DEFUZZIFICATION

- Defuzzification is a process of converting output fuzzy variable into a unique number.

- Defuzzification process has the capability to reduce a fuzzy set into a crisp single-valued quantity or into a crisp set;

- to convert a fuzzy matrix into a crisp matrix; or

- to convert a fuzzy number into a crisp number.

Λ-CUT OF A FUZZY SET A

- α-cut (λ-cut) of a fuzzy set A is set of all points x in X such that
- $\{(x \mid \mu_A(x) \geq \alpha\}$

universe $X = \{a, b, c, d, e, f\}$,

$$\underset{\sim}{A} = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$$

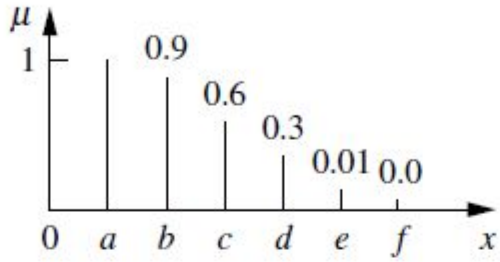


FIGURE 4.8
A discrete fuzzy set $\underset{\sim}{A}$.

This fuzzy set is shown schematically in Fig. 4.8. We can reduce this fuzzy set into several λ-cut sets, all of which are crisp. For example, we can define λ-cut sets for the values of $\lambda = 1$, 0.9, 0.6, 0.3, 0^+, and 0.

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$

$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0^+} = \{a, b, c, d, e\}, \quad A_0 = X$$

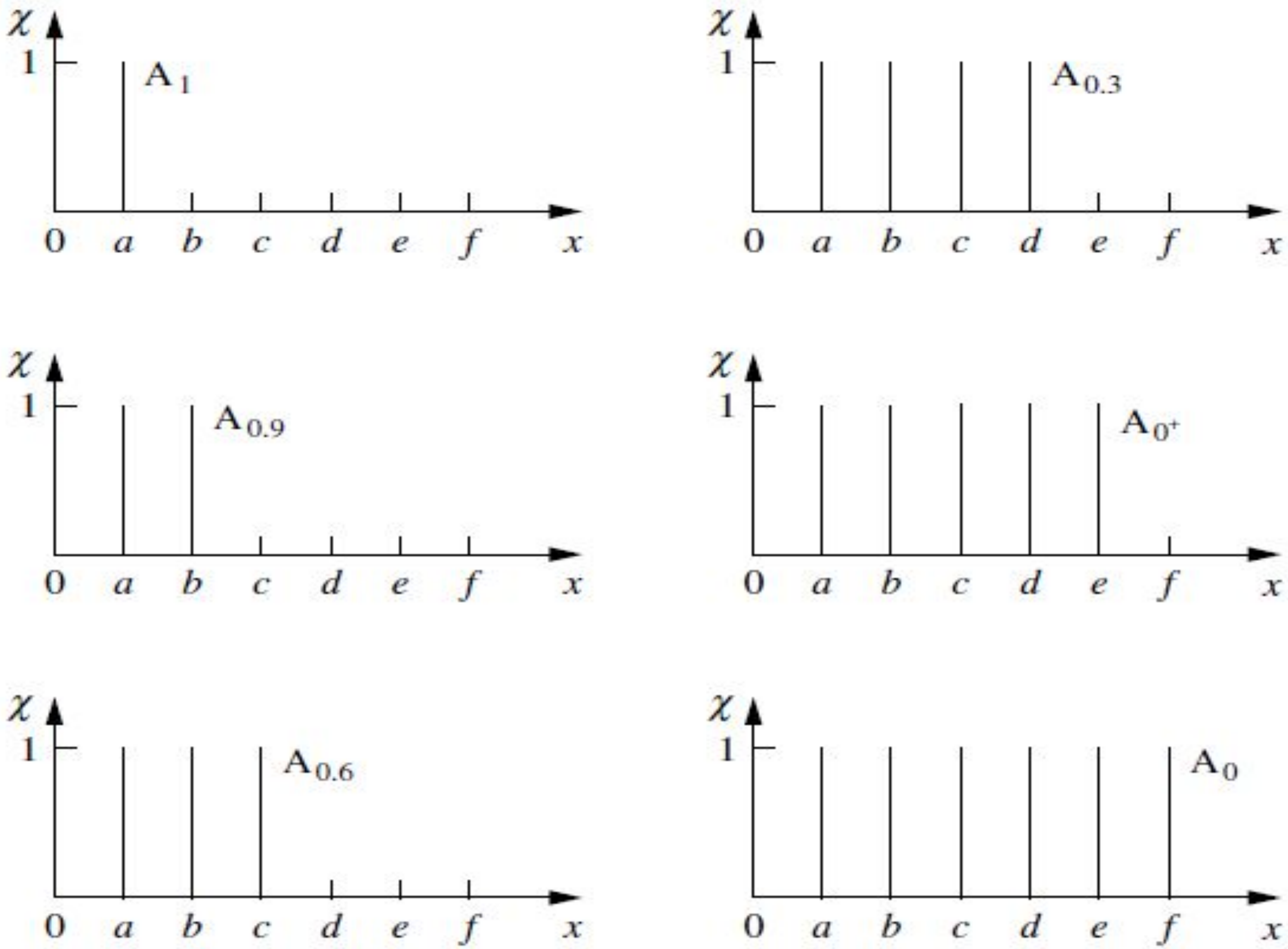


FIGURE 4.9
Lambda-cut sets for $\lambda = 1, 0.9, 0.6, 0.3, 0^+, 0$.

$$A_{0.9} = \left\{ \frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e} + \frac{0}{f} \right\} \qquad A_{0.25} = \left\{ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e} + \frac{0}{f} \right\}$$

λ-cut sets obey the following four very special properties:

1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$
2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$
3. $(\overline{\underset{\sim}{A}})_\lambda \neq \overline{A}_\lambda$ except for a value of $\lambda = 0.5$
4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$

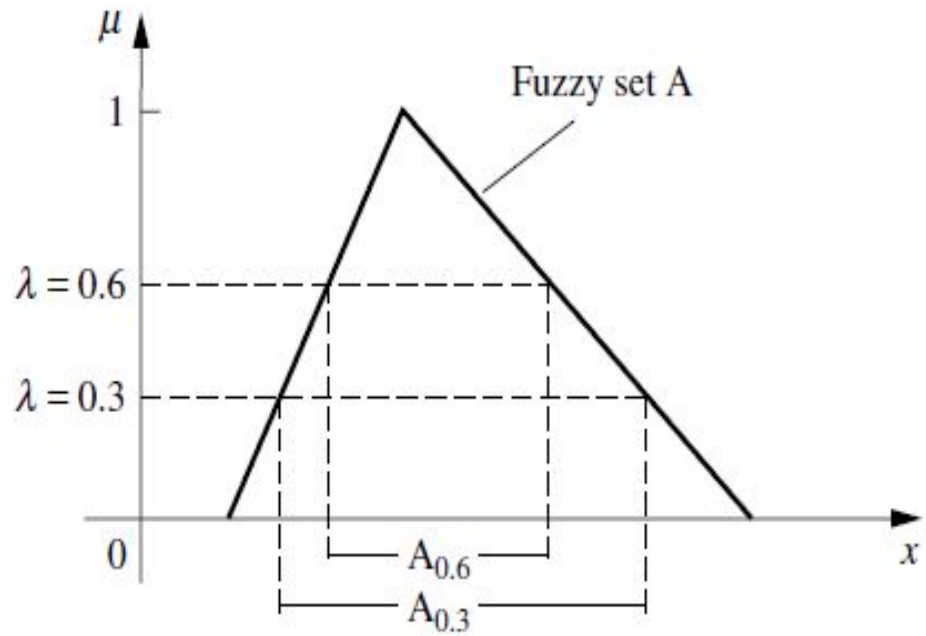


FIGURE 4.10
Two different λ-cut sets for a continuous-valued fuzzy set.

