不定积分竞赛题参考答案

1. $\int\frac{e^{\sqrt{x}}+\sin(2\sqrt{x}+1)}{2\sqrt{x}}dx \quad =\int[e^{\sqrt{x}}+\sin(2\sqrt{x}+1)]d\sqrt{x}=\int e^{t}+\sin(2t+1)dt$

$=e^{t}-\frac{1}{2}\cos(2t+1)+C=e^{\sqrt{x}}-\frac{1}{2}\cos(2\sqrt{x}+1)+C$

2. $\int\frac{1+\sin^{2}x}{1-\cos 2x}dx$

$=\int\frac{1+\sin^{2}x}{1-(1-2\sin^{2}x)}dx=\int\frac{1+\sin^{2}x}{2\sin^{2}x}dx=\int(\frac{1}{2\sin^{2}x}+\frac{1}{2})dx=-\frac{1}{2}\cot x+\frac{1}{2}x+C$

3. $\int\frac{1+2\cos x}{\sin^{2}x}dx$

$=\int(\frac{1}{\sin^{2}x}+\frac{2\cos x}{\sin^{2}x})dx=-\cot x+2\int\frac{1}{\sin^{2}x}d\sin x=-\cot x-\frac{2}{\sin x}+C$

4. $\int\frac{3x+2\arcsin x}{\sqrt{1-x^{2}}}dx$

$=\int(\frac{3x}{\sqrt{1-x^{2}}}+\frac{2\arcsin x}{\sqrt{1-x^{2}}})dx=-\frac{3}{2}\int\frac{1}{\sqrt{1-x^{2}}}d(1-x^{2})+2\int\arcsin x d\arcsin x$

$=-3\sqrt{1-x^{2}}+\arcsin^{2}x+C$

5. $\int\frac{x\arctan x^{2}-3x^{3}+2x}{1+x^{4}}dx$

$=\frac{1}{2}\int\frac{\arctan x^{2}-3x^{2}+2}{1+x^{4}}dx^{2}=\frac{1}{2}\int\frac{\arctan t-3t+2}{1+t^{2}}dt=\frac{1}{2}\int(\frac{\arctan t}{1+t^{2}}-\frac{3t}{1+t^{2}}+\frac{2}{1+t^{2}})dt$

$=\frac{1}{2}[\int\arctan t d\arctan t-\frac{3}{2}\int\frac{1}{1+t^{2}}d(1+t^{2})]+2\arctan t$

$=\frac{1}{4}\arctan^{2}t-\frac{3}{4}\ln(1+t^{2})+2\arctan t+C=\frac{1}{4}\arctan^{2}x^{2}-\frac{3}{4}\ln(1+x^{4})+2\arctan x^{2}+C$

6. 令 $\sqrt{x+3}=t$

$\int\frac{3dx}{(x+2)\sqrt{x+3}}=3\int\frac{2tdt}{(t^{2}-1)t}=3\int\frac{2}{t^{2}-1}dt=3\int[\frac{1}{t-1}-\frac{1}{t+1}]dt=3\ln\left|\frac{t-1}{t+1}\right|+C$

$=3\ln\left|\frac{\sqrt{x+3}-1}{\sqrt{x+3}+1}\right|+C$

7. $\int\frac{\sin^2 x\cos x}{1+4\sin^2 x}dx=\int\frac{\sin^2 x}{1+4\sin^2 x}d\sin x=\frac{1}{4}\int\frac{4\sin^2 x}{1+4\sin^2 x}d\sin x$

$=\frac{1}{4}\int\frac{1+4\sin^2 x-1}{1+4\sin^2 x}d\sin x=\frac{1}{4}\int\frac{1+4\sin^2 x-1}{1+4\sin^2 x}d\sin x=\frac{1}{4}\int 1-\frac{1}{1+4\sin^2 x}d\sin x$

$=\frac{1}{4}\sin x-\frac{1}{8}\arctan(2\sin x)+C$.

