Chapter 8. Velocity Potential and Stream Function

8.1. Velocity Potential

Velocity potential is an interesting and useful concept, particularly in combination with stream function. Velocity potential, ϕ, is defined as

$$\vec{u} = -\nabla\phi, \tag{8.1}$$

or equivalently,

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad \text{and} \quad w = -\frac{\partial\phi}{\partial z}. \tag{8.2}$$

According to (8.1), flow is from a higher potential toward a lower potential. As we discussed in a previous chapter, the direction of the flow is along the steepest descent of the velocity potential. In order to write the velocity field in terms of velocity potential as in (8.1), the velocity field should satisfy a specific property. By taking curl of (8.1), we have

$$\nabla\times\vec{u} = \vec{\omega} = -\nabla\times\left(\nabla\phi\right) \equiv 0. \tag{8.3}$$

If the vorticity of a flow is *not* zero, therefore, velocity potential does not exist for such a flow. Now, by taking divergence of (8.1), we have

$$\nabla\cdot\vec{u} = -\nabla\cdot\left(\nabla\phi\right) = -\nabla^2\phi. \tag{8.4}$$

For non-divergent flow, $\nabla\cdot\vec{u} = 0$, then, the corresponding velocity potential satisfies the Laplacian equation[☞], i.e.,

$$\nabla^2\phi = 0. \tag{8.5}$$

In cylindrical coordinate system, (8.1) can be written in a component form as

$$u_r = -\frac{\partial\phi}{\partial r}, \quad u_\theta = -\frac{\partial\phi}{r\partial\theta}, \quad \text{and} \quad u_z = -\frac{\partial\phi}{\partial z}. \tag{8.6}$$

(E1) Let us consider in Cartesian coordinates a uniform flow

$$\vec{u} = \left(U, 0, 0\right). \tag{8.7}$$

Then, the velocity potential is defined by

$$\phi = -Ux. \tag{8.8}$$

(E2) Let us consider a two-dimensional *point source* as an example. Obviously, only radial velocity is nonzero. Thus,

$$u_r = u_r(r), \quad \text{and} \quad u_\theta = 0. \tag{8.9}$$

[☞] The solutions of Laplacian equation have been heavily studied in different coordinate systems and can be found in many books on mathematical physics.

From the conservation of mass, the total mass flux leaving the smaller circle in Figure 8.1a should be identical with the total mass flux leaving the bigger circle. That is,

$$u_r(r=r_1)2\pi r_1 = u_r(r=r_2)2\pi r_2 = \text{const.}. \tag{8.10}$$

Thus,

$$u_r = \frac{m}{2\pi r}, \tag{8.11}$$

where m is a constant in (8.10) and represents the source strength. Then, the velocity potential should satisfy

$$\frac{\partial \phi}{\partial r} = -u_r = -\frac{m}{2\pi r}, \tag{8.12}$$

and

$$\frac{\partial \phi}{r\partial \theta} = 0. \tag{8.13}$$

By integrating (8.12), we have

$$\phi(r) = -\frac{m}{2\pi}\ln r + C, \tag{8.14}$$

which is plotted in red in Figure 8.1b.

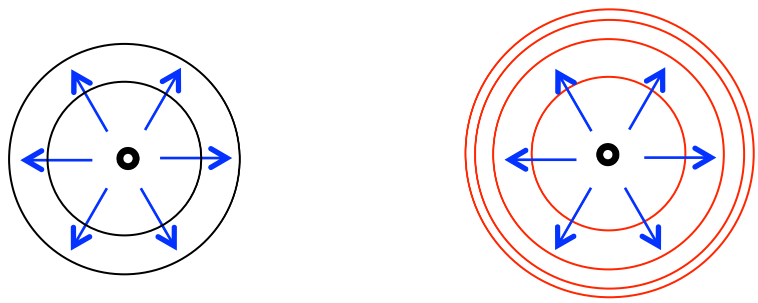

Figure 8.1. A two-dimensional point source (left) and its velocity potential (right; red circles).

(E3) Let us consider a two-dimensional velocity field

$$u_r = 0, \quad \text{and} \quad u_\theta = \frac{\Gamma}{r}, \tag{8.15}$$

where azimuthal velocity is inversely proportional to the radial distance from the origin (Figure 8.2a). It can be shown that

$$\nabla \times \vec{u} = \frac{1}{r}\left(\frac{\partial r u_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) = 0. \tag{8.16}$$

Thus, even though the flow is rotating around the center of a circle, the flow is actually irrotational as can be seen in (8.16). Thus, the flow is called the *irrotational vortex*. Then, the velocity potential should satisfy

$$\frac{\partial \phi}{\partial r} = 0 \ , \tag{8.17}$$

and

$$\frac{\partial \phi}{r \partial \theta} = -u_\theta = -\frac{\Gamma}{r} \ . \tag{8.18}$$