LAMBDA CUT FOR FUZZY RELATIONS

Let $\underset{\sim}{R}$ be a fuzzy relation where each row of the relational matrix is considered a fuzzy set. The jth row in a fuzzy relation matrix $\underset{\sim}{R}$ denotes a discrete membership function for a fuzzy set $\underset{\sim}{R}_j$. A fuzzy relation can be converted into a crisp relation in the following manner:

$$R_\lambda = \{(x, y) | \mu_{\underset{\sim}{R}}(x, y) \geq \lambda\}$$

where R_λ is a λ-cut relation of the fuzzy relation $\underset{\sim}{R}$.

LAMBDA CUT FOR FUZZY RELATIONS : Example

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

perform *λ-cut operations for the values of λ = 1, 0.9, 0. These crisp* relations are :

$$\lambda = 1,\ R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 0.9,\ R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 0,\ R_0 = \underset{\sim}{E}$ (whole relation; see Chapter 3)

LAMBDA CUT FOR FUZZY RELATIONS

λ-cuts on fuzzy relations obey certain properties, just as λ-cuts on fuzzy sets do

1. $(\underset{\sim}{R} \cup \underset{\sim}{S})_\lambda = R_\lambda \cup S_\lambda$
2. $(\underset{\sim}{R} \cap \underset{\sim}{S})_\lambda = R_\lambda \cap S_\lambda$
3. $(\overline{\underset{\sim}{R}})_\lambda \neq \overline{R}_\lambda$
4. For any $\lambda \leq \alpha, 0 \leq \alpha \leq 1$, then $R_\alpha \subseteq R_\lambda$

Examples:

The fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are defined as universe, $x = \{0, 1, 2, 3\}$, with the following membership fractions:

$$\mu_{\underset{\sim}{A}}(x) = \frac{2}{x+3},$$

$$\mu_{\underset{\sim}{B}}(x) = \frac{4x}{2(x+5)}.$$

Define the intervals along x-axis corresponding to the λ cut sets for each fuzzy set $\underset{\sim}{A}$ and $\underset{\sim}{B}$ for following values of λ. $\lambda = 0.2, 0.5, 0.6$.

METHODS OF DEFUZZIFICATION TO SCALARS

- Defuzzification is the process of conversion of a fuzzy quantity into a precise quantity, just as fuzzification is the conversion of a precise quantity to a fuzzy quantity.

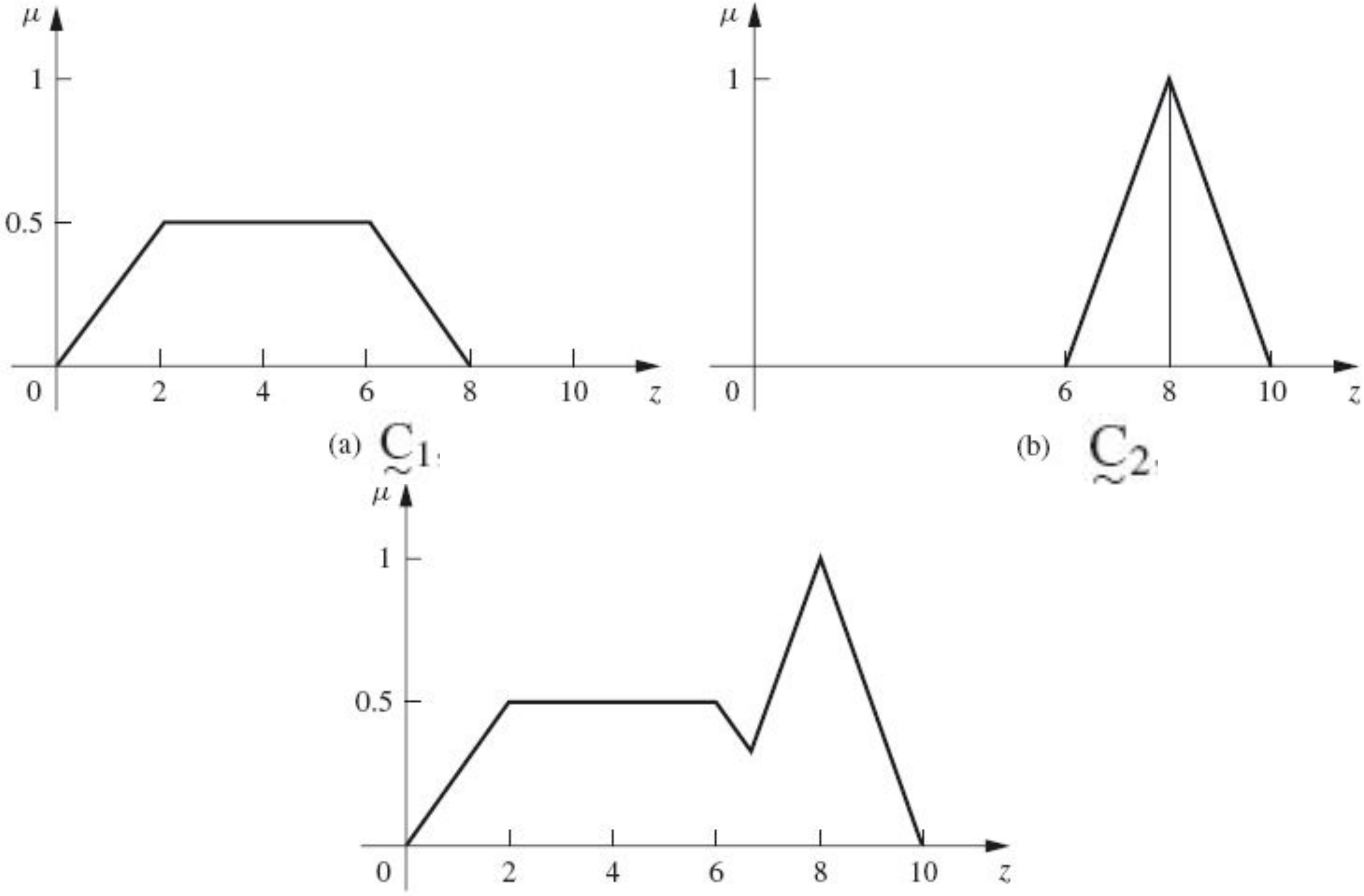
- The output of a fuzzy process can be the logical union of two or more fuzzy membership functions defined on the universe of discourse of the output variable.

- The union of these membership functions, that is, involves the max operator, which graphically is the outer envelope of the graphical shapes

$$\underset{\sim}{C}_k = \bigcup_{i=1}^{k} \underset{\sim}{C}_i = \underset{\sim}{C}.$$

METHODS OF DEFUZZIFICATION TO SCALARS

- For example:



(a) $\underset{\sim}{C}_1$.