8. $\int\frac{x^3}{\sqrt{4-x^2}}dx=\frac{1}{2}\int\frac{x^2}{\sqrt{4-x^2}}dx^2=\frac{1}{2}\int\frac{t}{\sqrt{4-t}}dt$

令 $\sqrt{4-t}=u$

所以 $\int\frac{x^3}{\sqrt{4-x^2}}dx$

=

$\frac{1}{2}\int\frac{t}{\sqrt{4-t}}dt=\frac{1}{2}\int\frac{4-u^2}{u}(-2udu)=\int(u^2-4)du=\frac{u^3}{3}-4u+C=\frac{1}{3}(4-x^2)^{\frac{3}{2}}-4\sqrt{4-x^2}+C$

或者：令 $x=2\sin t$

$\int\frac{x^3}{\sqrt{4-x^2}}dx=\int\frac{8\sin^3 t}{2\cos t}\cdot 2\cos tdt=8\int\sin^3 tdt=-8\int(1-\cos^2 t)d\cos t$

$=\frac{8}{3}\cos^3 t-8\cos t+C=\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3-8\cdot\frac{\sqrt{4-x^2}}{2}+C=\frac{1}{3}\left(\sqrt{4-x^2}\right)^{\frac{3}{2}}-4\sqrt{4-x^2}+C$

9. $\int\frac{\ln\cos x}{\cos^2 x}dx$

$=\int\ln\cos xd\tan x=\ln\cos x\cdot\tan x-\int\tan xd\ln\cos x=\ln\cos x\cdot\tan x-\int\tan x\frac{-\sin x}{\cos x}dx$

$=\ln\cos x\cdot\tan x+\int\tan^2 xdx=\ln\cos x\cdot\tan x+\int(\sec^2 x-1)dx$

$=\ln\cos x\cdot\tan x+\tan x-x+C$

10. $\int\frac{1+x+2\cos x}{\sin^2 x}dx$

$$=\int(\frac{1}{\sin^2 x}+\frac{x}{\sin^2 x}+\frac{2\cos x}{\sin^2 x})dx=-\cot x-\int xd\cot x+2\int\frac{1}{\sin^2 x}d\sin x$$

$$=-\cot x-[x\cot x-\int\cot xdx]-\frac{2}{\sin x}=-\cot x-x\cot x+\ln\left|\sin x\right|-\frac{2}{\sin x}+C$$

11. $\int(x+1)\log_2 xdx$

$$=\int\log_2 xd(\frac{x^2}{2}+x)=(\frac{x^2}{2}+x)\log_2 x-\int(\frac{x^2}{2}+x)d\log_2 x$$

$$=(\frac{x^2}{2}+x)\log_2 x-\int(\frac{x^2}{2}+x)\frac{1}{x\ln 2}dx$$

$$=(\frac{x^2}{2}+x)\log_2 x-\frac{1}{\ln 2}\int(\frac{x}{2}+1)dx=(\frac{x^2}{2}+x)\log_2 x-\frac{1}{\ln 2}(\frac{x^2}{4}+x)+C$$

12. $\int(3x+1)\sin 2xdx$

$$=\int(3x+1)d(-\frac{1}{2}\cos 2x)=-\frac{1}{2}(3x+1)\cos 2x+\frac{1}{2}\int\cos 2xd(3x+1)$$

$$=-\frac{1}{2}(3x+1)\cos 2x+\frac{3}{2}\int\cos 2xdx=-\frac{1}{2}(3x+1)\cos 2x+\frac{3}{4}\sin 2x+C$$

13. $\int x^3\arctan x^2dx$

$$=\frac{1}{2}\int x^2\arctan x^2dx^2=\frac{1}{2}\int t\arctan tdt=\frac{1}{4}\int\arctan tdt^2=\frac{1}{4}t^2\arctan t-\frac{1}{4}\int t^2d\arctan t$$

$$=\frac{1}{4}t^2\arctan t-\frac{1}{4}\int\frac{t^2}{1+t^2}dt=\frac{1}{4}t^2\arctan t-\frac{1}{4}\int\frac{1+t^2-1}{1+t^2}dt$$

$$=\frac{1}{4}t^2\arctan t-\frac{1}{4}t+\frac{1}{4}\arctan t+C=\frac{1}{4}x^4\arctan x^2-\frac{1}{4}x^2+\frac{1}{4}\arctan x^2+C$$

14. $\int e^x\arcsin(3e^x-2)dx$

$$=\frac{1}{3}\int\arcsin(3e^x-2)d(3e^x-2)=\frac{1}{3}\int\arcsin tdt=\frac{1}{3}t\arcsin t-\frac{1}{3}\int td\arcsin t$$

$$=\frac{1}{3}t\arcsin t-\frac{1}{3}\int\frac{t}{\sqrt{1-t^2}}dt=\frac{1}{3}t\arcsin t+\frac{1}{6}\int\frac{1}{\sqrt{1-t^2}}d(1-t^2)$$

$$=\frac{1}{3}t\arcsin t+\frac{1}{3}\sqrt{1-t^2}+C=\frac{1}{3}(3e^x-2)\arcsin(3e^x-2)+\frac{1}{3}\sqrt{1-(3e^x-2)^2}+C$$

