0.00	13.58	16.63	11.76
0.50	0.50	15.00	20.00	4.00	14.42	17.44	12.65
0.40	0.60	15.60	20.80	8.00	15.76	18.45	14.22
0.30	0.70	16.20	21.60	12.00	17.47	19.64	16.28
0.20	0.80	16.80	22.40	16.00	19.46	20.98	18.66
0.10	0.90	17.40	23.20	20.00	21.66	22.44	21.26
0.00	1.00	18.00	24.00	24.00	24.00	24.00	24.00
Minimum Variance Portfolio							
w_L			1.00	0.60	0.692	0.857	0.656
w_R			0.00	0.40	0.308	0.143	0.344
σ^2			256	0.00	177.23	246.86	135.00
σ (%)			16	0.00	13.31	15.71	11.62

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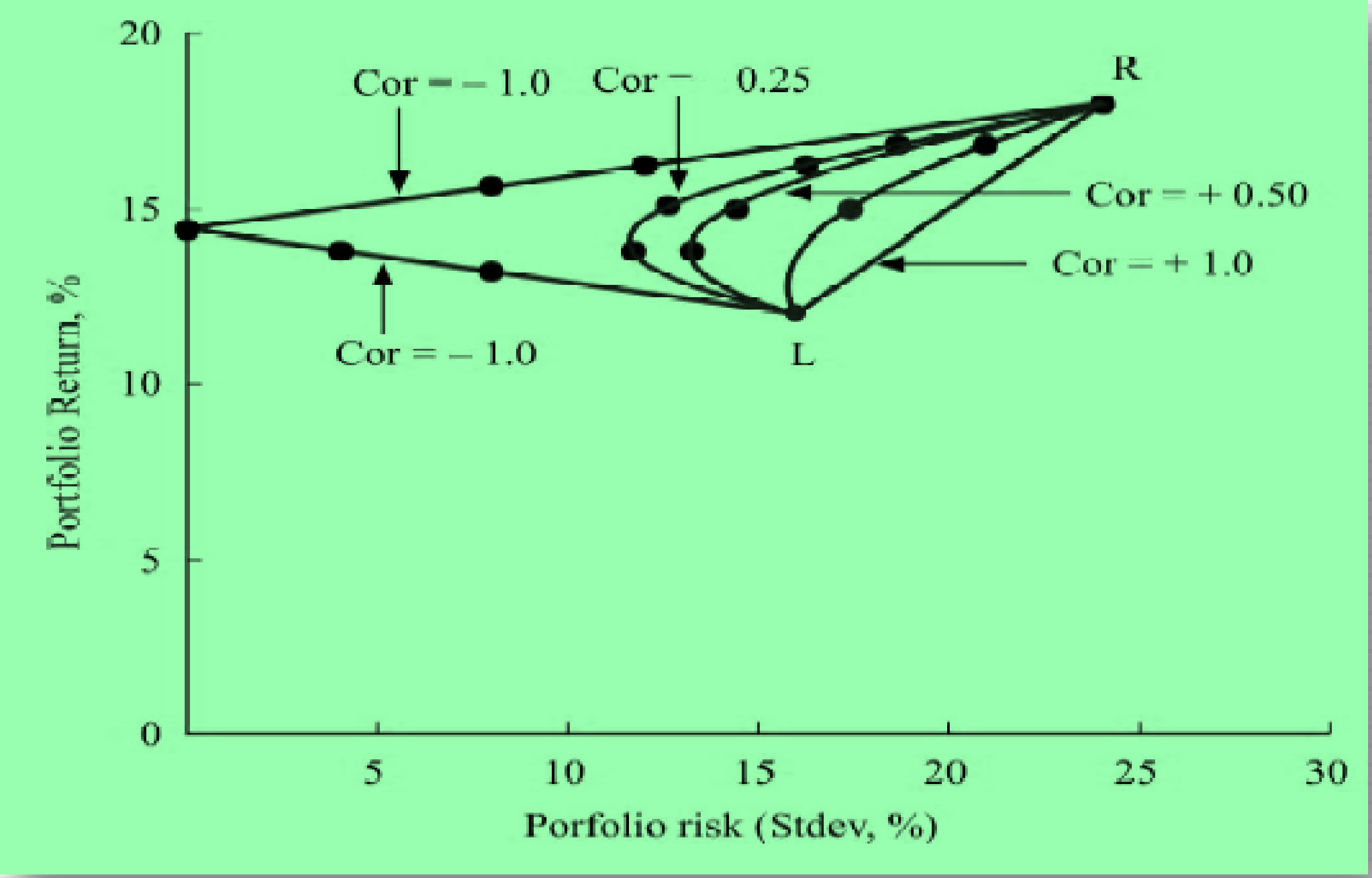
Investment opportunity sets given different correlations



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Mean-variance Criterion

- Inefficient portfolios- have lower return and higher risk

Investment Opportunity Set: The *n-Asset Case*

- An efficient portfolio is one that has the highest expected returns for a given level of risk.
- The efficient frontier is the frontier formed by the set of efficient portfolios.
- All other portfolios, which lie outside the efficient frontier, are inefficient portfolios.