(b) $\underset{\sim}{C}_2$.

(c) $\underset{\sim}{C} = \underset{\sim}{C}_1 \cup \underset{\sim}{C}_2$.

METHODS OF DEFUZZIFICATION TO SCALARS

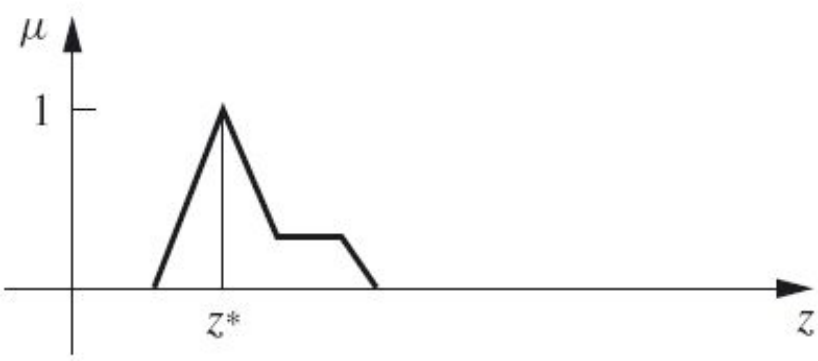
- Defuzzification methods include:
  - Max-membership principle,
  - Centroid method,
  - Weighted average method,
  - Mean-max membership,
  - Center of sums,
  - Center of largest area,
  - First of maxima, last of maxima.

MAX MEMBERSHIP METHOD

- Also known as the *height method*
- Fuzzy set with the largest membership value is selected.
- If two decisions have same membership max, use the average of the two.

$$\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z), \qquad \text{for all } z \in Z,$$

where z^* is the defuzzified value.



WEIGHTED AVERAGE METHOD

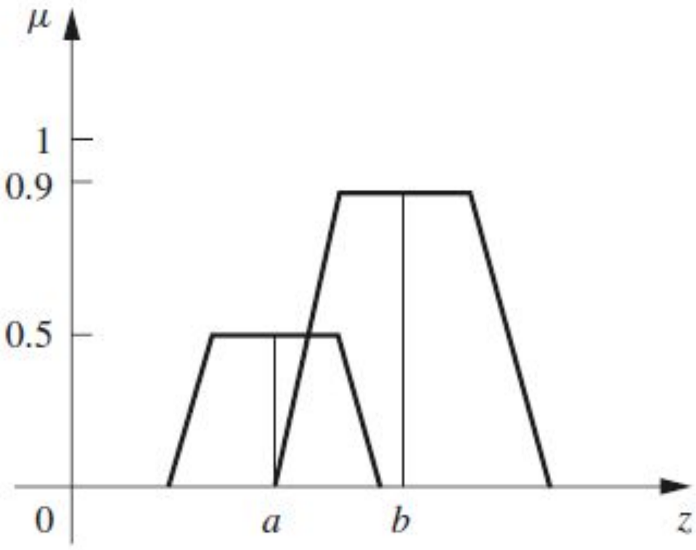
- Computationally efficient method
- Unfortunately, it is *usually restricted to symmetrical output membership functions*
- It is formed by weighting each membership function in the output by its respective maximum membership value

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{z}) \cdot \bar{z}}{\sum \mu_{\underset{\sim}{C}}(\bar{z})},$$

where $\sum$ denotes the algebraic sum and where $\bar{z}$ is the centroid of each symmetric membership function.

WEIGHTED AVERAGE METHOD

- Example:



$$z^* = \frac{a(0.5) + b(0.9)}{0.5 + 0.9}.$$

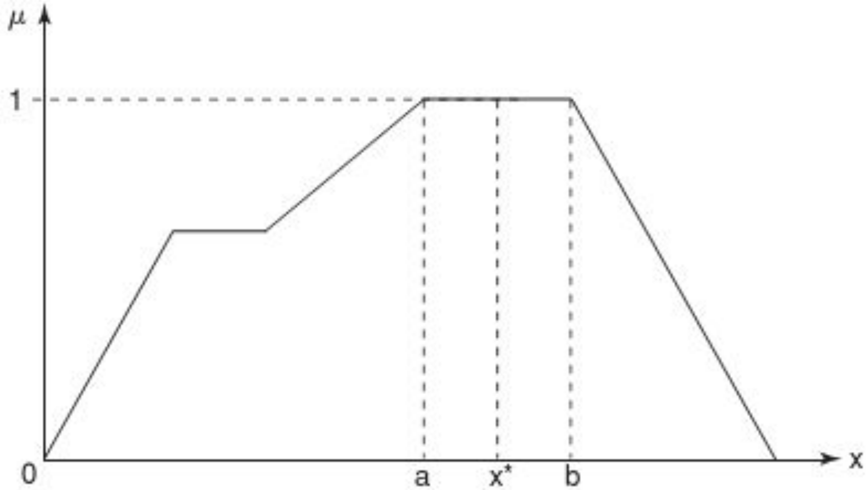
- Since the method is limited to symmetrical membership functions, the values *a* and *b are the means (centroids) of their respective shapes*

MEAN MAX MEMBERSHIP METHOD

- Also called *middle-of-maxima is closely related* to the first method, except that the locations of the maximum membership can be non-unique

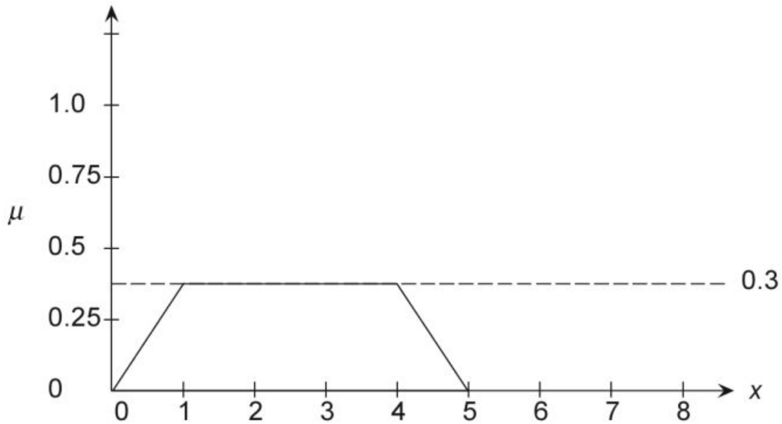
- The maximum membership can be a plateau rather than a single point

