FUZZY RULE BASE

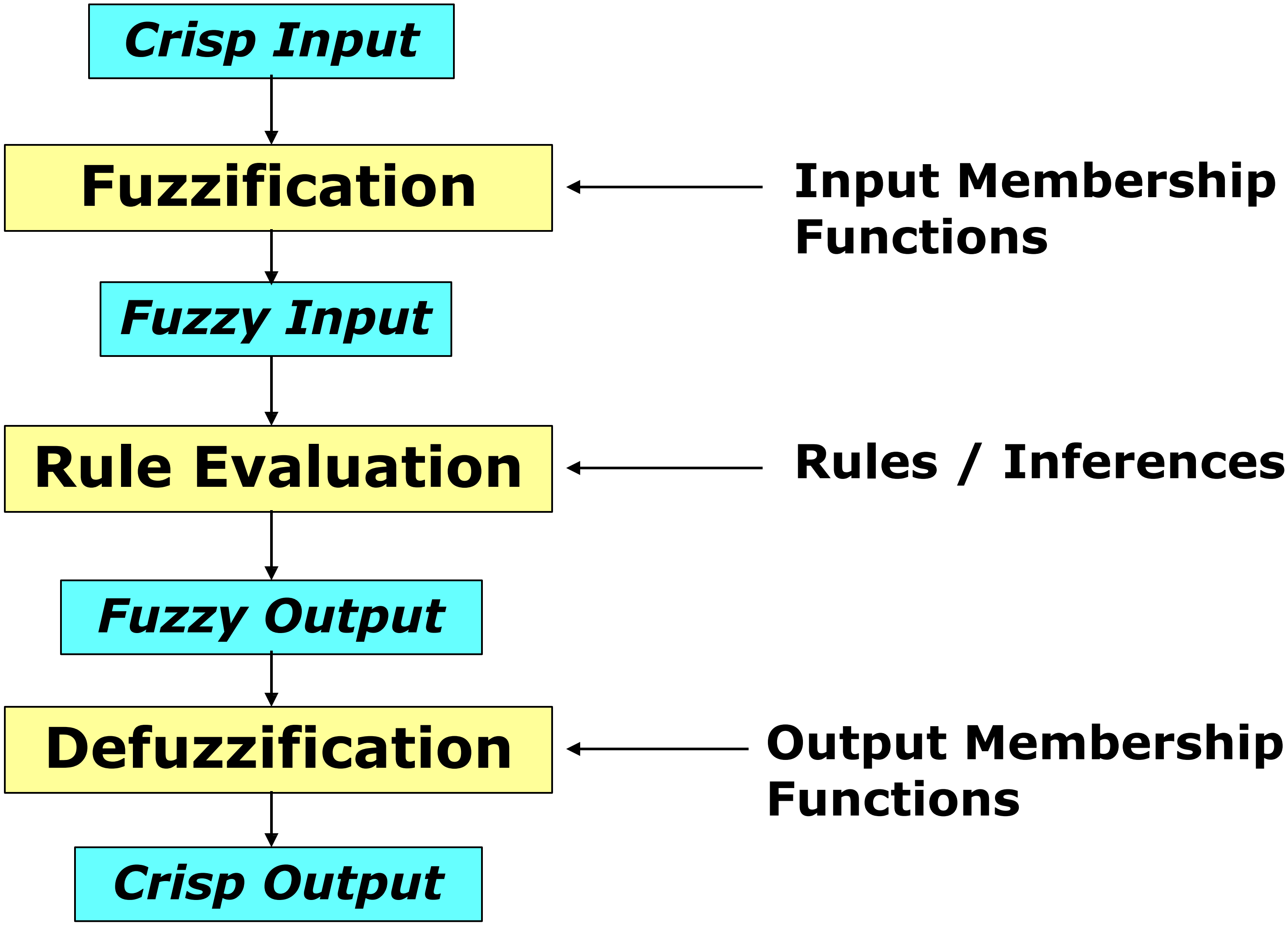
Fuzzy Logic Process

- It is one thing to compute, to reason, and to model with fuzzy information; it is another thing to apply the fuzzy results to the world around us.
- Despite the fact that the bulk of the information we assimilate every day is fuzzy, most of the actions or decisions implemented by humans or machines are crisp or binary.
- The decisions we make that require an action are binary, the hardware we use is binary, and certainly the computers we use are based on binary digital instructions.

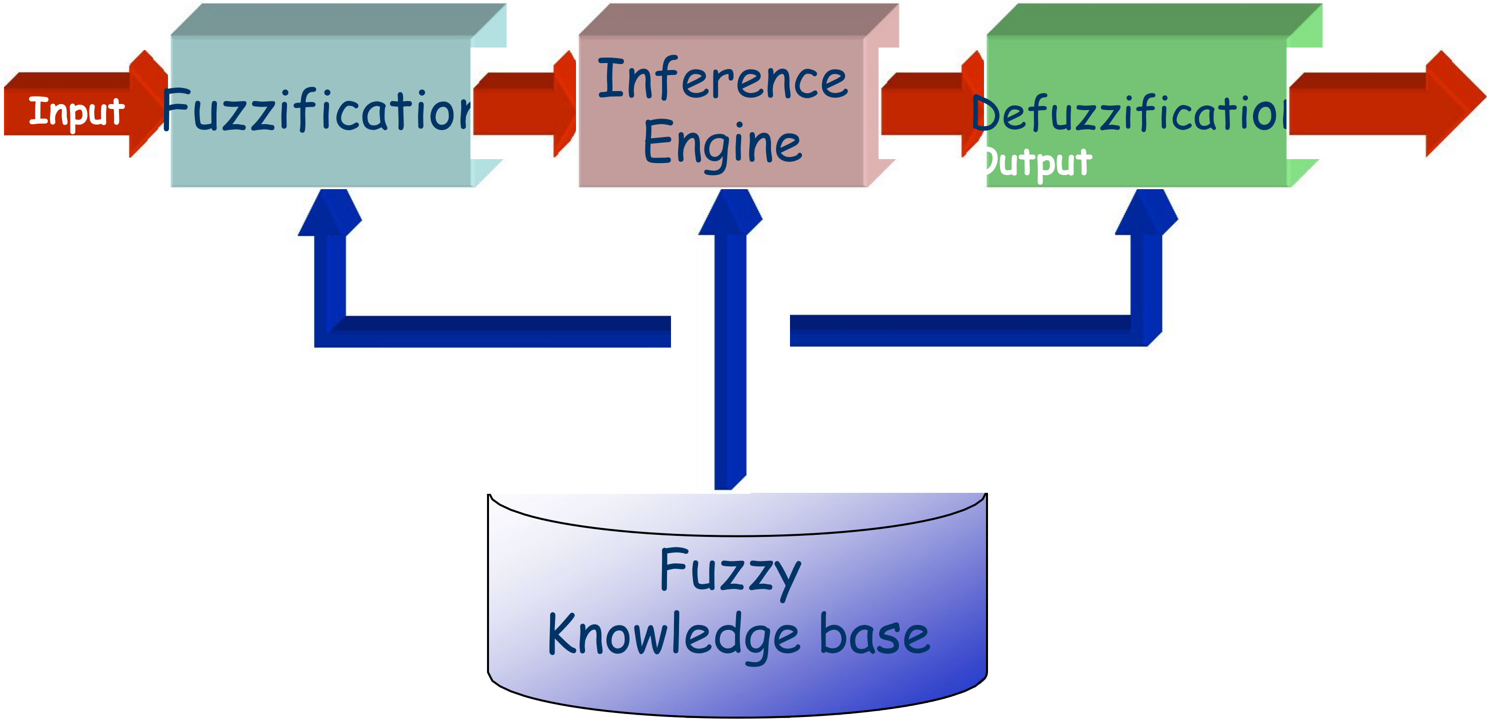
FUZZY LOGIC PROCESS

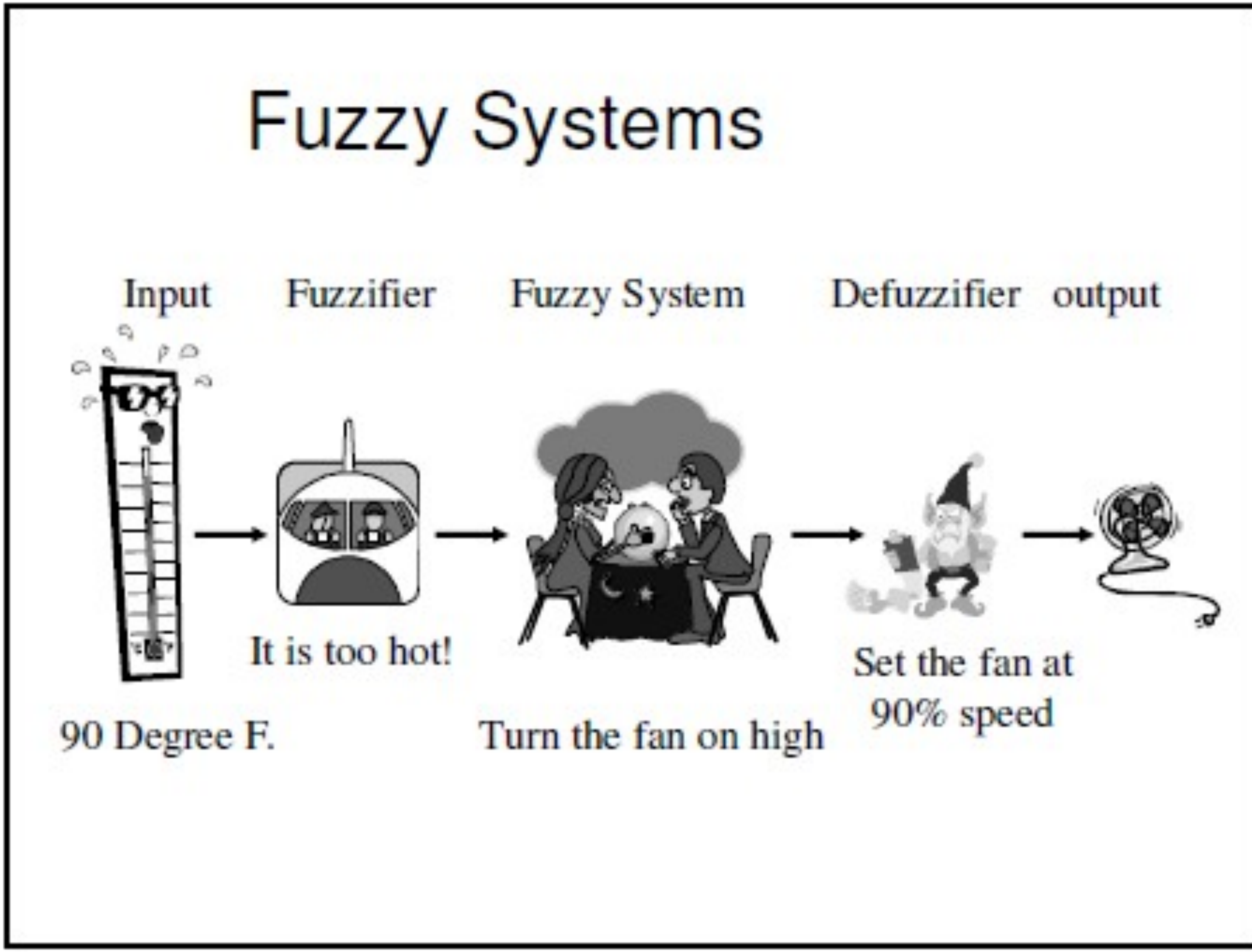
In giving instructions to an aircraft autopilot, it is not possible to turn the plane "slightly to the west"; an autopilot device does not understand the natural language of a human. We have to turn the plane by 15°, for example, a crisp number.

Operation of Fuzzy System



Fuzzy Systems




Fuzzy Systems
Input
Fuzzifier
Fuzzy System
Defuzzifier
output
It is too hot!
Set the fan at
90% speed
90 Degree F.
Turn the fan on high

FUZZY RULES AND REASONING

- Rules form the basis for the fuzzy logic to obtain the fuzzy output
- Rule-based system vs. Expert system
  - The rules comprising the rule-based system originates from sources other than that of human experts
- The rule-based form uses linguistic variables as its antecedents and consequents
- The antecedents express an inference or the inequality, which should be satisfied
- The consequents are those, which we can infer, and is the output if the antecedent inequality is satisfied
- The fuzzy rule-based system uses IF−THEN rule-based system, given by, *IF antecedent, THEN consequent*.

Classical propositions

- A simple proposition P is a linguistic, or declarative, statement
- Contained within a universe of elements, say X, that can be identified as being a collection of elements in X.
- The truth value of a proposition is the opposite of the truth value of its negation
- Truth Tables
  - Truth tables define logic functions of two propositions.
- The operations over the propositions
  - Let X and Y be two propositions, either of which can be true or false, and can be combined using the following five logical connectives
    1. Conjunction ( ): $\wedge$ X AND Y.
    2. Disjunction ( ): $\vee$ X OR Y. Negation
    3. (−): NOT X Implication or
    4. conditional ( ): $\Rightarrow$ IF X THEN Y.
    5. Bidirectional or equivalence ($\Leftrightarrow$): X IF AND ONLY IF Y.

Operations Over The Classical Propositions

Given a proposition $\mathrm{P} : x \in \mathrm{A}, \overline{\mathrm{P}} : x \notin \mathrm{A}$, we have the following for the logical connectives:

Disjunction

$$\begin{aligned}&\mathrm{P} \vee \mathrm{Q} : x \in \mathrm{A} \text{ or } x \in \mathrm{B};\\&\text{hence, } T(\mathrm{P} \vee \mathrm{Q}) = \max(T(\mathrm{P}), T(\mathrm{Q})).\end{aligned}$$

Conjunction

$$\begin{aligned}&\mathrm{P} \wedge \mathrm{Q} : x \in \mathrm{A} \text{ and } x \in \mathrm{B};\\&\text{hence, } T(\mathrm{P} \wedge \mathrm{Q}) = \min(T(\mathrm{P}), T(\mathrm{Q})).\end{aligned}$$

Negation

$$\text{If } T(\mathrm{P}) = 1, \text{ then } T(\overline{\mathrm{P}}) = 0; \text{ if } T(\mathrm{P}) = 0, \text{ then } T(\overline{\mathrm{P}}) = 1.$$

Implication

$$\begin{aligned}&(\mathrm{P} \longrightarrow \mathrm{Q}) : x \notin \mathrm{A} \text{ or } x \in \mathrm{B};\\&\text{hence, } T(\mathrm{P} \longrightarrow \mathrm{Q}) = T(\overline{\mathrm{P}} \cup \mathrm{Q}).\end{aligned}$$

Equivalence

$$(\mathrm{P} \longleftrightarrow \mathrm{Q}) : T(\mathrm{P} \longleftrightarrow \mathrm{Q}) = \begin{cases}1, & \text{for } T(\mathrm{P}) = T(\mathrm{Q})\\ 0, & \text{for } T(\mathrm{P}) \neq T(\mathrm{Q})\end{cases}.$$

Classical propositions

Truth table for various compound propositions.

