Notes on Difference Equations: Calculus Over the Naturals

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1 Motivation

Problems depending on a certain integer parameter n often can be solved with a recursion relation, where the nth answer can be calculated in terms of preceding or proceeding answers. Such recursion relations can be in the form of a difference equation, an equation that relates terms of a sequence a_n to other terms of that sequence like a_{n+1}. Take the following problem as an example to motivate the use of difference equations: find a closed form for

$$\int x^n \cos x \, dx.$$

Since cosine and sine are inherently intertwined, and since any sinusoid can be written as a sum of cosines and sines, let's generalize this problem to the following: find a closed form for

$$I_n = \begin{pmatrix} \int x^n \cos x \, dx \\ \int x^n \sin x \, dx \end{pmatrix}.$$

Notice that I_n is a sequence of functions of x. Then, using integration by parts and differentiating the x^n,

$$I_n = \begin{pmatrix} x^n \sin x - n \int x^{n-1} \sin x \, dx \\ -x^n \cos x + n \int x^{n-1} \cos x \, dx \end{pmatrix} = x^n I_0 + \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix} I_{n-1}, \tag{1}$$

where $I_0 = I_{n=0} = \begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}$. We now have a difference equation in I_n, but it involves a matrix, so it is more complicated than a scalar difference equation. One can resolve this by solving for each component of I_n individually, so let's set the cosine and sine components of the vector equal to a_n and b_n, respectively. We'll just solve for a_n for simplicity since b_n can be solved for in a very similar way. Then by Eq. 1,

$$a_n = x^n a_0 - n b_{n-1}; b_{n-1} = x^{n-1} b_0 + (n-1) a_{n-2},$$

and therefore

$$a_n = x^n a_0 - nx^{n-1} b_0 - n(n-1)a_{n-2}, \tag{2}$$

where $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = I_0$. This is a "second order" difference equation since it relates one term of the sequence not to the previous or next term but to the term two steps ahead or two steps behind. We fix this by making the following substitutions:

$$n = 2m; a_{2m} = c_m.$$

Eq. 2 becomes

$$c_m = x^2 m a_0 - 2mx^{2m-1} b_0 - 2m(2m-1)c_{m-1}.$$

Rearranging into what resembles standard form for a differential equation and shifting indices,

$$c_{m+1} + (2m+2)(2m+1)c_m = x^{2m+2} a_0 - (2m+2)x^{2m+1} b_0. \tag{3}$$

Now we have reduced our initial problem into a "first order" linear difference equation, which as we'll see is very solvable, just like a first order linear ordinary differential equation. In this equation, n is the independent variable, not x. Eq 3 is the difference equation that we'll aim to solve later in these notes by creating a method to solve them analogous to the way we would solve differential equations.

2 Discrete derivative

Recall from calculus that the derivative of a function $f(x)$ over the reals is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We would like to use the same definition for the discrete derivative of a sequence a_n, but there is one small problem. Our domain is over the integers, so h cannot go to zero. In fact, the smallest separation between the inputs n of two distinct values of the sequence is one. Therefore, we take h to be one in the definition of the discrete derivative of a sequence a_n, denoted by Δa_n:

$$\Delta a_n = \lim_{h \to 1} \frac{a_{n+h} - a_n}{h} = a_{n+1} - a_n. \tag{4}$$

This is also known as the forward difference operator. It satisfies linearity and a modified product rule as follows:

$$\begin{aligned}
\Delta(a_n b_n) = a_{n+1} b_{n+1} - a_n b_n &= a_{n+1} b_{n+1} - a_n b_{n+1} + a_n b_{n+1} - a_n b_n \\
&= (a_{n+1} - a_n) b_{n+1} + a_n (b_{n+1} - b_n) \\
&= (\Delta a_n) b_{n+1} + a_n \Delta b_n,
\end{aligned}$$

where I used the age-old trick of adding zero. What's strange about this formula is that it is not symmetric. In fact, if derived a different way, this same formula could instead equal

$$(\Delta b_n)a_{n+1} + b_n \Delta a_n.$$

I imagine this asymmetry could be exploited for algebraic simplifications. Though there is a nice product rule, I am unaware of a nice chain rule formula for the discrete derivative.