$$x^* = \frac{a+b}{2}$$

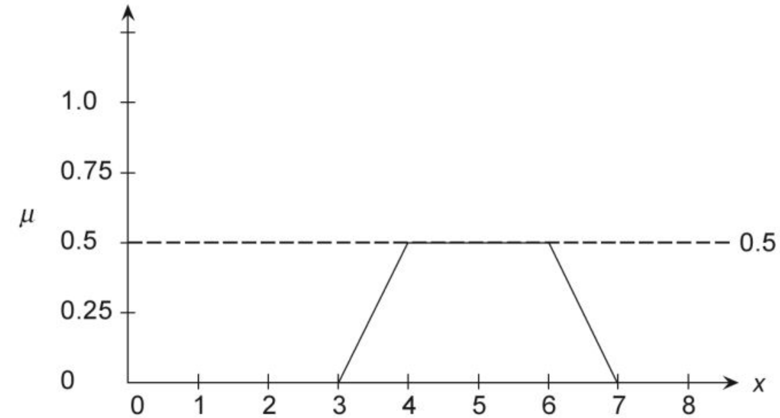


Weighted average and mean max membership METHOD

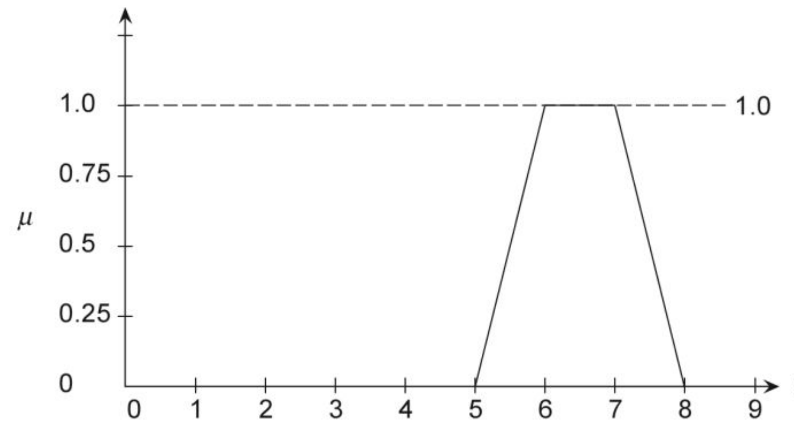
- Example:



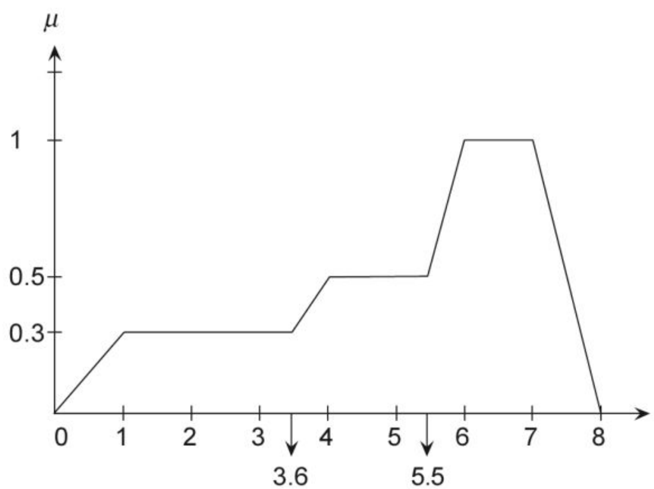
(a) $\tilde{A}_1$



(b) $\tilde{A}_2$



(c) $\tilde{A}_3$



Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

- Weighted average

$$z^* = \frac{(0.3 \times 2.5) + (0.5 \times 5) + (1 \times 6.5)}{0.3 + 0.5 + 1} = 5.41 \text{ meters}$$

- Mean max membership

$$z^* \quad (6 + 7)/2 = 6.5 \text{ meters}$$

CENTROID METHOD

- This method is also known as center-of-mass, center-of-area, or center-of-gravity method.
- It is the most commonly used defuzzification method.
- The defuzzified output x* is defined as

$$x^* = \frac{\int \mu(x)\, x\, dx}{\int \mu(x)\, dx}$$

for a continuous membership function, and

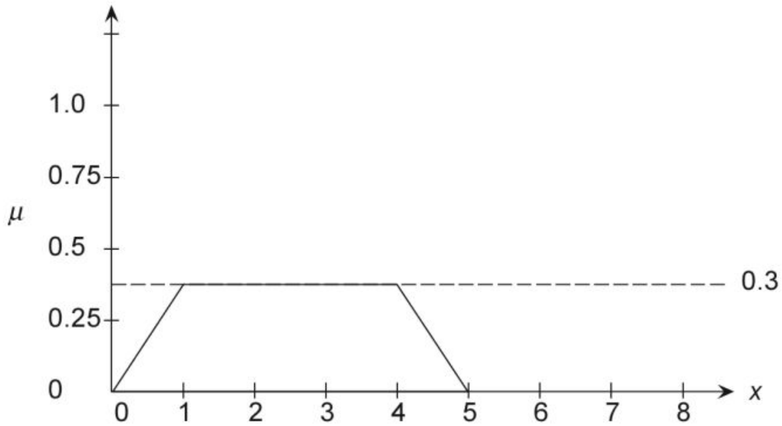
$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$

for a discrete membership function.

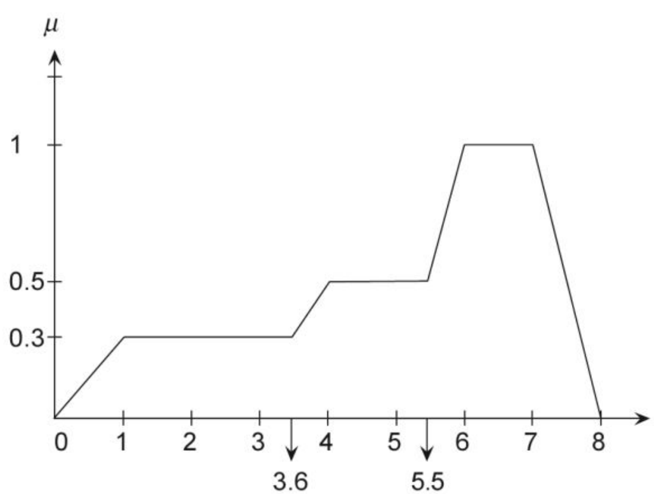
Here, n represents the number of elements in the sample, x_i's are the elements, and $\mu(x_i)$ is its membership function.

CENTROID METHOD

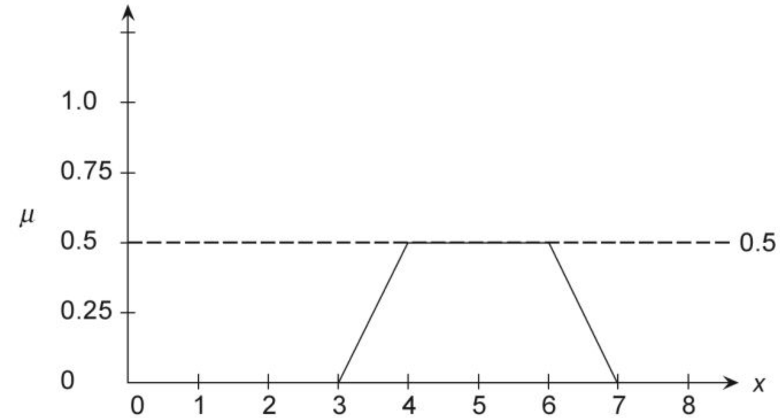
- Example:



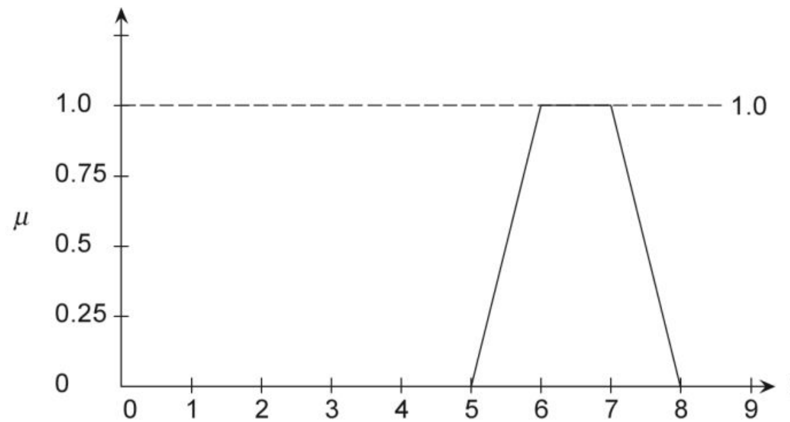
(a) $\tilde{A}_1$



Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.