P	Q	$\overline{P}$	$P \vee Q$	$P \wedge Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T (1)	T (1)	F (0)	T (1)	T (1)	T (1)	T (1)
T (1)	F (0)	F (0)	T (1)	F (0)	F (0)	F (0)
F (0)	T (1)	T (1)	T (1)	F (0)	T (1)	F (0)
F (0)	F (0)	T (1)	F (0)	F (0)	T (1)	T (1)

Fuzzy propositions

- The range of truth values of fuzzy propositions is not only {0, 1}, but [0, 1].
- The truth of a fuzzy proposition is a matter of degree.
- Fuzzy propositions are assigned to fuzzy sets.
- The logical connectives of negation, disjunction, conjunction, and implication are also defined for a fuzzy logic.

Negation

$$T(\overline{\underset{\sim}{P}}) = 1 - T(\underset{\sim}{P}).$$

Disjunction

$$\underset{\sim}{P} \vee \underset{\sim}{Q} : x \text{ is } \underset{\sim}{A} \text{ or } \underset{\sim}{B} \quad T(\underset{\sim}{P} \vee \underset{\sim}{Q}) = \max(T(\underset{\sim}{P}), T(\underset{\sim}{Q})).$$

Conjunction

$$\underset{\sim}{P} \wedge \underset{\sim}{Q} : x \text{ is } \underset{\sim}{A} \text{ and } \underset{\sim}{B} \quad T(\underset{\sim}{P} \wedge \underset{\sim}{Q}) = \min(T(\underset{\sim}{P}), T(\underset{\sim}{Q})).$$

Implication (Zadeh, 1973)

$$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} : x \text{ is } \underset{\sim}{A}, \text{ then } x \text{ is } \underset{\sim}{B}$$

$$T(\underset{\sim}{P} \rightarrow \underset{\sim}{Q}) = T(\overline{\underset{\sim}{P}} \vee \underset{\sim}{Q}) = \max(T(\overline{\underset{\sim}{P}}), T(\underset{\sim}{Q})).$$

Fuzzy Logic

the implication connective can be modeled in rule-based form;

$\underset{\sim}{P} \rightarrow \underset{\sim}{Q}$ is IF x is $\underset{\sim}{A}$, THEN y is $\underset{\sim}{B}$ and it is equivalent to the fuzzy relation $\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$

The membership function of $\underset{\sim}{R}$ is expressed by the following formula:

$$\mu_{\underset{\sim}{R}}(x, y) = \max[(\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)), (1 - \mu_{\underset{\sim}{A}}(x))].$$

When the logical conditional implication is of the compound form

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B}, \text{ ELSE } y \text{ is } \underset{\sim}{C},$$

then the equivalent fuzzy relation, $\underset{\sim}{R}$, is expressed as $\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times \underset{\sim}{C})$,

$$\mu_{\underset{\sim}{R}}(x, y) = \max[(\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)), ((1 - \mu_{\underset{\sim}{A}}(x)) \wedge \mu_{\underset{\sim}{C}}(y))].$$

Fuzzy Logic

Example: $X = \{1, 2, 3, 4\}$ $Y = \{1, 2, 3, 4, 5, 6\}$

$$\underset{\sim}{A} = \text{medium uniqueness} = \left\{\frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4}\right\}.$$

$$\underset{\sim}{B} = \text{medium market size} = \left\{\frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5}\right\}.$$

$$\underset{\sim}{C} = \text{diffuse market size} = \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6}\right\}.$$

Approximate reasoning

Rule 1: IF x is $\underset{\sim}{A}$, THEN y is $\underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ represent fuzzy propositions (sets).

Now suppose we introduce a new antecedent, say $\underset{\sim}{A}'$, and we consider the following rule:

Rule 2: IF x is $\underset{\sim}{A}'$, THEN y is $\underset{\sim}{B}'$.

From information derived from Rule 1, is it possible to derive the consequent in Rule 2, $\underset{\sim}{B}'$?

The answer is yes, and the procedure is fuzzy composition. The consequent $\underset{\sim}{B}'$ can be found from the composition operation, $\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$.

Example 5.9. Suppose we are evaluating a new invention to determine its commercial potential. We will use two metrics to make our decisions regarding the innovation of the idea. Our metrics are the "uniqueness" of the invention, denoted by a universe of novelty scales, $X = \{1, 2, 3, 4\}$, and the "market size" of the invention's commercial market, denoted on a universe of scaled market sizes, $Y = \{1, 2, 3, 4, 5, 6\}$. In both universes, the lowest numbers are the "highest uniqueness" and the "largest market," respectively. A new invention in your group, say a compressible liquid of very useful temperature and viscosity conditions, has just received scores of "medium uniqueness," denoted by fuzzy set $\underset{\sim}{A}$, and "medium market size," denoted fuzzy set $\underset{\sim}{B}$. We wish to determine the implication of such a result, that is, IF $\underset{\sim}{A}$, THEN $\underset{\sim}{B}$. We assign the invention the following fuzzy sets to represent its ratings:

$$\underset{\sim}{A} = \text{medium uniqueness} = \left\{\frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4}\right\}.$$

$$\underset{\sim}{B} = \text{medium market size} = \left\{\frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5}\right\}.$$

$$\underset{\sim}{C} = \text{diffuse market size} = \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6}\right\}.$$

$$\underset{\sim}{R} = \max(\underset{\sim}{A} \times \underset{\sim}{B}, \overline{\underset{\sim}{A}} \times Y)$$

$$
\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{bmatrix} \end{array},
$$

$$
\overline{\underset{\sim}{A}} \times Y = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array},
$$

$$
\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}.
$$

relation just developed, that is, $\underset{\sim}{R}$, describes the invention's commercial potential. We wish to know what market size would be associated with a uniqueness score of "almost high uniqueness." That is, with a new antecedent, $\underset{\sim}{A}'$, the consequent, $\underset{\sim}{B}'$, can be determined using composition. Let

$$\underset{\sim}{A}' = \text{almost high uniqueness} = \left\{ \frac{0.5}{1} + \frac{1}{2} + \frac{0.3}{3} + \frac{0}{4} \right\}.$$

Then, using the following max–min composition

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R} = \left\{ \frac{0.5}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.5}{6} \right\},$$

we get the fuzzy set describing the associated market size. In other words, the consequent is fairly diffuse, where there is no strong (or weak) membership value for any of the market size scores (i.e., no membership values near 0 or 1).

Example 5.11. For research on the human visual system, it is sometimes necessary to characterize the strength of response to a visual stimulus based on a magnetic field measurement or on an electrical potential measurement. When using magnetic field measurements, a typical experiment will require nearly 100 off/on presentations of the stimulus at one location to obtain useful data. If the researcher is attempting to map the visual cortex of the brain, several stimulus locations must be used in the experiments. When working with a new subject, a researcher will make preliminary measurements to determine if the type of stimulus being used evokes a good response in the subject. The magnetic measurements are in units of femtotesla (10^{-15} tesla). Therefore, the inputs and outputs are both measured in terms of magnetic units.

We will define inputs on the universe X = [0, 50, 100, 150, 200] femtotesla, and outputs on the universe Y = [0, 50, 100, 150, 200] femtotesla. We will define two fuzzy sets, two different stimuli, on universe X:

$$\underset{\sim}{W} = \text{“weak stimulus”} = \left\{\frac{1}{0} + \frac{0.9}{50} + \frac{0.3}{100} + \frac{0}{150} + \frac{0}{200}\right\} \subset X.$$

$$\underset{\sim}{M} = \text{“medium stimulus”} = \left\{\frac{0}{0} + \frac{0.4}{50} + \frac{1}{100} + \frac{0.4}{150} + \frac{0}{200}\right\} \subset X.$$

and one fuzzy set on the output universe Y,

$$\underset{\sim}{S} = \text{“severe response”} = \left\{\frac{0}{0} + \frac{0}{50} + \frac{0.5}{100} + \frac{0.9}{150} + \frac{1}{200}\right\} \subset Y.$$

We will construct the proposition IF "weak stimulus" THEN not "severe response," using classical implication.

$$\text{IF } \underset{\sim}{W} \text{ THEN } \overline{\underset{\sim}{S}} = \underset{\sim}{W} \rightarrow \overline{\underset{\sim}{S}} = (\underset{\sim}{W} \times \overline{\underset{\sim}{S}}) \cup (\overline{\underset{\sim}{W}} \times Y).$$

The complement of $\underset{\sim}{S}$ will then be

$$\overline{\underset{\sim}{S}} = \left\{ \frac{1}{0} + \frac{1}{50} + \frac{0.5}{100} + \frac{0.1}{150} + \frac{0}{200} \right\}.$$

$$\underset{\sim}{W} \times \overline{\underset{\sim}{S}} = \begin{bmatrix} 1 \\ 0.9 \\ 0.3 \\ 0 \\ 0 \end{bmatrix} [1\ \ 1\ \ 0.5\ \ 0.1\ \ 0] = \begin{array}{c c} & \begin{array}{ccccc} 0 & 50 & 100 & 150 & 200 \end{array} \\ \begin{array}{c} 0 \\ 50 \\ 100 \\ 150 \\ 200 \end{array} & \begin{bmatrix} 1 & 1 & 0.5 & 0.1 & 0 \\ 0.9 & 0.9 & 0.5 & 0.1 & 0 \\ 0.3 & 0.3 & 0.3 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$\overline{\underset{\sim}{W}} \times Y = \begin{bmatrix} 0 \\ .1 \\ .7 \\ 1 \\ 1 \end{bmatrix} [1\ 1\ 1\ 1\ 1] = \begin{array}{c} \\ 0 \\ 50 \\ 100 \\ 150 \\ 200 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 50 & 100 & 150 & 200 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$\underset{\sim}{R} = (\underset{\sim}{W} \times \overline{\underset{\sim}{S}}) \cup (\overline{\underset{\sim}{W}} \times Y) = \begin{array}{c} \\ 0 \\ 50 \\ 100 \\ 150 \\ 200 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 50 & 100 & 150 & 200 \end{array} \\ \begin{bmatrix} 1 & 1 & 0.5 & 0.1 & 0 \\ 0.9 & 0.9 & 0.5 & 0.1 & 0.1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

This relation $\underset{\sim}{R}$, then, expresses the knowledge embedded in the rule IF "weak stimuli" THEN not "severe response." Now, using a new antecedent (IF part) for the input $\underset{\sim}{M}$ = "medium stimuli" and a max–min composition, we can find another response on the Y universe to relate approximately to the new stimulus $\underset{\sim}{M}$, that is, to find $\underset{\sim}{M} \circ \underset{\sim}{R}$:

$$\underset{\sim}{M} \circ \underset{\sim}{R} = [0 \; 0.4 \; 1 \; 0.4 \; 0] \begin{array}{c} \begin{array}{ccccc} 0 & 50 & 100 & 150 & 200 \end{array} \\ \begin{bmatrix} 1 & 1 & 0.5 & 0.1 & 0 \\ 0.9 & 0.9 & 0.5 & 0.1 & 0.1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} = [0.7 \; 0.7 \; 0.7 \; 0.7 \; 0.7]$$

Example 5.13. Suppose you are a soils engineer and you wish to track the movement of soil particles under applied loading in an experimental apparatus that allows viewing of the soil motion. You are building pattern recognition software to enable a computer to monitor and detect the motions. However, there are some difficulties in "teaching" your software to view the motion. The tracked particle can be occluded by another particle. The occlusion can occur when a tracked particle is behind another particle, behind a mark on the camera's lens, or partially out of sight of the camera. We want to establish a relationship between particle occlusion, which is a poorly known phenomenon, and lens occlusion, which is quite well-known in photography. Let the membership functions

$$\underset{\sim}{A} = \left\{ \frac{0.1}{x_1} + \frac{0.9}{x_2} + \frac{0.0}{x_3} \right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{ \frac{0}{y_1} + \frac{1}{y_2} + \frac{0}{y_3} \right\}$$

describe fuzzy sets for a *tracked particle moderately occluded* behind another particle and a *lens mark associated with moderate image quality*, respectively. Fuzzy set $\underset{\sim}{A}$ is defined on a universe $X = \{x_1, x_2, x_3\}$ of tracked particle indicators and fuzzy set $\underset{\sim}{B}$ (note in this case that $\underset{\sim}{B}$ is a crisp singleton) is defined on a universe $Y = \{y_1, y_2, y_3\}$ of lens obstruction indices. A typical rule might be IF occlusion due to particle occlusion is moderate, THEN image quality will be similar to a moderate lens obstruction, or symbolically,

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B} \text{ or } (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y) = \underset{\sim}{R}$$

We can find the relation, $\underset{\sim}{R}$, as follows:

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \overline{\underset{\sim}{A}} \times \underset{\sim}{Y} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.9 & 0.9 & 0.9 \\ 0.1 & 0.1 & 0.1 \\ 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y) = \begin{bmatrix} 0.9 & 0.9 & 0.9 \\ 0.1 & 0.9 & 0.1 \\ 1 & 1 & 1 \end{bmatrix}.$$

This relation expresses in matrix form all the knowledge embedded in the implication. Let $\underset{\sim}{A}'$ be a fuzzy set, in which a tracked particle is behind a particle with *slightly more occlusion* than the particle expressed in the original antecedent $\underset{\sim}{A}$, which is given as

$$\underset{\sim}{A}' = \left\{ \frac{0.3}{x_1} + \frac{1.0}{x_2} + \frac{0.0}{x_3} \right\}.$$

We can find the associated membership of the image quality using max–min composition. For example, approximate reasoning will provide

$$\text{IF } x \text{ is } \underset{\sim}{A}', \text{ THEN } \underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R},$$

and we get

$$\underset{\sim}{B}' = [0.3 \;\; 1 \;\; 0] \circ \begin{bmatrix} 0.9 & 0.9 & 0.9 \\ 0.1 & 0.9 & 0.1 \\ 1 & 1 & 1 \end{bmatrix} = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} + \frac{0.3}{y_3} \right\}.$$

This image quality, $\underset{\sim}{B}'$, is more fuzzy than $\underset{\sim}{B}$, as indicated by the former's membership function.