By integrating (8.18), the velocity potential is given by (see Figure 8.2b)

$$\phi(\theta) = -\Gamma \theta + C \ . \tag{8.19}$$

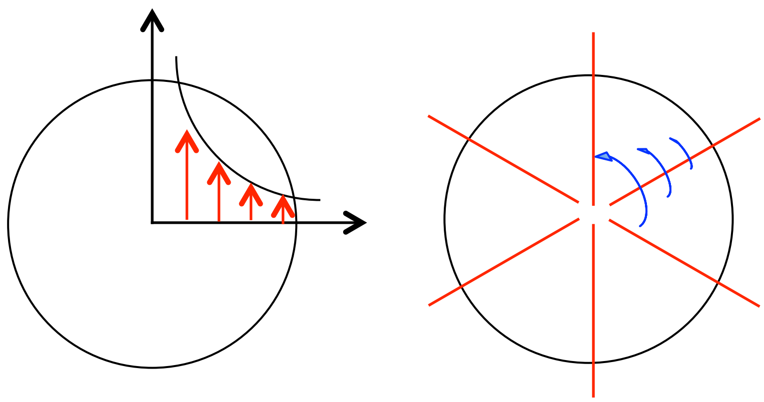

Figure 8.2. An irrotational vortex (left) and its velocity potential (right; red lines).

8.2. Stream Function

We have already discussed the concept of streamline. Streamlines are defined as lines that are tangent to velocity vectors at any instant (see Figure 8.3). A mathematical expression for streamline, then, is given by

$$\vec{v} \times d\vec{l} = 0 \ , \tag{8.20}$$

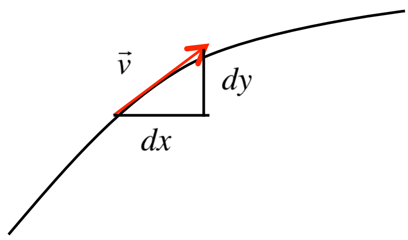

Figure 8.3. Streamline with a line segment $d\vec{l} = (dx, dy)$ and velocity vector $\vec{v}$.

or in component form,

$$\vec{v} \times d\vec{l} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{bmatrix} = 0, \tag{8.21}$$

where $d\vec{l} = (dx, dy, dz)$ is a line segment of the streamline. Then, we have from (8.21) that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{8.22}$$

(E4) Let a line be defined such that its line segment $d\vec{s} = (dx, dy, dz)$ is parallel to the flow $\vec{u} = (u, v, w)$.

(a) Show that the line segment is defined uniquely except for a constant multiplication factor by

$$c = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{8.23}$$

A mathematical statement that the line segment $d\vec{s} = (dx, dy, dz)$ is parallel to the flow $\vec{u} = (u, v, w)$ is

$$\vec{u} \times d\vec{s} = 0, \tag{8.24}$$

or equivalently,

$$\hat{i}(vdz - wdy) + \hat{j}(wdx - udz) + \hat{k}(udy - vdx) = 0. \tag{8.25}$$

Equation (8.25) means that each component of the vector should vanish so that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = c. \tag{8.26}$$

(b) Show that the line, ψ, defined by

$$\hat{k} \times \nabla\psi = \vec{u} \tag{8.27}$$

satisfies the property above.

Let us define a function ψ such that

$$\vec{u} = \hat{k} \times \nabla\psi. \tag{8.28}$$

Then,

$$\vec{u} \cdot \nabla\psi = \hat{k} \times \nabla\psi \cdot \nabla\psi = 0, \tag{8.29}$$

since $\hat{k} \times \nabla\psi$ is perpendicular to $\nabla\psi$.

Let us formally define two-dimensional stream function as

$$u = -\frac{\partial \psi}{\partial y}, \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial x}. \tag{8.30}$$

Equation (8.30) can be written as

$$\hat{k} \times \nabla \psi = \vec{u}, \tag{8.31}$$

which can be generalized in three dimensions as

$$\nabla \chi \times \nabla \psi = \vec{u}, \tag{8.32}$$

where $\nabla \chi$ denotes the unit normal direction of the surface, in which the two-dimensional velocity vector resides. Taking the divergence of (8.32), we have

$$\nabla \cdot \vec{u} = \nabla \cdot \left(\nabla \chi \times \nabla \psi \right) = \frac{\partial}{\partial x_i} \left(\varepsilon_{ijk} \frac{\partial \chi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \right)$$

$$= \varepsilon_{kij} \frac{\partial}{\partial x_i} \left(\frac{\partial \chi}{\partial x_j} \right) \frac{\partial \psi}{\partial x_k} + \frac{\partial \chi}{\partial x_j} \varepsilon_{kij} \frac{\partial}{\partial x_k} \left(\frac{\partial \psi}{\partial x_i} \right),$$

$$= \left(\nabla \times \nabla \chi \right) \cdot \nabla \psi + \nabla \chi \cdot \left(\nabla \times \nabla \psi \right) = 0. \tag{8.33}$$

Thus, stream function exists only for a non-divergent flow.

