INTRODUCTION TO FUZZY LOGIC, CLASSICAL SETS AND FUZZY SETS

FUZZY LOGIC

- Fuzzy logic is a way for getting computers to make decision like human.

- It uses fuzzy sets and fuzzy rules to model the world and to make decision about it.

FUZZY LOGIC

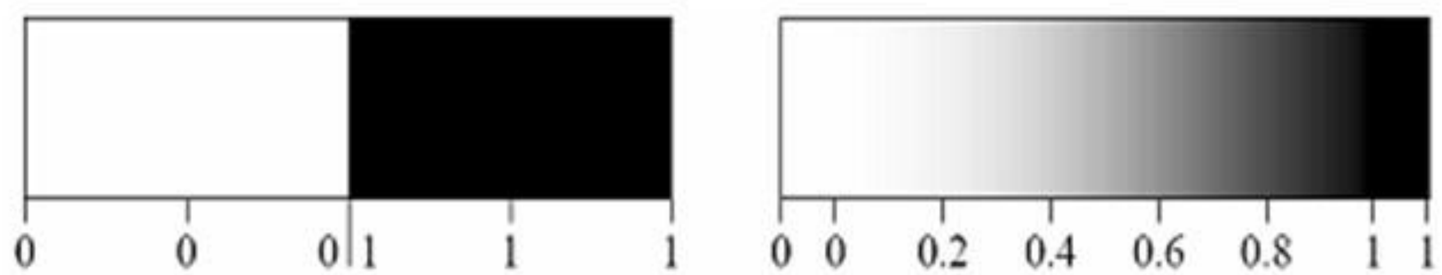
- Fuzzy logic is **the logic** underlying **approximate**, rather than exact, **modes of reasoning**.

- It is an extension of multivalued logic: **Everything**, including truth, **is a matter of degree**.

- It contains as special cases **not only** the classical two-value logic and multivalue logic systems, **but also** probabilistic logic.

- A proposition ***p*** has a **truth value**
  - 0 or 1 in two-value system,
  - element of a set T in multivalue system,
  - **Range over the fuzzy subsets of T** in fuzzy logic.

- Boolean logic uses sharp distinctions.
- Fuzzy logic reflects how people think.



(*a*) Boolean Logic. (*b*) Multi-valued Logic.

- Fuzzy logic is a set of mathematical principles for knowledge representation and reasoning based on degrees of membership.

TYPES AND MODELING OF UNCERTAINTY

Stochastic Uncertainty:

- The probability of hitting the target is 0.8

Lexical Uncertainty:

- "Tall Men", "Hot Days", or "Stable Currencies"
- We will probably have a successful business year.
- The experience of expert A shows that B is Likely to Occur. However, expert C is convinced This Is Not True.

Example: One finds in desert two bottles of fluids with the following labels:

- ✓ bottle 1: there is a probability of 5% that this bottle is poisoned.
- ✓ bottle 2: this bottle contains a liquid which belongs to the set of drinkable water with membership function value of 0.95.

FUZZY vs PROBABILITY

- Fuzzy ≠ Probability
- Probability deals with uncertainty and likelihood
- Fuzzy logic deals with ambiguity and vagueness

- Let L=set of all liquids
  - £ be the subset ={all drinkable liquids}
- Suppose you had been in desert (you must drink!) and you come up with two bottles marked C and A.
- **Bottle C is labeled $\mu_{£}(C)=0.95$ and bottle A is labeled $Pr[A \in £]=0.95$**

• C could contain swamp water, but would not contain any poison. Membership of 0.95 means that the contents of C are fairly similar to perfectly drinkable water.

• The probability that A is drinkable is 0.95, means that over a long run of experiments, the context of A are expected to be drinkable in about 95% of the trials. In other cases it may contain poison.

NEED OF FUZZY LOGIC

- Based on intuition and judgment.
- Fulfill the need for a mathematical model.
- Provides a smooth transition between members and nonmembers.
- Relatively simple, fast and adaptive.
- Less sensitive to system fluctuations.
- Can implement design objectives which are difficult to express mathematically, in linguistic or descriptive rules.