Linguistics Variables and Hedges

- Lotfi Zadeh introduced the concept of linguistic variable (or fuzzy variable) in 973, which allows computation with words in stead of numbers.

- Linguistic variables are variables with values that are words or sentences from natural language.
  - For example, the set of tall people, *tall is a linguistic variable.*

- *Sensory* inputs are linguistic variables, or nouns in a natural language,
  - For example, temperature, pressure, displacement, etc

Linguistics Variables and Hedges

- Linguistic variables can be divided into different categories:
  - *Quantification variables, e.g. all, most, many, none, etc.*
  - *Usuality variables, e.g. sometimes, frequently, always, seldom, etc.*
  - *Likelihood variables, e.g. possible, likely, certain, etc.*

- In natural language, nouns are frequently combined with adjectives for quantifications of these nouns.
- For example, in the phrase *very tall, the noun tall is quantified by the* adjective *very, indicating a person who is "taller" than tall. In fuzzy systems theory,*

Linguistics Variables and Hedges

- These adjectives are referred to as hedges.

- A hedge serves as a modifier of fuzzy values.

- In other words, the hedge *very changes the membership of elements of the set tall* to different membership values in the set *very tall.*

- *Hedges are implemented through* subjective definitions of mathematical functions, to transform membership values in a systematic manner.

Linguistics Variables and Hedges

Consider the set of tall people, and assume the membership function μ_{tall} *characterizes the degree of membership of elements* to the set *tall. Our task is to create a new set,* ***very_tall*** *of people that are very* tall.

- The hedge ***very*** *can be implemented as the square function.*

$$\mu_{very\_tall}(x) = \mu_{tall}(x)^2$$

Hence, if Peter belongs to the set tall with certainty 0.9, then he also belongs to the set *very tall with certainty 0.81.*

This makes sense according to our natural understanding of the phrase *very tall: Degree of membership to the set very tall should be less than membership to the set tall.*

Linguistics Variables and Hedges

consider the set sort of tall to represent all people that are ***sort_of_tall****, i.e. people that are shorter than* tall.

The hedge ***sort _of*** *can be implemented as the square root function,*

$$\mu_{sort\_of\_tall}(x) = (\mu_{tall}(x))^{1/2}$$

So, if Peter belongs to the set tall with degree 0.81, he belongs to the set *sort of tall with degree 0.9.*

Linguistics Variables and Hedges

1. ***Concentration hedges:*** (e.g. very), where the membership values get relatively smaller. That is, the membership values get more concentrated around points with higher membership degrees.

$$\boxed{\mu_{A'}(x) = \mu_A(x)^p, \quad for\ p > 1}$$

2. ***Dilation hedges***: (e.g. somewhat, sort of, generally), where membership values increases.

$$\boxed{\mu_{A'}(x) = \mu_A(x)^{1/p} \quad for\ p > 1}$$

3. ***Probabilistic hedges:*** which express probabilities, e.g. likely, not very likely, probably, etc.

Linguistics Variables and Hedges

4. ***Contrast intensification hedges:*** (e.g. extremely), where memberships lower than 1/2 are diminished, but memberships larger than 1/2 are elevated. This hedge is defined as,

$$\mu_{A'}(x) = \begin{cases} 2^{p-1}\mu_A(x)^p & \text{if } \mu_A(x) \leq 0.5 \\ 1 - 2^{p-1}(1-\mu_A(x))^p & \text{if } \mu_A(x) > 0.5 \end{cases}$$

which intensifies contrast.

5. ***Vague hedges:*** (e.g. seldom), are opposite to contrast intensification hedges, having membership values altered using

$$\mu_{A'}(x) = \begin{cases} \sqrt{\mu_A(x)/2} & \text{if } \mu_A(x) \leq 0.5 \\ 1 - \sqrt{(1-\mu_A(x))/2} & \text{if } \mu_A(x) > 0.5 \end{cases}$$

Vague hedges introduce more “fuzziness” into the set.

LINGUISTIC VARIABLES

- A linguistic variable is a fuzzy variable.

  - The linguistic variable speed ranges between 0 and 300 km/h and includes the fuzzy sets slow, very slow, fast, …
  - Fuzzy sets define the linguistic values.

- A linguistic hedges are qualifiers of a linguistic variable.

  - All purpose: very, quite, extremely
  - Probability: likely, unlikely
  - Quantifiers: most, several, few
  - Possibilities: almost impossible, quite possible

Define $\alpha = \int_{Y} \mu_{\alpha}(y)/y$, then

$$\text{“Very”}\ \alpha = \alpha^2 = \int_{Y} \frac{[\mu_{\alpha}(y)]^2}{y}.$$

$$\text{“Very, very”}\ \alpha = \alpha^4.$$

$$\text{“Plus”}\ \alpha = \alpha^{1.25}.$$

$$\text{“Slightly”}\ \alpha = \sqrt{\alpha} = \int_{Y} \frac{[\mu_{\alpha}(y)]^{0.5}}{y}.$$

$$\text{“Minus”}\alpha = \alpha^{0.75}.$$

$$\text{“intensify”}\alpha = \begin{cases} 2\mu_{\alpha}^2(y), & \text{for } 0 \le \mu_{\alpha}(y) \le 0.5; \\ 1 - 2[1 - \mu_{\alpha}(y)]^2, & \text{for } 0.5 \le \mu_{\alpha}(y) \le 1. \end{cases}$$

Example 5.14. Suppose we have a universe of integers, Y = {1, 2, 3, 4, 5}. We define t following linguistic terms as a mapping onto Y:

$$\text{“Small”} = \left\{\frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5}\right\}.$$

$$\text{“Large”} = \left\{\frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5}\right\}.$$

Now we modify these two linguistic terms with hedges,

$$\text{“Very small”} = \text{“small”}^2 \text{ (Equation (5.26))} = \left\{\frac{1}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5}\right\}.$$

$$\text{“Not very small”} = 1 - \text{“very small”} = \left\{\frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5}\right\}.$$

Then we construct a phrase, or a composite term:

$$\alpha = \text{“not very small and not very, very large,”}$$

which involves the following set-theoretic operations:

$$\alpha = \left(\frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5}\right) \cap \left(\frac{1}{1} + \frac{1}{2} + \frac{0.9}{3} + \frac{0.6}{4}\right) = \left(\frac{0.36}{2} + \frac{0.64}{3} + \frac{0.6}{4}\right).$$

$$\text{“Intensely small”} = \left\{\frac{1 - 2[1 - 1]^2}{1} + \frac{1 - 2[1 - 0.8]^2}{2}\right.$$
$$\left. + \frac{1 - 2[1 - 0.6]^2}{3} + \frac{2[0.4]^2}{4} + \frac{2[0.2]^2}{5}\right\}$$
$$= \left\{\frac{1}{1} + \frac{0.92}{2} + \frac{0.68}{3} + \frac{0.32}{4} + \frac{0.08}{5}\right\}.$$

FUZZY RULES

- A fuzzy rule is defined as the conditional statement of the form

  If x is A

  THEN y is B

where x and y are linguistic variables and A and B are linguistic values determined by fuzzy sets on the universes of discourse X and Y.

FUZZY RULES

- The decision-making process is based on rules with sentence conjunctives **AND, OR** and **ALSO**.
- Each rule corresponds to a fuzzy relation.
- Example:
  If (Distance x to second car is **SMALL) OR** (Distance y to obstacle is **CLOSE**) **AND** (speed v is **HIGH**) **THEN** (perform **LARGE** correction to steering angle θ) **ALSO** (make **MEDIUM** reduction in speed v).
- Three antecedents (or premises) in this example give rise to two outputs (consequences).

FORMATION OF FUZZY RULES

Three general forms are adopted for forming fuzzy rules. They are:

- Assignment statements,
- Conditional statements,
- Unconditional statements.

Assignment Statements

>> The assignment statement is found to restrict the value of a variable to a specific equality.

$$
\begin{aligned}
y &= \text{low}, \\
\text{Sky color} &= \text{blue}, \\
\text{Climate} &= \text{hot} \\
a &= 5 \\
p &= q + r \\
\text{Temperature} &= \text{high}
\end{aligned}
$$

Conditional Statements

>> In this statements, some specific conditions are mentioned, if the conditions are satisfied then it enters the following statements, called as restrictions.

IF y is very cool THEN stop.
IF A is high THEN B is low ELSE B is not low.
IF temperature is high THEN climate is hot.

$$\text{If } x = y \text{ Then both are equal,}$$

$$\text{If Mark } > 50 \text{ Then pass,}$$

$$\text{If Speed } > 1,500 \text{ Then stop.}$$

Unconditional Statements

>>There is no specific condition that hasto be satisfied in this form of statements.

Goto sum.

Stop.

Divide by a.

Turn the pressure low.

Canonical form – Fuzzy rule-based system

Rule 1:	IF condition C^1 THEN restriction R^1
Rule 2:	IF condition C^2 THEN restriction R^2
...	
Rule n:	IF condition C^n THEN restriction R^n

DECOMPOSITION OF FUZZY RULES

- A compound rule is a collection of several simple rules combined together.

- Any compound rule structure can be decomposed and reduced to number of simple canonical rules.

- There are various methods for decomposition of rules.

  - Multiple conjunctive antecedent,
  - Multiple disjunctive antecedent,
  - Conditional statements (with ELSE and UNLESS).

DECOMPOSITION OF FUZZY RULES

Multiple Conjunctive Antecedants

This uses fuzzy intersection operation. Since it involves linguistic "AND" connective

$$\text{IF } x \text{ is } \underset{\sim}{P}^1 \text{ AND } \underset{\sim}{P}^2 \cdots \text{ AND } \underset{\sim}{P}^n \text{ THEN } y \text{ is } \underset{\sim}{Q}^r,$$

where

$$\underset{\sim}{P}^r = \underset{\sim}{P}^1 \text{ AND } \underset{\sim}{P}^2 \cdots \text{ AND } \underset{\sim}{P}^n.$$

The membership for this can be

$$\mu_{\underset{\sim}{P}^r}(x) = \min\left[\mu_{\underset{\sim}{P}^1}(x), \mu_{\underset{\sim}{P}^2}(x), \ldots, \mu_{\underset{\sim}{P}^n}(x)\right].$$

Hence the rule can be
IF x is $\underset{\sim}{P}^r$ THEN $\underset{\sim}{Q}^r$.

DECOMPOSITION OF FUZZY RULES

Multiple disjunctive antecedent

This uses fuzzy union operations. It involves linguistic "OR" connections

IF x is $\underset{\sim}{P}^1$ OR $\underset{\sim}{P}^2 \cdots$ OR $\underset{\sim}{P}^n$ THEN y is $\underset{\sim}{Q}^r$,

where

$$\begin{aligned} \underset{\sim}{P}^r &= \underset{\sim}{P}^1 \text{ OR } \underset{\sim}{P}^2 \cdots \text{ OR } \underset{\sim}{P}^n \\ &= \underset{\sim}{P}^1 \cup \underset{\sim}{P}^2 \cdots \cup \underset{\sim}{P}^n . \end{aligned}$$

The membership for this can be

$$\mu_{\underset{\sim}{P}^r}(x) = \max\left[\mu_{\underset{\sim}{P}^1}(x), \mu_{\underset{\sim}{P}^2}(x), \ldots, \mu_{\underset{\sim}{P}^n}(x)\right].$$

Hence the rule can be

IF x is $\underset{\sim}{P}^r$ THEN y is $\underset{\sim}{Q}^r$.

DECOMPOSITION OF FUZZY RULES

Conditional Statements (With Else)

$$\text{(a) IF } \underset{\sim}{P}^1 \text{ THEN } \left(\underset{\sim}{Q}^1 \text{ ELSE } \underset{\sim}{Q}^2\right).$$

Considering this as one compound statement, splitting this into two canonical form rules, we get

IF $\underset{\sim}{P}^1$ THEN Q^1 OR IF NOT $\underset{\sim}{P}^1$ THEN Q^2.

$$\text{(b) IF } \underset{\sim}{P}^1 \text{ THEN } \left(Q^1 \text{ ELSE } \underset{\sim}{P}^2 \text{ THEN } \left(\underset{\sim}{Q}^2\right)\right).$$

The decomposition for this can be of the form

$$\text{IF } \underset{\sim}{P}^1 \text{ THEN } Q^1 \text{ OR}$$

$$\text{IF NOT } \underset{\sim}{P}^1 \text{ AND } \underset{\sim}{P}^2 \text{ THEN } Q^2.$$

DECOMPOSITION OF FUZZY RULES

Conditional Statements (Nested IF–THEN rules)

$$\text{IF } \underset{\sim}{P}^1 \text{ THEN } \left(\text{IF } \underset{\sim}{P}^2 \text{ THEN } (\underset{\sim}{Q}^1) \right).$$

This can be decomposed into

$$\text{IF } \underset{\sim}{P}^1 \text{ AND } \underset{\sim}{P}^2 \text{ THEN } \underset{\sim}{Q}^1.$$

DECOMPOSITION OF FUZZY RULES

Conditional Statements (With Unless)

IF $\underset{\sim}{A}_1$ (THEN $\underset{\sim}{B}_1$) UNLESS $\underset{\sim}{A}_2$
can be decomposed as
IF $\underset{\sim}{A}_1$ THEN $\underset{\sim}{B}_1$
OR
IF $\underset{\sim}{A}_2$ THEN NOT $\underset{\sim}{B}_1$

AGGREGATION OF FUZZY RULES

Aggregation of rules is the process of obtaining the overall consequents from the individual consequents provided by each rule.

- Conjunctive system of rules.
- Disjunctive system of rules.