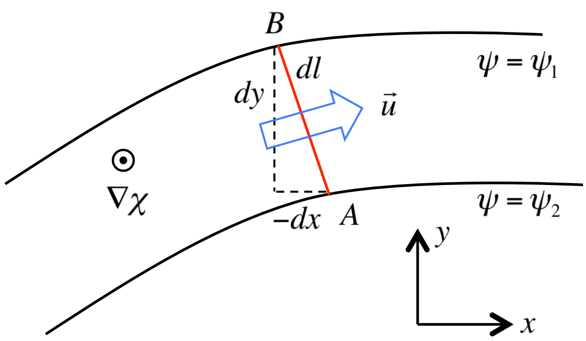

Figure 8.4. Flow between the $\psi = \psi_1$ and $\psi = \psi_2$ with $\psi_2 > \psi_1$.

Let us consider the simple case of two-dimensional flow on a plane with a normal direction $\nabla \chi$ as shown in Figure 8.4. Then, we can verify from (8.32) that the velocity vector in the direction with a higher value of stream function on the right-hand side, i.e., $\psi_2 > \psi_1$. The amount of flow across two points A and B is given by

$$dV = udy - vdx$$

$$= -\frac{\partial \psi}{\partial y} dy - \frac{\partial \psi}{\partial x} dx = -d\psi, \tag{8.34}$$

where $d\psi$ is the difference in the values of stream functions over the distance dl from A to B, i.e., ($-d\psi = \psi_A - \psi_B = \psi_2 - \psi_1 > 0$). Thus, the volume transport dV as depicted in Figure 8.4 is positive. Again, flow between two stream functions is in the direction with the larger value on the right-hand side of the flow.

For a two-dimensional flow, vorticity in the direction of $\nabla\chi$ is given by

$$\nabla\chi \cdot (\nabla \times \vec{u}) = \nabla\chi \cdot \left(\nabla \times (\nabla\chi \times \nabla\psi)\right)$$

$$= \frac{\partial\chi}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial\chi}{\partial x_l}\frac{\partial\psi}{\partial x_m}\right) = \frac{\partial\chi}{\partial x_i}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial\chi}{\partial x_l}\frac{\partial^2\psi}{\partial x_j \partial x_m}$$

$$= \frac{\partial\chi}{\partial x_i}\left(\frac{\partial\chi}{\partial x_i}\frac{\partial^2\psi}{\partial x_j \partial x_j} - \frac{\partial\chi}{\partial x_j}\frac{\partial^2\psi}{\partial x_i \partial x_j}\right)$$

$$= \frac{\partial\chi}{\partial x_i}\frac{\partial\chi}{\partial x_i}\frac{\partial^2\psi}{\partial x_j \partial x_j} - \frac{\partial\chi}{\partial x_i}\frac{\partial\chi}{\partial x_j}\frac{\partial^2\psi}{\partial x_i \partial x_j} = \nabla^2\psi\,. \tag{8.35}$$

Thus, for an irrotational flow, Laplacian of stream function is zero, i.e.,

$$\nabla^2\psi = 0\,. \tag{8.36}$$

For an irrotational, non-divergent flow, therefore, Laplacian of velocity potential (see (8.5)) and Laplacian of stream function both vanish.

(E5) A two-dimensional steady flow has velocity components

$$u = y \quad \text{and} \quad v = x\,. \tag{8.37}$$

Show that the streamlines are rectangular hyperbolas

$$x^2 - y^2 = \text{const}\,. \tag{8.38}$$

Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a 90° corner.

From the definition of stream function

$$v = \frac{\partial\psi}{\partial x} = x\,, \tag{8.39}$$

and

$$u = -\frac{\partial\psi}{\partial y} = y\,, \tag{8.40}$$

so that

$$\psi = \frac{1}{2}x^2 - \frac{1}{2}y^2 + C\,, \tag{8.41}$$

or equivalently,

$$x^2 - y^2 = \text{const}, \tag{8.42}$$

which represents the hyperbolic curves (Figure 8.5). Note also that

$$\hat{k} \cdot \nabla \times \vec{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 1 - 1 = 0 \,. \tag{8.43}$$

Thus the flow is *irrotational*.

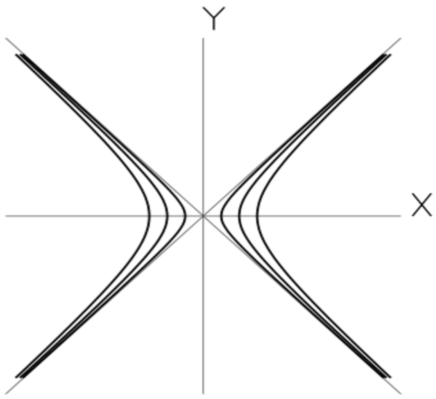

Figure 8.5. Streamlines in (8.42) for C=1, C=4, and C=9.

(E6) Find the stream function for a uniform flow of uniform strength U in the positive x direction.

Since

$$-U = -\frac{\partial \psi}{\partial y}, \tag{8.44}$$

we have

$$\psi = Uy + C \,. \tag{8.45}$$

(E7) Find the stream function for a source of strength m.