CLASSICAL SETS (CRISP SETS)

Conventional or crisp sets are Binary. An element either belongs to the set or does not.

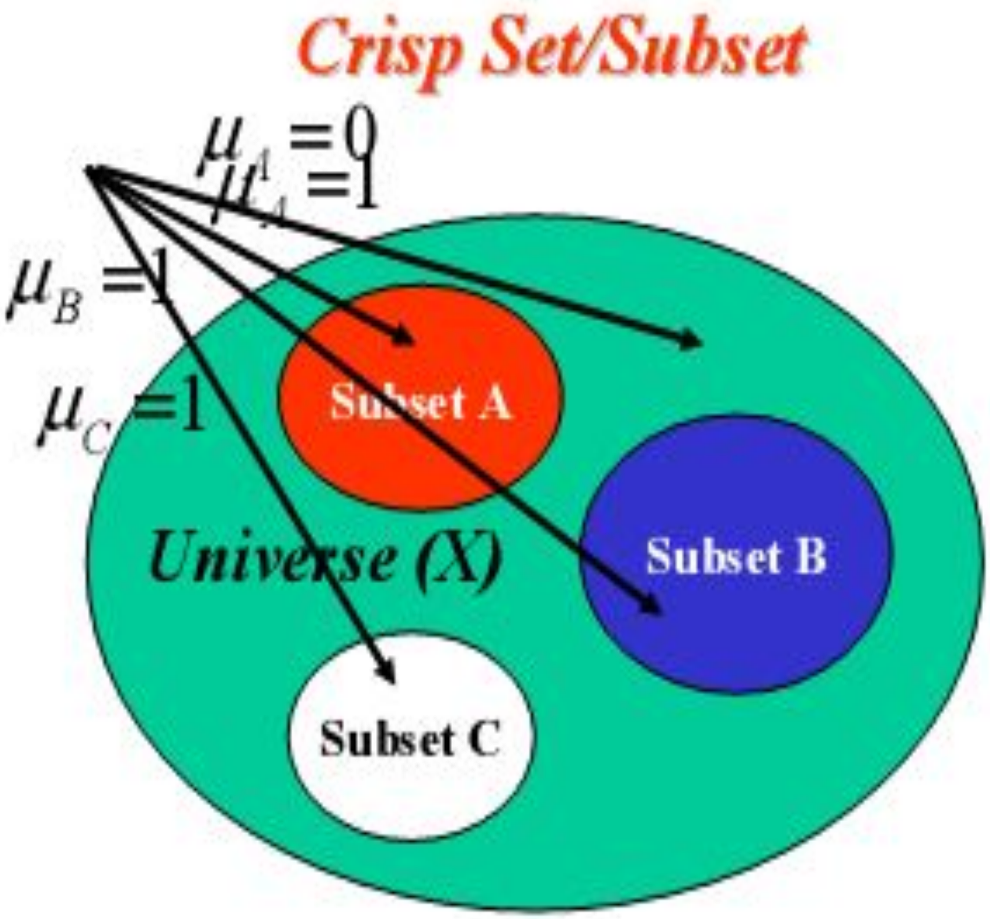
{True, False}

{1, 0}

Set of negative integeres

Set of students with grade above A

CRISP SETS



Crisp Set/Subset
$\mu_A = 0$
$\mu_A = 1$
$\mu_B = 1$
$\mu_C = 1$
Subset A
Universe (X)
Subset B
Subset C

OPERATIONS ON CRISP SETS

- UNION: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- INTERSECTION: $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- COMPLEMENT: $\bar{A} = \{x | x \notin A, x \in X\}$
- DIFFERENCE:

$$A|B \text{ or } (A - B) = \{x | x \in A \text{ and } x \notin B\}$$
$$= A - (A \cap B)$$

PROPERTIES OF CRISP SETS

The various properties of crisp sets are as follows:

1. Commutativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Idempotency

$$A \cup A = A$$
$$A \cap A = A$$

5. Transitivity

$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

6. Identity

$$A \cup \phi = A, \quad A \cap \phi = \phi$$
$$A \cap X = A, \quad A \cup X = X$$