(b) $\tilde{A}_2$



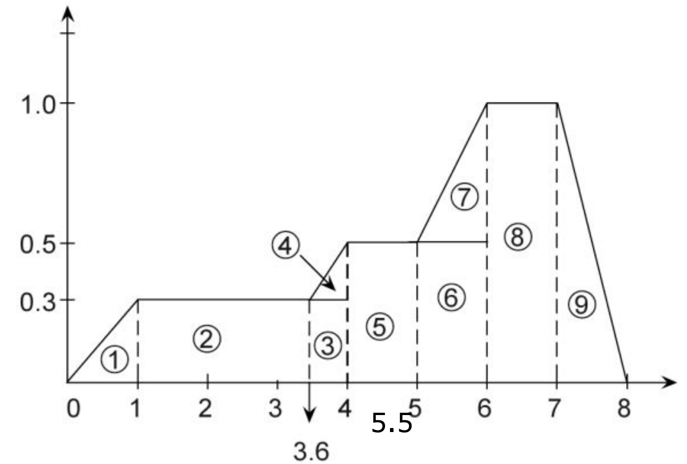
(c) $\tilde{A}_3$

CENTROID METHOD

$$x^* = \frac{\sum A\bar{x}}{\sum A}$$

$$x^* = 18.353 / 3.715$$

$$= 4.9$$



Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ viewed as area segments.

Computation of x^*

Area segment no.	Area (A)	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
2	$2.6 \times 0.3 = 0.78$	2.3	1.794
3	$0.3 \times 0.4 = 0.12$	3.8	0.456
4	$\frac{1}{2} \times 0.4 \times 0.2 = 0.04$	3.8667	0.1546
5	$1.5 \times 0.5 = 0.75$	4.75	3.5625
6	$1.5 \times 0.5 = 0.75$	5.75	1.4375
7	$\frac{1}{2} \times 0.5 \times 0.5 = 0.125$	5.833	0.729
8	$1 \times 1 = 1$	6.5	6.5
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

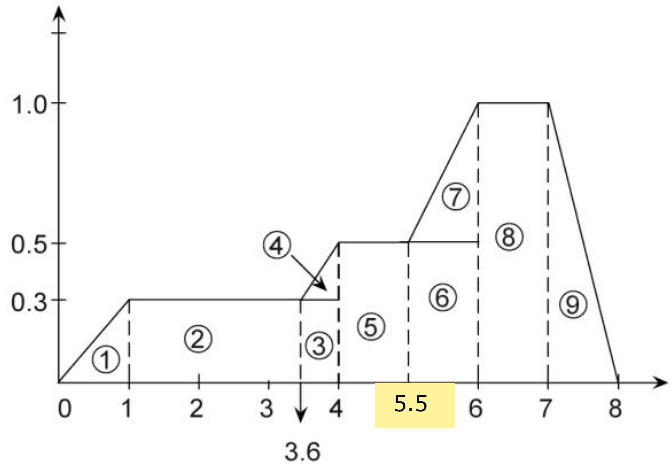
Area (A) shows the area of the segments of the aggregated fuzzy set and $\bar{x}$ shows the corresponding centroid.

CENTROID METHOD

$$x^* = \frac{\sum A\bar{x}}{\sum A}$$

$$x^* = 18.353 / 3.715$$

$$= 4.9$$



Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ viewed as area segments.

Computation of x^*

Area segment no.	Area (A)	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
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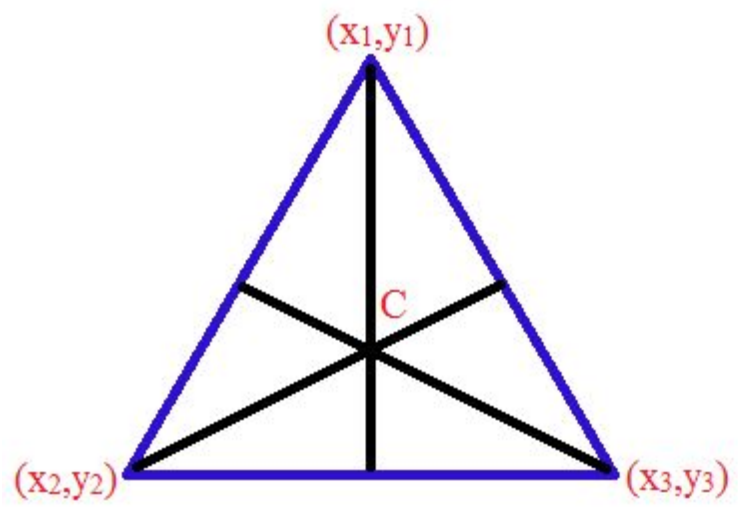
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

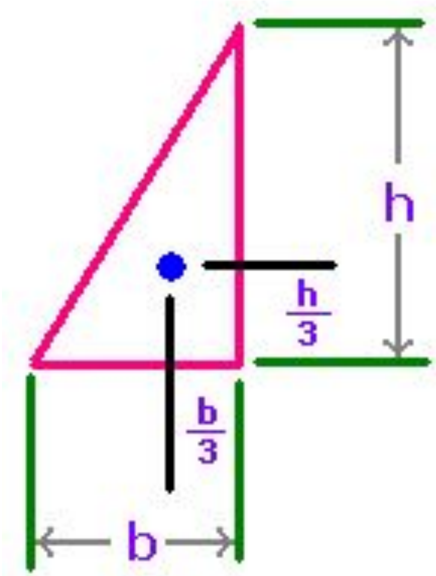
Area (A) shows the area of the segments of the aggregated fuzzy set and $\bar{x}$ shows the corresponding centroid.