Since

$$-\frac{\partial \psi}{r \partial \theta} = v_r = \frac{m}{2\pi r}, \tag{8.46}$$

we have

$$\psi = -\frac{m}{2\pi}\theta \,. \tag{8.47}$$

8.3. Vortex Filament and Vortex Tube

Vortex filament is a line that is everywhere parallel to vorticity vector. As in streamline, vortex filament, then, is defined by

$$\frac{dx}{\xi}=\frac{dx}{\eta}=\frac{dz}{\zeta}, \tag{8.48}$$

where the vorticity vector $\vec{\omega}=(\xi,\eta,\zeta)$. *Vortex tube* is a collection of vortex filaments such that they form a surface.

(E8) The velocity field of a flow in cylindrical coordinates (r,θ,z) is $\vec{u}=(u_r,u_\theta,u_z)=(0,arz,0)$ where a is a constant.

(a) Show that the vorticity components are $\vec{\omega}=(\omega_r,\omega_\theta,\omega_z)=(-ar,0,2az)$.

Vorticity is defined in terms of velocity as

$$\vec{\omega}=\nabla\times\vec{u}=\frac{1}{r}\begin{bmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ u_r & ru_\theta & u_z\end{bmatrix}=\frac{1}{r}\begin{bmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ 0 & ar^2z & 0\end{bmatrix}$$

$$=-\hat{r}\frac{\partial(ar^2z)}{r\partial z}+\hat{z}\frac{\partial(ar^2z)}{r\partial r}=(-ar,0,2az). \tag{8.49}$$

(b) Verify that $\nabla\cdot\vec{\omega}=0$.

Divergence of $\vec{\omega}$ can be evaluated from the vorticity components given in (8.49). That is,

$$\nabla\cdot\vec{\omega}=\frac{1}{r}\left[\frac{\partial(r\omega_r)}{\partial r}+\frac{\partial(\omega_\theta)}{\partial\theta}+\frac{\partial(r\omega_z)}{\partial z}\right]$$

$$=\frac{1}{r}\left[\frac{\partial(-ar^2)}{\partial r}+\frac{\partial(0)}{\partial\theta}+\frac{\partial(2arz)}{\partial z}\right]=-2a+2a=0. \tag{8.50}$$

(c) Sketch the streamlines and vortex lines in an rz plane. Show that the vortex lines are given by zr^2 =constant.

Equation for the streamlines are given by

$$\frac{u_r}{dr}=\frac{u_\theta}{rd\theta}=\frac{u_z}{dz}. \tag{8.51}$$

In the present case, there is only one component (u_θ) of velocity. Since streamlines are parallel to u_θ, each streamline is defined by a line for which $(r = C, z = D)$, where C and D are constants. Thus, streamlines look like a dot in the rz plane.

Equation for the vorticity lines in rz plane are given by

$$\frac{\omega_r}{dr} = \frac{\omega_z}{dz}, \tag{8.52}$$

which for the given vorticity field in (8.49) yields

$$\frac{-ar}{dr} = \frac{2az}{dz}, \quad \text{or equivalently,} \quad -d\ln r = 2d\ln z. \tag{8.53}$$

Therefore, the vorticity lines are given by

$$\ln r^{-1} = \ln z^2 + C, \quad \text{or equivalently,} \quad rz^2 = C. \tag{8.54}$$

Figure 8.6 shows the vorticity lines in the rz plane.

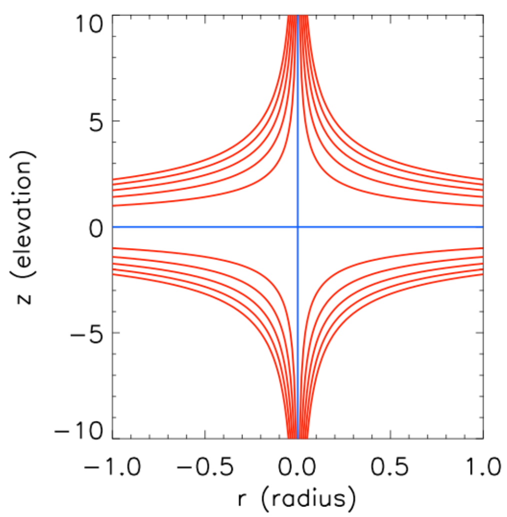

Figure 8.6. Vortex lines of Prob. (E8).

8.4. Superposition of Irrotational and Non-Divergent Flows

As we have already discussed, both stream function and velocity potential satisfy Laplace equation. Thus, the addition (superposition) of two irrotational and non-divergent flows results in irrotational and non-divergent flow. That is,

$$\nabla^2\phi = \nabla^2\phi_1 + \nabla^2\phi_2 = 0, \tag{8.55}$$

Let us consider a source and sink of equal strength on opposite sides of the origin at a distance of a from the center as shown in Figure 8.8. As we already derived in (12), velocity potentials for a source is given by