7. Involution (double negation)

$$\bar{\bar{A}} = A$$

8. Law of excluded middle

$$A \cup \bar{A} = X$$

9. Law of contradiction

$$A \cap \bar{A} = \phi$$

10. DeMorgan's law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Crisp set vs. fuzzy set

- The crisp set is defined in such a way as to partition the individuals in some given universe of discourse into two groups: members and nonmembers.
  - However, many classification concepts do not exhibit this characteristic.
  - For example, the set of tall people, expensive cars, or sunny days.

For example: the weather today

Sunny: If we define any cloud cover of 25% or less is sunny.
This means that a cloud cover of 26% is not sunny?
“Vagueness” should be introduced.

- A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.
  - For example: a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0%, 0.8 to a cloud cover of 20%, 0.4 to a cloud cover of 30%, and 0 to a cloud cover of 75%.

Fuzzy logic : it uses linguistic terms which humans often use to describe their actions. E.g. John is tall.

Tall

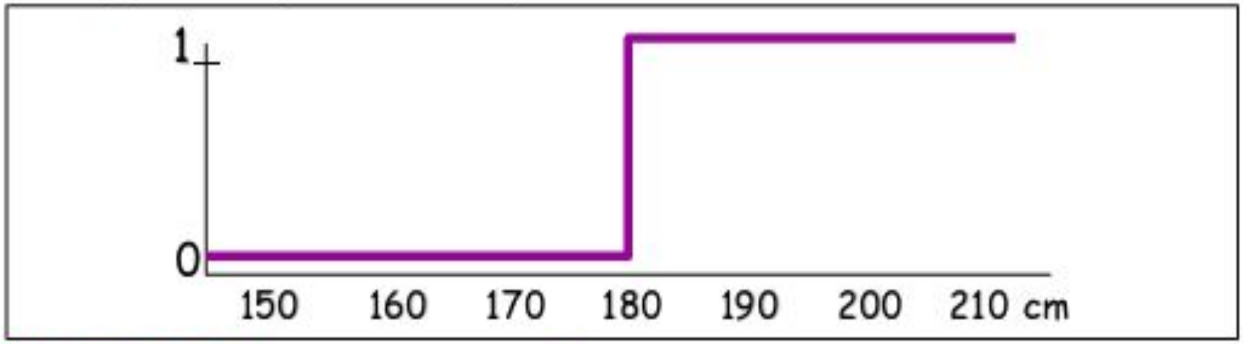
Tall or Short?

Short

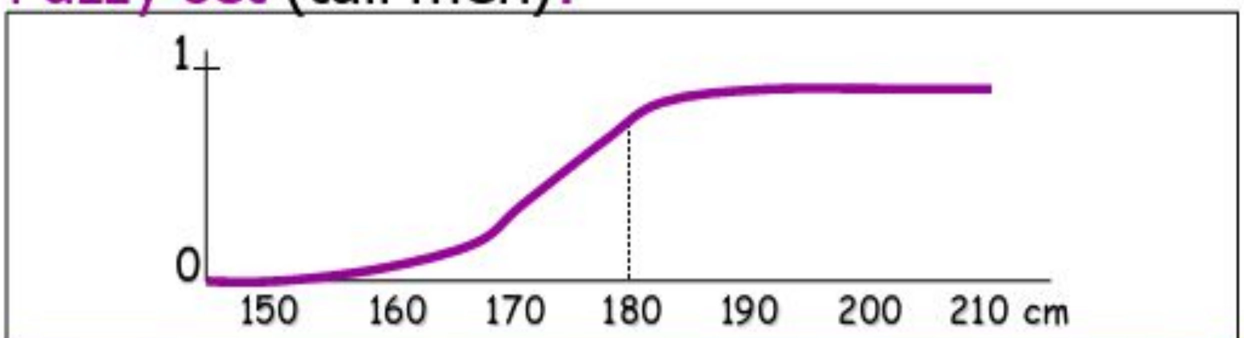
Fuzzy set

- Is a function f: domain → [0,1]