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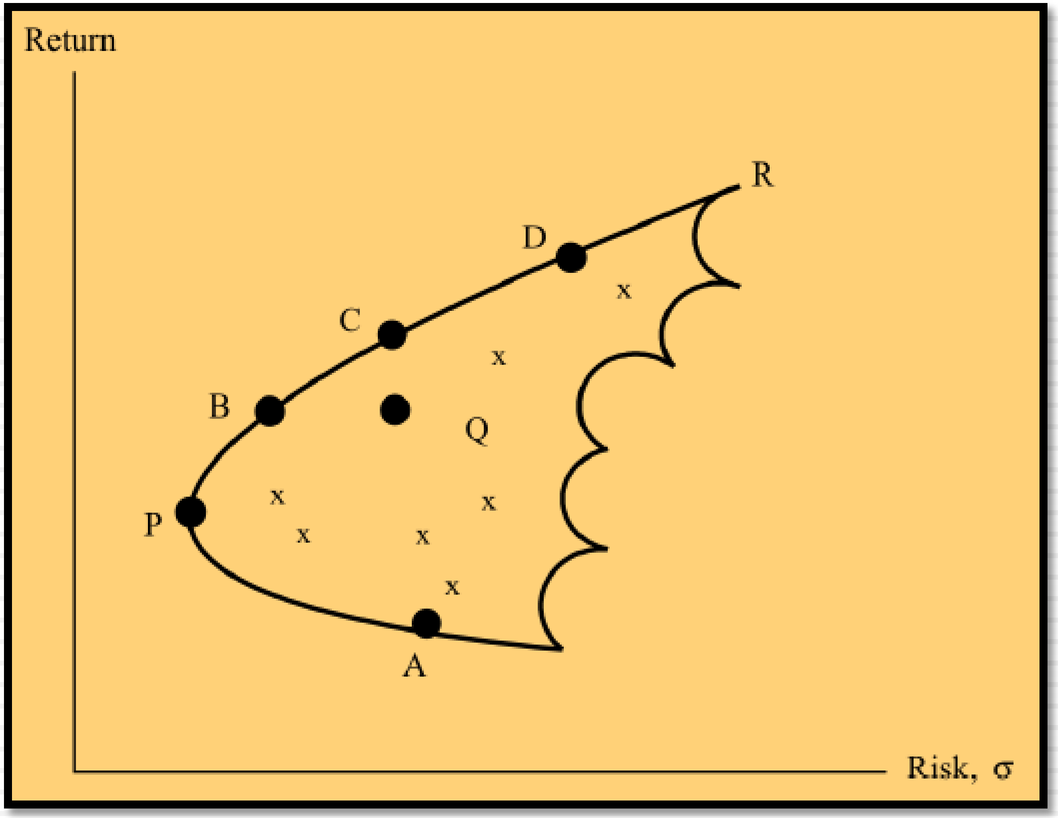
Efficient Portfolios of risky securities

An **efficient portfolio** is one that has the highest expected returns for a given level of risk. The **efficient frontier** is the frontier formed by the set of efficient portfolios. All other portfolios, which lie outside the efficient frontier, are **inefficient portfolios**.



PORTFOLIO RISK: THE *n-ASSET CASE*

- The calculation of risk becomes quite involved when a large number of assets or securities are combined to form a portfolio.

N-Asset Portfolio Risk Matrix

	Security X_1	Security X_2	Security X_3	.....	Security X_n
Security X_1	$X_1^2\sigma_1^2$	$X_1X_2\rho_{12}\sigma_1\sigma_2$	$X_1X_3\rho_{12}\sigma_1\sigma_3$	.....	$X_1X_n\rho_{1n}\sigma_1\sigma_n$
Security X_2	$X_2X_1\rho_{21}\sigma_2\sigma_1$	$X_2^2\sigma_2^2$	$X_2X_1\rho_{23}\sigma_2\sigma_3$	.....	$X_2X_n\rho_{2n}\sigma_2\sigma_n$
Security X_3	$X_3X_1\rho_{31}\sigma_3\sigma_1$	$X_3X_2\rho_{32}\sigma_3\sigma_2$	$X_3^2\sigma_3^2$	.....	$X_3X_n\rho_{3n}\sigma_3\sigma_n$
⋮	⋮	⋮	⋮	⋮	⋮
Security X_n	$X_nX_1\rho_{n1}\sigma_n\sigma_1$	$X_nX_2\rho_{n2}\sigma_n\sigma_2$	$X_nX_3\rho_{n3}\sigma_n\sigma_3$	.....	$X_n^2\sigma_n^2$

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- Based on the logic of the portfolio risk in a two security case, the portfolio risk in *n-security case can be* calculated.
- Let us assume a portfolio where all securities *(n) have equal weights, they have the same variance and* all covariance terms are equal.
- In this special case, the portfolio variance is given as follows:

$$\text{Portfolio variance} = \sigma_p^2 = n \; \frac{1}{n^2} \; \times \text{average variance}$$
$$+ n(n-1) \; \frac{1}{n^2} \; \times \text{average cov}$$

$$\text{Portfolio variance} = \left[\frac{1}{n}\right] \text{average variance}$$
$$+ \left[1 - \frac{1}{n}\right] \times \text{average covariance}$$

$$\text{Portfolio variance (when } n \to \infty) = \text{average covariance}$$

RISK DIVERSIFICATION: SYSTEMATIC AND UNSYSTEMATIC RISK

- When more and more securities are included in a portfolio, the risk of individual securities in the portfolio is reduced.
- This risk totally vanishes when the number of securities is very large.
- But the risk represented by covariance remains.
- Risk has two parts:
  1. Diversifiable (unsystematic)
  2. Non-diversifiable (systematic)

Systematic Risk

- Systematic risk arises on account of the economy-wide uncertainties and the tendency of individual securities to move together with changes in the market.
- This part of risk cannot be reduced through diversification.
- It is also known as market risk.
- Investors are exposed to market risk even when they hold well-diversified portfolios of securities.