AGGREGATION OF FUZZY RULES

Conjunctive system of rules

Conjunctive system of rules: For a system of rules to be jointly satisfied, the rules are connected by "and" connectives. Here, the aggregated output, y, is determined by the fuzzy intersection of all individual rule consequents, y_i, where $i = 1$ to n, as

$$y = y_1 \text{ and } y_2 \text{ and } \ldots \text{ and } y_n$$

or

$$y = y_1 \cap y_2 \cap y_3 \cap \cdots \cap y_n.$$

This aggregated output can be defined by the membership function

$$\mu_y(y) = \min[\mu_{y_1}(y),\ \mu_{y_2}(y),\ \ldots,\ \mu_{y_n}(y)] \text{ for } y \in Y.$$

AGGREGATION OF FUZZY RULES

Disjunctive system of rules

Disjunctive system of rules: In this case, the satisfaction of at least one rule is required. The rules are connected by "or" connectives. Here, the fuzzy union of all individual

rule contributions determines the aggregated output, as

$$y = y_1 \text{ or } y_2 \text{ or } \ldots \text{ or } y_n$$

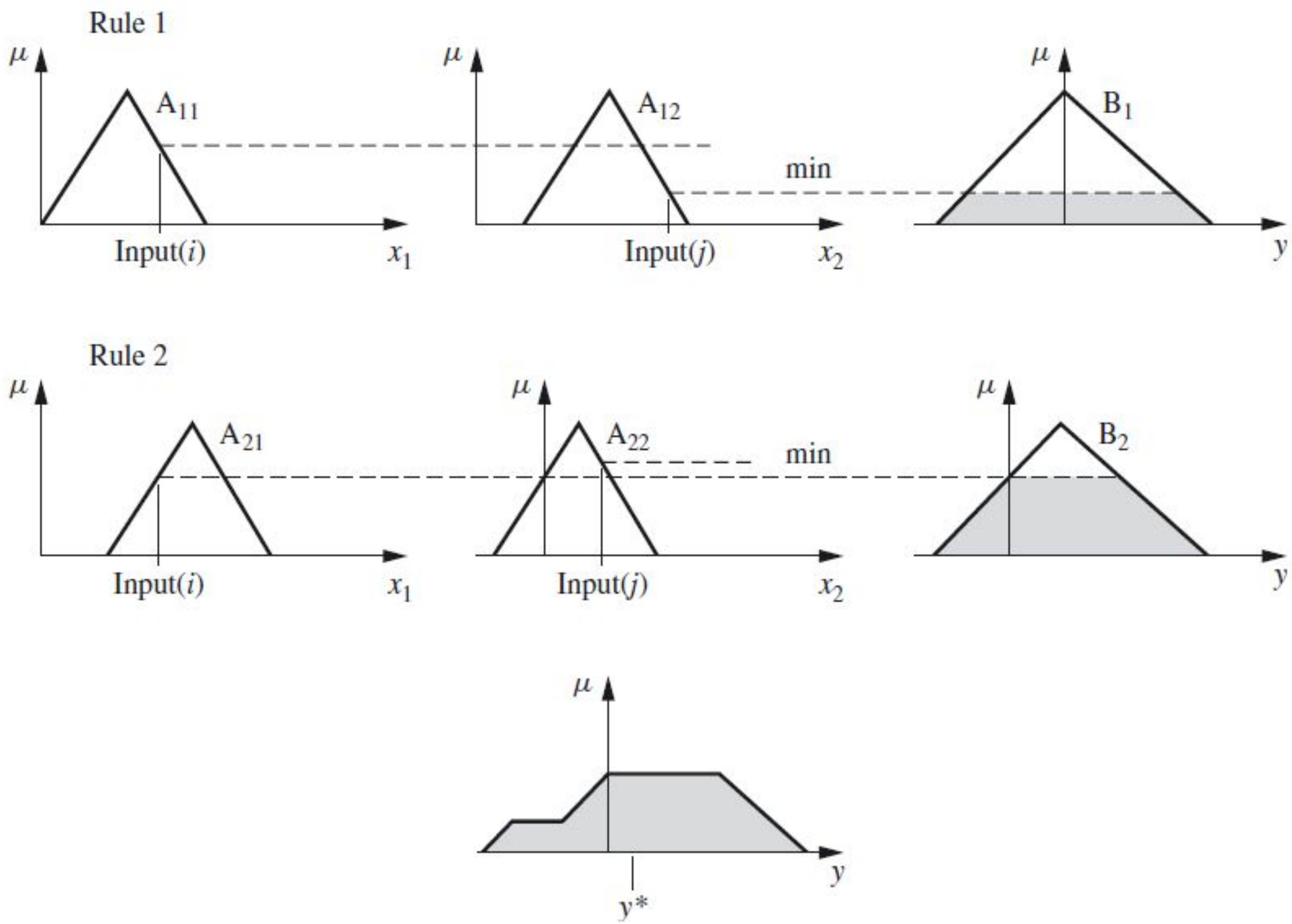
or

$$y = y_1 \cup y_2 \cup y_3 \cup \cdots \cup y_n.$$

Again it can be defined by the membership function

$$\mu_y(y) = \max\left[\mu_{y_1}(y),\ \mu_{y_2}(y),\ \ldots \mu_{y_n}(y)\right] \text{ for } y \in Y.$$

Graphical Mamdani (max–min) inference method with crisp inputs



Example 5.15. In mechanics, the energy of a moving body is called kinetic energy. If an object of mass m (kilograms) is moving with a velocity v (meters per second), then the kinetic energy k (in joules) is given by the equation $k = \frac{1}{2}mv^2$. Suppose we model the mass and velocity as inputs to a system (moving body) and the energy as output, then observe the system for a while and deduce the following two disjunctive rules of inference based on our observations:

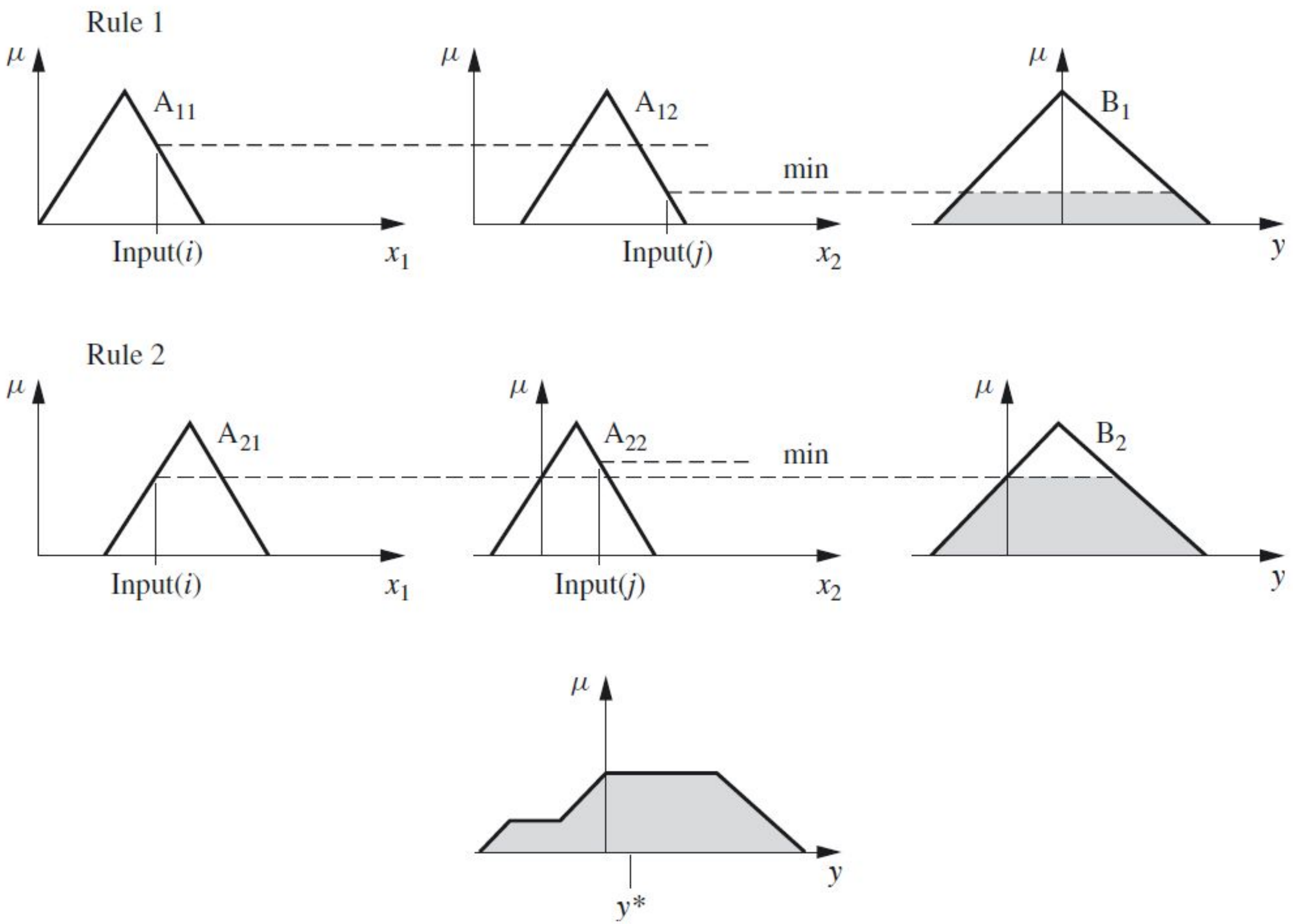
$$\text{Rule 1: IF } x_1 \text{ is } \underset{\sim}{A}_1^1 \text{ (small mass) } \textit{and } x_2 \text{ is } \underset{\sim}{A}_2^1 \text{ (high velocity),}$$

$$\text{THEN } y \text{ is } \underset{\sim}{B}^1 \text{ (medium energy).}$$

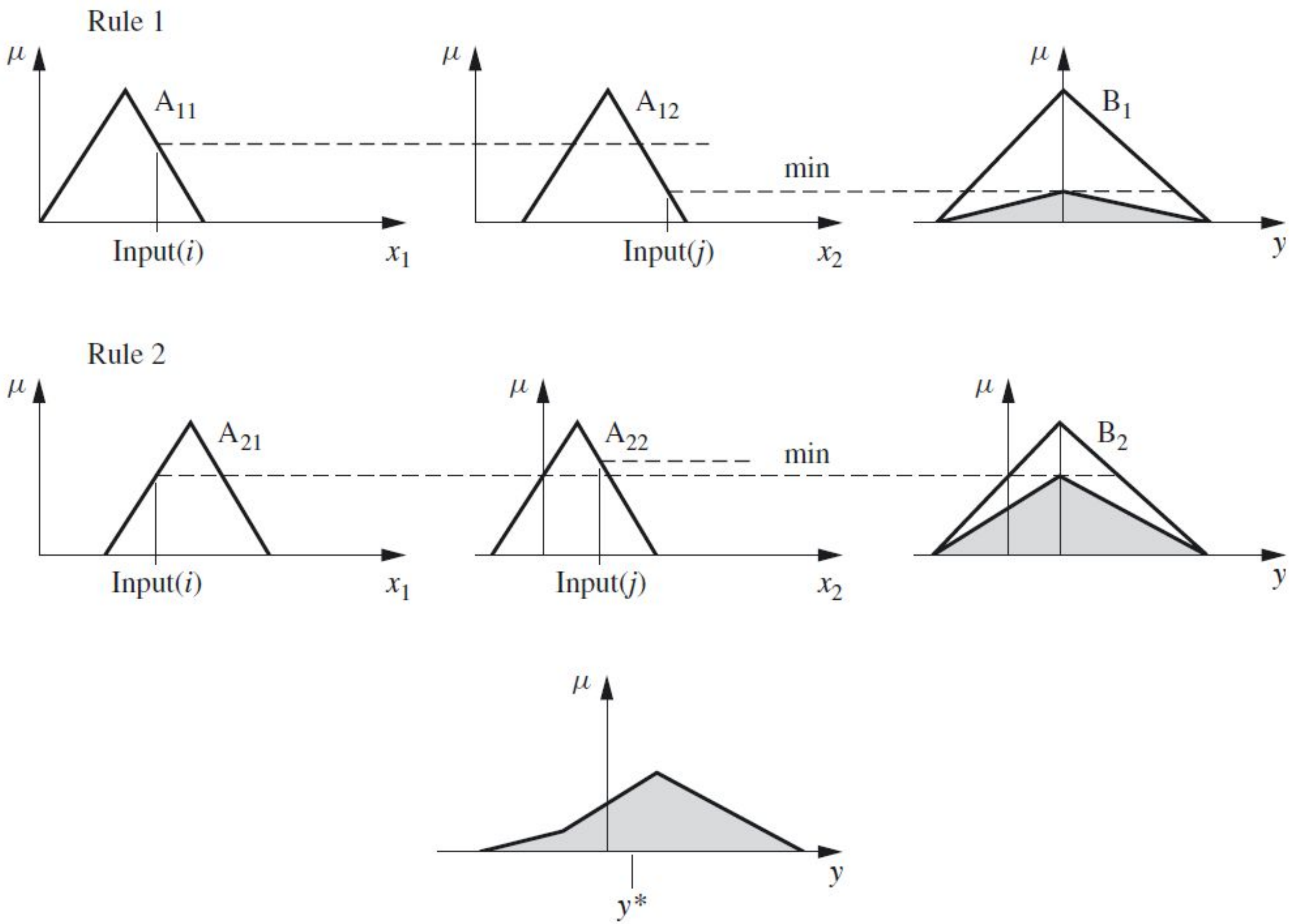
$$\text{Rule 2: IF} x_1 \text{ is } \underset{\sim}{A}_1^2 \text{ (large mass) } \textit{or } x_2 \text{ is } \underset{\sim}{A}_2^1 \text{ (high velocity),}$$