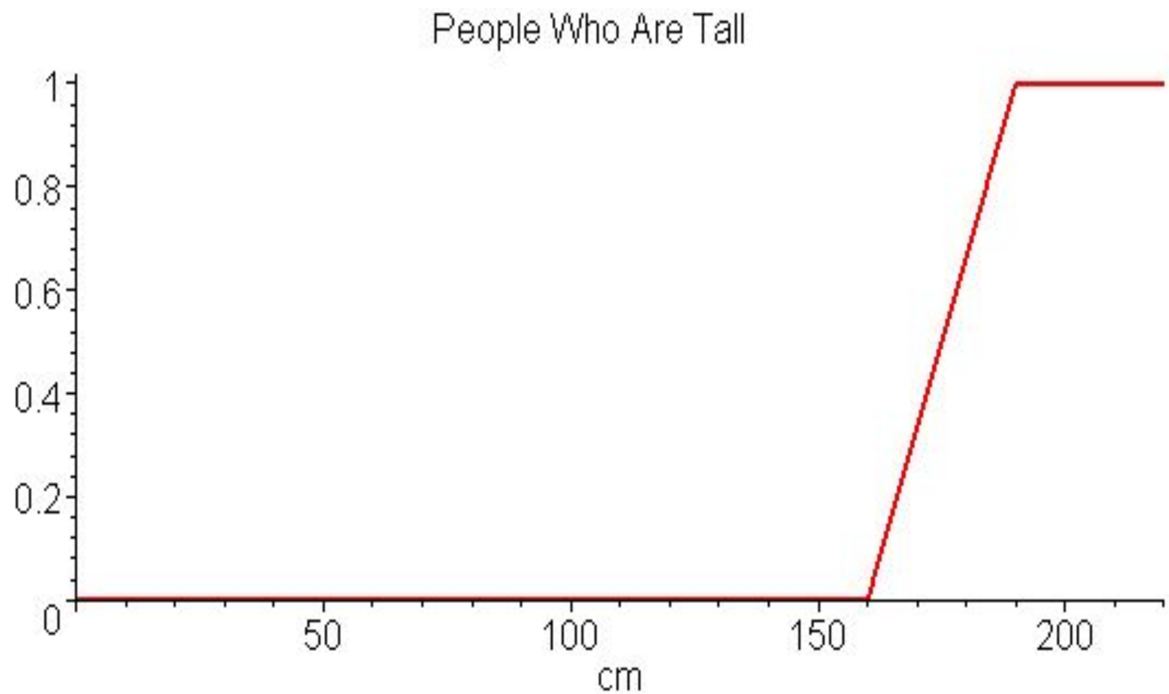
Crisp set (tall men):



Fuzzy set (tall men):



Someone over 190 cm is almost universally considered to be tall. Someone who is 180 cm may be considered to be *sort of tall* , while someone who is under 160 cm is not usually considered to be tall.


People Who Are Tall
1
0.8
0.6
0.4
0.2
0
50
100
150
200
cm

Fuzzy sets

- A membership function:
  - A characteristic function: the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set.
  - Larger values denote higher degrees of set membership.
- A set defined by membership functions is a fuzzy set.
- The most commonly used range of values of membership functions is the unit interval [0,1].
- The universal set X is always a crisp set.
- Notation:
  - The membership function of a fuzzy set A is denoted by μ_A: $\mu_A : X \rightarrow [0,1]$

$0 \leq \mu S(x) \leq 1$ ------ $\mu S(x)$ (or $\mu(S, x)$) is the **degree** of membership of x in set S

$\mu S(x) = 0$ x is not at all in S

$\mu S(x) = 1$ x is fully in S.

If $\mu S(x) = 0$ or 1, then the set S is **crisp**.