The inverse of the discrete derivative is the summation. Explicitly, let a_n be a sequence and $f(n$ a function such that

$$\Delta a_n = f(n).$$

Then

$$\sum_{k=m}^{n-1} \Delta a_k = a_{m+1} - a_m + a_{m+2} - a_{m+1} + \cdots + a_{n-1+1} - a_{n-1} = a_n - a_m = \sum_{k=m}^{n-1} f(k).$$

Or

$$a_n = a_m + \sum_{k=m}^{n-1} f(k).$$

3 Integrating factor

3.1 General method

Having introduced the discrete derivative, let's move on to explore a way to solve a linear "first order" difference equation. We will take inspiration from differential equations and attempt to develop a method to solve difference equations using an integrating factor. Recall from differential equations that a linear first order differential equation can be written as follows:

$$y'(x) + p(x)y(x) = q(x),$$

where y is the dependent variable and p and q are known functions that essentially parameterize the differential equation. We multiply this equation by an "integrating factor" $u(x)$ such that

$$y'(x)u(x) + p(x)y(x)u(x) = \frac{\mathrm{d}}{\mathrm{d}x}(y(x)u(x)).$$

This integrating factor turns out to be

$$u(x) = \exp \int p(x)\, dx,$$

where the constant of integration can just be ignored. From here, we can solve the differential equation pretty easily.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(y(x)u(x)) &= q(x)u(x)\\ y(x)u(x) &= \int q(x)u(x)\,dx\\ y(x) &= \frac{1}{u(x)}\int q(x)u(x)\,dx.\end{aligned} \tag{5}$$

Knowing this, we will try to create an integrating factor procedure that will work similarly for difference equations. Let's start with a simple example and try to guess the integrating factor. Say

$$a_{n+1} + (n+1)a_n = 0 \tag{6}$$

for some sequence a_n for $n \geq 0$. We want to multiply this through by some function of n so that it becomes of the form

$$\Delta b_n = 0.$$

Before we try to multiply through by something, let's guess b_n in such a way that Eq. 6 pops out. Say $b_n = \frac{a_n}{n!}$. Then

$$\begin{aligned}\Delta b_n = b_{n+1} - b_n &= \frac{a_{n+1}}{(n+1)!} - \frac{a_n}{n!}\\ &= \frac{a_{n+1} - (n+1)a_n}{(n+1)!}.\end{aligned}$$

The numerator is super close to Eq. 6 but that negative sign needs to turn into a plus. So let's fix our guess to $b_n = \frac{(-1)^n a_n}{n!}$. Then, after the dust settles,

$$\Delta b_n = \frac{(-1)^{n+1}}{(n+1)!}(a_{n+1} + (n+1)a_n) = \frac{(-1)^{n+1}}{(n+1)!}(\text{Eq.}6).$$

This means that the integrating factor for this equation is

$$u(n) = \frac{(-1)^{n+1}}{(n+1)!}.$$

Notice that $b_n = u(n-1)a_n$, so the integrating factor interestingly did not have the same input as it had in the equation. This is a takeaway we will use to solve the general case. The reason we don't encounter this problem in continuous differential equations is that the input is not shifted by one unit, but by a number h that approaches zero in the derivative's limit. We can now solve the difference equation.

$$\begin{aligned}\Delta(u(n-1)a_n) &= 0\\ \sum_{k=0}^{n-1}\Delta(u(n-1)a_n) = u(n-1)a_n - a_0 &= 0\\ a_n = \frac{a_0}{u(n-1)} &= (-1)^n n! a_0.\end{aligned}$$

You probably could have guessed this result just by looking at the original equation. By now, we can solve our motivation problem using the recursion from Eq. 3 and guessing an integrating factor. But let's go ahead and solve the general case. Let's solve the following difference equation that parallels a general first order linear differential equation:

$$a_{n+1} + p(n)a_n = q(n).$$

Let's multiply the equation through by a function $u(n)$, which we do not know explicitly yet.

$$a_{n+1}u(n) + p(n)a_n u(n) = q(n)u(n). \tag{7}$$

] We want the left hand side to become Δb_n, so let's use our takeaway from the example problem to see how we can do this. Let $b_n = u(n-1)a_n$. Then

$$\Delta b_n = u(n)a_{n+1} - u(n-1)a_n,$$

which we demand to be equal to the left-hand side of Eq. 7, so

$$u(n)a_{n+1} - u(n-1)a_n = a_{n+1}u(n) + p(n)a_n u(n)$$

for all n for any sequence a_n. Since a_n is not zero in general, we have

$$\begin{aligned} -u(n-1) &= p(n)u(n) \\ u(n) &= -\frac{u(n-1)}{p(n)} \\ u(n) &= u(0)\prod_{k=1}^{n} \frac{-1}{p(n)} \\ &= (-1)^n u(0) \prod_{k=1}^{n} \frac{1}{p(n)}. \end{aligned}$$