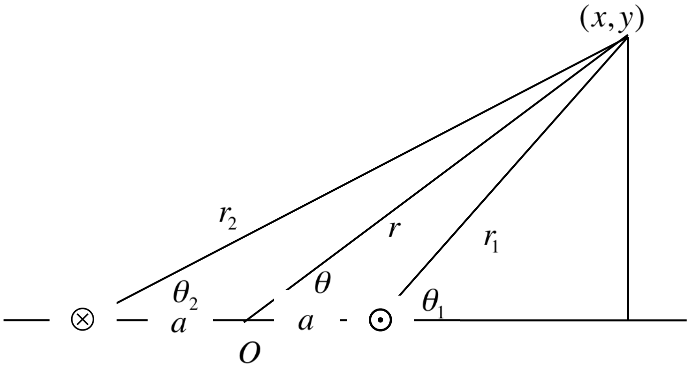

Figure 8.7. A source and a sink at a distance a from the origin.

$$\phi_{source} = -\frac{m}{2\pi r}, \tag{8.56}$$

where m is the strength of the source. In Figure 8.7, the radial distance of the source from the location $P(x,y)$ is given by

$$r_1 = \sqrt{(x-a)^2 + y^2} = \sqrt{(r\cos\theta - a)^2 + (r\sin\theta)^2}\ , \tag{8.57}$$

whereas the radial distance of the sink is given by

$$r_2 = \sqrt{(x+a)^2 + y^2} = \sqrt{(r\cos\theta + a)^2 + (r\sin\theta)^2}\ . \tag{8.58}$$

Thus, the velocity potential for a source and a sink combined is given by

$$\begin{aligned}
\phi &= -\frac{m}{2\pi}\ln r_1 + \frac{m}{2\pi}\ln r_2 \\
&= -\frac{m}{4\pi}\left[\ln\left((r\cos\theta - a)^2 + (r\sin\theta)^2\right) - \ln\left((r\cos\theta + a)^2 + (r\sin\theta)^2\right)\right] \\
&= -\frac{m}{4\pi}\left[\ln\left(r^2 - 2ra\cos\theta + a^2\right) - \ln\left(r^2 + 2ra\cos\theta + a^2\right)\right] \\
&= -\frac{m}{4\pi}\left[\ln\left(1 - \frac{2ra\cos\theta + a^2}{r^2}\right) - \ln\left(1 + \frac{2ra\cos\theta + a^2}{r^2}\right)\right] \\
&= -\frac{m}{4\pi}\left[\ln(1-\varepsilon) - \ln(1+\varepsilon)\right],
\end{aligned} \tag{8.59}$$

where

$$\varepsilon = \frac{2ra\cos\theta - a^2}{r^2}. \tag{8.60}$$

At an appreciable distance from the origin, that is, $|r| >> a$ (or, $|\varepsilon| << 1$), we can approximate the velocity potential as (see Figure 8.7)

$$\phi = -\frac{m}{4\pi}\left[\ln(1-\varepsilon)-\ln(1+\varepsilon)\right] \doteq \frac{m\varepsilon}{2\pi} \doteq \frac{m}{\pi}\frac{ra\cos\theta}{r^2} = \mu\frac{x}{x^2+y^2}, \tag{8.61}$$

where

$$\mu = \frac{ma}{\pi}. \tag{8.62}$$

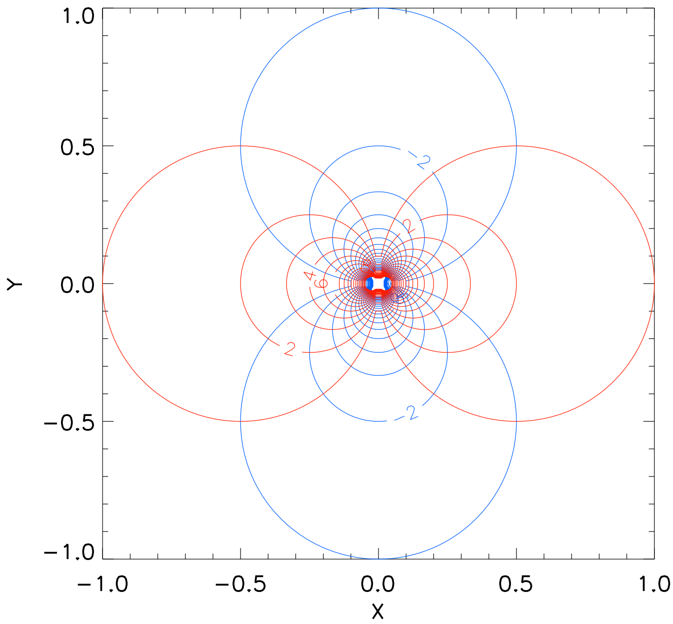

Figure 8.8. Velocity potential (red contours) and stream function (blue contours) for a doublet with the strength $\mu = 1$.

(Q1) Show that the stream function for the doublet depicted in Figure 8.8 is given by

$$\psi = -\mu\frac{y}{x^2+y^2}. \tag{8.63}$$

(E9) Let us consider a doublet in Figure 8.7 within a uniform flow of magnitude U in the negative x direction. Find the velocity potential and sketch it.