OPERATIONS ON FUZZY SETS

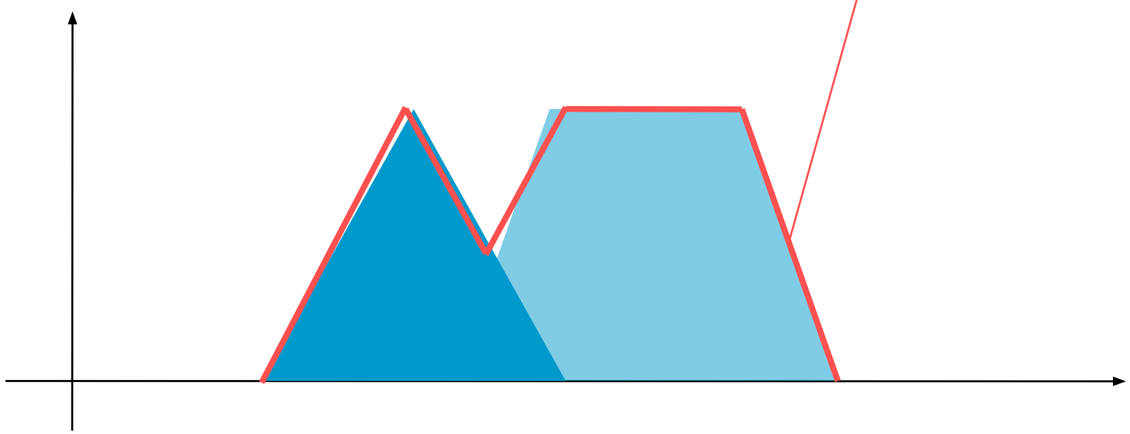
- Union: $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
- Intersection: $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
- Complement: $\mu_{\neg A}(x) = 1 - \mu_A(x)$

Fuzzy Union

- Union: The union the two sets *A* and *B* (*A*∪*B*) can be defined by the membership function $\mu_U(x)$

Union: $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

$\mu_U(x) = \max(\mu_A(x), \mu_B(x)), \ x \in X$



Fuzzy Union

- Fuzzy union ($\cup$): the union of two fuzzy sets is the maximum (MAX) of each element from two sets.
- E.g.
  - A = {1.0, 0.20, 0.75}
  - B = {0.2, 0.45, 0.50}
  - A $\cup$ B = {MAX(1.0, 0.2), MAX(0.20, 0.45), MAX(0.75, 0.50)}

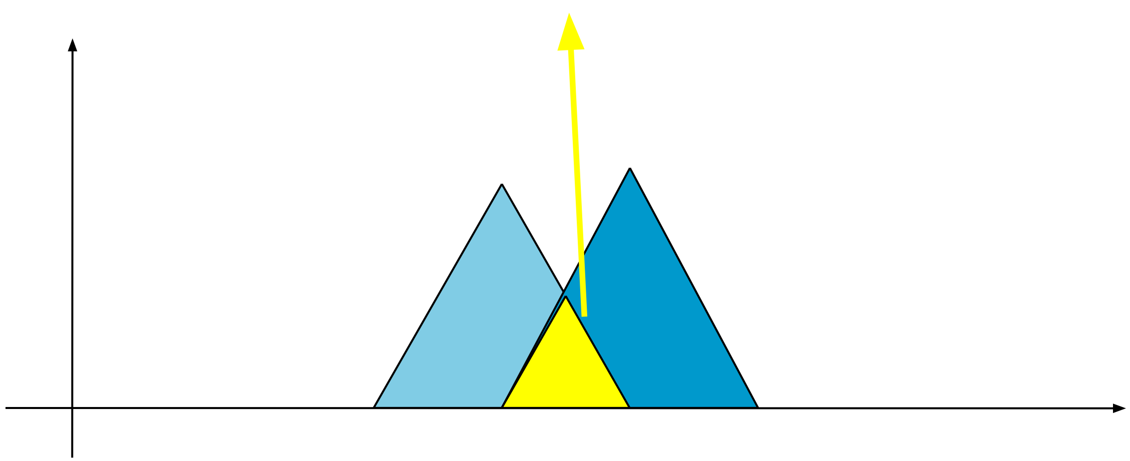
    = {1.0, 0.45, 0.75}

Intersection

- Intersection: the intersection of two sets *A* and *B* ($A \cap B$) can be defined by the membership function $\mu_{\cap}(x)$

