极限竞赛题答案

（每小题 5 分，共 100 分，考试时间 120 分钟）

（1—18 题每小题 4 分，19—23 每小题 7 分，共 100 分）

1. $\lim\limits_{x\to\infty}\dfrac{3x^2-2x\sin x}{2x^2+x\sin x}=\lim\limits_{x\to\infty}\dfrac{3-\dfrac{2\sin x}{x}}{2+\dfrac{\sin x}{x}}=\dfrac{3}{2}$

2. $\lim\limits_{x\to\infty}\dfrac{5x^2-3}{2x+1}\sin\dfrac{2}{x}=\lim\limits_{x\to\infty}\dfrac{5x^2-3}{2x+1}\cdot\dfrac{2}{x}=\lim\limits_{x\to\infty}\dfrac{10x^2-6}{2x^2+x}=\dfrac{10}{2}=5$

3. $\lim\limits_{x\to 0}\dfrac{\sin(4x)}{\sqrt{x+2}-\sqrt{2}}=\lim\limits_{x\to 0}\dfrac{4x}{\sqrt{x+2}-\sqrt{2}}=\lim\limits_{x\to 0}\dfrac{4x(\sqrt{x+2}+\sqrt{2})}{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}$

$=\lim\limits_{x\to 0}\dfrac{4x(\sqrt{x+2}+\sqrt{2})}{x}=\lim\limits_{x\to 0}4(\sqrt{x+2}+\sqrt{2})=8\sqrt{2}$

4. $\lim\limits_{x\to 2}\dfrac{\sin(x^2-4)}{x^2+x-6}=\lim\limits_{x\to 2}\dfrac{x^2-4}{x^2+x-6}=\lim\limits_{x\to 2}\dfrac{2x}{2x+1}=\dfrac{4}{5}$

5. $\lim\limits_{x\to 1}\dfrac{3x^3-2x^2-1}{\arcsin(x^2-1)}=\lim\limits_{x\to 1}\dfrac{3x^3-2x^2-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{9x^2-4x}{2x}=\dfrac{5}{2}$

6. $\lim\limits_{x\to 0}\dfrac{x-xe^x}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{x-xe^x}{x^2}=\lim\limits_{x\to 0}\dfrac{1-e^x}{x}=\lim\limits_{x\to 0}\dfrac{-x}{x}=-1$

7. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+5x}-\sqrt{1-3x}}{x^2+2x}=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+5x}-\sqrt{1-3x})(\sqrt{1+5x}+\sqrt{1-3x})}{(x^2+2x)(\sqrt{1+5x}+\sqrt{1-3x})}$

$=\lim\limits_{x\to 0}\dfrac{8x}{(x^2+2x)(\sqrt{1+5x}+\sqrt{1-3x})}=\lim\limits_{x\to 0}\dfrac{8}{(x+2)(\sqrt{1+5x}+\sqrt{1-3x})}=2$

8. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1+\tan x}}{x(1-\cos x)}=2\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1+\tan x}}{x^3}$

$=2\lim\limits_{x\to 0}\dfrac{(\sqrt{1+\sin x}-\sqrt{1+\tan x})(\sqrt{1+\sin x}+\sqrt{1+\tan x})}{x^3(\sqrt{1+\sin x}+\sqrt{1+\tan x})}$

$=\dfrac{2}{2}\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x^3}=\lim\limits_{x\to 0}\dfrac{(\cos x-1)\tan x}{x^3}=\lim\limits_{x\to 0}\dfrac{-\dfrac{x^2}{2}\cdot x}{x^3}=-\dfrac{1}{2}$

9. $\lim\limits_{x\to 0}\dfrac{e-e^{\cos x}}{\sqrt[3]{x^2+1}-1}=\lim\limits_{x\to 0}\dfrac{e-e^{\cos x}}{\frac{1}{3}x^2}=3\lim\limits_{x\to 0}\dfrac{e-e^{\cos x}}{x^2}=3\lim\limits_{x\to 0}\dfrac{e^{\cos x}\cdot\sin x}{2x}$

$$=3\lim_{x\to 0}\frac{e^{\cos x}\cdot x}{2x}=3\lim_{x\to 0}\frac{e^{\cos x}}{2}=\frac{3e}{2}$$

10. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{4x^2+x-1}+x+3}{\sqrt{x^2+\sin x}}=\lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{4x^2+x-1}}{x}+\frac{x}{x}+\frac{3}{x}}{\frac{\sqrt{x^2+\sin x}}{x}}$

$$=\lim_{x\to+\infty}\frac{\sqrt{\frac{4x^2+x-1}{x^2}}+1+\frac{3}{x}}{\sqrt{\frac{x^2+\sin x}{x^2}}}=\frac{2+1}{1}=3$$

11. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x\ln(1+\sin^2 x)}=\lim\limits_{x\to 0}\dfrac{\tan x(1-\cos x)}{x\cdot\sin^2 x}=\lim\limits_{x\to 0}\dfrac{x\cdot\frac{x^2}{2}}{x\cdot x^2}=\dfrac{1}{2}$

12. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}=\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}=\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}}{\cos x}=2$

13. $\lim\limits_{x\to+\infty}\ln(1+\dfrac{1}{x})\cdot\ln x=\lim\limits_{x\to+\infty}\dfrac{1}{x}\cdot\ln x=\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}}{1}=0$

14. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{e^x-1}\right)=\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x}{(e^x-1)\sin x}=\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x}{x^2}=\lim\limits_{x\to 0}\dfrac{e^x-\cos x}{2x}$

$$=\lim_{x\to 0}\frac{e^x+\sin x}{2}=\frac{1}{2}$$

15. $\lim\limits_{x\to+\infty}\left(3x-\sqrt{ax^2-x+1}\right)=\lim\limits_{x\to+\infty}\dfrac{(3x-\sqrt{ax^2-x+1})(3x+\sqrt{ax^2-x+1})}{3x+\sqrt{ax^2-x+1}}$

$$=\lim_{x\to+\infty}\frac{(9-a)x^2+x-1}{3x+\sqrt{ax^2-x+1}}=\frac{1}{6}$$

所以$\begin{cases}9-a=0\\ \dfrac{1}{3+\sqrt{a}}=\dfrac{1}{6}\end{cases}$，所以$a=9$

16. $\lim\limits_{x\to+\infty} x\left(\ln(x-2)-\ln(x+1)\right)=\lim\limits_{x\to+\infty} x\ln\dfrac{x-2}{x+1}=\lim\limits_{x\to+\infty}\ln(\dfrac{x-2}{x+1})^x=\lim\limits_{x\to+\infty}\ln(\dfrac{1-\frac{2}{x}}{1+\frac{1}{x}})^x$

$=\lim\limits_{x\to+\infty}\ln(\dfrac{1-\frac{2}{x}}{1+\frac{1}{x}})^x=\lim\limits_{x\to+\infty}\ln\dfrac{(1-\frac{2}{x})^x}{(1+\frac{1}{x})^x}=\lim\limits_{x\to+\infty}\ln\dfrac{e^{-2}}{e}=\ln e^{-3}=-3$

或者 $\lim\limits_{x\to+\infty} x\left(\ln(x-2)-\ln(x+1)\right)=\lim\limits_{x\to+\infty} x\ln\dfrac{x-2}{x+1}=\lim\limits_{x\to+\infty} x\ln\dfrac{x+1-3}{x+1}=\lim\limits_{x\to+\infty} x\ln(1-\dfrac{3}{x+1})$

$$=\lim_{x\to+\infty} x\cdot(-\frac{3}{x+1})=\lim_{x\to+\infty}(-\frac{3x}{x+1})=-3$$

17. $\lim\limits_{x\to0}\cot x\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)=\lim\limits_{x\to0}\dfrac{1}{\tan x}\cdot\dfrac{x-\sin x}{x\sin x}=\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3}=\lim\limits_{x\to0}\dfrac{1-\cos x}{3x^2}$

$$=\lim_{x\to0}\frac{\frac{1}{2}x^2}{3x^2}=\frac{1}{6}$$

18. $\lim\limits_{x\to0}(x+e^x)^{\frac{1}{x}}=\lim\limits_{x\to0}[1+(x+e^x-1)]^{\frac{1}{x}}=\lim\limits_{x\to0}[1+(x+e^x-1)]^{\frac{1}{x+e^x-1}\cdot\frac{x+e^x-1}{x}}$

$$=e^{\lim\limits_{x\to0}\frac{x+e^x-1}{x}}=e^{\lim\limits_{x\to0}\frac{1+e^x}{1}}=e^2$$