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Examples of Systematic Risk

- The government changes the *interest rate policy*. The *corporate tax* rate is increased.
- The government resorts to massive *deficit financing*.
- The *inflation* rate increases.
- The RBI promulgates a restrictive *credit policy*.
- The government relaxes the *foreign exchange* controls and announces full *convertibility* of the Indian rupee.
- The government withdraws tax on dividend payments by companies.
- The government eliminates or reduces the capital gain tax rate.

Unsystematic Risk

- Unsystematic risk arises from the unique uncertainties of individual securities.
- It is also called unique risk.
- These uncertainties are diversifiable if a large numbers of securities are combined to form well-diversified portfolios.
- Uncertainties of individual securities in a portfolio cancel out each other.
- Unsystematic risk can be totally reduced through diversification.

Examples of Unsystematic Risk

- The company workers declare strike.
- The R&D expert leaves the company.
- A formidable competitor enters the market.
- The company loses a big contract in a bid.
- The company makes a breakthrough in process innovation.
- The government increases custom duty on the material used by the company.
- The company is unable to obtain adequate quantity of raw material

Total Risk

- Total risk of an individual security is the variance (or standard deviation) of its return. It consists of two parts:

$$\text{Total risk of a security} = \text{Systematic risk} + \text{Unsystematic risk}$$

- Systematic risk is attributable to macroeconomic factors.

- Above equation can also be written as:

$$\text{Total risk} = \text{variance attributable to macroeconomic factors} + \text{(residual) variance attributable to firm-specific factors}$$

- Total risk is not relevant for an investor who holds a diversified portfolio.

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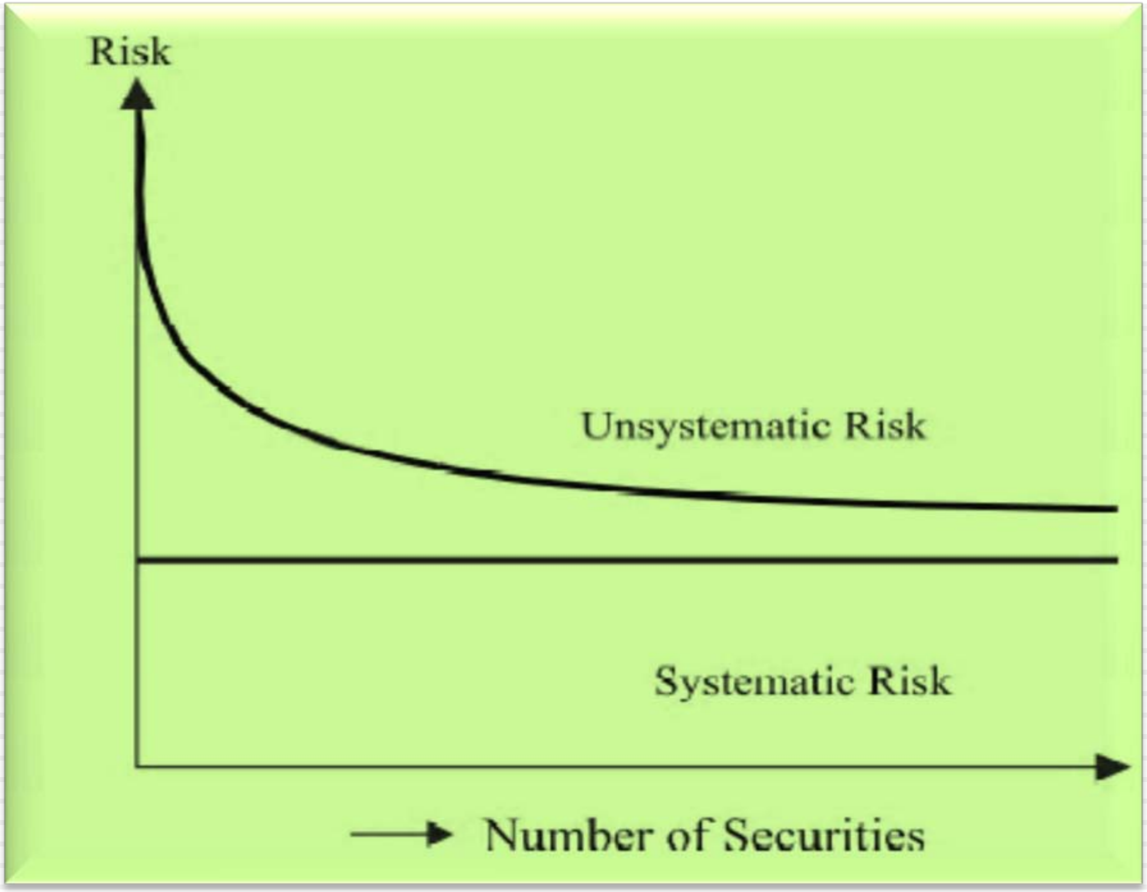
Systematic and unsystematic risk and number of securities