Intersection: $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

$\mu_{\cap}(x)=\min(\mu_A(x),\mu_B(x)),\ x \varepsilon X$

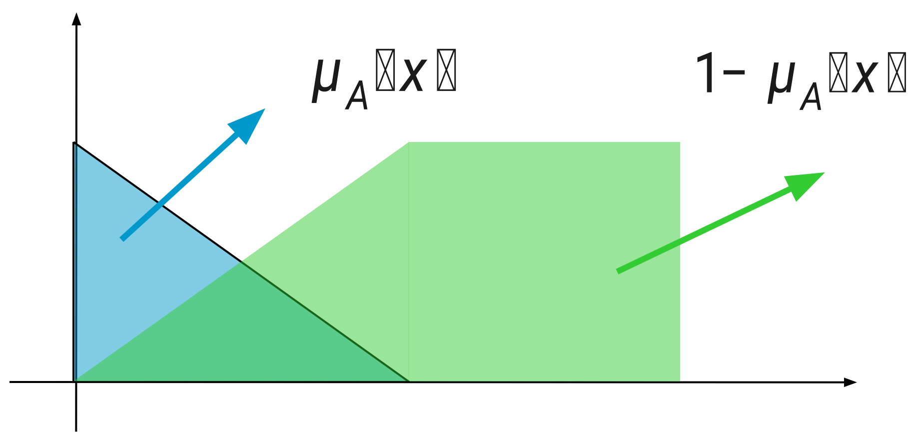


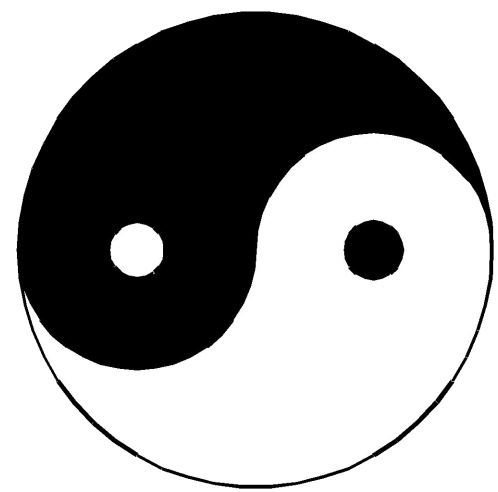
- Fuzzy intersection (∩): the intersection of two fuzzy sets is just the MIN of each element from the two sets.
- E.g.
  - $A \cap B = \{MIN(1.0, 0.2), MIN(0.20, 0.45), MIN(0.75, 0.50)\} = \{0.2, 0.20, 0.50\}$

Complement of a Fuzzy Set

- Complement: the complement of a fuzzy set *A* can be defined by the membership function $\mu_C(x)$

Complement: $\mu_{\neg A}(x) = 1 - \mu_A(x)$





Fuzzy Complement

- The *complement* of a fuzzy variable with DOM *x* is (1-x).
- Complement ($\_^c$): The *complement* of a fuzzy set is composed of all elements' *complement.*
- Example.
  - $A^c = \{1 – 1.0, 1 – 0.2, 1 – 0.75\} = \{0.0, 0.8, 0.25\}$