The velocity potential is given by

$$\phi = \phi_{doublet} + Ux = \mu\frac{x}{x^2+y^2} + Ux$$

$$= \left(\frac{\mu}{r} + Ur\right)\cos\theta. \tag{8.64}$$

The radial and azimuthal velocities are given by

$$v_r = -\frac{\partial \phi}{\partial r} = \left(\frac{\mu}{r^2} - U\right)\cos\theta \,, \tag{8.65}$$

and

$$v_r = -\frac{\partial \phi}{r\partial \theta} = \left(\frac{\mu}{r^2} + U\right)\sin\theta \,. \tag{8.66}$$

Thus, the radial velocity vanishes at $r = a$ if $\mu = Ua^2$, and $r = a$ appears as a solid pillar. Thus, this flow is called "flow past a cylinder". If we use $\mu = Ua^2$ the velocity potential can be rewritten as

$$\phi = U\left(1 + \frac{a^2}{r^2}\right) r\cos\theta = U\left(1 + \frac{a^2}{x^2 + y^2}\right) x \,. \tag{8.67}$$

Figure 8.9 is given by

$$\psi = U\left(1 + \frac{a^2}{r^2}\right) r\sin\theta = U\left(1 + \frac{a^2}{x^2 + y^2}\right) y \,. \tag{8.68}$$

In Figure 8.9, $r = 1$ is the location of the cylinder where radial velocity vanishes.

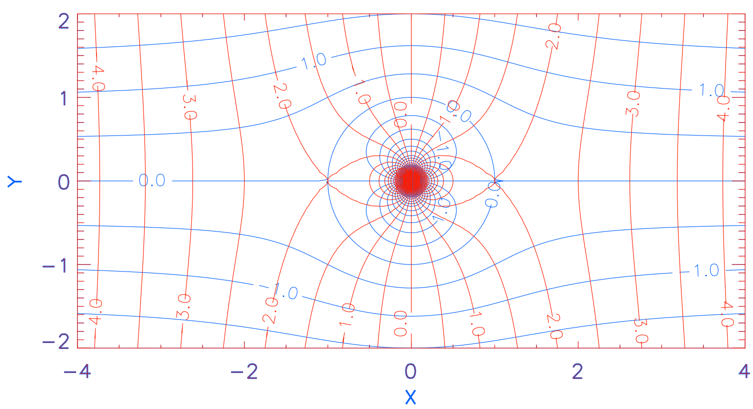

Figure 8.9. Velocity potential (red contours) and stream function (blue contours) of flow past a cylinder.

(Q2) Show that the stream function for the flow past a cylinder depicted in

(E10) Let us now consider a source within a uniform flow of strength U in the positive x direction. Then, the stream function of the flow is given by (see Figure 8.10)

$$\psi = \psi_{source} - Uy = -\frac{m}{2\pi}\theta - Uy = -\frac{m}{2\pi}\theta - Ur\sin\theta \,, \tag{8.69}$$

Note that the radial velocity is determined to be

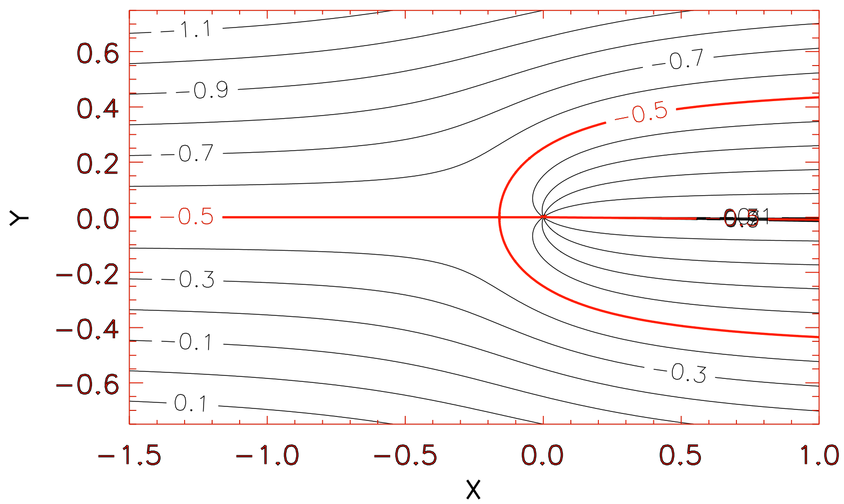

Figure 8.10. Stream function of a half body (uniform stream plus a source) in (78) for $m = 1$ and $U = 1$.

$$v_r = \frac{\partial\psi}{r\partial\theta} = -\frac{m}{2\pi r} - U\cos\theta . \tag{8.70}$$

A stagnation point is where the combined velocity is zero. Note that the radial velocity vanishes for $\theta = \pi$ (facing the uniform flow) when $r = \frac{m}{2\pi U}$. The value of the stream function at $r = \frac{m}{2\pi U}$ is given by $\psi = -\frac{m}{2}$. Thus, the $\psi = -0.5$ contour defines streamline passing through the stagnation point, i.e., stagnation streamline. In this example, we may consider the stagnation streamline as the surface of the solid body, which defines an impermeable surface. In the limit of $\theta \to 0$, we have

$$\lim_{\theta\to 0} y = \lim_{\theta\to 0}\left(r\sin\theta\right) = \lim_{\theta\to 0}\left(-\frac{m\theta}{2\pi U} - \frac{\psi}{U}\right) = -\lim_{\theta\to 0}\frac{m}{2U}\left(\frac{\theta}{\pi} - 1\right) = \frac{m}{2U} . \tag{8.71}$$