COMBINING A RISK-FREE ASSET AND A RISKY ASSET

- A risk-free asset or security has a zero variance or standard deviation.
- The risk-free security has no risk of default.
- The government treasury bills or bonds are approximate examples of the risk-free security as they have no risk of default.
- When risk-free and a risky asset are combined, the portfolio return is:

$$E(R_p) = wE(R_j) + (1 - w)R_f$$

- Where, R_f= risk-free security

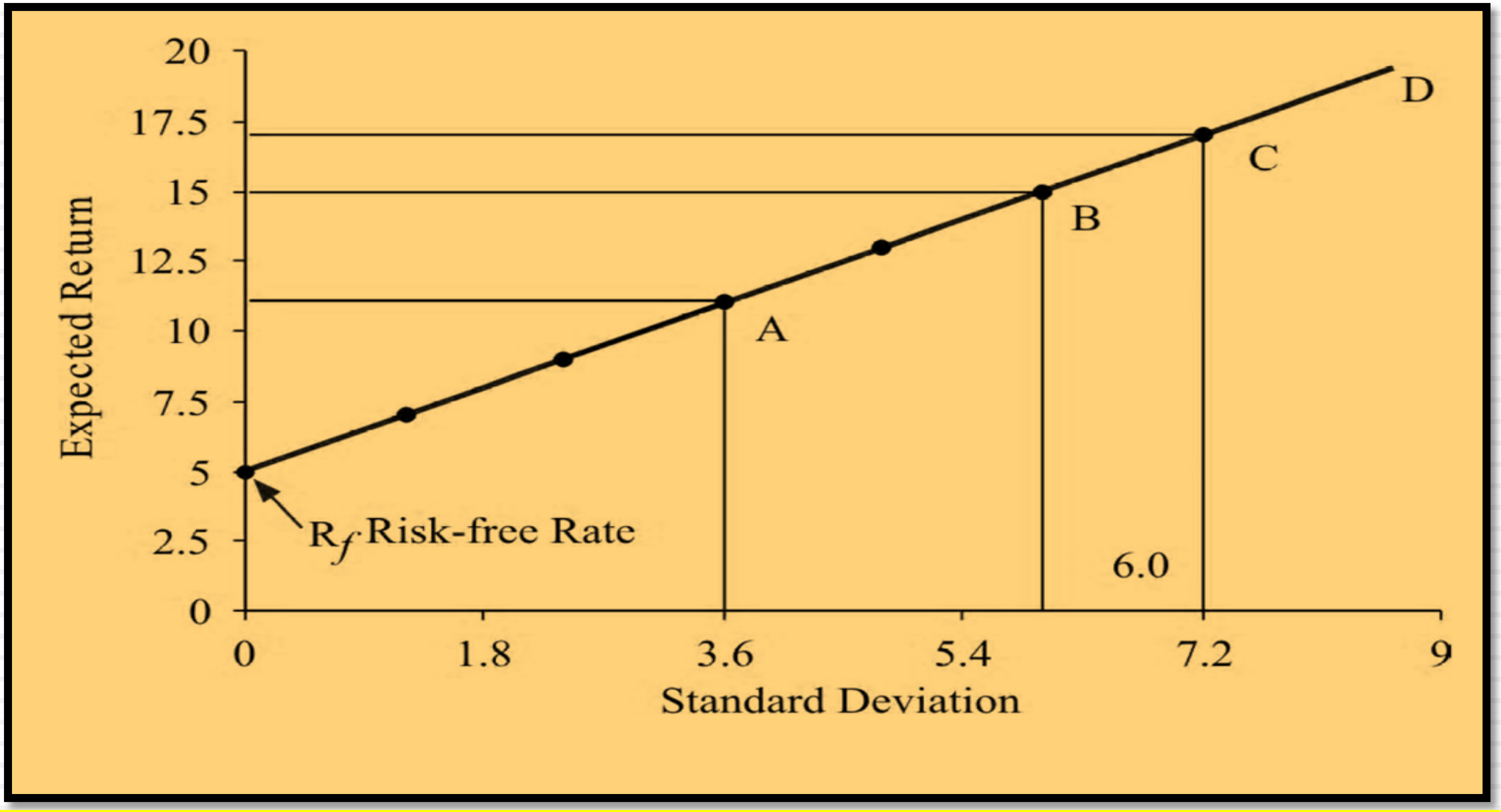
  R_j= risky security

A Risk-Free Asset and A Risky Asset: Example

Risk-return Analysis for a Portfolio of a Risky and a Risk-free Securities

Weights (%)		*Expected Return, R_p (%)*	*Standard Deviation (σ_p) (%)*
Risky security	*Risk-free security*		
120	– 20	17	7.2
100	0	15	6.0
80	20	13	4.8
60	40	11	3.6
40	60	9	2.4
20	80	7	1.2
0	100	5	0.0