Operations on Fuzzy Sets: Intersection

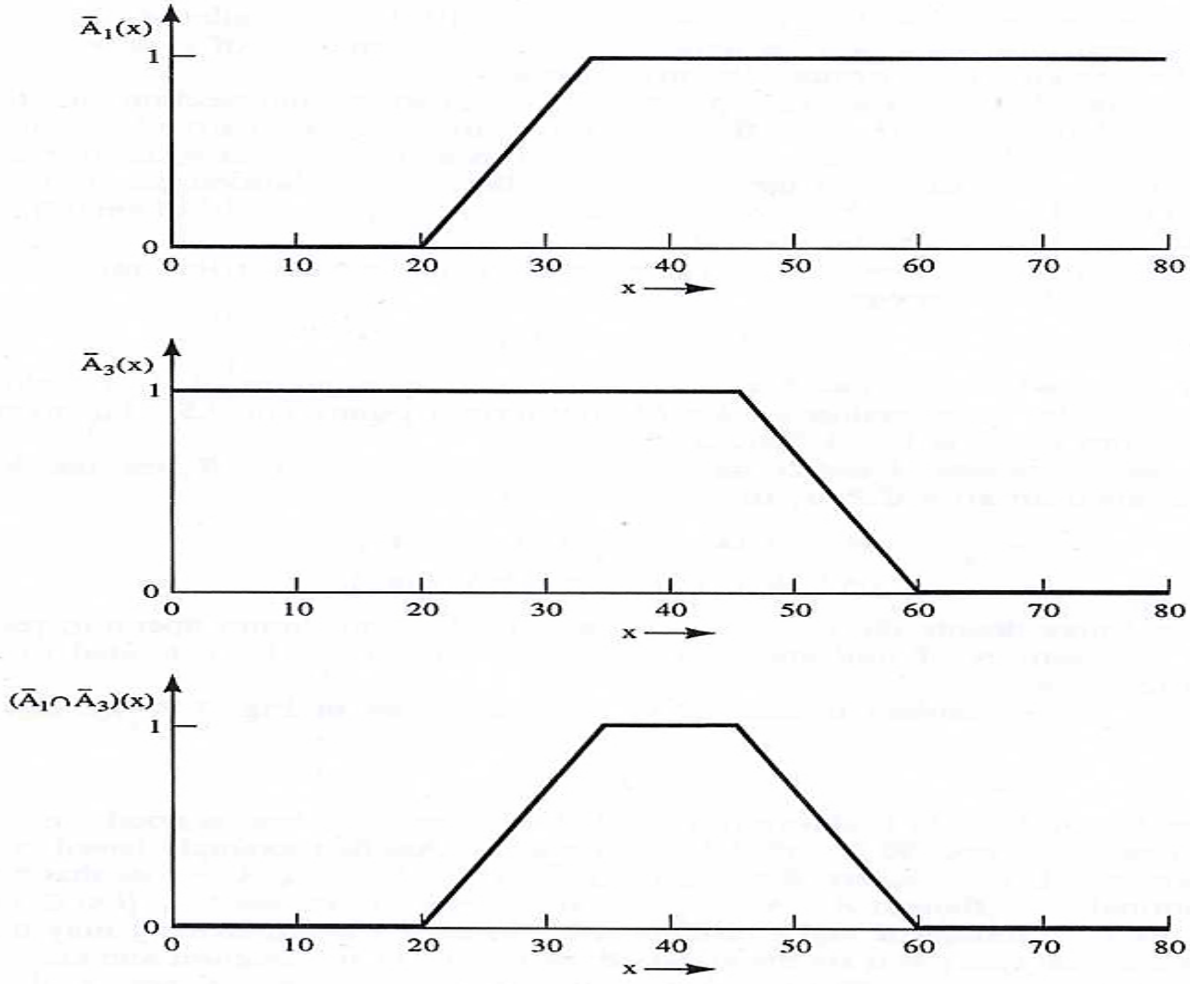


Illustration of standard operations on fuzzy sets (A_1, A_2, A_3).