$u(0)$ is a constant factor out front, so we can just ignore it as we did with the constant of integration from the integrating factor in the continuous differential equation. Then

$$u(n) = \frac{(-1)^n}{\prod_{k=1}^{n} p(n)}. \tag{8}$$

Now that $u(n)$ is known, we can solve the equation.

$$\begin{aligned} \Delta(u(n-1)a_n) &= q(n)u(n) \\ u(n-1)a_n - u(-1)a_0 &= \sum_{k=0}^{n-1} q(k)u(k) \\ a_n = a_h(n) &+ \frac{1}{u(n-1)} \sum_{k=0}^{n-1} q(k)u(k), \end{aligned}$$

where $a_h(n) = \frac{u(-1)a_0}{u(n-1)}$ is the solution to the homogeneous difference equation–the one in which $q(n) = 0$. The seed for a_n is included in $a_h(n)$. Notice the similarity of this equation to Eq. 5. One difference arises from the fact that we used bounds of summation in this derivation whereas in Eq. 5 we left the integral indefinite. Now, with our general solution to first order linear difference equations, let's solve our motivation problem.

3.2 The final answer via integrating factor

We are now ready to solve Eq. 3, reproduced below, for c_m.

$$c_{m+1} + (2m+2)(2m+1)c_m = x^{2m+2}a_0 - (2m+2)x^{2m+1}b_0.$$

In this case, $p(n) = (2m+2)(2m+1)$, so the integrating factor can be calculated as follows:

$$\begin{aligned} u(m) &= \frac{(-1)^m}{\prod_{k=1}^{m}(2k+2)(2k+1)} \\ &= \frac{(-1)^m}{(2m+2)(2m+1)(2m)(2m-1)\cdots(4)(3)} \\ &= \frac{2(-1)^m}{(2m+2)!} = \frac{2(-1)^m}{[2(m+1)]!}. \end{aligned}$$

Then, by our general procedure, Eq. 3 becomes

$$\Delta\left(\frac{2(-1)^{m-1}c_m}{(2m)!}\right) = \frac{2(-1)^m x^{2(m+1)}a_0}{[2(m+1)]!} - \frac{2(-1)^m x^{2m+1}b_0}{(2m+1)!}.$$

I emphasize factoring $2m+2$ into $2(m+1)$ in order to make obvious a shift of indices we will make. Now, we will sum both sides to cancel the difference operator, yielding

$$\frac{2(-1)^{m-1}c_m}{(2m)!} - (-2)c_0 = \sum_{k=0}^{m-1}\frac{2(-1)^k x^{2(k+1)}a_0}{[2(k+1)]!} - \sum_{k=0}^{m-1}\frac{2(-1)^k x^{2k+1}b_0}{(2k+1)!}. \quad (9)$$

Recall that -1 is not in the domain of the integrating factor, which is why $u(-1)$ disappears. The factor of two throughout the equation will cancel out. Next, we will simplify the right-hand side after cancelling out the factors of two, shifting indices to get

$$\begin{aligned} &-a_0\sum_{k=1}^{m}\frac{(-1)^k x^{2k}}{(2k)!} + b_0\sum_{k=1}^{m}\frac{(-1)^k x^{2k-1}}{(2k-1)!} \\ &= -a_0\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^m\frac{x^{2m}}{(2m)!}\right) + b_0\left(-x + \frac{x^3}{3!} - \cdots + (-1)^m\frac{x^{2m-1}}{(2m-1)!}\right). \end{aligned}$$

Notice that the bracketed polynomials resemble truncated Taylor series of $\cos x$ and $\sin x$ around $x = 0$. Therefore, the above equation becomes

$$-a_0(\cos(x, 2m) - 1) + b_0(-\sin(x, 2m-1)) = a_0(1 - \cos(x, 2m)) - b_0\sin(x, 2m-1),$$

where I added a second input to the cosine and sine functions to signify that they are Taylor series approximations around $x = 0$ whose highest power of x is x to the power of the second input. Recall that a_0 and b_0 were functions of x that were seeds to the original recursion in I_n, and

$$a_0 = \sin x; b_0 = -\cos x.$$

The right-hand side of Eq. 9 finally becomes

$$\sin x(1 - \cos(x, 2m)) + \cos x \sin(x, 2m - 1).$$

Next, since $c_m = a_{2m}$, $c_0 = a_0 = \sin x$, Eq. 9 becomes

$$\begin{aligned}
\frac{(-1)^{m-1}c_m}{(2m)!} + \sin x &= \sin x(1 - \cos(x, 2m)) + \cos x \sin(x, 2m - 1) \\
\frac{(-1)^m c_m}{(2m)!} &= \sin x \cos(x, 2m) - \cos x \sin(x, 2m - 1) \\
c_m &= (-1)^m (2m)!(\sin x \cos(x, 2m) - \cos x \sin(x, 2m - 1)) \\
a_{2m} &= (-1)^m (2m)!(\sin x \cos(x, 2m) - \cos x \sin(x, 2m - 1)).
\end{aligned}$$