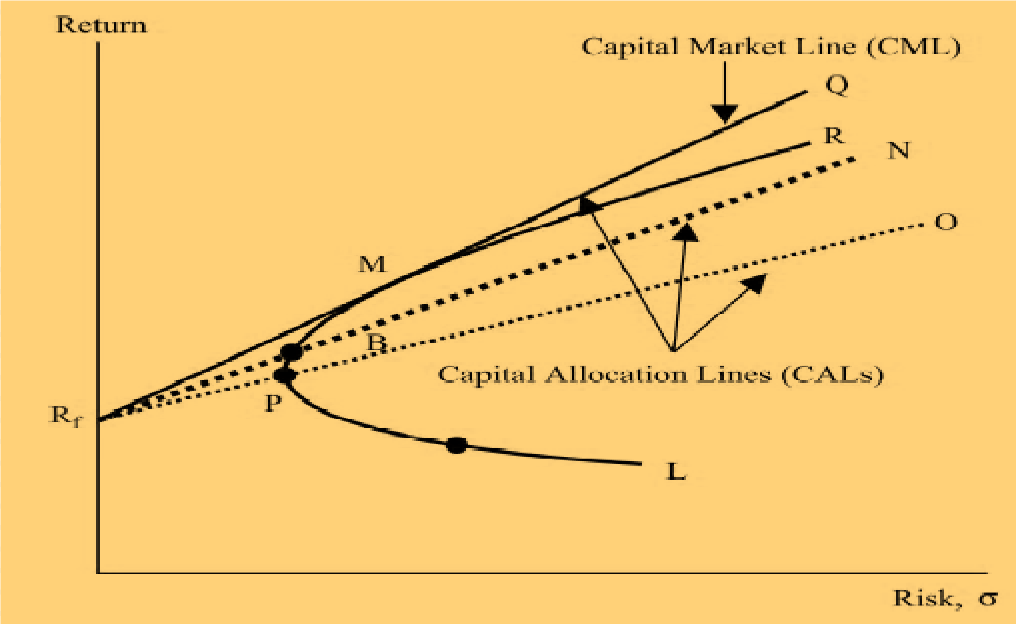
Borrowing and Lending



Risk-return relationship for portfolio of risky and risk-free securities

MULTIPLE RISKY ASSETS AND A RISK-FREE ASSET

- In a market situation, a large number of investors holding portfolios consisting of a risk-free security and multiple risky securities participate.



Risk-return relationship for portfolio of risky and risk-free securities

- We draw three lines from the risk-free rate (5%) to the three portfolios. Each line shows the manner in which capital is allocated. This line is called the **capital allocation line.**

- Portfolio *M* is the optimum risky portfolio, which can be combined with the risk-free asset.

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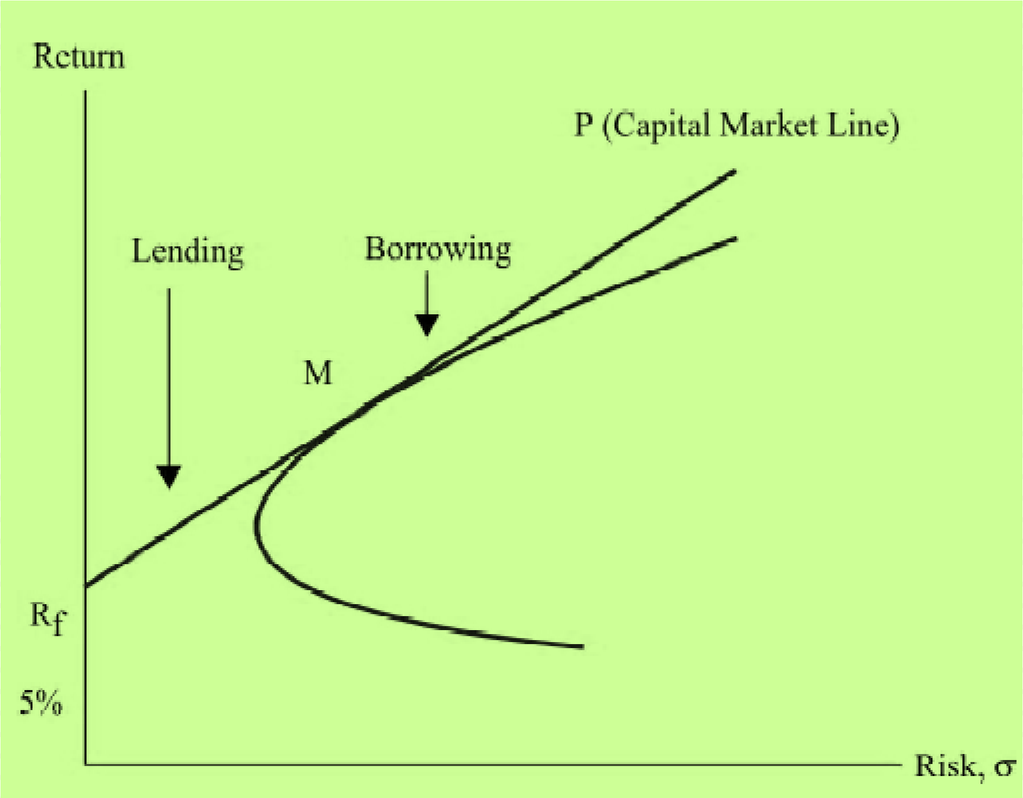


The capital market line

- **The capital market line (CML)** is an efficient set of risk-free and risky securities, and it shows the risk-return trade-off in the market equilibrium.

Separation Theory

- According to the separation theory, the choice of portfolio involves two separate steps.
- The first step involves the determination of the optimum risky portfolio.
- The second step concerns with the investor’s decision to form portfolio of the risk-free asset and the optimum risky portfolio depending on her risk preferences.