Operations on Fuzzy Sets: Union and Complement

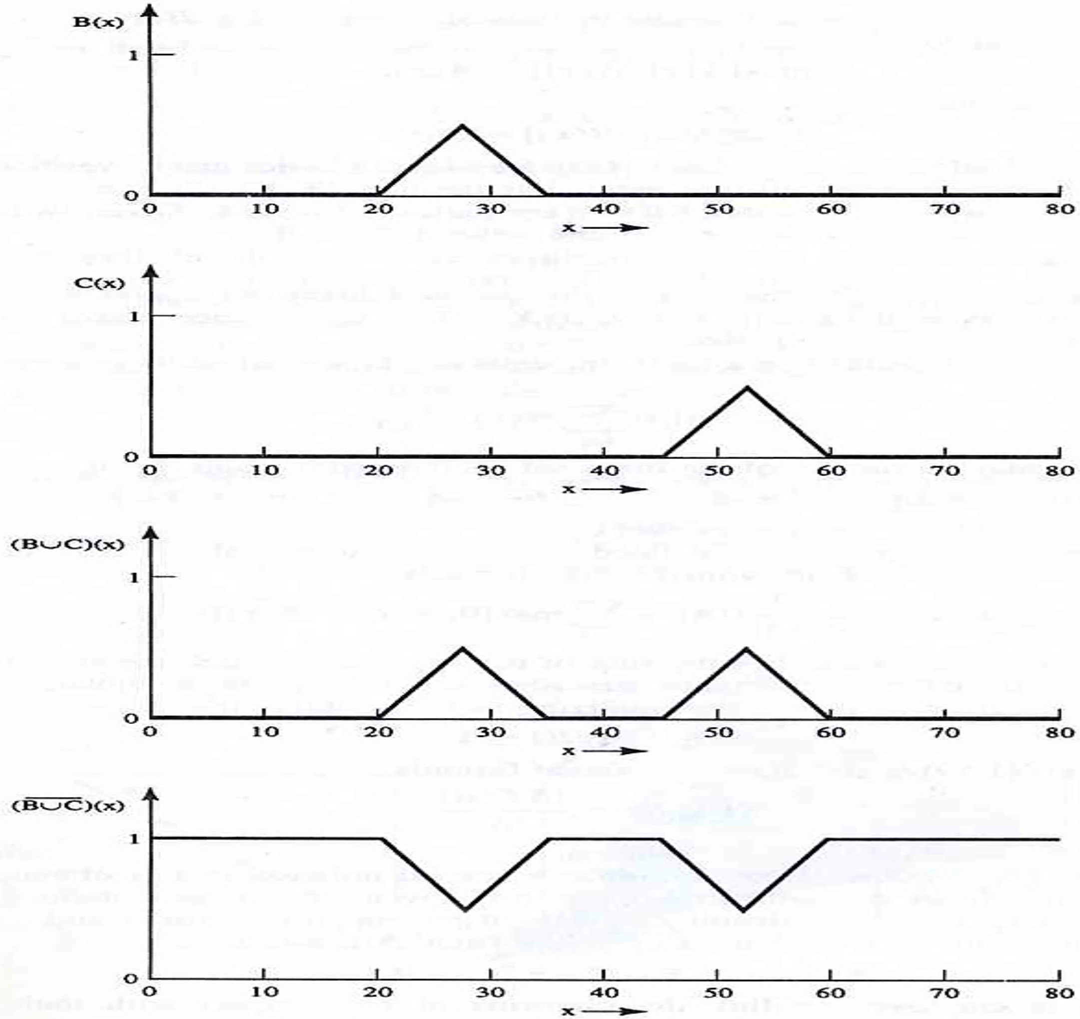


Illustration of standard operation on fuzzy sets $B = A_1 \cap A_2$ and $C = A_2 \cap A_3$

PROPERTIES OF FUZZY SETS

The same as for crisp sets

- Commutativity
- Associativity
- Distributivity
- Idempotency
- Identity
- De Morgan's Laws
- …

Except for the law of excluded middle and law of contradiction
For fuzzy set A

$$A \cup A' \neq U$$

$$A \cap A' \neq \emptyset$$

1. Commutativity

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

2. Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

3. Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

4. Idempotency

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

5. Identity

$$\underset{\sim}{A} \cup \phi = \underset{\sim}{A} \text{ and } \underset{\sim}{A} \cup U = U \text{(universal set)}$$
$$\underset{\sim}{A} \cap \phi = \phi \text{ and } \underset{\sim}{A} \cap U = \underset{\sim}{A}$$

6. Involution (double negation)

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

7. Transitivity

$$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

8. Demorgan's law

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$$
$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$$

SUMMARY

- The basic concepts of fuzzy logic has been discussed.
- An introduction to crisp sets and fuzzy sets has been included.
- The operations and properties of crisp sets and fuzzy sets are also dealt with.