$$\text{THEN } y \text{ is } \underset{\sim}{B}^2 \text{ (high energy).}$$

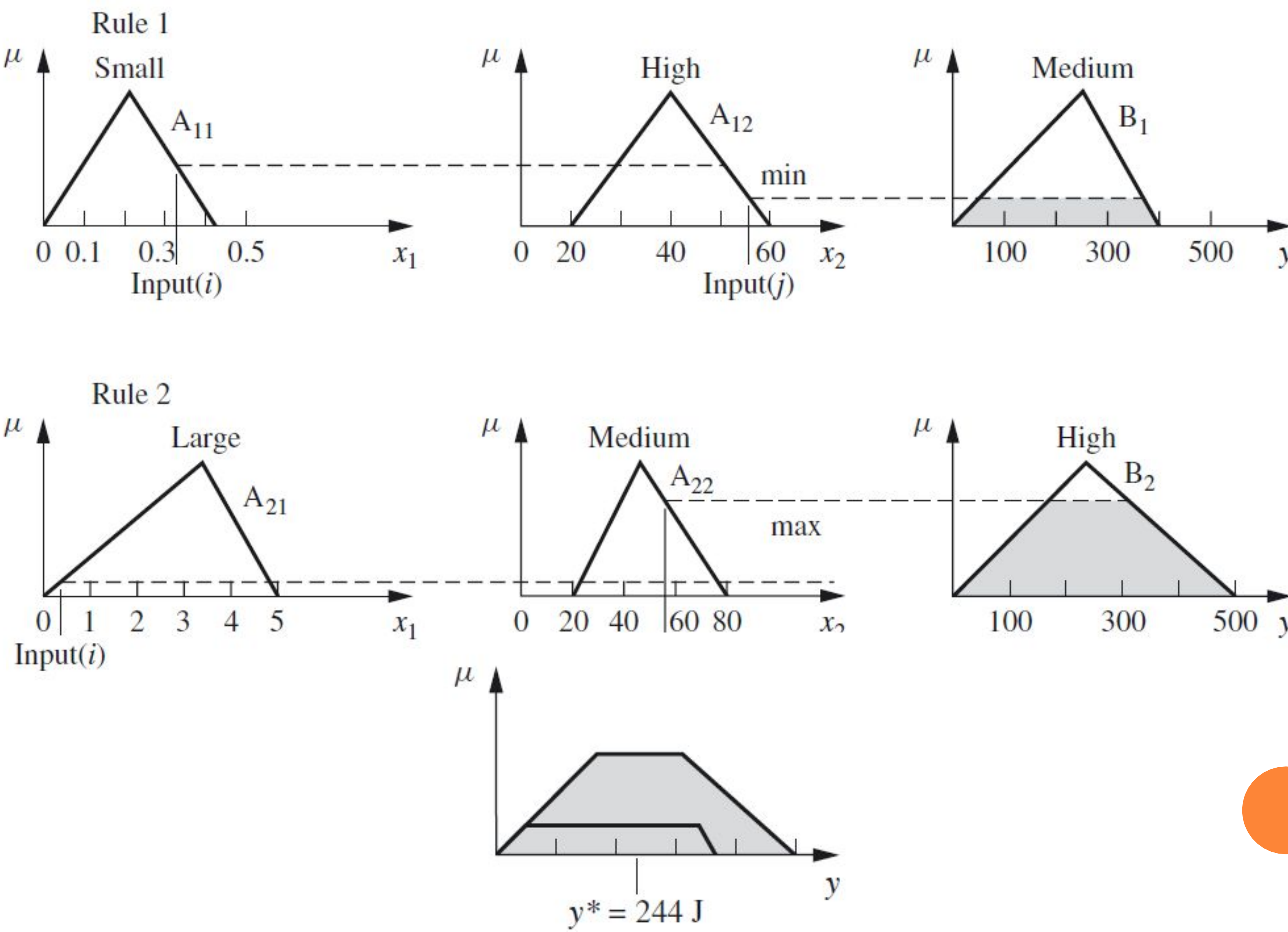
Fuzzy inference method using the case 1 graphical approach.



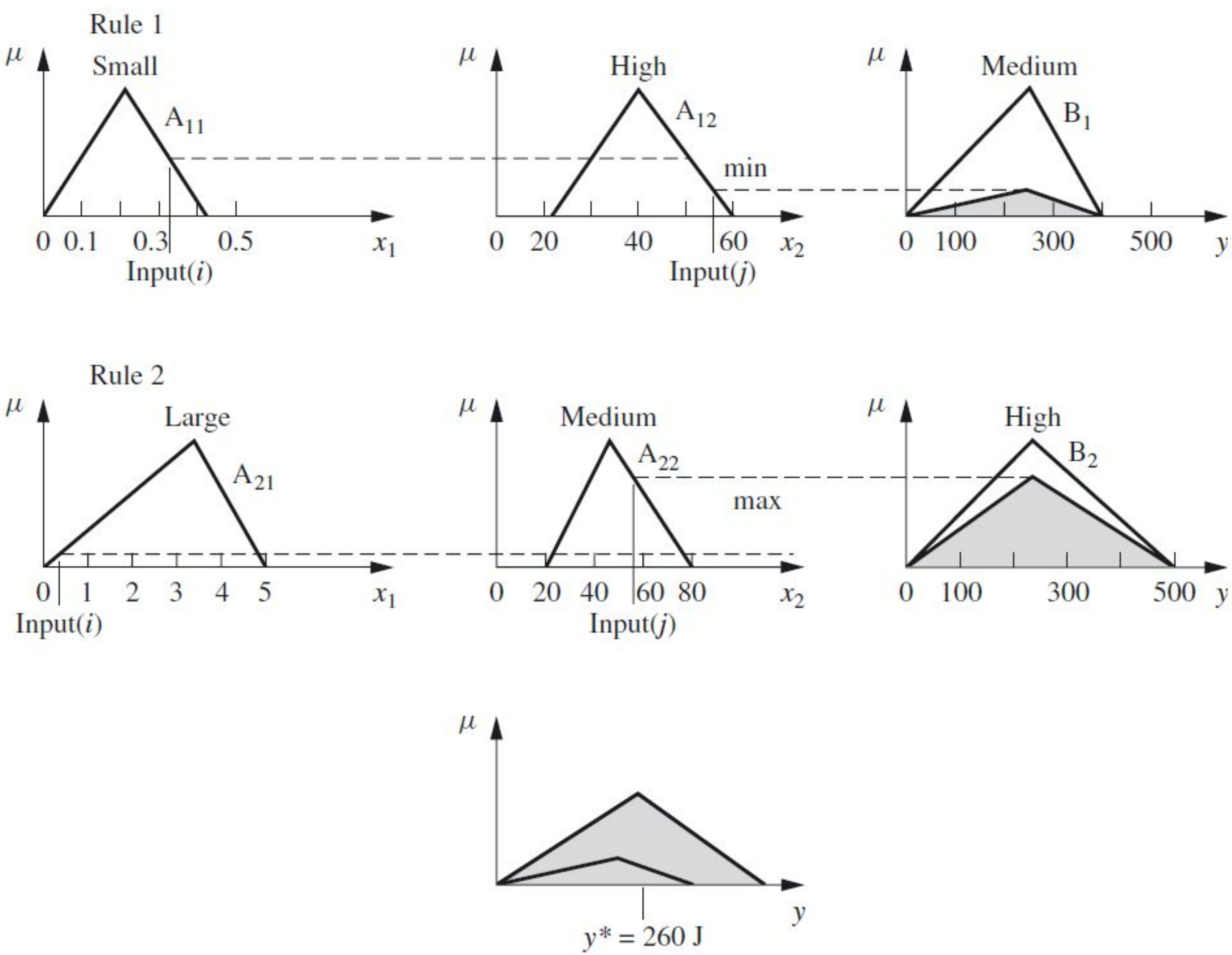
Graphical Mamdani (max–product) implication method with crisp inputs.



input(*i*) = 0.35 kg (mass) and input(*j*) = 55m/s (velocity)



input*(i)* = 0.35 kg *(*mass*)* and input*(j)* = 55m/s (velocity)



An example of a two-input, single-output **Sugeno model** with four rules is repeated from Jang *et al*. (1997):

IF X is small and Y is small, THEN $z = -x + y + 1$.

IF X is small and Y is large, THEN $z = -y + 3$.

IF X is large and Y is small, THEN $z = -x + 3$.

IF X is large and Y is large, THEN $z = x + y + 2$.

EXAMPLE

Rules

- If it's Sunny and Warm, drive Fast

  $$\text{Sunny(Cover)} \wedge \text{Warm(Temp)} \Rightarrow \text{Fast(Speed)}$$

- If it's Cloudy and Cool, drive Slow

  $$\text{Cloudy(Cover)} \wedge \text{Cool(Temp)} \Rightarrow \text{Slow(Speed)}$$

- Driving Speed is the combination of output of these rules...

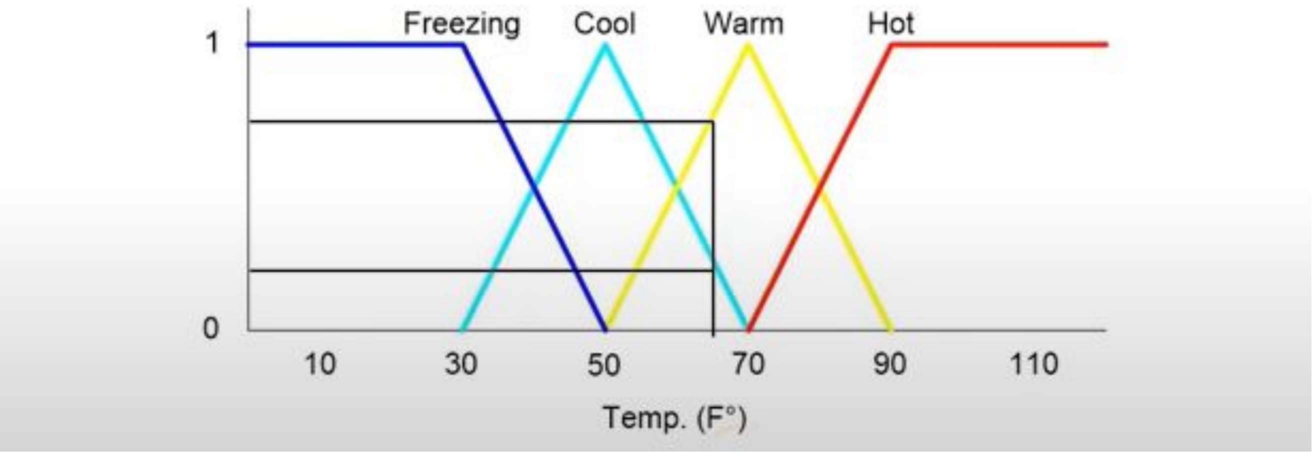
Query

- How fast will I go if it is
  - 65 F° (=18 ºC)
  - 25 % Cloud Cover ?

Fuzzification
Fuzzify Temp

- 65 F°

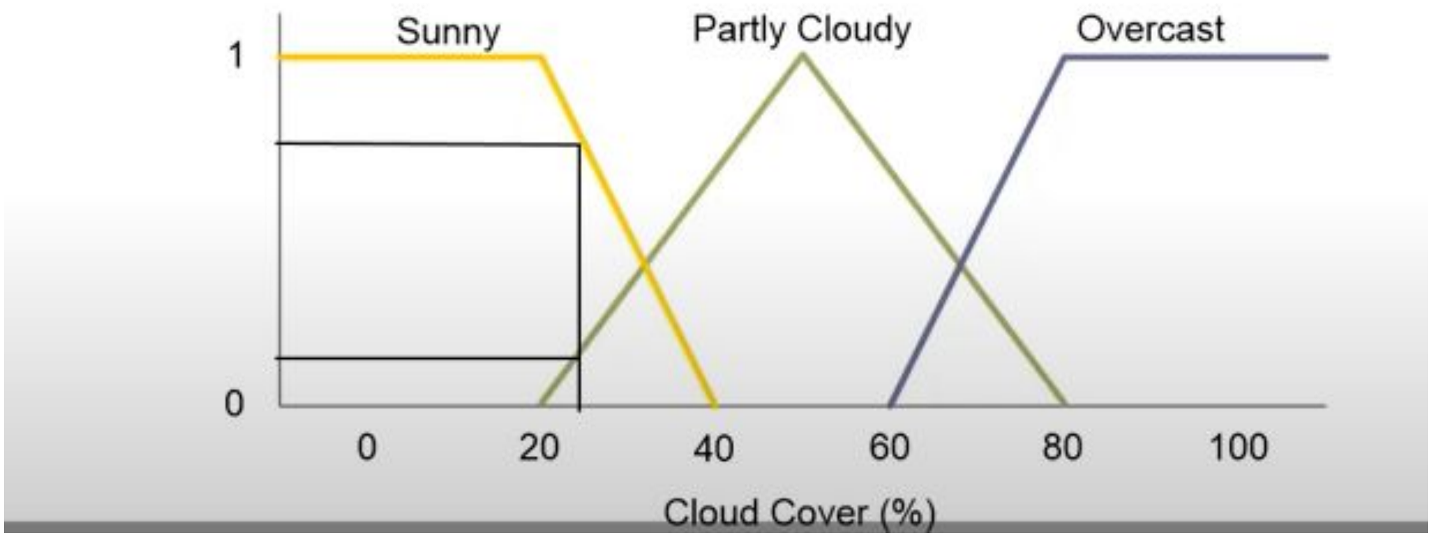
$\Rightarrow$ Cool = 0.3, Warm= 0.7



Fuzzify Cover

- 25% Cover $\Rightarrow$

$\text{Sunny} = 0.7, \text{Cloudy} = 0.1$



Rules Evaluation

- If it's Sunny and Warm, drive Fast

$$\text{Sunny(Cover)} \wedge \text{Warm(Temp)} \Rightarrow \text{Fast(Speed)}$$

$$0.7 \wedge 0.7 = 0.7$$

$$\Rightarrow \textbf{Fast = 0.7}$$

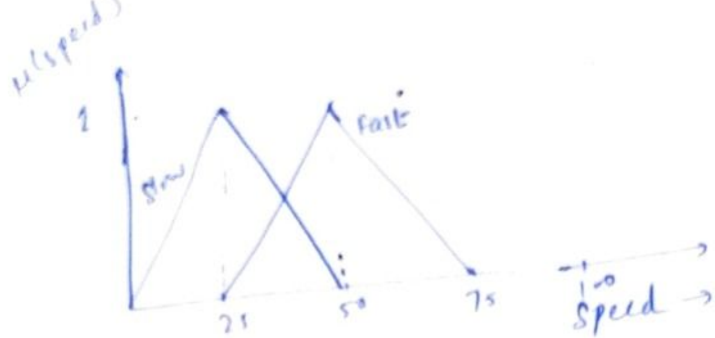
- If it's Cloudy and Cool, drive Slow

$$\text{Cloudy(Cover)} \wedge \text{Cool(Temp)} \Rightarrow \text{Slow(Speed)}$$

$$0.1 \wedge 0.3 = 0.1$$

$$\Rightarrow \textbf{Slow = 0.1}$$

Defuzzification (Find output)

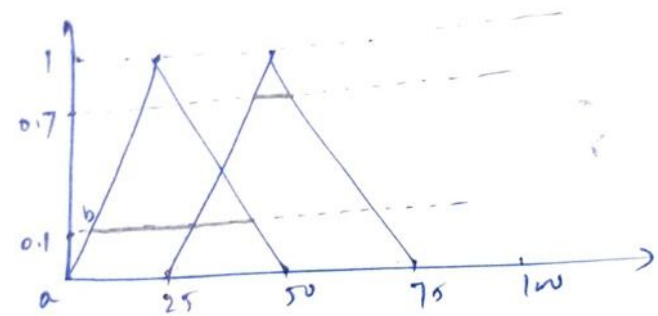


Rule 1 → Fast = 0.7

Rule 2 → Slow = 0.1

Slow = 0.1

Fast = 0.7



Defuzzification using Weighted Avg.

$$z^* = \frac{(25 \times 0.1) + (50 \times 0.7)}{0.1 + 0.7}$$

$\boxed{\text{Speed} = 46.87}$

Example 5.18. In heat exchanger design, a flexibility analysis requires the designer to determine if the size of the heat exchanger is either small or large. In order to quantify this linguistic vagueness of size, we form the general design equation for a heat exchanger, $Q = AU\Delta T_{\text{log mean}}$, where the heat transfer coefficient U and area A need to be determined.