Centroid and area calculation

- Centroid of Triangle



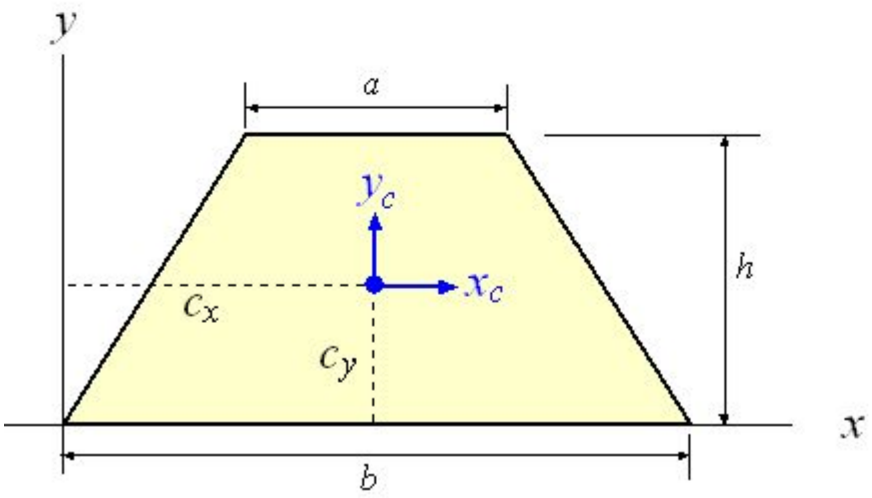
$$C = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$



Area = 1/2 *** Base * Height**

Centroid and area calculation

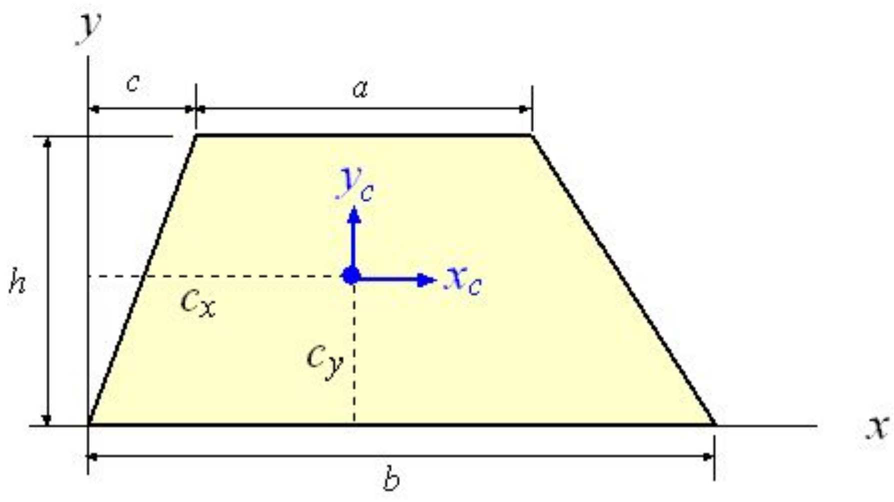
- Centroid of Isosceles Trapezoid



C_x	**C_y**	**Area**
$\frac{b}{2}$	$\frac{h}{3}\left(\frac{2a+b}{a+b}\right)$	$\frac{h}{2}(a+b)$

Centroid and area calculation

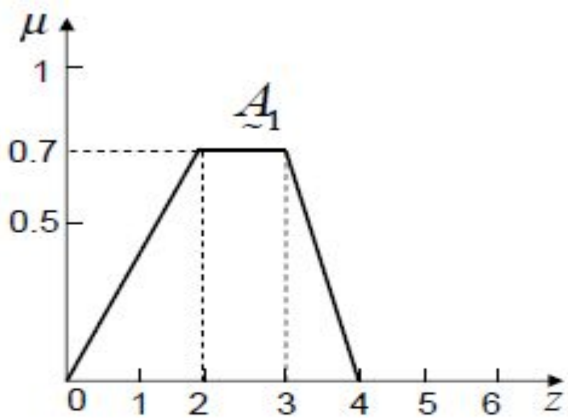
- Centroid of General Trapezoid

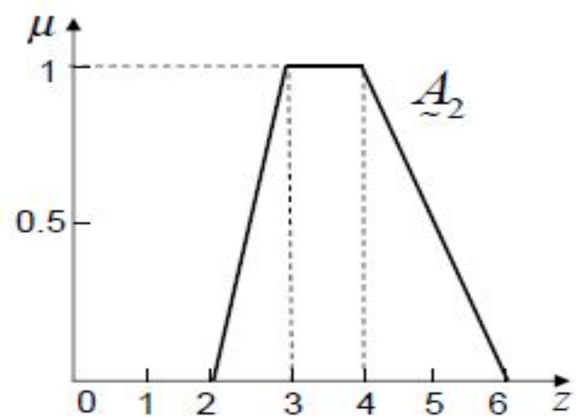


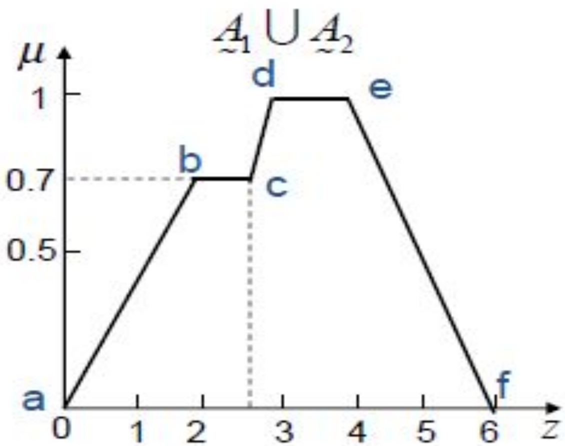
C_x	C_y	Area
$\dfrac{2ac + a^2 + cb + ab + b^2}{3(a+b)}$	$\dfrac{h}{3}\left(\dfrac{2a+b}{a+b}\right)$	$\dfrac{h}{2}(a+b)$

EXAMPLE

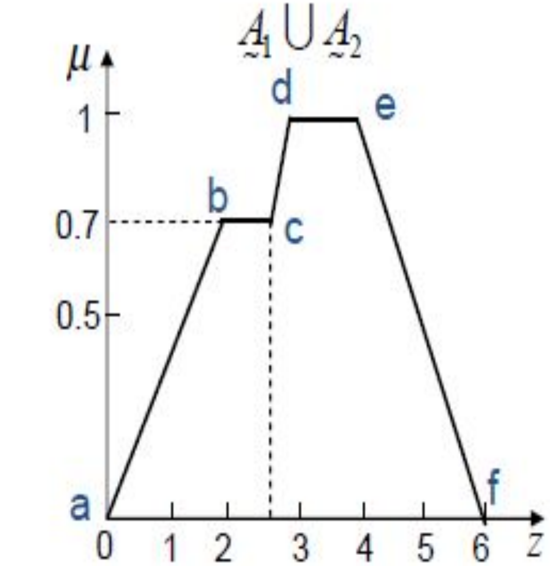
Find the defuzzified output value using different defuzzification methods for the given membership functions







$$\mu(z) = \begin{cases} 0.35z & 0 \le z < 2 \\ 0.7 & 2 \le z < 2.7 \\ z - 2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z + 3 & 4 \le z \le 6 \end{cases}$$



$$\mu(z)=\begin{cases} 0.35z & 0 \le z < 2 \\ 0.7 & 2 \le z < 2.7 \\ z-2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z+3 & 4 \le z \le 6 \end{cases}$$

(1) Maxima method

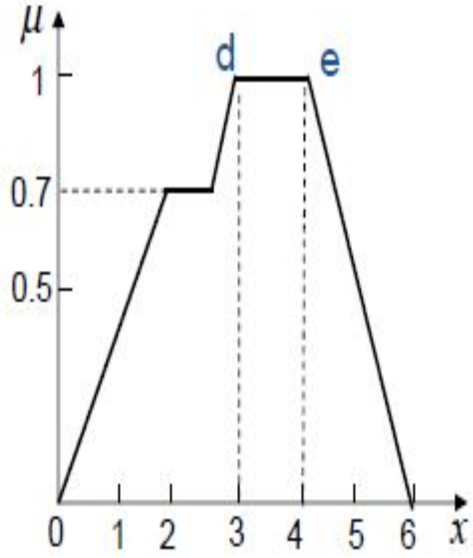
Not applicable since there is no a single maximum point.