Based on the Bernoulli theorem, then, we have

$$\begin{aligned} p_\infty + \frac{1}{2}\rho U^2 &= p + \frac{1}{2}\rho\left(v_r^2 + v_\theta^2\right) \\ &= p + \frac{1}{2}\rho\left[\left(\frac{m}{2\pi r} + U\cos\theta\right)^2 + \left(U\sin\theta\right)^2\right] \\ &= p + \frac{1}{2}\rho\left[\left(\frac{m}{2\pi r}\right)^2 + \frac{m}{\pi r}U\cos\theta + U^2\right]. \end{aligned} \tag{8.72}$$

Thus, pressure along the surface of the solid body is given by

$$p = p_\infty - \frac{1}{2}\rho\left[\left(\frac{m}{2\pi r}\right)^2 + \frac{mU}{\pi r}\cos\theta\right]. \tag{8.73}$$

Thus, the pressure distribution is symmetric about the x axis (i.e., $\cos(-\theta)=\cos\theta$), and is maximum at $\theta=\pi$ (stagnation point) and gradually decreases away from the stagnation point.

8.5. Complex Potential

As we have seen Figures 8.8 and 8.9, stream function and velocity potential for a two-dimensional flow are perpendicular to each other. Perpendicularity of the two functions can be shown by

$$\nabla\phi\cdot\nabla\psi=\left(-u,-v\right)\cdot\left(v,-u\right)=-uv+uv\equiv 0\,. \tag{8.74}$$

Thus, we can define a complex function

$$\varpi(z)=\phi(x,y)+i\psi(x,y)\,, \tag{8.75}$$

where $z=x+iy$. The function is called the *complex potential*. Note that

$$\begin{aligned}\frac{d\varpi}{dz}&=\lim_{\Delta z\to 0}\frac{\varpi(z+\Delta z)-\varpi(z)}{\Delta z}\\ &=\lim_{\Delta x\to 0}\frac{\varpi(z+\Delta x)-\varpi(z)}{\Delta x}=\lim_{\Delta y\to 0}\frac{\varpi(z+i\Delta y)-\varpi(z)}{i\Delta y}\,,\end{aligned} \tag{8.76}$$

since the derivative of a complex function should be independent of the path. Thus, we have

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\varpi(z+\Delta x)-\varpi(z)}{\Delta x}&=\lim_{\Delta x\to 0}\frac{\phi(x+\Delta x,y)+i\psi(x+\Delta x,y)-\phi(x,y)+i\psi(x,y)}{\Delta x}\\ &=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=-u+iv\,,\end{aligned} \tag{8.77}$$

$$\begin{aligned}\lim_{\Delta y\to 0}\frac{\varpi(z+i\Delta y)-\varpi(z)}{i\Delta y}&=\lim_{\Delta y\to 0}\frac{\phi(x,y+\Delta y)+i\psi(x,y+\Delta y)-\phi(x,y)+i\psi(x,y)}{i\Delta y}\\ &=-i\left(\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}\right)=-i\left(-v-iu\right)=-u+iv\,.\end{aligned} \tag{8.78}$$

This implies that

$$\frac{d\varpi}{dz}=-u+iv\,. \tag{8.79}$$

(E11) Derive the complex potential for a uniform flow U in the direction θ from the x axis.

Velocity field is given by

$$\vec{u}=\left(U\cos\theta,U\sin\theta\right). \tag{8.80}$$

Thus, the velocity potential and stream function should satisfy

$$U\cos\theta = -\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \text{and} \qquad U\sin\theta = -\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{8.81}$$

Thus, velocity potential and stream function are defined respectively by

$$\phi = -U\cos\theta x - U\sin\theta y, \tag{8.82}$$

$$\psi = -U\sin\theta x + U\cos\theta y. \tag{8.83}$$

Thus, the corresponding complex potential is given by

$$\begin{aligned}\varpi = \phi + i\psi &= -U(\cos\theta x + \sin\theta y) - iU(\sin\theta x - \cos\theta y) \\ &= -U(\cos\theta x - i\cos\theta y) - iU(\sin\theta x - i\sin\theta y) \\ &= -U(x - iy)(\cos\theta + i\sin\theta) = -Uze^{i\theta}.\end{aligned} \tag{8.84}$$