Slope of CML

- The slope of CML describes the best price of a given level of risk in equilibrium.
- The slope of CML is also referred to as the reward-to variability ratio.

$$\text{Slope of CML} = \frac{E(R_m) - R_f}{\sigma_m}$$

- The expected return on a portfolio on CML is defined by the following equation:

$$E(R_p) = R_f + \left[\frac{E(R_m) - R_f}{\sigma_m}\right]\sigma_p$$

CAPITAL ASSET PRICING MODEL (CAPM)

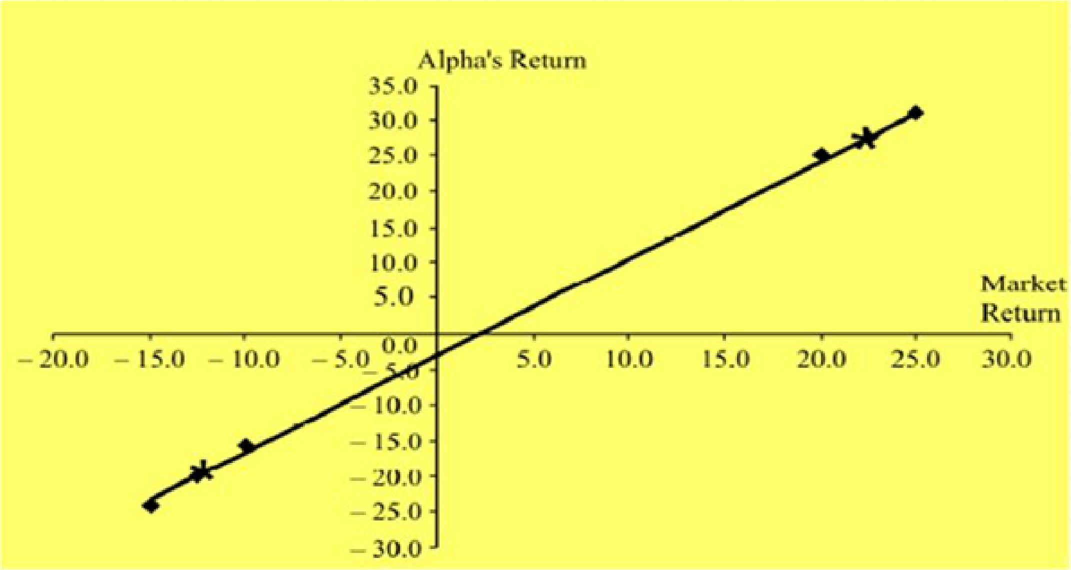
- The capital asset pricing model (CAPM) is a model that provides a framework to determine the required rate of return on an asset and indicates the relationship between return and risk of the asset.
- The required rate of return specified by CAPM helps in valuing an asset.
- One can also compare the expected (estimated) rate of return on an asset with its required rate of return and determine whether the asset is fairly valued.
- Under CAPM, the security market line (SML) exemplifies the relationship between an asset's risk and its required rate of return.

Assumptions of CAPM

- Market efficiency
- Risk aversion and mean-variance optimization
- Homogeneous expectations
- Single time period
- Risk-free rate