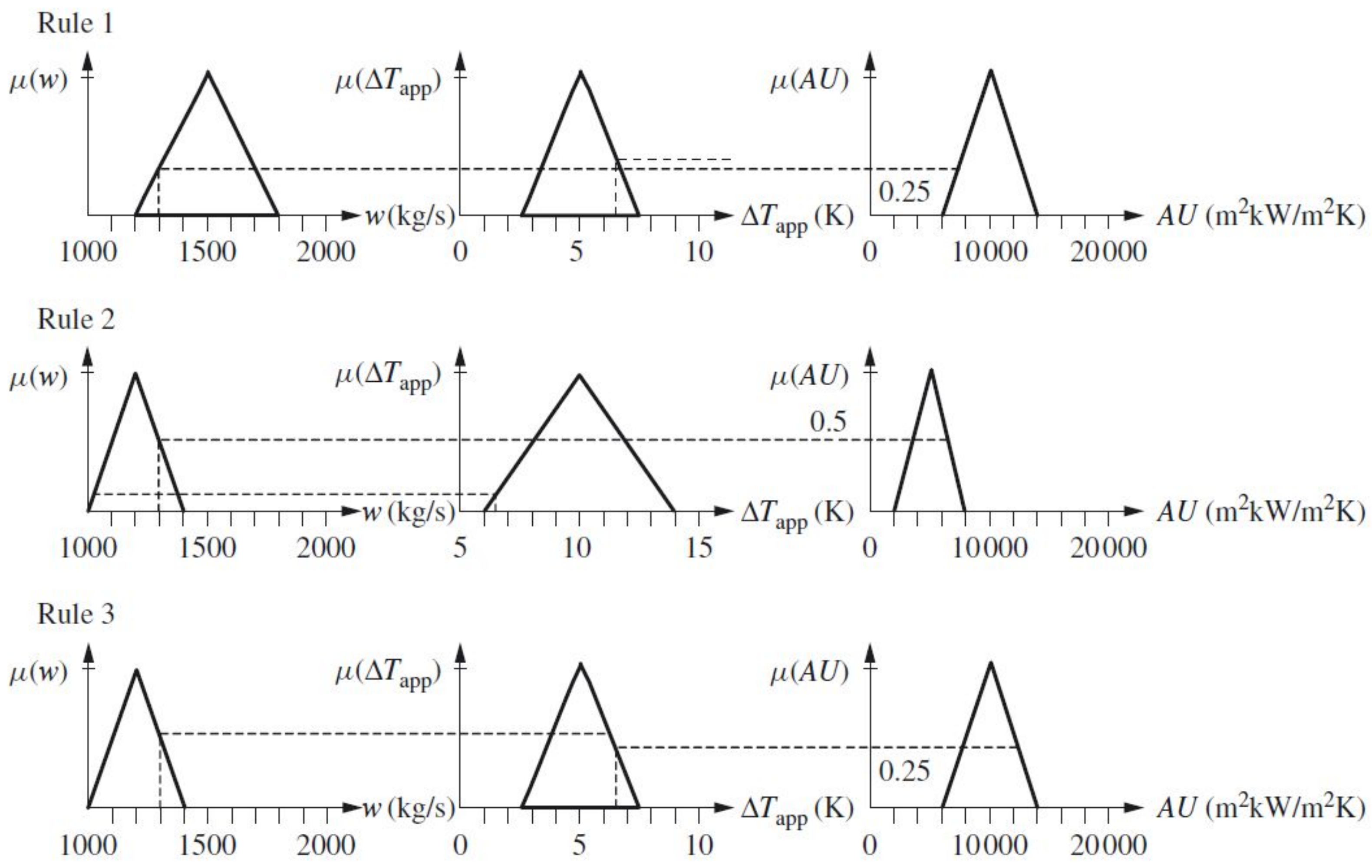
input two crisp values of benzene flow rate and temperature approach:
$w = 1300$ kg s−1 and Tapp = 6.5K.

Rule 1 : IF w is $\underset{\sim}{A}_1^1$(large flow rate) and ΔT_{app} is $\underset{\sim}{A}_2^1$(small approach),
THEN AU is $\underset{\sim}{B}^1$(large heat exchanger).

Rule 2 : IF w is $\underset{\sim}{A}_1^2$(small flow rate) or ΔT_{app} is $\underset{\sim}{A}_2^2$(large approach),
THEN AU is $\underset{\sim}{B}^1$(small heat exchanger).

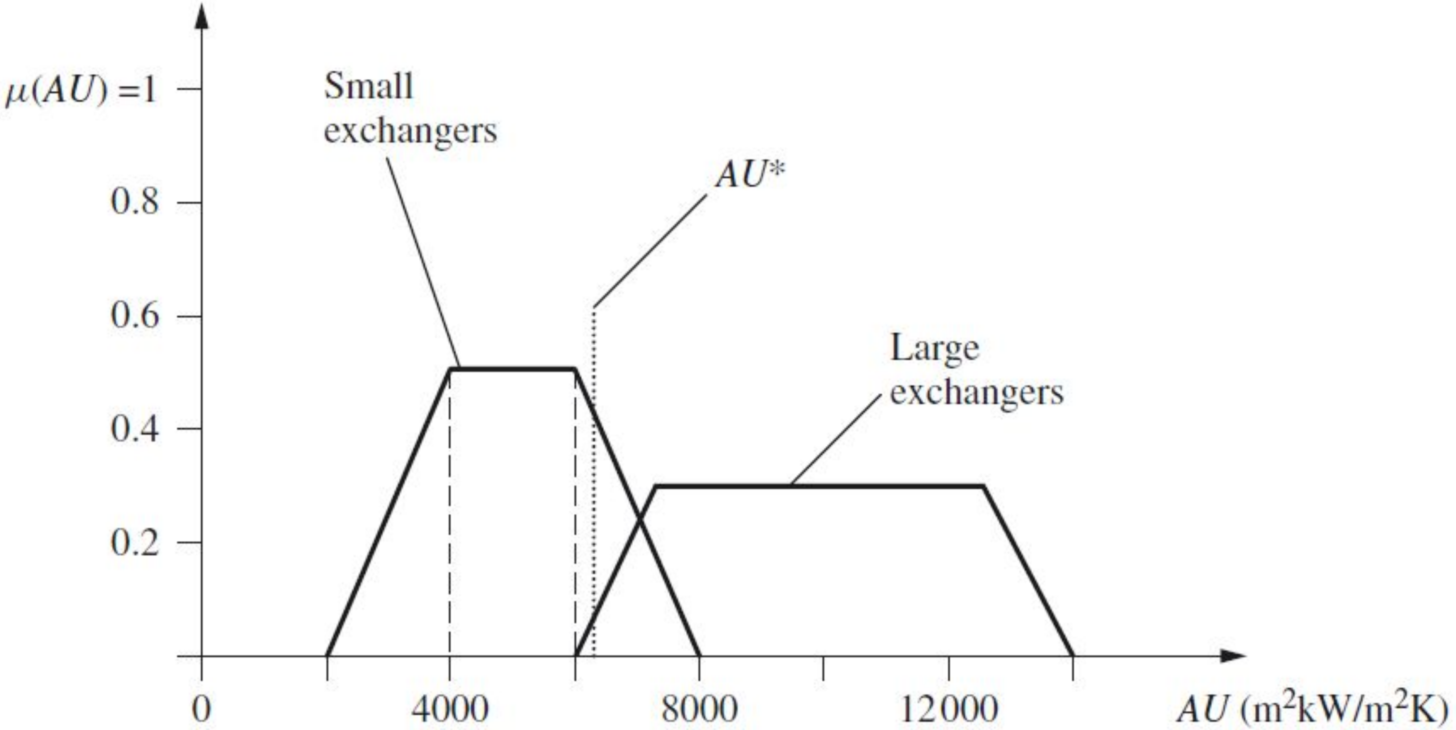
Rule 3 : IF w is $\underset{\sim}{A}_1^2$(small flow rate) and ΔT_{app} is $\underset{\sim}{A}_2^1$(small approach),
THENAU is $\underset{\sim}{B}^1$(large heat exchanger).

Using the max–min Mamdani implication method of inference



Result of the defuzzification step in the Mamdani method.

$$AU^* = 5000 * 0.5 + 10\,000 * 0.25)/(0.5 + 0.25) = 6666.67$$



For the Sugeno fuzzy method of inference, we have experience in heat exchanger design that gives the following expressions in a polynomial form for our two consequents (small and large heat exchangers):

$$AU_{\text{small}} = 3.4765w - 210.5\Delta T_{\text{app}} + 2103.$$

$$AU_{\text{large}} = 4.6925w - 526.2\Delta T_{\text{app}} + 2631.$$

$$\text{Rule 1:} \quad \mu(AU) = 0.25,$$

$$\text{Rule 2:} \quad \mu(AU) = 0.5,$$

$$\text{Rule 3:} \quad \mu(AU) = 0.25.$$

Then

$$AU_{\text{small}} = 5256 \text{ m}^2\text{kW/m}^2\text{K} \quad \text{and} \quad AU_{\text{large}} = 5311 \text{ m}^2\text{kW/m}^2\text{K}.$$

Finally, the defuzzified value of the heat exchange size is (using the weighted average method of defuzzification – see Figure 5.14)

$$AU^* = \frac{(5311 \text{ m}^2\text{kW/m}^2\text{K})(0.25) + (5256 \text{ m}^2\text{kW/m}^2\text{K})(0.5) + (5311 \text{ m}^2\text{kW/m}^2\text{K}(0.25)}{0.25 + 0.5 + 0.25}$$

$$= 5283.5 \text{ m}^2\text{kW/m}^2\text{K}.$$

Solar Pool Heater Example

- suppose we measure the pool water temp and the wind speed and we want to adjust the valve that sends water to the solar panels

- we have two input parameters

  temp

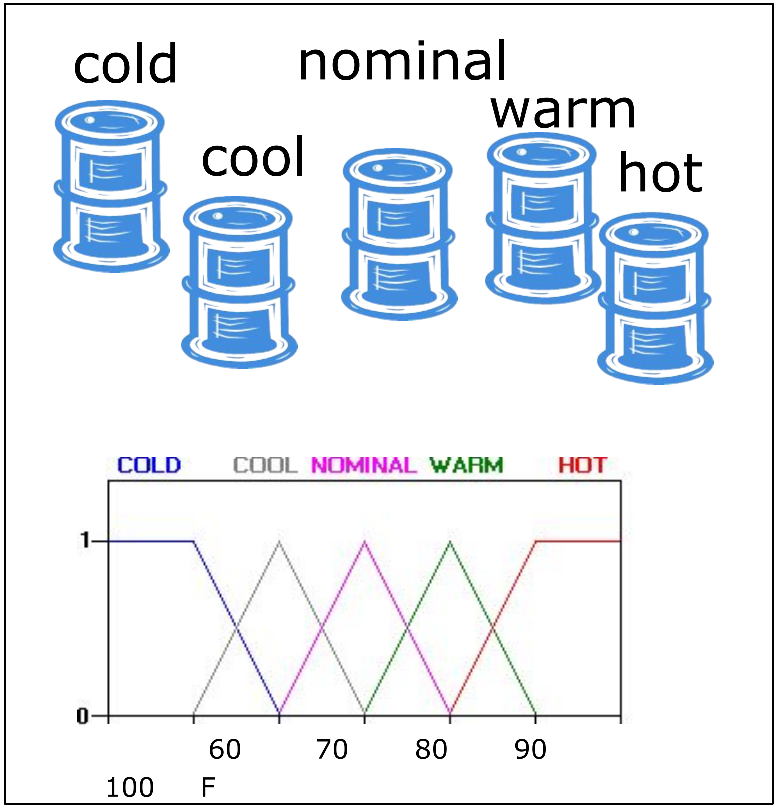
  wind_speed

- we have one output parameter

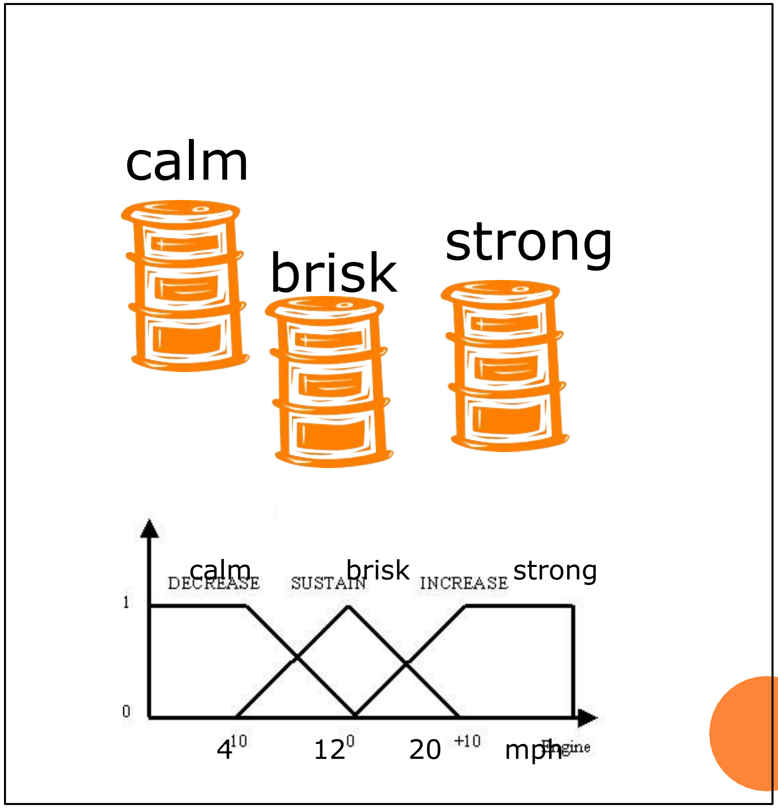
  change_in_valve

Solar Pool Heater Example

- set up membership functions for the inputs
  - for each input, decide on how many categories there will be and decide on their membership functions