So we get our final answer for the even integers as follows:

$$\int x^{2m} \cos x \, dx = (-1)^m (2m)!(\sin x \cos(x, 2m) - \cos x \sin(x, 2m - 1)) + C. \quad (10)$$

The final answer for odd integers and for the sine function can be found in a similar manner, resulting in a "closed form" for I_n. Unfortunately, our answer is not truly a closed form since it relies on a definition I made using the Taylor polynomials of cosine and sine.

4 Characteristic equation

Recall from continuous differential equations that you can find the homogeneous solution of a linear differential equation of arbitrary order with constant coefficients by guessing the ansatz $y = Ae^{rx}$. However, this method finds the general solution only for homogeneous equations with constant coefficients; for this reason, the motivation problem cannot be solved with the characteristic equation method alone.

This happens via the following. Say you have a linear differential equation whose homogeneous form is given by

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \quad (11)$$

Guess the ansatz $y = Ae^{rx}$, then for any k,

$$y^{(k)} = r^k A e^{rx} = r^k y.$$

Plugging this in to Eq. 11,

$$a_n r^n y + a_{n-1} r^{n-1} y + \cdots + a_1 r y + a_0 y = 0,$$

which applies for all x, so we can divide through by y assuming that y is not identically zero. This gives us a polynomial in r known as the characteristic equation for this differential equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

There are multiple roots of this equation and each root corresponds to a solution to the differential equation. The n possible roots correspond to the n possible exponentials that can be superposed to find the general solution to the homogeneous differential equation. If there are repeat roots, i.e. $r_i = r_j$ for some $i \neq j$, then a power of x must be multiplied onto the ith exponential to make the ith solution distinct from the jth, where the power of x depends on how many repeats of that root r_i there are.

Now, say we have a homogeneous linear difference equation of general order with constant coefficients. The dependent variable is the sequence a, and the coefficients are denoted by c_k. Then the difference equation takes the form

$$c_m a_{n+m} + c_{m-1} a_{n+m-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0. \tag{12}$$

When we guessed the ansatz $y = Ae^{rx}$, we were basically guessing $y = AR^x$ ($R = e^r$), but in a more convenient way since it's easier to work with e when taking derivatives. It turns out we can guess the same ansatz for difference equations; i.e., we can guess $a_n = Ar^n$. Then,

$$a_{n+k} = Ar^{n+k} = Ar^n r^k = a_n r^k$$

for any k. Plugging this in to Eq. 12,

$$c_m a_n r^m + c_{m-1} a_n r^{m-1} + \cdots + c_1 a_n r + c_0 a_n = 0.$$

Dividing through by a_n assuming that a_n is not identically zero, we get the following characteristic equation for our difference equation:

$$c_m r^m + c_{m-1} r^{m-1} + \cdots + c_1 r + c_0 = 0.$$

This can be solved to yield our (potentially complex) solutions $a_n = Ar^n$. Repeated roots work the same way as in differential equations. For example, consider the difference equation

$$a_{n+2} - 4a_{n+1} + 4a_n = 0.$$

Its characteristic equation is

$$r^2 - 4r + 4 = 0,$$

which has repeated roots $r = 2$ and $r = 2$. With the expectation that a_n will include a factor of $r^n = 2^n$, we can do a substitution where $b_n = \frac{a^n}{2^n}$ so $a_n = 2^n b_n$. Then the difference equation becomes

$$\begin{aligned}
4b_{n+2}2^n - 8b_{n+1}2^n + 4b_n 2^n &= 0 \\
b_{n+2} - 2b_{n+1} + b_n &= 0 \\
b_{n+2} - b_{n+1} - (b_{n+1} - b_n) &= 0 \\
\Delta b_{n+1} - \Delta b_n &= 0 \\
\Delta(\Delta b_n) &= 0.
\end{aligned}$$

This can be summed twice to yield

$$b_n = An + B$$

as the general solution for b_n, where A and B are constants of summation. Therefore, the general solution for a_n is

$$a_n = An2^n + B2^n.$$

This is what we expected; there was one extra $r = 2$, so one of the solutions had to be multiplied by n^1.

5 Discrete Laplace transform