(2) Centroid method $z^* = \dfrac{\int \mu_{Ç}(z)z\,dz}{\int \mu_{Ç}(z)\,dz}$

$$\text{Numerator} = \int_0^2 0.35\, z^2\, dz + \int_2^{2.7} 0.7Z\, dz + \int_{2.7}^{3} (z^2 - 2z)\, dz + \int_3^4 z\, dz + \int_4^6 (-0.5z^2 + 3z)\, dz = 10.98.$$

$$\text{Denominator} = \int_0^2 0.35\, z\, dz + \int_2^{2.7} 0.7\, dz + \int_{2.7}^{3} (z - 2)\, dz + \int_3^4 dz + \int_4^6 (-0.5z + 3)\, dz = 3.445.$$

$$z^* = \frac{\text{Numerator}}{\text{Demoninator}} = \frac{10.98}{3.445} = 3.187.$$



(3) Weighted average method

Not applicable since the membership functions are not symmetrical.

(4) Middle-of-maxima method

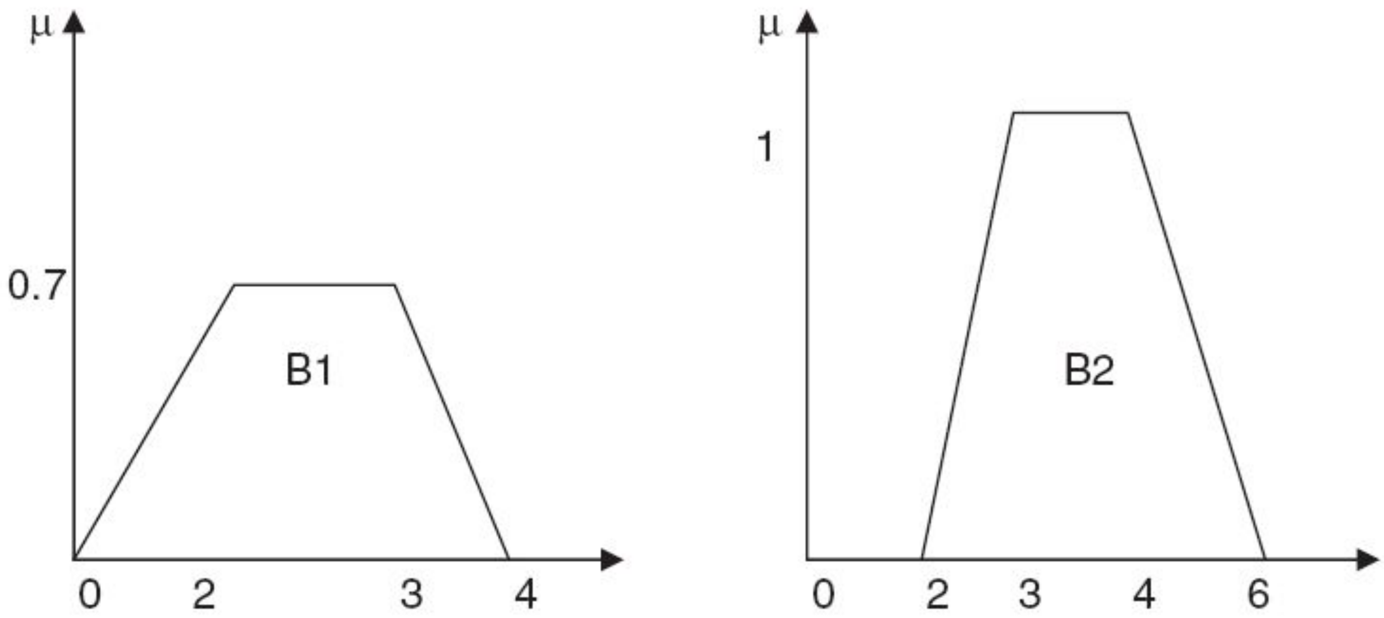
$$z^* = \frac{3+4}{2} = 3.5$$

$$\mu(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x-2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ -0.5x+3 & 4 \le x \le 6 \end{cases}$$

April 2007

EXERCISE

Two companies bid for a contract. The fuzzy set of two companies $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ is shown in the following figure. Find the defuzzified value z^* using different methods.



CENTER OF SUMS

- This method employs the algebraic sum of the individual fuzzy subsets instead of their unions

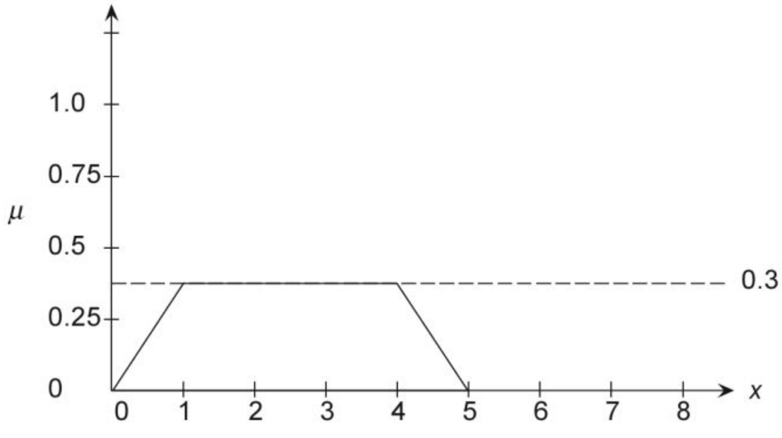
- The calculations here are very fast but the main drawback is that the intersecting areas are added twice

- The defuzzified value z* is given by

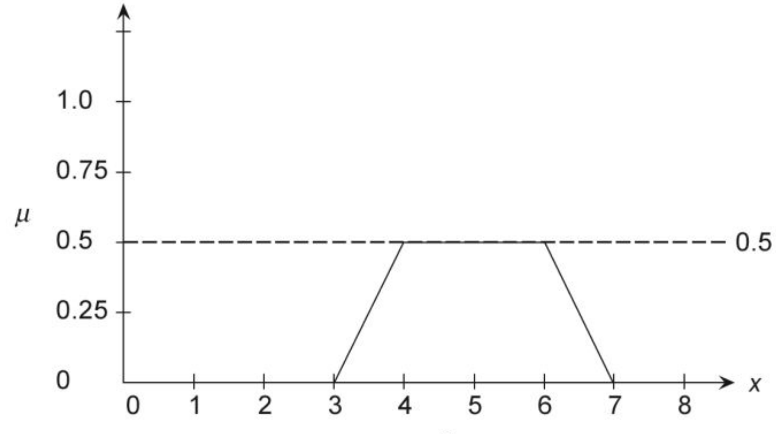
$$z^* = \frac{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z \bar{z}\, dz}{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z dz},$$