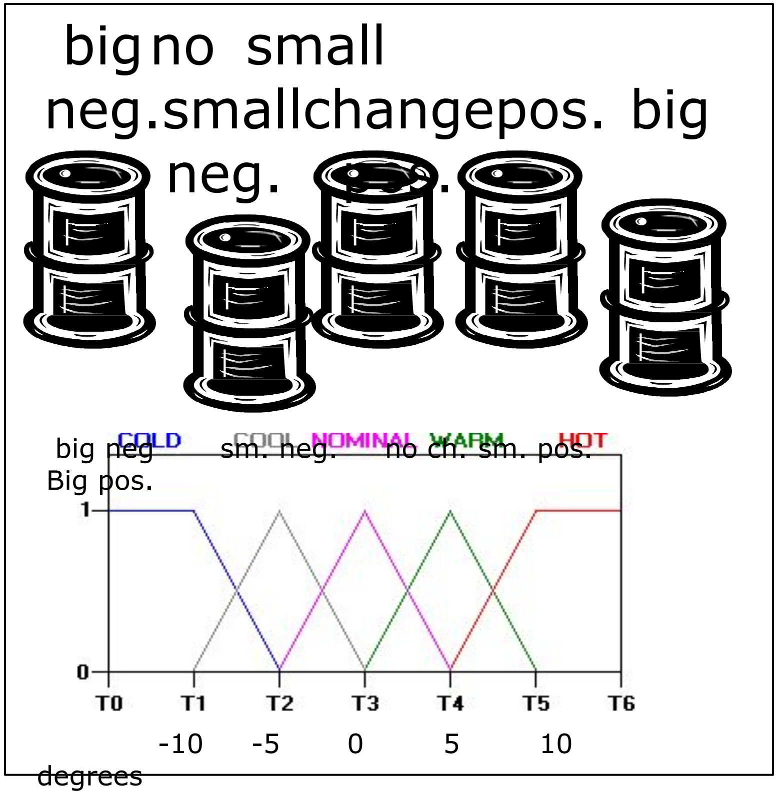
temp



wind_speed

Solar Pool Heater Example

- set up membership functions for the output(s)
  - for each output, decide on how many categories there will be and decide on their membership functions



change_in_valve

Solar Pool Heater Example

set up the rules

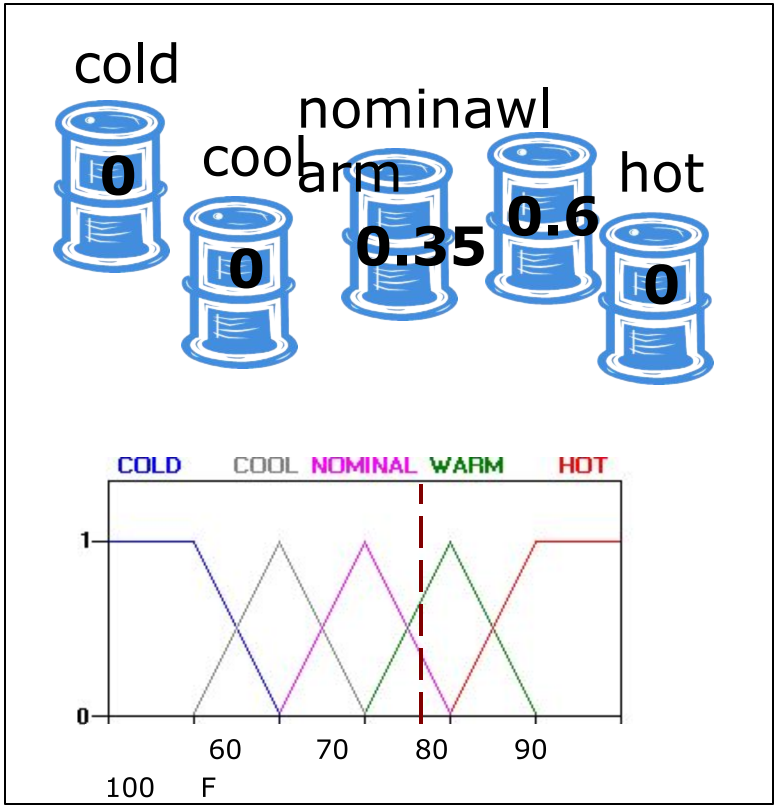
if (**temp** is hot) AND (**wind_speed** is calm) then (**change_in_valve** is big_negative)

if (**temp** is warm) AND (**wind_speed** is brisk) then (**change_in_valve** is small_negative)

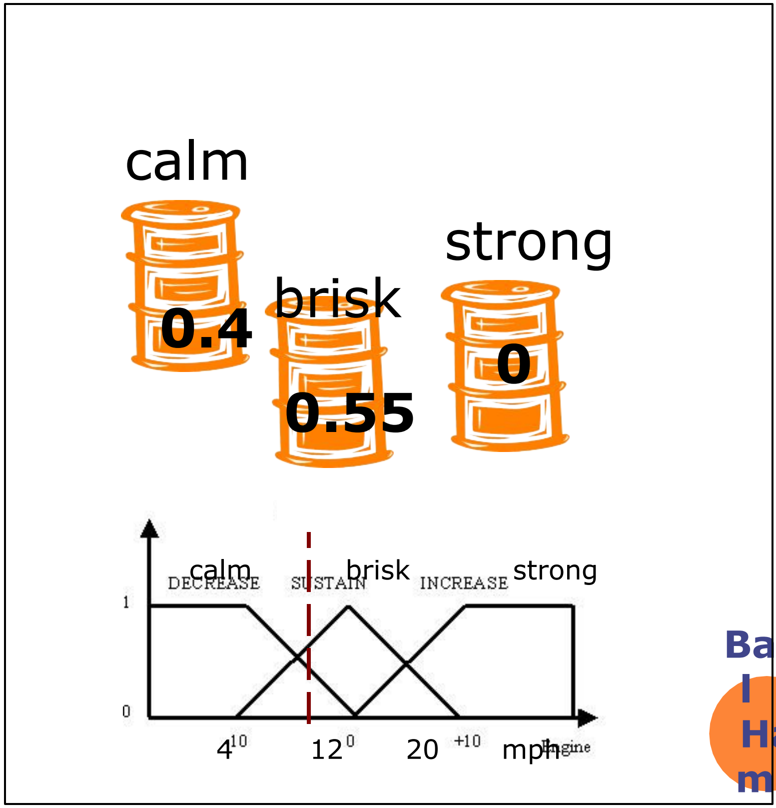
if (**temp** is nominal) OR (**temp** is warm) then (**change_in_valve** is no_change)

SOLAR POOL HEATER EXAMPLE

Fuzzify the inputs



temp = 87 F



wind_speed = 9 mph

Basil Hamed

Solar Pool Heater Example

0 0.4

- fire the rules

  if (**temp** is hot) AND (**wind_speed** is calm) then (**change_in_valve** is big_negative)

  0.6 0.55

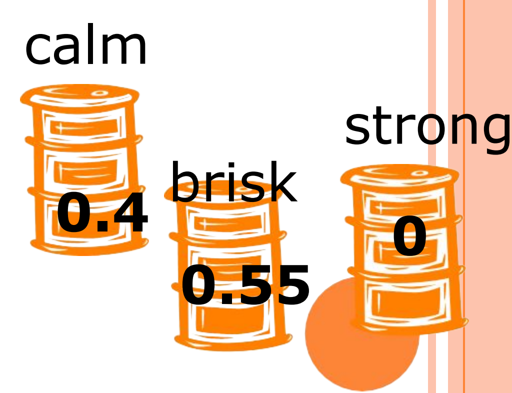
  0

  if (**temp** is warm) AND (**wind_speed** is brisk) then (**change_in_valve** is small_negative)

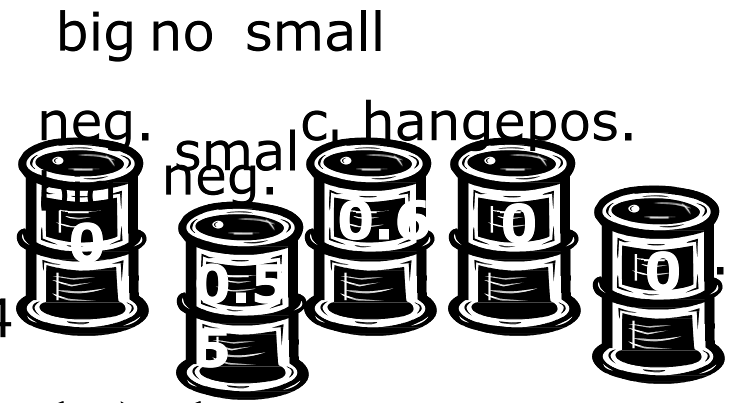
  0.35 0.55 0.6

  if (**temp** is nominal) OR (**temp** is warm) then (**change_in_valve** is no_change)

  0.6



Solar Pool Heater Example



Fire the rules

0 0.4

if (**temp** is hot) AND (**wind_speed** is calm) then
(**change_in_valve** is big_negative)

0.6 0 0.55

if (**temp** is warm) AND (**wind_speed** is
brisk)

then (**change_in_valve** is small_negative)

0.55

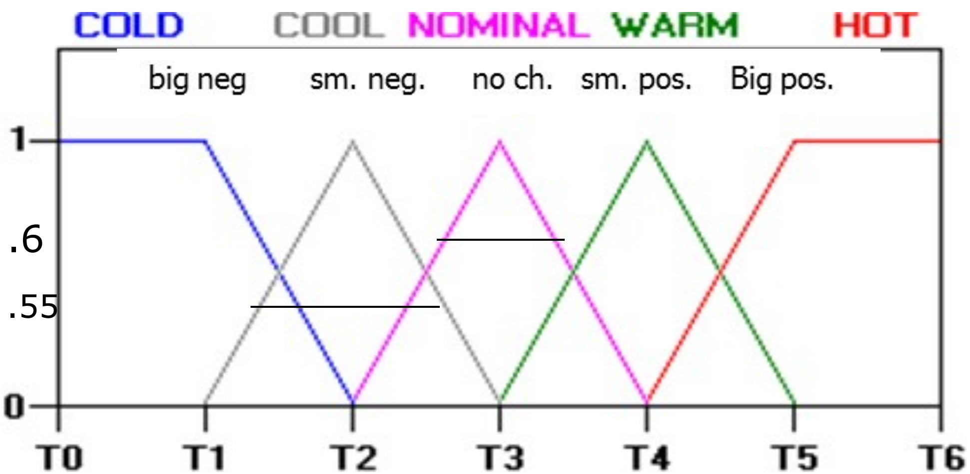
0.35 0.6

if (**temp** is nominal) OR (**temp** is warm)
then (**change_in_valve** is no_change)

0.6

Solar Pool Heater Example

Defuzzify the output(s)



- The cumulative fuzzy output is obtained by OR-ing the output from each rule.

FUZZY RULE - EXAMPLE

Rule 1: If height is short then weight is light.

Rule 2: If height is medium then weight is medium.

Rule 3: If height is tall then weight is heavy.

FUZZY RULE - EXAMPLE

Problem: Given

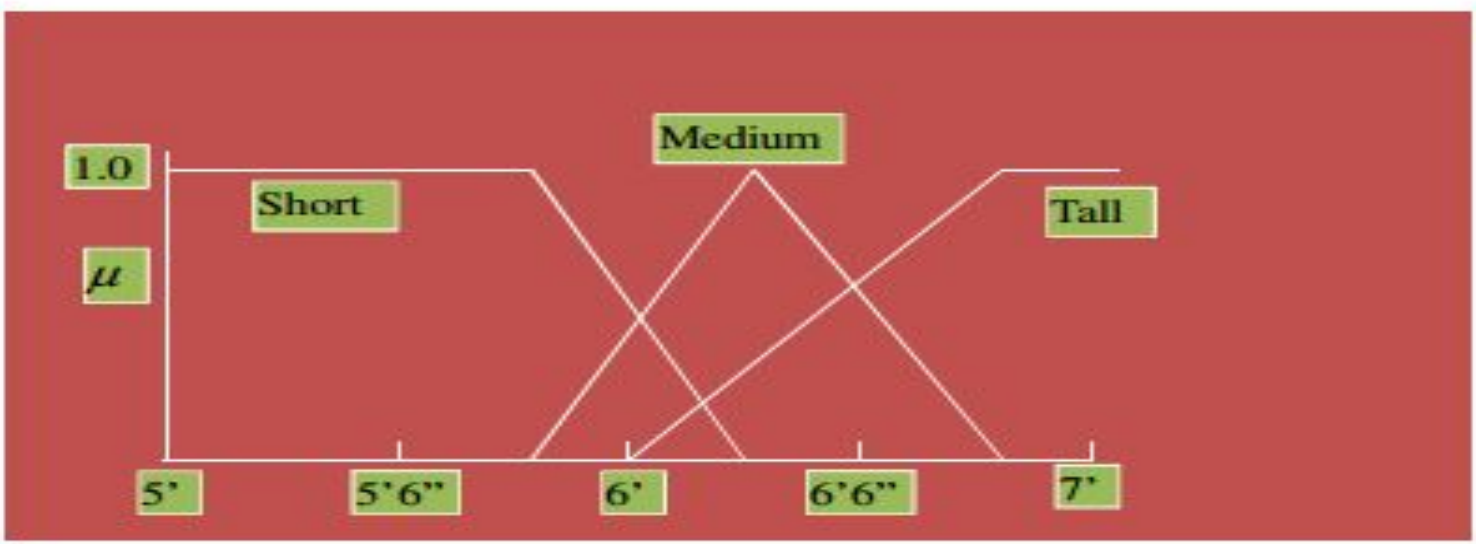
(a) membership functions for medium-height, tall, light, short, medium-weight and heavy;

(b) The three fuzzy rules;

(c) the fact that John’s height is 6’1”

estimate John’s weight.