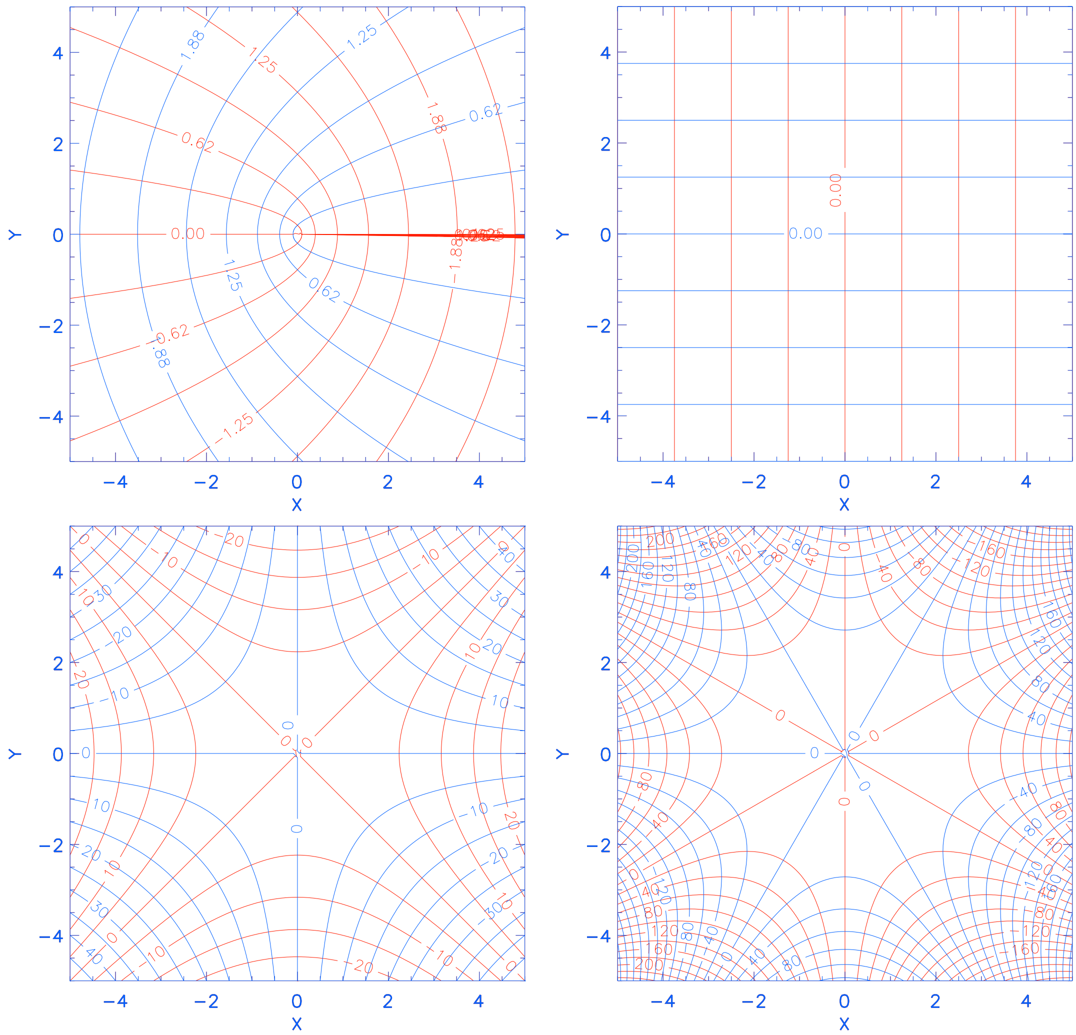

Figure 8.11. Velocity potential (red contours) and stream function (blue contours) of flow at an angle (a) $n = 1/2$, (b) $n = 1$, (c) $n = 2$, and (d) $n = 3$.

The stream function and velocity potential for (8.87) are shown in Figure 8.11 for $n = 1/2$, 1, 2, and 3. As can be seen in the figure, (8.87) represents complex potential for a flow at a corner with the angle of π/n.

(E12) Derive the velocity potential and stream function for a source of strength m.

The velocity

$$\phi = -\frac{m}{2\pi}\ln r\,, \tag{8.85}$$

$$\psi = -\frac{m}{2\pi}\theta\,. \tag{8.86}$$

Thus, the complex potential is given by

$$\varpi = \phi + i\psi = -\frac{m}{2\pi}\left(\ln r + i\theta\right) = -\frac{m}{2\pi}\ln\left(re^{i\theta}\right) = -\frac{mz}{2\pi}\,. \tag{8.87}$$

The velocity potential and the stream function of the flow are shown in Figure 8.11.

Let us consider the complex potential

$$\varpi = Az^n = A\left(re^{i\theta}\right)^n = Ar^n e^{in\theta} = Ar^n\cos n\theta + iAr^n\sin n\theta\,. \tag{8.88}$$

Then, the velocity potential and stream function are given by

$$\phi = Ar^n\cos n\theta\,, \quad \text{and} \quad \psi = Ar^n\sin n\theta\,. \tag{8.89}$$

Thus,

$$u_r = -\frac{\partial\phi}{\partial r} = -\frac{\partial\psi}{r\partial\theta} = -nAr^{n-1}\cos n\theta\,, \tag{8.90}$$

and

$$u_\theta = -\frac{\partial\phi}{r\partial\theta} = \frac{\partial\psi}{\partial r} = nAr^{n-1}\sin n\theta\,. \tag{8.91}$$