CENTER OF SUMS

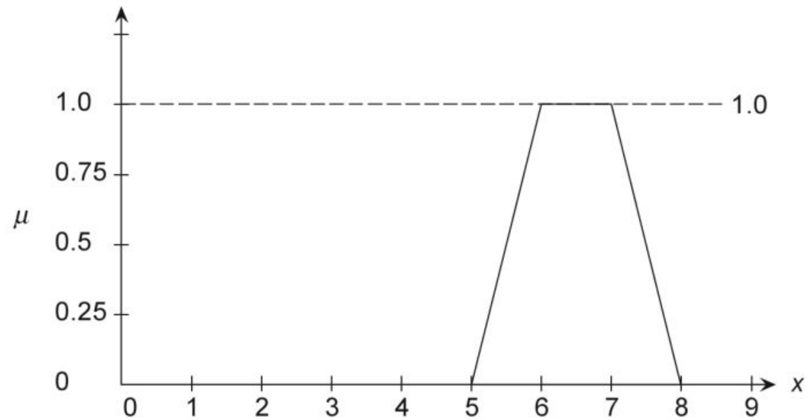
- Example:



(a) $\tilde{A}_1$



(b) $\tilde{A}_2$



(c) $\tilde{A}_3$

$$z^* = \frac{[2.5 \times 0.5 \times 0.3(3+5) + 5 \times 0.5 \times 0.5(2+4) + 6.5 \times 0.5 \times 1(3+1)]}{[0.5 \times 0.3(3+5) + 0.5 \times 0.5(2+4) + 0.5 \times 1(3+1)]}$$

$$= 5.0$$

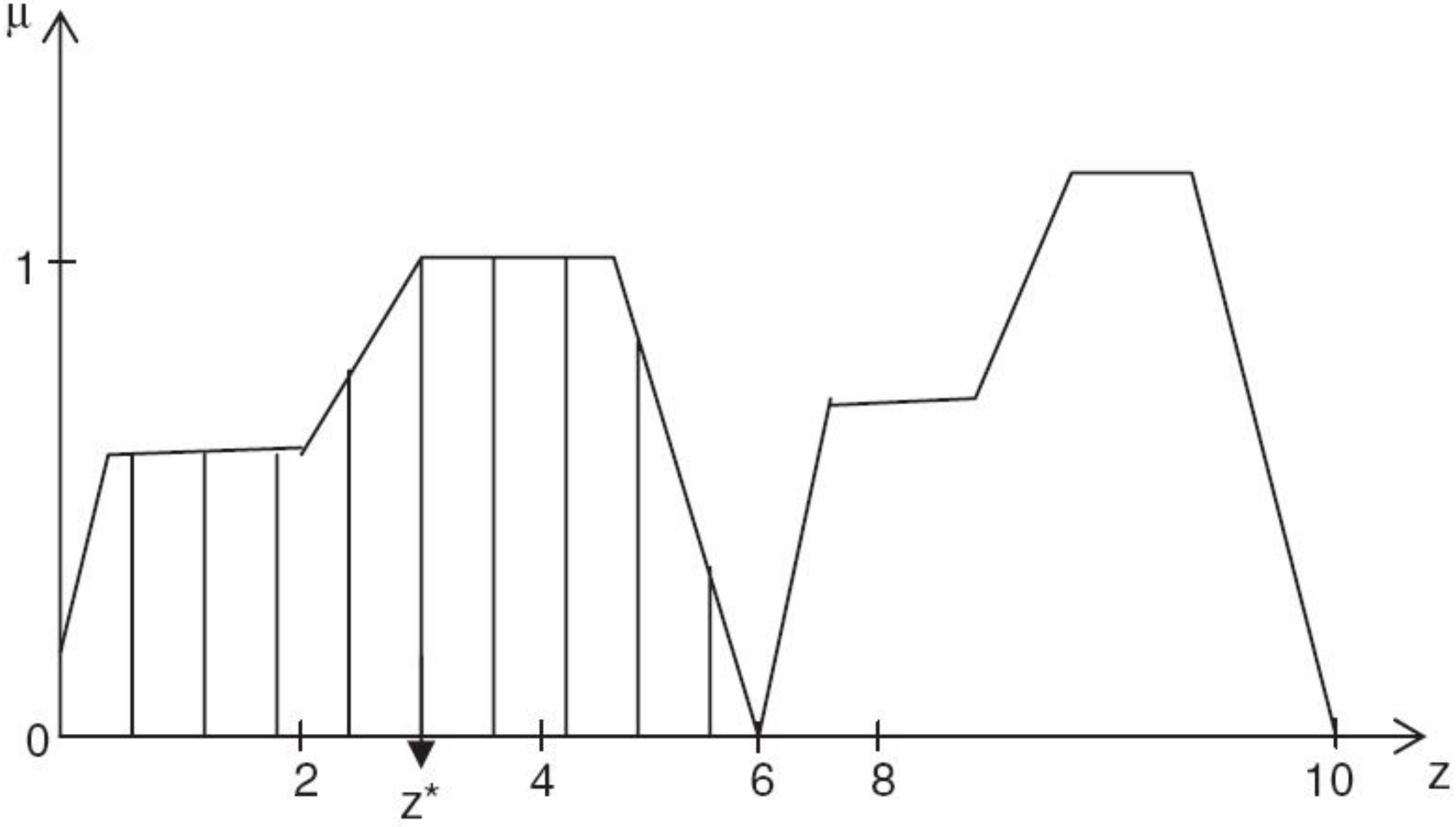
CENTER OF LARGEST AREA

If the output fuzzy set has at least two convex subregions, then the center of gravity (i.e., $z*$ is calculated using the centroid method) of the convex fuzzy sub-region with the largest area is used to obtain the defuzzified value $z*$ of the output.

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z) z \, dz}{\int \mu_{\underset{\sim}{C}_m}(z) \, dz},$$

where $\underset{\sim}{C}_m$ is the convex subregion that has the largest area making up $\underset{\sim}{C}_k$.

CENTER OF LARGEST AREA



FIRST OF MAXIMA (or LAST OF MAXIMA)

This method uses the overall output or union of all individual output fuzzy sets C_k to determine the smallest value of the domain with maximized membership degree in C_k.

First-of-Maxima

$$z_0 = \min\{z \mid C(z) = \max_w C(w)\}.$$

Last-of-Maxima

$$z_0 = \max\{z \mid C(z) = \max_w C(w)\}.$$

FIRST OF MAXIMA (LAST OF MAXIMA)

The steps used for obtaining crisp values are as follows:

1. Initially, the maximum height in the union is found:

$$\operatorname{hgt}(\underset{\sim}{c_i}) = \sup_{x \in X} \mu_{\underset{\sim}{c_i}}(x)$$

   where sup is supremum, i.e., the least upper bound.

2. Then the first of maxima is found:

$$x^* = \inf_{x \in X} \left\{ x \in X \,\middle|\, \mu_{\underset{\sim}{c_i}}(x) = \operatorname{hgt}(\underset{\sim}{c_i}) \right\}$$

   where inf is the infimum, i.e., the greatest lower bound.

3. After this the last maxima is found:

$$x^* = \sup_{x \in X} \left\{ x \in X \,\middle|\, \mu_{\underset{\sim}{c_i}}(x) = \operatorname{hgt}(\underset{\sim}{c_i}) \right\}$$

   where

   sup = supremum, i.e., the least upper bound

   inf = infimum, i.e., the greatest lower bound

FIRST OF MAXIMA (LAST OF MAXIMA)

- Example