Example: Short, Medium height and Tall



FUZZY RULE - EXAMPLE

Solution:

(1) From John's height we know that

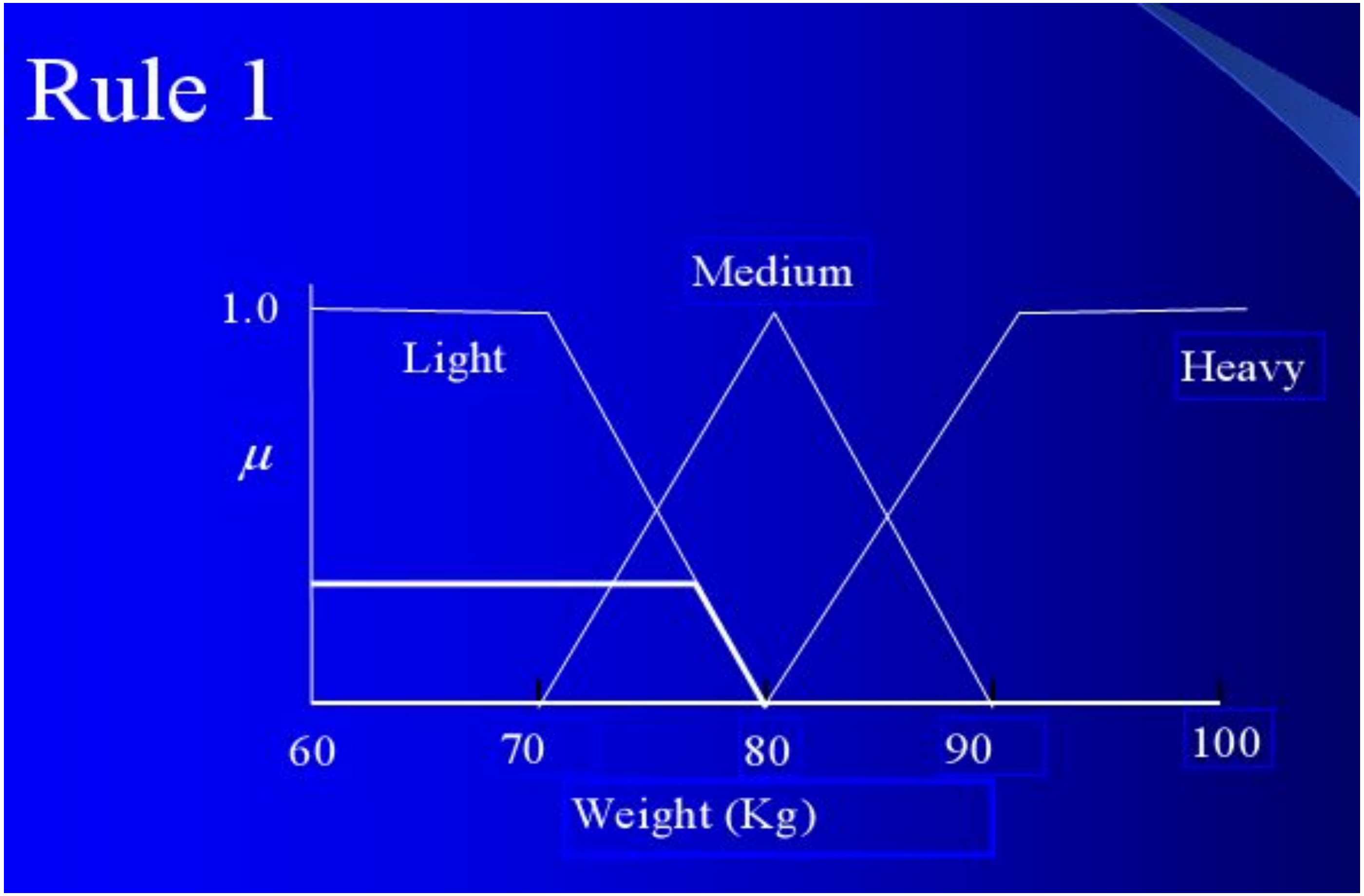
John is short (degree 0.3)

John is of medium height (degree 0.6).

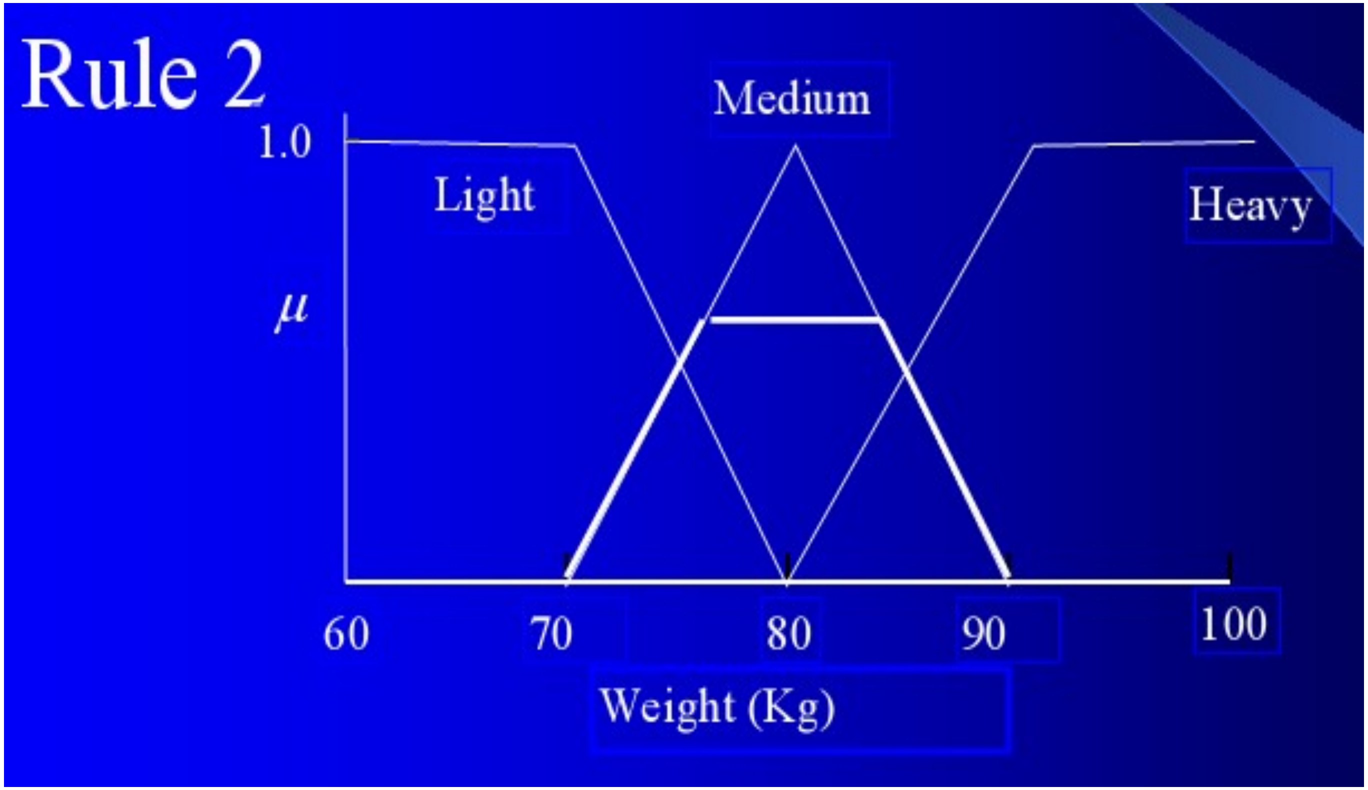
John is tall (degree 0.2).

(2) Each rule produces a fuzzy set as output by truncating the consequent membership function at the value of the antecedent membership.

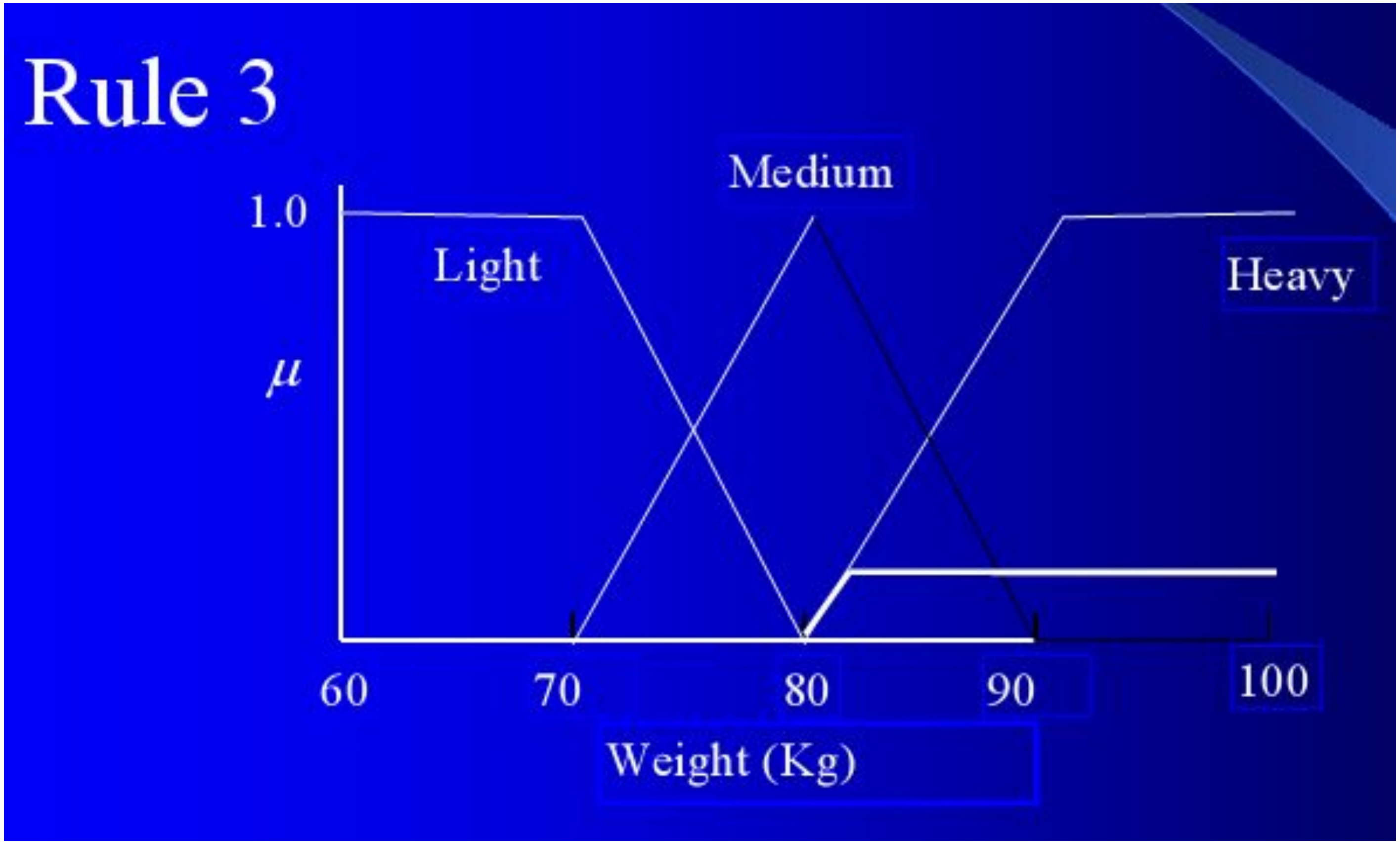
FUZZY RULE - EXAMPLE



FUZZY RULE - EXAMPLE



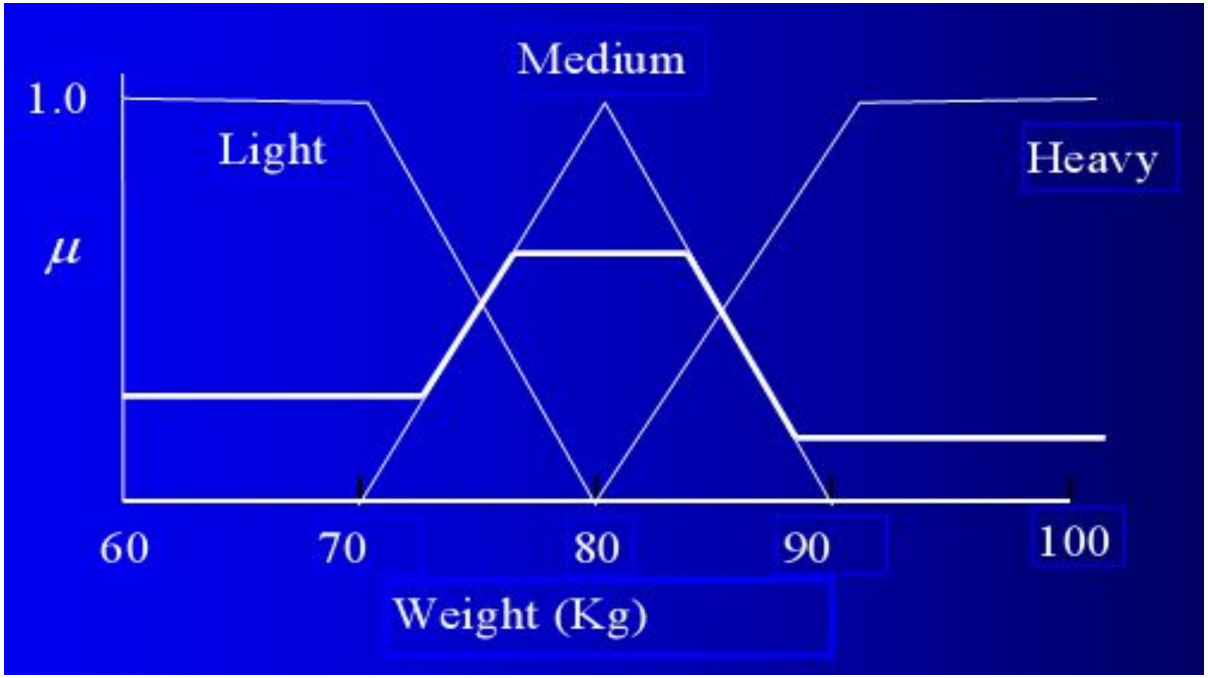
FUZZY RULE - EXAMPLE



FUZZY RULE - EXAMPLE

- The cumulative fuzzy output is obtained by OR-ing the output from each rule.

- Cumulative fuzzy output (weight at 6’1”).



FUZZY RULE - EXAMPLE

1. De-fuzzify to obtain a numerical estimate of the output.

2. Choose the middle of the range where the truth value is maximum.

3. John's weight = 80 Kg.