19. $\lim\limits_{n\to\infty}\dfrac{1}{n^2}\ln[f(1)\cdot f(2)\cdots f(n)]=\lim\limits_{n\to\infty}\dfrac{1}{n^2}\ln 2\cdot2^2\cdot2^3\cdot2^4\cdot\cdots\cdot2^n$

$=\lim\limits_{n\to\infty}\dfrac{1}{n^2}\ln 2^{1+2+3+\cdots+n}=\lim\limits_{n\to\infty}\dfrac{1}{n^2}\ln 2^{\frac{(1+n)n}{2}}=\lim\limits_{n\to\infty}\dfrac{1}{n^2}\cdot\dfrac{(1+n)n}{2}\ln 2=\lim\limits_{n\to\infty}\dfrac{n^2+n}{2n^2}\ln 2=\dfrac{1}{2}\ln 2$

20. $\lim\limits_{x\to0}\dfrac{\sin 2x}{\ln(a+x)}(\cos x-b)=4$，求 a,b

因为分子趋于零极限存在，所以分母趋于零，所以 $a=1$

所以 $\lim\limits_{x\to0}\dfrac{\sin 2x}{\ln(a+x)}(\cos x-b)=\lim\limits_{x\to0}\dfrac{\sin 2x}{\ln(1+x)}(\cos x-b)=\lim\limits_{x\to0}\dfrac{2x}{x}(\cos x-b)$

$$=2\lim_{x\to0}(\cos x-b)=2(1-b)=4$$

所以 $b=-1$，即 $a=1$， $b=-1$

21. $\lim\limits_{x\to-\infty}(4x+\sqrt{ax^2-bx-1})=1$， $a>0$，求 a,b

$$\lim_{x\to-\infty}(4x+\sqrt{ax^2-bx-1})=\lim_{x\to-\infty}\frac{(4x+\sqrt{ax^2-bx-1})\ \ (4x-\sqrt{ax^2-bx-1})}{4x-\sqrt{ax^2-bx-1}}$$

$$=\lim_{x\to-\infty}\frac{(16-a)x^2+bx+1}{4x-\sqrt{ax^2-bx-1}}=1$$，所以$16-a=0$，所以$a=16$

所以$\lim_{x\to-\infty}(4x+\sqrt{ax^2-bx-1})=\lim_{x\to-\infty}\frac{bx+1}{4x-\sqrt{16x^2-bx-1}}=\lim_{x\to-\infty}\frac{b+\frac{1}{x}}{4+\sqrt{\frac{16x^2-bx-1}{x^2}}}$

$$=\frac{b}{4+\sqrt{16}}=\frac{b}{8}=1$$

所以$b=8$.

22．已知极限 $\lim_{x\to+\infty}(\frac{x^2}{x+1}-x-a)=2$，求$a$.

$\lim_{x\to+\infty}(\frac{x^2}{x+1}-x-a)=\lim_{x\to+\infty}(\frac{x^2-x^2-x}{x+1}-a)=\lim_{x\to+\infty}(\frac{-x}{x+1}-a)=-1-a=2$，所以$a=-3$

23．若$\lim_{x\to1}f(x)$存在，且$f(x)=x^3+\frac{2x^2+1}{x+1}+2\lim_{x\to1}f(x)$，求$f(x)$

令$\lim_{x\to1}f(x)=a$，所以$f(x)=x^3+\frac{2x^2+1}{x+1}+2a$

所以$\lim_{x\to0}f(x)=\lim_{x\to0}(x^3+\frac{2x^2+1}{x+1}+2a)$，所以$a=1+2a$，所以$a=-1$

所以$f(x)=x^3+\frac{2x^2+1}{x+1}-2$