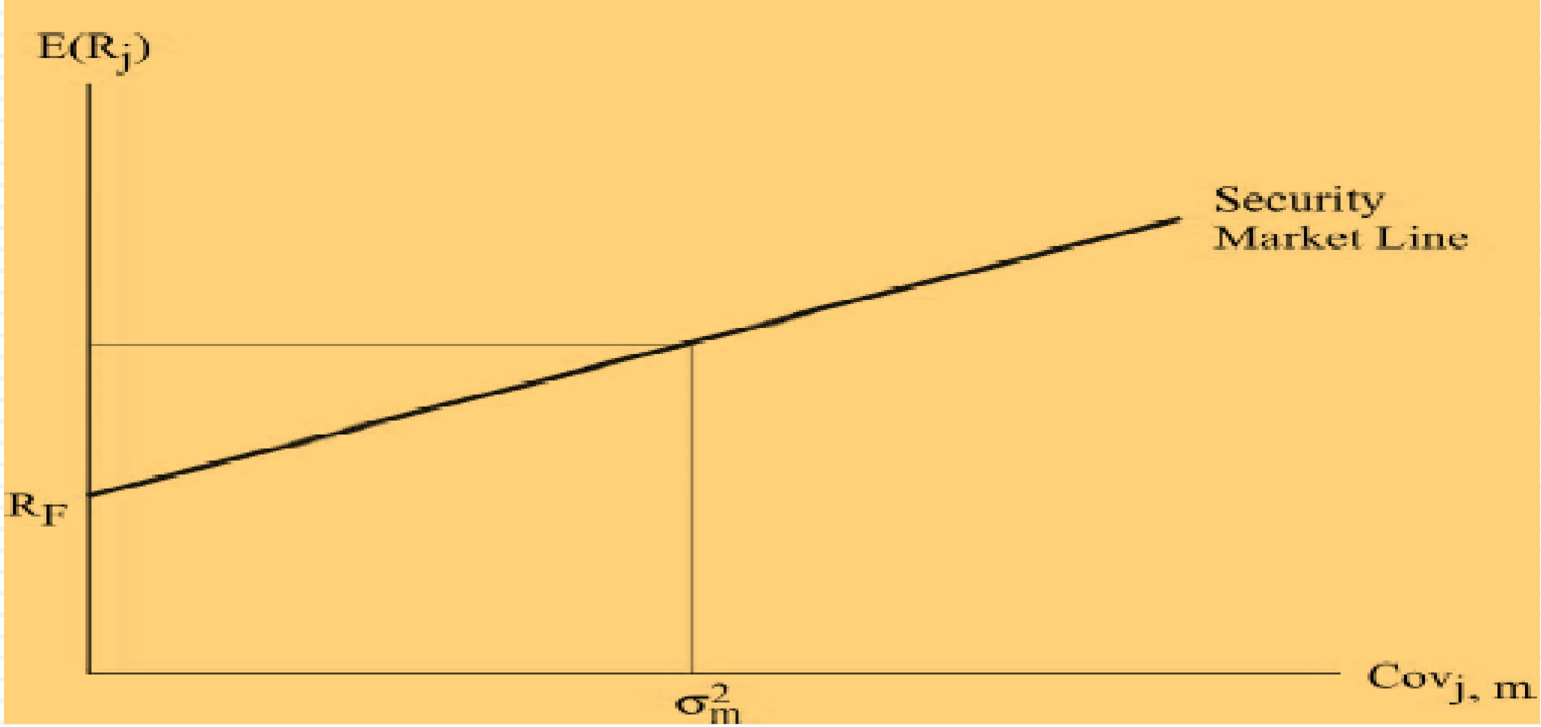
Characteristics Line

- The combinations of the expected returns points, when joined to form a line is called as **characteristics line.**
- The slope of the characteristics line is the sensitivity coefficient is referred to as beta.

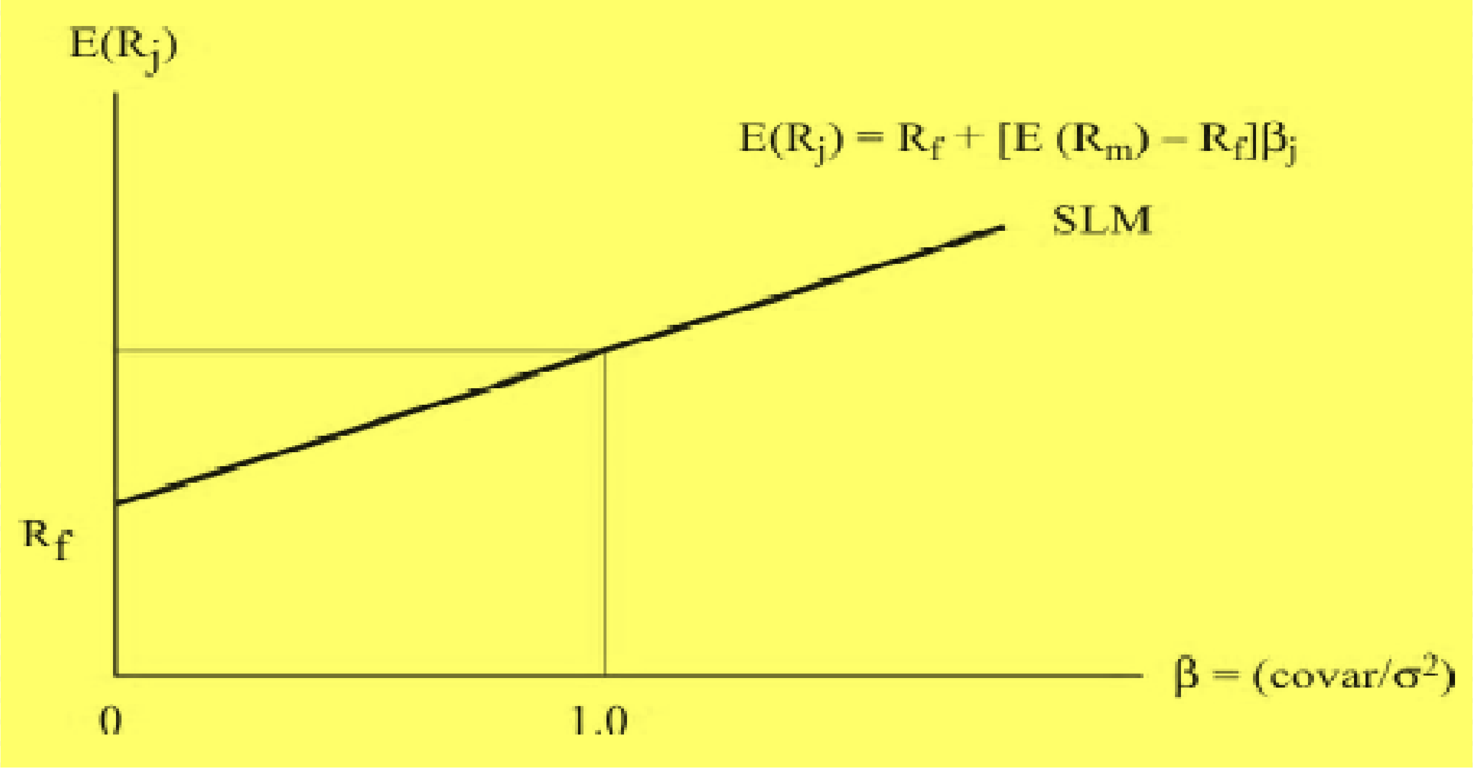


Characteristics Line: Market Return vs. Alpha's Return

Security Market Line (SML)



Security market line



Security market line with normalize systematic risk (β)

IMPLICATIONS AND RELEVANCE OF CAPM

Implications

- Investors will always combine a risk-free asset with a market portfolio of risky assets. They will invest in risky assets in proportion to their market value.