15. $\int \frac{xe^x}{(1+e^x)^2}dx$

$$=\int \frac{x}{(1+e^x)^2}d(1+e^x)=\int xd(-\frac{1}{1+e^x})=-\frac{x}{1+e^x}+\int \frac{1}{1+e^x}dx$$

$$=-\frac{x}{1+e^x}+\int \frac{1+e^x-e^x}{1+e^x}dx=-\frac{x}{1+e^x}+x-\int \frac{1}{1+e^x}d(1+e^x)=-\frac{x}{1+e^x}+x-\ln(1+e^x)+C$$

16. $\int e^{\sqrt[3]{4x+1}}\mathrm{d}x$

令 $\sqrt[3]{4x+1}=t$

$$\int e^{\sqrt[3]{4x+1}}dx=\frac{3}{4}\int e^t t^2 dt=\frac{3}{4}\int t^2 de^t=\frac{3}{4}t^2e^t-\frac{3}{4}\int e^t dt^2=\frac{3}{4}t^2e^t-\frac{3}{2}\int e^t tdt$$

$$=\frac{3}{4}t^2e^t-\frac{3}{2}\int tde^t=\frac{3}{4}t^2e^t-\frac{3}{2}(te^t-\int e^t dt)=\frac{3}{4}t^2e^t-\frac{3}{2}te^t+\frac{3}{2}e^t+C$$

$$=\frac{3}{4}(4x+1)^{\frac{2}{3}}e^{\sqrt[3]{4x+1}}-\frac{3}{2}\sqrt[3]{4x+1}e^{\sqrt[3]{4x+1}}+\frac{3}{2}e^{\sqrt[3]{4x+1}}+C$$

17. $\int \frac{x+1}{x^2-2x+5}dx$

$$=\frac{1}{2}\int \frac{2x+2}{x^2-2x+5}dx=\frac{1}{2}\int \frac{2x-2+4}{x^2-2x+5}dx=\frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx+2\int \frac{1}{x^2-2x+5}dx$$

$$=\frac{1}{2}\int \frac{1}{x^2-2x+5}d(x^2-2x+5)+2\int \frac{1}{4+(x-1)^2}dx$$

$$=\frac{1}{2}\ln\left|x^2-2x+5\right|+\int \frac{1}{1+(\frac{x-1}{2})^2}d\frac{x-1}{2}=\frac{1}{2}\ln\left|x^2-2x+5\right|+\arctan\frac{x-1}{2}+C$$

18. $\int \frac{x^2+5x+2}{x^2+4}dx=\int \frac{(x^2+4)+(5x-2)}{x^2+4}dx=x+5\int \frac{x}{x^2+4}dx-2\int \frac{1}{x^2+4}dx$

$$=x+\frac{5}{2}\int \frac{1}{x^2+4}d(x^2+4)-\int \frac{1}{1+(\frac{x}{2})^2}d\frac{x}{2}=x+\frac{5}{2}\ln(x^2+4)-\arctan\frac{x}{2}+C$$

19. $\int xe^{\sqrt{1-x^2}}dx=-\frac{1}{2}\int e^{\sqrt{1-x^2}}d(1-x^2)=-\frac{1}{2}\int e^{\sqrt{t}}dt$

令 $\sqrt{t}=u$

$$\int xe^{\sqrt{1-x^2}}dx = -\frac{1}{2}\int e^{\sqrt{t}}dt = -\int ue^u du$$

$$= -\int ude^u = -ue^u + e^u + C = -\sqrt{1-x^2}e^{\sqrt{1-x^2}} + e^{\sqrt{1-x^2}} + C$$

20. 已知$\int e^x f(x)dx = xe^x + C$，求$\int xf'(2x)dx$

$$e^x f(x) = e^x + xe^x, \therefore f(x) = 1 + x$$

所以$\int xf'(2x)dx = \frac{1}{2}\int xf'(2x)d2x = \frac{1}{2}\int xdf(2x) = \frac{1}{2}xf(2x) - \frac{1}{2}\int f(2x)dx$

$$= \frac{1}{2}x(1+2x) - \frac{1}{2}\int(1+2x)dx = \frac{1}{2}(x+2x^2) - \frac{1}{2}(x+x^2) + C = \frac{1}{2}x^2 + C$$