- Investors will be compensated only for that risk which they cannot diversify.

- Investors can expect returns from their investment according to the risk.

Limitations

- It is based on unrealistic assumptions.
- It is difficult to test the validity of CAPM.
- Betas do not remain stable over time.

THE ARBITRAGE PRICING THEORY (APT)

- The act of taking advantage of a price differential between two or more markets is referred to as arbitrage.
- The Arbitrage Pricing Theory (APT) describes the method of bringing a mispriced asset in line with its expected price.
- An asset is considered mispriced if its current price is different from the predicted price as per the model.
- The fundamental logic of APT is that investors always indulge in arbitrage whenever they find differences in the returns of assets with similar risk characteristics.

Concept of Return under APT

- In APT, the return of an asset is assumed to have two components: predictable (expected) and unpredictable (uncertain) return. Thus, return on asset *j* will be:

$$E(R_j) = R_f + UR$$

- Where R_f is the predictable return (risk-free return on a zero-beta asset) and *UR* is the unanticipated part of the return.

Concept of Risk under APT

- APT assumes that market risk can be caused by economic factors such as changes in gross domestic product, inflation, and the structure of interest rates and these factors could affects the firms differently.
- Therefore, under APT the sensitivity of the asset's return to each factor is estimated.
- For each firm, there will be as many betas as the number of factors.

$$E(R_j) = R_f + (\beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \cdots + \beta_n F_n) + UR_s$$

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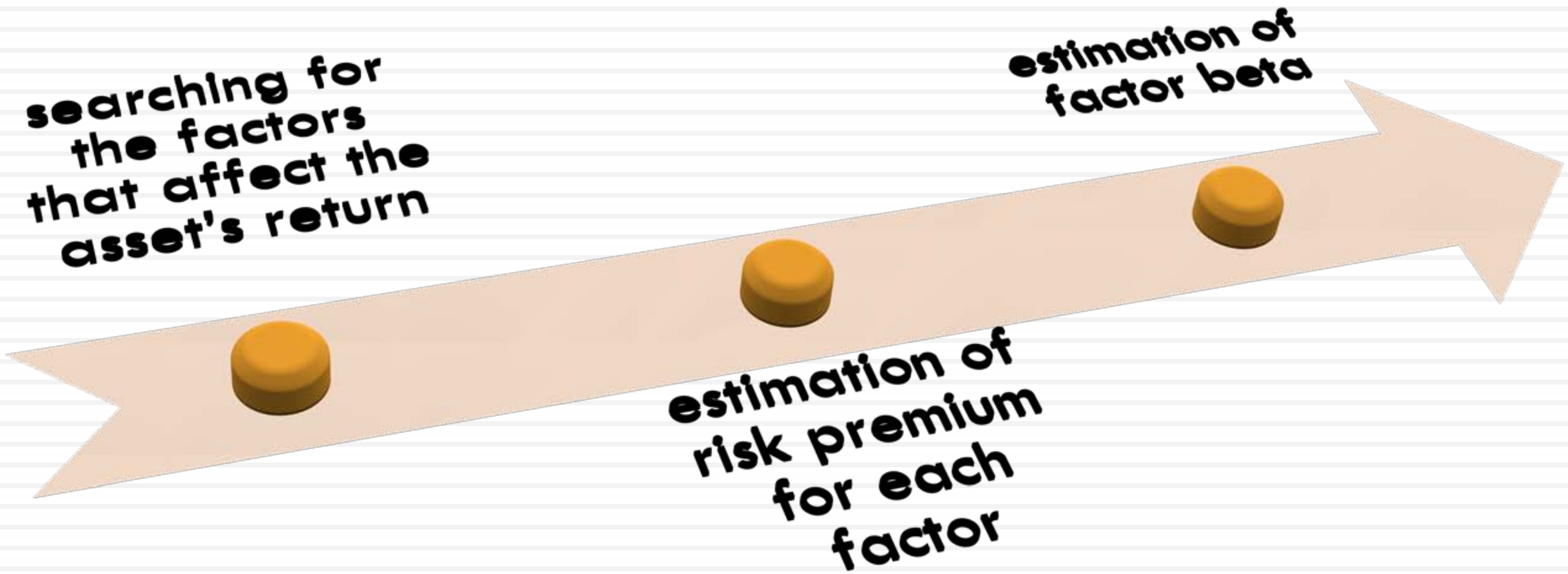
Steps in Calculating Expected Return under APT



Factors

Industrial production

Changes in default premium

Changes in the structure of interest rates

Inflation rate

Changes in the real rate of return

Risk premium

- Conceptually, it is the compensation, over and above, the risk-free rate of return that investors require for the risk contributed by the factor.

- One could use past data on the forecasted and actual values to determine the premium.

Factor beta

- The beta of the factor is the sensitivity of the asset's return to the changes in the factor.

- One can use regression approach to calculate the factor beta.